Module-2 : $\boxed{\text{Vector Calculus}}$

1) If $\vec{A} = xz^3 i - 2x^2y^2 j + 2yz^4 k$, find $\nabla \cdot \vec{A}$, $\nabla \times \vec{A}$, $\nabla \cdot (\nabla \times \vec{A})$ at $(1,-1,1)$

$\Rightarrow$ • $\text{div}.\vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left((xz^3) i - (2x^2y^2) j + (2yz^4) k\right)$

$$= \frac{\partial}{\partial x}(xz^3) - \frac{\partial}{\partial y}(2x^2y^2) + \frac{\partial}{\partial z}(2yz^4).$$

$$= z^3 - 4x^2y + 8yz^3 \quad | \text{ at } (1,-1,1) \ \text{div}.\vec{A} = -3$$

• $\text{Curl}.\vec{A} = \nabla \times \vec{A}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2y^2 & 2yz^4 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(2yz^4) + \frac{\partial}{\partial z}(2x^2y^2)\right] - j\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(-2x^2y^2) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= i\left[2z^4\right] - j\left[-3xz^2\right] + k\left[-4xy^2\right]$$

$$= 2z^4 i + 3xz^2 j - 4xy^2 k.$$

$$= 2(1)^4 i + 3(1)(1)^2 j - 4(1)(-1)^2 k$$

$$= 2i + 3j - 4k$$

$\nabla\cdot(\nabla\times\vec{A}) = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left(2z^4 i + 3xz^2 j - 4xy^2 k\right)$

$$= \frac{\partial}{\partial x}(2z^4) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-4xy^2)$$

$$= 0 + 0 + 0 = 0$$

) If $\vec{F} = (x+y+1)i + j - (x+y)k$, Show that $\vec{F}\cdot \text{curl}\,\vec{F} = 0$.

$\Rightarrow$ $\text{curl}\,\vec{F} = \nabla\times\vec{F}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & (1) & (-x-y) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right] - j\left[\frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial z}(x+y+1)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right]$$

$$= i(-1-0) - j(-1-0) + k(0-1)$$

$$= -i + j - k.$$

$\therefore \vec{F} \cdot \text{curl}\,\vec{F} = \left[(x+y+1)i + j - (x+y)k\right] \cdot (-i+j-k)$

$= (x+y+1)(-1) + (1)(1) + (x+y)(1)$

$= \cancel{-x} \cancel{-y} \cancel{-1} \cancel{+1} \cancel{+x} \cancel{+y}$

$= 0$

3) If $\vec{F} = \nabla(xy^3z^2)$, find div.$\vec{F}$ and curl $\vec{F}$ at $(1,-1,1)$.

$\Rightarrow$ Let $\phi = xy^3z^2$.

and $\vec{F} = \nabla\phi = \frac{\partial}{\partial x}\phi i + \frac{\partial \phi}{\partial y} j \; \frac{\partial \phi}{\partial z} k$.

$\vec{F} = y^3z^2 i + 3xy^2z^2 j + 2xy^3 z k$.

• div.$\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (y^3z^2 i + 3xy^2z^2 j + 2xy^3z k)$

$= \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$

$= 0 + 6xyz^2 + 2xy^3$

at $(1,-1,1)$, div.$\vec{F} = 6(1)(-1)(1)^2 + 2(1)(-1)^3$

div.$\vec{F} = -6-2 = -8$

$\cdot$ curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$

$$\nabla \times \vec{F} = i(6xy^2z - 6xy^2z) - j(2y^3z - 2y^3z)$$
$$+ k(3y^2z^2 - 3y^2z^2)$$
$$= i[6(1)(-1)^2(1) - 6(1)(-1)^2(1)] - j[2(-1)^3(1) - 2(-1)^3(1)] + k[3(-1)^2(1)^2 - 3(-1)^2(1)^2]$$

$$\nabla \times \vec{F} = \vec{0}$$

4) Show that $\vec{F} = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ is a conservative force field. [irrotational]. also find its scalar potential

$\Rightarrow \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy^2+yz) & (2x^2y+xz+2yz^2) & (2y^2z+xy) \end{vmatrix}$

$$= i(4yz + x - x - 4yz) - j(y - y) + k(4xy + z - 4xy - z)$$

$$\nabla \times \vec{F} = 0 + 0 + 0 = \vec{0}.$$

$\therefore \vec{F}$ is conservative.

Nao, $\nabla \phi = \vec{F}$.

$$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (2xy^2 + yz) i + (2x^2y + xz + 2yz^2) j + (2y^2z + xy) k.$$

$\Rightarrow$ 1) $\frac{\partial \phi}{\partial x} = 2xy^2 + yz.$

$$\int \frac{\partial \phi_1}{\partial x} dx = \int 2xy^2 + yz . dx + f(y,z).$$

$$\therefore \phi_1 = x^2y^2 + yzx + f(y,z) - ①$$

2) $$\int \frac{\partial \phi_2}{\partial y} dy = \int 2x^2y + xz + 2yz^2 dy + g(x,z)$$

$$\therefore \phi_2 = x^2y^2 + xzy + y^2z^2 + g(x,z) - ②$$

3) $$\int \frac{\partial \phi_3}{\partial z} dz = \int 2y^2z + xz . dz + h(x,y)$$

$$\phi_3 = y^2z^2 + xyz + h(x,y) - ③$$

To get unique value for ϕ,

Substitute -

$f(y,z) = y^2z^2$

$g(x,z) = 0$

$h(x,y) = x^2y^2$.

$\therefore$ Scalar potential, $\boxed{\phi = x^2y^2 + xyz + y^2z^2}$.

5) Find the value of constants a and b such that $\vec{F} = (axy + z^3)i + (3x^2 - z)j + (bxz^3 - y)k$. is irrotational. also find a scalar function ϕ such that $\vec{F} = \nabla\phi$.

$\Rightarrow$ Curl $\vec{F} = 0$

$\nabla \times \vec{F} = 0$

$$\therefore \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy + z^3) & (3x^2 - z) & (bxz^3 - y) \end{vmatrix} = \vec{0}$$

$$\Rightarrow i\left[\frac{\partial}{\partial y}(bxz^3 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - j\left[\frac{\partial}{\partial x}(bxz^3 - y) - \frac{\partial}{\partial z}(axy + z^3)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(axy + z^3)\right] = 0.$$

$$\Rightarrow i\cancelto{0}{(-1+1)} - j(bz^3 - 3z^2) + k(6x - ax) = 0$$

$\therefore \quad -bz^2 - 3z^2 = 0$

$bz^2 - 3z^2 = 0$

$bz^2 = 3z^2$

$b = \frac{3\cancel{z^2}}{\cancel{z^2}}$

$\therefore \boxed{b=3}$

$6x - ax = 0$

$ax = 6x$

$a = \frac{6\cancel{x}}{\cancel{x}}$

$\therefore \boxed{a=6}$

Substituting $a=6$ and $b=3$ in $\vec{F}$:

$\vec{F} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k.$

$\vec{F} = \nabla\phi$ or $\nabla\phi = \vec{F}$,

$$\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k.$$

1) $\int \frac{\partial\phi}{\partial x} dx = \int 6xy + z^3 \, dx + f(y,z)$

$\therefore \quad \phi = 3x^2y + z^3x + f(y,z) \quad — ①$

2) $\int \frac{\partial\phi}{\partial y} dy = \int 3x^2 - z \, dy + g(x,z)$

$\therefore \quad \phi = 3x^2y - zy + g(x,z) \quad — ②$

3) $\int \frac{\partial\phi}{\partial z} dz = \int 3xz^2 - y \, dz + h(x,y)$

$\therefore \phi = xz^3 - yz + h(x,y) \quad — ③$

To get unique expression for ϕ, let us choose

$$f(y,z) = -zy$$

$$g(x,z) = z^3 x$$

$$h(x,y) = 3x^2y.$$

$$\therefore \phi = 3x^2y + xz^3 - yz$$

Show that

6) $\vec{F} = 2xyz^2 i + (x^2z^2 + z\cos yz) j + (2x^2yz + y\cos yz) k$ is a potential field. & hence find its scalar potential.

$\Rightarrow$ $\text{curl}\,\vec{F} = \nabla \times \vec{F}$ (1)

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xyz^2) & (x^2z^2 + z\cos yz) & (2x^2yz + y\cos yz) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(x^2z^2 + z\cos yz)\right]$$

$$- j\left[\frac{\partial}{\partial x}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(2xyz^2)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(x^2z^2 + z\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right]$$

$$= i\left[\cancel{2x^2z} - \cancel{2y\sin yz} + \cancel{\cos yz} - \cancel{2x^2z} - \cancel{\cos yz} + \cancel{2y\sin yz}\right]$$

$-j\,(4xy\cancel{z}-\cancel{4}xyz) \quad +k\,(\cancel{2xz^2}-\cancel{2xz^2})$

$= 0i + 0j + 0k \rightarrow$ ~~potential field~~

$= \vec{0}$

$\therefore \vec{F}$ is a potential field.

$\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = 2xyz^2 i + (x^2z^2 + z\cos yz)j + (2x^2yz + y\cos yz)k.$$

1) $\int \frac{\partial\phi}{\partial x}dx = \int 2xyz^2\,dx + f(y,z)$

$\therefore \phi = x^2yz^2 + f(y,z) \quad —①$

2) $\int \frac{\partial\phi}{\partial y}dy = \int(x^2z^2 + z\cos yz)\,dy + g(x,z)$

$\therefore \phi = x^2z^2y + \sin y + y(x,z) \quad —②$

3) $\int \frac{\partial\phi}{\partial z}dz = \int(2x^2yz + y\cos yz)\,dz + h(x,y).$

$\therefore \phi = x^2\cancel{y}z^2 + \sin z + h(x,y) \quad —③$

~~to~~ get unique value for ϕ, from ①, ②, ③ we get,

$$\boxed{\phi = x^2yz^2 + \sin y + \sin z}$$

1) Express $\vec{A} = 2x\hat{i} - 3y^2\hat{j} + xz\hat{k}$ in the cylindrical polar coordinate system.

$\Rightarrow$ Let $\vec{A} = A_1\hat{e}_\rho + A_2\hat{e}_\phi + A_3\hat{e}_z$ — ①

where, A_1, A_2 and A_3 has to be determined.

$$\left.\begin{aligned} \vec{A}\cdot\hat{e}_\rho &= A_1 \\ \vec{A}\cdot\hat{e}_\phi &= A_2 \\ \vec{A}\cdot\hat{e}_z &= A_3 \end{aligned}\right\} \quad -\ ②$$

WKT, $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

$\vec{A} = 2x\hat{i} - 3y^2\hat{j} + xz\hat{k}$.

$\Rightarrow \vec{A} = 2\rho\cos\phi\,\hat{i} - 3\rho^2\sin^2\phi\,\hat{j} + \cancel{xzk}\,\rho\cos\phi\, z\,\hat{k}$. — ③

also, $\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$.

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$.

$\hat{e}_z = \hat{k}$.

$A_1 = \vec{A}\cdot\hat{e}_\rho = 2\rho\cos^2\phi - 3\rho^2\sin^3\phi$ [illegible]

$A_2 = \vec{A}\cdot\hat{e}_\phi = -2\rho\sin\phi\cos\phi - 3\rho^2\sin^2\phi\cos\phi$.

$A_3 = \vec{A}\cdot e_z =$ [illegible] $\rho\cos\phi\, z$.

Thus, $\vec{A} = A_1\hat{e}_\rho + A_2\hat{e}_\phi + A_3\hat{e}_z$ becomes,

$\vec{A} = (2\rho\cos^2\phi - 3\rho^2\sin^3\phi)\hat{e}_\rho - (2\rho\sin\phi\cos\phi - 3\rho^2\sin^2\phi\cos\phi)\hat{e}_\phi + (\rho\cos\phi z)\hat{e}_z$