16/09/2024.

# ROB - 501 - Home Work 3

1. $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

To determine which of the following are subsets of $\mathbb{R}^n$ with $n \geq 1$, are infact subspaces.

(a) $\{x \in \mathbb{R}^n : x_i \geq 0, i = 1, ---, n\}$;

Not a Subspace

Reason 1:- Not closed under Addition if two -ve numbers are added $x_i \leq 0$

Example:-

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$x_1 = -5 \Rightarrow x_1 \leq 0$.

$\therefore$ not closed under addition.

Reason 2:- Not closed under multiplication

Example:- if multiplied by a constant $-1$

(b) $\{x \in \mathbb{R}^n : x_1 = 0\}$

Is a Sub-Space

Reason's 1:- Zero Vector exists.

closed under vector Addition

$x_1 = 0$ ; $y_1 = 0$ $(x+y)_1 = 0$

$\alpha x_1 = 0 \Rightarrow$ closed under Scalar Multiplication.

(c) $\{x \in \mathbb{R}^n : x_1 x_2 = 0, \quad n \geq 2\}$:

Not a Subspace.

Reason:-

Not closed under Vector Addition.

Example

$$x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \quad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin X$$

$\therefore$ it is not a subspace of $X$.

(d) $\{x \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 0\}$

Is a Subspace

Reasons:-

Zero vector exists :- $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Bounded under Vector Addition

$$\alpha \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Bounded under ~~vector~~ scalar Multiplication.

(e) $\{x \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 1\}$;

Not a Subspace

Reason :- The zero vector does not belong to $\mathbb{R}^n$ condition $x_1 + x_2 + \cdots + x_n = 1$ does not satisfy. as $0 + 0 + \cdots + 0 \neq 1$

(F) $\{x \in \mathbb{R}^n : Ax = b, A \neq 0, b \neq 0\}$.

Not a Subspace

Reason :- The zero vector does not belong to $\mathbb{R}^n$ $b = 0$ for zero to belong to $\mathbb{R}^n$

(2) P.T $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$

proof :-
A vector $v$ in span $(S_1 \cup S_2)$ can be expressed as a linear combination of vectors from $S_1 \cup S_2$ :-

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n + b_1v_1 + b_2v_2 + \cdots + b_nv_n$$

where $u_i \in S_1$ & $v_j \in S_2$.

we can re-write the linear combination as

$$v = \underbrace{(a_1u_1 + a_2u_2 + \cdots + a_nu_n)}_{\text{is in span}(S_1)} + \underbrace{(b_1v_1 + b_2v_2 + \cdots + b_nv_n)}_{\text{is in span}(S_2)}$$

$$\Rightarrow v \in \text{Span}(S_1) + \text{Span}(S_2)$$

$\therefore$ Every vector $v \in \text{span}(S_1 \cup S_2)$ can be expressed as span $(S_1 \cup S_2)$

$\Rightarrow \text{Span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) + \text{Span}(S_2)$

$\text{ill}^{y}$

Let $v_1$ be a vector in the span($S_1$). $V_1$ can be represented as the linear combination of vectors from $S_1$ as :-

$$v_1 = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

where $u_i \in S_1$

Similarly,

let $v_2$ be a vector in the span($S_2$). $V_2$ can be represented as the linear combination of vectors from $S_2$ as :-

$$v_2 = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

where $v_i \in S_2$

Sum of vectors.

$$v = v_1 + v_2 = (a_1u_1 + a_2u_2 + \cdots + a_nu_n) + (b_1v_1 + b_2v_2 + \cdots + b_nv_n)$$

Since $u_i$ is in $S_1$ & $v_i$ is in $S_2$, this can be expressed as a linear combination of vectors from $S_1 \cup S_2$ proving the inclusion $\text{Span}(S_1) + \text{Span}(S_2) \subseteq \text{Span}(S_1 \cup S_2)$

$\therefore \text{span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$

(3) (a). $\left\{ \begin{array}{c} v^1 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{array}, \begin{array}{c} v^2 \\ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} v^3 \\ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \end{array} \right\}$;

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 3 & 0 & 9 \end{bmatrix} = 1\begin{vmatrix} 1 & 5 \\ 0 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}$$

$$1(9) - 2(18-15) + 1(-3)$$

$$= 9 - 2(3) + (-3)$$

$$= 9 - 6 - 3 = 0$$

$\therefore$ The det (A) i.e the column matrix is zero.

$\Rightarrow$ The above vectors are linearly dependent.

$$a_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 0 \quad \text{--- } (0)$$

$$a_1 + 2a_2 + a_3 = 0 \quad \text{--- } (1)$$
$$2a_1 + a_2 + 5a_3 = 0 \quad \text{--- } (2)$$
$$3a_1 + 0 + 9a_3 = 0 \quad \text{--- } (3)$$

$\Rightarrow 3a_1 = -9a_3$

$\boxed{a_1 = -3a_3}$

$\Rightarrow a_3 = -\frac{1}{3}a_1$

$-5a_1 - 10a_2 - \cancel{5a_3} = 0$
$+2a_1 + a_2 + \cancel{5a_3} = 0$

$-3a_1 - 9a_2 = 0$

$-3a_1 = 9a_2$

$a_1 = -3a_2 \quad \Rightarrow a_2 = -\frac{1}{3}a_1$

$a_1 = 1$; $a_2 = -\frac{1}{3}$; $a_3 = -\frac{1}{3}$

3(i) continued $eq^n$ (0) after substituting the values of $a_1, a_2, a_3$.
is as follows

~~$a_1 = 1$ ; $a_2 = \frac{1}{3}$ ; $a_3 = -3$~~ $a_1 = 1$ ; $a_2 = \frac{-1}{3}$ ; $a_3 = -\frac{1}{3}$

$a_1 v^1 + a_2 v^2 + a_3 v^3 = 0$ $\Rightarrow$ $a_1 v^1 = -a_2 v^2 - a_3 v^3$

$$1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 0$$

$$v^1 = +\frac{1}{3} \begin{bmatrix} v^2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} v^3 \end{bmatrix}$$

$$\boxed{v^1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}} \quad \text{--- } (1)$$

on simplifying eq (1) we indeed get $v^1$

$\boxed{\therefore v^1 \text{ is represented in terms of } v^1 \ \& \ v^2}$

$$\boxed{\text{hence } v^1 = \frac{1}{3} \begin{bmatrix} v^2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} v^3 \end{bmatrix}}$$

□

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

($v^1$, $v^2$, $v^3$, $v^4$)

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 4 & 0 & 1 \\ 3 & 5 & 6 & 1 \end{bmatrix}$$

det(A) does not exist as A is not a square matrix.

:. Method 2:-

take the linear combination of the vectors.

$a_1 v^1 + a_2 v^2 + a_3 v^3 + a_4 v^4 = 0$ — (1)

$= a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$a_1 + a_4 = 0$

$2a_1 + 4a_2 + a_4 = 0$

$3a_1 + 5a_2 + 6a_3 + a_4 = 0$

$a_1 = -a_4$

$\Rightarrow a_4 = -a_1$

$a_1 = -4a_2$

$a_2 = -\frac{1}{4} a_1$

$3a_1 + 5\left(-\frac{1}{4} a_1\right) + 6(a_3) - a_1 = 0$

$2a_1 - \frac{5}{4} a_1 + 6a_3 = 0$

$= \frac{3a_1}{4} + 6a_3 = 0$

$a_1 = \frac{(-\cancel{6}^{2} a_3) 4}{\cancel{3}} = -8a_3 \Rightarrow a_3 = -\frac{1}{8} a_1$

$\therefore \boxed{a_1 = 1, \ a_2 = -\frac{1}{4}, \ a_3 = -\frac{1}{8}, \ a_4 = -1}$

3(b) cont. $eq^n$ (0) after substituting the values of $a_1, a_2, a_3, a_4$ is as follows

$$a_1 = 1 \; ; \; a_2 = -\frac{1}{4} \; ; \; a_3 = -\frac{1}{8} \; , \; a_4 = -1$$

$$a_1 v^1 + a_2 v^2 + a_3 v^3 + a_4 v^4 = 0$$

$$a_1 v^1 = -a_2 v^2 - a_3 v^3 - a_4 v^4 \Rightarrow v^1 = \frac{1}{4}\left[v^2\right] + \frac{1}{8}\left[v^3\right] + v^4$$

$$v^1 = +\frac{1}{4}\begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad --- \; (1)$$

on simplifying eq (1) we get

$$v^1 = \begin{bmatrix} 0 \\ 1 \\ \frac{5}{4} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{6}{8} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v^1 = \begin{bmatrix} 0 + 0 + 1 \\ 1 + 0 + 1 \\ \frac{5}{4} + \frac{6}{8} + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \frac{24}{8} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which is the same value as $v^1$

hence $\boxed{v^1 = \frac{1}{4}\left[v^2\right] + \frac{1}{8}\left[v^3\right] + v^4}$ □

(4) To determine $\left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \right\} \subset \mathbb{R}^{2\times 2}$

$$a_1 v^1 + a_2 v^2 + a_3 v^3 = 0 \quad \text{—(0)}$$

$$\Rightarrow a_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & 2a_1 \\ 2a_1 & a_1 \end{bmatrix} + \begin{bmatrix} 2a_2 & a_2 \\ a_2 & a_2 \end{bmatrix} + \begin{bmatrix} 4a_3 & -a_3 \\ -a_3 & a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$a_1 + 2a_2 + 4a_3 = 0$ — (1)

$2a_1 + a_2 \cancel{+} - a_3 = 0$ — (2)

$2a_1 + a_2 - a_3 = 0$ — (3)

$a_1 + a_2 + a_3 = 0$ — (4)

(1),(2) $\rightarrow 9a_1 + 6a_2 = 0 \rightarrow$ (6)

(3),(4) $\rightarrow$

$3a_1 + 2a_2 = 0$ — (5)

~~$-3(3a_1 + 2a_2 = 0)$~~

~~$9a_1 + 6a_2 = 0$~~

~~$\Rightarrow -9a_1 - 6a_2 = 0$~~

~~$9a_1 + 6a_2 = 0$~~

$$\boxed{a_1 = -\frac{2}{3} a_2} \longrightarrow (7)$$

$$-\frac{2}{3}a_2 + a_2 + a_3 = 0$$

$$\frac{2}{3} - 1 = -\frac{1}{3}$$

$$a_3 = \frac{2}{3}a_2 - a_2 = -\frac{a_2}{3}$$

$$\Rightarrow \boxed{a_3 = -\frac{a_2}{3}} \longrightarrow (8)$$

if $a_2 = 1$
$a_1 = \frac{-2}{3}$
$a_3 = \frac{-1}{3}$

$\therefore a_1 \neq a_2 \neq a_3 \neq 0$ infact $a_1 = \frac{-2}{3}$, $a_2 = 1$ & $a_3 = \frac{-1}{3}$.

~~$\therefore$ eq$^n$ (o)~~ are the sol$^n$ that make eq (o) $= 0$
$\therefore$ The vectors are linearly dependent to one another.

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5. $(X, F)$ is a vector space & $S \subset X$ a subset (not necessarily a subspace)

claim :- if $Y$ is a subspace of $X$ & $S \subset Y$, then $\text{Span}\{S\} \subset Y$.

According to the claim, $Y$ is a subspace of $X$

$\Rightarrow$ $Y$ is closed under addition & scalar Multiplication.

$\text{Span}\{S\}$ is the set of linear combinations of vectors in $S$,
In other words, any vector $v$ in $\text{Span}\{S\}$ can be written as

$v = a_1 v^1 + a_2 v^2 + a_3 v^3 + \cdots + a_n v^n$

where $a_1, a_2, a_3, \ldots a_n \in F$ &

~~$v_1, v_2, v_3 \ldots v_n$~~ $v^1, v^2, v^3, v^4 \ldots v^n \in S$

Since $S \subset Y$ then $v^1, v^2, v^3, v^4 \ldots v^n$ belong to $Y$

$Y$ is closed under Scalar - Multiplication, so ~~$a_1 v_1, a_2 v_2$~~
$a_1 v^1, a_2 v^2, \ldots, a_i v^i$ belongs to $Y$

$\rightarrow$ $Y$ is closed under vector addition, so the sum
$a_1 v^1 + a_2 v^2 + \cdots + a_n v^n$ is also in $Y$.

$\therefore$ So any vector $v$ in $\text{Span}\{S\}$ is in $Y$
i.e $\text{Span}\{S\} \subset Y$ $\square$

6. let $(X, F)$ be a vector space & $V$ and $W$ be Subspaces of $X$.
prove the following
(a) $V \cap W = \{0\}$
(b) for every $x \in V+W$, there exists a unique $v \in V$, & $w \in W$
S.T $x = v + w$

$V + W := \{v+w \mid v \in V, w \in W\}$.

proof:- if $X = V \oplus W$, then it implies that $X = V + W$
suppose that $x \in V \cap W$ then there exists $v \in V$ S.T. ~~$V \neq 0$~~
$x = v + 0$ & on the other hand there is $w \in W$ S.T
$x = 0 + w$ $\therefore v = 0, w = 0$ so $V \cap W = \{0\}$.

Since $X = V + W$ for every $v \in V$. there exists ~~$v \in W$~~
$v \in V$ & $w \in W$ S.T $x = v + w$. Suppose there exists
other vectors $\tilde{v} \in V$ & $\tilde{w} \in W$ S.T $x = \tilde{v} + \tilde{w}$

then $0 = (v - \tilde{v}) + (w - \tilde{w}) \iff (v - \tilde{v}) = -(w - \tilde{w})$
$\therefore (w - \tilde{w}) \in W$ & so $(w - \tilde{w}) \in V \cap W$, Since $V \cap W = \{0\}$
we then conclude that $v = \tilde{v} \Rightarrow w = \tilde{w}$ Then $X = V \oplus W$
This establishes the theorem $\square$.