```
The co-varience matrix E is :
    4.0000   -2.7500
   -2.7500    2.0000


K_cap is :
   -2.0000    1.0000
    1.5000   -0.5000


The Best Linear Unbiased Estimate (BLUE) for x is:
    0.6194
    0.4591
```

```
The co-varience matrix E is :
    0.0679   -0.0260
   -0.0260    0.1129


K_cap is :
   -0.1050    0.0525    0.1895
    0.1872    0.1564   -0.1313


The Best Linear Unbiased Estimate (BLUE) for x is:
   -1.4303
    1.8791
```

```
The co-varience matrix E is :
    0.0487    0.0054
    0.0054    0.0618


K_cap is :
   -0.0540    0.1046    0.1480   -0.0518
    0.1041    0.0715   -0.0637    0.0843


The Best Linear Unbiased Estimate (BLUE) for x is:
   -1.2201
    1.5368
```

2(a) $\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\Sigma = \left[\begin{array}{cc|c} 2 & 2 & 1 \\ 2 & 4 & 2 \\ \hline 1 & 2 & 2 \end{array}\right]$

$\Sigma_{11} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$; $\Sigma_{12} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Sigma_{21} = \begin{bmatrix} 1 & 2 \end{bmatrix}$; $\Sigma_{22} = [2]$

$\mu_{xy} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$; $\mu_z = [1]$

$\mu_{1|2} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2)$

$\Rightarrow \mu_{xy/z} = \mu_{xy} + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_z)$

$\mu_{xy/z} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{1}{2} \left(z - 1 \right)$

$= \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} (0.5z - 0.5)$

$= \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5z - 0.5 \\ 2(0.5z - 0.5) \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5z - 0.5 \\ 1z - 1 \end{bmatrix}$

$$\boxed{\mu_{xy/z} = \begin{bmatrix} 0.5z - 1.5 \\ z - 1 \end{bmatrix}}$$

$$\Sigma_{xy|z} := \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\left(\frac{1}{2}\right)\begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\left(\frac{1}{2}\right)$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 0.5 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Sigma_{xy|z} = \begin{bmatrix} 1.5 & 1 \\ 1 & 2 \end{bmatrix}$$

(b) $\Sigma_{11} = 1.5$; $\Sigma_{12} = 1$; $\Sigma_{21} = 1$; $\Sigma_{22} = 2$

$$\mu_x = \mu_x + \Sigma_{12}\Sigma_{22}^{-1}(y - \mu_y) =$$

$$\Sigma_x = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$\mu_x = 0.5z - 1.5 + 1\left(\frac{1}{2}\right)(y - z + 1) = 0.5y - 1$$

$$\Sigma_{x|yz} = 1.5 - 1 \cdot \frac{1}{2} \cdot 1 = 1$$

$$\Rightarrow \mu_{x|yz} = 0.5y - 1 \; ; \; \Sigma_{x|yz} = 1$$

(c) $x|_{y=y,\, z=z}$

$\Sigma_{11} = [2]$, $\Sigma_{12} = [2\ 1]$

$\Sigma_{22} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$ $\Sigma_{21} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\cancel{\mu_{x|y,z} = \mu_x + \Sigma_{12}\, \Sigma_{22}^{-1} \left(\begin{bmatrix} y \\ z \end{bmatrix} - \mu_{yz} \right)}$$

$$\cancel{= -1 + [2\ 1]\, \frac{1}{2} \left\{ \begin{bmatrix} y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$$

$$\cancel{= -1 + [2\ 1]\, \frac{1}{2} \left\{ \begin{bmatrix} y \\ z-1 \end{bmatrix} \right\}}$$

$$\cancel{= -1 + [1\ 0.5] \begin{bmatrix} y \\ z-1 \end{bmatrix}}$$

$$\Sigma_{x|y,z} = 2 - [2\ 1]\, \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 - [2\ 1]\, \frac{1}{4} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= 1.$$

$$\mu_{y|y,z} = \mu_x + \Sigma_{12}\, \Sigma_{22}^{-1} \left(\begin{bmatrix} y \\ z \end{bmatrix} - \mu_{yz} \right)$$

$$= -1 + [2\ 1]\, \frac{1}{4} \begin{bmatrix} 2y - 2z + 2 \\ -2y + 4z - 4 \end{bmatrix} = 0.5y - 1$$

(d) The results of b, c are identical

↳ It does not matter if the distribution conditioned first to z & then to y or to both z, y.

problem 3. $S = \{y_1, \dots y_n\}$ → linearly independent, $M = \text{span}\{S\}$

$\hat{x}_k = \underset{m \in M}{\text{argmin}} \; \|x - m\|$

a) $y_{k+1} \perp M_k$ $\quad P_1: \exists \beta$ s.t $\hat{x}_{k+1} = \hat{x}_k + \beta y_{k+1}$

$x - \hat{x}_k \perp M$ by pre-projection theorm.

$x = \hat{x}_k + (x - \hat{x}_k) = \hat{x}_k + m^\perp$ where $m^\perp \in M_k^\perp$

$M_{k+1} = \text{Span}\{y_1, \dots y_n, y_{k+1}\}$ { after we get the new measurement $y_{k+1} \in M_k^\perp$

$y_{k+1} \perp M_k$; $y_{k+1} \in M_k^\perp$ then using gram - Schmit we can compute an orthoganal basis for $M^\perp$.

$B := \{y_{k+1}, v_1, v_2 \dots v_f\}$ $\Big| \Rightarrow m^\perp = \beta y_{k+1} + \beta_1 v_1 + \dots + \beta_f v_f$.

$x = \hat{x}_k + \beta y_{k+1} \in M_{k+1}$

$$\begin{cases} \{v_1, v_2, \dots v_f\} \perp \{y_1, y_2, \dots y_k\} \\ \{v_1, v_2, \dots v_f\} \perp y_{k+1} \end{cases}$$

Thus $\{v_1, v_2, \dots v_f\} \perp \text{Span}\{y_1, y_2, \dots, y_{k+1}\} = M_{k+1}$

$\Downarrow$

$m_{k+1}^\perp = \beta_1 v_1 + \dots + \beta_f v_f \in M_{k+1}^\perp$

$\Downarrow$

$x = m_{k+1} + m_{k+1}^\perp$ $\Big| \Rightarrow m_{k+1}^\perp = x - m_{k+1}$ $\Big| \Rightarrow (x - m_{k+1}) \perp M_{k+1}$

$x - \hat{x}_{k+1} \perp M_{k+1}$

$\cancel{\langle x - \hat{x}_{k+1} \rangle}$ $\langle x - \hat{x}_{k+1}, y_{k+1} \rangle = 0$

$\Rightarrow \langle \hat{x}_{k+1}, y_{k+1} \rangle = \langle x, y_{k+1} \rangle$

$\langle \hat{x}_k + \beta y_{k+1}, y_{k+1} \rangle = \langle x, y_{k+1} \rangle$

Since $y_{k+1} \perp M_k$; $\hat{x}_k \in M_k$: $\langle \hat{x}_k, y_{k+1} \rangle = 0$

$\Rightarrow \langle \hat{x}_k + \beta y_{k+1}, y_{k+1} \rangle = \beta \langle y_{k+1}, y_{k+1} \rangle = \langle x, y_{k+1} \rangle$

$$\Rightarrow \beta = \frac{\langle x, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle}$$

$$\hat{y}_{k+1|k} = \arg\min_{m \in M_k} \| y_{k+1} - m \|$$

$y_{k+1} - \hat{y}_{k+1|k} \perp M_k$; by pre-projection theorem.

$v = y_{k+1} - \hat{y}_{k+1|k}$, we get $v \perp \{y_1, y_2, \ldots y_k\}$

we already showed that

$\hat{x}_{k+1} = \hat{x}_k + \beta v$

$\beta = \frac{\langle x, v \rangle}{\langle v, v \rangle}$

$$\Rightarrow \begin{cases} \hat{x}_{k+1} = \hat{x}_k + \beta (y_{k+1} - \hat{y}_{k+1|k}) \\ \beta = \dfrac{\langle x, y_{k+1} - \hat{y}_{k+1|k} \rangle}{\langle y_{k+1} - y_{k+1|k}, y_{k+1} - y_{k+1|k} \rangle} \end{cases}$$

$M_{k+1} = \text{span}\{ y_1, y_2, \ldots y_k, y_{k+1} - \hat{y}_{k+1|k} \}$

$= M_{k+1}$.

$\hat{x}_{k+1} \in M_k$ must be the best estimator of x in M_{k+1}

since $\hat{y}_{k+1|k} \in M_k$; $\hat{y}_{k+1|k} \perp (y_{k+1} - \hat{y}_{k+1|k})$

$\hat{x}_{k+1} = \hat{x}_k + \beta (y_{k+1} - y_{k+1|k})$

$$\beta = \frac{\langle x, y_{k+1} \rangle - \langle x, \hat{y}_{k+1|k} \rangle}{\langle y_{k+1}, y_{k+1} \rangle - \langle y_{k+1}, \hat{y}_{k+1|k} \rangle}.$$

```
The co-varience matrix E is :
    0.2778   -0.0278
   -0.0278    0.1528


K_cap is :
    0.2222
    0.2778


The Minimum Variance Estimate for x is:
    0.3417
    0.4271
```

```
The co-varience matrix E is :
    0.1937   -0.0813
   -0.0813    0.1187


K_cap is :
   -0.0375    0.1375
    0.1125    0.0875


The Minimum Variance Estimate for x is:
    0.4504
    0.4963
```

```
The co-varience matrix E is :
    0.0545   -0.0105
   -0.0105    0.0828


K_cap is :
   -0.0695    0.0599    0.1461
    0.1287    0.1269   -0.0743


The Minimum Variance Estimate for x is:
   -1.0134
    1.2402
```

```
The co-varience matrix E is :
    0.0437    0.0072
    0.0072    0.0538


K_cap is :
   -0.0441    0.0962    0.1297   -0.0429
    0.0873    0.0674   -0.0474    0.0703


The Minimum Variance Estimate for x is:
   -1.0296
    1.2667
```

5(a) $\langle x, y\rangle = x^T y \rightarrow$ over determined system of eq^n

$\hat{x} = (A^T A)^{-1} A^T b \Rightarrow A\hat{x} = b$.

IIIy for $y = Ax$. we get.

$\hat{x} = (A^T A)^{-1} A^T y$

w.r.t $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \\ 0 & 6 \end{bmatrix}$; $y = \begin{bmatrix} 1.5377 \\ 3.6948 \\ -7.7193 \\ 7.3621 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 4 & 0 & 6 \end{bmatrix}$

$$\left[\begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \\ 0 & 6 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 4 & 0 & 6 \end{bmatrix} \times \begin{bmatrix} 1.5377 \\ 3.6948 \\ -7.7193 \\ 7.3621 \end{bmatrix}$$

$\hat{x} = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$

5(b) $Q = E\{\varepsilon\varepsilon^T\} = I$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \\ 0 & 6 \end{bmatrix} \qquad y = \begin{bmatrix} 1.5377 \\ 3.6948 \\ -7.7193 \\ 7.3621 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

w.k.T

$$\hat{k} = (A^T Q^{-1} A)^{-1} A^T Q^{-1}$$

$$\hat{x} = \hat{k} y$$

$$\hat{x} = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$$

5(c) $E(\varepsilon\varepsilon^T) = Q = I_{4\times4}$ & $P = E(xx^T) = 100\, I_{2\times2}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \\ 0 & 6 \end{bmatrix} ; \; y = \begin{bmatrix} 1.5377 \\ 3.6948 \\ -7.7193 \\ 7.3621 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = I_{4}$$

$= I_{4\times4}$.

$$P = 100 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 100 \times I_{2\times2}.$$

$$\hat{k} = PC^T(CPC^T + Q)^{-1}$$

$$\hat{x} = \hat{k} y.$$

$$\hat{x} = \begin{bmatrix} -1.3163 \\ 1.4365 \end{bmatrix}$$

5d Now $P = 10^6 \times I_{2\times2}$

Con.

$$\hat{x} = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$$

5d

Least squares	BLUE	MVE $P = 100 I_{2\times2}$ $Q = I_{4\times4}$	MVE $P = 10^6 I_{2\times2}$ $Q = I_{4\times4}$
$x = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$	$x = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$	$x = \begin{bmatrix} -1.3163 \\ 1.4365 \end{bmatrix}$	$x = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$

Most of the results match.

i.e; Value of $\hat{x}$ from Least Squares = Value of $\hat{x}$ from BLUE.

but at MVE; there is a small deviation but this deviation $(\delta) \to 0$ as $P \to \infty I$ (i.e $P = 10^6 I_{2\times2}$)

$\therefore$ even $\hat{x}$ with MVE matches with $\hat{x}$ from BLUE & least squares.

$\Rightarrow$ MVE reduces to BLUE when the co-variance of the unknown x becomes very large.

```
The co-varience matrix E is :
    0.0437    0.0072
    0.0072    0.0538


The Minimum Variance Estimate for x is:
   -0.8836
    1.0802
```