Problem - I

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & & & & \\ a_{n1} & \cdots & & \cdots & a_{nm} \end{bmatrix}$$

$$B_{m\times p} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ \vdots & & & & \\ b_{m1} & \cdots & & \cdots & b_{mp} \end{bmatrix}$$

$$AB = [AB]_{n\times p}$$

(a) prove that

$$AB = [Ab^1 \mid Ab^2 \mid \cdots \mid Ab^p] \quad \text{---} (0)$$

By definition of Matrix Multiplication :-

$$AB = \begin{bmatrix} \sum_{j=1}^{j=m} a_{1j} b_{j1} & \sum_{j=1}^{j=m} a_{1j} b_{j2} & \sum_{j=1}^{j=m} a_{1j} b_{jf} & \sum_{j=1}^{j=m} a_{1j} b_{jp} \\ \sum_{j=1}^{j=m} a_{2j} b_{j1} & \sum_{j=1}^{j=m} a_{2j} b_{j2} & \sum_{j=1}^{j=m} a_{2j} b_{jf} & \sum_{j=1}^{j=m} a_{2j} b_{jp} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{j=m} a_{nj} b_{j1} & \sum_{j=1}^{j=m} a_{nj} b_{jn} & \sum_{j=1}^{j=m} a_{nj} b_{jf} & \sum_{j=1}^{j=m} a_{nj} b_{jp} \end{bmatrix} \quad \text{---} (1)$$

where $f \in N \leq p$.

Now, lets note that for any $t \in N \leq p$ ~~and~~ the correspon-
-ding vector $b^t (m \times 1)$:

$$Ab^t = \begin{bmatrix} \sum_{j=1}^{j=m} a_{1j}\, b_{jt} \\ \sum_{j=1}^{j=m} a_{2j}\, b_{jt} \\ \vdots \\ \sum_{j=1}^{j=m} a_{nj}\, b_{jt} \end{bmatrix} \quad \text{--- (2)}$$

from (1) & (2) we can see that $col_t(AB) = Ab^t$
$t \in N \leq p$ --- (3)

∴ The same can be extrapolated

$$AB = \left[col_1(AB) \mid col_2(AB) \mid \cdots \mid col_p(AB) \right] \text{ --- (4)}$$

Now using the same logic as (3) in (4) we get

$$\boxed{AB = \left[Ab^1 \mid Ab^2 \mid \cdots \mid Ab^p \right] \text{ --- (0)}}$$

(b) P.T

$$AB = \begin{bmatrix} a^1 B \\ a^2 B \\ \vdots \\ a^n B \end{bmatrix} \text{ --- (0)}$$

$$AB = \begin{bmatrix} \sum_{j=1}^{j=m} a_{1j}\, b_{j1} & \sum_{j=1}^{j=m} a_{1j}\, b_{j2} & \cdots & \sum_{j=1}^{j=m} a_{1j}\, b_{jp} \\ \sum_{j=1}^{j=m} a_{tj}\, b_{j1} & \sum_{j=1}^{j=m} a_{tj}\, b_{j2} & \cdots & \sum_{j=1}^{j=m} a_{tj}\, b_{j3} \\ \sum_{j=1}^{j=m} a_{nj}\, b_{j1} & \cdots & \cdots & \sum_{j=1}^{j=m} a_{nj}\, b_{jp} \end{bmatrix}$$ where $t \in N \leq n$

Now note that for any such t:-

$$a^t B = \left[\sum_{j=1}^{j=m} a_{tj} b_{j1} \quad \sum_{j=1}^{j=m} a_{tj} b_{j2} \quad ----- \quad \sum_{j=1}^{j=m} a_{tj} b_{j3} \right]$$

from ① & ② we can see that:-

$$row_t (AB) = a^t B \;, \; q \in N \le n \quad --- ③$$

At the same time

$$AB = \begin{bmatrix} row_1 (AB) \\ row_2 (AB) \\ \vdots \\ row_n (AB) \end{bmatrix} \quad --- ④$$

After substitution of ③ in ④ we get.

$$\boxed{AB = \begin{bmatrix} a^1 B \\ a^2 B \\ \vdots \\ a^n B \end{bmatrix} \quad --- (0)}$$

Problem-1

(i) P.T

$$[AB]_{ij} = a^i b^j \quad - - - (0)$$

By Def. of matrix Multiplication

$$[AB]_{ij} = \sum_{k=1}^{k=m} a_{ik} \cdot b_{kj} \quad - - - (1)$$

So for a Vector $a^c (1 \times m)$ & $b^d (m \times 1)$, where $c \in N \leq n$; $d \in N \leq p$:

$$a^c b^d = M_{1\times1} = \sum_{k=1}^{k=m} a_{ck} b_{kd}$$

$i \in N \leq n$; $j \in N \leq p$ $\therefore$ for any combination of $i, j \; \exists \; c, d$ s.t $c = i$; $d = j$. The following will be true

$$a^{c=i} b^{d=j} = a^i b^j = \left[\sum_{k=1}^{k=m} a_{c=ik} b_{kd=j} \right]$$

$$= \sum_{k=1}^{k=m} a_{ik} b_{kj}$$

$$\sum_{k=1}^{k=m} a_{ik} b_{ki} = [AB] \rightarrow \text{By Definition.}$$

$$[AB]_{ij} = \sum_{k=1}^{k=m} a_{ik} b_{kj}$$

$$\Rightarrow \boxed{a^i b^j = [AB]_{ij} \quad - - - (0)}$$

Problem - 2

$$\text{Trace} = tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots a_{nn}$$

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\quad \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$

$$\Rightarrow \boxed{1 + 5 + 9 = 15}$$

(b) $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ $\quad x^T = \begin{bmatrix} x_1 & x_2 & x_3 & - & - & x_n \end{bmatrix}_{1 \times n}$

$$x \cdot x^T = \cancel{(x_1)^2 +} \begin{bmatrix} (x_1)^2 & x_1 x_2 & x_1 x_3 & \cdots & x_1 x_n \\ (x_2 x_1) & (x_2)^2 & x_2 x_3 & \cdots & x_2 x_n \\ \vdots & & & & \\ (x_n x_1) & - & - & - & (x_n)^2 \end{bmatrix}$$

$$\boxed{\text{Trace}(x \cdot x^T) = \sum_{i=1}^{n} (x_i)^2 = x_1^2 + x_2^2 + \cdots x_n^2}$$

(c) $$\underset{n \times m}{K_{nm}} = \begin{array}{c} \\ \left[\begin{array}{cccccc} k_{11} & k_{12} & \cdots & k_{1i} & \cdots & k_{1m} \\ k_{21} & k_{22} & \cdots & k_{2i} & \cdots & k_{2m} \\ \vdots & & & & & \\ k_{n1} & k_{n2} & \cdots & k_{ni} & \cdots & k_{nm} \end{array}\right] \end{array}$$

(column i labelled K^i)

$K^T = K_{m \times n}$ $\qquad Q_{n \times n}$

$K^T Q = \cancel{K_{m \times n}}\ (K^T Q)_{m \times n} \times K_{n \times m} \Rightarrow$

$$M_{m\times m} = (K^T Q K)_{m\times m} = M[i,j] = K^T Q K[i,j]$$

$$\text{Trace}(K^T Q K) = \sum_{i=1}^{m} \quad \boxed{M(i,j) = K_i^T Q K_j}$$

$K^T Q = K$

The trace is basically $\sum_{i=1}^{m} M(i,i)$

$$\Rightarrow \boxed{\sum_{i=1}^{m} M_{ii} = \sum_{i=1}^{m} K_i^T Q K_i}$$

Problem -3

A matrix is symmetrical if it is equal to its transpose $M^T = M$, hence a symmetrical matrix is a square Matrix.

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Eigen - value.

$$\det[M - \lambda I] = 0$$

$$\det\left\{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 0$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 1-0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - 1 = 0$$

$$6-5\lambda+\lambda^2-1=0 \quad \text{or} \quad \lambda^2-5\lambda+5=0$$

$$5-5\lambda+\lambda^2=0$$

$$\lambda=\frac{5}{2}\pm\frac{\sqrt{5}}{2}$$

(a) $$\boxed{\lambda_1=\frac{5+\sqrt{5}}{2}, \quad \lambda_2=\frac{5-\sqrt{5}}{2}}$$

$$\begin{bmatrix} 2-\left(\frac{5+\sqrt{5}}{2}\right) & 1 \\ 1 & 3-\left(\frac{5+\sqrt{5}}{2}\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} \frac{-1-\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{-1-\sqrt{5}}{2} x_1 + x_2 = 0$$

$$x_2 = \frac{1+\sqrt{5}}{2} x_1$$

so the eigine vector ~~component~~ for $\lambda_1 = \frac{5+\sqrt{5}}{2}$ is

$$x_1 \underbrace{\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}}_{\to v_1}$$

$$\boxed{v_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}}$$

$$\lambda_2 = \frac{5-\sqrt{5}}{2}$$

$$\begin{bmatrix} 2 - \left(\frac{5-\sqrt{5}}{2}\right) & 1 \\ 1 & 3 - \left(\frac{5-\sqrt{5}}{2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left(\frac{-1+\sqrt{5}}{2}\right) x_1 + x_2 = 0$$

$$x_2 = \frac{1-\sqrt{5}}{2} x_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} x_1$$

$$\boxed{v_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}}$$

$$v_1^T = \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$v_1^T v_2 = \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$1 + \frac{(1+\sqrt{5})(1-\sqrt{5})}{4}$$

$$1 + \frac{(1-5)}{4}$$

$$\frac{4-4}{4} = \boxed{0 = v_1^T v_2} \quad (b)$$

problem 3

(i) $M = A^T A$.

w.k.t $[AB]_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$

$$[A^T A]_{ij} = \sum_{k=1}^{n} a_{ki}\, a_{kj} \quad \text{--- (1)}$$

Then

$$[A^T A]_{ji} = \sum_{k=1}^{n} a^T_{jk}\, a_{ki} = \sum_{k=1}^{n} a_{kj}\, a_{ki} \quad \text{-- (2)}$$

Note that from (1) & (2) both the eq^n are identical

∴ for each combination of $i \in N < m$ & $j \in N < m$

$[A^T A]_{ij} = [A^T A]_{ji} \Rightarrow M = M^T$

Thus $A^T A$ is a symmetrical matrix by definition.

[17]:
```
import numpy as np

# Set random seed for reproducibility
np.random.seed(0)

# Parameters
num_matrices = 10
n_values = [5, 6, 7, 8, 9, 10, 10, 10, 10, 10]  # Ensure n >= m
m_values = [3, 3, 4, 4, 5, 5, 3, 4, 5, 6]       # Ensure some m >= 3

# Loop to generate matrices and perform calculations
for i in range(num_matrices):
    n, m = n_values[i], m_values[i]

    # Generate random matrix A (n x m)
    A = np.random.rand(n, m)

    # Form matrix M = A^T A (m x m)
    M = A.T @ A

    # Calculate eigenvalues and eigenvectors
    eigenvalues, eigenvectors = np.linalg.eig(M)

    # Print matrices and their respective eigenvalues and eigenvectors
    print(f"Matrix {i+1}: n={n}, m={m}")
    print(f"A = \n{A}\n")
    print(f"M = A^T A = \n{M}\n")
    print(f"Eigenvalues: {eigenvalues}\n")
    print(f"Eigenvectors: \n{eigenvectors}\n")
    print("-" * 50)
```

```
Matrix 1: n=5, m=3
A =
[[0.5488135  0.71518937 0.60276338]
 [0.54488318 0.4236548  0.64589411]
 [0.43758721 0.891773   0.96366276]
 [0.38344152 0.79172504 0.52889492]
 [0.56804456 0.92559664 0.07103606]]
```

```
M = A^T A =
[[1.25927853 1.84293681 1.34757994]
 [1.84293681 2.96979598 2.04858462]
 [1.34757994 2.04858462 1.99392477]]

Eigenvalues: [5.74777528 0.08022661 0.39499739]

Eigenvectors:
[[ 0.45321709  0.86443188 -0.2176047 ]
 [ 0.70394399 -0.49684152 -0.5075543 ]
 [ 0.54686117 -0.07685076  0.83368868]]

--------------------------------------------------
Matrix 2: n=6, m=3
A =
[[0.0871293  0.0202184  0.83261985]
 [0.77815675 0.87001215 0.97861834]
 [0.79915856 0.46147936 0.78052918]
 [0.11827443 0.63992102 0.14335329]
 [0.94466892 0.52184832 0.41466194]
 [0.26455561 0.77423369 0.45615033]]

M = A^T A =
[[2.22815173 1.82105067 1.98718103]
 [1.82105067 2.25155551 1.88973457]
 [1.98718103 1.88973457 2.66074328]]

Eigenvalues: [6.19038551 0.39150378 0.55856123]

Eigenvectors:
[[ 0.56245316  0.81900556 -0.11347396]
 [ 0.55435996 -0.47536084 -0.68316696]
 [ 0.61345862 -0.321344    0.7213921 ]]

--------------------------------------------------
Matrix 3: n=7, m=4
A =
[[0.56843395 0.0187898  0.6176355  0.61209572]
 [0.616934   0.94374808 0.6818203  0.3595079 ]
 [0.43703195 0.6976312  0.06022547 0.66676672]
 [0.67063787 0.21038256 0.1289263  0.31542835]
 [0.36371077 0.57019677 0.43860151 0.98837384]
 [0.10204481 0.20887676 0.16130952 0.65310833]
 [0.2532916  0.46631077 0.24442559 0.15896958]]

M = A^T A =
[[1.5513321  1.38570277 1.12240226 1.53905938]
```

```
 [1.38570277 2.00816319 1.12197212 1.65641988]
 [1.12240226 1.12197212 1.14473754 1.28170626]
 [1.53905938 1.65641988 1.28170626 2.47668466]]

Eigenvalues: [5.9765629  0.20556051 0.42405934 0.57473473]

Eigenvectors:
[[ 0.46954718  0.6746393  -0.55458865 -0.12968693]
 [ 0.5226466  -0.06726248  0.51820495 -0.67363189]
 [ 0.38856751 -0.73504604 -0.55330374 -0.05076991]
 [ 0.59613863  0.00670068  0.34314726  0.72582628]]

--------------------------------------------------
Matrix 4: n=8, m=4
A =
[[0.11037514 0.65632959 0.13818295 0.19658236]
 [0.36872517 0.82099323 0.09710128 0.83794491]
 [0.09609841 0.97645947 0.4686512  0.97676109]
 [0.60484552 0.73926358 0.03918779 0.28280696]
 [0.12019656 0.2961402  0.11872772 0.31798318]
 [0.41426299 0.0641475  0.69247212 0.56660145]
 [0.26538949 0.52324805 0.09394051 0.5759465 ]
 [0.9292962  0.31856895 0.66741038 0.13179786]]

M = A^T A =
[[1.64329798 1.41321822 1.06608383 1.14386079]
 [1.41321822 3.07187076 0.99835288 2.45366967]
 [1.06608383 0.99835288 1.19756816 1.14954979]
 [1.14386079 2.45366967 1.14954979 2.54607385]]

Eigenvalues: [6.63067591 1.15769234 0.13000107 0.54044142]

Eigenvectors:
[[ 0.38346341  0.67557454 -0.42573067 -0.4640132 ]
 [ 0.64281797 -0.40115626  0.4536784  -0.46907847]
 [ 0.31664136  0.54267044  0.57407545  0.52505659]
 [ 0.58264837 -0.29695366 -0.53232197  0.53756183]]

--------------------------------------------------
Matrix 5: n=9, m=5
A =
[[0.7163272  0.28940609 0.18319136 0.58651293 0.02010755]
 [0.82894003 0.00469548 0.67781654 0.27000797 0.73519402]
 [0.96218855 0.24875314 0.57615733 0.59204193 0.57225191]
 [0.22308163 0.95274901 0.44712538 0.84640867 0.69947928]
 [0.29743695 0.81379782 0.39650574 0.8811032  0.58127287]
 [0.88173536 0.69253159 0.72525428 0.50132438 0.95608363]
 [0.6439902  0.42385505 0.60639321 0.0191932  0.30157482]
```

```
 [0.66017354 0.29007761 0.61801543 0.4287687  0.13547406]
 [0.29828233 0.56996491 0.59087276 0.57432525 0.65320082]]

M = A^T A =
[[3.98128926 2.15024361 3.07938509 2.57327092 2.8248812 ]
 [2.15024361 2.78391175 2.22352759 2.64877495 2.49262925]
 [3.07938509 2.22352759 2.8068697  2.33894764 2.7209106 ]
 [2.57327092 2.64877495 2.33894764 3.02555247 2.5716384 ]
 [2.8248812  2.49262925 2.7209106  2.5716384  3.14560404]]

Eigenvalues: [13.4458203   1.46401362  0.52352622  0.08571307  0.22415401]

Eigenvectors:
[[-0.49087174 -0.66809929  0.37334522  0.41497625 -0.03311091]
 [-0.40628886  0.56471428  0.01794443  0.36299311 -0.61962984]
 [-0.4400934  -0.25160142 -0.24599654 -0.73450813 -0.37815081]
 [-0.43634623  0.40848354  0.5509919  -0.31514396  0.48972989]
 [-0.45814197  0.06766855 -0.70440398  0.23919199  0.48179781]]

--------------------------------------------------
Matrix 6: n=10, m=5
A =
[[0.65210327 0.43141844 0.8965466  0.36756187 0.43586493]
 [0.89192336 0.80619399 0.70388858 0.10022689 0.91948261]
 [0.7142413  0.99884701 0.1494483  0.86812606 0.16249293]
 [0.61555956 0.12381998 0.84800823 0.80731896 0.56910074]
 [0.4071833  0.069167   0.69742877 0.45354268 0.7220556 ]
 [0.86638233 0.97552151 0.85580334 0.01171408 0.35997806]
 [0.72999056 0.17162968 0.52103661 0.05433799 0.19999652]
 [0.01852179 0.7936977  0.22392469 0.34535168 0.92808129]
 [0.7044144  0.03183893 0.16469416 0.6214784  0.57722859]
 [0.23789282 0.934214   0.61396596 0.5356328  0.58990998]]

M = A^T A =
[[4.11225865 3.04802657 3.51320192 2.25217556 2.88672793]
 [3.04802657 4.33870593 2.93760058 2.05287386 2.90362373]
 [3.51320192 2.93760058 3.98521371 2.07763568 3.12580236]
 [2.25217556 2.05287386 2.07763568 2.5517504  2.1906663 ]
 [2.88672793 2.90362373 3.12580236 2.1906663  3.61917473]]

Eigenvalues: [14.74132657  0.39227323  0.89390176  1.42978921  1.14981264]

Eigenvectors:
[[ 0.48598496  0.52805757  0.50655925 -0.32342602 -0.35180556]
 [ 0.46921231 -0.09971312  0.10267164  0.85946413 -0.14379518]
 [ 0.4814979  -0.66847314 -0.24585572 -0.36996355 -0.35211558]
 [ 0.33162251 -0.26173673  0.42192491 -0.12935862  0.79168561]
 [ 0.44927132  0.44254948 -0.70312914 -0.05577045  0.32373493]]
```

```
--------------------------------------------------
Matrix 7: n=10, m=3
A =
[[0.73012203 0.311945   0.39822106]
 [0.20984375 0.18619301 0.94437239]
 [0.7395508  0.49045881 0.22741463]
 [0.25435648 0.05802916 0.43441663]
 [0.31179588 0.69634349 0.37775184]
 [0.17960368 0.02467873 0.06724963]
 [0.67939277 0.45369684 0.53657921]
 [0.89667129 0.99033895 0.21689698]
 [0.6630782  0.26332238 0.020651  ]
 [0.75837865 0.32001715 0.38346389]]

M = A^T A =
[[3.59862416 2.47940244 1.76099911]
 [2.47940244 2.21975974 1.28790833]
 [1.76099911 1.28790833 1.92050598]]

Eigenvalues: [6.51774048 0.33330869 0.88784072]

Eigenvectors:
[[-0.72100863 -0.62402583 -0.30122801]
 [-0.54439383  0.77908266 -0.31091086]
 [-0.42869793  0.06018274  0.90144114]]

--------------------------------------------------
Matrix 8: n=10, m=4
A =
[[0.58831711 0.83104846 0.62898184 0.87265066]
 [0.27354203 0.79804683 0.18563594 0.95279166]
 [0.68748828 0.21550768 0.94737059 0.73085581]
 [0.25394164 0.21331198 0.51820071 0.02566272]
 [0.20747008 0.42468547 0.37416998 0.46357542]
 [0.27762871 0.58678435 0.86385561 0.11753186]
 [0.51737911 0.13206811 0.71685968 0.3960597 ]
 [0.56542131 0.18327984 0.14484776 0.48805628]
 [0.35561274 0.94043195 0.76532525 0.74866362]
 [0.90371974 0.08342244 0.55219247 0.58447607]]

M = A^T A =
[[2.60874248 1.74234457 2.745154   2.68711108]
 [1.74234457 2.88654449 2.53838965 2.80898948]
 [2.745154   2.53838965 3.9078795  2.95675764]
 [2.68711108 2.80898948 2.95675764 3.73002752]]

Eigenvalues: [11.14279891  0.08592072  0.75683496  1.1476394 ]
```

```
Eigenvectors:
[[ 0.44136318  0.62225638  0.43769138 -0.475838  ]
 [ 0.44895901  0.4569273  -0.37204031  0.67174345]
 [ 0.54946828 -0.33027694 -0.60201701 -0.47600133]
 [ 0.54928951 -0.54307604  0.55460597  0.30945381]]

--------------------------------------------------
Matrix 9: n=10, m=5
A =
[[0.96193638 0.29214753 0.24082878 0.10029394 0.01642963]
 [0.92952932 0.66991655 0.78515291 0.28173011 0.58641017]
 [0.06395527 0.4856276  0.97749514 0.87650525 0.33815895]
 [0.96157015 0.23170163 0.94931882 0.9413777  0.79920259]
 [0.63044794 0.87428797 0.29302028 0.84894356 0.61787669]
 [0.01323686 0.34723352 0.14814086 0.98182939 0.47837031]
 [0.49739137 0.63947252 0.36858461 0.13690027 0.82211773]
 [0.18984791 0.51131898 0.22431703 0.09784448 0.86219152]
 [0.97291949 0.96083466 0.9065555  0.77404733 0.33314515]
 [0.08110139 0.40724117 0.23223414 0.13248763 0.05342718]]

M = A^T A =
[[4.35228377 3.09636308 3.25028912 2.71832092 2.64793326]
 [3.09636308 3.46802954 2.9146479  2.880215   2.76178171]
 [3.25028912 2.9146479  3.70092148 3.19490134 2.61638813]
 [2.71832092 2.880215   3.19490134 4.07359574 2.67168463]
 [2.64793326 2.76178171 2.61638813 2.67168463 3.24092476]]

Eigenvalues: [15.30563452  1.5101505   1.02159517  0.46350368  0.53487143]

Eigenvectors:
[[ 0.47202473  0.72475891 -0.2395493   0.4404089   0.02394389]
 [ 0.44176152  0.06315752  0.33795303 -0.35282538 -0.74977322]
 [ 0.45964833 -0.03154513 -0.39862102 -0.67975474  0.40836638]
 [ 0.45461505 -0.67471983 -0.33985173  0.44652779 -0.15228932]
 [ 0.40507774 -0.12038981  0.74431577  0.14177688  0.4973044 ]]

--------------------------------------------------
Matrix 10: n=10, m=6
A =
[[0.72559436 0.01142746 0.77058075 0.14694665 0.07952208 0.08960303]
 [0.67204781 0.24536721 0.42053947 0.55736879 0.86055117 0.72704426]
 [0.27032791 0.1314828  0.05537432 0.30159863 0.26211815 0.45614057]
 [0.68328134 0.69562545 0.28351885 0.37992696 0.18115096 0.78854551]
 [0.05684808 0.69699724 0.7786954  0.77740756 0.25942256 0.37381314]
 [0.58759964 0.2728219  0.3708528  0.19705428 0.45985588 0.0446123 ]
 [0.79979588 0.07695645 0.51883515 0.3068101  0.57754295 0.95943334]
 [0.64557024 0.03536244 0.43040244 0.51001685 0.53617749 0.68139251]
```

```
 [0.2775961  0.12886057 0.39267568 0.95640572 0.18713089 0.90398395]
 [0.54380595 0.45691142 0.88204141 0.45860396 0.72416764 0.39902532]]

M = A^T A =
[[3.29580993 1.25259528 2.59410617 2.07183602 2.36943623 2.93836697]
 [1.25259528 1.35430098 1.46918437 1.41242362 1.09721304 1.45737905]
 [2.59410617 1.46918437 2.98462311 2.30925023 2.10424752 2.42924444]
 [2.07183602 1.41242362 2.30925023 2.69003155 1.89323688 2.84441144]
 [2.36943623 1.09721304 2.10424752 1.89323688 2.30764008 2.39026934]
 [2.93836697 1.45737905 2.42924444 2.84441144 2.39026934 3.86943311]]

Eigenvalues: [13.46313801  1.11304557  1.0221109   0.08925612  0.39428615
0.42000202]

Eigenvectors:
[[-0.45294146 -0.58215631  0.2463975  -0.45544255 -0.33552634 -0.27426356]
 [-0.24012494  0.23901752 -0.4881832   0.00861392  0.17998046 -0.78384988]
 [-0.4265251  -0.24885752 -0.59316978  0.42820489 -0.30300641  0.3593377 ]
 [-0.40752593  0.552996   -0.14532928 -0.60432889 -0.00352781  0.37652568]
 [-0.37608916 -0.2873066   0.09848199  0.0641217   0.85752437  0.1638704 ]
 [-0.49773481  0.3920267   0.56417706  0.48970753 -0.16690036 -0.11229506]]

--------------------------------------------------
```

[16]:

```python
import numpy as np

# Set random seed for reproducibility
np.random.seed(0)

# Parameters
num_matrices = 10
n_values = [5, 6, 7, 8, 9, 10, 10, 10, 10, 10]  # Ensure n >= m
m_values = [3, 3, 4, 4, 5, 5, 3, 4, 5, 6]       # Ensure some m >= 3

# Loop to generate matrices and perform calculations
for i in range(num_matrices):
    n, m = n_values[i], m_values[i]

    # Generate random matrix A (n x m)
    A = np.random.rand(n, m)

    # Form matrix M = A^T A (m x m)
    M = A.T @ A

    # Calculate eigenvalues and eigenvectors
    eigenvalues, eigenvectors = np.linalg.eig(M)
```

```
    # Check orthogonality of a few eigenvectors
    print(f"Matrix {i+1}: n={n}, m={m}")
    for j in range(min(m, 3)):  # Check first few pairs only for simplicity
        for k in range(j + 1, min(m, 3)):
            inner_product = eigenvectors[:, j].T @ eigenvectors[:, k]
            print(f"Inner product of v_{j+1} and v_{k+1}: {inner_product:.4f}")

    # Sum of eigenvalues vs. trace of M
    sum_eigenvalues = np.sum(eigenvalues)
    trace_M = np.trace(M)
    print(f"Sum of eigenvalues: {sum_eigenvalues:.4f}, Trace of M: {trace_M:.
→4f}")

    # Product of eigenvalues vs. determinant of M
    product_eigenvalues = np.prod(eigenvalues)
    det_M = np.linalg.det(M)
    print(f"Product of eigenvalues: {product_eigenvalues:.4f}, Determinant of M:␣
→{det_M:.4f}")
    print("-" * 50)

# Summary of Observations
print("Observations:")
print("1. The eigenvalues are non-negative, as M is positive semi-definite (A^T␣
→A).")
print("2. The eigenvectors corresponding to different eigenvalues are␣
→orthogonal, as indicated by inner products close to zero.")
print("3. The sum of the eigenvalues equals the trace of M, consistent with␣
→properties of symmetric matrices.")
print("4. The product of the eigenvalues equals the determinant of M, consistent␣
→with properties of the determinant.")
```

```
Matrix 1: n=5, m=3
Inner product of v_1 and v_2: 0.0000
Inner product of v_1 and v_3: -0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 6.2230, Trace of M: 6.2230
Product of eigenvalues: 0.1821, Determinant of M: 0.1821
--------------------------------------------------
Matrix 2: n=6, m=3
Inner product of v_1 and v_2: -0.0000
Inner product of v_1 and v_3: 0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 7.1405, Trace of M: 7.1405
Product of eigenvalues: 1.3537, Determinant of M: 1.3537
--------------------------------------------------
Matrix 3: n=7, m=4
Inner product of v_1 and v_2: 0.0000
```

```
Inner product of v_1 and v_3: -0.0000
Inner product of v_2 and v_3: 0.0000
Sum of eigenvalues: 7.1809, Trace of M: 7.1809
Product of eigenvalues: 0.2994, Determinant of M: 0.2994
--------------------------------------------------
Matrix 4: n=8, m=4
Inner product of v_1 and v_2: -0.0000
Inner product of v_1 and v_3: -0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 8.4588, Trace of M: 8.4588
Product of eigenvalues: 0.5393, Determinant of M: 0.5393
--------------------------------------------------
Matrix 5: n=9, m=5
Inner product of v_1 and v_2: -0.0000
Inner product of v_1 and v_3: -0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 15.7432, Trace of M: 15.7432
Product of eigenvalues: 0.1980, Determinant of M: 0.1980
--------------------------------------------------
Matrix 6: n=10, m=5
Inner product of v_1 and v_2: -0.0000
Inner product of v_1 and v_3: -0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 18.6071, Trace of M: 18.6071
Product of eigenvalues: 8.4979, Determinant of M: 8.4979
--------------------------------------------------
Matrix 7: n=10, m=3
Inner product of v_1 and v_2: -0.0000
Inner product of v_1 and v_3: 0.0000
Inner product of v_2 and v_3: 0.0000
Sum of eigenvalues: 7.7389, Trace of M: 7.7389
Product of eigenvalues: 1.9288, Determinant of M: 1.9288
--------------------------------------------------
Matrix 8: n=10, m=4
Inner product of v_1 and v_2: -0.0000
Inner product of v_1 and v_3: -0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 13.1332, Trace of M: 13.1332
Product of eigenvalues: 0.8316, Determinant of M: 0.8316
--------------------------------------------------
Matrix 9: n=10, m=5
Inner product of v_1 and v_2: 0.0000
Inner product of v_1 and v_3: -0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 18.8358, Trace of M: 18.8358
Product of eigenvalues: 5.8540, Determinant of M: 5.8540
--------------------------------------------------
Matrix 10: n=10, m=6
```

```
Inner product of v_1 and v_2: -0.0000
Inner product of v_1 and v_3: 0.0000
Inner product of v_2 and v_3: -0.0000
Sum of eigenvalues: 16.5018, Trace of M: 16.5018
Product of eigenvalues: 0.2264, Determinant of M: 0.2264
--------------------------------------------------
Observations:
1. The eigenvalues are non-negative, as M is positive semi-definite (A^T A).
2. The eigenvectors corresponding to different eigenvalues are orthogonal, as
indicated by inner products close to zero.
3. The sum of the eigenvalues equals the trace of M, consistent with properties
of symmetric matrices.
4. The product of the eigenvalues equals the determinant of M, consistent with
properties of the determinant.
```

4(a) $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$\mu = 0, \quad \sigma = 1$

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x)^2}{2}}$

$\mu = 0, \quad \sigma = 3$

$f_X(x) = \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x)^2}{2(9)}}$

$f_X(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{x^2}{18}}$

To choose the range of x

$\mu - 3\sigma \leq x \leq \mu + 3\sigma$

case - 1 $(\mu = 0, \sigma = 1)$

$-3 \leq x \leq +3$

case - 2 $(\mu = 0, \sigma = 3)$

$-9 \leq x \leq 9$

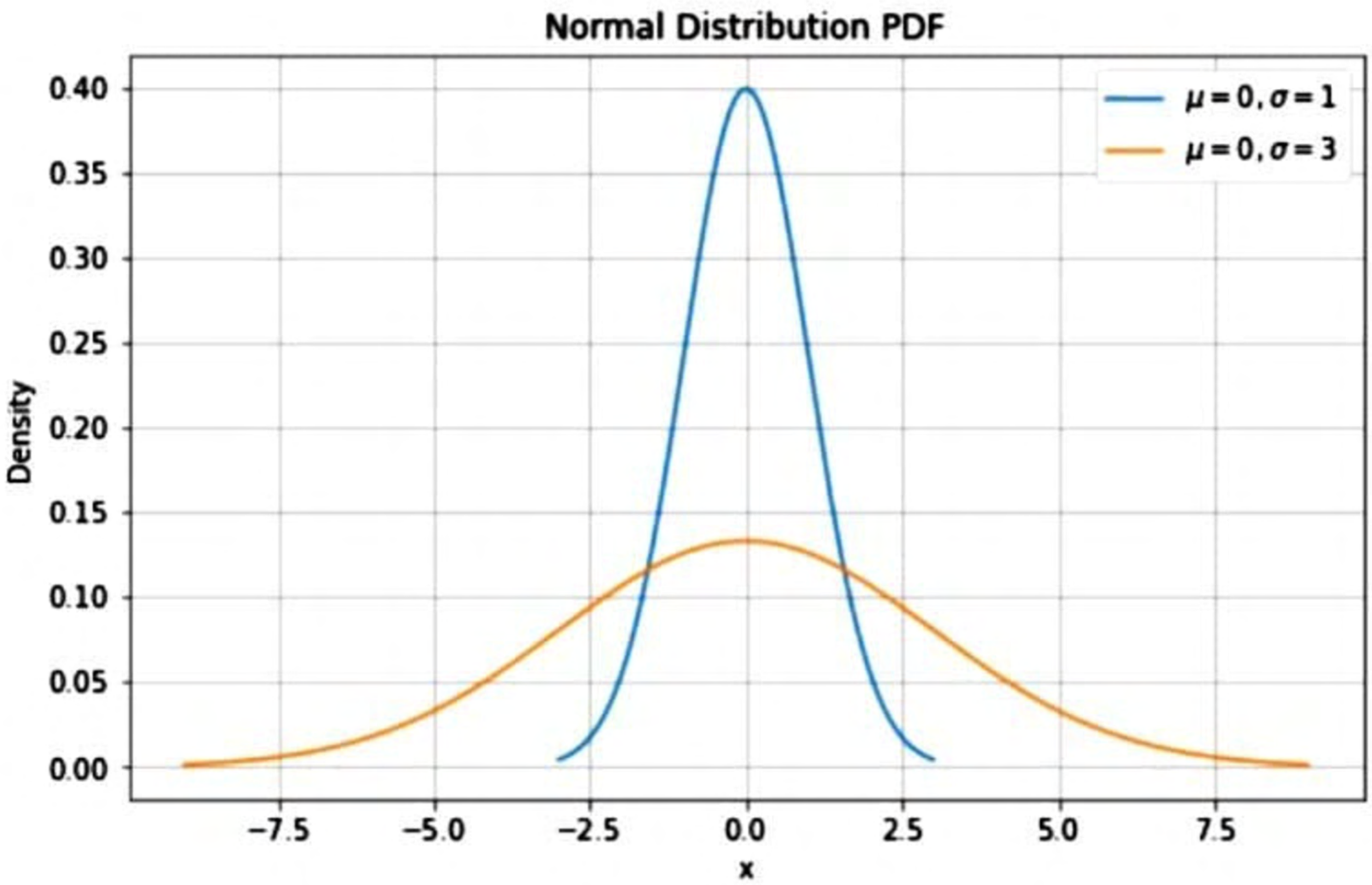
[12]:
```python
import numpy as np
import matplotlib.pyplot as plt

# Define the x values for both distributions
x1 = np.linspace(-3, 3, 400)  # Range for sigma = 1
x2 = np.linspace(-9, 9, 400)  # Range for sigma = 3

# Define the PDF for the normal distribution
def normal_pdf(x, mu, sigma):
    return (1 / (sigma * np.sqrt(2 * np.pi))) * np.exp(-0.5 * ((x - mu) / sigma)␣
 ↪** 2)

# Calculate the y values for both distributions
y1 = normal_pdf(x1, mu=0, sigma=1)
y2 = normal_pdf(x2, mu=0, sigma=3)

# Plot both distributions
plt.figure(figsize=(8, 5))
plt.plot(x1, y1, label=r'$\mu=0, \sigma=1$')
plt.plot(x2, y2, label=r'$\mu=0, \sigma=3$')
plt.xlabel('x')
plt.ylabel('Density')
plt.title('Normal Distribution PDF')
plt.legend()
plt.grid(True)
plt.show()
```



Normal Distribution PDF
$\mu = 0, \sigma = 1$
$\mu = 0, \sigma = 3$
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
Density
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
x

4. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

(b) for $\mu = 2$ & $\sigma = 5.0$, determine

(i) $P(x \geq 4)$

$$\int_4^\infty \frac{1}{5\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2(25)}}$$

$$P(x \geq 4) = \int_4^\infty \frac{1}{5\sqrt{2\pi}} e^{-\frac{(x-2)^2}{50}}$$

$$P(x \geq 4) = \frac{1}{5\sqrt{2\pi}} \int_4^\infty e^{-\frac{(x-2)^2}{50}} \Rightarrow P(x \geq 4) = 0.344$$

(ii) $P\{-2 \leq X \leq 4\}$

$$= \frac{1}{5\sqrt{2\pi}} \int_{-2}^{4} e^{-\frac{(x-2)^2}{50}}$$

$$P\{-2 \leq X \leq 4\} = 0.443566$$

(iii) $P(x \in A)$, where $A = [-2, 4] \cup [8, 100]$ = **0.558636**

$$\frac{1}{5\sqrt{2\pi}} \int_{-2}^{4} e^{-\frac{(x-2)^2}{50}} + \frac{1}{5\sqrt{2\pi}} \int_{8}^{100} e^{-\frac{(x-2)^2}{50}} \Rightarrow P(-2 \leq x \leq 4) + P(8$$

[10]:
```python
import numpy as np
from scipy import integrate

def integrand(x, mu=2):
    return (1 / (5.0 * np.sqrt(2*np.pi))) * np.exp(-((x - mu)**2) / 50)

# Define the integration limits
lower_limit = 4
upper_limit = np.inf  # Infinity

# Perform the integration
result, _ = integrate.quad(integrand, lower_limit, upper_limit)

# Print only the result
print(f"P(x>=4) = {result:.6f}")
```

```
P(x>=4) = 0.344578
```

[8]:
```python
import numpy as np
from scipy import integrate

def integrand(x, mu=2):
    return (1 / (5.0 * np.sqrt(2*np.pi))) * np.exp(-((x - mu)**2) / 50)
```

```
# Define the integration limits
lower_limit = -2
upper_limit = 4

# Perform the integration
result, _ = integrate.quad(integrand, lower_limit, upper_limit)

# Print only the result
print(f"P(-2 <= X <= 4) = {result:.6f}")
```

```
P(-2 <= X <= 4) = 0.443566
```

[7]:

```
import numpy as np
from scipy import integrate

def integrand(x, mu=2):
    return (1 / (5.0 * np.sqrt(2*np.pi))) * np.exp(-((x - mu)**2) / 50)

# Define the integration limits
lower_limit_1 = -2
upper_limit_2 = 4
lower_limit_3 = 8
upper_limit_4 = 100

# Perform the integrations
result1, _ = integrate.quad(integrand, lower_limit_1, upper_limit_2)
result2, _ = integrate.quad(integrand, lower_limit_3, upper_limit_4)

# Sum the results
total_result = result1 + result2

# Print only the total result
print(f"Total integral: {total_result:.6f}")
```

```
Total integral: 0.558636
```

4(c) $Y = a\mu + b$; $Y : (a\sigma)^2$

$\Rightarrow y = 2x + 4$

from (b) $\Rightarrow$ $\sigma = 5$; $\mu = 2$

$\sigma^2 = 25$,

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$f_y(x) = \frac{1}{\ \sqrt{2\pi}}$

$\mu_y = a\mu + b = (2)(2) + 4$

$\mu_y = 8$

$\sigma_y = a\sigma = (2 \times 5) \Rightarrow 10^2 = 100$

$\sigma_y = \sqrt{100} = 10$

$\mu \to \mu_y = 8$; $\sigma \to \sigma_y = 10$

The term $\frac{1}{\sigma_y\sqrt{2\pi}} = \frac{1}{10\sqrt{2\pi}}$

Co-efficient

exponent :- The term $\frac{-(y-\mu_y)^2}{2\sigma_y^2}$

$$= -\frac{(y-8)^2}{2 \cdot 100} = -\frac{(y-8)^2}{200}$$

Thus, the density function $\boxed{f_x(x) = \frac{1}{10\sqrt{2\pi}} e^{\left(-\frac{(y-8)^2}{200}\right)}}$

5(a) The total integral of $f_{xy}(x,y)$ over its entire domain must be equal to 1

$$\iint F_{xy}(x,y) \cdot dx \cdot dy = 1$$

$$\Rightarrow \int_0^1 \int_0^2 K(x+y)^2 \cdot dx \cdot dy = 1$$

$$K \int_0^1 \int_0^2 (x^2 + 2xy + y^2) \cdot dx \cdot dy = 1 \quad (1)$$

$\Rightarrow$ let us integrate by y first (i.e x in a

$$\int_0^2 x^2 \cdot dy + \int_0^2 2xy \cdot dy + \int_0^2 y^2 \cdot dy$$

$$\Rightarrow x^2 \cdot y\Big]_0^2 + \frac{\cancel{2}xy^2}{\cancel{2}}\Big]_0^2 + \frac{y^3}{3}\Big]_0^2$$

$$x^2(2-0) + x(4-0) + \left(\frac{8-0}{3}\right)$$

$$K \int_0^1 2x^2 + 4x + \frac{8}{3} \cdot dx \quad \text{(integrating wrt}$$

$$K\left[\int_0^1 2x^2 dx + \int_0^1 4x \, dx + \int_0^1 \frac{8}{3} \cdot dx\right] = 1$$

$$= k\left[\frac{2\cdot x^3}{3}\Big]_0^1 + \frac{4x^2}{2}\Big]_0^1 + \frac{8}{3}x\Big]_0^1\right]$$

$$= K\left[2\cdot\left(\frac{1}{3}\right) + 4\left(\frac{1}{2}\right) + \frac{8}{3}(1)\right]$$

$$= K\left[\frac{2}{3} + 2 + \frac{8}{3}\right]$$

$$= K\left[\frac{10}{3} + 2\right] = 1$$

nstant)

$$= K\left[\frac{16}{3}\right] = 1$$

$$\boxed{K = \frac{3}{16}}$$

(b) The marginal density of x, denoted as $F_x(x)$ is found by integrating the joint density over all possible values of y:-

$$\Rightarrow F_X(x) = \int_0^2 F_{xy}(x,y)\cdot dy$$

$$\Rightarrow \int_0^2 K(x+y)^2\cdot dy = \int_0^2 \frac{3}{16}(x+y)^2\cdot dy$$

$$\Rightarrow \frac{3}{16}\int_0^2 x^2 + 2xy + y^2\cdot dy.$$

$$\Rightarrow \frac{3}{16}\left[x^2y\Big]_0^2 + \frac{2xy^2}{2}\Big]_0^2 + \frac{y^3}{3}\Big]_0^2\right] = F_x(x)$$

$$\frac{3}{16}\left[x^2(2) + \cancel{x}x\frac{(4-0)}{\cancel{2}} + \frac{8}{3}\right]$$

$$\frac{3}{16}\left[2x^2 + 4x + \frac{8}{3}\right]$$

$$\boxed{F_x(x) = \frac{3}{8}x^2 + \frac{3}{4}x + \frac{1}{2}, \quad (0 \le x \le 1)}$$

IIIly the marginal distribution of y is given as.

$$F_y(y) = \int_0^1 k(x+y)^2 \cdot dx$$

$$= \frac{3}{16}\int_0^1 x^2 + 2xy + y^2 \cdot dx$$

$$= \frac{3}{16}\left[\int_0^1 x^2 \cdot dx + \int_0^1 2xy \cdot dx + \int_0^1 y^2 \cdot dx\right]$$

$$= \frac{3}{16}\left[\left.\frac{x^3}{3}\right]_0^1 + \left.\frac{2x^2}{2}y\right]_0^1 + \left.y^2(x)\right]_0^1\right]$$

$$F_y(y) = \frac{3}{16}\left[\frac{(1)^3-(0)^3}{3} + \cancel{2}\left[(1)^2 - (0)^2\right]y + y^2[1-0]\right]$$

$$F_y(y) = \frac{3}{16}\left[\frac{1}{3} + y + y^2\right]$$

$$f_Y(y) = \frac{1}{16} + \frac{3y}{16} + \frac{3y^2}{16}$$ at $0 \le y \le 2$

(c) To find the conditional distribution of X given Y=y we need to determine the conditional density function $f_{X/Y}(x/y)$. The conditional density is defined as:-

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad \text{— ①}$$

The joint distribution is given as:-

$$f_{XY}(x,y) = \frac{3}{16}(x+y)^2 \quad \text{for } 0 \le x \le 1 \ \& \ 0 \le y \le 2 \quad \text{— ②}$$

The marginal distribution of Y is given as:-

$$f_Y(y) = \frac{1}{16} + \frac{3y}{16} + \frac{3y^2}{16} \quad 0 \le y \le 2. \quad \text{— ③}$$

$$f_{X/Y}(x/y) = \frac{\frac{3}{\cancel{16}}(x+y)^2}{\frac{1}{\cancel{16}}\left[1+3y+3y^2\right]} = \frac{3(x+y)^2}{1+3y+3y^2}.$$

$$\int_0^1 f_{X/Y}(x/y) \cdot dx \quad \left[\begin{array}{l} \text{As } Y=y; \\ X;(0 \le x \le 1) \end{array}\right]$$

$$\Rightarrow \int_0^1 \frac{3(x+y)^2}{1+3y+3y^2} \cdot dx$$ {Note y is a constant}

let $x+y=z$ when $x=0;\ z=y$

$dx=dz$ when $x=1\ z=1+y$

$$\Rightarrow \frac{3}{1+3y+3y^2}\int_{y}^{1+y} z^2 \cdot dz$$

$$\frac{\cancel{3}}{(1+3y+3y^2)}\,\frac{z^3}{\cancel{3}}\Big]_y^{1+y}$$

$$\Rightarrow \frac{(1+y)^3-y^3}{(1+3y+3y^2)}$$

$$\Rightarrow \frac{(1+3y+3y^2+\cancel{y^3})-\cancel{(y^3)}}{1+3y+3y^2} = 1.$$

Thus the normalized condition is satisfied

$\therefore$ The conditional density of X given $Y=y$ is:-

$$\boxed{f_{X/Y}(x/y) = \frac{3(x+y)^2}{1+3y+3y^2},\quad 0\le x\le 1,\ 0\le y\le}$$

The above function correctly describes the probability distribution of X given a particular value of Y.

(6) $C(x_1, x_2) = x_1^2 + x_2^2$ —— (0)

Constraint $\Rightarrow x_1 + 3x_2 = 4 \Rightarrow x_1 + 3x_2 - 4 = 0$

$$F(x_1, x_2, \lambda) = x_1^2 + x_2^2 - \lambda(x_1 + 3x_2 - 4)$$

$$F(x_1, x_2, \lambda) = x_1^2 + x_2^2 - \lambda x_1 - 3\lambda x_2 + 4\lambda$$

F_{x_1} (partial derivative wrt x_1)

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial x_1} = 2x_1 + 0 - \lambda \quad \text{--- (1)}$$

F_{x_2} (partial derivative wrt x_2)

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial x_2} = 2x_2 - 3\lambda \quad \text{--- (2)}$$

F_λ (partial derivative wrt λ)

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial \lambda} = -x_1 - 3x_2 + 4 \quad \text{--- (3)}$$

set $F_{x_1} = 0$; $F_{x_2} = 0$; $F_\lambda = 0$

$\Rightarrow 2x_1 + 0 - \lambda = 0 \Rightarrow 2x_1 = \lambda \Rightarrow \boxed{x_1 = \frac{\lambda}{2}}$ —— (4)

$2x_2 - 3\lambda = 0 \Rightarrow \boxed{x_2 = \frac{3\lambda}{2}}$ -- (5)

$-x_1 - 3x_2 + 4 = 0 \longrightarrow$ (6)

Substituting (4) & (5) in (6) we get

$$-\left(\frac{\lambda}{2}\right) - 3\left(\frac{3\lambda}{2}\right) + 4 = 0$$

$$= \frac{-\lambda}{2} - \frac{9\lambda}{2} + 4 = 0$$

$$-\frac{\lambda}{2} - \frac{9\lambda}{2} + 4 = 0$$

$$-\frac{10\lambda}{2} + 4 = 0$$

$$\frac{10}{2}\lambda = 4$$

$$\boxed{\lambda = \frac{4}{5}} \text{ ——— } (7)$$

sub (7) in (4) & (5) we get

$$x_1 = \frac{\left(\frac{4}{5}\right)}{2} \; ; \quad x_2 = \frac{3\left(\frac{4}{5}\right)}{2}$$

$$\boxed{x_1 = \frac{2}{5}} \to (8) \; ; \quad \boxed{x_2 = \frac{6}{5}} \to (9)$$

subⁿ. (8) & (9) in (0) we get.

$$C\left(\frac{2}{5}, \frac{6}{5}\right) = \left(\frac{2}{5}\right)^2 + \left(\frac{6}{5}\right)^2 = \frac{4+36}{25}$$

$$= \frac{40}{25}$$

$$= \frac{8}{5}.$$

∴ The min of $x_1^2 + x_2^2$ subjected to the Constrain $x_1 + 3x_2 = 4$ is $\frac{8}{5}$ and occurs at the point $\left(\frac{2}{5}, \frac{6}{5}\right)$