ROB-501 - Home Work - 2

Problem-1 Nagation of Statements.

(a) $(P \wedge Q) \Rightarrow \sim(P \wedge Q) \Rightarrow \sim P \vee \sim Q$

Truth Table.

P	~P	Q	~Q	P ∧ Q	~P ∨ ~Q
T	F	T	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	T	F	T	F	T

→ (Table 1)

∴ The column 5 & 6 of Table (1) shows that the $\sim(P \wedge Q)$ is indeed $\sim P \vee \sim Q$.

(b) $(P \vee Q) \Rightarrow \sim(P \vee Q) \Rightarrow \sim P \wedge \sim Q$.

P	~P	Q	~Q	P ∧ Q	~P ∨ ~Q.
T	F	T	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	T	F	T	F	T

→ (Table 2)

∴ The column 5 & 6 of table 1 shows that the $\sim(P \vee Q)$ is indeed $\sim P \wedge \sim Q$.

Problem 2

(a) For every integer n, $2n+1$ is odd

Mathematical form :- $\forall n \in \mathbb{Z}$, $2n+1$ is odd

Nagation:- For some integer n, $2n+1$ is not odd.

Mathematical form:- $\exists n \in \mathbb{Z}$, S.T $2n+1$ is not odd

(b) For some integer n, 2^n+1 is prime

Mathematical form :- $\exists n \in \mathbb{Z}$, 2^n+1 is prime.

Nagation :- For all integers n, $2n+1$ is not prime

Mathematical form:- $\forall n \in \mathbb{Z}$, 2^n+1 is not prime

(c) $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$. Statement:- $\exists v \in \mathbb{R}^n, v \neq 0$ S.T $Av = \lambda v$

Nagation $\rightarrow$ $\cancel{\exists}$ $\forall v \in \mathbb{R}^n$, $v \neq 0$ S.T $Av \neq \lambda v$

(d) let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function, statement:- $\forall \eta > 0, \exists \delta > 0$, S.T $|x| \le \delta \Rightarrow |f(x)| \le \eta |x|$

~~Nagation $\rightarrow$ $\exists \eta > 0, \forall \delta > 0$, ST $|x| \ge \delta \Rightarrow |f(x)| \ge \eta |x|$~~

Nagation $\rightarrow$ $\exists \eta > 0, \forall \delta > 0, \exists x$ with $|x| \le \delta$ & $|f(x)| > \eta |x|$

(Include every thing for the nagation)

Problem 3: let us assume $\sqrt{7}$ is Rational (i.e proof by contra-diction)

if $\sqrt{7}$ is Rational : then

$\sqrt{7} = \frac{p}{q}$ where, p & q have no common factors. p & q ≠ 0 --(0)

[Note - This is the definition of a Rational Number]

$$(\sqrt{7})^2 = \left(\frac{p}{q}\right)^2$$

$$\Rightarrow 7 = \frac{p^2}{q^2} \quad --- (1)$$

$$\Rightarrow \cancel{7q} \; q^2 = \frac{p^2}{7}$$

~~Now in the problem~~

i.e 7 is dividing p^2 then we can assume that 7 divides p, as well

$$\Rightarrow p = 7k \quad (\text{where } k \neq 0)$$

$$\therefore p^2 = 49k^2$$

$$q^2 = \frac{49k^2}{7} = 7k^2 \quad -- (2)$$

(2) implies that q is also divisible by 7.
that mean both p & q have a common factors ~~or~~ that is 7.

The result we have obtained contradicts our original assumption that $\sqrt{7} = \frac{p}{q}$ (where p & q have no common factors)

but our result is showing that 7 is a common factor. That means our original assumption is wrong

$\Rightarrow \sqrt{7}$ is not a Rational Number

$\Rightarrow \sqrt{7}$ is Irrational

Problem-5 Prove that for all integers $n \geq 1$ $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

let $P(n) = \sum_{k=1}^{n} \frac{1}{k+1} = \frac{n}{n+1}$

lets prove this by Induction.

(1) Base Case :- $k=1$ $P(1) = \frac{1}{(1)(1+1)} = \frac{1}{2}$

$\frac{n}{n+1}$; $n=1$ $\Rightarrow \frac{1}{1+1} = \frac{1}{2}$

$\Rightarrow$ LHS = RHS [$\therefore$ The base case is True]

(2) ~~Base~~ $k=2$ (base case 2 ; $n=2$)

$\Rightarrow p(2) = \frac{1}{1*(1+1)} + \frac{1}{(2)(2+1)} \Rightarrow \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$n=2$

$= \frac{2}{2+1} = \frac{2}{3}$

$\Rightarrow$ LHS = RHS ($\therefore$ base case 2 is true)

(3) $k = 3$ (base case 3, $n=3$)

$P(3) = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12}$

$= \frac{2}{3} + \frac{1}{12} = \frac{8+1}{12} = \frac{9}{12} = \frac{3}{4}$

$n=3$

$\Rightarrow \frac{3}{3+1} = \frac{3}{4}$ ($\therefore$ LHS = RHS) $\Rightarrow$ base case 3 is true.

Now lets Assume $\Rightarrow P(n=Q)$ to be true.

$$\Rightarrow P(Q) = \sum_{K=1}^{Q} \frac{1}{K(K+1)} = \frac{Q}{Q+1}$$

$$\Rightarrow \frac{1}{(1)(1+1)} + \frac{1}{(1)(2+1)} + \frac{1}{(1)(3+1)} + \cdots + \frac{1}{(Q)(Q+1)}$$

$$= \frac{Q}{Q+1} \quad \text{---} \quad 1$$

Now lets consider $P(n=Q+1)$

$$\Rightarrow P(Q+1) = \sum_{K=1}^{Q+1} \frac{1}{(K)(K+1)} = \frac{Q+1}{(Q+1)+1} = \frac{Q+1}{Q+2}$$

$$\Rightarrow \frac{1}{1(1+1)} + \frac{1}{(1)(2+1)} + \frac{1}{(1)(3+1)} + \cdots + \frac{1}{(Q)(Q+1)}$$

$$+ \frac{1}{(Q+1)(Q+2)}$$

From 1 W.K.T $\frac{1}{1(1+1)} + \frac{1}{(1)(2+1)} + \frac{1}{(1)(3+1)} + \cdots + \frac{1}{(Q)(Q+1)}$

$$= \frac{Q}{Q+1}$$

$$\Rightarrow \frac{Q}{(Q+1)} + \frac{1}{(Q+1)(Q+2)}$$

$$\frac{1}{(Q+1)}\left[Q+\frac{1}{(Q+2)}\right]$$

$$=\left[\frac{Q(Q+2)+1}{(Q+2)}\right]\frac{1}{(Q+1)}$$

$$=\frac{Q^2+2Q+1}{(Q+2)(Q+1)}$$

w.k.t

$\Rightarrow Q^2+2Q+1=(Q+1)^2$

$$\Rightarrow \frac{\cancel{(Q+1)^2}}{(Q+2)\cancel{(Q+1)}} \Rightarrow \frac{Q+1}{Q+2}$$

$\therefore$ we have proved that $P(Q+1)=\sum_{k=1}^{Q+1}\frac{1}{(k)(k+1)}=\frac{Q+1}{Q+2}$

$\therefore$ we have proved by Induction that ~~for~~

$$\boxed{\forall n\in\mathbb{Z},\ n\geq 1,\ \sum_{k=1}^{n}\frac{1}{(k)(k+1)}=\frac{n}{n+1}}$$

Problem-4 if A is a Square Matrix, if det(A) = 0, then A is not invertable.

Definition of the inverse of a Matrix.
The inverse of a Matrix A is denoted as A^{-1}
& $AA^{-1} = I = A^{-1}A$.

W.K.T if a Matrix is invertable then its inverse exists.

$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

if $\det(A) = 0$, then $\frac{1}{\det(A)} = \frac{1}{0}$ ⇒ undefined.

∴ if det(A) = 0 then A is not invertable.

W.K.T $\det(AB) = \det(A) \cdot \det(B)$.
Lets Assume that A is Actuall invertable.
$\Rightarrow AA^{-1} = I = A^{-1}A$.

$\Rightarrow \det(I) = I$; —— (1)
$\det(A) = 0$ —— (2)

$\Rightarrow \det(I) = \det(A^{-1}) \det(A)$

W.K.T from (1) $\det(I) = 1$

$\det(A^{-1}) \cdot 0 = 0$

⇒ LHS ≠ RHS
∴ $\det(I) \neq \det(A^{-1}) \det(A)$
∴ our Assumption that A is invertable is false
⇒ A is non-invertable if $|\det(A)| = 0$

problem(6)

(a) Prove that for all integers $n \geq 12$, there exists a non-negative integers k_1 & k_2 S.T $n = k_1 4 + k_2 5$.

Base cases

→ let start with the least value of n; $n = 12$

(1) $12 = 3 \times 4 + 0 \times 5 = 12$
(2) $13 = 2 \times 4 + 5 \times 1 = 13$
(3) $14 = 1 \times 4 + 2 \times 5 = 14$
(4) $15 = 3 \times 5 + 0 \times 4 = 15$
(5) $16 = 4 \times 4 + 0 \times 5 = 16$.

So for $n = 12, 13, 14, 15, 16$ the statement holds good, these are the base-cases.

Induction hypothesis. - - - (1)

lets assume that the statement is true for all integers $n \geq 12$ upto an integer k

i.e for every $n = 12; 13; \ldots; k$ there exists a non-negative integer k_1 & k_2 S.T $n = k_1 4 + k_2 5$

Inductive Step.

we want to show the statement holds for $k+1 = n$

by (1) there exists k_1 & k_2 S.T $n = k_1 4 + k_2 5$.

if $k \geq 12$ W.K.T $k = k_1 4 + k_2 5$ & because $k \geq 12$ we can subtract 4 or 5 from k & still represent the difference as a product of 4 or 5 so adding 1 to k must be a value that can be written as a product of 4 or 5. Since the method holds for any $k \geq 12$ the induction step is complete.

$\therefore$ for $n \geq 12$, there exists non-negative integers k_1 & k_2 s.t $n = k_1 4 + k_2 5$.

To check if the statement is true for $n \geq 8$.
lets check for values below 12 as for $n \geq 12$ we have already proved the integers can be represented as $k_1 4 + k_2 5$.

$n = 8$: $8 = 2 \times 4 + 0 \times 5 = 8$
$n = 9$: $9 = 4 \times 1 + 5 \times 1 = 9$
$n = 10$; $10 = 5 \times 2 + 4 \times 0 = 10$
$n = 11$; $11 = NA$ $\rightarrow$ its impossible to represent 11 as $k_1 4 + k_2 5$.

(b) Prove that, for all even integers $n \geq 6$, there exists non-negative integers k_1 & k_2 s.t $n = k_1 3 + k_2 5$.

lets consider the Base cases.

(1) $n = 6$; $6 = 2 \times 3 + 0 \times 5 = 6$ (As it is the least value)
(2) $8 = 5 \times 1 + 3 \times 1 = 8$
(3) $10 = 5 \times 2 + 3 \times 0 = 10$
(4) $12 = 3 \times 4 + 5 \times 0 = 12$.
(5) $14 = 3 \times 3 + 5 \times 1 = 12$

These our the base cases & for all the cases the statement $n = k_1 3 + k_2 5$ holds good.

now lets assume that for $n \geq 6$ to some value k, the statement : $n = k_1 3 + k_2 5$ holds good & all these numbers are even.

$k = k_1 3 + k_2 5$ $\rightarrow$ (induction step)

Now, we have to prove that it holds good for $k+2$.

$\therefore$ for $n = k+2$ (As n is even)

$\Rightarrow \quad k+2 = \underbrace{(k_1 3 + k_2 5)}_{k} + 2$

Now we can represent the above expression as follows.

$\Rightarrow \quad k+2 = (k_1 - 1) \times 3 + (k_2 + 1) \times 5$

$\therefore$ we are able to represent $k+2$ as some form of multiples of 3 & 5.

$\therefore$ The statement holds true for $n = k+2$, even.

Thus for all values of $n \geq 6$ there exists non-negative integers k_1 & k_2 s.t $n = k_1 3 + 5 k_2$

(7) we need to

(a). Maximize $f(x) = x^T M x$ subjected to the constraint $g(x) = x^T x - 1 = 0$

$L(x, \lambda) = x^T M x - \lambda(x^T x - 1) = 0$

$\frac{\partial}{\partial x}(L(x, \lambda)) = \frac{\partial}{\partial x}(x^T M x) - \frac{\partial}{\partial x}(\lambda(x^T x - 1)) = 0$

$\Rightarrow M \frac{\partial}{\partial x}(x^T x) - \lambda \frac{\partial}{\partial x}(x^T x) + 0 = 0$

$2xM - \lambda(2x) = 0$

$\Rightarrow \quad \boxed{Mx = \lambda x} \quad \text{---} \quad (1)$

$\rightarrow$ std eigen value problem. it says that x must be an eigen vector of matrix M & λ the corresponding eigen value

$$\frac{\partial}{\partial\lambda}\left(L(x,\lambda)\right) = \frac{\partial}{\partial\lambda}\left(x^T M x - \lambda(x^T x - 1)\right)$$

$$= \cancel{\frac{\partial}{\partial\lambda}(x^T M x)}^{\,0} - (x^T x - 1)\cdot\cancel{\frac{\partial}{\partial\lambda}(\lambda)}^{\,1}$$

$$\frac{\partial}{\partial\lambda}\left(L(x,\lambda)\right) = -(x^T x - 1) = 0$$

This gives the constraint $x^T x = 1$ meaning x is a unit vector.

From (1) (ie $Mx = \lambda x$) λ is the eigen value correspon-ding to the eigen vector x.

so solving this will give the eigen value of M

$\Rightarrow$ lets consider $F(x) = x^T M x$

$F(x) = x^T M x = \lambda$ (λ is the eigen value corresponding to the eigen vector x) ~~i.e.~~

so the max value of $F(x)$ will be the max value of λ

& the min. value of $F(x)$ will be the min. value of λ

Conclusion

(a) The max value of $x^T M x$ subject to $x^T x = 1$ is λ_{max} where λ_{max} is the largest eigen value of M, & the correspon-ding vector x is the eigenvector associated with λ_{max}

(b) The min value of $x^T M x$ subjected to $x^T x = 1$ is λ_{min} where λ_{min} is the min eigen value of M, & the corres. vector x is the eigen vector associated with λ_{min}