ROB 501 - Mathematics for Robotics
HW #3

Due 23:59 on Thursday, Sept. 19, 2024
To be submitted on Canvas

Preliminaries: Read Chapter 4 of Nagy. Selected chapters of the textbook *Linear Algebra* by Gabriel Nagy are available under `Files` $\to$ `Handouts` $\to$ `Supplementary Material` $\to$ `02_LinearAlgebraAndGeometry.pdf` on Canvas.

1. Nagy, Page 117, Prob. 4.1.1. Note that the x_i are components of the vector, namely

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Remark: Be very brief when giving reasons. For example (a) Not a subspace. *Reason:* Not closed under multiplication by a constant, such as -1.

4.1.1.- Determine which of the following subsets of $\mathbb{R}^n$, with $n \geqslant 1$, are in fact subspaces. Justify your answers.

(a) $\{\mathsf{x} \in \mathbb{R}^n \; : \; x_i \geqslant 0 \quad i = 1, \cdots, n\}$;
(b) $\{\mathsf{x} \in \mathbb{R}^n \; : \; x_1 = 0\}$;
(c) $\{\mathsf{x} \in \mathbb{R}^n \; : \; x_1 x_2 = 0 \quad n \geqslant 2\}$;
(d) $\{\mathsf{x} \in \mathbb{R}^n \; : \; x_1 + \cdots + x_n = 0\}$;
(e) $\{\mathsf{x} \in \mathbb{R}^n \; : \; x_1 + \cdots + x_n = 1\}$;
(f) $\{\mathsf{x} \in \mathbb{R}^n \; : \; \mathsf{A}\mathsf{x} = \mathsf{b}, \; \mathsf{A} \neq 0, \; \mathsf{b} \neq 0\}$.

Figure 1: Q 1

2. Nagy, Page 117, Prob. 4.1.5 (denote the field by $\mathcal{F}$).

4.1.5.- Given two finite subsets S_1, S_2 in a vector space V, show that

$$\mathrm{Span}(S_1 \cup S_2) = \mathrm{Span}(S_1) + \mathrm{Span}(S_2).$$

Figure 2: Q 2

3. Nagy, Page 121, Prob. 4.2.1 (the field is $\mathbb{R}$)

4.2.1.- Determine which of the following sets is linearly independent. For those who are linearly dependent, express one vector as a linear combination of the other vectors in the set.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \right\}$;

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$;

(c) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Figure 3: Q 3

4. Nagy, Page 121, Prob. 4.2.5 (the field is $\mathbb{R}$) (Note: there is a typo. The last part should be "show linearly independent OR dependent.")

4.2.5.- Determine whether the set

$$\left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \right\} \subset \mathbb{R}^{2,2}$$

is linearly independent of dependent.

Figure 4: Q 4

5. Let $(X, \mathcal{F})$ be a vector space and $S \subset X$ a subset (not necessarily a subspace). Prove the following **Claim:** If Y is a subspace of X and $S \subset Y$, then $\text{span}\{S\} \subset Y$.

   Remark: Usually the result is stated as "$\text{span}\{S\}$ is the smallest subspace of X that contains S". The claim is a restatement of this in a form that will make it easier for you to see what needs to be shown.

6. Let $(X, \mathcal{F})$ be a vector space and V and W subspaces of X. Prove the following **Claim:** The following two statements are equivalent:

   (a) $V \cap W = \{0\}$

   (b) For every $x \in V + W$, there exist unique $v \in V$ and $w \in W$ such that $x = v + w$.

   Remark: $V + W := \{v + w \mid v \in V,\ w \in W\}$ and is called the *sum* of V and W. When $V \cap W = \{0\}$, one writes $V \oplus W$ and calls it a *direct sum*. The intersection cannot be empty because the zero vector is an element of every subspace!

Hints

Hints: Prob. 2 It is not important that S_1 and S_2 have a finite number of elements. You need to show a double inclusion, namely

$$\text{span}\{S_1 \cup S_2\} \subset \text{span}\{S_1\} + \text{span}\{S_2\}, \text{ and}$$

$$\text{span}\{S_1\} + \text{span}\{S_2\} \subset \text{span}\{S_1 \cup S_2\}.$$

The main thing is to carefully apply the definition of "span". What does an element of $\text{span}\{S_1\}$ look like, etc.

Hints: Prob. 4 Form a general linear combination of the matrices and set it to the zero matrix. Realize that this gives you a set of simultaneous equations for the coefficients you used in your linear combination (due to the matrices being symmetric, you'll get three equations). Now, check if there exists a nontrivial solution to your equations.

Hints: Prob. 5 If $S_1 \subset S_2$, then how is $\text{span}\{S_1\}$ related to $\text{span}\{S_2\}$? What is the span of a subspace?

Hints: Prob. 6

- You need to show that (a) implies (b) and that (b) implies (a). That is what is meant by equivalent.
- The result is proven in Nagy, Chapter 4. You can copy the proof, using our notation. His vocabulary is slightly different from ours, but that is not important. I am assigning the problem just to force you to read the result and (hopefully) understand it. We'll come back to it in a week or two.
- What does the uniqueness part mean? It means that if $v_1, v_2 \in V$ and $w_1, w_2 \in W$ are such that

  $$v_1 + w_1 = v_2 + w_2$$

  then $v_1 = v_2$ and $w_1 = w_2$.