ROB-501 HW-7.

(1) let V, W be a vector space over the field F.
$\dim(V) = n$

As $\ker T \subset V$ {There exists a basis for it} $\dim \ker T = k$
& let $K := \{v_1, \ldots, v_k\} \subset \ker(T)$

→ Now we can extend K with $(n-k)$ linearly independent vectors $w_1, \ldots, w_{n-k}$ to form a full basis of V.

let

$S := \{w_1, \ldots, w_{n-k}\} \subset V \setminus \ker(T)$

s.t

$B := K \cup S = \{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\} \subset V$
$\operatorname{Im} T = \operatorname{span} T(B) = \operatorname{span}\{T(v_1), \ldots, T(v_k), T(w_1), \ldots, T(w_{n-k})\}$
$\operatorname{span}\{T(w_1), \ldots, T(w_{n-k})\} = \operatorname{span} T(S)$

$T(S)$ is the basis for $\operatorname{Im} T$

Suppose $T(S)$ is not linearly independent, let
$$\sum_{j=1}^{n-k} \alpha_j T(w_j) = 0_W$$
for some $\alpha_j \in F$

$$T\left(\sum_{j=1}^{n-k} \alpha_j w_j\right) = 0_W \Rightarrow \left(\sum_{j=1}^{n-k} \alpha_j w_j\right) \in \ker T$$

$= \operatorname{span} K \subset V$

This is a contradiction as B is a basis, unless $\sum_{j=1}^{n-k} \alpha_j = 0$

This shows that $T(S)$ is linearly independent & basis of $\operatorname{Im} T$

To summarize, we have K, a basis for $\ker T$, & $T(S)$ a basis for $\operatorname{Im}(T)$

Finally we can state that:-

$$\operatorname{Rank}(T) + \operatorname{Nullity}(T) = \dim(\operatorname{Im} T) + \dim \ker T$$
$$= |T(S)| + |K| = n-k + k = n = \dim V$$

(2) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$v^1 = \begin{bmatrix} -0.5000 \\ -0.7071 \\ 0.5000 \end{bmatrix}$ $\lambda_1 = 0.5858$

$v^2 = \begin{bmatrix} 0.7071 \\ -0.000 \\ 0.7071 \end{bmatrix}$ $\lambda_2 = 2.000$

$v^3 = \begin{bmatrix} 0.5000 \\ -0.7071 \\ -0.5000 \end{bmatrix}$ $\lambda_3 = 3.4142$

$$O = \begin{bmatrix} -0.5000 & 0.7071 & 0.5000 \\ -0.7071 & -0.000 & -0.7071 \\ 0.5000 & 0.7071 & -0.5000 \end{bmatrix}$$

$$O^T = \begin{bmatrix} -0.5000 & -0.7071 & 0.5000 \\ 0.7071 & -0.000 & 0.7071 \\ 0.5000 & -0.7071 & -0.5000 \end{bmatrix}$$

Now $OO^T = I \Rightarrow O$ is orthogonal.

$$\Lambda = \begin{bmatrix} 0.5858 & 0 & 0 \\ 0 & 2.000 & 0 \\ 0 & 0 & 3.4142 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -0.5000 & 0.7071 & 0.5000 \\ -0.7071 & -0.000 & -0.7071 \\ 0.5000 & 0.7071 & -0.5000 \end{bmatrix} \begin{bmatrix} 0.5858 & 0 & 0 \\ 0 & 2.000 & 0 \\ 0 & 0 & 3.4142 \end{bmatrix} \begin{bmatrix} -0.5000 & -0.7071 & 0.5000 \\ 0.7071 & -0.000 & 0.7071 \\ 0.5000 & -0.7071 & -0.500 \end{bmatrix}$$

```
The smallest n such that A_k has at least 100 independent columns is: 34
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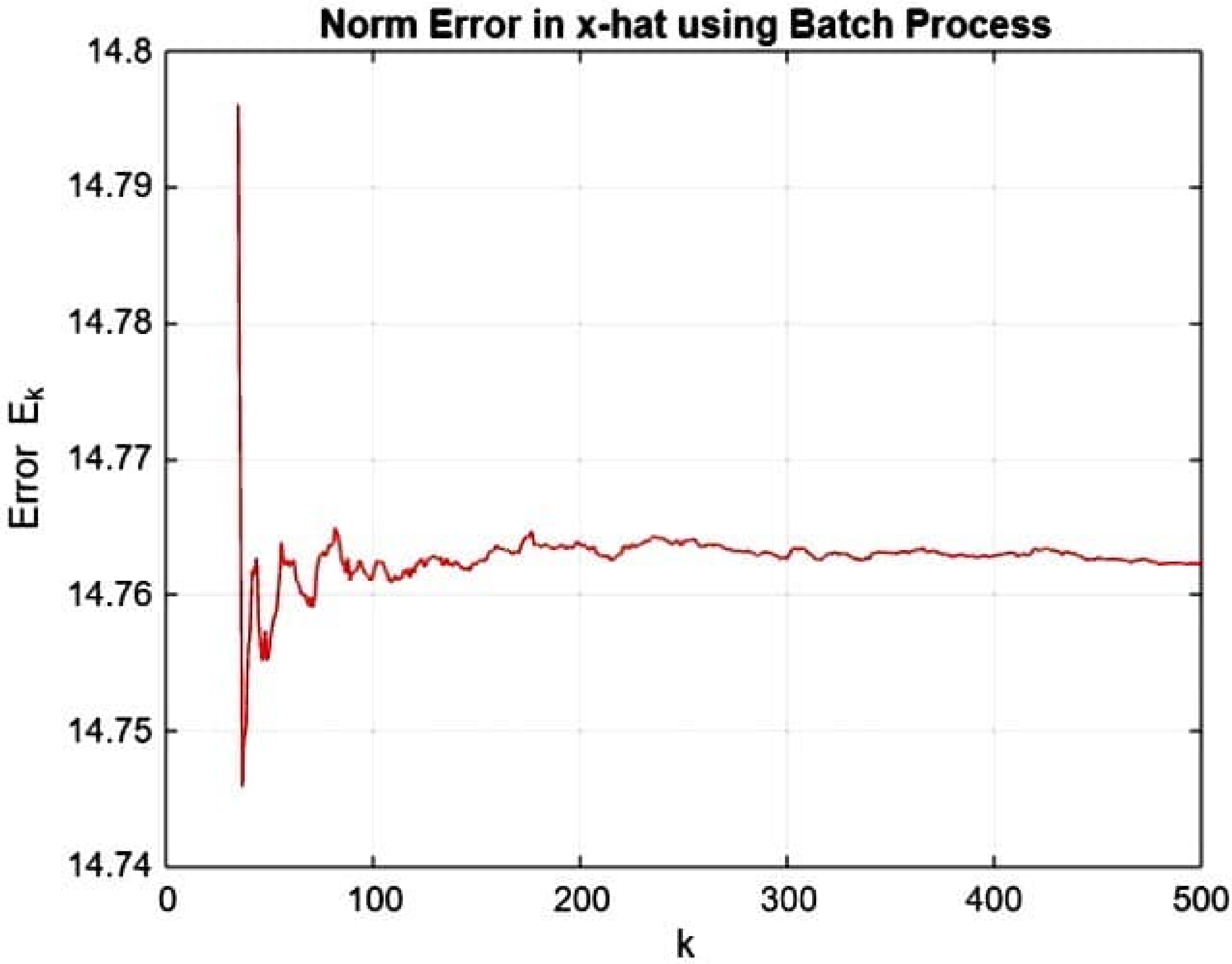
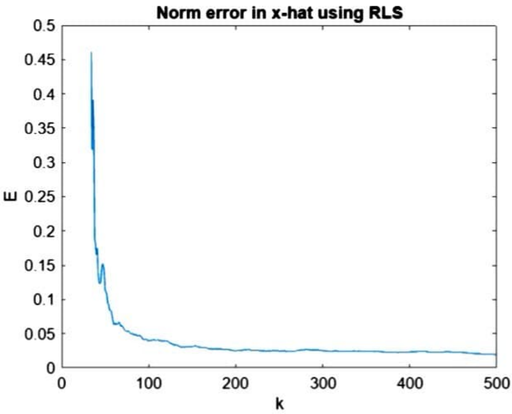


```
%HW 7 Q4c
tic
close all
clear
clc
load("DataHW07_Prob4.mat")
S = eye(3);
temp = cell2mat(C(1));
temp_1 = size(temp);
n = ceil(temp_1(2)/temp_1(1));


for i = 1:n
    if (i == 1)
        Qk = cell2mat(C(1))'*eye(3)*cell2mat(C(1));
        Tk = cell2mat(C(1))'*eye(3)*cell2mat(y(1));
    else
        Qk = Qk + cell2mat(C(i))'*eye(3)*cell2mat(C(i));
        Tk = Tk + cell2mat(C(i))'*eye(3)*cell2mat(y(i));
    end
end
xk = inv(Qk)*Tk;
for k = n:N-1
    Q1 = Qk + cell2mat(C(k+1))'*eye(3)*cell2mat(C(k+1));
    Qk = Q1;
    K_k_1 = inv(Q1) * cell2mat(C(k+1))' * eye(3);
    temp = K_k_1*(cell2mat(y(k+1)) - (cell2mat(C(k+1))*xk));
    x_k_k = xk + temp(:,1);
    xk = x_k_k;

    Ek_temp = x_k_k - cell2mat(x_actual(k));
    Ek_temp1 = Ek_temp.^2;
    if (k == n)
    Ek = sum(Ek_temp1)^0.5;
    else
    Ek = [Ek;(sum(Ek_temp1))^0.5];
    end
end
plot(n:N-1,Ek)
title('Norm error in x-hat using RLS')
xlabel('k')
ylabel('E')
```


```
toc
```

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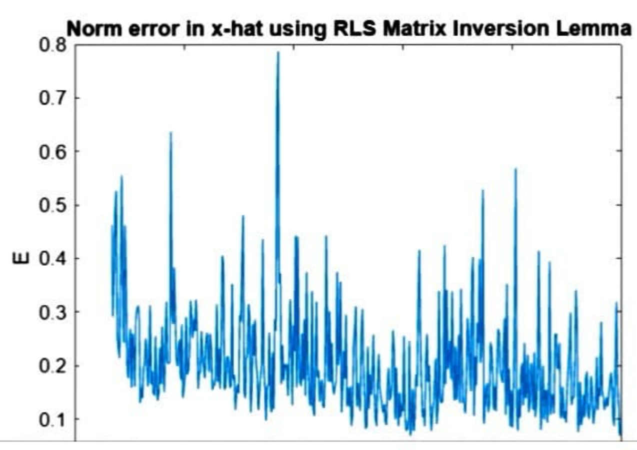
Elapsed time is 0.606846 seconds.
```

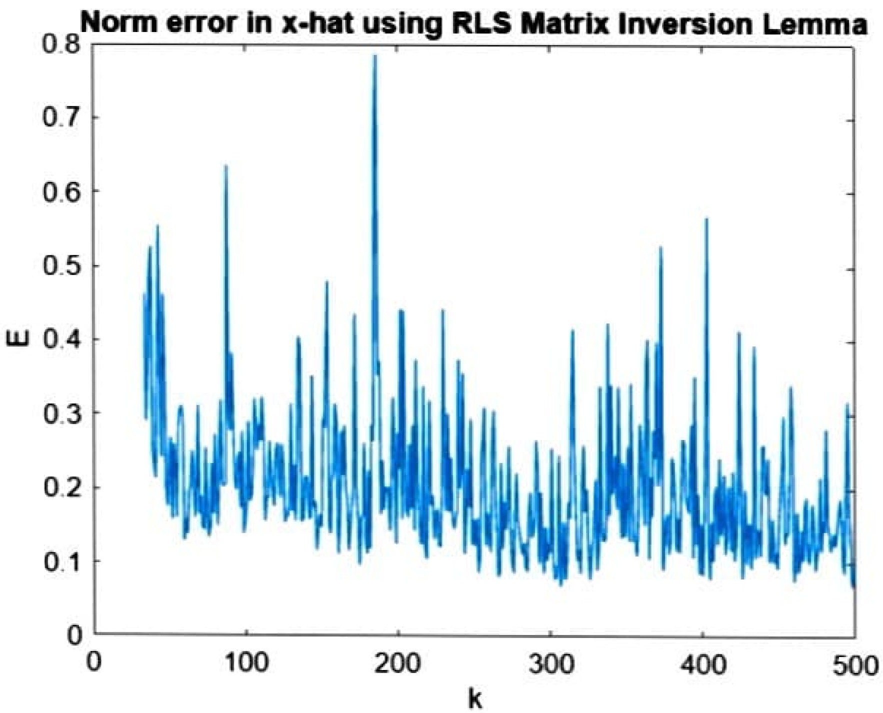
```
%HW 7 Q4d
tic
close all
clear
clc
load("DataHW07_Prob4.mat")
temp = cell2mat(C(1));
temp_1 = size(temp);
n = ceil(temp_1(2)/temp_1(1));
S = eye(3);
for i = 1:n
    if (i == 1)
        Qk = cell2mat(C(1))'*eye(3)*cell2mat(C(1));
        Tk = cell2mat(C(1))'*eye(3)*cell2mat(y(1));
    else
        Qk = Qk + cell2mat(C(i))'*eye(3)*cell2mat(C(i));
        Tk = Tk + cell2mat(C(i))'*eye(3)*cell2mat(y(i));
    end
end
Pk = inv(Qk);
x_k_hat = Pk*Tk;

for k = n:N-1
    Pk1 = Pk - Pk*cell2mat(C(k+1))'*inv(inv(S)...
        + cell2mat(C(k+1))*Pk*cell2mat(C(k+1))')*cell2mat(C(k+1))*Pk;
    Kk1 = Pk1 * cell2mat(C(k+1))' * S;
    Xk1 =x_k_hat + Kk1*(cell2mat(y(k+1))-cell2mat(C(k+1))*x_k_hat);
    x_k_hat = Xk1;

    Ek_temp = x_k_hat - cell2mat(x_actual(k));
    Ek_temp1 = Ek_temp.^2;
    if (k == n)
    Ek = sum(Ek_temp1)^0.5;
    else
    Ek = [Ek;(sum(Ek_temp1))^0.5];
    end
end
plot(n:N-1,Ek)
title('Norm error in x-hat using RLS Matrix Inversion Lemma')
xlabel('k')
ylabel('E')
```

1

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toc
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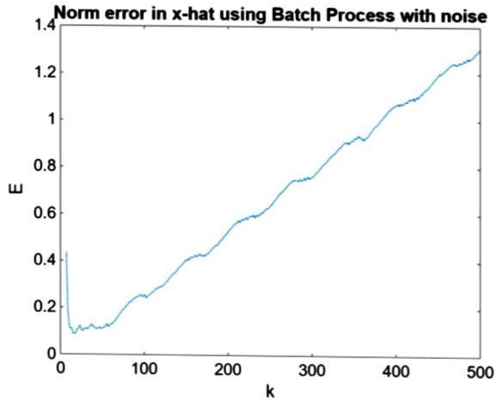
```
%HW 7 Q5a
tic
close all
clear
clc
load("DataHW07_Prob5.mat")
temp = cell2mat(C(1));
temp_1 = size(temp);
n = ceil(temp_1(2)/temp_1(1));
for i = 1:n-1
    if (i == 1)
        Ak = cell2mat(C(i));
        Yk = cell2mat(y(i));
    else
        Ak = [Ak;cell2mat(C(i))];
        Yk = [Yk;cell2mat(y(i))];
    end
end

for k = n:N
    Ak = [Ak;cell2mat(C(k))];
    Yk = [Yk;cell2mat(y(k))];
    Rk = eye(3*k);
    xk_hat = inv(Ak'*Rk*Ak) * (Ak'*Rk*Yk);
    Ek_temp = xk_hat - cell2mat(x_actual(k));
    Ek_temp1 = Ek_temp.^2;
    if (k == n)
    Ek = sum(Ek_temp1)^0.5;
    else
    Ek = [Ek;(sum(Ek_temp1))^0.5];
    end
end
plot(n:N,Ek)
title('Norm error in x-hat using Batch Process with noise')
xlabel('k')
ylabel('E')
```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toc
```

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(3) By solving the code on Matlab we get:

$$\Lambda = O^T A O = \begin{bmatrix} 2.000 & 0 & 0 \\ 0 & -1.0001 & +0\cancel{.0001} \\ 0 & 0 & 2.0001 \end{bmatrix}$$

(6) (a) The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ is +ve semi definite but the eigen values are unique the eigen values are:-

$\lambda_1 = 0.000$

$\lambda_2 = 10.000$

$$A = \underbrace{\begin{bmatrix} -0.948 & 0.316 \\ 0.3162 & 0.9487 \end{bmatrix}}_{O} \underbrace{\begin{bmatrix} 0.000 & 0 \\ 0 & 10.000 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -0.948 & 0.316 \\ 0.316 & 0.948 \end{bmatrix}}_{O^T}$$

(b) eigen values are unique & all eigen values are +ve

$\lambda_1 = 0.1856$

$\lambda_2 = 1.0842$

$\lambda_3 = 44.73$

∴ The matrix $A = \begin{bmatrix} 6 & 10 & 11 \\ 10 & 19 & 19 \\ 11 & 19 & 21 \end{bmatrix}$ is +ve definite

8.

$$A = \underbrace{\begin{bmatrix} -0.875 & 0.3270 & 0.3583 \\ -0.0244 & -0.7674 & 0.6407 \\ 0.4845 & 0.5515 & 0.6791 \end{bmatrix}}_{Q} \times \underbrace{\begin{bmatrix} 0.1856 & 0 & 0 \\ 0.000 & 1.0842 & 0 \\ 0 & 0 & 44.7307 \end{bmatrix}}_{\Lambda} \times \underbrace{\begin{bmatrix} -0.875 & -0.0244 & 0.4845 \\ 0.3270 & -0.7674 & 0.5515 \\ 0.3583 & 0.6407 & 0.6791 \end{bmatrix}}_{Q^T}.$$

9 (c) The Eigen values of these matrix $A = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 4 & 7 \\ 6 & 7 & 10 \end{bmatrix}$ are:-

$\cancel{\lambda_1 = 0.1856}$

$\cancel{\lambda_2 = 1.0842}$

$\cancel{\lambda_3 = 44}$

$\lambda_1 = -2.9165$

$\lambda_2 = 0.000$

$\lambda_3 = 32.9165$

$\therefore$ it is not +ve definite.

(5) (a) $A = \left[\begin{array}{c|c} 1 & 3 \\ \hline 3 & 8 \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$A = A^T$; $B^T = C$; $D = D^T$

$D = 8$; $D > 0$;

$A - BD^{-1}B^T > 0$

$[1] - [3] \times \frac{1}{[8]} \times (3)$

$\Rightarrow 1 - \frac{9}{8} = 1 - 1.125 = -0.125$

$\Rightarrow -0.125 < 0$.

$\therefore$ The Matrix A is not positive definite.

(b) $A = \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 4 & 7 \\ \hline 6 & 7 & 10 \end{array}\right]$ $A = A^T$; $C = B^T$; $D = D^T$ $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$D > 0$ (As $D = 10$)

$A - BD^{-1}B^T \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 6 \\ 7 \end{bmatrix}\begin{bmatrix} 6 & 7 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3.6 & 4.2 \\ 4.2 & 4.9 \end{bmatrix}$

$$\Rightarrow \begin{array}{cc} A & B \\ \left[\begin{array}{c|c} -2.6 & -4.2 \\ \hline -4.2 & -0.9 \end{array}\right] \\ C & D \end{array} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$D < 0$; As $D = -0.9$

Since $D < 0$ The matrix A fails the test & hence it is not +ve definite

7(c) $A = \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 5 & 7 \\ \hline 6 & 7 & a \end{array}\right]$ $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$A = A^T$

$B^T = C$

$D = D^T$

~~$D > 0$~~

~~$\Rightarrow a > 0$.~~

$A > 0$;

$$\Rightarrow \begin{array}{cc} A' & B' \\ \left[\begin{array}{c|c} 1 & 2 \\ \hline 2 & 5 \end{array}\right] \\ C' & D' \end{array}$$

$A' = A'^T$

$D' = D'^T$

$B'^T = C'$

$\Rightarrow A' > 0$; $5 - [2][1][2] = 1$

$\Rightarrow 1 > 0$.

$\therefore \boxed{A > 0}$.

$D - B^T A^{-1} B = a - [6\ 7] \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} > 0$

$\Rightarrow a > [6\ 7] \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

$$\begin{bmatrix} 30-144 & -12+7 \\ & \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 16\times 6 & -5\times 7 \end{bmatrix}$$

$$= 61$$

$$\boxed{a > 61}$$

(6) ~~(5)~~ (a) $\underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b}$ (using std inner product on $\mathbb{R}^3$)

$$Ax - b = 0$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{x} = A^T (A \cancel{S^{-1}} A^T)^{-1} b$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 3 & 8 \\ 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix}$$

computing $\hat{x}$ we get

$$\hat{x} = \begin{bmatrix} 1 & 3 \\ 3 & 8 \\ 2 & 4 \end{bmatrix} \left[\begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 8 \\ 2 & 4 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -0.0952 \\ 0.0476 \\ 0.4762 \end{bmatrix}$$

8(b)

$$\langle x, y \rangle = x^T \underbrace{\begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix}}_{S} y$$

→ the inner product is different now.

$$S = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix}$$

$\hat{x} = S^{-1} A^T (A S^{-1} A^T)^{-1} b$. {because now the inner product has changed}

$$S^{-1} = \begin{bmatrix} 8.25 & -2.000 & -4.2500 \\ -2.000 & 1.000 & 1.0000 \\ -4.2500 & 1.000 & 2.25 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -0.6497 \\ 0.3248 \\ 0.3376 \end{bmatrix}$$

6(a) continued

$$\Lambda^{1/2} = \begin{bmatrix} 0 & 0 \\ 0 & 3.16 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} -0.948 & 0.316 \\ 0.3162 & 0.9482 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3.16 \end{bmatrix} \begin{bmatrix} -0.948 & 0.316 \\ 0.316 & 0.948 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} 0.3159 & 0.9474 \\ 0.9474 & 2.8411 \end{bmatrix}$$

6(b)

$$A^{1/2} = \begin{bmatrix} -0.8745 & 0.3270 & 0.3583 \\ -0.0244 & -0.7674 & 0.6407 \\ 0.4845 & 0.5515 & 0.6391 \end{bmatrix} \begin{bmatrix} \sqrt{0.19} & 0 & 0 \\ 0 & \sqrt{1.08} & 0 \\ 0 & 0 & \sqrt{44.73} \end{bmatrix} \begin{bmatrix} -0.8 & 0.32 & 0.35 \\ 0.024 & 0.78 & 0.64 \\ 0.4845 & 0.5 & 0.67 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} 1.30 & 1.28 & 1.63 \\ 1.28 & 3.36 & 2.46 \\ 1.63 & 2.46 & 3.50 \end{bmatrix}$$