# ROB-501 - HW · #4

(1)

$$S = \left\{ \overset{w_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}}, \overset{w_2}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}}, \overset{w_3}{\begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}}, \overset{w_4}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \overset{w_5}{\begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}} \right\}$$

The field is $\mathbb{R}$ & the vector space is $\mathbb{R}^4$

Step 1

To find the no. of vectors that are linearly dependent.

$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$ → This is linearly independent as no vectors in set S can sup. this vector.

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ → This vector is also linearly independent as no vectors in set S can sup. this vector.

$$\begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} = 4 \times \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} - 2 \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

∴ The vector $w_3 = \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}$ is linearly dependent.

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ → This vector is also linearly independent as no vectors in set S can sup this vector.

$$\begin{bmatrix}3\\3\\0\\6\end{bmatrix} \Rightarrow \begin{bmatrix}1\\2\\-1\\3\end{bmatrix} + \begin{bmatrix}1\\0\\0\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\0\\6\end{bmatrix}$$

∴ This vector is linearly dependent.

⇒ we can omit the vectors that are linearly dependent.

hence now the subset of set (S) can be written as

$$S^* = \left\{ \begin{bmatrix}1\\2\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$$

Now the set $S^*$ is linearly independent.

& they originate from the set S.

∴ The Rank of set 'S' is equal to 3

Note → Rank of a vector space is the no. of linearly independent vectors present in that space.

∴ Rank = 3 (As 3 - linearly independent vectors are present)

Now if these 3 - linearly independent vectors in the set $S^*$ ~~form~~ span the whole vector space, linearly independent & $S^* \subset S \subseteq V$ then we can tell that the vectors in $S^*$ form the basis of S

(i) $S^* \subset S$ (✓) {Proved → As $S^*$ was obtained from S}

(ii) $S^*$ is linearly independent.

To prove $S^*$ is linearly independent $\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 = 0$

$\alpha_1 = \alpha_2 = \alpha_3 = 0$

lets consider this ~~case~~ senario

$$S^* = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 0$ --- (1)

$2\alpha_1 + 0 + \alpha_3 = 0$ --- (2)

$-\alpha_1 + 0 + \alpha_3 = 0$ --- (3)

$3\alpha_1 + 2\alpha_2 + \alpha_3 = 0$ --- (4)

$\alpha_1 = \alpha_3$ --- (5)

$2\alpha_3 + \alpha_2 = 0$ --- (6)

for eq$^n$ (6) to be $= 0$ $\alpha_3 = 0$ ; $\alpha_2 = 0$

since $\alpha_3 = 0$ ; $\Rightarrow \alpha_1 = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ The vectors in $S^*$ are linearly independent.

(iii) Span$\{S^*\}$ $\Rightarrow$ is the linear combination of the vectors in the set $S^*$ & this linear combination can also represent the vectors in $S$. (This is already shown)

$\Rightarrow$ Span$\{S^*\}$ $\Rightarrow$ Span$\{S\}$

$\therefore$ The vectors in $S^*$ are indeed the basis of $S$.

$\Rightarrow$ The rank of $S^* = 3$ = rank of $S$ is the dimension of the Span$\{S\}$ ***.

$\therefore$ $\boxed{\text{The dimension of Span}\{S\} = 3}$

2. $v = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$ $u_{1s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_{2s} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $u_{3s} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\alpha_1 u_{1s} + \alpha_2 u_{2s} + \alpha_3 u_{3s} = v$$

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 8 \quad \text{---} \; (1)$$

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 7 \quad \text{---} \; (2)$$

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 4 \quad \text{---} \; (3)$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}}_{B}$$

$$\cancel{\alpha} \quad A\alpha = B$$

$$\alpha = \cancel{BA^{-1}} \; A^{-1}B$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 16-7 \\ -8+14-4 \\ -7+4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\boxed{\alpha_1 = 9 \; ; \alpha_2 = 2 ; \; -3 = \alpha_3}$$ ***

(3) $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$u_{1s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $u_{2s} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $u_{3s} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\cancel{e_1 = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{---} (1)$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \beta_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{---} (2)$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \xi_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \xi_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \xi_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{----} \ 3$$

Solving eq (1), (2), & (3) we get.

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1.0 \end{bmatrix}$$

$\therefore$ The matrix $P = \begin{bmatrix} \alpha_1 & \beta_1 & \xi_1 \\ \alpha_2 & \beta_2 & \xi_2 \\ \alpha_3 & \beta_3 & \xi_3 \end{bmatrix}$

$$\left\{ P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right\}$$

(5)

(a) if a Sub-space has to be considered as a basis of a Vector space it has to satisfy the following conditions.

(i) The subspace has to be a part of the Vector space

$\therefore$ it is given that $M \subset \mathbb{R}^{2\times 2}$. hence this condition is satisfied.

(ii) The Elements in the sub-space must be linearly independent.

$$\Rightarrow M = \{M_1, M_2, M_3, M_4\}$$

$$= \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

have to be linearly independent

$$\Rightarrow \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = 0 \quad ; \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

$$\alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0$$

$$\cancel{\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4 = 0} \qquad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\cancel{\alpha_1 + \alpha_2 + \alpha_3}$$

$$\begin{bmatrix} 0 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{bmatrix} + \begin{bmatrix} \alpha_4 & 0 \\ 0 & -\alpha_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_3+\alpha_4 & \alpha_1-\alpha_2 \\ \alpha_1+\alpha_2 & \alpha_3-\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\alpha_3 + \alpha_4 = 0$ -- ①

$\alpha_1 - \alpha_2 = 0 \Rightarrow \boxed{\alpha_1 = \alpha_2}$ -- ②

$\alpha_1 + \alpha_2 = 0$ -- ③

$\alpha_3 - \alpha_4 = 0 \Rightarrow \boxed{\alpha_3 = \alpha_4}$ -- ④

if eq$^n$ ① & ③ have to be true then

$\boxed{\alpha_1 = \alpha_2 = 0}$ ; $\boxed{\alpha_3 = \alpha_4 = 0}$

$\Rightarrow \boxed{\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0}$

$\therefore$ The set of matrices in M are linearly independent

(iii) Span{M} must includ all the elements of $\mathbb{R}^{2\times2}$ for it to be considered as a basis of $\mathbb{R}^{2\times2}$

As the elements of M are linearly independent as shown from (ii) we can get any matrix in $\mathbb{R}^{2\times2}$ by using the linear-combination $\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4$

$\therefore$ span{M} contains every element in $\mathbb{R}^{2\times2}$ ;

$\Rightarrow M = \{M_1, M_2, M_3, M_4\}$ is the basis of $\mathbb{R}^{2\times2}$ //

5(b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$M = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

$A = \alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \alpha_3 + \alpha_4 & \alpha_1 - \alpha_2 \\ \alpha_1 + \alpha_2 & \alpha_3 - \alpha_4 \end{bmatrix}$

$\alpha_3 + \alpha_4 = 1$ $\qquad$ $\alpha_1 - \alpha_2 = 2$

$\alpha_1 + \alpha_2 = 3$ $\qquad$ $\alpha_3 - \alpha_4 = 4$

$2\alpha_3 = 5$ ; $\boxed{\alpha_3 = \frac{5}{2}}$

$2\alpha_1 = 5$ ; $\boxed{\alpha_1 = \frac{5}{2}}$

$\frac{5}{2} - \alpha_2 = 2$

$-\alpha_2 = 2 - \frac{5}{2} \Rightarrow -\alpha_2 = -\frac{1}{2} \Rightarrow \boxed{\alpha_2 = \frac{1}{2}}$

$-\alpha_4 = 4 - \frac{5}{2} \Rightarrow -\alpha_4 = \frac{3}{2} \Rightarrow \boxed{\alpha_4 = \frac{-3}{2}}$

$$\therefore \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ 5/2 \\ -3/2 \end{bmatrix}$$

* * *

F)

(6) (a) $S = \{ P_0 = 1,\ P_1 = x,\ P_2 = x^2 \}$

$r(x) = 2 + 3x - x^2$

$$r(x) = \alpha_1 (P_0) + \alpha_2 (P_1) + \alpha_3 (P_2)$$

$$2 + 3x - x^2 = \alpha_1 (1) + \alpha_2 (x) + \alpha_3 (x^2)$$

~~(b) New basis $q_0 = 1,\ q_1 = 1-x,\ q_2 = x+x^2$~~

$$2 + 3x - x^2 = (\alpha_1) + x(\alpha_2) + x^2(\alpha_3)$$

$$\cancel{2 + 3x - x^2} = (\alpha_1 - 2) + x(\alpha_2 - 3) + x^2(\alpha_3 + 1) = 0$$

$$\alpha_1 - 2 = 0;\quad \alpha_2 - 3 = 0;\quad \alpha_3 + 1 = 0$$

$$= \boxed{\alpha_1 = 2;\quad \alpha_2 = 3,\quad \alpha_3 = -1}$$

(b) $r(x) = 2 + 3x - x^2$

New basis $q_0 = 1;\ q_1 = 1 - x,\ q_2 = x + x^2$

$$2 + 3x - x^2 = q_0(\alpha_1) + q_1(\alpha_2) + q_2(\alpha_3)$$

$$2 + 3x - x^2 = \alpha_1 + \alpha_2(1-x) + \alpha_3(x + x^2)$$

$2+3x-x^2 = \alpha_1 + \alpha_2 - \alpha_2 x + \alpha_3 x + \alpha_3 x^2$

$2+3x-x^2 = \alpha_3 x^2 + (\alpha_3 - \alpha_2)x + (\alpha_1+\alpha_2)$

$\cancel{\alpha_3 = 2 \; ; \; (\alpha_3 - \alpha_2) = 3}$

$\alpha_1 + \alpha_2 = 2$

$(\alpha_3 - \alpha_2) = 3$

$\alpha_3 = -1$

$(-1) - (\alpha_2) = 3$

$-\alpha_2 = 4 \Rightarrow \boxed{\alpha_2 = -4}$

$\boxed{\alpha_1 = 6}$

$\boxed{\alpha_3 = -1}$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$$

(7) let $F = \mathbb{R}$ + let $X$ be the set of $2\times2$ matricies with real coreffecients. Define $L: X \to X$ by

$$L(M) = 2(M + M^T)$$

(a) To show $L$ is a linear operator.

lets consider $M = \alpha M_1 + \beta M_2$

$$L(\alpha M_1 + \beta M_2) = 2(\{\alpha M_1 + \beta M_2\} + \{\alpha M_1 + \beta M_2\}^T)$$

$$= 2(\{\alpha M_1 + \beta M_2\} + \alpha M_1^T + \beta M_2^T)$$

$$= 2(\alpha(M_1 + M_1^T) + \beta(M_2 + M_2^T))$$

$$= \alpha\left[2(M_1 + M_1^T)\right] + \beta\left[2(M_2 + M_2^T)\right]$$

$$= \alpha L(M_1) + \beta L(M_2)$$

$$\therefore \boxed{L(\alpha M_1 + \beta M_2) = \alpha L(M_1) + \beta L(M_2)}$$

↳ This shows that $L$ is a linear operator.

(b) The std. basis which are given are :-

$$E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$L(M) = 2(M + M^T)$$

$$= L(E^{11}) = 2\left[E^{11} + (E^{11})^T\right]$$

$$= 2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$= \cancel{2}4\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 0\end{bmatrix}$$

$$\left[L(E^{11})\right]_E = \begin{bmatrix}4\\0\\0\\0\end{bmatrix} \longleftarrow ①$$

III'ly

$$L(E^{12}) = 2\left[E^{12} + \left[E^{12}\right]^T\right]$$

$$= 2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$$

$$\left[L(E^{12})\right]_E = \begin{bmatrix}0\\2\\2\\0\end{bmatrix}$$

$$L(E^{21}) = 2\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 2\\2 & 0\end{bmatrix} \Rightarrow \left[L(E^{21})\right]_E = \begin{bmatrix}0\\2\\2\\0\end{bmatrix}$$

$$L(E^{22}) = 2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + 2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 4\end{bmatrix} \Rightarrow L(E^{22}) = \begin{bmatrix}0\\0\\0\\4\end{bmatrix}$$

$$A = \left[ L(E^{11}) \;\middle|\; L(E^{12}) \;\middle|\; L(E^{21}) \;\middle|\; L(E^{22}) \right]$$

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

8 (a) $F \in \mathbb{C}$, $A \in \mathbb{C}^{n \times n}$, $L : \mathbb{C}^n \to \mathbb{C}^n$, $[L(x)]_B = A[x]_B$

$[L(x)]_E = \hat{A}[x]_E$.

Changing the basis from B to E, we get

$[x]_E = P[x]_B$

$[L(x)]_E = \hat{A} P [x]_B$ — ①

Now $[L(x)]_E$ is a vector with basis E & $[L(x)]_B$ is the same vector with basis B

$\Rightarrow$ ~~$[L(x)]_B = P[L(x)]_E$~~ $[L(x)]_E = P[L(x)]_B$

~~$[L(x)]_E = P^{-1}[L(x)]_B$~~ $\Rightarrow [L(x)]_B = P^{-1}[L(x)]_E$.

$[L(x)]_B = P^{-1}\left[\hat{A} P[x]_B\right]$

$[L(x)]_B = \cancel{P^{-1}P\hat{A}}\ (P^{-1}\hat{A}P)\left[P[x]_B\right]$

$[L(x)]_B = A\left[P[x]\right]_B$

$\Rightarrow \boxed{A = P^{-1}\hat{A}P}$

→ $P$ is an Indentity matrix over the field $\mathbb{C}^n$ in order to prove this.

lets find the change of basis from $B$ to $E$.

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} j, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} j \cdots\cdots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} j \right\}$$

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} j \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ j \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ j \end{bmatrix} \right\}$$

$$P = \Big[ [B_1]_E \,\Big|\, [B_2]_E \,\Big|\, [B_3]_E \cdots\cdots \Big|\, [B_m]_E \Big]$$

$$= \big[ e_1 \,|\, e_2 \,|\, \cdots\cdots \,|\, e_n \big] = I.$$

Since $P$ is the Identity matrix.

$$A = P^{-1} \hat{A} P \quad \text{becomes} \quad \boxed{A = \hat{A}}$$

This is the relation b/t $A$ & $\hat{A}$.

8(b) Let us Assume that the Eigen vectors we have obtained from these unique Eigen Values are

$$v = \{v^1, v^2, v^3, \ldots v^n\}$$ → these are the eigen values of A.

Then by definition $[L(x)]_B = A[v^i]_B = \lambda_i [v^i]_B$.

where $v^i$ is the $i^{th}$ eigen vector.

$\lambda_i$ is the $i^{th}$ eigen value.

Now lets consider the change of basis.

$$v_B = \{[v^1]_B ; [v^2]_B ; \ldots [v^n]_B\}$$

$$[v^i]_B = [e^i]_E.$$

→ Rep the eigen vector's in the Std. base.

$$\mathcal{E} = \{e^1, e^2, e^3, \ldots e^n\}$$

By def. of $\hat{A}$, since L is a linear - operator :-

$$\hat{A} = \left[ L(e^1) \mid L(e^2) \mid \ldots \mid L(e^n) \right]$$

$$= \left[ [[L(v^1)]_B]_E \mid [[L(v^2)]_B]_E \ldots \mid [[L(v^n)]_B]_E \right]$$

$$\Rightarrow \left[ [\lambda_1 [v^1]_B]_E \mid [\lambda_2 [v^2]_B]_E \ldots \mid [\lambda_2 [v^n]_B]_E \right]$$

$$\Rightarrow \underbrace{\left[\, e^1 \,|\, e^2 \,|\, \cdots \,|\, e^n \,\right]}_{I} \operatorname{dig}(\lambda_1, \lambda_2, \cdots \lambda_n)$$

$$\Rightarrow \hat{A} = \operatorname{dig}(\lambda_1, \lambda_2, \cdots \lambda_n)$$

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

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