ROB 501 - Mathematics for Robotics
HW #11

Due 23:59 on Thursday, Dec. 06, 2024
To be submitted on Canvas

Remarks: Last two questions are optional since we will cover those subjects in the last two lectures. You will get extra credit if you solve them.
Definition: Let $(\mathcal{X}, \mathbb{R})$ be a vector space. Two norms $||\cdot|| : \mathcal{X} \to [0, \infty)$ and $|||\cdot||| : \mathcal{X} \to [0, \infty)$ are equivalent if there exist positive constants K_1 and K_2 such that, for all $x \in \mathcal{X}$,

$$K_1|||x||| \leq ||x|| \leq K_2|||x|||.$$

Remark: It follows from the definition of equivalent norms that $\frac{1}{K_2}||x|| \leq |||x||| \leq \frac{1}{K_1}||x||$.

Theorem: A vector space $(\mathcal{X}, \mathbb{R})$ is finite dimensional if, and only if, all norms defined on it are equivalent.

1. Consider the normed space $(\mathbb{R}, \mathbb{R}, |\cdot|)$, the set of reals equipped with the absolute value as the norm. Consider the set $S \subset \mathbb{R}$ given as $S = (-2, 3) \cup (3, 4) \cup \{5\} \cup [7, \infty)$. For parts (a)-(b)-(c), provide the set being asked for (e.g., if you were asked about the interior of $\mathbb{R}$, the answer would be $\mathbb{R}$).

   (a) Compute the interior $\mathring{S}$ of S.

   (b) Compute the set of all limit points of S.

   (c) Compute the boundary ∂S of S.

2. For any $x \in \mathbb{R}^n$, show that

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$$
$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$$
$$\|x\|_\infty \leq \|x\|_1 \leq n\,\|x\|_\infty$$

   In the above, you may wish to try the case $x \in \mathbb{R}^2$ first. In fact, working correctly the problem for $x \in \mathbb{R}^2$ will earn full credit.

3. Assume that $||\cdot||$ and $|||\cdot|||$ are equivalent norms on $(\mathcal{X}, \mathbb{R})$, with K_1 and K_2 defined in the definition.

   (a) Let $B_a(x_0)$ be an open ball of radius $a > 0$ about x_0 in the norm $||\cdot||$ and let $\widetilde{B}_r(x_0)$ be an open ball of radius $r > 0$ about x_0 in the norm $|||\cdot|||$. Show that

$$\widetilde{B}_{\frac{a}{K_2}}(x_0) \subset B_a(x_0) \subset \widetilde{B}_{\frac{a}{K_1}}(x_0)$$

   (b) Show that a set P is open in $(\mathcal{X}, \mathbb{R}, ||\cdot||)$ if, and only if, it is open in $(\mathcal{X}, \mathbb{R}, |||\cdot|||)$. In other words, equivalent norms define the same open[1] sets.

[1] Because a set is closed if, and only if its complement is open, we see that equivalent norms also define the same closed sets.

(c) Show that a sequence (x_n) is Cauchy in $(\mathcal{X}, \mathbb{R}, ||\cdot||)$ if, and only if, it is Cauchy in $(\mathcal{X}, \mathbb{R}, |||\cdot|||)$. In other words, equivalent norms define the same Cauchy sequences and similarly, the same convergent sequences.

4. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at x_0, and if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$. Conversely, if f is discontinuous at x_0, then there exists a sequence (x_n) such that $\lim_{n\to\infty} x_n = x_0$, but the sequence defined by $y_n = f(x_n)$ does not converge to $f(x_0)$.

5. Find x_0 such that $F(x_0) = 0$, where

$$F(x_1, x_2) = F(x) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} x - xx^\top \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Is the answer unique? Try several initial guesses and see what you get. Note that

$$xx^\top = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix}.$$

6. The following are some TRUE/FALSE style questions that might be similar to what will be in the exam. Please state whether the statements are TRUE or FALSE together with a short justification.

(a) A robot part has a projected lifetime X in days that is modeled as $X \sim exp(0.001)$, where $exp(0.001)$ is the exponential distribution with rate parameter $\lambda = 0.001$. Then the probability that it will fail within one year is less than 0.31.

(b) We have a system $y = Cx + \epsilon$, where $y, \epsilon \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $C \in \mathbb{R}^{m\times n}$, where $\epsilon \sim N(\mu, Q)$ is the error, and C is the observation model with linearly independent columns. Then $\hat{x} = Ky$ is an unbiased estimator of x if $KC = I \in \mathbb{R}^{n\times n}$.

(c) Consider random variables X_1 and X_2 with $\text{cov}(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then X_1 and X_2 are independent.

7. **(OPTIONAL)** Use the `quadprog` command in MATLAB to solve the following problems;

(a) The under determined problem

$$\widehat{x} = \arg \min_{A_{eq}x = b_{eq}, A_{in}x \le b_{in}} ||x||_2,$$

where

$$A_{eq} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } b_{eq} = \begin{bmatrix} 3 \end{bmatrix}.$$

$$A_{in} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \text{ and } b_{in} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

(b) A cost function with an offset

$$\widehat{x} = \arg \min_{A_{in}x \le b_{in}} (x - x_0)^\top Q(x - x_0),$$

where

$$Q = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 8 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$A_{in} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \text{ and } b_{in} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

8. **(OPTIONAL)** Use the `linprog` command in MATLAB to solve the over determined problem

$$\widehat{x} = \arg\ \min ||Ax - b||_1,$$

for

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 7 \\ 5 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 2 \\ 12 \end{bmatrix}.$$

Repeat the problem using the ∞-norm (i.e., $||\cdot||_\infty = \max\{|x_1|, |x_2|\}$) and note that the answers are not the same.

Remark: Equivalent norms do not give rise to the same optimization problems! So, if you seek to prove that something converges and you are working in a finite-dimensional normed space, you can use whatever norm you wish to prove convergence. However, for an optimization problem, the natural notion of distance is often imposed by the "physics" of the problem, and thus you may not have a choice.

Remark: Another way to access the optimization tools in MATLAB is through `optimtool`. Type it in a command window, and it will open a GUI.

Hints

Hints: Prob. 2 When working with the 2-norm, it is often easier to work with its square. For example, proving

$$\|x\|_2 \leq \|x\|_1$$

is equivalent to proving

$$\|x\|_2^2 \leq \|x\|_1^2,$$

that is

$$|x_1|^2 + |x_2|^2 \leq (|x_1| + |x_2|)^2,$$

which is clearly true! Each of the inequalities requires a clever arrangement of terms. Just play with the them and do the best you can. What is important here is to become aware that bounds relating the most common norms on $\mathbb{R}^n$ are well known.

Handy Inequality: $2ab \leq \left(a^2 + b^2\right)$. It comes from the fact that $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$ and rearranging terms.

Hints: Prob. 3 (a) It is just a matter of applying inequalities. Write down what it means for $x \in B_a(x_0)$ and relate that to $|||x - x_0|||$. For (b), look at the definition of an open set.

Hints: Prob. 4 Proving that $(f(x_n)) \to f(x_0)$ when $x_n \to x_0$ and f is continuous at x_0 is a matter of applying the definitions. First apply the definition of continuity to find a condition on x and x_0 to ensure that $f(x)$ is within ϵ of $f(x_0)$. Then apply the definition of convergence of the sequence (x_n) to make that condition hold true for x_n and x_0.

The other direction is not nearly as easy. What makes it hard is that you have to write down what it means for a sequence NOT to converge and what it means for a function NOT to be continuous at a point. I will give you these properties, but not more help, even in office hours.

Sequence does not converge: y_n does not converge to y_0 if there exists some $\epsilon > 0$ such that, for all $N < \infty$, there exists $n \geq N$ such that $|y_n - y_0| \geq \epsilon$.

Discontinuous at a point: f is discontinuous at x_0 if there exists some $\epsilon > 0$ such that, for all $\delta > 0$, there exists $x \in B_\delta(x_0)$ such that $|f(x) - f(x_0)| \geq \epsilon$.

What happens if you set $\delta = \frac{1}{n}$ and choose x_n so that $|x_n - x_0| < \frac{1}{n}$, but ....I cannot do it all for you!

Hints: Prob. 5 Newton-Raphson Algorithm.

Hints: Prob. 6 For (a), you can find the definition of the exponential distribution (and its density function) here: `https://en.wikipedia.org/wiki/Exponential_distribution`

Hints: Prob. 7 For (b), note that $(x-x_0)^\top Q(x-x_0) = x^\top Qx - 2x_0^\top Qx + x_0^\top Qx_0^\top$. Also note that adding or subtracting a constant changes the VALUE of the function being minimized, but does NOT change the ARGUMENT of the MINIMUM.

If we want to use Python instead of MATLAB for this problem, we can utilize the tools from the `scipy` library. In particular, we can solve this quadratic program as a general minimization problem using the function `scipy.optimize.minimize`. This function takes two argument: a function representing the objective (a function f mapping x to $||x||_2$ for part a) and an initial guess (for example $x_0 = [0, 0, 0, 0]$).

Hints: Prob. 8 You need the lecture notes to know how to set these up as linear programming problems! If we want to use Python instead of `LINPROG` in MATLAB, we can use the similar `scipy` function `scipy.optimize.linprog`.