ROB 501 HW #5

1(a) $A_3 = \begin{bmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 3-\lambda & 10 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$\det[A - \lambda I] = 0$

$\Rightarrow (3-\lambda)\left[(2-\lambda)(1-\lambda) - 0\right] - 10\left[0-0\right] + 0(0)$

$\Rightarrow \det(A - \lambda I) = (3-\lambda)(2-\lambda)(1-\lambda) = 0.$

$\lambda = 3; \quad \lambda = 2, \quad \lambda = 1$

$Av = \lambda v.$

$(A - \lambda)v = 0$

$\Rightarrow \begin{bmatrix} 3-\lambda & 10 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Case - 1

$\lambda = 3$

$$\begin{bmatrix} 0 & 10 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$10 x_2 = 0$ - - ①

$-x_2 + 4x_3 = 0$ - - ②.

$-2x_3 = 0$

$x_2 = x_3 = 0$

x_1 can be any arbitary value. let $x_1 = 1$

$$\Rightarrow x_2' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ ③

Case - 2

$\lambda = 2$

$$\begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $x_1 + 10x_2 = 0$

$4x_3 = 0$

$1 x_3 = 0$

$x_3 = 0$

$x_1 = -10x_2$

$x_2 = 1$; $x_1 = -10$.

$$\Rightarrow x^2 = \begin{bmatrix} -10 \\ 1 \\ 0 \end{bmatrix} \quad \text{--- --- } (4)$$

Case - 3

$\lambda = 1$

$$\begin{bmatrix} 2 & 10 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$2x_1 + 10x_2 = 0$

$1x_2 + 4x_3 = 0$

$x_2 = -4x_3$

$2x_1 + 10(-4x_3) = 0$

$\Rightarrow 2x_1 = 40x_3$

$x_1 = 20x_3$

$$\begin{bmatrix} 20 \\ -4 \\ 1 \end{bmatrix} = x^3 \quad \text{- - - } (5)$$

$\therefore$ The eigen vectors are.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ -4 \\ 1 \end{bmatrix}$$

To find if these vectors are linearly independent lets consider the following vectors.

$$x^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x^2 = \begin{bmatrix} -10 \\ 1 \\ 0 \end{bmatrix}, \quad x^3 = \begin{bmatrix} 20 \\ -4 \\ 1 \end{bmatrix}$$

$$a_1 x^1 + a_2 x^2 + a_3 x^3 = 0$$

$a_1 = a_2 = a_3 = 0$ (must satisfy this condition) if x^1, x^2 & x^3 have to be linearly independent.

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -10 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 20 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ~~$a_1 +$~~ $a_1 - 10a_2 + 20a_3 = 0$ --- ①

$0 + a_2 - 4a_3 = 0$ --- ②

$a_3 = 0$ --- ③

from eq^n ①, ② & ③
$a_1 = a_2 = a_3 = 0$
$\Rightarrow$ The e- vectors are linearly independent.

(b) $$A_4 = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|A_4 - \lambda I| = 0 = \begin{vmatrix} 5-\lambda & 1 & 1 \\ 0 & 5-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$(5-\lambda)(5+\lambda)(2-\lambda) = 0$

$\lambda_1 = 5$, $\cancel{\lambda_2 = 5}$, $\lambda_2 = 2$.

$\therefore$ The Eigen values are repeated.

Case - 1

$\lambda_1 = 5$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 + x_3 = 0$

$3x_3 = 0$

$-3x_3 = 0$

$x_3 = 0$; $x_2 = 0$; $x_1 \Rightarrow$ can be any arbitrary value

let $x_1 = 1$

$\Rightarrow$ e-vector $= x^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x^3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ {Any repeated vector that satisfies the eqn}

Case - 2

$\lambda_2 = 2$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad x^2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$3x_1 + x_2 + x_3 = 0$

$3x_2 + 3x_3 = 0 \Rightarrow x_2 = -x_3$

$\therefore$ The Eigen ~~values~~ vectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Here these vectors are not linearly independent.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = 1(0) + 1(0) + 0 = 0$$

$\text{Det}(A) = 0$ {Thus linearly dependent}.

$\therefore$ They cannot form a basis for $\mathbb{R}^3$.

(2) To show two square Matrix A & B are ~~said to be~~ similar if there exists an invertible matrix P.
S.T $B = P^{-1}AP$, if A & B are similar then they have the same characteristic eq^n:

Basically the problem is asking us to show.

$$|A - \lambda I| = |B - \lambda I|$$

lets consider $\det(\lambda I - B) = 0$

~~with~~ lets assume $B = P^{-1}AP$

$$\det(\lambda I - P^{-1}AP) = 0$$

$$\det(\lambda I P P^{-1} - P^{-1}AP) = 0$$

$$\det(P^{-1}(\lambda I P - AP)) = 0$$

$$\det(P^{-1}P(\lambda I - A)) = 0$$

$P^{-1}P = I$

$\Rightarrow \det\left(I(\lambda I - A) \right) = 0$

$\Rightarrow \det(\lambda I^2 - AI) = 0 \quad \{ (\lambda I)(I) = \lambda I \}$

$\Rightarrow \det(\lambda I - A) = 0 \quad \left\{ \begin{array}{l} \text{since } \lambda I^2 = (\lambda I) \\ I \text{ is the identity matrix} \end{array} \right\}$

$AI = A$ as I is the identity matrix

$\therefore$ we have proved that $\det(\lambda I - B) = \det(\lambda I - A)$

$\Rightarrow$ The characteristic polynomial of B is the same as that of A, Since Eigen values are the roots of the characteristic eq^n, Similar matricies have the same eigen values

(3) $A_3 = \begin{bmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

The eigen values of this matrix A_3 is $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$

The eigen values can be represented as a diagonal matrix $\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

in order to show A_3 is similar to Λ

we have to show $A_3 = P^{-1} \Lambda P$ (or) $\Lambda = P^{-1} A_3 P$

$P = [x^1 \mid x^2 \mid x^3] = \begin{bmatrix} 1 & -10 & 20 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

$\Lambda = P^{-1} A P$ can be re-written as

$P\Lambda = A_3 P$

$$P\Lambda = \begin{bmatrix} 1 & -10 & 20 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 10 & 1 \end{bmatrix}$$

$$\cancel{PA} \quad P\Lambda = \begin{bmatrix} 3 & -20 & 20 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{--- } 1$$

$$\cancel{PA} \quad A_3 P = \begin{bmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -10 & 20 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_3 P = \begin{bmatrix} 3 & \overset{-20}{-30+10} & \overset{20}{60-40} \\ 0 & 2 & 0 \\ 0 & & \end{bmatrix} \quad \text{--- } 2$$

$\therefore$ 1 = 2 $\Rightarrow P\Lambda = A_3 P$

$\Rightarrow \Lambda = P^{-1} A_3 P \quad \therefore$ A & Λ are similar Matrices.

4(b) The α-hat values are :-

-0.0370

2.2158

2.8901

-11.2271

7.6978

-1.5396

4(c) The α-hat values are :-

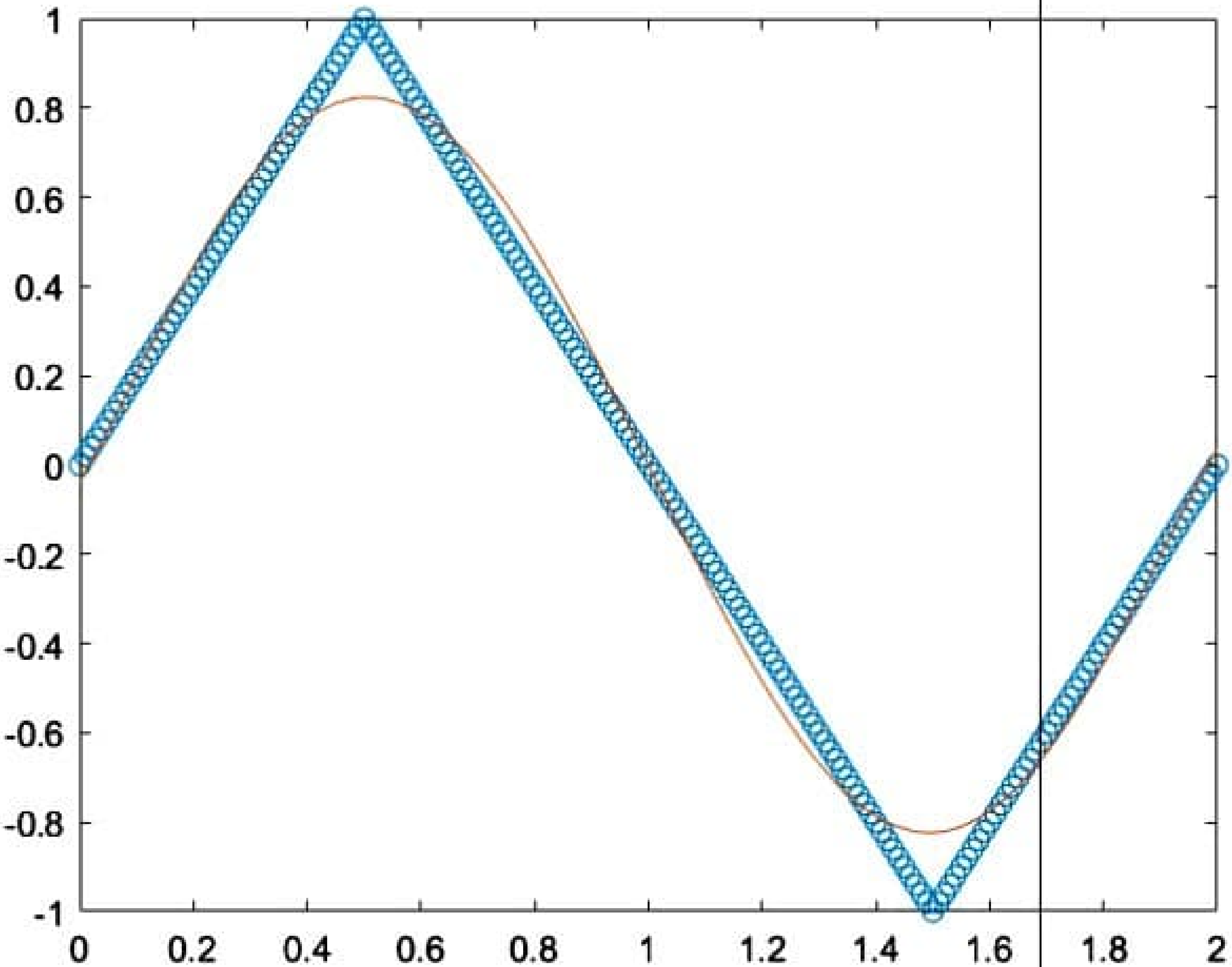
0.8106

0.0000

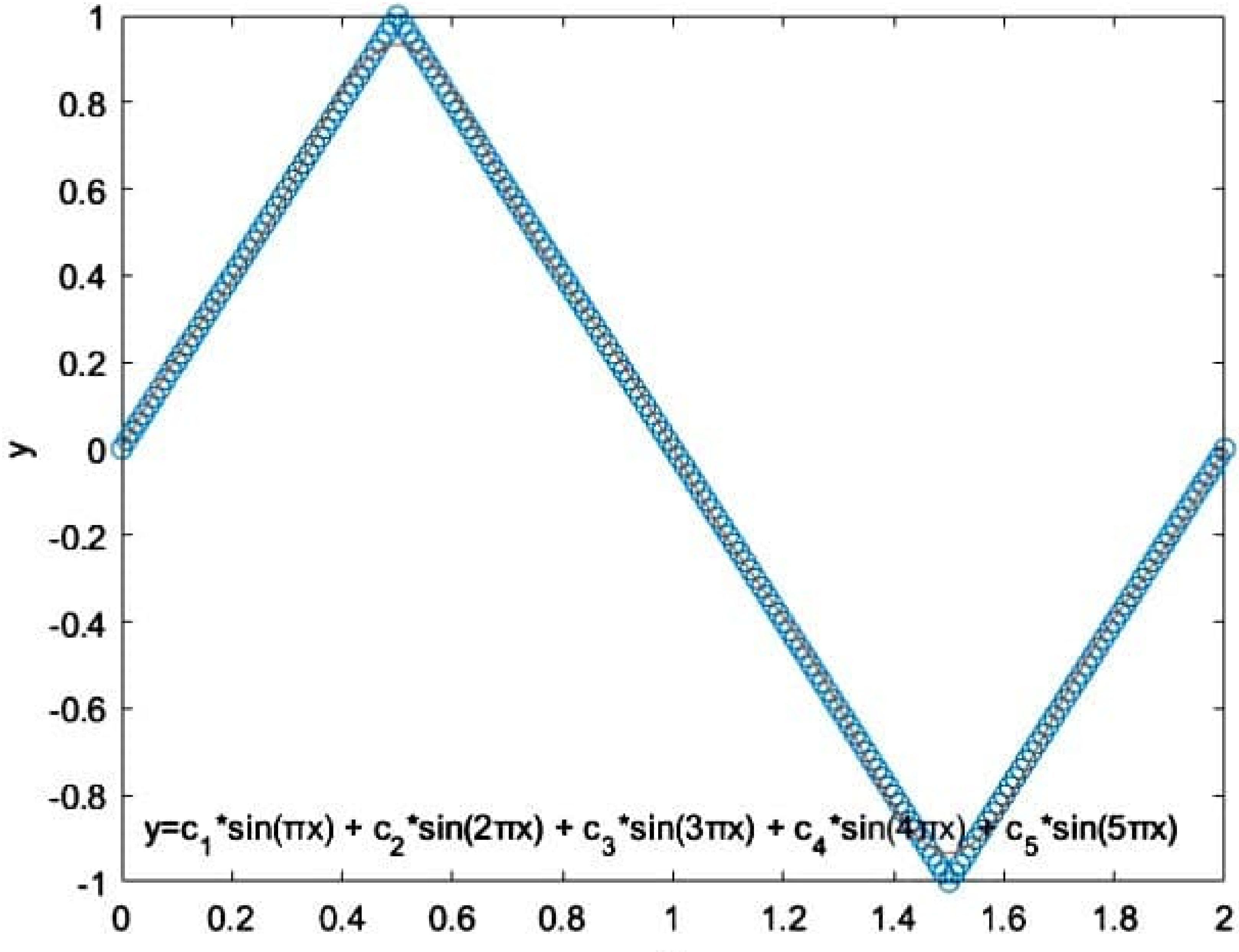
-0.0901

0.0000

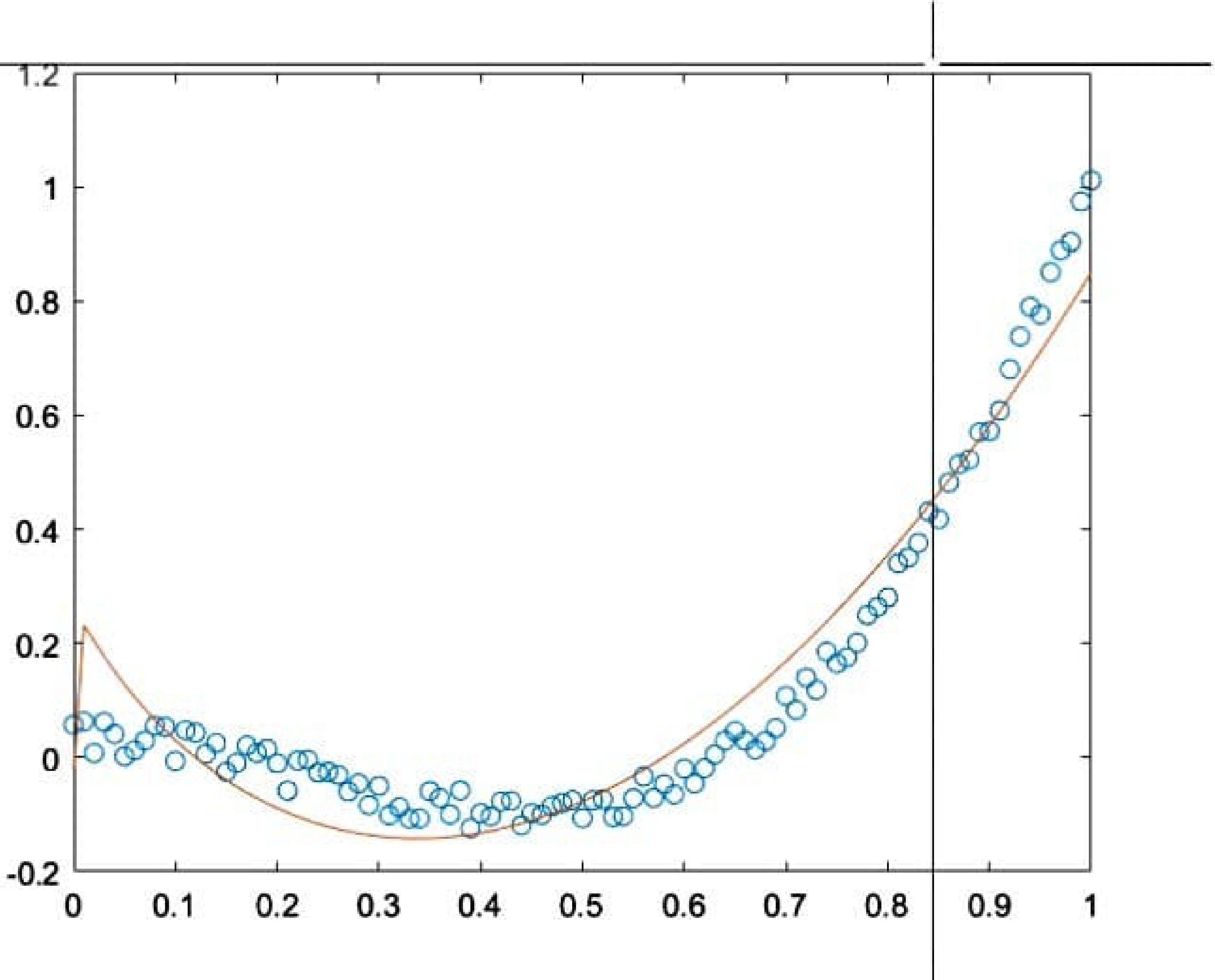
0.0325



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



y=c1*sin(πx) + c2*sin(2πx) + c3*sin(3πx) + c4*sin(4πx) + c5*sin(5πx)
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

(5) There are two definitions of the inner product.

Def - 1 $\langle x, y\rangle = x^T \bar{y}$

Def - 2 $\langle x, y\rangle = \bar{x}^T y$.

To show both these definitions verify the following properties

(1) conjugate symmetry

$\langle x, y\rangle = \overline{\langle y, x\rangle}$

(2) $\langle x, (ay + bz)\rangle = a\langle x, y\rangle + b\langle x, z\rangle$ (Linearity)

(3) $\langle x, x\rangle \geq 0$ & $\langle x, x\rangle = 0$ iff $x = 0$ (positive definitive-ness)

Def - 1 $\langle x, y\rangle = x^T \bar{y}$ — — — ①

(1) conjugate symmetry.

$\langle y, x\rangle = y^T \bar{x}$

$\overline{\langle y, x\rangle} = \overline{y^T \bar{x}} \Rightarrow \overline{y^T}\,\bar{\bar{x}} = \bar{y}^T x$ (using the property of complex conjugates $\overline{A^T} = \bar{A}^T$ & $\bar{\bar{z}} = z$ for any complex number z)

$\Rightarrow \overline{\langle y, x\rangle} = \bar{y}^T x$. — — ②

Now to show $x^T \bar{y} = \bar{y}^T x$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $x^T = [x_1, x_2, \ldots, x_n]$

$$x^T\bar{y} = [x_1, x_2, x_3, \text{---} x_n]\begin{bmatrix}\bar{y}_1\\ \bar{y}_2\\ \vdots\\ \bar{y}_n\end{bmatrix}$$

$$x^T\bar{y} = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 + \text{---} + x_n\bar{y}_n \quad \text{—(1)}$$

my calculating

$\bar{y}^T x$

$$\bar{y}^T = [\bar{y}_1 \; \bar{y}_2 \; \bar{y}_3 \text{---} \bar{y}_n]$$

$$\bar{y}^T x = [\bar{y}_1 \; \bar{y}_2 \; \bar{y}_3 \text{---} \bar{y}_n]\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix}$$

$$= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \bar{y}_3 x_3 + \text{---} + \bar{y}_n x_n$$

$$\bar{y}^T x = \langle y, x\rangle \text{—(2)}$$

w.r.t complex multiplication is commutative

$\Rightarrow a, b \in \mathbb{C}$

$a \times b = b \times a.$

$\Rightarrow$ (1) = (2). $\{$ As $x_i\bar{y}_i = \bar{y}_i x_i$ $[i = 1, 2, \text{--} n]\}$

$\Rightarrow \sum_{i=1}^{n} x_i\bar{y}_i = \sum_{i=1} \bar{y}_i x_i$

$\therefore x^T\bar{y} = \bar{y}^T x.$

$\Rightarrow \langle x, y\rangle = \overline{\langle y, x\rangle}$

(2) Linearity

$$\langle ax + bz, y\rangle = (ax + bz)^T \bar{y} = ax^T \bar{y} + bz^T \bar{y}$$

$$= a\langle x, y\rangle + b\langle z, y\rangle .$$

(3) Positive definiteness :-

$$\langle x, x\rangle = x^T \bar{x} = \sum_{i=1}^{n} |x_i|^2 \geq 0$$

$$\& \ \langle x, x\rangle = 0 \iff x = 0 .$$

III'y the same has to be proved for Def - 2

$$\langle x, y\rangle = \bar{x}^T y \quad - - - (1)$$

$$\langle y, x\rangle = \bar{y}^T x \quad \overline{\langle y, x\rangle} = \overline{\bar{y}^T x} = \bar{\bar{y}}^T \bar{x} = y^T \bar{x} \quad - - (2)$$

From ① & ② it is evident that $\bar{x}^T y = \bar{y}^T \bar{x}$

To prove this:-

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ | \\ | \\ | \\ \bar{x}_n \end{bmatrix} \qquad \bar{x}^T = [\bar{x}_1 \ \bar{x}_2 \ \bar{x}_3 \ - - - \ \bar{x}_n]$$

$$\Rightarrow \bar{x}^T y = \bar{x}_1 y_1 + \bar{x}_2 y_2 + - - - \bar{x}_n y_n \longrightarrow ③$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ | \\ | \\ y_n \end{bmatrix}$$

$||^{ly}$

$$y^T = [y_1 \ y_2 \ - - - \ y_n] \quad ; \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ | \\ | \\ \bar{x}_n \end{bmatrix}$$

$$y^T \bar{x} = y_1 \bar{x}_1 + y_2 \bar{x}_2 + - - y_n \bar{x}_n \longrightarrow ④$$

from ③ & ④ we have proved that
$\bar{x}^T y = y^T \bar{x}$

(2) Linearity $(a, b \in \mathbb{C})$:

$$\langle ax + bz, y \rangle = \left(\overline{ax + bz}\right)^T y = \left(a\bar{x} + b\bar{z}\right)^T y$$

$$= a\bar{x}^T y + b\bar{z}^T y = a\langle x, y \rangle + b\langle z, y \rangle.$$

$\therefore$ def-2 satisfies linearity.

(3) +ve Definiteness

$$\langle x, x \rangle = \bar{x}^T x = \sum_{i=1}^{n} |x_i|^2 \geq 0$$

$$\& \quad \langle x, x \rangle = 0 \iff x = 0.$$

$\therefore$ both the definitions satisfy the properties of an inner product in the complex vector space $\mathbb{C}^n$

$\Rightarrow$ both definitions are valid inner products when the field $F = \mathbb{C}$.

(7) First lets show these are orthogonal vectors

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1); \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\langle P_0(x), P_3(x) \rangle = \int_{-1}^{1} (1)\left(\frac{1}{2}(5x^3 - 3x)\right)$$

$$\frac{1}{2}\int_{-1}^{1} 5x^3 - 3x = \frac{1}{2}\left[\left[\frac{5x^4}{4}\right]_{-1}^{1} - \left[\frac{3x^2}{2}\right]_{-1}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{5}{4}\left[\cancelto{0}{(1)^4 - (-1)^4}\right] - \frac{3}{2}\left[\cancelto{0}{(1)^2 - (-1)^2}\right]\right]$$

$\langle P_0(x), P_3(x) \rangle = 0$ $\therefore$ These basis are orthogonal to each other.

$$\langle P_1(x), P_2(x)\rangle = \int_{-1}^{+1} P_1(x) \cdot P_2(x)$$

$$= \int_{-1}^{+1} (x)\left(\frac{1}{2} 3x^2-1\right)$$

$$= \frac{1}{2}\int_{-1}^{+1} 3x^3 - x$$

$$= \frac{1}{2}\left[\left[\frac{3x^4}{4}\right]_{-1}^{+1} - \left[\frac{x^2}{2}\right]_{-1}^{+1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{3}{4}\right)\left[\cancel{(1)^4 - (-1)^4}\right]^{\to 0} - \frac{1}{2}\left[\cancel{(1)^2 - (-1)^2}\right]^{\to 0}\right]$$

$$= 0$$

$\therefore$ The ~~vecto~~ $P_1(x)$ & $P_2(x)$ are orthogonal to each other.

Now we have to show that these polynomials form the basis of P_3 [Note - we have proved that these polynomials are orthogonal to each other]

(1) prove the polynomials be lin. independent to prove they are the basis

lets consider α_0 ; α_1 ; α_2 ; α_3

The linear combination is given as.

$$\alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 = 0$$

if $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$ Then these

~~$\alpha_0(1) + \alpha$~~ polynomials are linearly independent

$$\alpha_0(1) + \alpha_1(x) + \alpha_2\left(\frac{1}{2}(3x^2-1)\right) + \alpha_4\left(\frac{1}{2}(5x^3-3x)\right)$$

$$\alpha_0 + x\alpha_1 + \frac{\alpha_2}{2}(3x^2) - \frac{\alpha_2}{2} + \alpha_4\left(\frac{5}{2}x^3\right) - \alpha_4(3x) = 0$$

$$\left(-3\alpha_4 + \alpha_1\right)x + \left(\frac{5}{4}\alpha_4\right)x^3 + \left(\frac{3}{2}\alpha_2\right)x^2 + \left(\alpha_0 - \frac{\alpha_2}{2}\right) = 0 \quad \text{--- (1)}$$

$\therefore$ from (1) we get

$\Rightarrow -3\alpha_4 + \alpha_1 = 0$ --- (2)

$\frac{5}{4}\alpha_4 = 0$ --- (3)

$\frac{3}{2}\alpha_2 = 0$ --- (4)

$\alpha_0 - \frac{\alpha_2}{2} = 0$ --- (5)

$\alpha_4 = \alpha_2 = 0$ {from (3) & (4)}

if $\alpha_4 = 0$; $\alpha_1 = 0$ {from (2)}

if $\alpha_2 = 0$; $\alpha_0 = 0$ {from (5)}

$\therefore$ we have proved that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

hence these polynomials are linearly independent.

Each of these polynomials have to belong to $\mathbb{P}_3$

$P_0 = 0x^3 + 0x^2 + 0x + 1 \Rightarrow$ highest order $= 3 \Rightarrow \in \mathbb{P}_3$.

$P_1 = 0x^3 + 0x^2 + x + 0 \Rightarrow$ highest order $= 3 \Rightarrow \in \mathbb{P}_3$

$P_2 = 0x^3 + \frac{1}{2}(3)x^2 + 0 - \frac{1}{2} \Rightarrow$ highest order $= 3 \Rightarrow \in \mathbb{P}_3$

$P_3 = \frac{5}{2}x^3 + 0x^2 - \frac{3}{2}x + 0 \Rightarrow$ highest order $= 3 \Rightarrow \in \mathbb{P}_3$.

(2) $\therefore$ all of these polynomials belong to $\mathbb{P}_3$

(3) Span $\{P_0, P_1, P_2, P_3\}$ contains all the Elements of $\mathbb{P}_3$.

lets consider the natural basis of $\mathbb{P}_3$

$\Rightarrow \{1, x, x^2, x^3\}$

The highest order is (4) & are linearly independent

i.e $\dim(\mathbb{P}^3) = 4$

& the basis we have

$\{1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x)\}$ s The higher order is (4) & are linearly independent $\longrightarrow$ As shown in (1)

$\Rightarrow$ Span $\{P_0, P_1, P_2; P_3\}$ contains all the Elements of $\mathbb{P}_3$

8(b) $A = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ $C = 0.25$ $D = B^T$

$\Rightarrow D = \begin{bmatrix} 3 & 0 & 2 & 0 & 1 \end{bmatrix}$

$BCD = 0.25 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}_{5\times1} \begin{bmatrix} 3 & 0 & 2 & 0 & 1 \end{bmatrix}_{1\times5}$

$$BD = \begin{bmatrix} 9 & 0 & 6 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$0.25\,[BD] = \begin{bmatrix} 9/4 & 0 & 6/4 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \\ 6/4 & 0 & 4/4 & 0 & 2/4 \\ 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 2/4 & 0 & 1/4 \end{bmatrix}$$

$A + 0.25\,[BD]$

$$= \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 9/4 & 0 & 6/4 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \\ 6/4 & 0 & 4/4 & 0 & 2/4 \\ 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 2/4 & 0 & 1/4 \end{bmatrix}$$

$$A + 0.25[BD] = \begin{bmatrix} 9/4 + 0.5 & 0 & 6/4 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 0 \\ 6/4 & 0 & 1+1 & 0 & 2/4 \\ 0 & 0 & 0 & 0.5 & 0 \\ 3/4 & 0 & 2/4 & 0 & 1/4+1 \end{bmatrix}.$$

$$A + 0.25[BD] = \begin{bmatrix} 2.75 & 0 & 1.5 & 0 & 0.75 \\ 0 & 0 & 0 & 0 & 0 \\ 1.5 & 0 & 2 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0.75 & 0 & 0.5 & 0 & 1.25 \end{bmatrix}$$

$$\left[A + 0.25[BD]\right]^{-1} = \begin{bmatrix} 0.6667 & 0 & -0.444 & 0 & -0.2222 \\ 0 & 1.000 & 0 & 0 & 0 \\ -0.444 & 0 & 0.8519 & 0 & -0.0741 \\ 0 & 0 & 0 & 2.000 & 0 \\ -0.2222 & 0 & -0.0741 & 0 & 0.9630 \end{bmatrix}.$$

8(a) Identity - 1

$$(I+P)^{-1} = (I+P)^{-1}(I+P-P)$$

$$= I-(I+P)^{-1}P \quad \text{--- ①}$$

Identity 2

$$P+PQP = P(I+QP) = (I+PQ)P$$

$$(I+PQ)^{-1}P = P(I+QP)^{-1} \quad \text{--- ②}$$

$$(A+BCD)^{-1} = \left[A(I+A^{-1}BCD)\right]^{-1}$$

$$= \left[I+A^{-1}BCD\right]^{-1}A^{-1}$$

$$= \left[I-[I-A^{-1}BCD]^{-1}A^{-1}BCD\right]A^{-1} \quad \text{using ①}$$

$$= A^{-1}-(I+A^{-1}BCD)^{-1}A^{-1}BCDA^{-1}$$

Repeatedly using (2) in sequence now produces.

$$(A+BCD)^{-1} = A^{-1}-\left(I+A^{-1}BCD\right]^{-1}A^{-1}BCDA^{-1} \quad \text{--- ③}$$

$$= A^{-1}-A^{-1}\left[I+BCDA^{-1}\right]^{-1}BCDA^{-1} \quad \text{--- ④}$$

$$= A^{-1}-A^{-1}B\left(I+CDA^{-1}B\right)^{-1}CDA^{-1} \quad \text{--- ⑤}$$

$$= A^{-1}-A^{-1}BC\left[I+DA^{-1}BC\right]^{-1}DA^{-1} \quad \text{--- ⑥}$$

$$= A^{-1}-A^{-1}BCD\left[I+A^{-1}BCD\right]^{-1}A^{-1} \quad \text{--- ⑦}$$

$$= A^{-1}-A^{-1}BCDA^{-1}\left(I+BCDA^{-1}\right)^{-1} \quad \text{--- ⑧}$$

if (c) is also invertable from (5)

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B\left[I + CDA^{-1}B\right]^{-1}CDA^{-1}$$

$$= A^{-1} - A^{-1}B\left[C^{-1} + DA^{-1}B\right]^{-1}DA^{-1} \quad \text{--- (9)}.$$

(9) (a) $f(x)$ is a norm if

1. $\|x\| > 0 \ \forall x \in X \setminus \{0\}$; $\|x\| = 0 \iff x = 0$
2. $\|x+y\| \le \|x\| + \|y\|$ $\forall x, y \in X$
3. $\|ax\| = |a|\,\|x\|$; $\forall x \in X$, $a \in F$.

(1) Since A is +ve definite

$\forall x \in \mathbb{R}^n$, $x \neq 0 \implies x^T A x > 0$ → by def^n of A

$\Downarrow$

$\forall x \in \mathbb{R}^n$, $x \neq 0 \implies (x^T A x)^{1/2} > 0$
If $x = 0$, then $(0^T A 0)^{1/2} = 0$

$\implies \begin{cases} f(x) > 0 \ \forall x \in X \\ f(x) = 0 \iff x = 0 \end{cases}$

(2) P: $g(x,y) = x^T A y$ is an inner product

$A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n \implies x^T A y \in \mathbb{R}$

$x^T A y = (x^T A y)^T = y^T (x^T A)^T = y^T A^T x$

given A is symmetric i.e. $A^T = A$

$\Downarrow$

$x^T A y = y^T A^T x = y^T A x$

$g(x,y) = x^T A y = g(y,x) = y^T A x.$

$\Downarrow$

$g(x,y) = g(y,x)$

$= a g(x, z) + b y(y, z)$

$= g(ax + by, z) = a g(x, t) + b g(y, z)$

(3) $g(x, x) > 0 \;\forall x \in X, x \neq 0$
$g(x, x) = 0 \iff x = 0$ $\Big\}$ → shown in (1)

All in All we have shown

1. $g(x, y) = g(y, x)$ → for real x-y
2. $g(ax + by, z) = a g(x, z) + b g(y, z)$
3. $g(x, x) > 0 \;\forall x \in X, x \neq 0$
   $g(x, x) = 0 \iff x = 0$

(4)

$g(x, y) = (x^T A y)^{1/2}$ is an Norm by definition.

if we replace A with 2A the new function becomes:-

$$g(x) = (x^T (2A) x)^{1/2} = \sqrt{2} \left(x^T A x\right)^{1/2} = \sqrt{2} f(x).$$

So we are scaling the original function by $\sqrt{2}$
i.e only the magnitude is changed

⇒ All the ~~thes~~ three properties satisfy

$\|x\| > 0 \;\forall x \in X / \{0\}$

$\|x + y\| \leq \|x\| + \|y\| \;\forall x, y \in X$

$\|ax\| = |a| \|x\| \;\forall x \in X, a \in F$.

∴ The function is still a Norm.

9(a) continued

$$f(x+y) = ((x+y)^T A (x+y))^{1/2} = (x^T A x + y^T A x + x^T A y + y^T A y)^{1/2}$$

Since $y^T A x$ is a scalar $y^T A x = (y^T A x)^T = (Ax)^T y = x^T A y$

$$f(x+y) = (x^T A x + x^T A y + x^T A y + y^T A y)^{1/2}$$

$$f(x+y) = (x^T A x + 2x^T A y + y^T A y)^{1/2}$$

The lemma from above tells us that $x^T A y$ is an inner product

$\Rightarrow$ using Cauchy-Schwarts Inequality for inner products

$$(g(x,y)) \leq g^{1/2}(x,x) \cdot g^{1/2}(y,y)$$

$$(x^T A y) \leq (x^T A x)^{1/2} \cdot (y^T A y)^{1/2}$$

$$\underbrace{(x^T A x + 2x^T A y + y^T A y)^{1/2}}_{F(x+y)} \leq (x^T A x + 2(x^T A x)^{1/2}(y^T A y)^{1/2} + y^T A y)^{1/2}$$

9(b) Yes, it is a norm, called induced norm. Here we only prove that it satisfies triangular inequality since non-negativity and scalability is trivial. $\forall A, B \in X$.

$$f_V(A+B) = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|(A+B)x\|_V}{\|x\|_V} \le \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax + Bx\|_V}{\|x\|_V}$$

$$\le \sup \frac{\|Ax\|_V + \|Bx\|_V}{\|x\|_V}$$ {triangular inequality in normed space $(\mathbb{R}^n, \mathbb{R}, \|\cdot\|_V)$}

$$\le \sup \frac{\|Ax\|_V}{\|x\|_V} + \sup \frac{\|Bx\|_V}{\|x\|_V} = f_V(A) + f_V(B)$$

for an induced norm $\|A\|_V = \sup \frac{\|Ax\|_V}{\|x\|_V}$

$= \sup \left\| A \frac{x}{\|x\|_V} \right\|_V$ if we let $y = \frac{x}{\|x\|_V}$, then

$\|y\|_V = 1$ & $\|A\|_V = \sup \|Ay\|_V$. Now lets calculate

1

$F_1(A) =$

$F_\infty(A) =$

⇒ using the function $FV(A) = \sup (x \neq 0)$

1. for $\|A\|_1$, $V = 1$, so we use the 1-vector norm for both $\|Ax\|_1$ & $\|x\|_1$

2. $\|x\|_1 = \sum |x_i|$ (sum of absolute values of elements in x)

3. $\|Ax\|_1 = \sum | \sum a_{ij} x_j$ (sum of absolute values of elements in Ax)

The sup occurs when x is chosen to max. the largest column sum of A.

$\|A\|_1 = \max (1 \le j \le n) \sum i=1 \text{ to } m |a_{ij}|$

This means $\|A\|_1$ is the max. absolute column sum of A.

The ∞ norm of Matrix A is defined as the max. absolute row sum of A

1. for $\|A\|^\infty$, $V = \infty$, so we use the ∞-vector norm for both $\|Ax\|^\infty$ & $\|x\|^\infty$

2. $\|x\|^\infty = \max \|x_i\|$ (max. absolute value of element in x)

3. $\|Ax\|^\infty = \max \| \sum a_{ij} x_j$ (max absolute value of element in Ax)

⇒ $\|A\|^\infty$ is the max. absolute row sum of A.