Date :- 23/11/2024

ROB. 501 HW - 11

(1) $S = (-2,3) \cup (3,4) \cup \{5\} \cup [7,\infty)$ in the normed space $(\mathbb{R}, |\cdot|)$

(a) S° of S: $(-2,3)$ - open interval, $\Rightarrow$ All points are interior points

$(3,4)$ - open interval $\Rightarrow$ All points are interior points

$\{5\}$ - No interior point $\Rightarrow$ it cannot contain a neighbourhood

$[7,\infty) \Rightarrow (7,\infty)$ {open set in the interior ϕ of a set}

$(-2,3) \cup (3,4) \cup (7,\infty) \rightarrow$ (a)

(b) limit point of sets is a point where every neighborhood intersects S in at least one point other than itself.

$(-2,3) \rightarrow$ All points are limit points $[-2,3]$

$(3,4) \rightarrow$ All points are limit points $[3,4]$

$[7,\infty) \rightarrow$ All points are limit points $[7,\infty)$

$\{5\} \rightarrow 5$ is not a limit point

$\therefore$ The ans b is $[-2,3] \cup [3,4] \cup [7,\infty)$

$\rightarrow$ (b)

(c) For $(-2,3) \cup (3,4)$: The boundaries are the Endpoints $-2, 3, 4$ as neighborhoods around these points intersect S and its complement

For $\{5\}$: The point 5 is in the boundary since any neighborhood of 5 intersects S at 5

For $[7,\infty)$: 7 is a boundary point; all other points are interior points.

$\partial S = \{-2, 3, 4, 5, 7\} \rightarrow$ (c)

(2) To show when $x \in \mathbb{R}^n$.

a. $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$

b. $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$

c. $\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$

let $x \in \mathbb{R}^2$

(a) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$

$\|x\|_2 \leq \|x\|_1$

$\|x\|_2^2 \leq \|x\|_1^2$

$= \left[\sqrt{(x_1)^2 + (x_2)^2}\right]^2 \leq \left[|x_1|^2 + |x_2|^2\right]^2 \Rightarrow x_1^2 + x_2^2 \leq x_1^2 + x_2^2 + 2|x_1||x_2|$

$\therefore \|x\|_2 \leq \|x\|_1$ ---- (0)

$\|x_1\| \le \sqrt{n}\,\|x\|_2$

$= \|x_1\|^2 \le \left[\sqrt{2}\,\|x\|_2\right]^2 \quad x \in \mathbb{R}^2 \implies n=2$

$= \|x_1\|^2 \le 2\|x\|_2^2$

$= |x_1|^2 + |x_2|^2 + 2|x_1||x_2| \le 2\left[x_1^2 + x_2^2\right]$

$\hookrightarrow$ To prove this we can re-arrange it as follows.

$|x_1|^2 + |x_2|^2 + 2|x_1||x_2|$ - - - (1)

$(x_1^2 + x_2^2) + (x_1^2 + x_2^2)$ - - - (2)

Now lets compare (1) & (2)

it is visible that $2|x_1||x_2| \le (x_1)^2 + (x_2)^2$

$\implies (1) \le (2)$

$\therefore \|x_1\| \le \sqrt{n}\,\|x\|_2$ - - - (3)

$\therefore$ from (0) & (3) we get

$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2.$ $\square$

(b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$

let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $x \in \mathbb{R}^2$, also $|x_1| > |x_2|$

$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$

$\|x\|_\infty^2 = (x_1)^2$

$\|x\|_2^2 = \left[\sqrt{|x_1|^2 + |x_2|^2}\right]^2 = |x_1|^2 + |x_2|^2.$

$\Rightarrow \|x\|_\infty^2 \le \|x\|_2^2$

$\therefore \|x\|_\infty \le \|x\|_2$. ------- (1)

lets consider

$\|x\|_2 \le \sqrt{n}\,\|x\|_\infty$

To prove

$(\|x\|_2)^2 \le n\,\|x\|_\infty^2$

$\Rightarrow \|x\|_2^2 = |x_1|^2 + |x_2|^2$

$n\,\|x\|_\infty^2 = n|x_1|^2 = n x_1^2 \quad n=2$ as $x \in \mathbb{R}^2$.

$\cancel{2}$. $\Rightarrow 2\|x\|_\infty^2 = 2|x_1|^2$ { Assumed $|x_1| > |x_2|$ }

↳ This can be rearranged as $|x_1|^2 + |x_1|^2$.

w.r.t $|x_1| > |x_2|$

$\Rightarrow 2\|x\|_\infty^2 \ge \|x\|_2^2$.

$\therefore \|x\|_2^2 \le 2\|x\|_\infty^2$ ------ (2)

Combining (1) & (2) we get

$\|x\|_\infty \le \|x\|_2 \le \sqrt{2}\,\|x\|_\infty$

Extrapolating it to $x \in \mathbb{R}^n$ we get

$$\boxed{\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty.}$$

* *

(c) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$ Assume $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $|x_1| > |x_2|$

$x \in \mathbb{R}^2$

$\|x\|_\infty = |x_1|$

$n = 2$

$\|x\|_1 = |x_1| + |x_2|$

$\Rightarrow \|x\|_\infty \le \|x\|_1$ - - - (1)

consider

$\|x\|_1 \le 2\|x\|_\infty$ $\{n = 2\}$

$|x_1| + |x_2| = \|x\|_1$

$2|x_1| = 2\|x\|_\infty$

$\Rightarrow |x_1| + |x_1|$

$|x_1| > |x_2|$

$\Rightarrow \|x\|_1 \le 2\|x\|_\infty$ - - - (2)

from (1) + (2) we get $\|x\|_\infty \le \|x\|_1 \le 2\|x\|_\infty$.

Extrapolating it to $x \in \mathbb{R}^n$ we get

$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$.

(3) TO Show

$\tilde{B}_{\frac{a}{k_2}}(x_0) \subseteq B_a(x_0) \subseteq \tilde{B}_{\frac{a}{k_1}}(x_0)$

Def. of open Balls.

$\|\cdot\|$: $B_a(x_0) = \{x \in X : \|x - x_0\| < a\}$.

For $|||\cdot|||$: $\tilde{B}_a(x_0) = \{x \in X : |||x - x_0||| < a\}$.

3 (a) $\tilde{B}_{\frac{a}{k_2}}(x_0) \Rightarrow |||x - x_0||| < \frac{a}{k_2}$

$\because \frac{1}{k_2} ||x - x_0|| \leq |||x - x_0||| \Rightarrow ||x - x_0|| \leq k_2 |||x - x_0||| < a \; (k_2 \cdot \frac{a}{k_2} = a)$

$\Rightarrow ||x - x_0|| < a$

$\therefore ||x - x_0|| < a \Rightarrow B_a(x_0) \quad \therefore \tilde{B}_{\frac{a}{k_2}}(x_0) < B_a(x_0)$ --- ①

$B_a(x_0) \Rightarrow ||x - x_0|| < a, \because k_1 |||x - x_0||| \leq ||x - x_0||$

$\Rightarrow |||x - x_0||| \leq \frac{1}{k_1} ||x - x_0|| < \frac{a}{k_1} \Rightarrow \tilde{B}_{\frac{a}{k_1}}(x_0)$

$\therefore B_a(x_0) \subset \tilde{B}_{\frac{a}{k_1}}(x_0)$ --- ②

from ① & ② $\tilde{B}_{\frac{a}{k_2}}(x_0) \subset B_a(x_0) \subset \tilde{B}_{\frac{a}{k_1}}(x_0)$ $\square$.

(b) W.k.T $\frac{1}{k_2} ||x|| \leq |||x|||$ --- ①

$|||x||| \leq \frac{1}{k_1} ||x||$ --- ②

P is open in $(X, \mathbb{R}, ||\cdot||) \Leftrightarrow P$ is open in $(X, \mathbb{R}, |||\cdot|||)$

$\Rightarrow P = P^o = \{x \in X \mid d(x, \sim P) > 0\}$

$\Rightarrow \exists \varepsilon > 0, \forall y \in \sim P$, s.t $||x - y|| \geq \varepsilon$

by ①, $||x - y|| \geq \varepsilon \Rightarrow \frac{\varepsilon}{k_2} \leq \frac{1}{k_2} ||x - y|| \leq |||x - y|||$

take $\varepsilon' = \frac{\varepsilon}{k_2} \Rightarrow |||x - y||| \geq \varepsilon'$

$\Rightarrow \exists \varepsilon' > 0, \forall y \in \sim P$ s.t $|||x-y||| \geq \varepsilon' \Rightarrow P$ is open in $(X, \mathbb{R}, |||\cdot|||)$

$\Leftarrow$ $P = P^{\circ} = \{x \in X \mid d(x, \sim P) > 0\}$

$\Rightarrow \exists \varepsilon > 0, \forall y \in \sim P$, s.t $|||x-y||| > \varepsilon$

by ② $|||x-y||| \geq \varepsilon$, $||x-y|| \geq k_1 |||x-y||| \geq k_1 \varepsilon$

take $\varepsilon' = k_1 \varepsilon \Rightarrow \exists \varepsilon', \forall y \in \sim P$ s.t $||x-y|| \geq \varepsilon'$

$\Rightarrow P$ is open in $(X, \mathbb{R}, ||\cdot||)$ □.

ⓒ we need to prove (x_n) is Cauchy in $(X, \mathbb{R}, ||\cdot||)$

$\Leftrightarrow$ x_n is cauchy in $(X, \mathbb{R}, |||\cdot|||)$

from def^n $\frac{1}{k_2} ||x|| \leq |||x|||$ --- ①

$|||x||| \leq \frac{1}{k_1} ||x||$ --- ②

$\Rightarrow$ $\forall \varepsilon > 0, \exists N(\varepsilon) < \infty$, s.t $\forall n, m > N$, $||x_m - x_n|| < \varepsilon$

from ② $|||x_m - x_n||| \leq \frac{1}{k_1} ||x_m - x_n|| \leq \frac{\varepsilon}{k_1}$

Take $\varepsilon' = \frac{\varepsilon}{k_1}$ $\because \varepsilon > 0, k_1 > 0 \Rightarrow \varepsilon' > 0$

$\forall \varepsilon' > 0$ $\exists N(\varepsilon') < \infty$, s.t $\forall n, m > N$

$|||x_m - x_n||| < \varepsilon'$

$\Leftarrow$ $\forall \varepsilon > 0, \exists N(\varepsilon) < \infty$, s.t $\forall n, m > N$ $|||x_m - x_n||| < \varepsilon$

from ①, $||x_m - x_n|| \leq k_2 |||x_m - x_n||| \leq k_2 \varepsilon$

take $\varepsilon' = k_2 \varepsilon$ $\because \varepsilon > 0, k_2 > 0 \Rightarrow \varepsilon' > 0$

$\Rightarrow \forall \varepsilon' > 0$, $\exists N(\varepsilon') < \infty$, s.t $\forall n, m > N$, $||x_m - x_n|| < \varepsilon'$

(4) if $f: \mathbb{R} \to \mathbb{R}$ is continuous at x_0, then $\lim_{n \to \infty} x_n = x_0$

$\Rightarrow \lim_{n \to \infty} f(x_n) = f(x_0)$: (To prove)

Proof:-

f is continuous at x_0 if:-

$\forall \varepsilon > 0, \exists \delta > 0$ s.t $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$

lets assume x_n is a sequence s.t

$\lim_{n \to \infty} x_n = x_0$

for every $\delta > 0$, There exists an $N \in \mathbb{N}$ ST

$n > N \Rightarrow |x_n - x_0| < \delta$

W.K.T f is continuous at x_0 using that we get

$|f(x_n) - f(x_0)| < \varepsilon$

$\therefore$ for an $\varepsilon > 0$ There exists $N \in \mathbb{N}$. s.t

$n > N \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$

$\Rightarrow \lim_{n \to \infty} f(x_n) = f(x_0)$.

$\square$.

if f is not continuous at x_0 if $\exists$ $\varepsilon > 0$ s.t $\exists \delta > 0$, there exists x satisfying.

$\underline{|x - x_0| < \delta}$ but $\underline{|f(x) - f(x_0)| \not< \varepsilon}$

$\hookrightarrow$ Sequence converges

$\hookrightarrow$ The fun^n doesn't converge.

$\lim_{n \to \infty} x_n = x_0$ but $|f(x_n) - f(x_0)| \geq \varepsilon$ for some fixed $\varepsilon > 0$

Since $|f(x_n) - f(x_0)| \geq \varepsilon$, the sequence does not converge to $f(x_0)$.

→ This shows that if f is discontinuous at x_0, we can find a sequence $\{x_n\}$ that converges to δ $\square$.

(5) $F(x_1, x_2) = F(x) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} x - x x^T \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$x x^T = \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_2 x_1 & x_2^2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \underline{F(x)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_2 x_1 & x_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4x_1 + 3x_2 \\ 2x_1 + 1x_2 \end{bmatrix} - \begin{bmatrix} x_1^2 + 2x_1 x_2 \\ x_2 x_1 + 2x_2^2 \end{bmatrix}$$

$$\underline{\underline{F(x)}} = \begin{bmatrix} 3 + 4x_1 + 3x_2 - x_1^2 - 2x_1 x_2 \\ 4 + 2x_1 + 1x_2 - x_2 x_1 - 2x_2^2 \end{bmatrix}$$

$$J_F(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\underline{J_F(x)} = \begin{bmatrix} 4 - 2x_1 - 2x_2 & 3 - 2x_1 \\ 2 - x_2 & 1 - x_1 - 4x_2 \end{bmatrix}$$

$\Delta x = -J_F(x_k)^{-1} F(x_k)$ $\cancel{x_k = (0,0)^T}$. $x_k = [x_1 \ x_2]^T$

$x_1 = x_0 + \Delta x.$ $x_0 = [0,0]^T$ {Assuming an in. value}

```
Starting point: [0 0]
Starting point [0 0] converged in 8 iterations.
Solution: [-2.22682999  1.45781962]
F(solution): [-1.77635684e-15 -8.88178420e-16]

Starting point: [0 1]
Starting point [0 1] converged in 6 iterations.
Solution: [-2.22682999  1.45781962]
F(solution): [ 3.64153152e-14 -4.70734562e-14]

Starting point: [1 0]
Starting point [1 0] converged in 7 iterations.
Solution: [ 0.11819328 -1.25155111]
F(solution): [ 1.11022302e-16 -8.88178420e-16]

Starting point: [1 1]
Starting point [1 1] converged in 10 iterations.
Solution: [ 0.11819328 -1.25155111]
F(solution): [ 1.11022302e-16 -4.44089210e-16]

Starting point: [-1 -1]
Starting point [-1 -1] converged in 5 iterations.
Solution: [ 0.11819328 -1.25155111]
F(solution): [-3.88578059e-16 -8.88178420e-16]

Starting point: [-1  0]
Starting point [-1  0] converged in 8 iterations.
Solution: [ 0.11819328 -1.25155111]
F(solution): [ 1.66533454e-16 -8.88178420e-16]

Starting point: [ 0 -1]
Starting point [ 0 -1] converged in 4 iterations.
Solution: [ 0.11819328 -1.25155111]
F(solution): [ 4.82947016e-15 -6.66133815e-15]
```

Observation.

↳ with Different Initial guesses, different solutions were obtained.

i.e It can be concluded that Newton-Raphson is sensitive to the Starting point

6\. (a) $\boxed{T}$ $x \sim exp(0.001)$, $P(x < x \mid x = 365) = \int_0^{365} \lambda e^{-\lambda x} = $

↳ TRUE

$$= -e^{-\lambda x}\Big|_0^{365} = -e^{-365\lambda} + 1 = 0.3058 \approx 0.31$$

(b) $\boxed{F}$ for unbiased estimators $E\{\hat{x}\} = x$

↳ FALSE

$$E\{\hat{x}\} = kcx + E\{k\varepsilon\}$$

$$= x + E\{k\varepsilon\}$$ since ε is not zero mean.

$\Rightarrow$ the estimator is biased.

(c) $\boxed{F}$

↳ False

$$\operatorname{cov}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \operatorname{cov}(x_1, x_1) & \operatorname{cov}(x_1, x_2) \\ \operatorname{cov}(x_2, x_1) & \operatorname{cov}(x_2, x_2) \end{bmatrix}$$

Since $\operatorname{cov}(x_1, x_2) = \operatorname{cov}(x_2, x_1) = 0$, x_1, x_2 are uncorrelated

but it doesn't imply independence b/t x_1 & x_2