ROB-599 - Control's - Assignment-6

02/12/2024.

1.) $x_1 = \theta_l$, $x_2 = \dot{\theta}_l$, $x_3 = \theta_m$, $x_4 = \dot{\theta}_m$

$$X = [x_1 \; x_2 \; x_3 \; x_4]^T$$

$$\dot{X} = Ax + Bu \quad \text{------ (1)}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} \qquad u = -kx. \quad \text{--- (2)}$$

(a) $K \in \mathbb{R}^4$ $\quad K = [k_1 \; k_2 \; k_3 \; k_4]$ --- (3)

sub (2) in (1) we get

$$\dot{X} = Ax + B(-kx)$$

$$\dot{x} = (A - Bk)X.$$

$$A_{cl} = A - Bk$$

$$\Rightarrow \dot{x} = A_{cl} X$$

$$A_{cl} = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m} \end{bmatrix}}^{A} - \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} [k_1 \; k_2 \; k_3 \; k_4]}^{BK}$$

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_1}{J_m} & \frac{k_2}{J_m} & \frac{k_3}{J_m} & \frac{k_4}{J_m} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(k-k_1)}{J_m} & -\frac{k_2}{J_m} & -\frac{(k+k_3)}{J_m} & -\frac{(B_m+k}{J_m} \end{bmatrix}$$

The State-Space Eq^n of this system when $u = -Kx$ where $K \in \mathbb{R}^4$ looks as follows:-

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(k-k_1)}{J_m} & -\frac{k_2}{J_m} & -\frac{(k+k_3)}{J_m} & -\frac{(B_m+k_4)}{J_m} \end{bmatrix}}_{A_{cl} = A - BK} x$$

(b) $J_m = J_l = 0.0097\ kgm^2$ $b_m = b_l = 0.04169\ Ns\text{-}m^{-1}$, $k = 100 Nm\ rad^{-1}$

$Q = diag([1, 0.1, 1, 0.1])$ be a 4×4 digonal matrix, $R = 1$ To find k which minimizes the cost $J = \int_0^\infty [x^T(t) Q x(t) + Ru^2(t)].dt$

in order to do this we have to use the LQR function on Matlab

$K = [-3.2737 \quad 0.0307 \quad 4.6879 \quad 0.3973]$

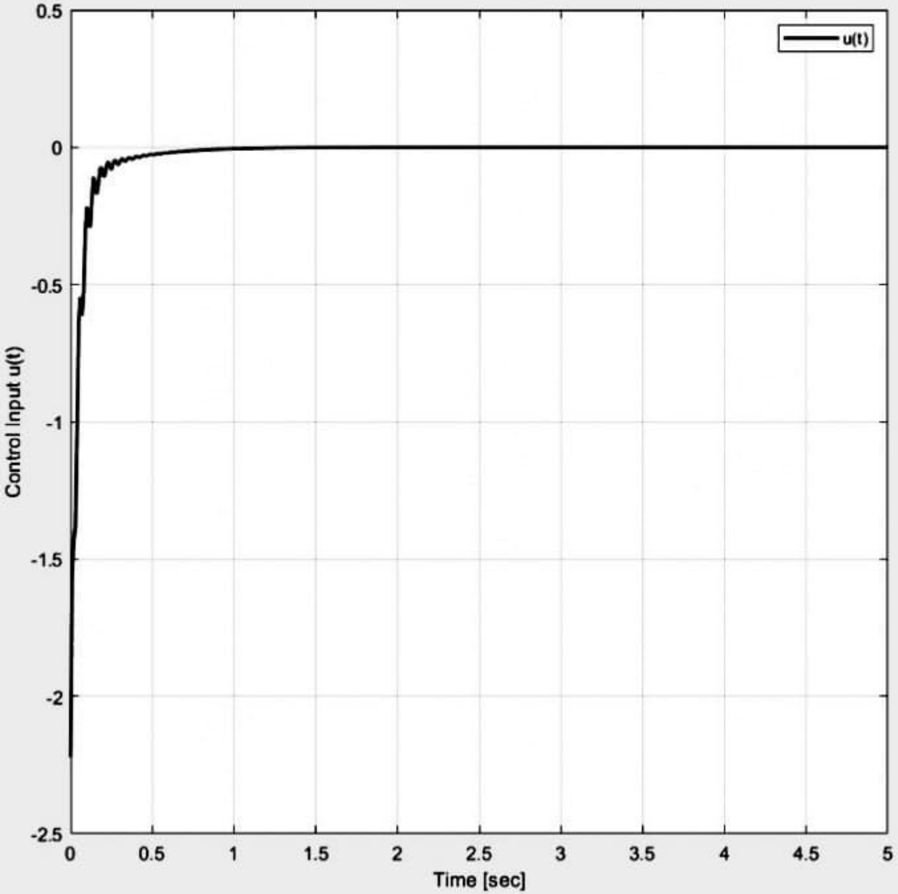
(c) $Q = dig([1, 0.1, 1, 0.1])$ $R = 0.1$ Add all the parameters are the same then the feed back gain k that min the cost function is

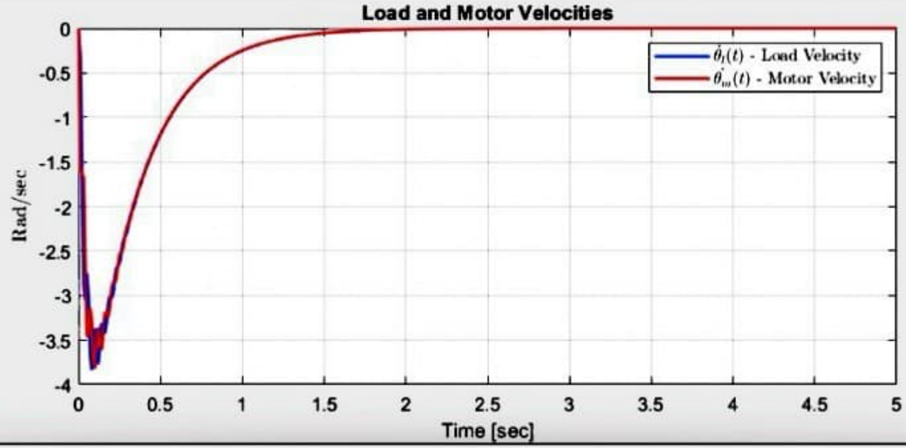
$J = \int_0^\infty [x^T(t) Q x(t) + Ru^2(t)].dt$ is

$k = [-36.5581 \quad 0.0901 \quad 41.0302 \quad 1.2991]$

(d) $Q = dig([5, 0.1, 5, 0.1])$ $R = 0.1$ And all the parameters are the same then the feed back gain k that min. the cost function

$J = \int_0^\infty [x^T(t) Q x(t) + Ru^2(t)].dt$ is $K = [-36.0517 \quad 0.1392 \quad 46.0517 \quad 1.335]$


u(t)
Control Input u(t)
Time [sec]


Load and Motor Velocities
$\dot{\theta}_l(t)$ - Load Velocity
$\dot{\theta}_m(t)$ - Motor Velocity
Rad/sec
Time [sec]

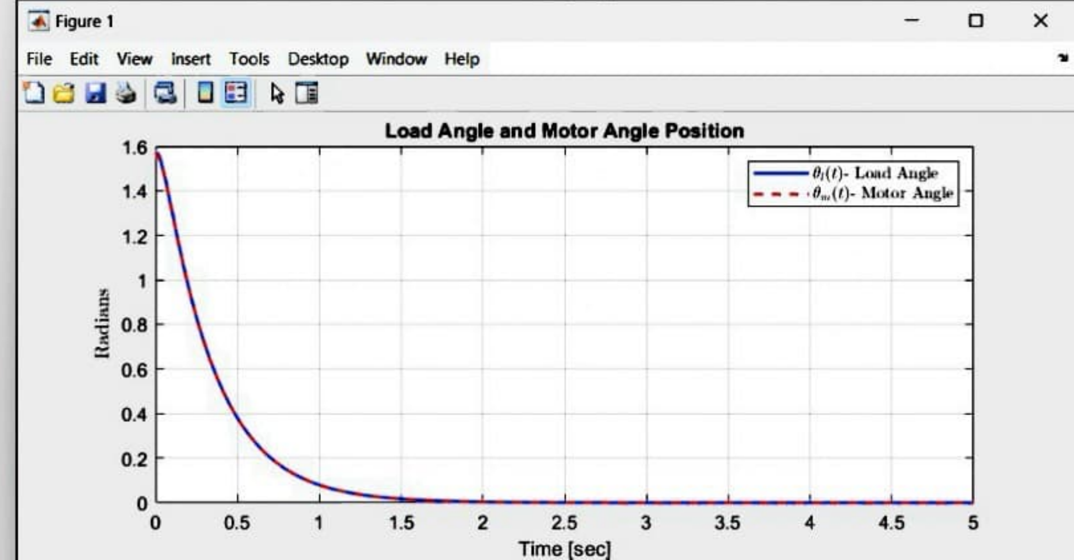
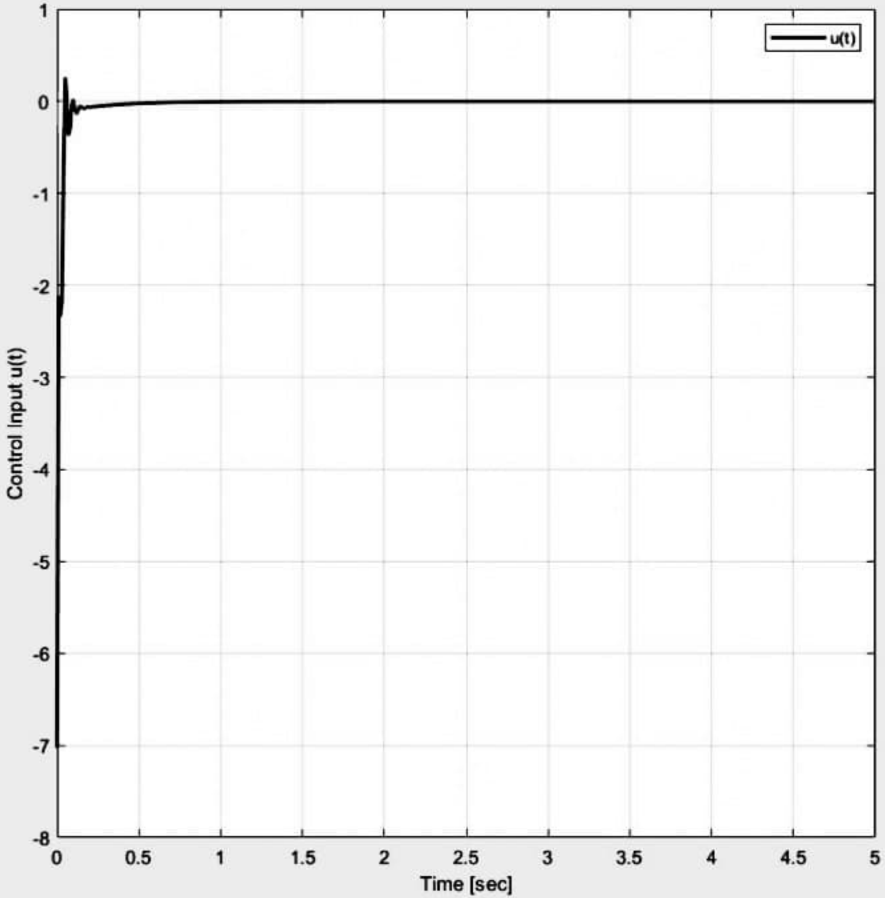
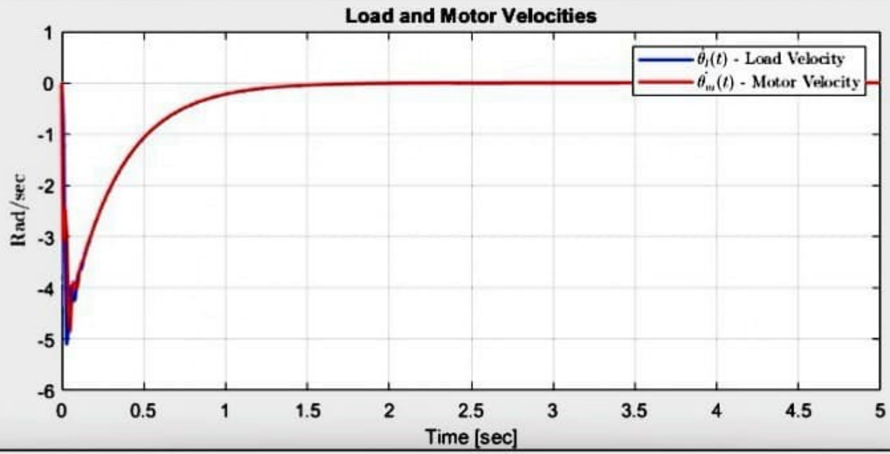


Figure 1
File Edit View Insert Tools Desktop Window Help
Load Angle and Motor Angle Position
$\theta_l(t)$- Load Angle
$\theta_m(t)$- Motor Angle
Radians
Time [sec]



u(t)
Control Input u(t)
Time [sec]



Load and Motor Velocities
$\dot{\theta}_l(t)$ - Load Velocity
$\dot{\theta}_m(t)$ - Motor Velocity
Rad/sec
Time [sec]

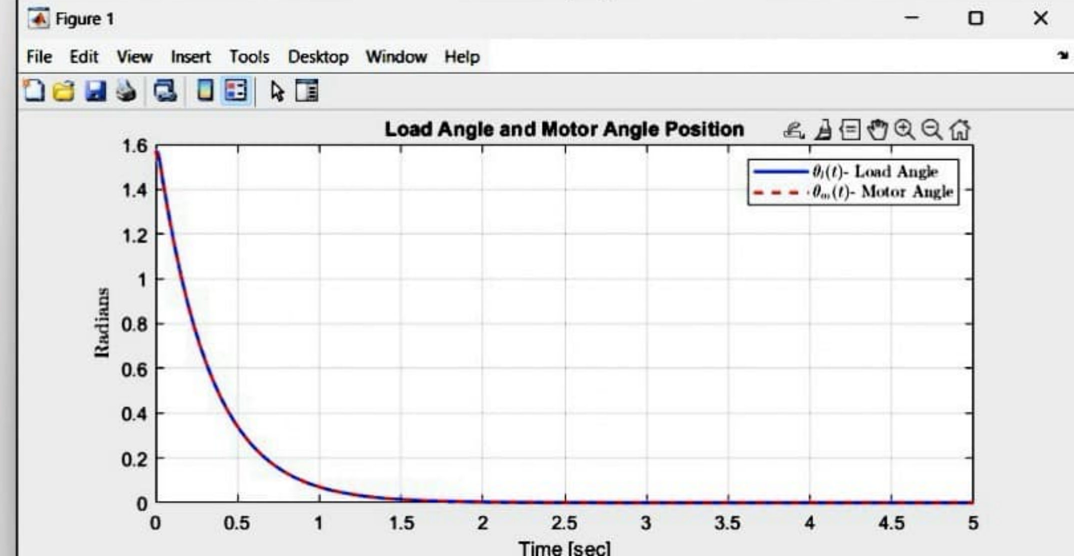
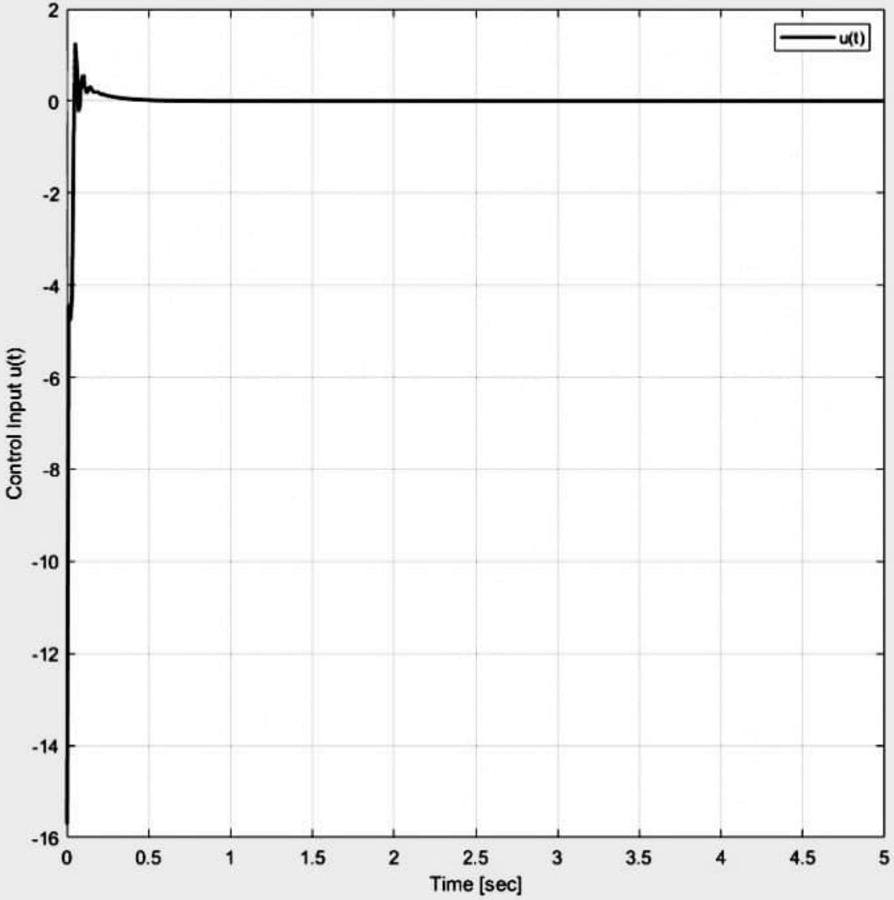
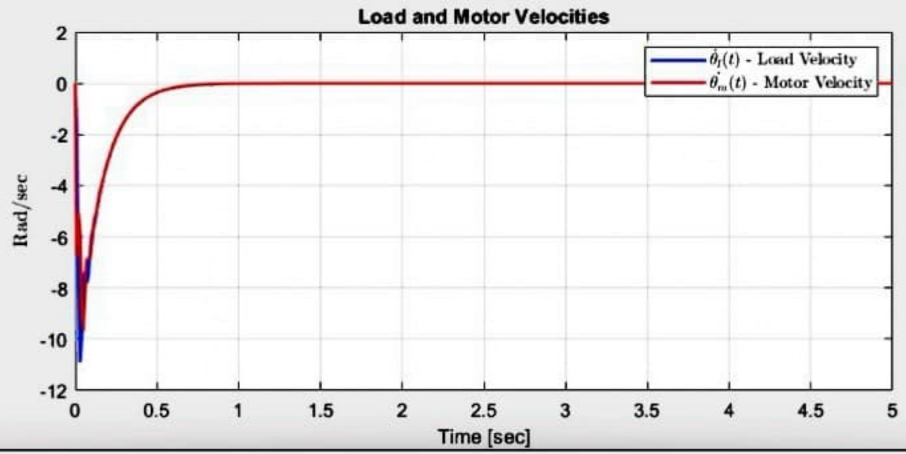


Figure 1
File Edit View Insert Tools Desktop Window Help
Load Angle and Motor Angle Position
$\theta_l(t)$- Load Angle
$\theta_m(t)$- Motor Angle
Radians
Time [sec]


u(t)
Control Input u(t)
Time [sec]


Load and Motor Velocities
$\dot{\theta}_l(t)$ - Load Velocity
$\dot{\theta}_m(t)$ - Motor Velocity
Rad/sec
Time [sec]

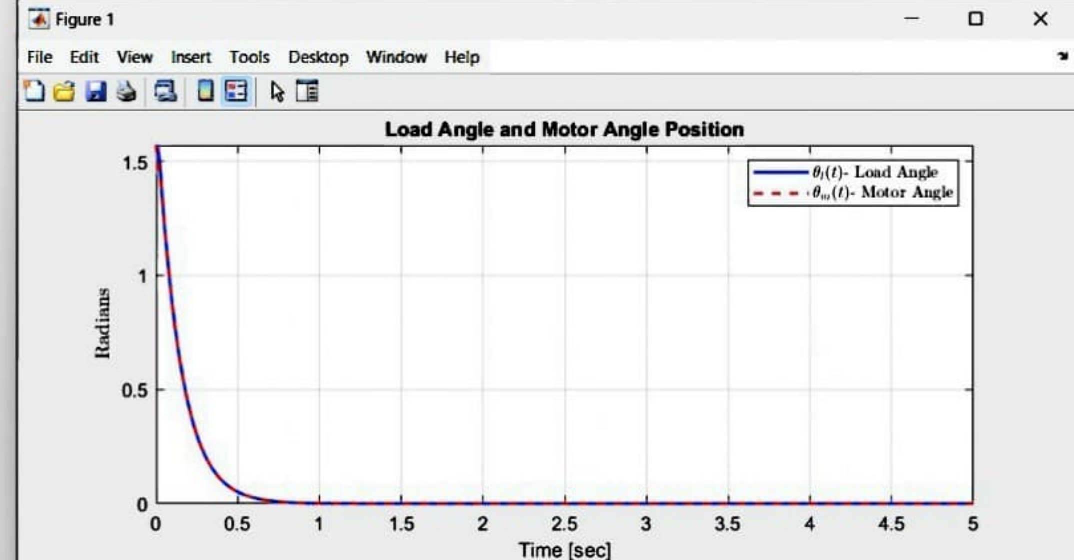


Figure 1
File Edit View Insert Tools Desktop Window Help
Load Angle and Motor Angle Position
$\theta_l(t)$- Load Angle
$\theta_m(t)$- Motor Angle
Radians
Time [sec]

(e) Case 1

$Q = diag([1, 0.1, 1, 0.1])$ & $R = 1$

→ The control effort is more con. due to high value of R ($R=1$)

→ max control input reaches around -2.2

→ Smoother response with less aggressive control action

Case 2

$Q = diag([1, 0.1, 1, 0.1])$ & $R = 0.1$

→ Reducing R to 0.1 allows for more aggressive control

→ Initial control input spike reaches about -2.5

→ Faster settling time compared to Case I

Case 3

$Q = diag([5, 0.1, 5, 0.1])$ & $R = 0.1$

→ Increased state penalties (5 instead of 1) while keeping $R = 0.1$

→ Most aggressive control response

→ Largest initail control input spike

→ Fastest settling time among all cases.

These differences occur because:-

1. R matrix penalizes control effort - smaller R allows larger control inputs

2. Q matrix penalizes state errors - larger Q value forces faster state convergences

3. The ratio Q/R determines the balance b/t state regulation & control effort

In LQR design, increasing Q relative to R results in more aggressive control actions to minimize state errors quickly, while increasing R relative to Q produces more conservative control actions to minimize control effort.

2(a). $m\ddot{x} + b\dot{x} + kx = u$ - - - (0)

where m, b, & k are the mass, damping co-efficient & spring stiffness

$x \rightarrow$ Displacement & $u \rightarrow$ External force

$$\dot{x} = Ax + Bu$$

$m = 1kg$, $b = 2 Ns/m$, $k = 1 N/m$

$u = -Kx$, where $K = [5 \quad 1]$ $x_0 = 1$; $\dot{x}_0 = 0$, Simulate the closed loop system response. plot x & $\dot{x}$

first let find the state space rep. of the System.

$$\left.\begin{aligned} x &= x_1, \quad \dot{x} = x_2 \\ \dot{x} &= \dot{x}_1 = x_2 \\ \ddot{x} &= \dot{x}_2 \end{aligned}\right\} \quad \text{- - - - (1)}$$

Sub (1) in (0) we get.

$$m\dot{x}_2 + bx_2 + kx_1 = u$$

$$\dot{x}_2 = \frac{u - bx_2 - kx_1}{m}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{u - bx_2 - kx_1}{m} \end{bmatrix}$$

$$\dot{x} = f(x, u)$$

$$\left\{ \dot{x} = \left[\frac{\partial f}{\partial x} = A\right]x + \left[\frac{\partial f}{\partial u} = B\right] \right\}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} = A.$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad \text{- - - - (2)}$$

given $K = [5 \quad 1]$ & $u = -Kx$ Sub^n these values in (2) we get

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} [5 \quad 1] x \quad \text{---- (3)}$$

$k = 1 N/m$, $b = 2 NS/m$, $m = 1 kg$. (Sub^n these values in (3) we get)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x \cancel{+} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [5 \quad 1] x.$$

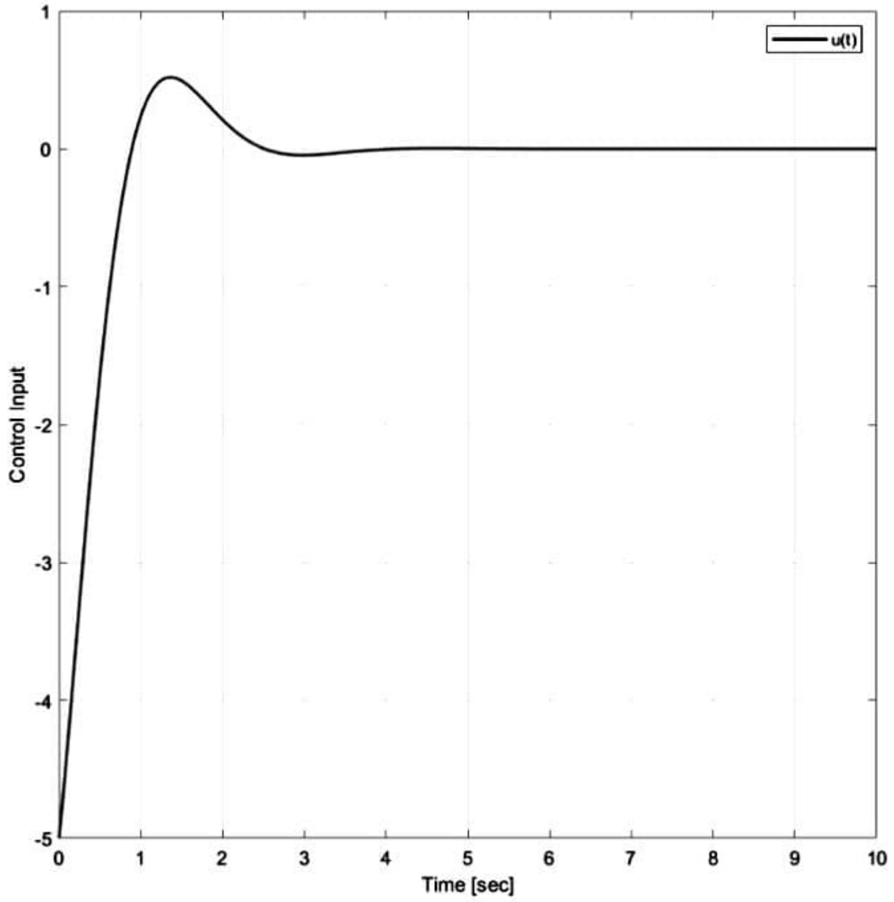
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x \cancel{+} - \begin{bmatrix} 0 & 0 \\ 5 & 1 \end{bmatrix} x$$

$$\cancel{\dot{x} = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} x}$$ ~~---- State space rep. of $\dot{x}$ in terms of x.~~

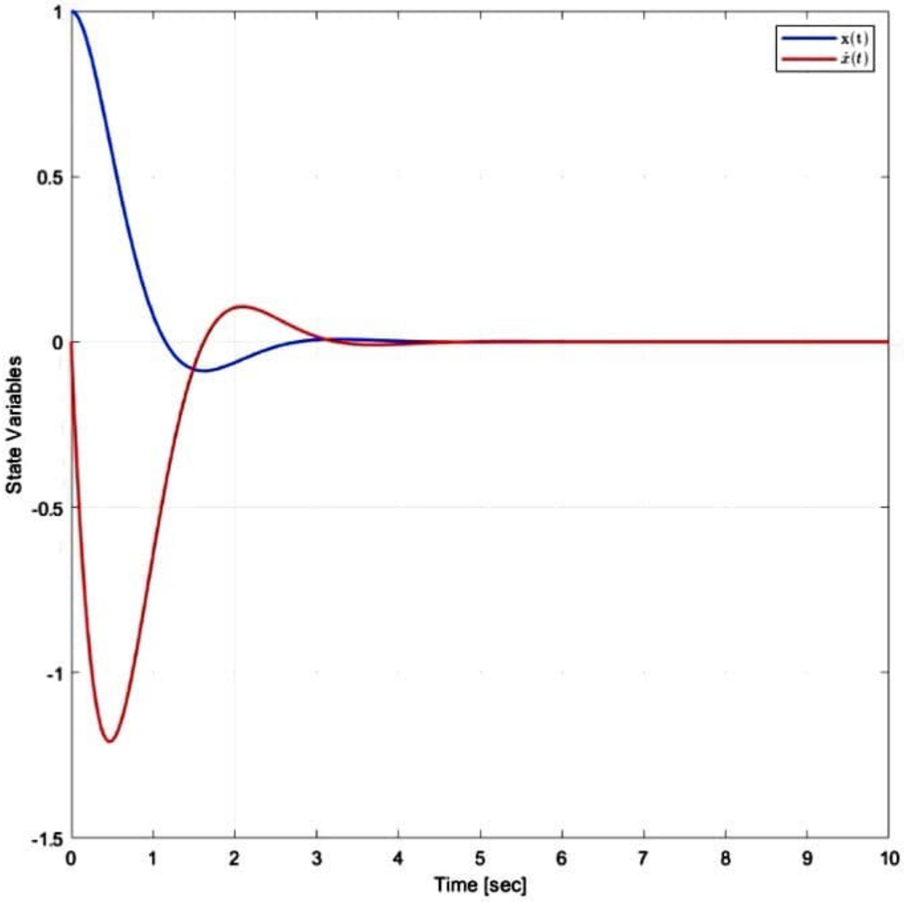
~~The plot of the states are as follows:-~~

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -3 \end{bmatrix} x$$

The plot of the states are as follows:-



u(t)
Control Input
Time [sec]



x(t)
$\dot{x}(t)$
State Variables
Time [sec]

2(b) $y = Cx$

$C = [1 \;\; 0] \qquad L = [5 \;\; -1]^T \Big\}$ ---- (1)

Now $u = -K\hat{x}$ [the estimated state from the state observer] ---- (2)

$[\hat{x}_0, \dot{\hat{x}}_0] = [1.2, 0.2]$ ---- (3)

Since full measurements are not available the state-space eq^n becomes

$\dot{\hat{x}} = A\hat{x} + Bu + L(y_m - \hat{y}_m)$, $\hat{y}_m = C_m\hat{x}$ ---- (4)

Sub (1) & (2) in eq^n (4) we get.

$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x}) + L(y - C\hat{x})$

$\Rightarrow \dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly.$

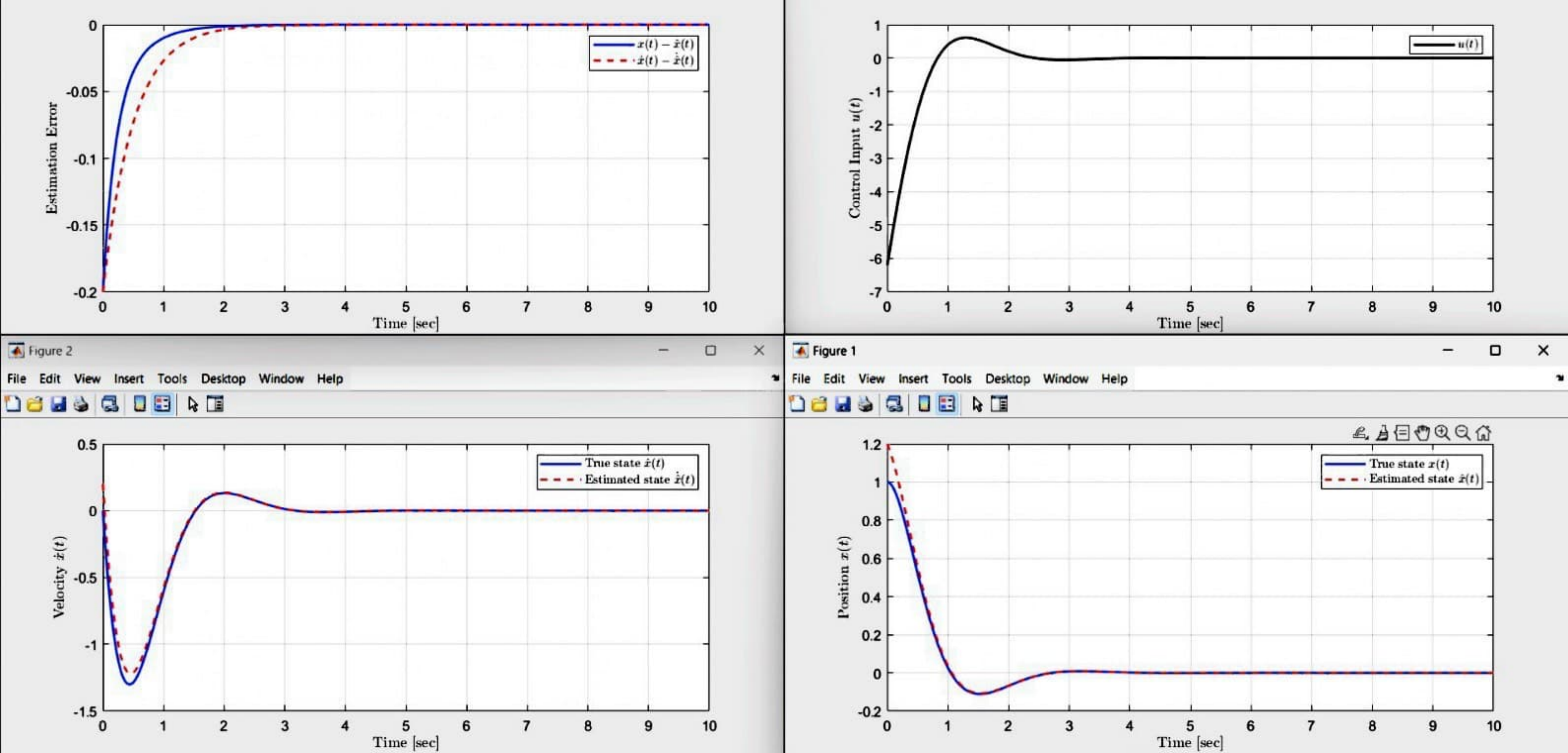
$\dot{\hat{x}} = (A - BK - LC)\hat{x} + L(Cx)$ {As $y = Cx$ from (1)}

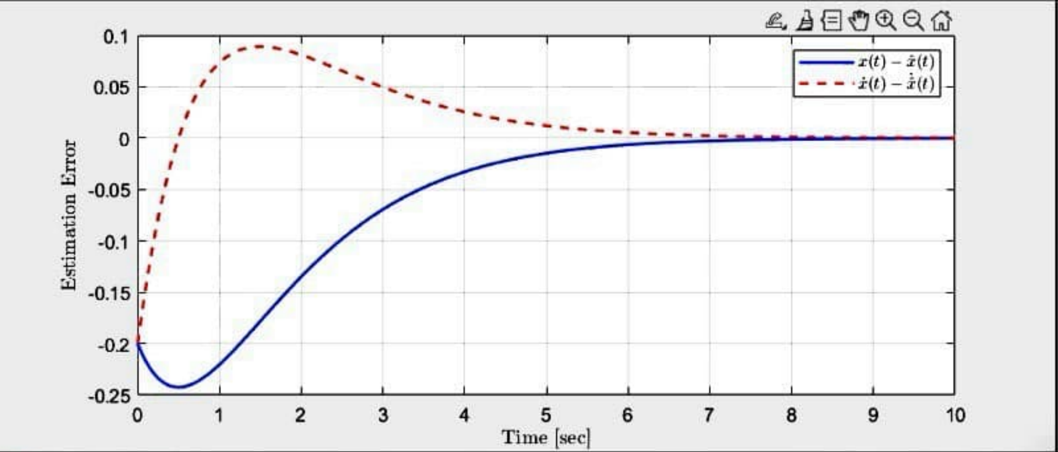
$\Rightarrow \dot{\hat{x}} = (A - BK - LC)\hat{x} + (LC)x$

True state is $x_0 = [1, 0]$ & $\hat{x}_0 = [1.2, 0.2]$

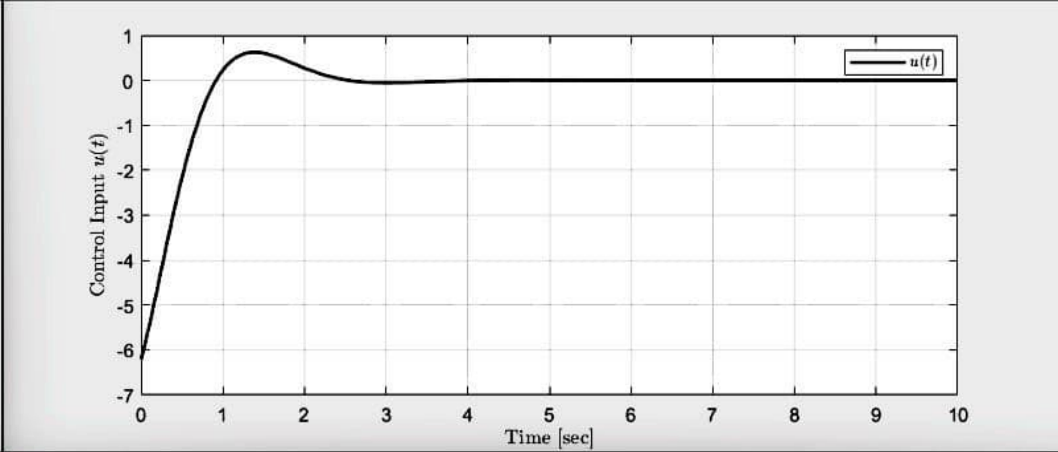
using this state space eq^n we can find the system response.

The code for it is as follows:-


Estimation Error
Time [sec]
$x(t) - \hat{x}(t)$
$\dot{x}(t) - \dot{\hat{x}}(t)$
Control Input $u(t)$
Time [sec]
$u(t)$
Figure 2
File Edit View Insert Tools Desktop Window Help
Velocity $\dot{x}(t)$
Time [sec]
True state $\dot{x}(t)$
Estimated state $\dot{\hat{x}}(t)$
Figure 1
File Edit View Insert Tools Desktop Window Help
Position $x(t)$
Time [sec]
True state $x(t)$
Estimated state $\hat{x}(t)$



Estimation Error
Time [sec]
$x(t) - \hat{x}(t)$
$\dot{x}(t) - \dot{\hat{x}}(t)$



Control Input $u(t)$
Time [sec]
$u(t)$

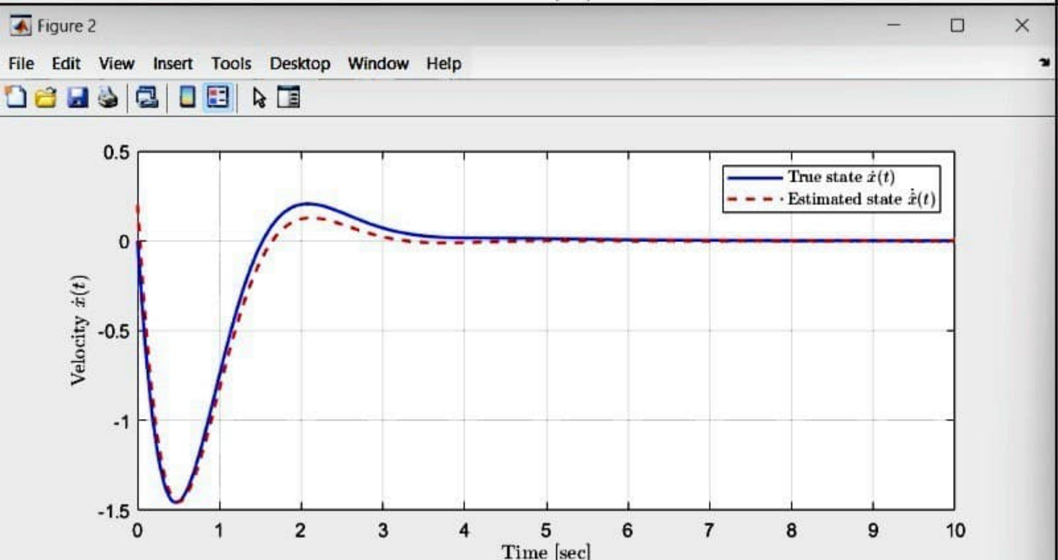


Figure 2
File Edit View Insert Tools Desktop Window Help
Velocity $\dot{x}(t)$
Time [sec]
True state $\dot{x}(t)$
Estimated state $\dot{\hat{x}}(t)$

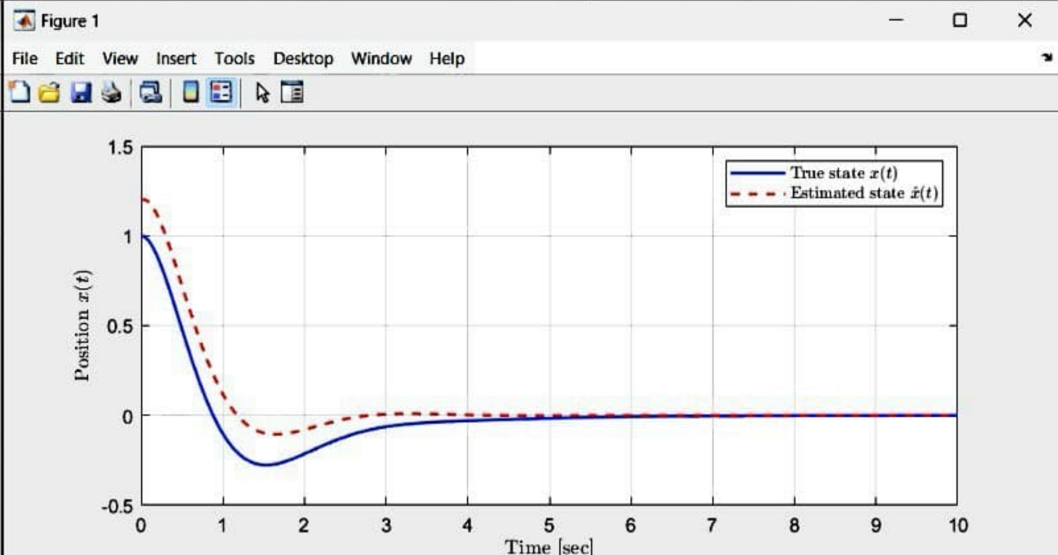


Figure 1
File Edit View Insert Tools Desktop Window Help
Position $x(t)$
Time [sec]
True state $x(t)$
Estimated state $\hat{x}(t)$

2ⓒ The key difference b/t these plots lies in how the eigen values of A-LC affect the state estimation performance.

Plot from 2ⓑ { eigenvalues : -5, -2 } :-

This plot the state observer has constructed using $L = [5\ -1]^T$ and it shows the following behaviour :-

→ Faster convergence of the estimation error to zero.

→ More aggressive initial response

→ Better tracking of the true state

→ Larger initial overshoot in the estimation

Plot from 2ⓒ { eigenvalues :- $-1.0007 \pm 0.00316i$ } :-

These plots are obtained from the LQE designed observer ~~state~~ & they show the following behaviour :-

→ Slower convergence to the true state

→ Less aggressive response

→ More oscillatory behavior due to complex eigen values

→ smaller initial overshoot by longer settling time

The reason why the first plot performs better is as follows :-

The eigen values of (A-LC) determine the observer dynamics & how quickly the estimated states converge to the true states.

→ The eigenvalues of 2ⓑ are placed farther left in the complex plane, resulting in faster error dynamics & quicker convergence of the ~~estimal~~ estimated state to the true state.

→ 2ⓑ has real eigen values & 2ⓒ has both real & complex eigen values, hence 2ⓑ avoiding the oscillatory behavior which is seen in 2ⓒ

→ The wider separation of eigen values (-5 v/s -2) in 2ⓑ allows for a combination of fast initial response & good steady state behaviour.

Note:- While LQE typically provides optimal performance for systems with noise in this case the manually placed poles at -5 & -2 provide better performance because they prioritize faster state estimation over noise reduction.

2(d) There is a change in performance when noise is introduced & reveals the fundamental design b/t manually chosen poles & LQE design the explanation is as follows :-

LQE v/s Manual Design under Noise :-

As Explained Earlier LQE performs better under noise because:-

→ it accounts for both measurement noise & process noise in its design.

→ It provides an optimal balance b/t noise rejection & state Estimation

→ The slower poles $(-1.0007 \pm 0.0316i)$ act as low-pass filter. effectively rejecting high-frequency measurement noise.

Why the Manual Design Fails.

The manually chosen poles $(-5, -2)$ perform poorly because:-

→ Fast poles make the observer sensitive to measurement noise

→ higher gain $L = [5 \ -1]^T$ amplifies the measurement noise

→ The aggressive response attempts to track the noisy measurements too closely, leading to poor state Estimation

The observer dynamics follows this eq^n with noise:-

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y + z - C\hat{x})$$

↳ The Lz term directly injects noise into the state Estimate. with Large "L" value (Faster poles)

→ The noise term Lz has a large magnitude

→ The Estimation error never fully converges due to con. noise injection

→ The observer becomes more sensitive to high-frequency components of the noise.

But LQE mitigates all this and the advantages of LQE are as follows

→ The LQE design

→ min. the expected value of the estimation error co-variance

→ places poles based on the relative mag. of the measurement + process noise.

→ creates natural trade-off b/t quick response & noise rejection.

~~for~~ because of all these reasons LQE designed observers show better noise rejection, more consistent state estimation, smaller estimation error despite slower convergence, more robust performence in practical application with sensor noise.