

R0B - 599 / 003

Assignment - 2

28/09/2020.

$$(D_a) (\ddot{J} + m l^2) \ddot{\theta} + \gamma \dot{\theta} - m g l \sin \theta = l \cos \theta F$$

As it is a 2nd order diff eqn around θ there are two first
order ODE's.

$$\theta = x_1, \quad \dot{\theta} = x_2, \quad F = u$$

$$\dot{\theta} = \dot{x}_1 = x_2$$

$$\ddot{\theta} = \ddot{x}_2$$

$$(\ddot{J} + m l^2) \ddot{x}_2 + \gamma x_2 - m g l \sin(x_1) = l \cos(x_1) u$$

$$x_2 = \frac{l \cos(x_1) u + m g l \sin(x_1) - \gamma x_2}{(\ddot{J} + m l^2)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{l \cos(x_1) u + m g l \sin(x_1) - \gamma x_2}{(\ddot{J} + m l^2)} \end{bmatrix} = A$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left[\begin{array}{c} x_2 \\ \frac{l \cos(x_1) u + m g l \sin(x_1) - \gamma x_2}{\ddot{J} + m l^2} \end{array} \right] = f(x_1, u)$$

$$\text{taking } \ddot{J} = \ddot{J} + m l^2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{l \cos(x_1) u + m g l \sin(x_1) - \gamma x_2}{\ddot{J}} \end{bmatrix}$$

usec/col/2

S - triangular A

200/202/203

$$A = \left. \frac{\partial F}{\partial x} \right|_{x_1=x_2=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

very out too with 3 hours
feelings
yours

$$N=7, \quad x=0, \quad x=0$$

unknowns

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial x_2} & -\frac{\partial f_1}{\partial x_2} \end{bmatrix} \quad \begin{array}{l} \text{shear } \dot{x} = 0 \\ \text{tension } x = 0 \end{array}$$

approx.

$$N(\dot{x})_{\text{act}} = \frac{-\text{Shear } \dot{x} + \text{Tension } x}{J_t} = \frac{-8}{J_t} \quad \begin{array}{l} \text{act} \\ \text{shear } \dot{x} + \text{tension } x \end{array}$$

$$\dot{x} = (\omega) \dot{x}_{\text{act}} + N(x) \dot{x}_{\text{act}} = \dot{x} \quad (\omega J_t + C)$$

$$A = \begin{bmatrix} 0 & 1 \cdot \dot{x} \\ -(\omega) \dot{x}_{\text{act}} + N(\dot{x})_{\text{act}} & N(\dot{x})_{\text{act}} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{x} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{x} \end{bmatrix} = \frac{mgl}{J_t} \quad \begin{bmatrix} \dot{x} \\ \dot{x} \end{bmatrix}$$

$$B = \left. \frac{\partial F}{\partial w} \right|_{w=0} = (\omega) \dot{x}_{\text{act}} \begin{bmatrix} \frac{\partial f_1}{\partial w} + \omega \frac{\partial f_1}{\partial \dot{x}} \\ \frac{\partial f_2}{\partial w} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{x} \end{bmatrix}$$

$$\frac{\partial f_2}{\partial w} = 2C \quad \begin{array}{l} \text{act} \\ w=0 \end{array}$$

$$B = \begin{bmatrix} 0 \\ \frac{+f_2(x_1) h}{J_t} \end{bmatrix} \quad \begin{array}{l} w=0 \\ x_1=0 \\ x_2=0 \end{array} = \begin{bmatrix} 0 \\ \frac{1}{J_t} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ mgl/J_E & -\gamma/J_E \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \ddot{\theta}_E \end{bmatrix} u_S$$

x_S

A_S

x_S

B_S

$$J_E(\ddot{\theta}_E) - 0.006\ddot{\theta}_E - (mgl/g)^2 x_S^2 = mgl\sin\theta + \ddot{\theta}_E(g\cos\theta)$$

$$(A_S - \lambda I) = \begin{bmatrix} 0 & 1 \\ mgl/J_E & -\gamma/J_E \end{bmatrix}$$

$$mgl\sin\theta + \ddot{\theta}_E(g\cos\theta) + mgl\sin\theta - \ddot{\theta}_E(g\cos\theta)$$

$$m = 0.2 \text{ kg}, \quad l = 0.3 \text{ m}, \quad \gamma = 0.1 \text{ N/m sec/deg} \quad g = 9.81 \text{ m/sec}^2$$

$$J = 0.006 \text{ kg m}^2$$

$$J_E = J + ml^2 = 0.006 + (0.2)(0.3)^2 = 0.024$$

$$A_S = \begin{bmatrix} 0 & 1 & 1 \\ (0.2)(9.81)(0.3) & -\frac{(0.1)}{0.024} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 24.525 - 4.16 & \end{bmatrix}$$

$$q\dot{x}_1\theta + (l\dot{x}_2)\cos\theta + (ml^2\ddot{\theta}_E)\sin\theta = \ddot{\theta}_E(g\cos\theta) + ml\ddot{\theta}_E \quad (1)$$

$$|A_S - \lambda I| = 0$$

$$\begin{bmatrix} 0 \\ 12.5 \end{bmatrix}$$

$$(20 = 3\pi + 0)$$

$$0 - \lambda \quad 1 \quad 0$$

$$-20 - \lambda$$

$$(20 - \lambda)(12.5 - \lambda) + 24.525\lambda - 4.165\lambda^2 = 0 \quad (2)$$

$$\lambda^2 - 32.5\lambda + 24.525 = 0$$

$$\lambda_1 = -7.45135 \quad ; \quad \lambda_2 = 3.29135$$

The system is unstable at the θ_E^* point as one of the λ values are +ve, it will cause the system to deviate over time.

$$16) u(t) = k_p(\theta_{cmd} - \theta) - k_d \dot{\theta}$$

$$\Rightarrow (J + ml^2)\ddot{\theta} + r\dot{\theta} - mgl \sin\theta = l \cos(\theta) \cdot [k_p(\theta_{cmd} - \theta) - k_d \dot{\theta}]$$

$$(J + ml^2)\ddot{\theta} + r\dot{\theta} - mgl \sin\theta = l \cos(\theta) (k_p \theta_{cmd}) - l \cos(\theta) k_p \cdot \dot{\theta} - k_d (\cos(\theta))(\dot{\theta})$$

$$(J + ml^2)\ddot{\theta} + r\dot{\theta} - mgl \sin\theta + l \cos(\theta) k_p + k_d \cos\theta \cdot \dot{\theta}$$

$$= u_3 \theta = x_1 \cdot (s \cdot 0)(c \cdot 0) + d \cdot 0 \cdot 0 = s \cdot 0 + 0 = 0$$

$$\dot{\theta} = \dot{x}_1 = x_2$$

$$\ddot{\theta} = \dot{x}_2 = \begin{bmatrix} 1 & 0 \\ 0 & (1 \cdot 0) - (s \cdot 0)(c \cdot 0)(s \cdot 0) \end{bmatrix} = 2A$$

$$\begin{array}{l} f_{x_1} \\ \cancel{f_{x_2}} \end{array} \quad J \in \dot{x}_2 + r x_2 - mgl \sin(x_1) + l \cos(x_1) \cdot x_1 k_p + k_d \cos(x_1) \cdot x_2 = l \cos(x_1) k_p \theta_{cmd}.$$

$$(J + ml^2 = \bar{J}_t)$$

$$\ddot{x}_2 = \dot{x}_1 \cdot 2A$$

$$\dot{x}_2 = l \cos(x_1) k_p \theta_{cmd} - 8 x_2^2 + mgl \sin(x_1) + l \cos(x_1)$$

$$- l \cos(x_1) \cdot x_1 k_p - k_d \cos(x_1) \cdot x_2$$

$$\underline{0 = \cos(x_1) \cdot x_1 k_p + l \cos(x_1) \cdot x_2}$$

$$2 \sin x_1 \cos x_1 = s \cdot k \quad ; \quad 2 \sin x_1 \cos x_1 = 1 \cdot k$$

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$$2 \sin x_1 \cos x_1 = s \cdot k \quad ; \quad 2 \sin x_1 \cos x_1 = 1 \cdot k$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -l \cos(x_1) k_p \theta_{cmd} - \gamma x_2 + mgl \sin(x_1) \\ -l \cos(x_1) x_1 k_p - k_d l \cos(x_1) \cdot x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial (F_1(x,u))}{\partial x_1} & \frac{\partial (F_1(x,u))}{\partial x_2} \\ \frac{\partial (F_2(x,u))}{\partial x_1} & \frac{\partial (F_2(x,u))}{\partial x_2} \end{bmatrix} \Big|_{x_1=x_2=0}$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} - \frac{l k_p}{J_t} & -\frac{\gamma}{J_t} - \frac{k_d l}{J_t} \end{bmatrix}$$

$$B = \frac{\partial(F)}{\partial \theta_{cmd}} \Big|_{x_1=x_2=0} = \begin{bmatrix} 0 & 1 \\ l \cos(x_1) k_p \end{bmatrix} \Big|_{x_1=x_2=0} = \begin{bmatrix} 0 \\ l k_p \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} - \frac{l k_p}{J_t} & -\frac{\gamma}{J_t} - \frac{k_d l}{J_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{l k_p}{J_t} \end{bmatrix} \theta_{cmd}$$

Subⁿ the values of $m = 0.2 \text{ kg}$, $J = 0.006 \text{ kg m}^2$,

$d = 0.3 \text{ m}$, $C = 0.1 \text{ N sec/m}$, $\gamma = 0.1 \text{ N m sec/rad}$

$$g = 9.81 \text{ m/sec}^2$$

$$= \begin{bmatrix} 0 & 1 \\ \left(\frac{0.2 \times 9.81 \times 0.3}{0.024} - (0.3) kp \right) & \left(-\frac{0.1}{0.024} - kd \frac{(0.3)}{0.024} \right) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ (2u \cdot 525 - 12.5 kp) & (-4.16 - 12.5 kp) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0.3 kp \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2u \cdot 525 - 12.5 kp \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -4.16 - 12.5 kp & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.6 & 12.5 kp \end{bmatrix} \theta_{cmd}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.6 \theta_{cmd} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -4.16 - 12.5 kp & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(C) \quad A = \begin{bmatrix} 0.25 + j1.12 & 24.525 - 12.5kp \\ 24.525 - 12.5kp & 4.16 - 12.5kd \end{bmatrix}$$

$$|S\bar{I} - A| = \begin{bmatrix} S + 4.16 \\ 0 + 12.5kp \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 24.525 - 12.5kp \end{bmatrix} = 4.16 - 12.5kp$$

$$S^2 = S(S + 4.16) = 85.88 \cdot 0 - 12.5kp$$

$$-24.525 + 12.5kp$$

$$S^2 + S(4.16 + 12.5kp) + (-24.525 + 12.5kp)$$

$$a_0 = 1 ; a_1 = (4.16 + 12.5kp) ; a_2 = (-24.525 + 12.5kp)$$

$$\Delta = (q(2.51 + 26.25kp))^2 - (a_1)^2 = 85.88^2$$

using the Routh-Hurwitz method to determine the value of

k_p & k_d such that $\Delta < 0$

$$\Delta = \begin{bmatrix} a_1 & 0 \\ a_0 & a_2 \end{bmatrix} = \begin{bmatrix} 4.16 + 12.5kp & 0 \\ -24.525 + 12.5kp & 85.88 \end{bmatrix}$$

$$\Delta > 0$$

$$\Rightarrow 4.16 + 12.5kp > 0$$

$$\Rightarrow kp > -\frac{4.16}{12.5} \Rightarrow kp > -0.3328$$

$$\Delta_2 = a_1 a_2 > 0 \quad \Delta_2 = (4.16 + 12.5 k_D) (24.525 + 12.5 k_P) > 0$$

$$\Rightarrow -24.525 + 12.5 k_P > 0$$

$$\Rightarrow \boxed{k_P > 1.962} \quad (A-12)$$

\rightarrow The ranges of k_P & k_D for the system to be stable is

$$\boxed{k_D > -0.3328} \quad \Rightarrow \quad \boxed{k_P > 1.962}$$

(d) In order to find the values of k_P & k_D using percentage overshoot & settling time lets consider the characteristic equation $s^2 + s(4.16 + 12.5 k_D) + 24.525 + 12.5 k_P = 0$ to calculate the Routh-Hurwitz matrix.

$$(4.16 + 12.5 k_D) = 0 \quad (4.16 + 12.5 k_D) = 10 \quad (1 = 0)$$

$$s^2 + s(4.16 + 12.5 k_D) + 24.525 + 12.5 k_P = 0 \quad \dots \quad (1)$$

for unknown variables of overshoot & settling time we get

lets compare it to the std. characteristic polynomial of a 2nd order diff. equation.

$$s^2 + 2\zeta n_s s + \omega_n^2 = 0 \quad (2)$$

$$2\zeta n_s s = 4.16 + 12.5 k_D \quad \dots \quad (3)$$

$$\omega_n^2 = 24.525 + 12.5 k_P \quad (4)$$

$$24.525 - 4.16 = 20.36 \quad \zeta = \frac{4.16}{20.36} = 0.2036$$

W.L.T. I - stability of linear system, part 1

$$\eta_S = S_{\text{sec}}, \quad P_0 = 12\%$$

$$P_0 = e^{(-\xi \pi / \sqrt{1-\xi^2})} \Rightarrow \xi = -\frac{a}{\sqrt{1-a^2}}, \quad a = \log_e \left(\frac{P_0}{100} \right)$$

$$\tau_S = \frac{4}{\xi \omega_n} \Rightarrow \omega_n = \frac{4}{\xi \tau_S}$$

$$\text{now } a = \log_e \left(\frac{12}{100} \right) = -2.120$$

$$\xi = -\frac{(-2.120)}{\sqrt{(3.14)^2 + (-2.120)^2}} = 0.5595$$

$$\omega_n = \frac{4}{(0.5595)(5)} = 1.4298$$

$$(1.4298)^2 = 12.5 k_p - 24.525$$

$$\therefore 1.0 = q_1, \quad 2 = q_2$$

$$2(0.5595) \left[k_p = 0.2048 \right] = 12.5 k_p - 24.525$$

$$2(0.5595) \left[k_p = -0.2048 \right] = 12.5 k_p - 24.525$$

$$-2.56 = 12.5 k_p \quad \text{or} \quad k_p = -0.2048$$

$$2(0.5595) \left[k_D = -0.2048 \right] = 24.525 \quad \text{and} \quad k_D = -0.2048$$

∴ for a percent overshoot of 12% & a settling time of 5 sec

for a 2% band width $k_p = 0.2255$, $k_D = -0.2048$.

Problem (2)

(a) The linearized model in problem I in the form $\dot{x} = Ax + Bu$ is as follows:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ mg/J_t & -8/J_t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ d/J_t \end{bmatrix} u_s$$

$$x_1 = x_1 \quad \text{(as)} \quad \text{--- (a)}$$

$$\dot{x}_2 = \frac{mg}{J_t} x_1 - \frac{8}{J_t} x_2 + \frac{d}{J_t} u_s \quad \text{--- (b)}$$

$$u(t) = k_p(\theta_{cmd} - \theta) + k_E \int^t (\theta_{cmd} - \theta(t)) dt - k_D \dot{\theta} \quad \text{--- (1)}$$

$$\dot{x}_1 = (\theta_{cmd} - \theta(t)) \quad \text{--- (2) from (1)}$$

$$\theta(t) = \theta_0 = x_1 ; \quad \dot{\theta} = \dot{x}_1 = x_2 ; \quad x_3 = \theta$$

$$\dot{\theta} = x_2 ; \quad \ddot{\theta} = \dot{x}_2 = x_3 = \theta$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ mg/J_t & -8/J_t & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ d/J_t \\ 0 \end{bmatrix} u_s$$

by sub" (2) in (1) we get.

$$u_s = k_p(\theta_{cmd} - \theta) + k_E \int^t (\dot{x}_2) dt - k_D \dot{\theta}$$

$$(k_E \cdot 2 \cdot t + 0) = (2k_E \cdot t) (R/18.8) \quad \text{S}$$

$$u(t) = k_p(\theta_{cmd} - \theta) + k_E x_2 - k_D \dot{\theta} \quad \text{--- (3)}$$

$$(W.K.T \quad \dot{\theta} = x_2 = \dot{x}_1 ; \quad \theta_0 = x_1)$$

Sub these values in (3) & we get $\theta = 0$.

$$u(t) = k_p(\theta_{cmd} - \theta) + k_E x_2 + k_D \dot{x}_2 \quad \text{--- (4)}$$

Subⁿ ④ in (b) we get the following expression
for x_2

$u(t)$ replaces us

$$\dot{x}_2 = \frac{mgl}{J_t} (x_1) - \frac{\gamma}{J_t} (x_2) + \frac{l}{J_t} \left(k_p (\theta_{cmd} - x_1) + k_s x_1 - k_d x_2 \right)$$

$$\dot{x}_2 = \left[\frac{mgl}{J_t} - \frac{l}{J_t} k_p \right] x_1 + \left[-\frac{\gamma}{J_t} - \frac{l}{J_t} k_d \right] x_2 + \frac{l}{J_t} (k_p \theta_{cmd})$$

$$+ \frac{l}{J_t} x_1 k_s$$

$$(x_3 = x_1)$$

$$\dot{x}_1 = x_2$$

$$\Rightarrow \dot{x}_2 = \left[\frac{mgl}{J_t} - \frac{l}{J_t} k_p \right] x_1 + \left[-\frac{\gamma}{J_t} - \frac{l}{J_t} k_d \right] x_2 + \frac{l}{J_t} (k_p \theta_{cmd})$$

$$+ \frac{l}{J_t} k_s x_3$$

$$\dot{x}_3 = \theta_{cmd} - x_1$$

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{bmatrix}$$

$A =$

$$A = \begin{bmatrix} 0 & 1 & (d)D \\ \frac{mgl - lkp}{J_t} & \frac{-\gamma - lkp}{J_t} & \frac{0 + lkp}{J_t} \\ -1 & 0 & 0 \end{bmatrix}$$

$(b) \sin \theta \dot{\theta} + x \left[\omega^2 - \frac{\gamma}{l} - \right] + x \left[\frac{1}{l} \sin(\theta - 180^\circ) \right] = \ddot{x}$

$B =$

$$B = \begin{bmatrix} \frac{\partial F_1}{\partial \theta_{cmd}} \\ \frac{\partial F_2}{\partial \theta_{cmd}} \\ \frac{\partial F_3}{\partial \theta_{cmd}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{l \cdot kp}{J_t} \\ \frac{1}{l} \end{bmatrix}$$

$(b) \sin \theta \dot{\theta} + x \left[\omega^2 - \frac{\gamma}{l} - \right] + x \left[\frac{1}{l} \sin(\theta - 180^\circ) \right] = \ddot{x}$

Writing it in the following form
 $\ddot{x} = A_{c,d} x_c + B_{e,f} \theta_{cmd}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{mgl - lkp}{J_t} & \frac{-\gamma - lkp}{J_t} & \frac{lkp}{J_t} \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{l \cdot kp}{J_t} \\ 1 \end{bmatrix}$$

$\ddot{x} = b_{c,d} x_c + b_{e,f} \theta_{cmd}$

$$\frac{16}{245} \quad \frac{76}{245} \quad \frac{76}{245}$$

Given $m = 0.2 \text{ kg}$, $I = 0.006 \text{ kg m}^2$, $d = 0.3 \text{ m}$, $c = 0.1 \text{ N sec/m}$, $\gamma = 0.1 \text{ Nm sec/rad}$, $g = 9.81 \text{ m/sec}^2$

$$J_E = M^2 + \gamma = (0.2)(0.3)^2 + 0.006 = 0.024$$

(Sub the above values in eq ⑤ we get.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0.2 \cdot 81 + 0.1 \cdot 0 & 0 \\ 24 \cdot 825 - 12 \cdot 5 \text{ kp} & 12 \cdot 5 \text{ kp} \\ -1 & 0.2(0.3) + 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \cdot 5 \text{ kp} \\ 0 \end{bmatrix} \theta_{\text{cmd}}$$

(b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 24 \cdot 825 - 12 \cdot 5 \text{ kp} & -4 \cdot 16 - 12 \cdot 5 \text{ kp} & 12 \cdot 5 \text{ kp} \\ -1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \theta_{\text{cmd}}$

$$|S_I - A| = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 24 \cdot 825 - 12 \cdot 5 \text{ kp} & -4 \cdot 16 - 12 \cdot 5 \text{ kp} & 12 \cdot 5 \text{ kp} \\ -1 & 0 & 0 \end{bmatrix}$$

$$|S_I - A| = \begin{bmatrix} S & 0 & 0 \\ -24 \cdot 825 + 12 \cdot 5 \text{ kp} & S + 4 \cdot 16 + 12 \cdot 5 \text{ kp} & -12 \cdot 5 \text{ kp} \\ + 2 \cdot 81 & 0 & S \end{bmatrix} = J_E$$

The characteristic polynomial of this matrix is

$$fs [s(s + (4.16 + 12.5k_D))] + 1 \left(s(-24.525 + 12.5k_F) \right)$$

$$= s^2(s + (4.16 + 12.5k_D)) + s(-24.525 + 12.5k_F)$$

$$= s^3 + (4.16 + 12.5k_D)s^2 + s(-24.525 + 12.5k_F)$$

$$= s^3 + (4.16 + 12.5k_D)s^2 + s(-24.525 + 12.5k_F) + 12.5k_F$$

$$a_0 = 1; \quad a_1 = 4.16 + 12.5k_D \quad a_2 = -24.525 + 12.5k_F$$

$$a_3 = 12.5k_F, \quad a_4 = 0, \quad a_5 = 0$$

$$J_A = \begin{bmatrix} a_1 & a_3 & a_5 \\ 0 & 0 & 0 \\ 0 & a_2 & a_4 \end{bmatrix} = \begin{bmatrix} 4.16 + 12.5k_D & 12.5k_F & 0 \\ 0 & -24.525 + 12.5k_F & 0 \\ 0 & 0 & 0 \end{bmatrix} = (A - \beta I)$$

$$J_L = \begin{bmatrix} 4.16 + 12.5k_D & 12.5k_F & 0 \\ 0 & -24.525 + 12.5k_F & 0 \\ 0 & 0 & 0 \end{bmatrix} = (A - \beta I)$$

$$J_L = \begin{bmatrix} 4.16 + 12.5k_D & 12.5k_F & 0 \\ 0 & -24.525 + 12.5k_F & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta_1 > 0$$

$$\Rightarrow 4 \cdot 16 + 12.5 k_D > 0$$

$$\Rightarrow \underline{k_D > -0.3328}$$

$$\Delta_2 > 0$$

$$\Rightarrow (4 \cdot 16 + 12.5 k_D)(-24 \cdot 525 + 12.5 k_I) - 12.5 k_I^2 > 0$$

$$(4 \cdot 16 + 12.5 k_D)(-24 \cdot 525 + 12.5 k_I) > 12.5 k_I^2.$$

$$\Rightarrow \underline{(4 \cdot 16 + 12.5 k_D)(-24 \cdot 525 + 12.5 k_I) > k_I^2}.$$

$$\Rightarrow \cancel{(0.3328 + k_D)} \neq 0 \text{ (not possible)}$$

$$\Rightarrow \underline{4 \cdot 16(-24 \cdot 525) + 4 \cdot 16(12.5 k_I) + 12.5 k_D[-24 \cdot 525 + 12.5 k_I]}$$

$$\Rightarrow \underline{\frac{-102.024}{12.5} + 4 \cdot 16 k_I + (-24 \cdot 525 k_D) + 12.5 k_I k_D} > k_I^2$$

$$\Rightarrow 4 \cdot 16 k_I - 24 \cdot 525 k_D - 8 \cdot 16 k_I^2 + 12.5 k_I k_D > k_I^2$$

$$\Rightarrow k_I^2 - 4 \cdot 16 k_I - 24 \cdot 525 k_D + 12.5 k_I k_D - 8 \cdot 16 k_I^2$$

$$k_I \rightarrow 1962$$

$$\Delta_3 > 0$$

$$\Rightarrow \Delta_3 = a_3 \Delta_2$$

$$\Rightarrow a_3 > 0$$

$$\text{w.r.t } a_3 = 12.5 k_I.$$

$$\Rightarrow k_I > 0.$$

$$J < 282.81 + (q12.81 + 282.116) (a12.5k_I + d1N) < 0$$

$$\text{if } k_2 > 0 \quad \& \quad \underbrace{(4.16 + 12.5 k_D)}_{J < 282.81 + 282.116} (-24.525 + 12.5 k_P) > k_2 \\ \text{or } J < (q12.81 + 282.116) > 0$$

$$\Rightarrow (-24.525 + 12.5 k_P) > 0$$

$$J < (q12.81 + 282.116) (a12.5k_I + d1N) < 0$$

$$\Rightarrow k_P > 1.962.81$$

\therefore The values of k_P , k_D & k_I (for the system to be stable) are as follows.

$$k_P > 1.962 (q12.81) d1N + (282.116) d1N <$$

$$k_D > -0.3328$$

$$2.61$$

$$0 < k_I < (4.16 k_P - 24.525 k_D + 12.5 k_P k_D - 8.16192)$$

$$q12.81 + (q12.82.116) + q12.81 N + p10.801 <$$

$$2.61 < 3$$

$$2.61 < q12.81 + 2.61 + 282.116 + q12.82.116 <$$

$$2.61 < q12.81 + 2.61 + q12.82.116 - q12.81 N <$$

Ans

3(a) The linearized model is given as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_E} & -\frac{g}{J_E} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{d}{J_E} \end{bmatrix} u_s$$

now if a PD controller is introduced.

$$u = k_p(\theta_{cmd} - \theta) + k_d(\dot{\theta} - \omega) \quad \{ w \text{ is the disturbance in the angular velocity} \}$$

with $\theta = x_1$, $\dot{\theta} = x_2 \Rightarrow \theta = \cos \theta \Rightarrow \dot{\theta} = -\sin \theta \cdot \ddot{\theta}$ \Rightarrow $w \Rightarrow$ is a sve disturbance

$$\dot{\theta} = \dot{x}_1 = x_2$$

So when the PD controller is introduced the linear system takes the following form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_E} - \frac{dk_p}{J_E} - \frac{g}{J_E} + \frac{dk_d}{J_E} & -\frac{g}{J_E} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{d}{J_E} k_p - \frac{d}{J_E} k_d & \frac{d}{J_E} \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

By sub. the values of given in Q-1 we get

$$A = \begin{bmatrix} 0 & 1 \\ 24.525 & -4.16 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 12.5 k_p + 12.5 k_d & \end{bmatrix}$$

$$k_p = 5 ; k_D = 0.1$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -37.975 & -5.41 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 62.5 + 1.25 \end{bmatrix}$$

$$\text{mit } w_{\text{kip}} \} x_{ss} = A x_{ss} + B w$$

Lösungen folgen mit

$$x_{ss} = -A^{-1}Bw$$

$$x = y = \theta$$

$$\Rightarrow w = \begin{bmatrix} \theta_{\text{cmd}} \\ w_{\text{kip}} \end{bmatrix}$$

$$\Rightarrow x_{ss} = -A^{-1}B \begin{bmatrix} \theta_{\text{cmd}} \\ w \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta_{ss} \\ \ddot{\theta}_{ss} \end{bmatrix} = -A^{-1}B \begin{bmatrix} \theta_{\text{cmd}} \\ w \end{bmatrix} \quad (1)$$

aus der Gleichung für $\ddot{\theta}_{ss}$ folgt

$$A^{-1} = \begin{bmatrix} -0.1425 & -0.0283 \\ 1.8 & 0 \end{bmatrix}$$

1	2	3	4
1.8	0	0.0283	-0.1425

By Sub these values in ① we get

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = - \begin{bmatrix} -0.1425 & -0.0263 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & +1.25 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$= \cancel{0 - (0.0263)(62.5)}$$

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = - \begin{bmatrix} -0.0263 \times 62.5 & -1.25 + 0.0263 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = \begin{bmatrix} -64.375 + \theta_{cmd} + 0.032875(w) \\ 0 \end{bmatrix}$$

by subⁿ the values of $\theta_{cmd} = \frac{\pi}{3}$ & $w = 3 \frac{R_e}{3}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ we get} \\ \begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = \begin{bmatrix} 1.819 \\ -62.183 \end{bmatrix}$$

as $t \rightarrow \infty \quad \theta_{ss} \rightarrow 1.62183 - 1.819$

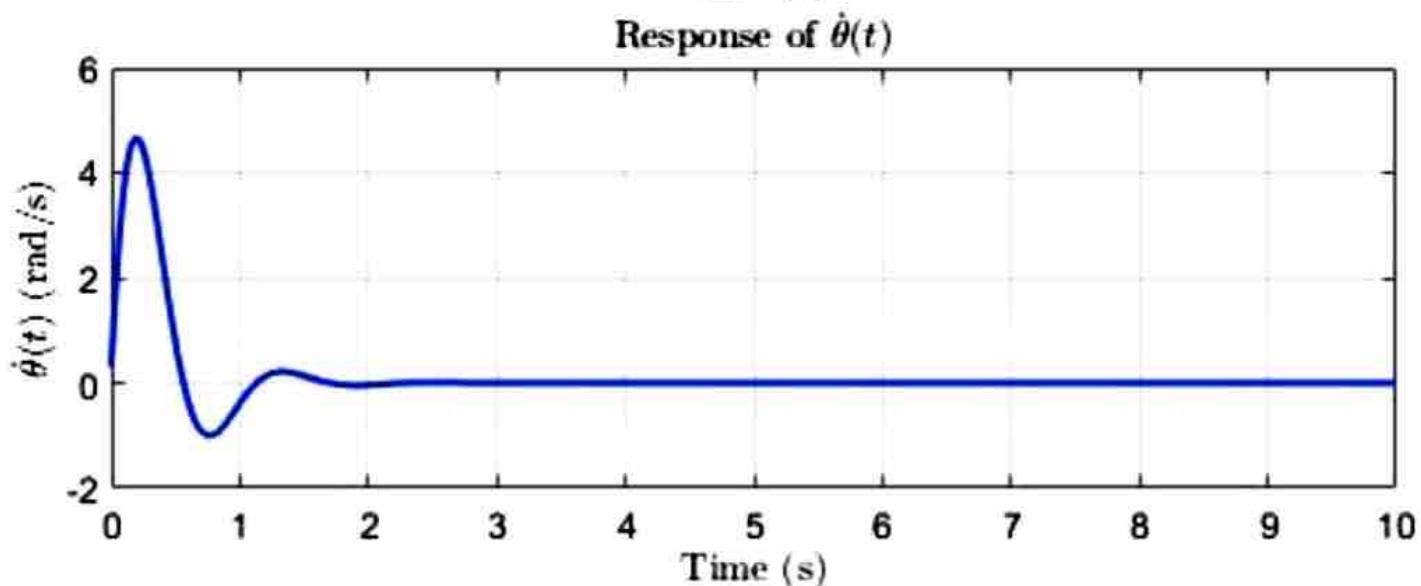
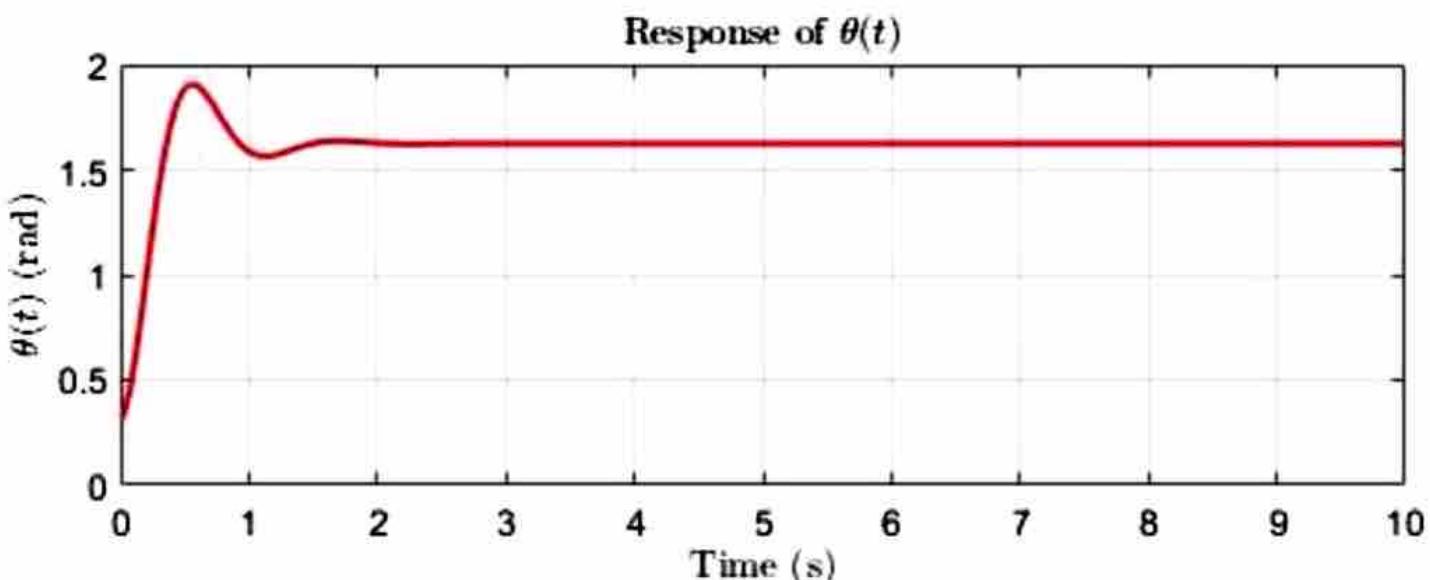
Steady-state value of the state variables are:

1.8222

0

steady-state value of theta is :

1.8222



3a if $w \rightarrow +ve$ fire disturbance is +ve } then

$$(continued) \quad u = k_p(\theta_{cmd} - \theta) - k_d(\dot{\theta} + w)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 12.5k_p & -12.5k_d \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2w \cdot 52.5 & 0 \\ -12.5k_p & -12.5k_d \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 37.925 & 0 \\ -37.925 & -5.41 \end{bmatrix}$$

$$x_{ss} = -A^{-1}B \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$x_{ss} = - \begin{bmatrix} 2.0 & 1.425 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$x_{ss} = \begin{bmatrix} 0.1425 & 0.0283 \\ -1 & 62.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$x_{ss} = \begin{bmatrix} 0.10263 & 62.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0.0283 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$X_{SS} = \begin{bmatrix} 0.0263 \times 62.5 \text{ Ond} & 0.0263 \times 0 \\ 0 & -10.25 \times 0.0263 W \\ 0 & 0 \end{bmatrix}$$

$$X_{SS} = \begin{bmatrix} 1.6247 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_{SS} \\ \dot{\theta}_{SS} \\ \ddot{\theta}_{SS} \end{bmatrix}$$

$\Rightarrow \theta_{SS} \rightarrow 1.6247$ as $t \rightarrow \infty$
 $\dot{\theta}_{SS} \rightarrow 0$

$$\begin{bmatrix} 1.6247 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.6247 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

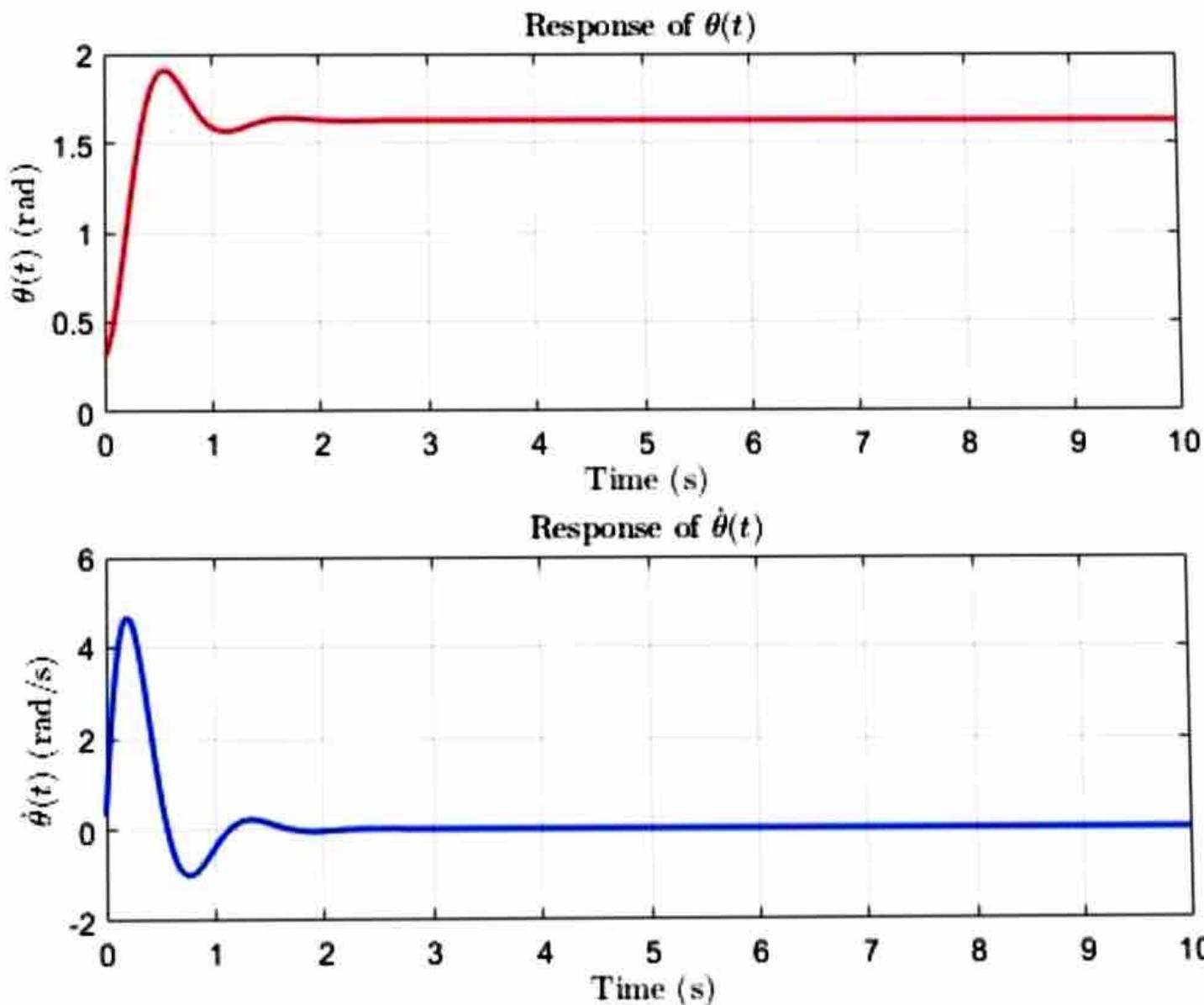
Steady-state value of the state variables are:

1.6247

0

steady-state value of theta is :

1.6247



2(b) The linearized model is given as follows:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_e} & -\frac{g}{J_e} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ d/J_e \end{bmatrix} \text{ ms}$$

when a PID controller & a disturbance w is introduced to this system

$$K_P(\theta_{cmd} - \theta) + K_I \int_0^t (\theta_{cmd} - \theta(s)) dt + K_D(\dot{\theta} - w)$$

{Taking a
+ve disturbance}

The linear model becomes:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \left(\frac{mgl}{J_e} - \frac{K_P}{J_e}\right) & -\frac{g}{J_e} + \frac{K_P}{J_e} & \frac{dK_I}{J_e} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$(i) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{dK_P}{J_e} \\ -\frac{dK_D}{J_e} \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

Sub all the values in

$$K_P = 5$$

$$K_D = 0.1$$

$$K_I = 2.3562$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.41 & 29.4325 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 62.5 & +1.25 \\ 1.25 & 0 \end{bmatrix}$$

W.X.T

$$0 = Ax_{ss} + Bu_{ss}$$

$$\Rightarrow x_{ss} = -A^{-1}Bu_{ss}$$

$$\begin{bmatrix} \dot{\theta}_{ss} \\ \ddot{\theta}_{ss} \\ x_I \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.412345 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 62.5 & +1.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\theta}_{ss} \\ \ddot{\theta}_{ss} \\ x_I \end{bmatrix} = -\begin{bmatrix} 0 & 0 & -1.000 \\ 1.000 & 0 & 0 \\ 0.1837 & 0.0340 & -1.2894 \end{bmatrix} \begin{bmatrix} \dot{\theta} & 0 \\ 62.5 & -1.25 \\ 201.78 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\theta}_{ss} \\ \ddot{\theta}_{ss} \\ x_I \end{bmatrix} = -\begin{bmatrix} -1.000 & 0 & 0 \\ 0 & 0 & \omega \\ 0.0340 & (+1.25 \times 0.0340) & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} \dot{\theta}_{ss} \\ \ddot{\theta}_{ss} \\ x_I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -0.8356 & -0.0425 & 1 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \\ 1 \end{bmatrix}$$

by sub $\theta_{cmd} = \frac{\pi}{2}$ & $\omega = 3 \text{ rad/sec}$

$$\begin{bmatrix} \dot{\theta}_{ss} \\ \ddot{\theta}_{ss} \\ x_I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.8356 \theta_{cmd} & -0.0425 \omega & 1 \end{bmatrix} \begin{bmatrix} 0.047 \\ 0 \\ 1 \end{bmatrix}$$

as $t \rightarrow \infty$

$\dot{\theta}_{ss} \rightarrow 0.047$.

$x_I \rightarrow -0.993$

Steady-state value of the state variables are:

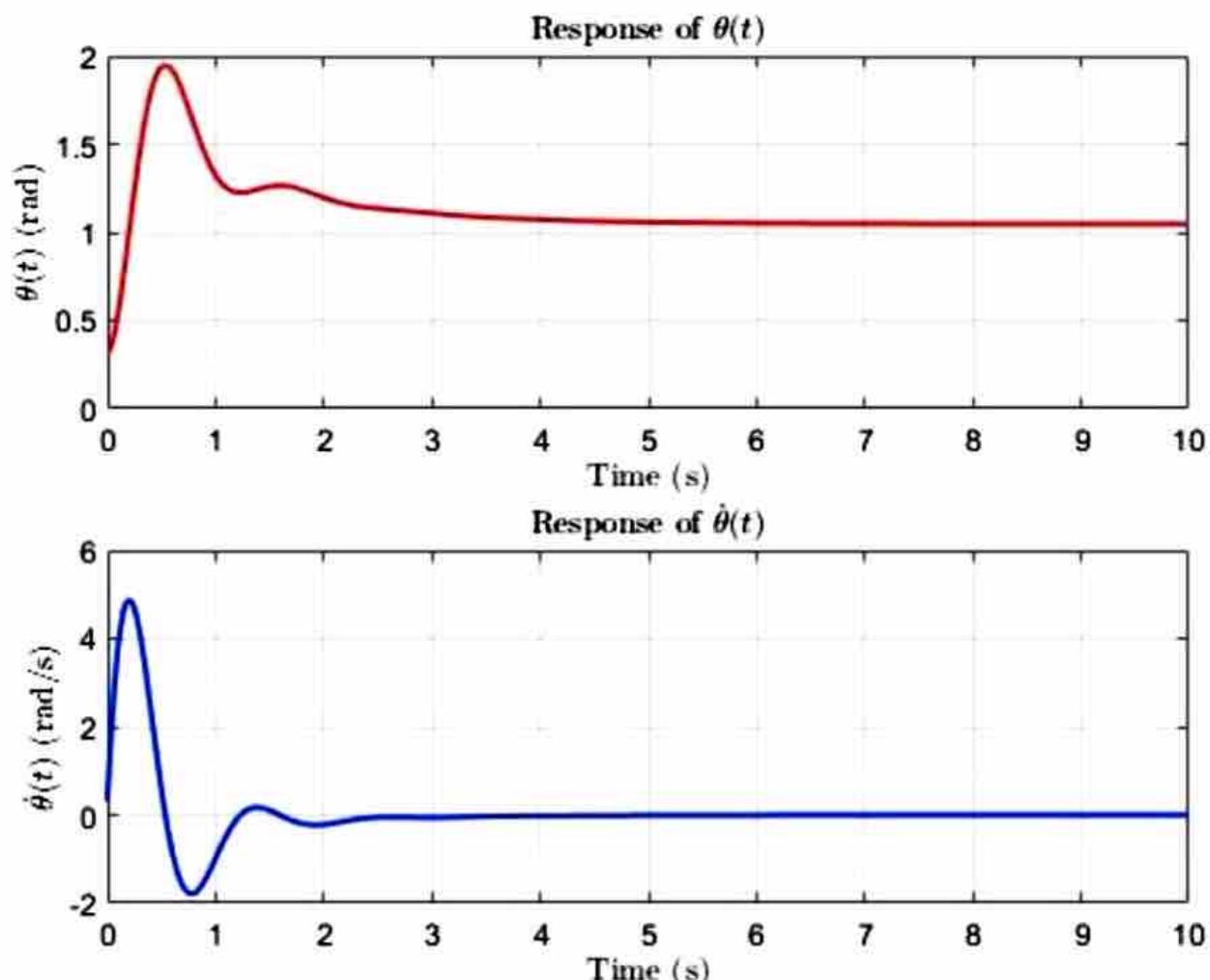
1.0472

0

-0.9993

steady-state value of theta is

1.0472



3(b) if $w = \omega$ & the disturbance is 'ave' then
 continued $w = \omega_0 (\theta_{md} - \theta) + k_d \int_0^t \theta_{md} - \theta(t) dt$
 $- k_d (\theta + w)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.41 & 29.4525 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \\ 1 & 0 \end{bmatrix}$$

$$x_{ss} = -A^{-1}B \begin{bmatrix} \theta_{md} \\ w \end{bmatrix}$$

$$x_{ss} = - \begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.41 & 29.4525 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{md} \\ w \end{bmatrix}$$

$$x_{ss} = - \begin{bmatrix} 0 & 0 & -1.000 \\ 1.000 & 0 & 0 \\ 0.1837 & 0.0340 & -1.2894 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{md} \\ w \end{bmatrix}$$

On simplification we get

$$X_{SS} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -0.8356 & +0.0425 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$\text{w.r.t } \theta_{cmd} = \frac{\pi}{3}$$

$$X_{SS} = \begin{bmatrix} \theta_{cmd} \\ 0 \\ -0.8356 \theta_{cmd} \\ + 0.0425 w \end{bmatrix}$$

$$+ w = 3 \text{ rad/sec}$$

$$\Rightarrow X_{SS} = \begin{bmatrix} 1.0466 \\ 0 \\ -0.7447 \end{bmatrix}$$

$$\theta_{SS} \rightarrow \cancel{-0.7447} \quad 1.0466 \quad \text{as } t \rightarrow \infty.$$

$$x_2 \rightarrow \cancel{-0.7447}$$

Steady-state value of the state variables are:

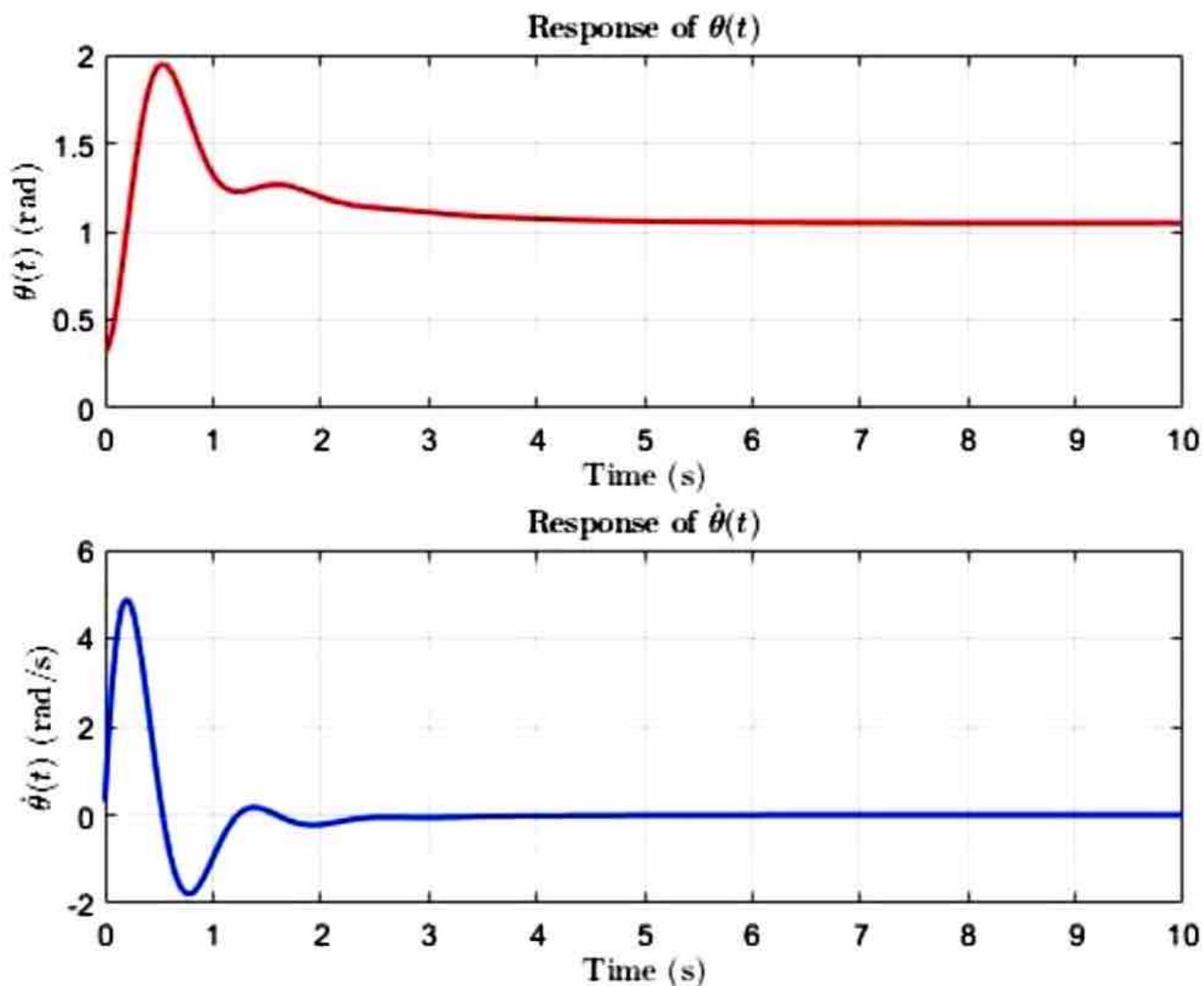
1.0472

0

-0.7447

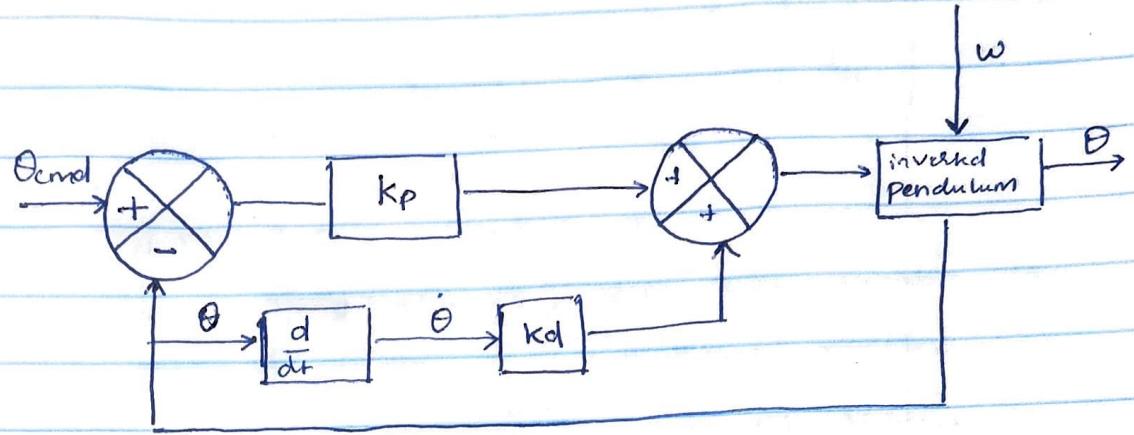
steady-state value of theta is

1.0472



3(c)

Feed - Back PD controller :-



Feed Back PID controller .

