ROB 498/599 Fall 2024: Assignment #6

Due on Dec 6th, at 11:59 pm, on Gradescope.

Submission Instructions: Submit a PDF file with written answers to all questions, images of plots (included on the PDF). Paste your code to the end of the document. Label the document **[uniquename]_ROB498_Assignment6.pdf** and upload to Gradescope. **No late submissions will be accepted.**

Problem 1 Consider the series elastic actuator (SEA) system we discussed in Assignment 4. Let the state variables be

$$x_1 = \theta_l, \quad x_2 = \dot{\theta}_l, \quad x_3 = \theta_m, \quad x_4 = \dot{\theta}_m \ ,$$

and the state vector be $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, then the state-space representation of this system is

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \ ,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_l} & -\frac{B_l}{J_l} & \frac{k}{J_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m} \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_m} \end{bmatrix} .$$

(a) Assume we use a constant-gain, full-state feedback with the form

$$u = -\mathbf{K}\mathbf{x} \ ,$$

where $\mathbf{K} \in \mathbb{R}^4$. Write out the state-space representation of the closed-loop system.

(b) Use the same system parameters we used in Assignment 4, i.e., $J_m = J_l = 0.0097\,\mathrm{kg} \cdot \mathrm{m}^2, b_m = b_l = 0.04169\,\mathrm{Ns} \cdot \mathrm{m}^{-1}, k = 100\,\mathrm{Nm} \cdot \mathrm{rad}^{-1}$. Let $Q = \mathrm{diag}([1, 0.1, 1, 0.1])$ be a 4×4 diagonal matrix, and $R = 1$. Find the constant feedback gain matrix K that minimizes the cost function

$$J = \int_0^\infty \left[\mathbf{x}^T(t) Q \mathbf{x}(t) + R u^2(t) \right] dt \ . \tag{1}$$

(c) Use the same system parameters. Let $Q = \mathrm{diag}([1, 0.1, 1, 0.1])$ be a 4×4 diagonal matrix, and $R = 0.1$. Find the constant feedback gain K that minimizes the cost function (1).

(d) Use the same system parameters. Let $Q = \mathrm{diag}([5, 0.1, 5, 0.1])$ be a 4×4 diagonal matrix, and $R = 0.1$. Find the constant feedback gain K that minimizes the cost function (1).

(e) Let the initial condition be $\mathbf{x}_0 = \begin{bmatrix} \pi/2 & 0 & \pi/2 & 0 \end{bmatrix}^T$. Simulate the closed-loop system response with the feedback gains you obtained in (b), (c), and (d) for 5 seconds. Plot the state variables $\mathbf{x}$, and control input u. Compare the results, what are the differences and why?

Problem 2 Consider a mass-spring-damper system described by the equation of motion

$$m\ddot{x} + b\dot{x} + kx = u \ ,$$

where m, b, k are the mass, damping coefficient, and spring stiffness. x represents the displacement from the origin, and u represents the external force. This can be written in the state-space form

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \ ,$$

where $\mathbf{x}$ is the state vector. In this problem, use $m = 1$ kg, $b = 2$ Ns/m, $k = 1$ N/m.

(a) Suppose we have full-state measurement, and the external force u is determined by

$$u = -K\mathbf{x} \ , \tag{2}$$

where $K = \begin{bmatrix} 5 & 1 \end{bmatrix}$. Suppose the initial condition is $x_0 = 1$, $\dot{x}_0 = 0$, simulate the closed-loop system response. Plot x and $\dot{x}$.

(b) The feedback controller used in (a) requires full-state feedback. However, in practice, full-state measurements may not be available. Suppose the only measurement you have is position, i.e.,

$$y = C\mathbf{x} \ ,$$

where $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Construct a state observer with $L = \begin{bmatrix} 5 & -1 \end{bmatrix}^T$. Repeat simulation in (a), but replace $\mathbf{x}$ in (2) with $\hat{\mathbf{x}}$, the estimated state from your state observer. Additionally, since you need to "guess" the initial condition, let $\hat{x}_0 = 1.2$ and $\dot{\hat{x}}_0 = 0.2$. Plot the results of your simulation.

(c) Construct another state observer, but this time let L be the gain computed using `lqe` function in MATLAB. Let $G = I$ be a 2×2 identity matrix, Q be a 2×2 diagonal matrix with diagonal entries $q_1 = q_2 = 10^{-4}$, and $R = 0.1$ (If you are wondering the reason we choose these values, check the next part). Simulate the system response with initial condition $\hat{x}_0 = 1.2, \dot{\hat{x}}_0 = 0.2$. Plot the result and compare with results of (b). Can you explain the difference?

Hint: Compare the eigenvalues of $(A - LC)$.

(d) From (b) and (c), it seems that using L obtained from `lqe` doesn't give the best results. However, in practice, measurement always comes with noise. Sensors always have additive white Gaussian noise caused by electron motion, with Gaussian distributed amplitude.

Assume we have a precise model of the system dynamics (this implies small process noise), but the position measurement has additive white Gaussian noise, with mean $\mu = 0$ and standard deviation $\sigma = 0.1$. The position measurement can be expressed as

$$\tilde{y} = y + z \ ,$$

where y is the ground-truth position, and $z \sim \mathcal{N}(\mu, \sigma^2)$ is the noise. Repeat (b) and (c) and compare the results. What are the differences and how can you explain it? You don't need to submit any plots.

Hint: You can generate a random number from a standard normal distribution $z \sim \mathcal{N}(0, 1)$ using `randn` function in MATLAB.