RoB-599/003 Assignment-2 28/03/2024.

(1)a. $(J+ml^2)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin\theta = l\cos\theta\, F$

As it is a 2nd order diff eq^n around θ there are two first order ODE's.

$\theta = x_1 \qquad \dot{\theta} = x_2 \quad , \quad F = u$

$\dot{\theta} = \dot{x}_1 = x_2$

$\ddot{\theta} = \dot{x}_2$

$$(J+ml^2)\dot{x}_2 + \gamma x_2 - mgl\sin(x_1) = l\cos(x_1)u$$

$$\dot{x}_2 = \frac{l\cos(x_1)u + mgl\sin(x_1) - \gamma x_2}{(J+ml^2)}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{l\cos(x_1)u + mgl\sin(x_1) - \gamma x_2}{(J+ml^2)}\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{\dot{x}} = \overbrace{\begin{bmatrix} x_2 \\ \dfrac{l\cos(x_1)u}{J_t} + \dfrac{mgl}{J_t}\sin(x_1) - \dfrac{\gamma}{J_t}x_2\end{bmatrix}}^{f(x,u)}$$

taking $J_t = J + ml^2$

$$A = \frac{\partial F}{\partial x}\bigg|_{x_1 = x_2 = 0} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\[2ex] \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x_1 = x_2 = 0}$$

$$A = \begin{bmatrix} 0 & 1 \\[1ex] \dfrac{-\overset{\to 0}{\sin(x_1)}\, l u + \overset{\to 1}{\cos(x_1)}\, m g l}{J_t} & \dfrac{-\gamma}{J_t} \end{bmatrix}\Bigg|_{x_1 = x_2 = 0}$$

$$A = \begin{bmatrix} 0 & 1 \\[1ex] \dfrac{mgl}{J_t} & \dfrac{-\gamma}{J_t} \end{bmatrix}$$

$$B = \frac{\partial F}{\partial u}\bigg|_{u=0} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\[2ex] \dfrac{\partial f_2}{\partial u} \end{bmatrix}\Bigg|_{u=0}$$

$$B = \begin{bmatrix} 0 \\[1ex] \dfrac{+\cos(x_1)\, l}{J_t} \end{bmatrix}_{\substack{u=0 \\ x_1=0 \\ x_2=0}} = \begin{bmatrix} 0 \\[1ex] \dfrac{l}{J_t} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}_s} = \underbrace{\begin{bmatrix} 0 & 1 \\ mgl/J_t & -\gamma/J_t \end{bmatrix}}_{A_s} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x_s} + \underbrace{\begin{bmatrix} 0 \\ l/J_t \end{bmatrix}}_{B_s} u_s$$

$$A_s = \begin{bmatrix} 0 & 1 \\ mgl/J_t & -\gamma/J_t \end{bmatrix}$$

$m = 0.2\,kg$, $l = 0.3\,m$, $\gamma = 0.1\,N/m\ sec/rad$ $g = 9.81\,m/sec^2$

$J = 0.006\ kgm^2$

$J_t = J + ml^2 = 0.006 + (0.2)(0.3)^2 = 0.024$

$$A_s = \begin{bmatrix} 0 & 1 \\ \frac{(0.2)(9.81)(0.3)}{0.024} & -\frac{(0.1)}{0.024} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 24.525 & -4.16 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 0 \\ 12.5 \end{bmatrix}$$

$$|A_s - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & 1 \\ 24.525 & -4.16-\lambda \end{vmatrix} = 0$$

$$4.16\lambda + \lambda^2 - 24.525 = 0$$

$\lambda_1 = -7.45135$; $\lambda_2 = 3.29135$

The system is unstable at the eq^n point as one of the λ values are +ve, it will cause the system to deviate over time.

1(b) $u(t) = k_P(\theta_{cmd} - \theta) - k_D \dot{\theta}$

$$\Rightarrow (J + ml^2)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin\theta = l\cos(\theta)\cdot\left[k_P(\theta_{cmd} - \theta) - k_D(\dot{\theta})\right]$$

$$(J + ml^2)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin\theta = l\cos\theta\,(k_P\theta_{cmd}) - l\cos\theta\cdot(\theta)k_P - k_D(l\cos\theta)(\dot{\theta})$$

$$(J + ml^2)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin\theta + l\cos\theta(\theta)k_P + k_D\, l\cos\theta\cdot\dot{\theta} = l\cos\theta\,(k_P\theta_{cmd})$$

$\theta = x_1$

$\dot{\theta} = \dot{x}_1 = x_2$

$\ddot{\theta} = \dot{x}_2$

$$J_t\dot{x}_2 + \gamma x_2 - mgl\sin(x_1) + l\cos(x_1)\cdot x_1 k_P + k_D\, l\cos(x_1)\cdot x_2 = l\cos(x_1)\,k_P\theta_{cmd}.$$

$\cancel{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

$(J + ml^2 = J_t)$

$\cancel{J_t\ddot{\theta}}$ J_t

$$\dot{x}_2 = \frac{l\cos(x_1)\,k_P\theta_{cmd} - \gamma x_2 + mgl\sin(x_1) - l\cos(x_1)\cdot x_1 k_P - k_D\, l\cos(x_1)\cdot x_2}{J_t}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{l\cos(x_1) k_p \theta_{cmd} - \gamma x_2 + mgl\sin(x_1) - l\cos(x_1) x_1 k_p - k_D l \cos(x_1)\cdot x_2}{J_t} \end{bmatrix}$$

$$A = \begin{bmatrix} \dfrac{\partial(f_1(x,u))}{\partial x_1} & \dfrac{\partial(f_1(x,u))}{\partial x_2} \\ \dfrac{\partial(f_2(x,u))}{\partial x_1} & \dfrac{\partial(f_2(x,u))}{\partial x_2} \end{bmatrix}\Bigg|_{x_1 = x_2 = 0}$$

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{mgl}{J_t} - \dfrac{lk_p}{J_t} & -\dfrac{\gamma}{J_t} - \dfrac{k_D l}{J_t} \end{bmatrix}$$

$$B = \frac{\partial(f)}{\partial \theta_{cmd}}\Bigg|_{x_1 = x_2 = 0} = \begin{bmatrix} 0 \\ \dfrac{l\cos(x_1) k_p}{J_t} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{l k_p}{J_t} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{mgl}{J_t} - \dfrac{lk_p}{J_t} & -\dfrac{\gamma}{J_t} - \dfrac{k_D l}{J_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{l k_p}{J_t} \end{bmatrix} \theta_{cmd}$$

Sub^n the values of $m = 0.2kg$, $J = 0.006\ kg\,m^2$, $l = 0.3m$, $c = 0.1\ N\ sec/m$, $\gamma = 0.1\ N\ m\ sec/rad$, $g = 9.81\ m/sec^2$

$$= \begin{bmatrix} 0 & 1 \\ \left(\frac{(0.2)(9.81)(0.3)}{0.024} - \frac{(0.3)kp}{0.024}\right) & \left(-\frac{0.1}{0.024} - \frac{kD(0.3)}{0.024}\right) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ (24.525 - 12.5kp) & (-4.16 - 12.5kp) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{0.3\,kp}{0.024} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 24.525 - 12.5kp & (-4.16 - 12.5kp) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 12.5kp \end{bmatrix} \theta_{cmd}$$

1(c)

$$A = \begin{bmatrix} 0 & 1 \\ 24.525 - 12.5k_P & (-4.16 - 12.5k_D) \end{bmatrix}$$

$$|SI - A| = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 24.525 - 12.5k_P & -4.16 - 12.5k_D \end{bmatrix}$$

$$= \begin{bmatrix} S & -1 \\ -24.525 + 12.5k_P & S + (4.16 + 12.5k_D) \end{bmatrix}$$

$$s^2 + S(4.16 + 12.5k_D) + (-24.525 + 12.5k_P)$$

$a_0 = 1$; $a_1 = (4.16 + 12.5k_D)$; $a_2 = (-24.525 + 12.5k_P)$

Using the Routh-Hurwitz Matrix to determine the value of K_P & K_D

$$H = \begin{bmatrix} a_1 & 0 \\ a_0 & a_2 \end{bmatrix} = \begin{bmatrix} 4.16 + 12.5k_D & 0 \\ 1 & -24.525 + 12.5k_P \end{bmatrix}$$

$\Delta_1 > 0$

$\Rightarrow 4.16 + 12.5k_D > 0$

$\Rightarrow k_D > \frac{-4.16}{12.5} \Rightarrow \boxed{k_D > -0.3328.}$

$\Delta_2 = a_1 a_2 > 0 \quad \Delta_2 = (4.16 + 12.5 k_D)(-24.525 + 12.5 k_P) > 0$

$\Rightarrow -24.525 + 12.5 k_P > 0$

$\Rightarrow \boxed{k_P > 1.962}$

$\therefore$ The ranges of k_P & k_D for the system to be stable is

$\boxed{k_D > -0.3328}$ & $\boxed{k_P > 1.962}$

1(d) In order to find the values of k_P & k_D using percentage overshoot & settling time lets consider the characteristic eq^n we get to calculate the Roth-hurwitz matrix.

$$s^2 + s(4.16 + 12.5 k_D) + (-24.525 + 12.5 k_P) \quad --- (1)$$

lets compare it to the std. characteristic polynomial of a 2nd order diff equation.

$$s^2 + 2\xi\omega_n s + \omega_n^2 \quad --- (2)$$

comparing (1) + (2) we get

$$2\xi\omega_n = 4.16 + 12.5 k_D \quad --- (3)$$

$$\omega_n^2 = -24.525 + 12.5 k_P \quad (4)$$

W.K.T

$T_s = 5\,sec$, $PO = 12\%$

$$PO = e^{(-\xi\pi/\sqrt{1-\xi^2})} \Rightarrow \xi = \frac{-a}{\sqrt{\pi^2 + a^2}} \quad , a = \log_e\left(\frac{PO}{100}\right)$$

$$T_s = \frac{4}{\xi w_n} \Rightarrow w_n = \frac{4}{\xi T_s}$$

$$a = \log_e\left(\frac{12}{100}\right) = -2.120$$

$$\xi = \frac{-(-2.120)}{\sqrt{(3.14)^2 + (-2.120)^2}} = 0.5595$$

$$w_n = \frac{4}{(0.5595)(5)} = 1.4298$$

$$(1.4298)^2 = 12.5\,k_P - 24.525$$

$$\boxed{k_P = 2.1255}$$

$$2\xi w_n = 4.16 + 12.5(k_D)$$

$$2(0.5595)(1.4298) = 4.16 + 12.5\,k_D$$

$$-2.56 = 12.5\,k_D$$

$$\boxed{k_D = -0.2048}$$

$\therefore$ for a percent overshoot of 12% & a settling time of 5 sec for a 2% band width $k_P = 2.1255$, $k_D = -0.2048$.

Problem (2)

(a). The linerized model in problem I in the form $\dot{x}_s = A_s x_s + B_s u_s$ is as follows:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ mgl/J_t & -\gamma/J_t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ l/J_t \end{bmatrix} u_s$$

$$\dot{x}_1 = x_1 \quad \text{---} \ (a)$$

$$\dot{x}_2 = \frac{mgl}{J_t} x_1 - \frac{\gamma}{J_t} x_2 + \frac{l}{J_t} u_s \quad \text{---} \ (b)$$

$$u(t) = k_P(\theta_{cmd} - \theta) + k_I \int_0^t (\theta_{cmd} - \theta(t)) \cdot dt - k_D \dot{\theta} \quad \text{---} \ (1)$$

$$\dot{x}_I = (\theta_{cmd} - \theta(t)) \quad \text{---} \ (2)$$

$\theta(t) = \theta_0 = x_1$; $\dot{\theta} = \dot{x}_1 = x_2$; $x_3 = x_I$

$\dot{\theta} = x_2$; $\ddot{\theta} = \dot{x}_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

by sub^n (2) in (1) we get.

$$u(t) = k_P(\theta_{cmd} - \theta) + k_I \int_0^t (\dot{x}_I) \cdot dt - k_D \dot{\theta}$$

$$u(t) = k_P(\theta_{cmd} - \theta) + k_I x_I - k_D \dot{\theta} \quad \text{---} \ (3)$$

(WKT $\dot{\theta} = x_2 = \dot{x}_1$; $\theta_0 = x_1$)

Sub these values in (3) we get

$$u(t) = k_P(\theta_{cmd} - x_1) + k_I x_I - k_D x_2 \quad \text{---} \ (4)$$

Sub^n (4) in (b) we get the following expression for $\dot{x}_2$

u(t) replaces u_s

$$\dot{x}_2 = \frac{mgl}{J_t}(x_1) - \frac{\gamma}{J_t}(x_2) + \frac{l}{J_t}\left(k_p(\theta_{cmd} - \cancel{x_1}) + k_I x_I - k_D \cancel{x_2}\right)$$

$$\dot{x}_2 = \left[\frac{mgl}{J_t} - \frac{l}{J_t}k_p\right]x_1 + \left[-\frac{\gamma}{J_t} - \frac{l\,k_D}{J_t}\right]x_2 + \frac{l}{J_t}(k_p\theta_{cmd}) + \frac{l}{J_t}x_I k_I$$

$(x_3 = x_I)$

$$\dot{x}_1 = x_2$$

$$\Rightarrow \dot{x}_2 = \left[\frac{mgl}{J_t} - \frac{l}{J_t}k_p\right]x_1 + \left[-\frac{\gamma}{J_t} - \frac{l\,k_D}{J_t}\right]x_2 + \frac{l}{J_t}(k_p\theta_{cmd}) + \frac{l k_I x_3}{J_t}$$

$$\dot{x}_3 = \theta_{cmd} - x_1$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{mgl}{J_t} - \frac{l k_P}{J_t} & -\frac{\gamma}{J_t} - \frac{l k_D}{J_t} & 0 + \frac{l k_I}{J_t} \\ -1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial F_1}{\partial \theta_{cmd}} \\ \frac{\partial F_2}{\partial \theta_{cmd}} \\ \frac{\partial F_3}{\partial \theta_{cmd}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{l}{J_t} k_P \\ 1 \end{bmatrix}$$

writing it in the following form

$$\dot{x}_e = A_{e,cl} X_e + B_{e,cl}\, \theta_{cmd}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ mgl/J_t - \frac{l k_P}{J_t} & -\frac{\gamma}{J_t} - \frac{l k_D}{J_t} & \frac{l k_I}{J_t} \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{l k_P}{J_t} \\ 1 \end{bmatrix} \theta_{cmd}$$

w.k.t $m = 0.2\ kg$, $J = 0.006\ kg\,m^2$, $l = 0.3m$,

$c = 0.1\ N\ sec/m$, $\gamma = 0.1\ Nm\ sec/rad$, $g = 9.81\ m/sec^2$

$J_t = ml^2 + J = (0.2)(0.3)^2 + 0.006 = 0.024$

Sub the above values in eq ⑤ we get.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 24.525-12.5k_P & \left(-4.16 - 12.5k_D\right) & 12.5k_I \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 12.5k_P \\ 1 \end{bmatrix} \theta_{cmd}$$

(b) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 24.525-12.5k_P & -4.16-12.5k_D & 12.5k_I \\ -1 & 0 & 0 \end{bmatrix} \cancel{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 12.5k_P \\ 1 \end{bmatrix} \theta_{cmd}}$$

$$|sI - A| = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 24.525-12.5k_P & -4.16-12.5k_D & 12.5k_I \\ -1 & 0 & 0 \end{bmatrix}$$

$$|sI - A| = \begin{bmatrix} s & 1 & 0 \\ -24.525+12.5k_P & s+4.16+12.5k_D & -12.5k_I \\ +1 & 0 & s \end{bmatrix}$$

The characteristic polynomial of this matrix is

$$+s\left[s\left(s+(4.16+12.5k_D)\right)\right]+1\left(s(-24.525+12.5k_p)+12.5k_I\right)$$

$$=s^2\left(s+(4.16+12.5k_D)\right)+s(-24.525+12.5k_p)+12.5k_I$$

$$=s^3+(4.16+12.5k_D)s^2+s(-24.525+12.5k_p)+12.5k_I.$$

$a_0=1$; $a_1=4.16+12.5k_D$ $\quad a_2=-24.525+12.5k_p$

$a_3=12.5k_I.$, $a_5=0$, $a_4=0$

$$H=\begin{bmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix} = \begin{bmatrix} \cancel{4.16+125} & \cancel{12.5k_I} & \cancel{0} \\ & & \\ & & \end{bmatrix}$$

$$H=\begin{bmatrix} 4.16+12.5k_D & 12.5k_I & 0 \\ 1 & -24.525+12.5k_p & 0 \\ 0 & 4.16+12.5k_D & 12.5k_I \end{bmatrix}$$

$\Delta_1 > 0$

$\Rightarrow 4.16 + 12.5k_D > 0$

$\Rightarrow \boxed{k_D > -0.3328}$

$\Delta_2 > 0$

$\Rightarrow (4.16 + 12.5k_D)(-24.525 + 12.5k_P) - 12.5k_I > 0$

$(4.16 + 12.5k_D)(-24.525 + 12.5k_P) > 12.5k_I$

$\Rightarrow \dfrac{(4.16 + 12.5k_D)(-24.525 + 12.5k_P)}{12.5} > k_I$

$\Rightarrow \cancel{(0.3328 + k_D)}$

$\Rightarrow \dfrac{4.16(-24.525) + 4.16(12.5k_P) + 12.5k_D[-24.525 + 12.5k_P]}{12.5}$

$\Rightarrow \dfrac{-102.024}{12.5} + 4.16k_P + (-24.525k_D) + 12.5k_Pk_D > k_I$

$\Rightarrow 4.16k_P - 24.525k_D - 8.16192 + 12.5k_Pk_D > k_I$

$\Rightarrow k_I < 4.16k_P - 24.525k_D + 12.5k_Pk_D - 8.16192$

~~$k_P > 1.962$~~

$\Delta_3 > 0$

$\Rightarrow \Delta_3 = a_3 \Delta_2$ w.k.t $\Delta_2 > 0$

$\Rightarrow a_3 > 0$

w.k.t $a_3 = 12.5 k_I$.

$\Rightarrow k_I > 0$.

if $k_I > 0$ & $\underbrace{(4.16 + 12.5 k_D)(-24.525 + 12.5 k_p)}_{>0} > k_I$

$\Rightarrow (-24.525 + 12.5 k_p) > 0$

$\Rightarrow k_p > 1.962$

$\therefore$ The values of k_p, k_D & k_I for the system to be stable are as follows.

$k_p > 1.962$

$k_D > -0.3328$

$0 < k_I < (4.16 k_p - 24.525 k_D + 12.5 k_p k_D - 8.16192)$

3(a) The linearized model is given as follows:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} & -\frac{\gamma}{J_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ l/J_t \end{bmatrix} u_s$$

now if a PD controller is introduced.

$u = kp(\theta_{cmd} - \theta) \cancel{+} kD(\dot{\theta} - w)$ { w is the disturbance in the angular velocity }

w => is a +ve disturbance

with $\theta = x_1$

$\dot{\theta} = \dot{x}_1 = x_2$

So when the PD controller is introduced the linear system takes the following form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} - \frac{lkp}{J_t} & -\frac{\gamma}{J_t} + \frac{lkD}{J_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{l}{J_t} kp & -\frac{lkD}{J_t} \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

By sub the values of given in Q.1 we get

$$A = \begin{bmatrix} 0 & 1 \\ 24.525 - 12.5kp & -4.16 - 12.5\,kD \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 12.5\,kp & +12.5kD \end{bmatrix}$$

$$k_P = 5 \quad ; \quad k_D = 0.1$$

$$A = \begin{bmatrix} 0 & 1 \\ -37.925 & -5.41 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 62.5 & 1.25 \end{bmatrix}$$

$\cancel{A^{-1}}$ W.K.T $\} \; \dot{X}_{ss} = A X_{ss} + B w$

$$X_{ss} = -A^{-1} B w$$

$$\Rightarrow w = \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$\Rightarrow X_{ss} = -A^{-1} B \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = -A^{-1} B \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix} \quad \text{--- (1)}$$

$$A^{-1} = \begin{bmatrix} -0.1425 & -0.0263 \\ 1 & 0 \end{bmatrix}$$

By sub these values in 1 we get

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = -\begin{bmatrix} -0.1425 & -0.0263 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & +1.25 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\cancel{= -\begin{bmatrix} 0 - (0.0263)(62.5) \\ \end{bmatrix}}$$

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = -\begin{bmatrix} -0.0263 \times 62.5 & \cancel{+}-1.25 \times 0.0263 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = \begin{bmatrix} 1.64375\,\theta_{cmd} + 0.032875\,(\omega) \\ 0 \end{bmatrix}$$

By sub^n the values of $\theta_{cmd} = \frac{\pi}{3}$ & $\omega = 3$ Ra

we get

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix} = \begin{bmatrix} 1.819 \;\; \cancel{1.62183} \\ 0 \end{bmatrix}$$

as $t \to \infty$ $\theta_{ss} \to$ ~~1.62183~~ 1.819

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Steady-state value of the state variables are:
    1.8222
         0

steady-state value of theta is :
    1.8222
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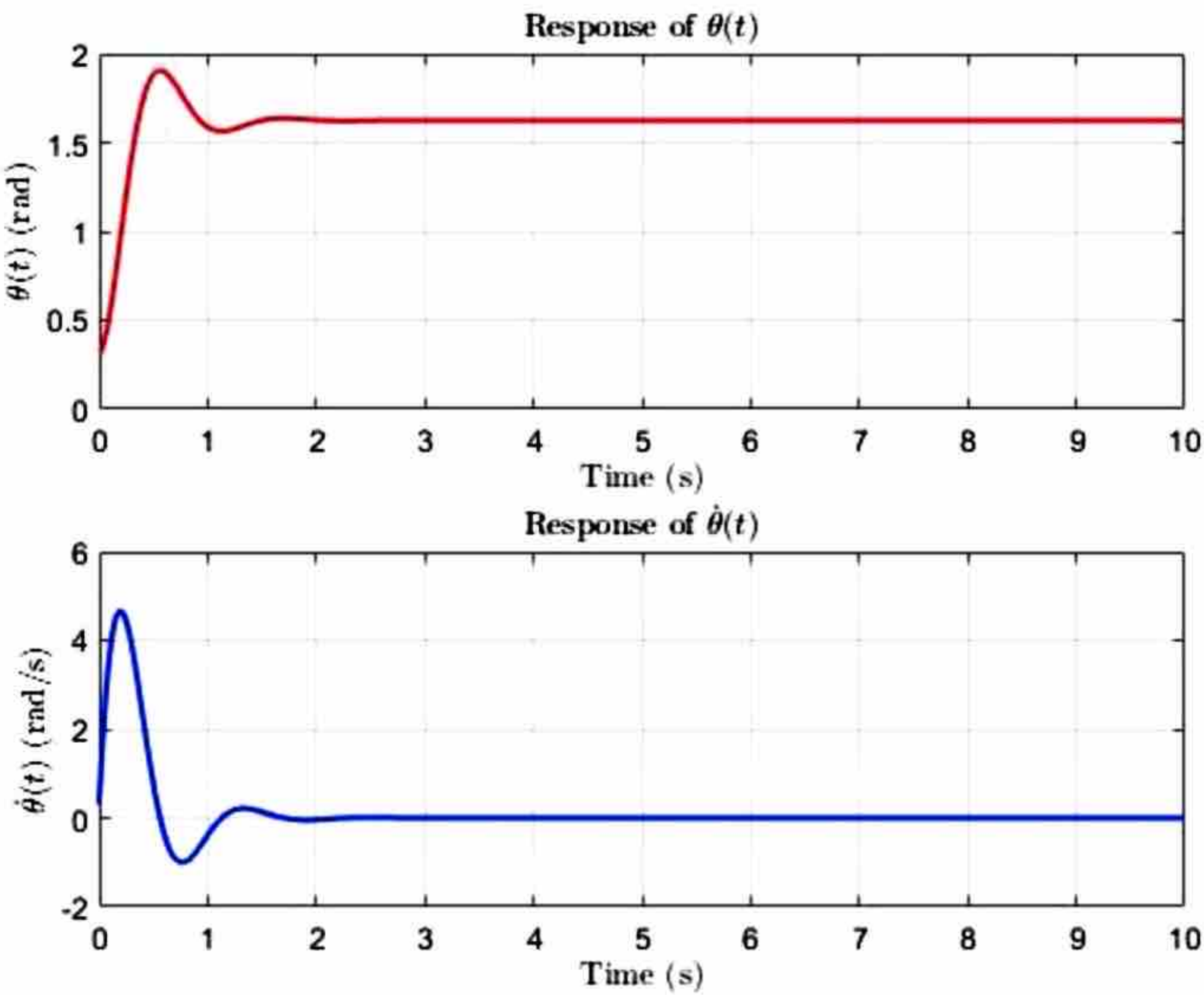


3a (continued)

if $w \rightarrow +ve$ {i.e disturbance is +ve}, then

$$u = k_P(\theta_{cmd} - \theta) - k_D(\dot{\theta} + w)$$

$$A = \begin{bmatrix} 0 & 1 \\ (24.525 - 12.5k_P) & (-4.16 - 12.5k_D) \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 12.5k_P & -12.5k_D \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -37.975 & -5.41 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \end{bmatrix}$$

$$x_{ss} = -A^{-1}B \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$x_{ss} = -\begin{bmatrix} -0.1425 & -0.0263 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$x_{ss} = \begin{bmatrix} 0.1425 & 0.0263 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$x_{ss} = \begin{bmatrix} 0.0263 \times 62.5 & -1.25 \times 0.0263 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

$$X_{ss} = \begin{bmatrix} 0.0263 \times 62.5\ \theta_{cmd} \\ -1.25 \times 0.0263 w \\ 0 \end{bmatrix}$$

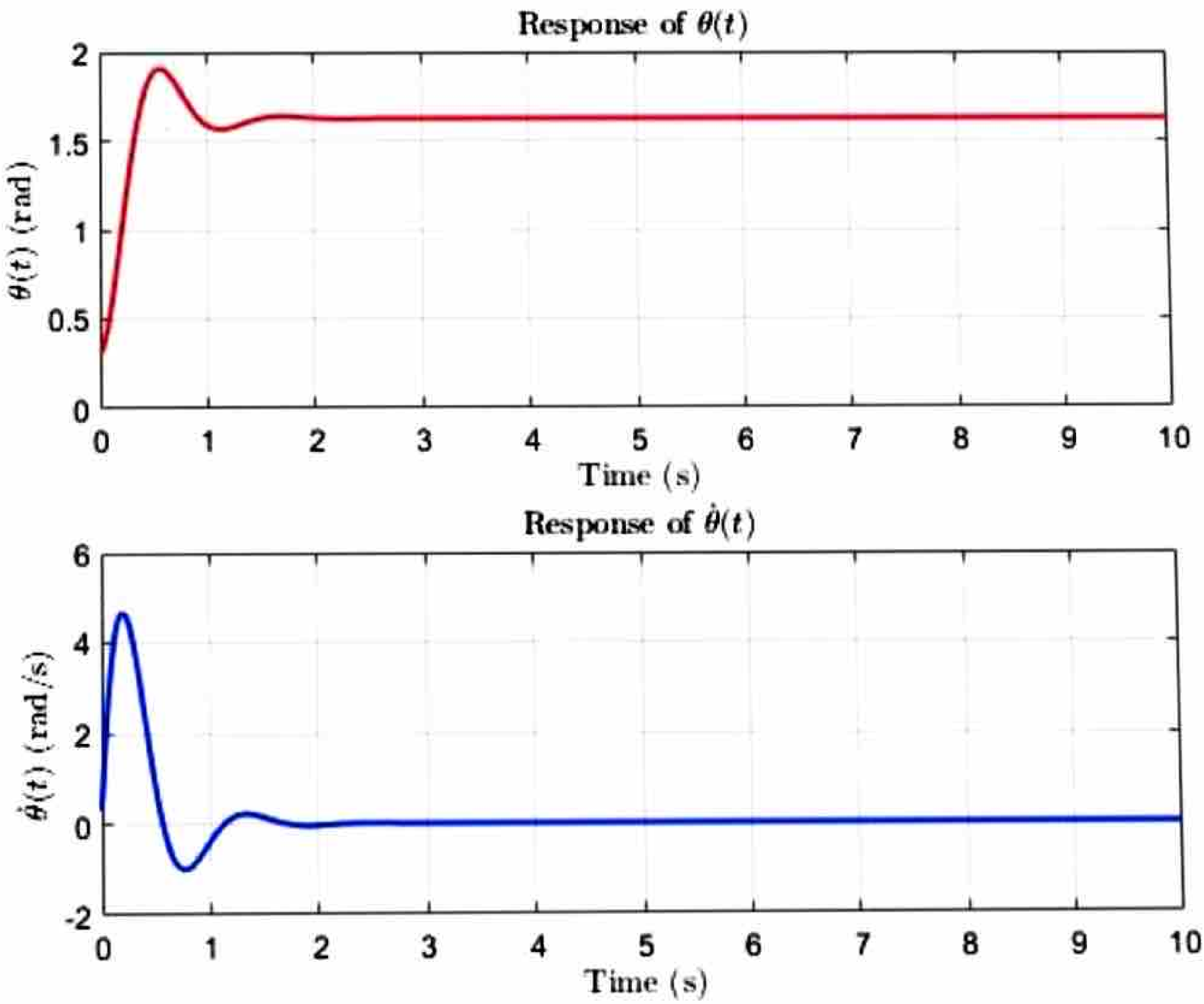
$$X_{ss} = \begin{bmatrix} 1.6247 \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \end{bmatrix}$$

$$\Rightarrow \quad \theta_{ss} \to 1.6247 \quad \text{as } t \to \infty.$$

$$\dot{\theta}_{ss} \to 0$$

```
Steady-state value of the state variables are:
    1.6247
         0

steady-state value of theta is :
    1.6247
```



3(b) The linearized model is given as follows:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} & \frac{-\gamma}{J_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ l/J_t \end{bmatrix} u_s$$

when a PID controller & a disturbance w is introduced to this system

$$K_P(\theta_{cmd} - \theta) + K_I \int_0^t (\theta_{cmd} - \theta(t))\,dt - K_D(\dot{\theta} - w)$$

{Taking a +ve disturbance}

The linear model becomes:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \left(\frac{mgl}{J_t} - \frac{lk_P}{J_t}\right) & -\frac{\gamma}{J_t} + \frac{lk_D}{J_t} & \frac{lk_I}{J_t} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ \frac{lk_P}{J_t} & -\frac{lk_D}{J_t} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ w \end{bmatrix}$$

Subⁿ all the values

$k_P = 5$

$k_D = 0.1$

$K_I = 2.3562$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.41 & 29.4525 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 62.5 & +1.25 \\ 1 & 0 \end{bmatrix}$$

w.r.t

$$0 = A x_{ss} + B u_{ss}$$

$$\therefore \Rightarrow \quad x_{ss} = -A^{-1} B u_{ss}$$

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \\ x_I \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.41 & 23.45 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 62.5 & +1.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \\ x_I \end{bmatrix} = -\begin{bmatrix} 0 & 0 & -1.000 \\ 1.000 & 0 & 0 \\ 0.1837 & 0.0340 & -1.2894 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \\ x_I \end{bmatrix} = -\begin{bmatrix} -1.000 & 0 \\ 0 & 0 \\ (0.0340 \times 62.5 - 1.2894) \times 1 & (+1.25 \times 0.0340) \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \\ x_I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -0.8356 & -0.0425 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

by sub $\theta_{cmd} = \frac{\pi}{3}$ & $\omega = 3$ rad/sec

$$\begin{bmatrix} \theta_{ss} \\ \dot{\theta}_{ss} \\ x_I \end{bmatrix} = \begin{bmatrix} 1\,\theta_{cmd} \\ 0 \\ -0.8356\,\theta_{cmd} - 0.0425\,\omega \end{bmatrix} = \begin{bmatrix} 1.047 \\ 0 \\ \cancel{-0.8299} \\ \cancel{-0.747}\ -0.993 \end{bmatrix}$$

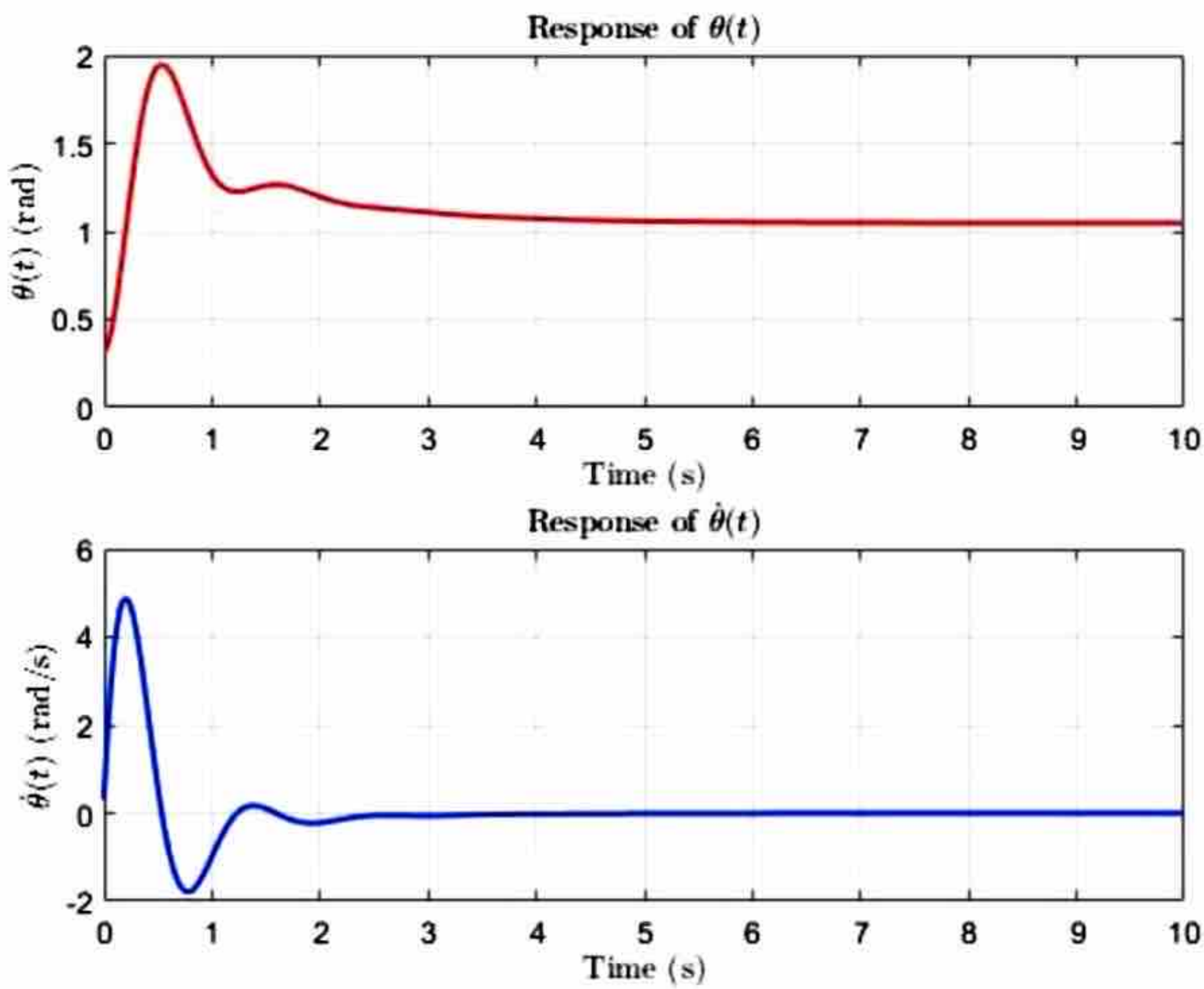
as $t \to \infty$

$\theta_{ss} \to 1.047$.

$x_I \to -0.993$

```
Steady-state value of the state variables are:
    1.0472
         0
   -0.9993

steady-state value of theta is
    1.0472
```



3(b) continued

if $\omega = +ve$ {i.e disturbance is +ve} then

$$u = k_p(\theta_{cmd} - \theta) + k_I \int_0^t \theta_{cmd} - \theta(t) \cdot dt - k_D(\dot{\theta} + \omega)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.41 & 29.4525 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \\ 1 & 0 \end{bmatrix}$$

$$x_{ss} = -A^{-1}B \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$x_{ss} = -\begin{bmatrix} 0 & 1 & 0 \\ -37.975 & -5.41 & 29.4525 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

$$x_{ss} = -\begin{bmatrix} 0 & 0 & -1.000 \\ 1.000 & 0 & 0 \\ 0.1837 & 0.0340 & -1.2894 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 62.5 & -1.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

On simplification we get

$$x_{ss} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -0.8356 & +0.0425 \end{bmatrix} \begin{bmatrix} \theta_{cmd} \\ \omega \end{bmatrix}$$

w.r.t $\theta_{cmd} = \frac{\pi}{3}$

& $\omega = 3$ rad/sec

$$x_{ss} = \begin{bmatrix} \theta_{cmd} \\ 0 \\ -0.8356\,\theta_{cmd} + 0.0425\,\omega \end{bmatrix}$$

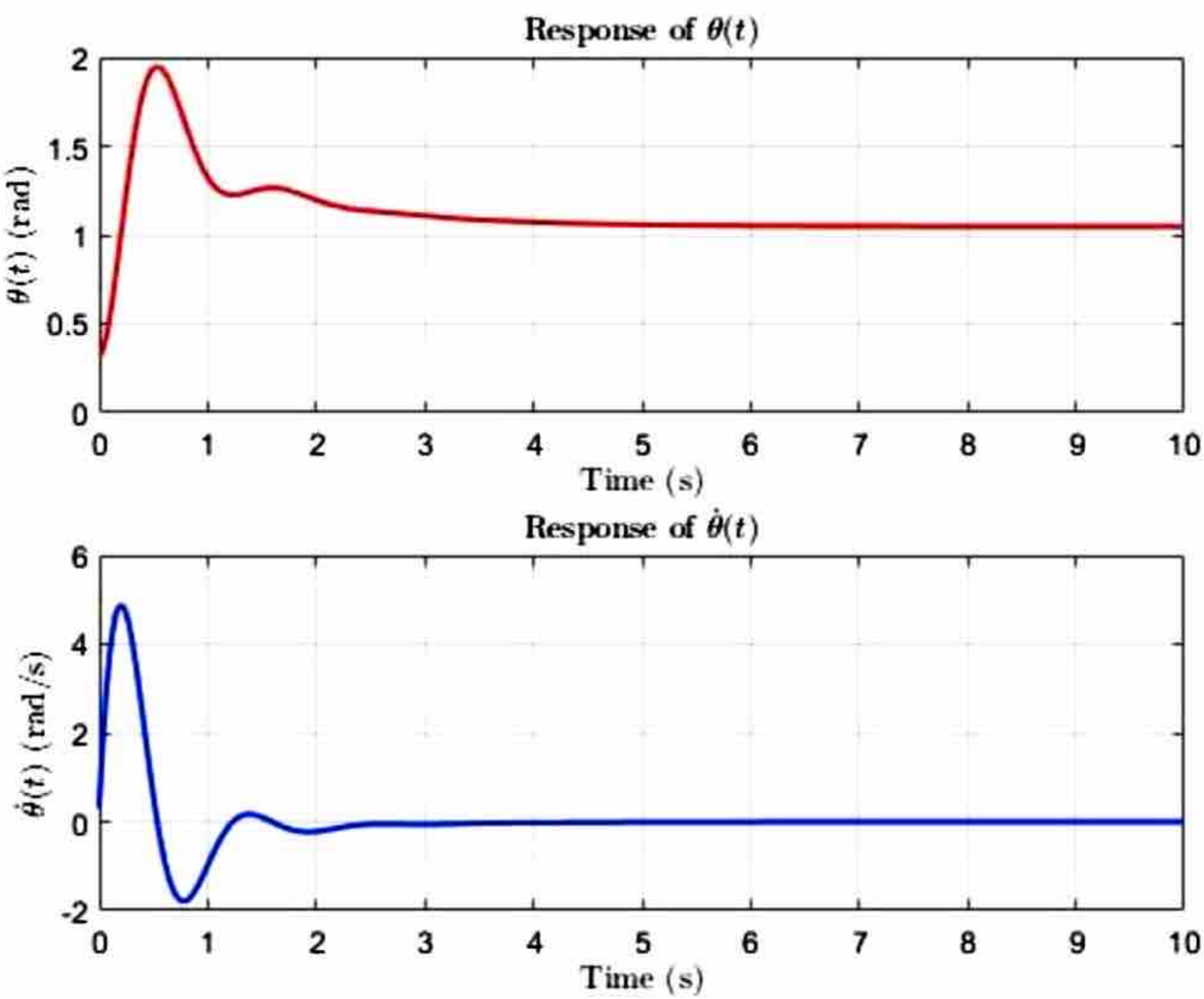
$$\Rightarrow x_{ss} = \begin{bmatrix} 1.0466 \\ 0 \\ -0.7447 \end{bmatrix}$$

$\theta_{ss} \longrightarrow \cancel{-0.7447}\ 1.0466$ as $t \to \infty$.

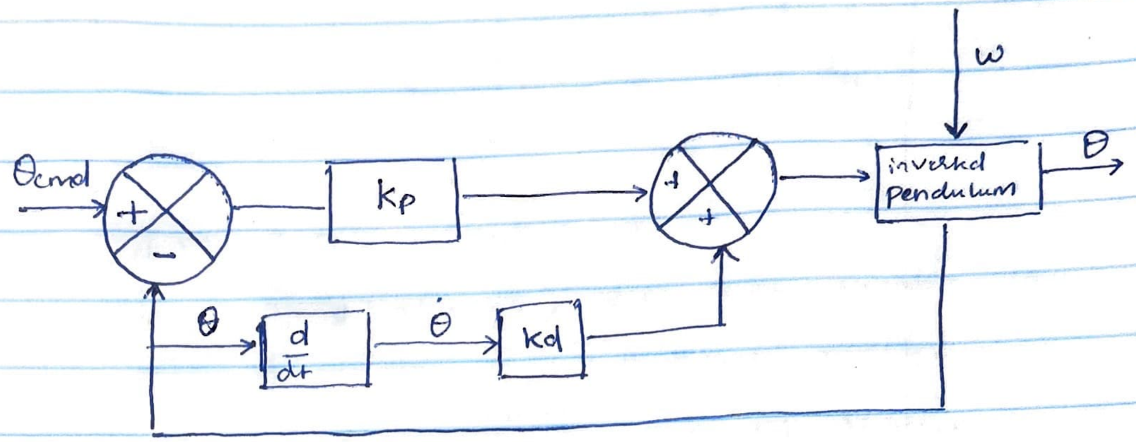
$x_3 \longrightarrow -0.7447$

```
Steady-state value of the state variables are:
    1.0472
         0
   -0.7447

steady-state value of theta is
    1.0472
```



3(c) Feed-Back PD controller:-



Feed Back PID controller.