Homework Assignment N°1

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Contents

1 Exercise 1: Geometry

1.1 Part a

We have:

$$w_0 + 3w_1 + w_2 = 0$$
$$w_0 - w_1 + 2w_2 = 0$$

Normalize by w_2 it gives

$$W_0 + 3W_1 = -1$$
$$W_0 - W_1 = -2$$

Where $W_0 = \frac{w_0}{w_2}$ and $W_1 = \frac{w_1}{w_2}$
Cramer's rule gives us the results:

$$W_0 = \frac{(-1 \times -1) - (3 \times -2)}{(1 \times -1) - (1 \times 3)} = -\frac{7}{4}$$

$$W_1 = \frac{(1 \times -2) - (-1 \times 1)}{(1 \times -1) - (1 \times 3)} = \frac{1}{4}$$

Thus, we can chose $w_2 = 1$ and then

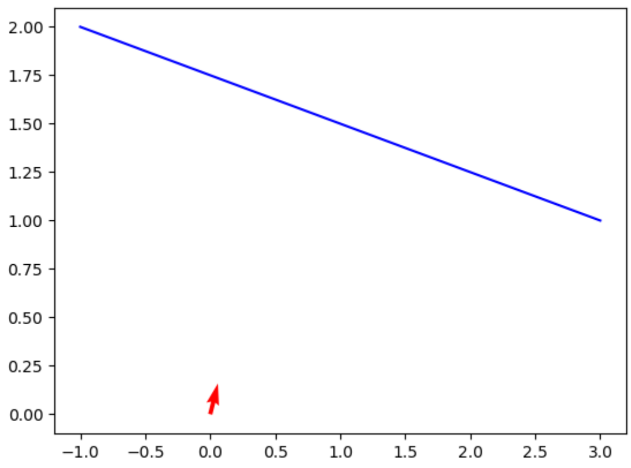
$$w_0 = -\frac{7}{4}$$
$$w_1 = \frac{1}{4}$$
$$w_2 = 1$$

We can check our first system is solved with this solution. Hence the equation of the line is the following:

$$-\frac{7}{4} + \frac{x_1}{4} + x_2 = 0$$

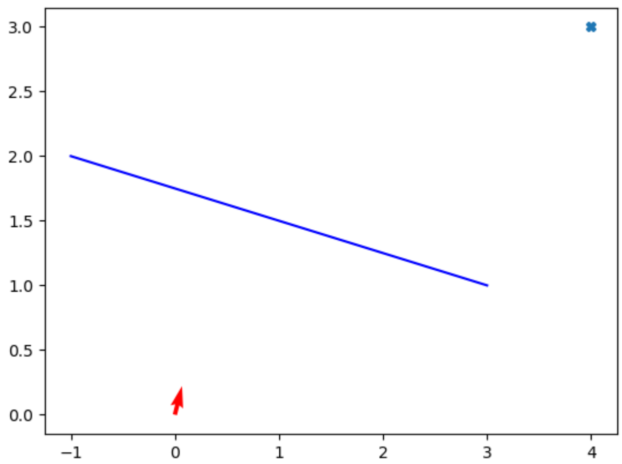
1.2 Part b

Our figure is the following:



1.3 Part c

The point $(4,3)$ is in the positive side of the line, the most simple «proof» is this plot:



It shows our point «above» the blue line, hence it is in the positive side of the line. For this point, because we saw it was on the positive side, $w_0+w_1x_1+w_2x_2$ should give a positive result.

$$-\frac{7}{4}+\frac{4}{4}+3=\frac{9}{4}>0$$

1.4 Part d

First we must notice that (w_1, w_2) is orthogonal to our line. To figure out the direction vector of the line, we can rewrite its equation like this

$$-w_1 x - w_0 = w_2 y$$

Hence its direction vector is $(w_2, -w_1)$ and now we can proove that (w_1, w_2) is orthogonal to the line by showing that

$$(w_1, w_2) \cdot (w_2, -w_1) = 0$$

Now we know the perpendicular direction to the line, we can compute the distance between the point and the line.

$$d_{line}(P) = |[(x_1, x_2) - (3, 1)] \cdot (w_1, w_2)| \, / \, \|(w_1, w_2)\|$$

$$d_{line}(P) = |x_1 w_1 + x_2 w_2 - 3w_1 - 1w_2| \, / \, \|(w_1, w_2)\|$$

Or because $w_0 + 3w_1 + 1w_2 = 0$ ($(3, 1)$ is on the line) we can add $w_0 + 3w_1 + 1w_2$ without changing the value. Hence:

$$d_{line}(P) = \frac{|w_0 + x_1 w_1 + x_2 w_2|}{\|(w_1, w_2)\|}$$

1.5 Part e

We use what we learned from part d:

$$d_{line}((0, 0)) = \frac{|w_0|}{\|(w_1, w_2)\|}$$

Hence we have

$$d_{line}((0, 0)) = \frac{|w_0|}{\sqrt{{w_1}^2 + {w_2}^2}} = \frac{\frac{7}{4}}{\sqrt{\frac{1}{4}^2 + 1}} \approx 1.7$$

2 Exercice 2: Calculus

$$\sigma(y) = \frac{1}{1 + e^{-y}}$$

2.1 Part a

Calculus says $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, then:

$$\sigma'(y) = \frac{e^{-y}}{(1 + e^{-y})^2}$$

$$\sigma'(y) = \frac{(1 + e^{-y}) - 1}{(1 + e^{-y})^2}$$

$$\sigma'(y) = \sigma(y) \times \frac{(1 + e^{-y}) - 1}{1 + e^{-y}}$$

$$\sigma'(y) = \sigma(y) \times (1 - \sigma(y))$$

2.2 Part b

We know that $\lim_{y \to \infty} e^{-y} = 0$ thus because the denominator does not tend to zero:

$$\lim_{y \to \infty} \sigma(y) = \frac{1}{1 + \lim_{y \to \infty} e^{-y}} = 1$$

and now we know the limit exists and its value:

$$\lim_{y \to \infty} \sigma'(y) = \lim_{y \to \infty} \sigma(y) \times (1 - \sigma(y)) = 1 \times (1 - 1) = 0$$

2.3 Part c

$$\frac{\partial \sigma}{\partial x_1} = \frac{\partial \sigma}{\partial y} \times \frac{\partial y}{\partial x_1}$$

$$\frac{\partial \sigma}{\partial x_1} = (\sigma \times (1 - \sigma)) \times w_1$$

Hence:

$$\frac{\partial \sigma}{\partial x_1}(y) = \sigma(y)(1 - \sigma(y) \times w_1$$

2.4 Part d

Same as in Part c:
For w_1

$$\frac{\partial \sigma}{\partial w_1} = \frac{\partial \sigma}{\partial y} \times \frac{\partial y}{\partial w_1}$$

$$\frac{\partial \sigma}{\partial w_1} = (\sigma \times (1 - \sigma)) \times x_1$$

Hence:

$$\frac{\partial \sigma}{\partial w_1}(y) = \sigma(y)(1 - \sigma(y) \times x_1$$

Same procedure for w_2 and w_0:
For w_2

$$\frac{\partial \sigma}{\partial w_2}(y) = \sigma(y)(1 - \sigma(y) \times x_2$$

For w_0

$$\frac{\partial \sigma}{\partial w_0}(y) = \sigma(y)(1 - \sigma(y) \times 1$$

Finally we can compute the gradient:

$$\nabla_w \sigma(y) = \begin{bmatrix} \frac{\partial \sigma}{\partial w_0} \\ \\ \frac{\partial \sigma}{\partial w_1} \\ \\ \frac{\partial \sigma}{\partial w_2} \end{bmatrix} (y) = \begin{bmatrix} \sigma(y)(1 - \sigma(y) \\ \\ \sigma(y)(1 - \sigma(y))x_1 \\ \\ \sigma(y)(1 - \sigma(y))x_2 \end{bmatrix} = \sigma(y)(1 - \sigma(y))x$$

Where $x = \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix}^\top$

2.5 Part e

When $x_1 > 0$, if $w_1 \gg 1$ (and assuming w_1 larger than w_0 and w_2) then $y = w_0 + x_1w_1 + x_2w_2 \gg 1$
Then what we showed in Part b applies:

$$\lim_{y\to\infty} \frac{\partial\sigma}{\partial w_1}(y) = \lim_{y\to\infty} \sigma(y)(1-\sigma(y))x_1 = 0 \times x_1 = 0$$

When $x_1 < 0$, under the same assumptions, we have $\lim_{w_1\to\infty} y = -\infty$. That's why we first need to find the value of $\lim_{y\to-\infty} \sigma'(y)$

$$\lim_{y\to-\infty} \sigma(y) = \lim_{y\to-\infty} \frac{1}{1+e^{-y}} = 0$$

$$\lim_{y\to-\infty} \sigma'(y) = \lim_{y\to-\infty} \sigma(y)(1-\sigma(y)) = 0$$

Then

$$\lim_{y\to-\infty} \frac{\partial\sigma}{\partial w_1}(y) = \lim_{y\to-\infty} \sigma'(y) \times x_1 = 0 \times x_1 = 0$$

3 Exercise 3

3.1 Part a

To compute the mean, we sum the data and divide by the cardinal of the set:

$$\bar{X} = \frac{\sum_{x_i in X} x_i}{|X|} = \begin{bmatrix} 0 & \frac{3}{2} \end{bmatrix}$$

3.2 Part b

$$cov(X) = E\left[(X_i - \bar{X}_i)(X_j - \bar{X}_j)\right] = E\left[X_iX_j\right] - \bar{X}_i\bar{X}_j$$

This formula applies if this data is exact data. If it was some sample data, we would need to correct the bias to correctly predict the whole dataset covariance (hopefully without bias). Because the statement does not specify whether this is a sample of a larger dataset or not, we will use no correction and apply the previous formula.
Hence:

$$cov(X) = \begin{bmatrix} 4.5 & 5.5 \\ 5.5 & 8.75 \end{bmatrix}$$

3.3 Part c

The formula for correlation is the following:

$$corr(X,Y) = \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}$$

$$corr(X_1,X_2) = \frac{cov(X_1,X_2)}{\sigma(X_1)\sigma(X_2)} \approx \frac{5.5}{2.12 \times 2.96} = 0.87$$

3.4 Part d

The eigenvalues of the covariance matrix are:

$$\lambda_0 \approx 0.7288 \quad , \quad u_0 \approx \begin{bmatrix} -0.8247 \\ 0.5655 \end{bmatrix}$$

$$\lambda_1 \approx 12.5212 \quad , \quad u_1 \approx \begin{bmatrix} -0.5655 \\ -0.8247 \end{bmatrix}$$

We can check they verify this identity:

$$A \cdot u_i = \lambda_i u_i$$

If we transformed the data points into coordinates with respect to the eigen vectors, then the covariance matrix should be diagonal. And the values inside the diagonal should be the eigenvalues.