**Indian Institute of Technology Kharagpur**
**Department of Computer Science and Engineering**

**Foundations of Computing Science (CS60005)** **Autumn Semester, 2022-2023**

**Class Test 1** **01-Sep-2022 (Thursday), 17:30–18:30** **Maximum Marks: 20**

**Instructions:**

- *Write your answers in the answer booklet provided to you in the examination hall.*
- *There are a total of THREE questions, having 6 marks, 7 marks and 7 marks, respectively.*
- *Answer ALL the questions (or as many as you can) mentioning the question numbers clearly.*
- *Be brief and precise. Write the answers for all parts of a question together.*
- *If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.*
- *Write all the proofs/deductions in mathematically/logically precise language.*
  *Unclear and/or dubious statements would be severely penalized.*

**Q1.** Let $\texttt{Cat}(x)$, $\texttt{Dog}(x)$, $\texttt{Striped}(x)$, $\texttt{Friends}(x, y)$, $\texttt{Equal}(x, y)$ and $\texttt{ShortTempered}(x)$ be predicates to be evaluated over the set of all animals. The predicates have the following obvious interpretations.

- $\texttt{Cat}(x)$ (or $\texttt{Dog}(x)$) evaluates to $\texttt{True}$ iff $x$ is a cat (or dog).
- $\texttt{Striped}(x)$ evaluates to $\texttt{True}$ iff the coat of $x$ is striped.
- $\texttt{Friends}(x, y)$ evaluates to $\texttt{True}$ iff $x$ and $y$ are friends.
  Obviously, $\texttt{Friends}(x, y)$ also implies $\texttt{Friends}(y, x)$.
- $\texttt{Equal}(x, y)$ evaluates to $\texttt{True}$ iff $x$ and $y$ are one and the same animal.
- $\texttt{ShortTempered}(x)$ evaluates to $\texttt{True}$ iff $x$ is short-tempered.

Express the following English language sentences as predicate logic sentences (formulae without free variables). You may assume that the domain for evaluating the truth of these sentences is always the set of all animals (which could include animals other than cats and dogs as well).

**(a)** There is a short-tempered dog who is not friendly with any other dog, but is friendly with at least one striped cat. **(2)**

**(b)** Every cat that is not striped is friendly with at least one dog that is not a friend of any striped cat. **(2)**

**(c)** Every short-tempered cat is friendly with one and only one striped dog. **(2)**

**Q2.** Let $\mathbb{Z}$ be the set of all integers. Define a relation $R$ on $\mathbb{N}$ (the set of positive integers) as follows:

$$\forall a, b \in \mathbb{N},\ a\ R\ b \text{ if and only if } \exists i \in \mathbb{Z},\ \frac{a}{b} = 2^i.$$

**(a)** Prove that $R$ is an equivalence relation. **(3)**

**(b)** List the equivalence classes defined by $R$ on $\mathbb{N}$. **(2)**

**(c)** Prove / Disprove: $R$ is a partial order. **(2)**

**Q3.** Consider the real intervals $(0, 1)$ and $[0, 1]$. A function $f : (0, 1) \to [0, 1]$ is defined as follows. Take $x \in (0, 1)$. Find (the unique) $n \in \mathbb{N}$ such that $\frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}$. Define $f(x) = \frac{3-2^n x}{2^n}$.

**(a)** Prove / Disprove: $f$ is injective. **(4)**

**(b)** Prove / Disprove: $f$ is surjective. **(3)**

# Indian Institute of Technology Kharagpur
## Department of Computer Science and Engineering

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(a) Prove that $R$ is an equivalence relation. **(3)**

(b) List the equivalence classes defined by $R$ on $\mathbb{N}$. **(2)**

(c) Prove / Disprove: $R$ is a partial order. **(2)**

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(a) Prove / Disprove: $f$ is injective. **(4)**

(b) Prove / Disprove: $f$ is surjective. **(3)**

(B) $(X \cup Y \cup Z) \setminus Z = X \cup Y$
(C) $(X \cap Y) \setminus Z = (X \setminus Z) \cap (Y \setminus Z)$
(D) $(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z)$

(e) Let $A = \{a, b, c, d\}$. How many binary operations $f$ on $A$ are there such that $f(a, b)$ is either $c$ or $d$?
(A) $15^4$
(B) $4^{15}$
(C) $2 \times 4^{15}$
(D) $2 \times 15^4$

(f) Define a relation $\rho$ on $\mathbb{Z}$ (the set of integers) such that $a \, \rho \, b$ if and only if either $a = b$ or $|a - b| > 1$. Which of the following statements about $\rho$ is/are incorrect?
(A) $\rho$ is reflexive
(B) $\rho$ is symmetric
(C) $\rho$ is antisymmetric
(D) $\rho$ is transitive

(g) Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denotes the set of natural numbers, integers and rational numbers, respectively. Which of the following sets is/are countable?
(A) $\mathbb{N} \times \mathbb{N}$
(B) $\mathbb{Z} \times \mathbb{Q}$
(C) $\mathbb{Q} \setminus \mathbb{N}$
(D) $\mathbb{Q} \times \mathbb{Q}$

(h) What is the inverse of an element $a$ in the group $G = \{a \in \mathbb{R} \mid a > 0\}$ ($\mathbb{R}$ denotes the set of real numbers) under the operation $\odot$ defined by $a \odot b = a^{\ln b}$?
(A) $1/a$
(B) $1/e^{\ln a}$
(C) $e^{1/\ln a}$
(D) $1/\ln a$

(i) Consider the set $\mathbb{Z}$ of all integers with the following operations, where $s$ and $t$ are constant integers.

$$\begin{aligned} a \oplus b &= a + b + s, \\ a \odot b &= a + b + tab. \end{aligned}$$

Which of the following is/are a necessary and sufficient condition for $(\mathbb{Z}, \oplus, \odot)$ to be a ring?
(A) $st = 1$.
(B) $s = t = -1$.
(C) $s$ and $t$ can have any values.
(D) $s$ and $t$ can have any values with $t \neq 0$.

(j) Which of the following functions $f : \mathbb{Z} \to \mathbb{Z}$ ($\mathbb{Z}$ denotes the set of integers) is/are a homomorphism of the ring $(\mathbb{Z}, +, \cdot)$ to itself?
(A) $f(x) = 1$
(B) $f(x) = x$
(C) $f(x) = 2x$
(D) $f(x) = x^2$

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**Foundations of Computing Science (CS60005)**

**Class Test 2** **Date: 05-Nov-2022 (Saturday) 10:30AM – 11:30AM** **Autumn Semester, 2022-2023**

**Marks: 20**

***Instructions:***

- *There are THREE questions. Answer ALL questions.*
- Write your answers in the answer booklet provided to you in the examination hall.
- Keep you answers brief and precise. Write solutions for all parts of a question together.
- Precisel state all assumptions you make.
- Sketchy proofs and claims without proper reasoning will be given no credit.

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1. Prove or disprove the following statements.
   (a) Every infinite regular set contains a subset that is not recursively enumerable. $\boxed{4}$
   (b) Every infinite *r.e.* set contains an infinite recursive subset.
   **Hint:** A set is recursive iff there exists an enumeration machine enumerating its strings in non-decreasing order of length. $\boxed{6}$

2. Consider the languge $\{(\mathcal{M}, x, p) \mid \mathcal{M}$ on input $x$ visits state $p$ during the computation$\}$. (Here, $p \in Q$ with $Q$ being the set of states of $\mathcal{M}$ and $x \in \Sigma^*$ where $\Sigma$ is the input alphabet of the Turing machine $\mathcal{M}$.) Is this language decidable? Justify. $\boxed{5}$

3. Let $\mathsf{REG} = \{\mathcal{M} \mid \mathcal{M}$ is a TM and $L(\mathcal{M})$ is a regular set$\}$. One of the following is true. Identify which one and justify your answer. $\boxed{5}$
   (a) $\mathsf{REG}$ is recursive.
   (b) $\mathsf{REG}$ is *r.e.* and $\neg\mathsf{REG}$ is not *r.e.*
   (c) $\mathsf{REG}$ is not *r.e.* and $\neg\mathsf{REG}$ is *r.e.*
   (d) Neither $\mathsf{REG}$ nor $\neg\mathsf{REG}$ is *r.e.*

**Foundations of Computing Science (CS60005)** **Autumn Semester, 2022-2023**

**Mid-Semester Examination** **Duration: 2 Hours** **Maximum Marks: 60**

**Instructions:**

- *Write your answers in the answer booklet provided to you in the examination hall.*
- *There are a total of SIX questions, each having 10 marks.*
- *Answer ALL the questions (or as many as you can) mentioning the question numbers clearly.*
- *Be brief and precise. Write the answers for all parts of a question together.*
- *If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.*
- *Write all the proofs/deductions in mathematically/logically precise language. Unclear and/or dubious statements would be severely penalized.*

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**Q1.** Your task is to (logically) solve a murder-mystery on behalf of Sherlock Holmes which appeared in the novel "A Study in Scarlet" by Sir Arthur Conan Doyle. The arguments (simplified from the novel) go as follows.

$F_1$ : There was a murder. If it was not done for robbery, then either it was a political assassination, or it might be for a woman.

$F_2$ : In case of robbery, usually something is taken.

$F_3$ : However, nothing was taken from the murderer's place.

$F_4$ : Political assassins leave the place immediately after their assassination work gets completed.

$F_5$ : On the contrary, the assassin left his/her tracks all over the murderer's place.

$F_6$ : For an assassin, to leave tracks all over the murderer's place indicates that (s)he was there all the time (for long duration).

Your goal is to (logically) find the reason for the murder. Please frame the above arguments logically (using propositional logic) and formally derive the solution. *Present your answer as asked in the following parts.*

(a) Write all propositions with English meaning (statements) that you have used. **(2)**

(b) Build suitable propositional logic formula to encode each of the *six* statements above. **(3)**

(c) Show all deduction steps (with the name of the rules you apply) to derive the goal (mystery). **(4)**

(d) Conclude what was the reason for the murder. **(1)**

**Q2.** Let the relation $\sigma$ on $\mathbb{N}$ (the set of natural numbers) consist only of the following tuples:

$$\sigma = \Big\{(n,n) \mid n \in \mathbb{N}\Big\} \bigcup \Big\{(2n, 2n-1) \mid n \in \mathbb{N}\Big\} \bigcup \Big\{(2n, 2n+1) \mid n \in \mathbb{N}\Big\}.$$

**(a)** Prove that $\sigma$ is a partial order. (6)

**(b)** Is $\sigma$ a total (that is, linear) order? (2)

**(c)** Is $\mathbb{N}$ a lattice under $\sigma$? (2)

**Q3.** Let $S$ be the set of all infinite bit sequences. In the class, $S$ has been proven to be uncountable (using diagonization argument). The $n$-th element of a sequence $\alpha \in S$ is denoted by $\alpha(n)$ for $n \geq 0$.

Prove the countability / uncountability of each of the following subsets of $S$.

**(a)** $T_1 = \Big\{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0, \text{ for } \underline{\text{some}}\ n \geq 0\Big\}$. (5)

**(b)** $T_2 = \Big\{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0, \text{ for } \underline{\text{no}}\ n \geq 0\Big\}$. (5)

Note: Solve the above parts independently, that is, do not use the result of any part in the other.

**Q4.** Let $R = \mathbb{Z} \times \mathbb{Z}$ (Cartesian product between set of integers). Define addition and multiplication on $R$ as:

$$\begin{aligned} (a,b) + (c,d) &= (a+c,\ b+d), \text{ and} \\ (a,b) \cdot (c,d) &= (ac + ad + bc,\ 2ac + bd). \end{aligned}$$

**(a)** Verify that $R$ is a ring under these two operations. (7)

**(b)** Prove that $R$ is a commutative ring with unity. (3)

**Q5.** Let $L$ be a language over an alphabet $\Sigma$. Recall that a string $x$ is called a prefix of a string $y$ if $y = xz$ for some string $z$. For example, all the prefixes of $abbab$ are $\varepsilon, a, ab, abb, abba, abbab$. From $L$, we generate the language $\texttt{dupPrefix}(L)$ by duplicating prefixes of strings in $L$. More precisely, we define,

$$\texttt{dupPrefix}(L) = \Big\{xy \mid y \in L, \text{ and } x \text{ is a prefix of } y\Big\}.$$

Prove / Disprove:

**(a)** If $L$ is regular, then $\texttt{dupPrefix}(L)$ must also be regular. (5)

**(b)** If $L$ is not regular, then $\texttt{dupPrefix}(L)$ must also be non-regular. (5)

**Q6.** Consider the following language $L_1$ over the alphabet $\{a, b, \#\}$.

$$L_1 = \Big\{x\#y \mid x, y \in \{a,b\}^*,\ x \neq y,\ |x| = |y|\Big\}.$$

Here, $|w|$ denotes the length of the string $w$. Prove / Disprove: $L_1$ is context-free. (10)

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**Foundations of Computing Science (CS60005)** | **Autumn Semester, 2022-2023**

**End-Semester Examination** | **Date: 18-Nov-2022, 2:00 PM – 5:00 PM** | **Marks: 100**

**Instructions:**

- Write your answers in the answer booklet provided to you in the examination hall.
- There are a total of FIVE questions, each having 20 marks.
- Answer ALL the questions (or as many as you can) mentioning the question numbers clearly.
- Be brief and precise. Write the answers for all parts of a question together.
- State any results you use or assumptions you make.
- Write all the proofs/deductions in mathematically/logically precise language.
- Sketchy proofs or claims without reasoning receive no credit.

1. Only write down all the correct choice(s) / option(s) for each of the following questions. $\boxed{10 \times 2 = 20}$

**Note:** *No justification is required for your given option(s); but a penalty of $\frac{1}{2}$ per question will be deducted for guessing wrong, including not making all choices (in case of multiple correct options).*

(a) The propositional logic statement $[(p \vee q \vee r) \rightarrow s]$ is equivalent to which one/more of the following?
  (A) $\neg s \rightarrow (p \vee q \vee r)$
  (B) $\neg s \rightarrow (p \wedge q \wedge r)$
  (C) $(p \rightarrow s) \wedge (q \rightarrow s) \wedge (r \rightarrow s)$
  (D) $(p \rightarrow s) \vee (q \rightarrow s) \vee (r \rightarrow s)$

? (b) According to political experts, "A person who is a radical (*radical*) is elected (*elected*) if (s)he is conservative (*conservative*), but otherwise is not elected". Which of the following is/are the correct logical representation(s) of the above statement made by political experts?
  (A) $(radical \wedge elected) \leftrightarrow conservative$
  (B) $radical \rightarrow (elected \leftrightarrow conservative)$
  ✓ (C) $radical \rightarrow ((conservative \rightarrow elected) \vee \neg elected)$
  (D) $(conservative \rightarrow (radical \wedge elected)) \vee \neg elected$

(c) Let $P$ and $Q$ are two predicates, and $F_1 \Leftrightarrow F_2$ symbolically denotes equivalence of two predicate logic formulas, $F_1$ and $F_2$ (that is, $F_1$ if and only if $F_2$). Which of the following is/are incorrect?
  (A) $\forall x\ [P(x) \vee Q(x)] \Leftrightarrow \forall x\ P(x) \vee \forall x\ Q(x)$
  (B) $\exists x\ [P(x) \vee Q(x)] \Leftrightarrow \exists x\ P(x) \vee \exists x\ Q(x)$
  (C) $\forall x\ [P(x) \wedge Q(x)] \Leftrightarrow \forall x\ P(x) \wedge \forall x\ Q(x)$
  (D) $\exists x\ [P(x) \wedge Q(x)] \Leftrightarrow \exists x\ P(x) \wedge \exists x\ Q(x)$

(d) Let $X, Y, Z$ be sets. Which of the following formulas is/are wrong?
  (A) $(X \cap Y \cap Z) \setminus Z = \emptyset$