CS60050
Machine Learning

Logistic Regression

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Logistic Regression

for Classification

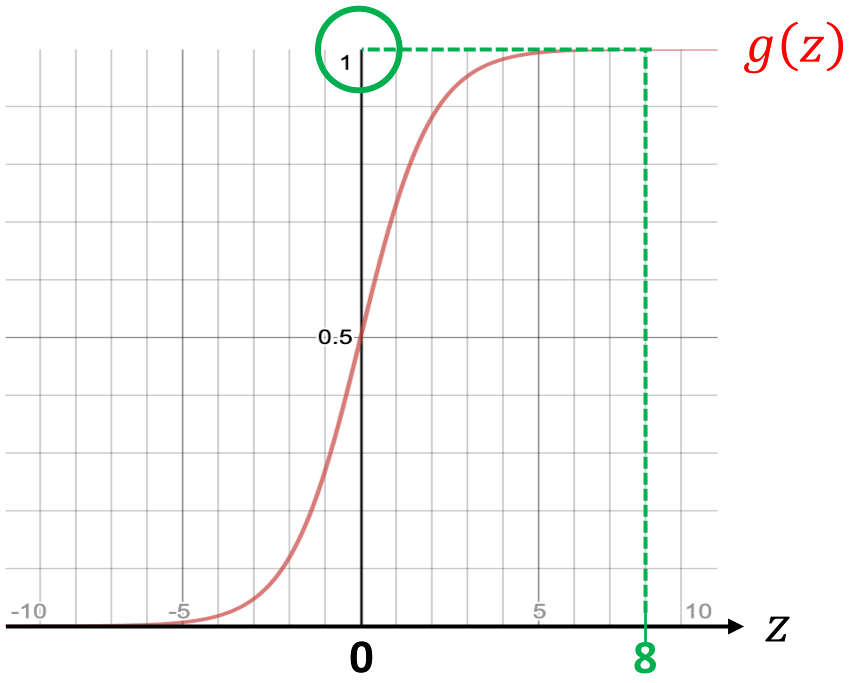
Regression vs. Classification

We want the possible outputs of $\boldsymbol{f_{\theta}(x)} = \boldsymbol{\theta^T x}$ to be discrete-valued

Use an ***activation function*** (e.g., ***sigmoid or logistic function***)

$$g(z) = \frac{1}{1+e^{-z}}$$

$z \in \mathbb{R}, but$
$g(z) \in [0,1]$



If y = **1**, we want $g(z) \approx 1$ (i.e., we want a correct prediction)
For this to happen, $\boldsymbol{z} \gg \boldsymbol{0}$

If y = **0**, we want $g(z) \approx 0$ (i.e., we want a correct prediction)
For this to happen, $\boldsymbol{z} \ll \boldsymbol{0}$

Classification

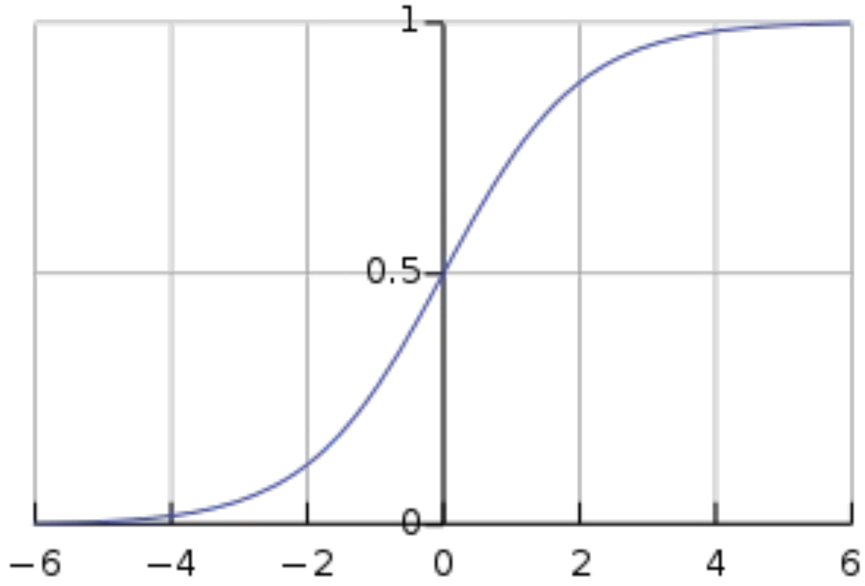
$$\boldsymbol{x} = [\boldsymbol{x}_0, \dots, \boldsymbol{x}_m]$$

$$g(\boldsymbol{\theta}^T \boldsymbol{x})$$

$$\boldsymbol{h}_{\boldsymbol{\theta}}(\boldsymbol{x}) \in [0,1]$$

$$h_\theta(x) = g(\theta^\top x)$$

$$g(z) = \frac{1}{1 + e^{-z}}$$



Thresholding:
predict "y = 1" if $h_\theta(x) \geq 0.5$

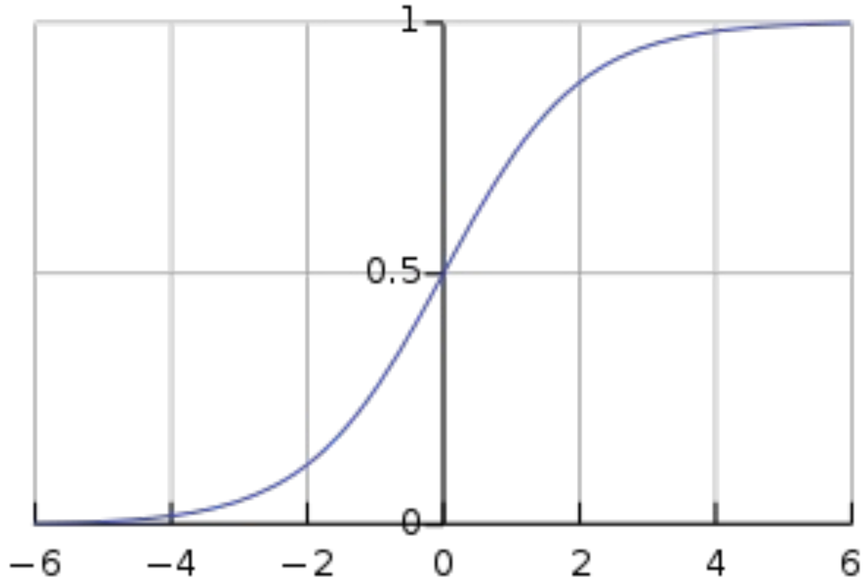
predict "y = 0" if $h_\theta(x) < 0.5$

Classification

$\boldsymbol{x} = [\boldsymbol{x_0}, \ldots, \boldsymbol{x_m}]$ → $\boldsymbol{g(\theta^T x)}$ → $\boldsymbol{h_\theta(x)} \in [0,1]$

$h_\theta(x) = g(\theta^\top x)$

$g(z) = \dfrac{1}{1 + e^{-z}}$



Thresholding:

predict "y = 1" if $h_\theta(x) \geq 0.5$

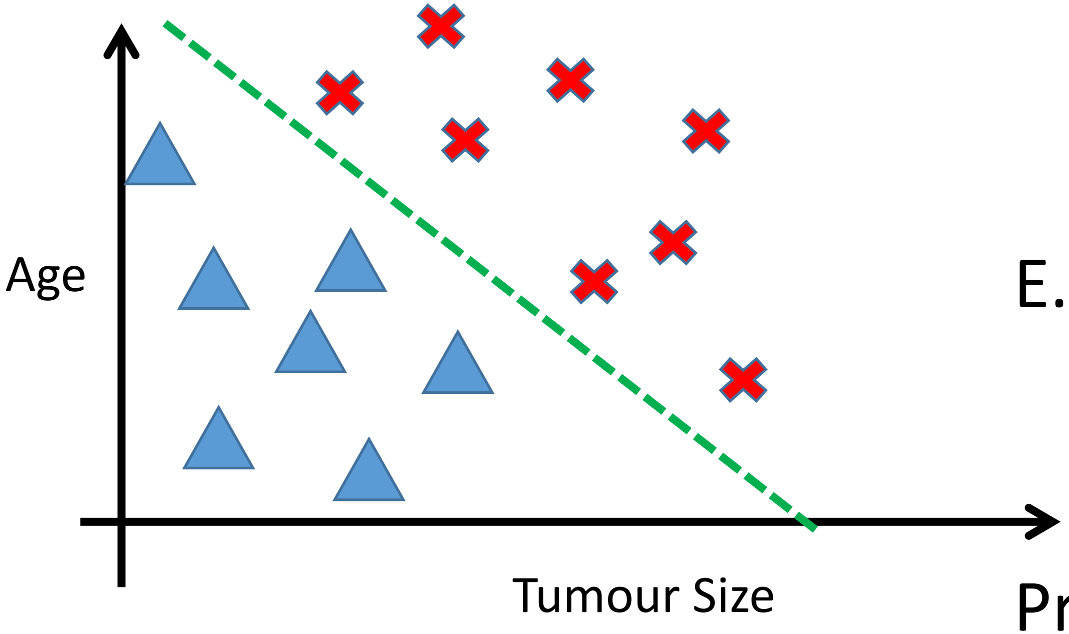
$\boldsymbol{z = \theta^\top x \geq 0}$

predict "y = 0" if $h_\theta(x) < 0.5$

$\boldsymbol{z = \theta^\top x < 0}$

Alternative Interpretation: $h_\theta(x) =$ estimated probability that $y = 1$ on input x

Decision boundary



$$f_\theta(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2)$$

E.g., $\theta_0 = -3, \theta_1 = 1, \ \theta_2 = 1$

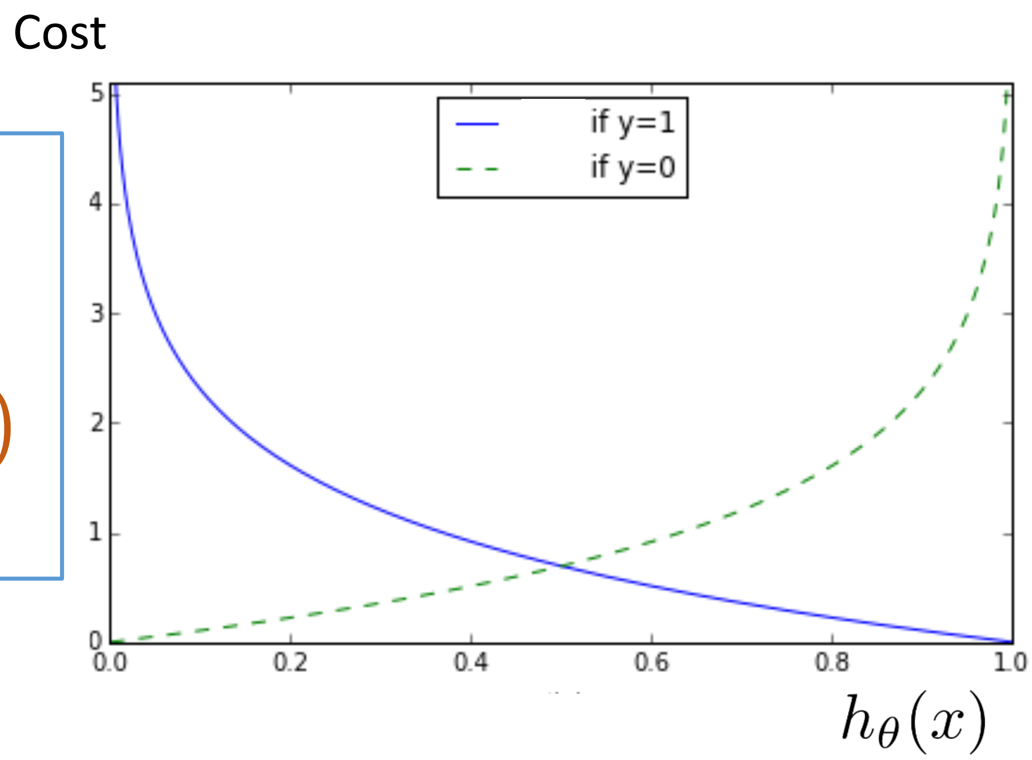
Predict "$y = 1$" if $-3 + x_1 + x_2 \geq 0$

Cost function for Logistic Regression

Logistic Regression

$$\text{Cost}(h_\theta(x), y) = \begin{cases} -\log(h_\theta(x)) & \text{if } y = 1 \\ -\log(1 - h_\theta(x)) & \text{if } y = 0 \end{cases}$$

$$= -\boldsymbol{y}\ \mathbf{log}(\boldsymbol{h_\theta}(\mathbf{x})) - (\mathbf{1} - \mathbf{y})\ \mathbf{log}(\mathbf{1} - \boldsymbol{h_\theta}(\boldsymbol{x}))$$



$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} \text{Cost}(\boldsymbol{h}_\theta(x^{(i)}), y^{(i)}))$$

$$= -\frac{1}{m}\left[\sum_{i=1}^{m} y^{(i)}\ \log\left(\boldsymbol{h}_\theta\left(x^{(i)}\right)\right) + (1 - y^{(i)}) \log\left(1 - \boldsymbol{h}_\theta\left(x^{(i)}\right)\right)\right]$$

Gradient descent

$$J(\theta) = -\frac{1}{m}\left[\sum_{i=1}^{m} y^{(i)}\ \log\left(h_\theta\left(x^{(i)}\right)\right) + (1 - y^{(i)})\log\left(1 - h_\theta\left(x^{(i)}\right)\right)\right]$$

Goal: $\min_{\theta} loss(\theta)$

Good news: Convex function!
Bad news: No analytical solution

Gradient descent

$$loss(\theta) = -\frac{1}{m}\left[\sum_{i=1}^{m} y^{(i)}\ \log\left(h_\theta\left(x^{(i)}\right)\right) + (1 - y^{(i)})\log\left(1 - h_\theta\left(x^{(i)}\right)\right)\right]$$

$$\boxed{\frac{\partial}{\partial \theta_j} loss(\theta) = \frac{1}{m}\sum_{i=1}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right) x_j^{(i)}}$$

Gradient descent

Repeat {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} loss(\theta)$$

}

(Simultaneously update all θ_j)

$$\boxed{\frac{\partial}{\partial \theta_j} l(\theta) = \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right) x_j^{(i)}}$$

Gradient descent for Linear Regression

Repeat {

$$\theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right) x_j^{(i)}$$

}

$$\boxed{h_\theta(x) = \theta^\top x}$$

Gradient descent for Logistic Regression

Repeat {

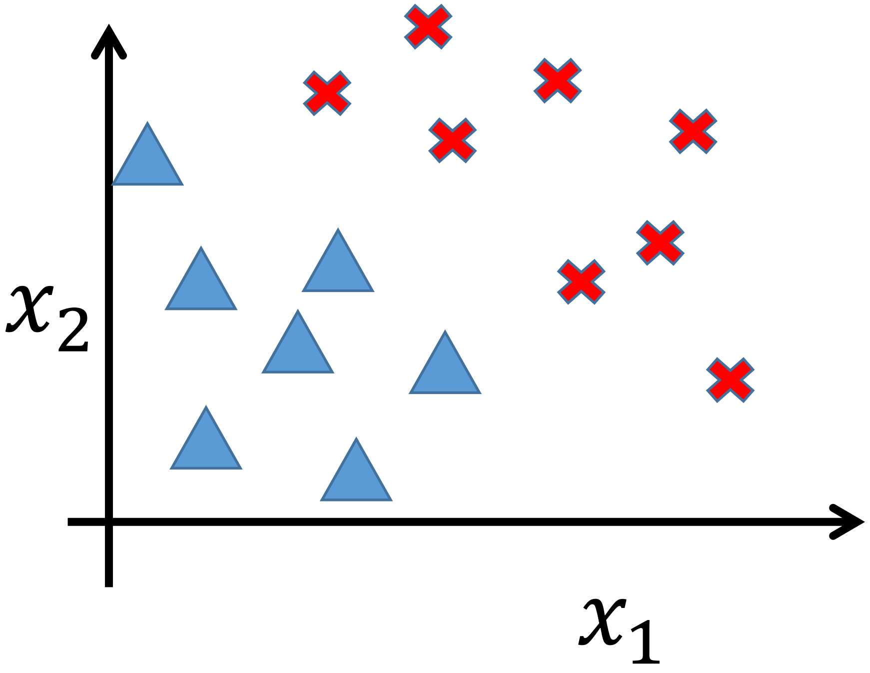
$$\theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right) x_j^{(i)}$$

}

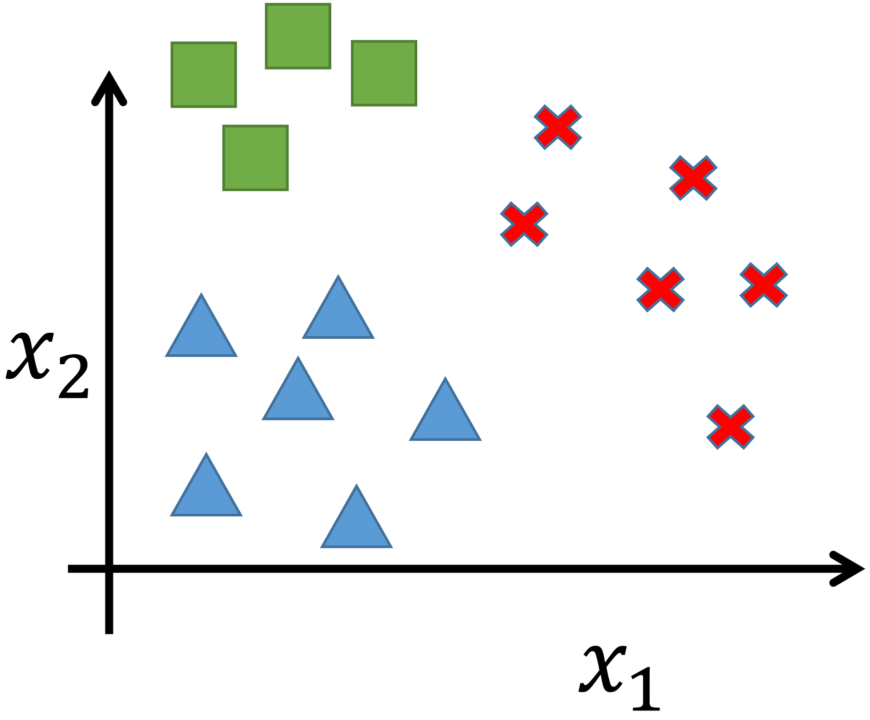
$$\boxed{h_\theta(x) = \frac{1}{1 + e^{-\theta^\top x}}}$$

Multiclass classification

Binary classification



Multiclass classification



Multi-class Classification

- Multi-class Classification: y can take on K different values $\{1,2,\ldots,k\}$
- $f_\theta(x)$ estimates the probability of belonging to each class

$$P(y=k|x,\theta) \propto \exp\left(\theta_k^T x\right)$$

$$\theta = \begin{bmatrix} \vdots & \vdots & \vdots \\ \theta_1 & \theta_2 & \theta_k \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$P(y=k|x,\theta) = \frac{\exp\left(\theta_k^T x\right)}{\sum_{j=1}^{K} \exp\left(\theta_j^T x\right)}$$

$$\boxed{\mathrm{J}(\theta) = -\left[\sum_{i=1}^{m} \sum_{j=1}^{K} 1\{y^{(i)} = k\} \log \frac{\exp\left(\theta_k^T x^{(i)}\right)}{\sum_{j=1}^{K} \exp\left(\theta_j^T x^{(i)}\right)}\right]}$$