Tutorial 2
Sets, Relations and Functions

1. Show the following.
   (a) If $a \in \{\{b\}\}$, then $b \in a$.
   (b) If $\mathcal{C} = \{\{x\} \mid b \in B$, then $\cup_{X \in \mathcal{C}} X = B$.
2. Prove that for all sets A, B the following statements are equivalent:
   (a) $A \subseteq B$
   (b) $A \setminus B = \emptyset$.
   (c) $A \cup B = B$.
   (d) $A \cap B = A$.
3. For two sets A, B, the *symmetric difference* $A\Delta B$ is defined as $(A \setminus B) \cup (B \setminus A)$. Prove the following for all sets A, B, C.
   (a) $A\Delta B = (A \cup B) \setminus (A \cap B)$
   (b) If $A \setminus C = B \setminus C$, then $A\Delta B \subseteq C$.
   (c) $A\Delta B = \emptyset$ if and only if $A = B$.
4. Let A be the set of all cities in India. Define a binary relation $\mathcal{R}$ on A as follows: for $x, y \in A$, $(x, y) \in \mathcal{R}$ if the distance between x and y is atmost 400 km. Determine whether or not R is reflexive, asymmetric, transitive, anti-symmetric or irreflexive.
5. A relation $\mathcal{R}$ is *circular* if $(x, y), (y, z) \in \mathcal{R}$ implies $(z, x) \in \mathcal{R}$. Prove that $\mathcal{R}$ is an equivalence relation if and only if it is both circular and reflexive.
6. Let A be a set and $\mathcal{R}$ be an equivalence relation on A. Denote by $A/\mathcal{R}$ (read A quotiented by $\mathcal{R}$) the set of equivalence classes under $\mathcal{R}$. For $a \in A$, $[a]_{\mathcal{R}}$ denotes the equivalence class to which a belongs.

   Let $f : A \to B$ be a function and let $\mathcal{R}$ be an equivalence relation on B. Define a relation $\mathcal{T}$ on A as follows: for all $a, a' \in A$, $(a, a') \in \mathcal{T}$ if and only if $(f(a), f(a')) \in \mathcal{R}$.
   (a) Is $\mathcal{T}$ an equivalence relation on A?
   (b) Does $\mathcal{T}$ define a partial order on A?
   (c) Define a map $\bar{f} : A/\mathcal{T} \to B/\mathcal{R}$ as $[a]_{\mathcal{T}} = [f(a)]_{\mathcal{R}}$. Show that $\bar{f}$ is well-defined.
   (d) Prove that $\bar{f}$ is injective.
7. Define a relation $\mathcal{R}$ on $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $((a, b), (c, d)) \in \mathcal{R}$ if and only if $ad = bc$. Prove that $\mathcal{R}$ is an equivalence relation. Show that $A/\mathcal{R}$ is essentially $\mathbb{Q}$.
8. Let Σ be an alphabet which is totally ordered i.e., for every $a, b \in \Sigma$, either $a \preceq b$ or $b \preceq a$. Consider the set Σ^*. We define the *lexicographic ordering* $\preceq$ on Σ^* as follows. Let $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_m$ be two strings in Σ^* with $\{x_i\}_i, \{y_j\}_j \in \Sigma$. We say $x \preceq y$ if: $n < m$ or $n = m$ and there exists an index $i \in \{1, 2, \ldots, n\}$ such that $x_j = y_j$ for all $1 \leq j \leq i - 1$ and $x_i \preceq y_i$.
   (a) Show that $(\Sigma^*, \preceq)$ is a partial order. Is it a total order?

(b) Does there exist a least element? What is it?

(c) What is the greatest element?

(d) Consider the subset $A = \{x \in \Sigma^* \mid l_1 \leq |x| \leq l_2\}$ where $\ell_1, \ell_2 \in \mathbb{Z}^+$ with $\ell_1 \leq \ell_2$. What are the minimal/maximal elements of A? Are there least/greatest elements?

(e) For the set A defined above, write down 2 different upper bounds and lower bounds. Is there a least upper bound and a greatest lower bound? What are they?

9. Let S be a set. A characteristic function over S is a function $\chi : S \to \{0, 1\}$. If $A \subseteq S$, then the characteristic function of A is $\chi_A : S \to \{0, 1\}$ given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Prove that there exists a bijection between 2^S and the set of all Boolean function on S i.e., $(S \to \{0, 1\})$.

10. Which of the following are bijections? Justify your answer.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, where $f(n) = (-1)^{|n|} n$ for every $n \in \mathbb{Z}$.

(b) $f : \mathbb{Z} \to \mathbb{Z}_{19}$ where $f(n) = n \bmod 19$ for every $n \in \mathbb{Z}$.

(c) $f : \mathbb{R} \to \mathbb{C}$ where $f(x) = x^3$ for all $x \in \mathbb{R}$.

(d) $f : \mathbb{N} \to \mathbb{N}$, where

$$f(n) = \begin{cases} n + 1 & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases}$$