Tutorial 3
Set Cardinality and Algebraic Structures

COUNTABLE AND UNCOUNTABLE SETS

1. Suppose that $A \subseteq B$ and A is uncountable. Show that $A \sim B$ (i.e., A and B are equipotent).

2. Suppose that A is uncountable and B is a countable subset of A. Prove or disprove: $A \sim A \setminus B$.

3. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Show that $[a, b) \times [c, d)$ is equipotent with $[0, 1)$.

4. Let $\mathbb{Z}[x]$ denote the set of all polynomials in variable x with integer coefficients.
   (a) Prove that $\mathbb{Z}[x]$ is countable.
   (b) $a \in \mathbb{C}$ is called *algebraic* if a is the root of some non-zero polynomial $f(x) \in \mathbb{Z}[x]$. Let $\mathbb{A}$ be the set of all algebraic numbers. Is $\mathbb{A}$ countable?

5. Let $f : S \to \mathbb{N}$ be a one-one correspondence of set S with $\mathbb{N}$. Define a relation $\mathcal{R}_f$ on S as:
$$\mathcal{R}_f = \{(a, b) \in S^2 \mid f(a) \leq f(b)\}.$$
   Prove that $\mathcal{R}_f$ is a linear ordering on S such that every element of S has only finitely many predecessors under $\mathcal{R}_f$.

6. A set $S \subseteq R$ is called *bounded* if S has both an upper bound and a lower bound. Provide examples for
   (a) Countable bounded subset of $\mathbb{R}$.
   (b) Uncountable bounded subset of $\mathbb{R}$.

7. Answer whether the following sets are countable or uncountable.
   (a) Set of all bounded subsets of $\mathbb{Z}$.
   (b) Set of all bounded subsets of $\mathbb{Q}$.

ALGEBRAIC STRUCTURES

1. Let $(S, \circ)$ and $(T, \star)$ be two algebraic systems. A function $f : S \to T$ is called a *homomorphism* if got any $s_1, s_2 \in S$, we have
$$f(s_1 \circ s_2) = f(s_1) \star f(s_2).$$
   f is called
   - an *epimorphism* if it is onto,
   - a *monomorphism* if it is one-one,
   - and an *isomorphism* if it is a bijection.

   (a) Define a homomorphism from $(\mathbb{N}, +)$ to $(\mathbb{Z}_4, +_4)$. Determine whether the map you define is an epimorphism, monomorphism or both.
   (b) Consider the algebraic system $(T = \{1, -1, i, -i\}, \cdot)$ (here, $\cdot$ is multiplication). Show that $(T, \cdot)$ is a group.

(c) Show that $(S_4, \cdot)$ is isomorphic to $(\mathbb{Z}_4, +_4)$.

2. Show that the following systems are semi-groups. Are any of them monoids?
   (a) $(2^X, \cup)$ where X is a finite set.
   (b) $(2^X, \cap)$ where X is a finite set.
   (c) $(\mathbb{Z}^+, \max)$ where for $x, y \in \mathbb{Z}^+$, $\max(x, y)$ is the maximum of x and y.
   (d) $(\mathbb{N}, \max)$.

3. Let G be the set of all points on the hyperbola $xy = 1$, along with the point $(0, \infty)$ at infinity. Define $(a, \frac{1}{a}) + (b, \frac{1}{b}) = (a + b, \frac{1}{a+b})$. Is G a group under this operation? Is it Abelian?

4. Consider $I = [0, 1) \subseteq \mathbb{R}$. Define an binary operation $\star$ on I as follows: for $x, y \in I$, $x \star y = x + y - \lceil x + y \rceil$ where $\lceil x \rceil$ denotes the greatest integer samller than or equal to x. Show that $(I, \star)$ is an Abelian group.

5. Let R be a commutative ring with identity. Show that $R[x]$, the set of all polynomials in x with coefficients from R, is a commutative ring with identity under polynomial addition and multiplication operations.

6. Let $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Clearly $S \subset \mathbb{R}$. Show that S is a field.