Prof. Faouzi Bellili — ECE Department (University of Manitoba)

ECE–4260 Communication Systems

Assignment#2 (Winter 2025)

Remarks:

- This assignment is due on **February 12th, 2025**. Please scan your answers and upload them to UM Learn. Do not e-mail pdf documents.
- **Make sure to compile a single PDF document that contains all your answers, i.e., do not submit separate pages and make sure to number your pages.**
- **Presenting clean and well-justified answers helps you get full marks for questions.**
- Assignments submitted after the due date will not be marked.
- Clearly show the solution method. Marks are awarded for the method and not the final answer itself.

1. **Problem 1 (10 points)**: Consider an LTI system whose response to the signal $x_1(t)$ depicted in Fig. 1(a) is the signal $y_1(t)$ illustrated in Fig. 1(b).

   1.1) Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted in Fig. 1(c). *You must show the key points on both the horizental and vertical axes.* (**5 points**):

   1.2) Determine and sketch carefully the response of the system to the input $x_3(t)$ depicted in Fig. 1(d). *You must show the key points on both the horizental and vertical axes.* (**5 points**):

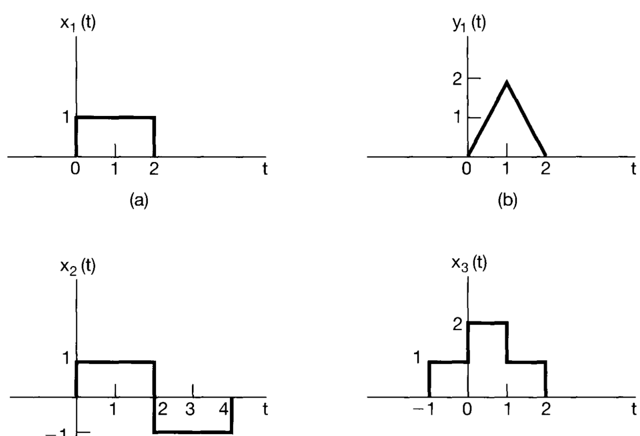

Figure 1: Plot of the signals used in Problem 1.

2. **Problem 2 (15 points)**: Consider an LTI system whose impulse response is given by $h(t) = \frac{\sin(4[t-1])}{\pi(t-1)}$ Determine the output of this system to each of the following input signals:

   2.1) Input signal $x_1(t) = \frac{\sin(4[t+1])}{\pi(t+1)}$ (**7.5 points**)

   2.2) Input signal $x_2(t) = \left(\frac{\sin(2t)}{\pi t}\right)^2$ (**7.5 points**)

   Hint: Express $h(t)$, $x_1(t)$ and $x_2(t)$ in terms of the sinc(.) function.

3. **Problem 3 (15 points)**: A periodic, quadratic function and some surprising applications.

Let $g(t)$ be a *periodic* function (or signal) of fundamental period $T = 2$ with

$$g(t) = t^2 \quad \text{if} \quad 0 \leq t \leq 2 \tag{1}$$

Here is the picture

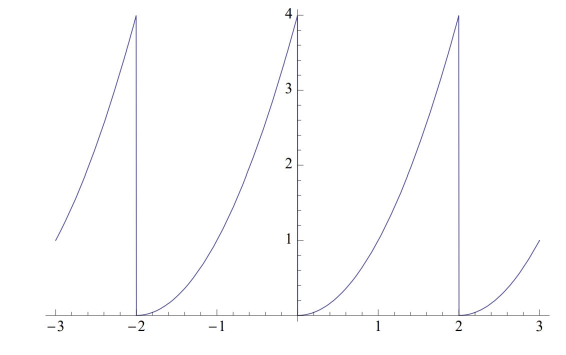

Figure 2: Plot of $g(t)$ which obviously extends from $-\infty$ to $+\infty$ since it is periodic.

Let the Fourier series expansion of $g(t)$ be:

$$g(t) = \sum_{-\infty}^{+\infty} g_n e^{j2\pi \frac{n}{T} t} \tag{2}$$

3.1) Show that the Fourier series coefficients involved in (2) are given by (**5 points**):

$$g_n = \frac{2(1 + j\pi n)}{\pi^2 n^2} \quad \text{for } n \neq 0 \qquad \text{and} \qquad g_0 = \frac{4}{3}. \tag{3}$$

3.2) Show the following identity (**5 points**):

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{4}$$

Hint: You should evaluate the right-hand side of eq. (2) at $t = 0$ and recall that the series converges to the average value of $g(.)$ at 0 which is $\frac{1}{2}[g(0^-) + g(0^+)] = 2$ instead of converging to $g(0) = 0$ (This is because the periodic function has a discontinuity at $t = 0$). You need to manipulate the sum a little further to arrive to the required identity in eq. (4).

3.3) Show the following identity (**5 points**):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}. \tag{5}$$

Hint: Consider evaluating eq. (2) at another adequate point $t_0 \neq 0$.

4. **Problem 4 (10 points)**: Find the signal $x(t)$ whose Fourier transform $X(f)$ is pictured in Fig. 3.

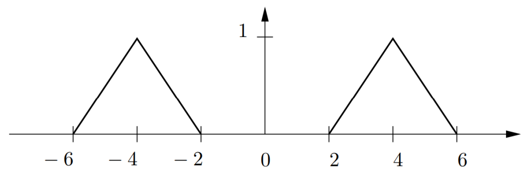

Figure 3: Plot of $X(f)$.

5. **Problem 5 (10 points)**: Fourier transforms and Fourier coefficients.

Suppose the signal $x(t)$ is zero outside the interval $-\tau/2 \leq t \leq \tau/2$ for some $\tau > 0$ and denote its Fourier transfrom by $X(f)$. We form a function $g(t)$ which is a periodic version of $x(t)$ with period $T \geq \tau$ by the formula

$$g(t) = \sum_{k=-\infty}^{+\infty} x(t - kT). \tag{6}$$

5.1) Show that $g(t)$ is indeed periodic with period T. (**2 points**).

5.2) Pick any shape for the signal $x(t)$ and sketch the plot of $g(t)$ over the interval $-2T - \frac{\tau}{2} \leq t \leq 2T + \frac{\tau}{2}$ (**2 points**).

The Fourier series expansion of $g(t)$ is given by

$$g(t) = \sum_{n=-\infty}^{+\infty} g_n e^{j2\pi \frac{n}{T} t}. \tag{7}$$

5.3) Show that the Fourier series coefficients of $g(t)$ are obtained from $X(f)$ as follows (**6 points**):

$$g_n = \frac{1}{T} X(\tfrac{n}{T}). \tag{8}$$

6. **Problem 6 (40 points)**

Consider the functions $g(x)$ and $h(x)$, shown in Fig. 4. Let $G(f)$ and $H(f)$ denote the fourier transforms of $g(x)$ and $h(x)$, respectively.

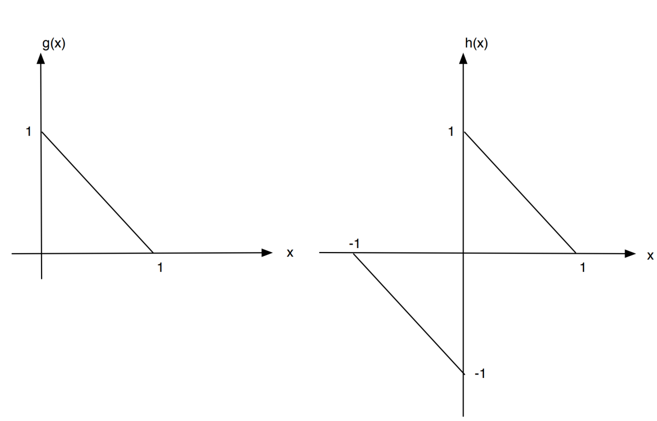

Figure 4: Plot of the functions $g(x)$ and $h(x)$.

In general, Fourier transforms are complex-valued functions of the frequency f. Let $G(f) = |G(f)|e^{j\theta_G(f)}$ and $H(f) = |H(f)|e^{j\theta_H(f)}$ be the Euler representations of the two Fourier transforms.

6.1) What are the two possible values of $\theta_H(f)$, i.e., the phase of $H(f)$? Express your answer in radians. (**5 points**)

6.2) Evaluate the following integral (**5 points**):

$$\int_{-\infty}^{+\infty} G(f)\cos(\pi f)df. \tag{9}$$

6.3) Evaluate the following integral (**5 points**):

$$\int_{-\infty}^{+\infty} H(f)e^{j4\pi f}df. \tag{10}$$

The function $g(x)$ can be decomposed into its *even* and *odd* parts as follows:

$$g(x) = g_e(x) + g_o(x), \tag{11}$$

where

$$g_e(x) = \tfrac{1}{2}[g(x) + g(-x)], \tag{12}$$
$$g_o(x) = \tfrac{1}{2}[g(x) - g(-x)]. \tag{13}$$

6.4) Plot $g_e(x)$ and $g_o(x)$. (**5 points**).

6.5) Without performing any integration, find the real part of $G(f)$? Explain your reasoning. (**5 points**). **Hint**: You must use the fact $g_e(x)$ is *even* and $g_o(x)$ is *odd*.

6.6) Using integration, show that (**5 points**):

$$G(f) = \frac{\text{sinc}(f)^2}{2} + j\left[\frac{\text{sinc}(2f) - 1}{2\pi f}\right] \tag{14}$$

6.7) Without performing any integration, find $H(f)$? Explain your reasoning. (**5 points**)

6.8) Let $\varphi(x)$ be the periodized version of $h(x)$ with period $T = 2$. Without performing any integration, find the Fourier series coefficents, φ_n, of the periodic signal $\varphi(x)$. (**5 points**).