Prof. Faouzi Bellili — ECE Department (University of Manitoba)

ECE–4260 Communication Systems

Assignment#3 (Winter 2025)

Remarks:

- This assignment is due on **February 28th, 2025**. Please scan your answers and upload them to UM Learn. Do not e-mail pdf documents.
- **Make sure to compile a single PDF document that contains all your answers, i.e., do not submit separate pages and make sure to number your pages.**
- **Presenting clean and well-justified answers helps you get full marks for questions.**
- Assignments submitted after the due date will not be marked.
- Clearly show the solution method. Marks are awarded for the method and not the final answer itself.

1. **Problem 1 (30 points)**: Application of Parseval's Theorem.

   Let $g_1(t)$ be $g_2(t)$ be two (possibly complex-valued) signals with Fourier transforms $G_1(f)$ and $G_2(f)$, respectively.

   1.1) Show that (**10 points**)
   $$\int_{-\infty}^{+\infty} g_1(t)g_2^*(t)dt \;=\; \int_{-\infty}^{+\infty} G_1(f)G_2^*(f)df \tag{1}$$

   1.2) Explain how we can obtain Parseval's Theorem from (1). (**10 points**).

   1.3) Using Parseval's Theorem, show that for any $k > 0$ we have (**10 points**):
   $$\int_{-\infty}^{+\infty} \mathrm{sinc}^2(kt)dt \;=\; \frac{1}{k}. \tag{2}$$

2. **Problem 2 (40 points)**: Correlation of signals and energy spectral density.

Part I: An important concept in many communications applications is the correlation between two signals. Let $x(t)$ and $y(t)$ be two signals; then the correlation function is defined as
$$r_{xy}(t) \;=\; x(t) * y^*(-t). \tag{3}$$

The function $r_{xx}(t)$ is usually referred to as the *autocorrelation* function of the signal $x(t)$, while $r_{xy}(t)$ is often called a *cross-correlation* function.

2.1) Show that (**5 points**):
$$r_{xy}(t) \;=\; \int_{-\infty}^{+\infty} x(\tau)y^*(\tau - t)d\tau \;=\; \int_{-\infty}^{+\infty} y(\tau)^* x(\tau + t)d\tau \tag{4}$$

2.2) Show that $r_{xy}(t) = r_{yx}(-t)^*$. (**5 points**)

2.3) Suppose that $y(t) = x(t + T)$. Express $r_{xy}(t)$ and $r_{yy}(t)$ in terms of $r_{xx}(t)$. (**5 points**).

Part II: The transformation $y(t) = x(t+T)$ is the input-output relationship $y(t) = h_0(t) * x(t)$ of a particular LTI system whose impulse response $h_0(t) = \delta(t+T)$. In this part, we will generalize the study to an arbitrary impulse response $h(t)$ and we will rather rely on the enrgy spectral density (ESD) to simplify the calculation. The cross-ESD of any two signals $x(t)$ and $y(t)$, denoted as $\Psi_{xy}(f)$, is defined as the Fourier transform (FT) of their cross-correlation function $r_{xy}(t)$, i.e.:

$$\Psi_{xy}(f) \triangleq \text{FT}\{r_{xy}(t)\} = \int_{-\infty}^{+\infty} r_{xy}(t) e^{-j2\pi f t} dt. \tag{5}$$

Obviously, $\Psi_{xx}(f) = \text{FT}\{r_{xx}(t)\}$ is the ESD of $x(t)$ and $\Psi_{yy}(f) = \text{FT}\{r_{yy}(t)\}$ is the ESD of $y(t)$.

2.4) What is the relationship between $\Psi_{xy}(f)$ and $\Psi_{yx}(f)$? (**5 points**):

2.5) Find an expression for $\Psi_{xy}(f)$ in terms of $X(f)$ and $Y(f)$. (**5 points**)

2.6) Show that $\Psi_{xx}(f)$ is real and positive for every f. (**5 points**).

2.7) Suppose now that $x(t)$ and $y(t)$ are, respectively, the input and output of an LTI system with impulse response $h(t) \longleftrightarrow H(f)$. Find expressions for $\Psi_{xy}(f)$ and $\Psi_{yy}(f)$ in terms of $\Psi_{xx}(f)$ and $H(f)$. (**5 points**).

2.8) Deduce the expressions of $r_{xy}(t)$ and $r_{yy}(t)$ in terms of $h(t)$, $r_{hh}(t)$, and $r_{xx}(t)$. (**5 points**).

3. **Problem 3 (30 points):**

You are asked to design a DSB-SC modulator to generate a modulated signal $z(t) = k\, m(t) \cos(2\pi f_c t)$, where $m(t)$ is a signal band-limited to B Hz and k is some constant. Fig. 1 shows a DSB-SC modulator available in the stock room.

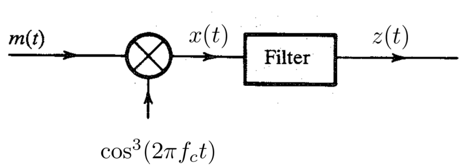

Figure 1: The DSB-SC modulator available to you.

In Fig. 1, the carrier generator available generates not $\cos(2\pi f_c t)$, but $\cos^3(2\pi f_c t)$. In the following you will explain whether you would be able to generate the desired modulated signal using only this equipment provided that you have the flexibility to design any kind of filter you like. Without loss of generality, assume that the message signal is given by:

$$m(t) = B \,\text{sinc}^2(Bt), \tag{6}$$

where $B > 0$.

3.1) Determine and sketch $X(f)$, the Fourier transform of $x(t)$. (**7.5 points**):

3.2) Design an adequate filter, $H(f)$, that you can use in Fig. 1.
Sketch $H(f)$ in a separate figure. (**7.5 points**).

3.3) What is the minimum usable value for f_c? Justify your answer (**7.5 points**).

3.4) Design a receiver that uses the available oscillator $\cos^3(2\pi f_c t)$ to recover the message signal $m(t)$ from $z(t)$. (**7.5 points**).