Assignment 2

Rajdeep Gill 7934493

ECE 4260 A01

February 14, 2025

1 Problem 1

Given that it is an LTI system, the responses to the two new inputs can be derived from the given response.

1.1 We can express $x_2(t)$ as:

$$\begin{aligned} x_2(t) &= x_1(t) - x(t-2) \\ \implies y_2(t) &= y_1(t) - y(t-2) \\ &= 2\Lambda(t-1) - 2\Lambda(t-2) \end{aligned}$$

1.2 We can express $x_3(t)$ as:

$$\begin{aligned} x_3(t) &= x(t+1) + x(t) \\ \implies y_3(t) &= y(t+1) + y(t) \\ &= 2\Lambda(t) + 2\Lambda(t-1) \end{aligned}$$

The plots of the inputs and outputs can be seen in Figure 1.

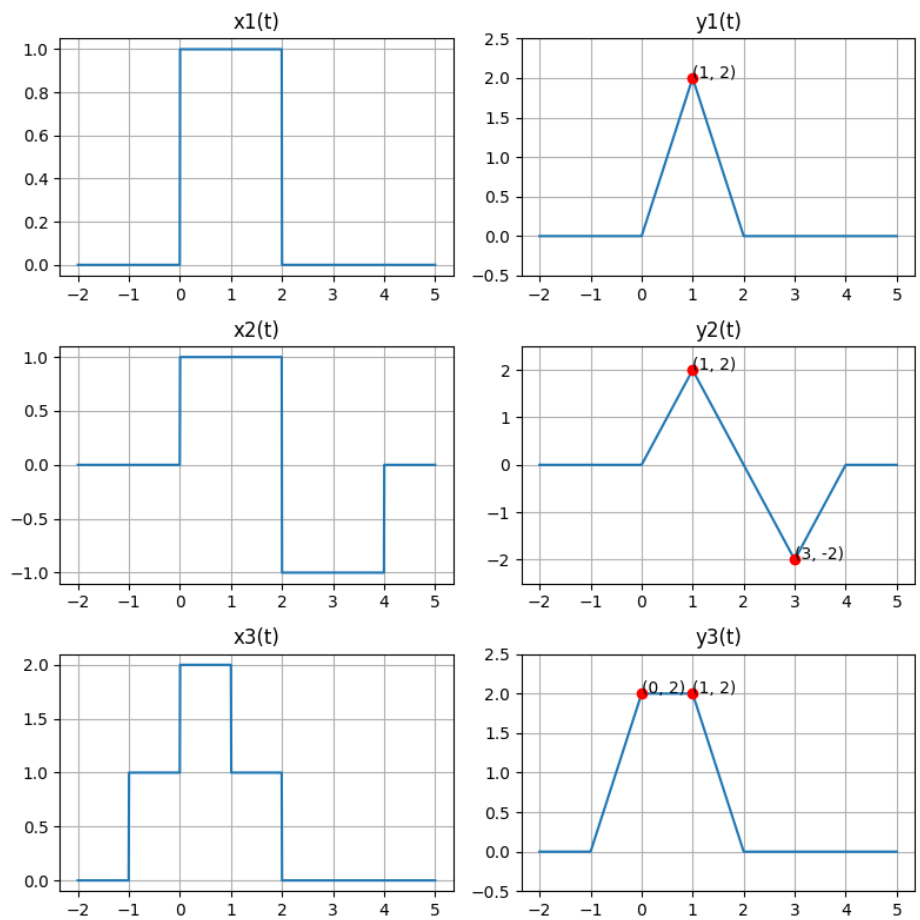

Figure 1: Plots of the inputs and outputs

2 Problem 2

We can express $h(t)$ as a sinc function:

$$\begin{aligned}
\mathrm{sinc}(t) &= \frac{\sin(\pi t)}{\pi t} \\
\implies \frac{\sin(4[t-1])}{\pi(t-1)} &= \frac{\sin(\pi\frac{4[t-1]}{\pi})}{\pi(t-1)} \\
&= \frac{\frac{4}{\pi}\sin(\pi\frac{4[t-1]}{\pi})}{\pi\frac{4}{\pi}(t-1)} \\
&= \frac{4}{\pi}\mathrm{sinc}\left(\frac{4[t-1]}{\pi}\right)
\end{aligned}$$

In frequency domain, using the tables, we find that the impulse response of a sinc function is a rectangle function. We also apply the time scaling property and time shifting property of the fourier transform:

$$\begin{aligned}
\text{sinc(t)} &\xrightarrow{\mathcal{F}} \mathrm{rect}(f) \\
\mathrm{sinc}(a(t-t_0)) &\xrightarrow{\mathcal{F}} \frac{1}{|a|}\mathrm{rect}(f/a) \times e^{-j2\pi f t_0}
\end{aligned}$$

We have that $a = 4/\pi$ and $t_0 = 1$. Thus, the fourier transform of $h(t)$ is:

$$H(f) = \mathrm{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f}$$

2.1 We can rewrite $x_1(t)$ as follows:

$$x_1(t) = \frac{\sin(4[t+1])}{\pi(t+1)} = \frac{\frac{4}{\pi}\sin\left(\pi\frac{4[t+1]}{\pi}\right)}{\pi\frac{4}{\pi}(t+1)} = \frac{4}{\pi}\mathrm{sinc}\left(\frac{4[t+1]}{\pi}\right)$$

Similarly the fourier transform of $x_1(t)$ is:

$$X_1(f) = \frac{\pi}{4}\mathrm{rect}\left(\frac{\pi f}{4}\right) \times e^{j2\pi f}$$

The output of the system is given by:

$$\begin{aligned}
Y_1(f) &= H(f) \cdot X_1(f) \\
&= \left(\frac{\pi}{4}\mathrm{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f}\right) \times \left(\frac{\pi}{4}\mathrm{rect}\left(\frac{\pi f}{4}\right) \times e^{j2\pi f}\right) \\
&= \frac{\pi}{4}\mathrm{rect}\left(\frac{\pi f}{4}\right)
\end{aligned}$$

Taking the inverse fourier transform of the rect function, with the time scaling:

$$\mathcal{F}^{-1}\left\{\text{rect}\left(\frac{f}{a}\right)\right\} = |a|\text{sinc}(at)$$

$$\text{Where } a = 4/\pi$$

$$\implies \text{rect}\left(\frac{\pi f}{4}\right) \xrightarrow{\mathcal{F}^{-1}} \frac{4}{\pi}\text{sinc}\left(\frac{4t}{\pi}\right)$$

Thus, the output of the system is:

$$\begin{aligned} y_1(t) &= \frac{\pi}{4} \times \frac{4}{\pi}\text{sinc}\left(\frac{4t}{\pi}\right) \\ &= \boxed{\text{sinc}\left(\frac{4t}{\pi}\right)} \end{aligned}$$

2.2 We can rewrite $x_2(t)$ as follows:

$$\begin{aligned} x_2(t) &= \left(\frac{\sin(2t)}{\pi t}\right)^2 \\ &= \left(\frac{\frac{2}{\pi}\sin\left(\pi\frac{2t}{\pi}\right)}{\pi \times \frac{2t}{\pi}}\right)^2 \\ &= \frac{4}{\pi^2}\text{sinc}^2\left(\frac{2t}{\pi}\right) \\ &= \frac{4}{\pi^2}\text{sinc}\left(\frac{2t}{\pi}\right) \times \text{sinc}\left(\frac{2t}{\pi}\right) \end{aligned}$$

Since multiplication in the time domain is convolution in the frequency domain, we can find the fourier transform of $x_2(t)$ as follows:

$$\text{sinc}\left(\frac{2t}{\pi}\right) \xrightarrow{\mathcal{F}} \frac{\pi}{2}\text{rect}\left(\frac{\pi f}{2}\right)$$

The convolution of two rectangular functions of the same width is a triangle function. Thus, the fourier transform of $x_2(t)$ is:

$$\begin{aligned} X_2(f) &= \frac{4}{\pi^2}\left(\frac{\pi}{2}\text{rect}\left(\frac{\pi f}{2}\right) * \frac{\pi}{2}\text{rect}\left(\frac{\pi f}{2}\right)\right) \\ &= \text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right) \end{aligned}$$

The output in the frequency domain is:

$$\begin{aligned} Y_2(f) &= H(f) \cdot X_2(f) \\ &= \frac{\pi}{4}\left(\text{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f}\right) \times \left(\text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right)\right) \end{aligned}$$

Since the rectangle function is 1 for $|f| \leq 1/2$, mutliplying by the rectangular function on the outside is essentially mutliplying by 1. Thus, the output in the frequency domain is:

$$Y_2(f) = \left(\frac{\pi}{4}\text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right)\right) \times e^{-j2\pi f}$$

The convolution in the frequency domain is equivalent to multiplication in the time domain. Thus, the output in the time domain is:

$$\begin{aligned}
\mathcal{F}^{-1}\left(\frac{\pi}{4}\text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right)\right) &= \frac{\pi}{4}\mathcal{F}^{-1}\left(\text{rect}\left(\frac{\pi f}{2}\right)\right) \times \mathcal{F}^{-1}\left(\text{rect}\left(\frac{\pi f}{2}\right)\right) \\
&= \frac{\pi}{4}\left(\frac{2}{\pi}\text{sinc}\left(\frac{2t}{\pi}\right)\right) \times \left(\frac{2}{\pi}\text{sinc}\left(\frac{2t}{\pi}\right)\right) \\
&= \frac{1}{\pi}\left(\text{sinc}\left(\frac{2t}{\pi}\right)\right)^2
\end{aligned}$$

The multiplication by the exponential term in the frequency domain is equivalent to a time shift, $t_0 = 1$, in the time domain. Thus, the output in the time domain is:

$$y_2(t) = \boxed{\frac{1}{\pi}\left(\text{sinc}\left(\frac{2(t-1)}{\pi}\right)\right)^2}$$

3 Problem 3

3.1 We can find the fourier series coefficients as follows:

$$\begin{aligned}
g_n &= \frac{1}{T}\int_0^T g(t)e^{-j2\pi nt/T}dt \\
&= \frac{1}{2}\int_0^2 t^2 e^{-j\pi nt}dt
\end{aligned}$$

We can solve this integral by doing integration by parts twice:

$$\int t^2 e^{at}dt = \frac{t^2e^{at}}{a} - \frac{2te^{at}}{a^2} - \frac{2e^{at}}{a^3}$$

We have that $a = -j\pi n$:

$$
\begin{aligned}
2g_n &= \left\{ \frac{t^2 e^{-j\pi n t}}{-j\pi n} - \frac{2te^{-j\pi n t}}{(-j\pi n)^2} - \frac{2e^{-j\pi n t}}{(-j\pi n)^3} \right\} \Bigg|_0^2 \\
&= \left(\frac{4e^{-2j\pi n}}{-j\pi n} - \frac{4e^{-2j\pi n}}{(-j\pi n)^2} - \frac{2e^{-2j\pi n}}{(-j\pi n)^3} \right) - \left(-\frac{2}{(-j\pi n)^3} \right) \\
g_n &= \frac{2e^{-2j\pi n}}{-j\pi n} - \frac{2e^{-2j\pi n}}{(-j\pi n)^2} - \frac{e^{-2j\pi n}}{(-j\pi n)^3} + \frac{1}{(-j\pi n)^3} \\
&= \frac{j2e^{-2j\pi n}}{\pi n} + \frac{2e^{-2j\pi n}}{\pi^2 n^2} + \frac{je^{-2j\pi n}}{\pi^3 n^3} - \frac{j}{\pi^3 n^3} \\
\text{Since n is an integer, } e^{-2j\pi n} &= 1 \\
&= \frac{j2}{\pi n} + \frac{2}{\pi^2 n^2} + \frac{j}{\pi^3 n^3} - \frac{j}{\pi^3 n^3} \\
&= \frac{(\pi n)2j + 2}{\pi^2 n^2} \\
&= \frac{2(1 + j\pi n)}{\pi^2 n^2}, \quad \text{for } n \neq 0
\end{aligned}
$$

When n = 0, we have that:

$$
\begin{aligned}
g_0 &= \frac{1}{T} \int_0^T g(t) dt \\
&= \frac{1}{2} \int_0^2 t^2 dt \\
&= \frac{1}{2} \left(\frac{t^3}{3} \right) \Bigg|_0^2 \\
&= \frac{4}{3}
\end{aligned}
$$

3.2 We can equate the series at $t = 0$, to show the idenity:

$$
\begin{aligned}
g(0) &= \sum_{n=-\infty}^{\infty} g_n e^{j2\pi n 0} \\
2 &= \frac{4}{3} + \sum_{n=-\infty}^{-1} g_n + \sum_{n=1}^{\infty} g_n
\end{aligned}
$$

Since this is a real signal, we have that $g_n = g_{-n}^*$. Where, g_n is:

$$
g_n = \frac{2(1 + j\pi n)}{\pi^2 n^2} = \frac{2}{\pi^2 n^2} + \frac{2j}{\pi n}
$$

$$\begin{aligned}
2 - \frac{4}{3} &= \sum_{n=1}^{\infty} g_n + \sum_{n=1}^{\infty} g_{-n}^* \\
\frac{2}{3} &= \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} + \frac{2j}{\pi n} \right) + \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} - \frac{2j}{\pi n} \right) \\
\frac{2}{3} &= 2 \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \\
\frac{1}{3} &= \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

And thus, the identity is proven.

3.3 We now equate at another point, $t = 1$:

$$g(1) = 1 = \sum_{n=-\infty}^{\infty} g_n e^{j\pi n}$$

We recognize that the exponential can be simplified:

$$\begin{aligned}
e^{j\pi n} &= \cos(\pi n) + j \sin(\pi n) \\
&= (-1)^n
\end{aligned}$$

Thus, the sum is:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} g_n (-1)^n &= 1 \\
\frac{4}{3} + \sum_{n=1}^{\infty} g_n (-1)^n + \sum_{n=1}^{\infty} g_{-n}^* (-1)^n &= 1 \\
\frac{4}{3} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} + \frac{2j}{\pi n} \right) (-1)^n + \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} - \frac{2j}{\pi n} \right) (-1)^n &= 1 \\
\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} (-1)^n &= 1 - \frac{4}{3} \\
\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2} &= -\frac{1}{12} \\
\sum_{n} = 1^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}
\end{aligned}$$

And thus, the identity is proven.

4 Problem 4

The fourier transform provided is an ampltiude modulated signal with a carrier frequency of $f = 4$. The fourier transform of the base signal is:

$$X(f) = 2\Lambda\left(\frac{f}{2}\right)$$

We know that the inverse transform of a triangular function is a sinc^2 function. That is:

$$\Lambda(f) \xrightarrow{\mathcal{F}^{-1}} \text{sinc}^2(t)$$

Applying the proper scaling:

$$2\Lambda\left(\frac{f}{2}\right) \xrightarrow{\mathcal{F}^{-1}} 4\text{sinc}^2(2t)$$

This signal is modulated by a cosine function and the plotted fourier transform has two peaks, centered at 4 and -4. Therefore, we have that the signal is:

$$x(t) = 4\text{sinc}^2(2t) \times \cos(2\pi(4)t)$$

5 Problem 5

5.1 We can show $g(t)$ is periodic with period T as follows:

$$g(t) = \sum_{k=-\infty}^{\infty} x(t - kT) \stackrel{?}{=} g(t+T)$$

$$\begin{aligned}
g(t+T) &= \sum_{k=-\infty}^{\infty} x(t + T - kT) \\
&= \sum_{k=-\infty}^{\infty} x(t + (1-k)T) \\
\text{Let } j &= 1 - k \\
&= \sum_{j=-\infty}^{\infty} x(t + jT) \\
&= g(t)
\end{aligned}$$

5.2 Let $x(t) = t^2$, then the plot of $g(t)$ over the interval $[-2T - \frac{\tau}{2}, 2T + \frac{\tau}{2}]$ is shown in Figure 2.

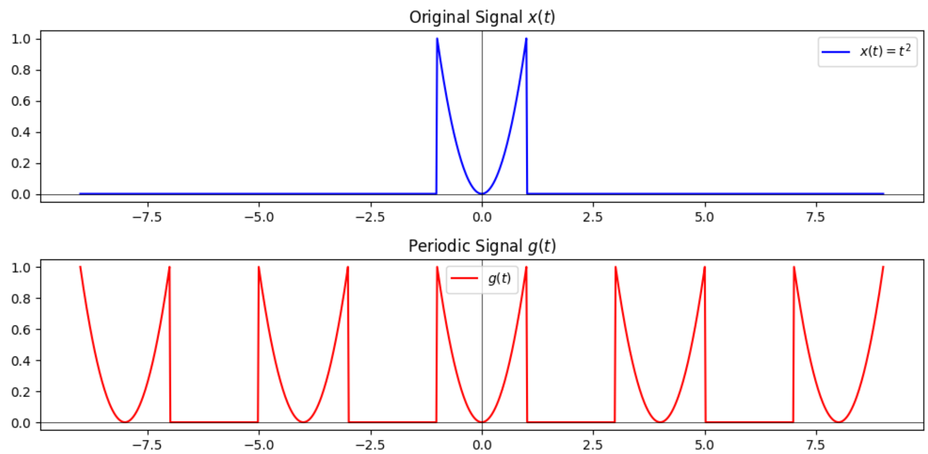

Figure 2: Plot of $x(t)$ and $g(t)$, The selected parameters are $T = 4$ and $\tau = 2$

5.3 The fourier series coefficients of $g(t)$ are:

$$g_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)e^{-j2\pi nt/T}dt$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} x(t-kT)e^{-j2\pi nt/T}dt$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\tau/2}^{\tau/2} x(t-kT)e^{-j2\pi nt/T}dt, \quad \text{Let } u = t-kT$$

The bounds can be set to $-\tau/2, \tau/2$ as $x(t)$ is 0 outside of this range

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{\tau/2}^{-\tau/2} x(u)e^{-j2\pi n(u+kT)/T}du$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{-j2\pi nk}\int_{\tau/2}^{-\tau/2} x(u)e^{-j2\pi nu/T}du$$

We know that for any integer $n, k, e^{-j2\pi nk} = 1$

$$= \frac{1}{T}\int_{\tau/2}^{-\tau/2} x(u)e^{-j2\pi(\frac{n}{T})u}du$$
$$= \frac{1}{T}X\left(\frac{n}{T}\right)$$

6 Problem 6

6.1 Since $h(t)$ is an odd function, the fourier transform will be purely imaginary, and will have a phase of $\pm\frac{\pi}{2}$.

6.2 Evaluating the following integral:

$$\begin{aligned}\int_{-\infty}^{\infty} G(f)\cos(\pi f)df &= \int_{-\infty}^{\infty} G(f)\left(\frac{e^{j\pi f}+e^{-j\pi f}}{2}\right)df \\ &= \frac{1}{2}\int_{-\infty}^{\infty} G(f)e^{j\pi f}df + \frac{1}{2}\int_{-\infty}^{\infty} G(f)e^{-j\pi f}df \\ &= \frac{1}{2}\int_{-\infty}^{\infty} G(f)e^{j2\pi f(\frac{1}{2})}df + \frac{1}{2}\int_{-\infty}^{\infty} G(f)e^{j2\pi f(-\frac{1}{2})}df \\ &= \frac{1}{2}\left(g\left(\frac{1}{2}\right)+g\left(-\frac{1}{2}\right)\right) \\ &= \frac{1}{4}\end{aligned}$$

6.3 Evaluating the following integral:

$$\begin{aligned}\int_{-\infty}^{\infty} H(f)e^{j4\pi f}df &= \int_{-\infty}^{\infty} H(f)e^{j2\pi f(2)}df \\ &= h(2) \\ &= 0\end{aligned}$$

6.4 The plot of the odd and even parts of $g(x)$ are shown in Figure 3.

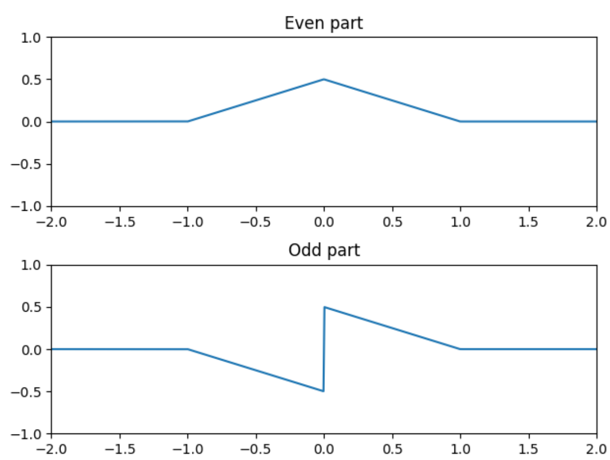

Figure 3: Plot of the odd and even parts of $g(x)$

6.5 The real part of $G(f)$ is fourier transform of the even part of $g(t)$:

$$\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} (g_e(t) + g_o(t))e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} g_e(t)e^{-j2\pi ft}dt + \int_{-\infty}^{\infty} g_o(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} g_e(t)cos(2\pi ft)dt + j\int_{-\infty}^{\infty} g_o(t)sin(2\pi ft)dt \\
&= \mathcal{F}\{g_e(t)\} + j\mathcal{F}\{g_o(t)\}
\end{aligned}$$

In Figure 3 we see that the even part of $g(t)$ is a triangle function, which has a fourier transform of a sinc^2 function. Which is:

$$g_e(t) = \frac{1}{2}\Lambda(t) \implies \text{Re}(G(f)) = \frac{1}{2}\text{sinc}^2(f)$$

6.6 We can find the fourier transform of $g(t)$ as follows:

$$\begin{aligned}
G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt &= \int_0^1 (1-t)e^{-j2\pi ft}dt \\
&= \int_0^1 e^{-j2\pi ft}dt - \int_0^1 te^{-j2\pi ft}dt \\
&= \left.\frac{e^{-j2\pi ft}}{-j2\pi f}\right|_0^1 - \left(\left.\frac{te^{-j2\pi ft}}{-j2\pi f}\right|_0^1 - \int_0^1 \frac{e^{-j2\pi ft}}{-j2\pi f}dt\right) \\
&= \frac{1-e^{-j2\pi f}}{j2\pi f} - \left(\frac{e^{-j2\pi f}}{-j2\pi f} - \left.\left(\frac{e^{-j2\pi ft}}{(-j2\pi f)^2}\right)\right|_0^1\right) \\
&= \frac{1}{j2\pi f} - \frac{e^{-j2\pi f}}{j2\pi f} + \frac{e^{-j2\pi f}}{j2\pi f} + \frac{e^{-j2\pi f}-1}{(-j2\pi f)^2} \\
&= \frac{-j}{2\pi f} + \frac{1-\cos(2\pi f) + j\sin(2\pi f)}{(2\pi f)^2} \\
\text{Using the idenity: } 1-\cos(2x) &= 2\sin^2(x) \\
&= \frac{-j}{2\pi f} + \frac{2\sin^2(\pi f) + j\sin(2\pi f)}{(2\pi f)^2} \\
&= \frac{2\sin^2(\pi f)}{4(\pi f)^2} + j\left(\frac{\sin(2\pi f)}{(2\pi f)^2} - \frac{1}{2\pi f}\right) \\
&= \frac{\text{sinc}(f)^2}{2} + j\left(\frac{\text{sinc}(2f)}{2\pi f} - \frac{1}{2\pi f}\right) \\
&= \boxed{\frac{\text{sinc}(f)^2}{2} + j\left(\frac{\text{sinc}(2f)-1}{2\pi f}\right)}
\end{aligned}$$

6.7 We can see that $h(t) = 2g_o(t)$, and thus the fourier transform of $h(t)$ is:

$$H(f) = j2\text{Im}(G(f)) = j\frac{\text{sinc}(2f)-1}{\pi f}$$

6.8 Given that $\varphi(x)$ is the periodized version of $h(x)$ with period $T = 2$, we can use the result in **5.3** to find the fourier series coefficients φ_n:

$$\begin{aligned}\varphi_n = \frac{1}{T}H\left(\frac{n}{T}\right) = \frac{1}{2}H\left(\frac{n}{2}\right) &= \frac{j}{2}\frac{\operatorname{sinc}(n)-1}{\pi\frac{n}{2}} \\ &= j\left(\frac{\operatorname{sinc}(n)-1}{n\pi}\right)\end{aligned}$$