Assignment 4

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1 Problem 1

1.1 Part I

1.1 Show the following is true:

$$\int_0^T g(x)\,dx = \int_{-T/2}^{T/2} g(x)\,dx$$

Using the fact that $g(x) = g(x+T)$, we can write the integral as:

$$\begin{aligned}\int_0^T g(x)\,dx &= \int_0^{T/2} g(x)\,dx + \int_{T/2}^{T} g(x)\,dx \\ &= \int_0^{T/2} g(x)\,dx + \int_{T/2}^{T} g(x-T)\,dx \\ &= \int_0^{T/2} g(x)\,dx + \int_{-T/2}^{0} g(x)\,dx \\ &= \int_{-T/2}^{T/2} g(x)\,dx\end{aligned}$$

1.2 We can deduce that $J_n(y)$ is also given by:

$$J_n(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(y\sin(x)-nx)}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{j(y\sin(x)-nx)}\,dx$$

We can show this with the result from 1.1, as the function we are integrating over is 2π periodic. That is:

$$\begin{aligned}f(x,y) &= e^{j(y\sin(x)-nx)} \\ f(x+2\pi,y) &= e^{j(y\sin(x+2\pi)-n(x+2\pi))} = e^{j(y\sin(x)-nx)} = f(x,y)\end{aligned}$$

Thus, the integral goes to:

$$\begin{aligned}J_n(y) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(y\sin(x)-nx)}\,dx \\ &= \frac{1}{2\pi}\int_{-T/2}^{T/2} f(x,y)\,dx \\ &= \frac{1}{2\pi}\int_0^{T} f(x,y)\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{j(y\sin(x)-nx)}\,dx\end{aligned}$$

1.3 We can deduce from (6) the following:

$$J_{-n}(y) = (-1)^n J_n(y)$$

We start with the definition of $J_{-n}(y)$:

$$\begin{aligned}
J_{-n}(y) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(y\sin(x)+nx)}\,dx, \quad \text{Let } t = x+\pi \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{j(y\sin(t-\pi)+n(t-\pi))}\,dt \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{j(-y\sin(t)+nt)}e^{-j\pi n}\,dt \\
&= \frac{(-1)^n}{2\pi}\int_{0}^{2\pi} e^{-j(y\sin(t)-nt)}\,dt, \quad \text{Let } u = -t+2\pi \\
&= \frac{(-1)^n}{2\pi}\int_{0}^{2\pi} e^{-j(y\sin(2\pi-u)-n(2\pi-u))}\,du \\
&= \frac{(-1)^n}{2\pi}\int_{0}^{2\pi} e^{-j(-y\sin(u)+nu)}e^{jn2\pi}\,du \\
&= \frac{(-1)^n}{2\pi}\int_{0}^{2\pi} e^{j(y\sin(u)-nu)}\,du = (-1)^n J_n(y)
\end{aligned}$$

1.4 We can show that $J_n(y)$ is real-valued for all $y \in \mathbb{R}$:

$$\begin{aligned}
J_n(y)^* &= \left(\frac{1}{2\pi}\int_{0}^{2\pi} e^{j(y\sin(x)-nx)}\,dx\right)^* \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{-j(y\sin(x)-nx)}\,dx \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{j(y\sin(-x)-n(-x))}\,dx, \quad \text{Let } u = -x+2\pi \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{j(y\sin(u+2\pi)-n(u+2\pi))}\,du \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} e^{j(y\sin(u)-nu)}\,du = J_n(y)
\end{aligned}$$

Since the complex conjugate of $J_n(y)$ is equal to $J_n(y)$, $J_n(y)$ is real-valued for all $y \in \mathbb{R}$.

1.5 We can find the deviation ratio as follows. We are given the equations for the frequency deviation, β

$$\beta = \frac{\Delta f}{B}$$
$$\text{Where } \Delta f = \frac{k_f m_p}{2\pi}$$

We have our message signal, $m(t)$ as:

$$m(t) = \alpha\cos(\omega_m t)$$

Which has a maxmimum value of α, and a bandwidth of $\omega_m/2\pi$. We can find the deviation ratio, β as:

$$\beta = \frac{\Delta f}{B} = \frac{k_f\alpha}{2\pi\frac{\omega_m}{2\pi}} = \frac{k_f\alpha}{\omega_m}$$

1.6 We can show the FM-modulated signal corresponding to $m(t)$ is given by:

$$x_{FM}(t) = A\cos\left(\omega_c t + k_f a(t)\right)$$

We assume initially that $a(-\infty) = 0$, and we can find $a(t)$ as:

$$\begin{aligned} a(t) &= \int_{-\infty}^{t} m(\tau)\,d\tau = \int_{-\infty}^{t} \alpha\cos(\omega_m \tau)\,d\tau \\ &= \frac{\alpha}{\omega_m}\sin(\omega_m t) \end{aligned}$$

We also recognize that:

$$k_f a(t) = \frac{\alpha k_f}{\omega_m}\sin(\omega_m t) = \beta\sin(\omega_m t)$$

Thus, we have:

$$x_{FM}(t) = A\cos\left(\omega_c t + \beta\sin(\omega_m t)\right)$$

We know that $\cos(\theta)$ is the real part of $e^{j\theta}$, and we can write the FM-modulated signal as:

$$\begin{aligned} x_{FM}(t) &= \Re\left\{Ae^{j(\omega_c t + \beta\sin(\omega_m t))}\right\} \\ &= A\,\Re\left\{e^{j\omega_c t}e^{j\beta\sin(\omega_m t)}\right\} \\ &= A\,\Re\left\{e^{j\omega_c t}z(t)\right\} \end{aligned}$$

Where $z(t) = e^{j\beta\sin(\omega_m t)}$.

1.7 We can show that the signal $z(t)$ is periodic with period $T = 2\pi/\omega_m$:

$$\begin{aligned} z(t + 2\pi/\omega_m) &= e^{j\beta\sin(\omega_m(t+2\pi/\omega_m))} \\ &= e^{j\beta\sin(\omega_m t + 2\pi)} \\ &= e^{j\beta\sin(\omega_m t)} \\ &= z(t) \end{aligned}$$

Thus, $z(t)$ is periodic with period $T = 2\pi/\omega_m$.

1.8 We can show the relationship between the fourier series coefficients of $z(t)$ and $J_n(\beta)$ as follows. We start with the definition of the fourier series coefficients of $z(t)$:

$$\begin{aligned} z_n &= \frac{\omega_m}{2\pi}\int_{\pi/\omega_m}^{\pi/\omega_m} z(t)e^{-jn\omega_m t}\,dt \\ &= \frac{\omega_m}{2\pi}\int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta\sin(\omega_m t)}e^{-jn\omega_m t}\,dt \\ &= \frac{\omega_m}{2\pi}\int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j(\beta\sin(\omega_m t) - n\omega_m t)}\,dt, \quad \text{Let } x = \omega_m t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin(x) - nx)}\,dx = J_n(\beta) \end{aligned}$$

1.9 We can show the FM-modulated signal $x_{FM}(t)$ can be written as:

$$x_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left(2\pi \left[f_c + nf_m\right] t\right)$$

Using the results from 1.6 and 1.7, we have:

$$\begin{aligned}
x_{FM}(t) &= A\,\Re\left\{e^{j\omega_c t} z(t)\right\} \\
&= A\,\Re\left\{e^{j\omega_c t} \sum_{n=-\infty}^{\infty} z_n e^{jn\omega_m t}\right\} \\
&= A\,\Re\left\{\sum_{n=-\infty}^{\infty} z_n e^{j(\omega_c + n\omega_m)t}\right\} \\
&= A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left(\omega_c t + n\omega_m t\right) \\
&= A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left(2\pi \left[f_c + nf_m\right] t\right)
\end{aligned}$$

1.10 Plugging in the provided values into $x_{FM}(t)$, we have:

$$\beta = \frac{k_f \alpha}{\omega_m} = \frac{2\pi \times 10^5 \cdot 6}{2\pi \times 300 \times 10^3} = 2$$

$$x_{FM}(t) = 2 \sum_{n=-\infty}^{\infty} J_n(2) \cos\left(2\pi \left[f_c + n \cdot 300 \times 10^3\right] t\right)$$

The fourier transform of $x_{FM}(t)$ is given by:

$$X_{FM}(f) = \sum_{n=-\infty}^{\infty} J_n(2) \left(\delta(f - (f_c + n \cdot 300 \times 10^3)) + \delta(f + (f_c + n \cdot 300 \times 10^3))\right)$$

Assuming that $J_n(2)$ is negligible for $n > 3$, we can plot from $n = -3$ to $n = 3$. We will also utilize the fact that $J_{-n}(y) = (-1)^n J_n(y)$.

$$\begin{aligned}
&n = -3: \quad -J_3(2) \left(\delta(f + (f_c - 900 \times 10^3)) + \delta(f - (f_c - 900 \times 10^3))\right) \\
&n = -2: \quad J_2(2) \left(\delta(f + (f_c - 600 \times 10^3)) + \delta(f - (f_c + 600 \times 10^3))\right) \\
&n = -1: \quad -J_1(2) \left(\delta(f + (f_c - 300 \times 10^3)) + \delta(f - (f_c - 300 \times 10^3))\right) \\
&n = 0: \quad J_0(2) \left(\delta(f + f_c) + \delta(f - f_c)\right) \\
&n = 1: \quad J_1(2) \left(\delta(f + (f_c + 300 \times 10^3)) + \delta(f - (f_c + 300 \times 10^3))\right) \\
&n = 2: \quad J_2(2) \left(\delta(f + (f_c + 600 \times 10^3)) + \delta(f - (f_c + 600 \times 10^3))\right) \\
&n = 3: \quad J_3(2) \left(\delta(f + (f_c + 900 \times 10^3)) + \delta(f - (f_c + 900 \times 10^3))\right)
\end{aligned}$$

Plotting the magnitude, $|X_{FM}(f)|$, we get the following result seen in Figure 1.

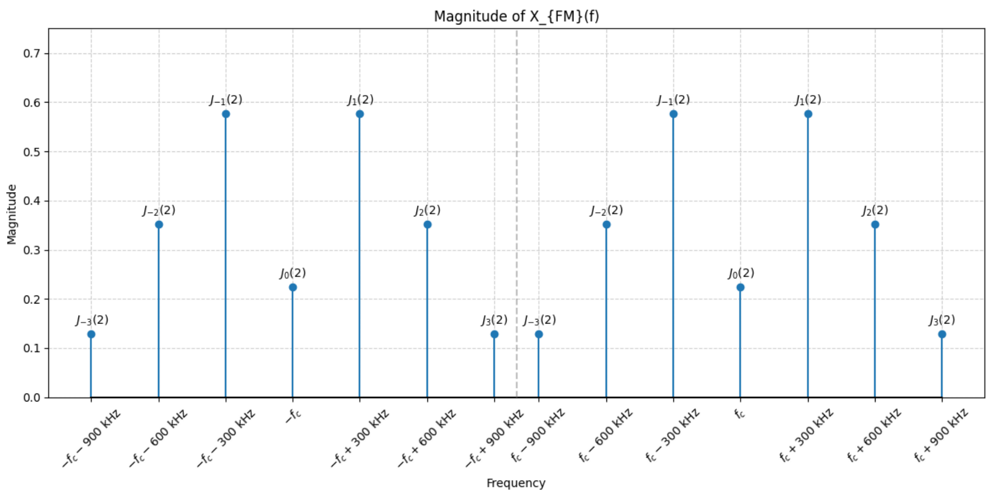

Figure 1: Fourier Transform of $x_{FM}(t)$

2 Problem 2

2.1 Determine and sketch $G_+(f)$

Since $g_+(t)$ is defined as:

$$g_+(t) = g(t) + j\hat{g}(t), \quad \text{where } \hat{g}(t) = \frac{1}{\pi t} * g(t)$$

The fourier transform of $g_+(t)$ is given by:

$$\begin{aligned}\mathcal{F}\{g_+(t)\} &= \mathcal{F}\{g(t)\} + j\mathcal{F}\{\hat{g}(t)\} \\ &= G(f) + jG(f)H(f)\end{aligned}$$

Where $H(f)$ is the fourier transform of $\frac{1}{\pi t}$, which is given by:

$$H(f) = \frac{1}{j}\text{sgn}(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$

Therefore we have:

$$\begin{aligned}G_+(f) &= G(f) + jG(f)H(f) = G(f) + \text{sgn}(f)G(f) \\ &= \begin{cases} 2G(f) & f > 0 \\ 0 & f < 0 \end{cases}\end{aligned}$$

The plot of $G_+(f)$ can be seen in Figure 2.

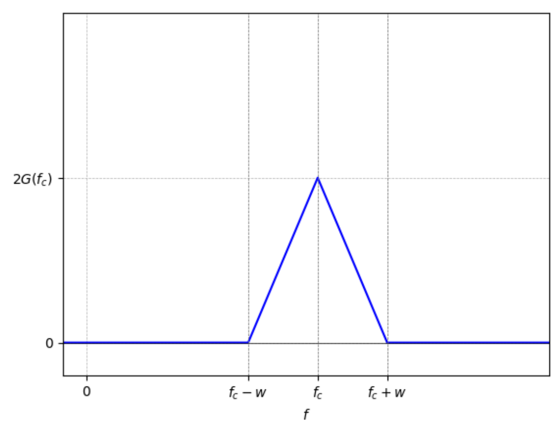

Figure 2: Plot of $G_+(f)$

2.2 Given that $\tilde{g}(t) = g_+(t)e^{-j2\pi f_c t}$, the fourier transform of $\tilde{g}(t)$ is given by:

$$\begin{aligned}\text{Using: } \mathcal{F}\{g(t)e^{j2\pi f_0 t}\} &= G(f - f_0) \\ \mathcal{F}\{\tilde{g}(t)\} &= G_+(f + f_c)\end{aligned}$$

The plot of $\tilde{G}(f)$ can be seen in Figure 3. It is just a shifted version of $G_+(f)$, with the triangular pulse centered at $f = 0$.

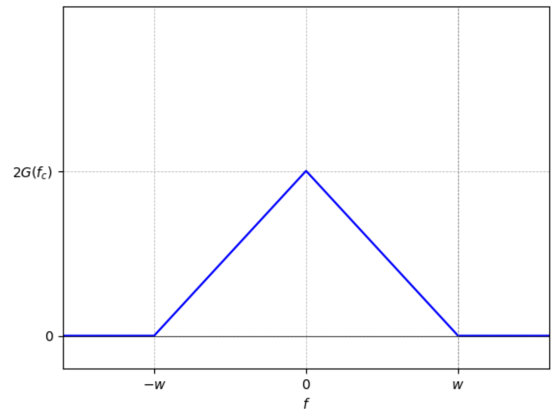

Figure 3: Plot of $\tilde{G}(f)$

2.3 We can show that:

$$g(t) = \Re\left\{\tilde{g}(t)e^{j2\pi f_c t}\right\}$$

We expand the right side of the equation:

$$\begin{aligned}\Re\left\{\tilde{g}(t)e^{j2\pi f_c t}\right\} &= \Re\left\{g_+(t)e^{-j2\pi f_c t}e^{j2\pi f_c t}\right\} \\ &= \Re\left\{g_+(t)\right\} \\ &= \Re\left\{g(t) + j\hat{g}(t)\right\} = g(t)\end{aligned}$$

Thus, we have shown that $g(t) = \Re\left\{\tilde{g}(t)e^{j2\pi f_c t}\right\}$.

2.4 We can find the complex envelopes $\tilde{v}(t), \tilde{h}(t)$ as follows.

Using the result from 2.3 we can write $v(t)$ and find the envelope $\tilde{v}(t)$:

$$\begin{aligned}v(t) = g(t)s(t) = \Re\left\{g(t)\left(e^{j(2\pi f_c t - \pi k t^2)}\right)\right\} = \Re\left\{g(t)e^{j2\pi f_c t}e^{-j\pi k t^2}\right\} = \Re\left\{\tilde{v}(t)e^{j2\pi f_c t}\right\} \\ \implies \tilde{v}(t) = g(t)e^{-j\pi k t^2}\end{aligned}$$

Similarily for $h(t)$:

$$\begin{aligned}h(t) = \cos(2\pi f_c t + \pi k t^2) = \Re\left\{e^{j(2\pi f_c t + \pi k t^2)}\right\} = \Re\left\{e^{j2\pi f_c t}e^{j\pi k t^2}\right\} = \Re\left\{\tilde{h}(t)e^{j2\pi f_c t}\right\} \\ \implies \tilde{h}(t) = e^{j\pi k t^2}\end{aligned}$$

2.5 We can find the complex envelope $\tilde{z}(t)$ as follows:

$$\begin{aligned}\tilde{z}(t) = \tilde{v}(t) * \tilde{h}(t) &= g(t)e^{-j\pi kt^2} * e^{j\pi kt^2} \\ &= \int_{-\infty}^{\infty} g(\tau)e^{-j\pi k\tau^2} e^{j\pi k(t-\tau)^2}\, d\tau \\ &= \int_{-\infty}^{\infty} g(\tau)e^{-j\pi k\tau^2} e^{j\pi kt^2} e^{-j2\pi kt\tau} e^{j\pi k\tau^2}\, d\tau \\ &= e^{j\pi kt^2} \int_{-\infty}^{\infty} g(\tau)e^{-j2\pi(kt)\tau}\, d\tau \\ &= e^{j\pi kt^2} G(kt)\end{aligned}$$

3 Problem 3

3.1 We can show the relationship between the fourier coefficents of $g(t)$ and $\dot{g}(t)$ as follows. The relationship is given by:

$$\dot{g}_n = jn\omega_0 g_n$$

We start with the definition of $\dot{g}(t)$:

$$\begin{aligned}\dot{g}(t) = \frac{d}{dt}g(t) &= \frac{d}{dt}\sum_{n=-\infty}^{\infty} g_n e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} jn\omega_0 g_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \dot{g}_n e^{jn\omega_0 t} \\ \implies jn\omega_0 g_n &= \dot{g}_n\end{aligned}$$

3.2 Given that $m(t)$ are triangular pulses, with the time period of 2×10^{-4} seconds, the derivative, $\dot{m}(t)$, will be square pulses, with the same time period. We can find the fourier coefficents of $\dot{m}(t)$ and then relate them to the fourier coefficents of $m(t)$ using the relationship from 3.1.

We have that $m(t)$ is given by:

$$\begin{aligned}m(t) &= \begin{cases} 2 \cdot 10^4 t + 1 & -10^{-4} \le t < 0 \\ -2 \cdot 10^4 t + 1 & 0 \le t < 10^{-4} \end{cases} \\ m(t) &= m(t + 2 \times 10^{-4})\end{aligned}$$

Therefore, we can find $\dot{m}(t)$ as:

$$\begin{aligned}\dot{m}(t) &= \begin{cases} 2 \cdot 10^4 & -10^{-4} \le t < 0 \\ -2 \cdot 10^4 & 0 \le t < 10^{-4} \end{cases} \\ \dot{m}(t) &= \dot{m}(t + 2 \times 10^{-4})\end{aligned}$$

The fourier series of $\dot{m}(t)$ is given by:

$$\begin{aligned}
\dot{m}_n &= \frac{1}{T}\int_T \dot{m}(t)e^{-j2\pi\frac{n}{T}t}\,dt \\
&= \frac{1}{2\times 10^{-4}}\int_{-10^{-4}}^{10^{-4}} \dot{m}(t)e^{-j2\pi\frac{n}{2\times 10^{-4}}t}\,dt \\
&= \frac{1}{2\times 10^{-4}}\int_{-10^{-4}}^{0} 2\cdot 10^4 e^{-j\pi n 10^4 t}\,dt + \frac{1}{2\times 10^{-4}}\int_{0}^{10^{-4}} -2\cdot 10^4 e^{-j\pi n 10^4 t}\,dt
\end{aligned}$$

The first integral evaluates to:

$$\begin{aligned}
2\times 10^4\int_{-10^{-4}}^{0} e^{-j\pi n 10^4 t}\,dt &= \frac{2\times 10^4}{-j\pi n 10^4}\left(e^{-j\pi n 10^4 t}\right)\Big|_{-10^{-4}}^{0} \\
&= \frac{-2}{j\pi n}\left(1-e^{j\pi n}\right) \\
&= \frac{2\cdot(-1)^n-2}{j\pi n}
\end{aligned}$$

The second integral evaluates to:

$$\begin{aligned}
-2\times 10^4\int_{0}^{10^{-4}} e^{-j\pi n 10^4 t}\,dt &= \frac{-2\times 10^4}{-j\pi n 10^4}\left(e^{-j\pi n 10^4 t}\right)\Big|_{0}^{10^{-4}} \\
&= \frac{2}{j\pi n}\left(e^{-j\pi n}-1\right) \\
&= \frac{2\cdot(-1)^n-2}{j\pi n}
\end{aligned}$$

Putting it all together, we have:

$$\begin{aligned}
\dot{m}_n &= \frac{1}{2\times 10^{-4}}\left(\frac{2\cdot(-1)^n-2}{j\pi n}+\frac{2\cdot(-1)^n-2}{j\pi n}\right) = \frac{4\cdot(-1)^n-4}{j\pi n}\cdot\frac{1}{2\times 10^{-4}} \\
&= \frac{2\cdot(-1)^n-2}{j\pi n}\cdot 10^4
\end{aligned}$$

Using the relationship from 3.1, we have:

$$\begin{aligned}
m_n &= \frac{\dot{m}_n}{jn\omega_0}, \quad \text{Where } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\times 10^{-4}} = 10^4\pi \\
m_n &= \frac{1}{jn\omega_0}\cdot\frac{2\cdot(-1)^n-2}{j\pi n}\cdot 10^4 = \frac{2-2\cdot(-1)^n}{\pi^2 n^2} \\
m_n &= \begin{cases} \frac{4}{\pi^2 n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}$$

For $n=0$, we have:

$$\dot{m}_0 = \frac{1}{2\times 10^{-4}}\int_{-10^{-4}}^{10^{-4}} \dot{m}(t)\,dt = 0$$

This is $\dot{m}(t)$ is an odd function, and the integral of an odd function over a symmetric interval is 0.

3.3 We can find the power, P_m, of $m(t)$ using integration as follows:

$$P_m = \frac{1}{T}\int_T |m(t)|^2\,dt = \frac{1}{2\times 10^{-4}}\left(\int_{-10^{-4}}^{0} (2\cdot 10^4 t+1)^2\,dt + \int_0^{10^{-4}} (-2\cdot 10^4 t+1)^2\,dt\right)$$

The first integral evaluates to:

$$\begin{aligned}
\frac{1}{2\times 10^{-4}}\int_{-10^{-4}}^{0} (2\cdot 10^4 t+1)^2\,dt &= \frac{1}{2\times 10^{-4}}\int_{-10^{-4}}^{0} (4\cdot 10^8 t^2 + 4\cdot 10^4 t + 1)\,dt \\
&= \frac{1}{2\times 10^{-4}}\left(\frac{4\cdot 10^8}{3}t^3 + 2\cdot 10^4 t^2 + t\right)\Big|_{-10^{-4}}^{0} \\
&= \frac{1}{2\times 10^{-4}}\left(\frac{4\cdot 10^8}{3}10^{-12} - 2\cdot 10^4\cdot 10^{-8} + 10^{-4}\right) \\
&= \frac{1}{2\times 10^{-4}}\left(\frac{4}{3} - 2 + 1\right)\times 10^{-4} \\
&= \frac{1}{6}
\end{aligned}$$

Similarily, the second integral evaluates to:

$$\begin{aligned}
\frac{1}{2\times 10^{-4}}\int_{0}^{10^{-4}} (-2\cdot 10^4 t+1)^2\,dt &= \frac{1}{2\times 10^{-4}}\int_{0}^{10^{-4}} (4\cdot 10^8 t^2 - 4\cdot 10^4 t + 1)\,dt \\
&= \frac{1}{2\times 10^{-4}}\left(\frac{4\cdot 10^8}{3}t^3 - 2\cdot 10^4 t^2 + t\right)\Big|_{0}^{10^{-4}} \\
&= \frac{1}{2\times 10^{-4}}\left(\frac{4\cdot 10^8}{3}10^{-12} - 2\cdot 10^4\cdot 10^{-8} + 10^{-4}\right) \\
&= \frac{1}{2\times 10^{-4}}\left(\frac{4}{3} - 2 + 1\right)\times 10^{-4} \\
&= \frac{1}{6}
\end{aligned}$$

Therefore we have:

$$P_m = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

3.4 Using Parseval's theorem, we can find the power of $m(t)$ using the fourier coefficients as follows:

$$\begin{aligned}
P_m = \sum_{n=-\infty}^{\infty} |m_n|^2 = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left|\frac{4}{\pi^2 n^2}\right|^2 &= \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{16}{\pi^4 n^4} \\
&= 2\times\left(\frac{16}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\right) \\
&= \frac{32}{\pi^4}\times\frac{\pi^4}{96} \\
&= \frac{1}{3}
\end{aligned}$$

3.5 We first find the bandwidth associated with the frequency-modulated signal $x_{FM}(t)$. We have that the instantaneous frequency deviation is:

$$f_i(t) = f_c + \frac{k_f}{2\pi} m(t)$$

From this, we can find the min and max, $f_{\min}, f_{\max}$:

$$\begin{aligned} f_{\min} &= f_c + \frac{2\pi \times 10^5}{2\pi} \cdot (-1) = f_c - 10^5 \\ f_{\max} &= f_c + \frac{2\pi \times 10^5}{2\pi} \cdot (1) = f_c + 10^5 \\ B_{\text{FM}} &= f_{\max} - f_{\min} = 2 \times 10^5 = 200 \text{ kHz} \end{aligned}$$

Similarily, we can find the bandwidth associated with the phase-modulated signal $x_{PM}(t)$. We have that the instantaneous phase deviation is:

$$f_i(t) = f_c + \frac{k_p}{2\pi} \dot{m}(t)$$

From this, we can find the min and max, $f_{\min}, f_{\max}$:

$$\begin{aligned} f_{\min} &= f_c + \frac{5\pi}{2\pi} \cdot (-2 \times 10^4) = f_c - 5 \times 10^4 \\ f_{\max} &= f_c + \frac{5\pi}{2\pi} \cdot (2 \times 10^4) = f_c + 5 \times 10^4 \\ B_{\text{PM}} &= f_{\max} - f_{\min} = 10^5 = 100 \text{ kHz} \end{aligned}$$

3.6 If we double the amplitude of the message signal, the bandwidth of the frequency-modulated signal and phase-modulated signal will also double. Let $m'(t) = 2m(t)$, then we have:

$$\begin{aligned} f_{\min} &= f_c + \frac{2\pi \times 10^5}{2\pi} \cdot (-2) = f_c - 2 \times 10^5 \\ f_{\max} &= f_c + \frac{2\pi \times 10^5}{2\pi} \cdot (2) = f_c + 2 \times 10^5 \\ B'_{\text{FM}} &= f_{\max} - f_{\min} = 4 \times 10^5 = 400 \text{ kHz} = 2 \times 200 \text{ kHz} = 2B_{\text{FM}} \end{aligned}$$

Similarily for the phase-modulated signal:

$$\begin{aligned} f_{\min} &= f_c + \frac{5\pi}{2\pi} \cdot (-4 \times 10^4) = f_c - 10^5 \\ f_{\max} &= f_c + \frac{5\pi}{2\pi} \cdot (4 \times 10^4) = f_c + 10^5 \\ B'_{\text{PM}} &= f_{\max} - f_{\min} = 2 \times 10^5 = 200 \text{ kHz} = 2 \times 100 \text{ kHz} = 2B_{\text{PM}} \end{aligned}$$