Assignment 5

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ECE 4260 A01

April 8, 2025

1 Problem 1

1.1 We can find the expectation of $f_U(u)$ as follows:

$$\mathbb{E}\left\{e^{-j2\pi fU}\right\}$$

We have that $g(x) = e^{-j2\pi fx}$ so we use the property:

$$\begin{aligned}\mathbb{E}\left\{g(X)\right\} &= \int_{-\infty}^{\infty} e^{-j2\pi fx} f_U(u)du \\ &= F_U(f)\end{aligned}$$

1.2 We know that $f_{Z_n}(z_n)$ is the sum of the n independent random variables $X_1, X_2, \ldots, X_n$, thus the PDF will be a convolution of the PDFs of the individual random variables:

$$\begin{aligned}Z_2 &= X_1 + X_2 \\ f_{Z_2}(z_2) &= f_{X_1}(x_1) * f_{X_2}(x_2)\end{aligned}$$

Now we take the expectation similar to 1.1:

$$\begin{aligned}\mathbb{E}\left\{e^{-j2\pi fZ_2}\right\} &= \int_{-\infty}^{\infty} e^{-j2\pi fz} f_{Z_2}(z)dz \\ &= F_{Z_2}(f)\end{aligned}$$

Now applying the property of the convolution, and the fact they are both indentically distributed:

$$\begin{aligned}F_{Z_2}(f) &= F_{X_1}(f)F_{X_2}(f) \\ &= F_X(f)F_X(f) \\ &= F_X(f)^2\end{aligned}$$

1.3 We can deduce the following:

$$F_{Z_n}(f) = F_X(f)^n$$

Using mathematical induction, we can show that this is true for all n. First we have our base case, $n = 1$:

$$\begin{aligned}F_{Z_1}(f) = \mathbb{E}\left\{e^{-j2\pi fZ_1}\right\} &= \int_{-\infty}^{\infty} e^{-j2\pi fz} f_{Z_1}(z)dz \\ &= \int_{-\infty}^{\infty} e^{-j2\pi fz} f_X(x)dx \\ &= F_X(f)\end{aligned}$$

We also showcased the case for $n = 2$ in the previous part. Now we assume that this is true for $n = k$, that is:

$$F_{Z_k}(f) = F_X(f)^k$$

Now we show that this is true for $n = k + 1$. We know that we can represent Z_{k+1} as:

$$\begin{aligned} Z_{k+1} &= Z_k + X_n \\ f_{Z_{k+1}}(z) &= f_{Z_k}(z) * f_{X_n}(x) \end{aligned}$$

Taking the expectation of Z_{k+1}:

$$\begin{aligned} \mathbb{E}\left\{e^{-j2\pi f Z_{k+1}}\right\} &= \int_{-\infty}^{\infty} e^{-j2\pi f z} f_{Z_{k+1}}(z) dz \\ &= F_{Z_{k+1}}(f) \end{aligned}$$

Using the convolution property:

$$\begin{aligned} F_{Z_{k+1}}(f) &= F_{Z_k}(f) F_{X_n}(f) \\ &= F_X(f)^k F_X(f) \\ &= F_X(f)^{k+1} \end{aligned}$$

Thus we have shown that this is true for $n = k + 1$. Therefore, by the principle of mathematical induction, we have shown that this is true for all $n \geq 1$.

1.4 We know that S_n is defined as the normalized sum of the n independent random variables:

$$\begin{aligned} S_n &= \frac{1}{\sqrt{n}}(X_1 + X_2 + \ldots + X_n) \\ &= \frac{1}{\sqrt{n}} Z_n \end{aligned}$$

We can now use the property in equation (44):

$$\begin{aligned} \mathbb{E}\left\{e^{-j2\pi f S_n}\right\} &= \mathbb{E}\left\{e^{-j2\pi f \frac{1}{\sqrt{n}} Z_n}\right\} \\ &= \int_{-\infty}^{\infty} e^{-j2\pi \frac{f}{\sqrt{n}} z} f_{Z_n}(z) dz \\ &= F_{Z_n}\left(\frac{f}{\sqrt{n}}\right) \end{aligned}$$

1.5 The order-2 Taylor series expansion of the given exponential is:

$$\begin{aligned} e^{-j2\pi \frac{f}{\sqrt{n}} x} &= 1 - j2\pi \frac{f}{\sqrt{n}} x + \frac{1}{2}\left(-j2\pi \frac{f}{\sqrt{n}} x\right)^2 + \dots \\ &\approx 1 - \frac{j2\pi f}{\sqrt{n}} x - \frac{2\pi^2 f^2}{n} x^2 \end{aligned}$$

1.6 We can now estimate the fourier transform of $F_X(f/\sqrt{n})$ using (13) and the order-2 Taylor series expansion:

$$\begin{aligned} F_X f/\sqrt{n} &= \int_{-\infty}^{\infty} e^{-j2\pi \frac{f}{\sqrt{n}} x} f_X(x) dx \\ &\approx \int_{-\infty}^{\infty} \left(1 - \frac{j2\pi f}{\sqrt{n}} x - \frac{2\pi^2 f^2}{n} x^2\right) f_X(x) dx \\ &\approx 1 - \frac{j2\pi f}{\sqrt{n}} \int_{-\infty}^{\infty} x f_X(x) dx - \frac{2\pi^2 f^2}{n} \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &\approx 1 - \frac{2\pi^2 f^2}{n} \end{aligned}$$

1.7 We can find the approximation of the Fourier transform of S_n as follows, using (10), (15) and (16):

$$\begin{aligned} F_{S_n}(f) &= \left(F_X\left(\frac{f}{\sqrt{n}}\right)\right)^n \\ &\approx \left(1 - \frac{2\pi^2 f^2}{n}\right)^n \\ &\approx e^{-2\pi^2 f^2} \quad \text{as } n \longrightarrow \infty \end{aligned}$$

1.8 Now we can find inverse Fourier transform of $F_{S_n}(f)$. We first recognize that:

$$e^{-\pi x^2} \leftrightarrow e^{-\pi f^2}$$

In our case, we have a scaled version, so we also apply the scaling property:

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Rearranging $F_{S_n}(f)$, we can find the scaling factor needed to apply the above property:

$$F_{S_n}(f) \approx e^{-\pi(\sqrt{2\pi} f)^2}$$

Using $a = \sqrt{2\pi}$, we can find the inverse Fourier transform:

$$f_{S_n}(t) \approx \frac{1}{\sqrt{2\pi}} e^{-\pi t^2} \text{as } n \longrightarrow \infty$$

2 Problem 2

2.1 We first show that the length of the two basis vectors is equal to 1:

$$\begin{aligned}
\|\Psi_1(t)\| &= \left\| \frac{s_1(t)}{\|s_1(t)\|} \right\| = \frac{1}{\|s_1(t)\|} \|s_1(t)\| = 1 \\
\|\Psi_2(t)\| &= \left\| \frac{d_2(t)}{\|d_2(t)\|} \right\| = \frac{1}{\|d_2(t)\|} \|d_2(t)\| = 1
\end{aligned}$$

We can show that these two vectors are orthogonal to each other:

$$\begin{aligned}
\langle \Psi_1(t), \Psi_2(t) \rangle &= \langle \Psi_1(t), \frac{d_2(t)}{\|d_2(t)\|} \rangle \\
&= \langle \Psi_1(t), \frac{1}{\|d_2(t)\|} (s_2(t) - \alpha_{21}\psi_1(t)) \rangle \\
&= \frac{1}{\|d_2(t)\|} [\alpha_{21} - \alpha_{21}(1)] \\
&= 0
\end{aligned}$$

We utilize the fact that $\alpha_{21} = \langle s_2(t), \Psi_1(t) \rangle$ and that the length of $\Psi_1(t) = 1$.

2.2 We can express the two signals $s_1(t)$ and $s_2(t)$ in terms of the basis vectors $\Psi_1(t)$ and $\Psi_2(t)$:

$$\begin{aligned}
s_1(t) &= \|s_1(t)\| \Psi_1(t) \\
s_2(t) &= d_2(t) + \alpha_{21}\Psi_1(t) \\
&= \|d_2(t)\| \Psi_2(t) + \alpha_{21}\Psi_1(t)
\end{aligned}$$

We know that from (25) that $\|s_1(t)\| = \sqrt{\mathcal{E}_1}$, then we can write:

$$s_1(t) = \sqrt{\mathcal{E}_1}\Psi_1(t)$$

We can also calculate the length of $d_2(t)$ as follows:

$$\begin{aligned}
\langle d_2(t), d_2(t) \rangle &= \int_{-\infty}^{\infty} d_2(t) d_2(t) dt \\
&= \int_{-\infty}^{\infty} (s_2(t) - \alpha_{21}\Psi_1(t))^2 \, dt \\
&= \int_{-\infty}^{\infty} s_2(t)s_2(t)dt - 2\alpha_{21} \int_{-\infty}^{\infty} s_2(t)\Psi_1(t)dt + \alpha_{21}^2 \int_{-\infty}^{\infty} \Psi_1(t)\Psi_1(t)dt \\
&= \mathcal{E}_2 - 2\alpha_{21} \times \alpha_{21} + \alpha_{21}^2 \times 1 \\
&= \mathcal{E}_2 - \alpha_{21}^2 \implies \|d_2(t)\| = \sqrt{\mathcal{E}_2 - \alpha_{21}^2}
\end{aligned}$$

Now with the results from above, we express these two signals in terms of the basis vectors in their geomtric form:

$$
\begin{aligned}
s_1(t) = \sqrt{\mathcal{E}_1}\Psi_1(t) &\longleftrightarrow \mathbf{s_1} = \begin{bmatrix} \sqrt{\mathcal{E}_1} \\ 0 \end{bmatrix} \\
s_2(t) = \sqrt{\mathcal{E}_2 + \alpha_{21}^2}\Psi_2(t) + \alpha_{21}\Psi_1(t) &\longleftrightarrow \mathbf{s_2} = \begin{bmatrix} \alpha_{21} \\ \sqrt{\mathcal{E}_2 - \alpha_{21}^2} \end{bmatrix}
\end{aligned}
$$

2.3 We can show the following identities:

a. $\|s_1(t)\|^2 = \|\mathbf{s_1}\|^2$

$$
\begin{aligned}
\|s_1(t)\|^2 &= \int_{-\infty}^{\infty} s_1(t)s_1(t)dt \\
&= \sqrt{\mathcal{E}_1}^2 = \mathcal{E}_1 \\
\|\mathbf{s_1}\|^2 &= \left\| \begin{bmatrix} \sqrt{\mathcal{E}_1} \\ 0 \end{bmatrix} \right\|^2 = \left(\sqrt{\left(\sqrt{\mathcal{E}_1}\right)^2 + 0^2} \right)^2 = \mathcal{E}_1 \\
\implies \|s_1(t)\|^2 &= \|\mathbf{s_1}\|^2
\end{aligned}
$$

b. $\|s_2(t)\|^2 = \|\mathbf{s_2}\|^2$

$$
\begin{aligned}
\|s_2(t)\|^2 &= \int_{-\infty}^{\infty} s_2(t)s_2(t)dt \\
&= \sqrt{\mathcal{E}_2}^2 = \mathcal{E}_2 \\
\|\mathbf{s_2}\|^2 &= \left\| \begin{bmatrix} \alpha_{21} \\ \sqrt{\mathcal{E}_2 - \alpha_{21}^2} \end{bmatrix} \right\|^2 = \alpha_{21}^2 + \sqrt{\mathcal{E}_2 - \alpha_{21}^2}^2 = \alpha_{21}^2 + \mathcal{E}_2 - \alpha_{21}^2 = \mathcal{E}_2 \\
\implies \|s_2(t)\|^2 &= \|\mathbf{s_2}\|^2
\end{aligned}
$$

c. $\|s_2(t) - s_1(t)\|^2 = \|\mathbf{s_2} - \mathbf{s_1}\|^2$

$$\begin{aligned}
\|s_2(t) - s_1(t)\|^2 &= \int_{-\infty}^{\infty} (s_2(t) - s_1(t))^2 \, dt \\
&= \int_{-\infty}^{\infty} \left(s_2^2(t) - 2s_2(t)s_1(t) + s_1^2(t)\right) \, dt \\
&= \mathcal{E}_2 + \mathcal{E}_1 - 2\int_{-\infty}^{\infty} s_2(t)s_1(t)dt \\
&= \mathcal{E}_2 + \mathcal{E}_1 - 2\int_{-\infty}^{\infty} \left(\alpha_{21}\Psi_1(t) + \sqrt{\mathcal{E}_2 - \alpha_{21}^2}\Psi_2(t)\right)\left(\sqrt{\mathcal{E}_1}\Psi_1(t)\right) dt \\
&= \mathcal{E}_2 + \mathcal{E}_1 - 2\Bigg(\alpha_{21}\sqrt{\mathcal{E}_1}\int_{-\infty}^{\infty} \Psi_1(t)\Psi_1(t)dt \\
&\quad + \sqrt{\mathcal{E}_2 - \alpha_{21}^2}\int_{-\infty}^{\infty} \Psi_2(t)\Psi_1(t)dt\Bigg) \\
&= \mathcal{E}_2 + \mathcal{E}_1 - 2\alpha_{21}\sqrt{\mathcal{E}_1} \\
\|\mathbf{s_2} - \mathbf{s_1}\|^2 &= \left\| \begin{bmatrix} \alpha_{21} \\ \sqrt{\mathcal{E}_2 - \alpha_{21}^2} \end{bmatrix} - \begin{bmatrix} \sqrt{\mathcal{E}_1} \\ 0 \end{bmatrix} \right\|^2 \\
&= \left\| \begin{bmatrix} \alpha_{21} - \sqrt{\mathcal{E}_1} \\ \sqrt{\mathcal{E}_2 - \alpha_{21}^2} \end{bmatrix} \right\|^2 \\
&= (\alpha_{21} - \sqrt{\mathcal{E}_1})^2 + \left(\sqrt{\mathcal{E}_2 - \alpha_{21}^2}\right)^2 \\
&= \alpha_{21}^2 - 2\alpha_{21}\sqrt{\mathcal{E}_1} + \mathcal{E}_1 + \mathcal{E}_2 - \alpha_{21}^2 \\
&= \mathcal{E}_1 + \mathcal{E}_2 - 2\alpha_{21}\sqrt{\mathcal{E}_1} \\
&\implies \|s_2(t) - s_1(t)\|^2 = \|\mathbf{s_2} - \mathbf{s_1}\|^2
\end{aligned}$$

d. $\langle s_2(t), s_1(t)\rangle = \langle \mathbf{s_2}, \mathbf{s_1}\rangle$

$$
\begin{aligned}
\langle s_2(t), s_1(t)\rangle &= \int_{-\infty}^{\infty} s_2(t)s_1(t)dt \\
&= \int_{-\infty}^{\infty} \left(\alpha_{21}\Psi_1(t) + \sqrt{\mathcal{E}_2 - \alpha_{21}^2}\Psi_2(t)\right)\left(\sqrt{\mathcal{E}_1}\Psi_1(t)\right)dt \\
&= \alpha_{21}\sqrt{\mathcal{E}_1}\int_{-\infty}^{\infty}\Psi_1(t)\Psi_1(t)dt + \sqrt{\mathcal{E}_2 - \alpha_{21}^2}\int_{-\infty}^{\infty}\Psi_2(t)\Psi_1(t)dt \\
&= \alpha_{21}\sqrt{\mathcal{E}_1} + \sqrt{\mathcal{E}_2 - \alpha_{21}^2}\cdot 0 \\
&= \alpha_{21}\sqrt{\mathcal{E}_1} \\
\langle \mathbf{s_2}, \mathbf{s_1}\rangle &= \mathbf{s_2}^T\mathbf{s_1} \\
&= \begin{bmatrix}\alpha_{21} & \sqrt{\mathcal{E}_2 - \alpha_{21}^2}\end{bmatrix}\begin{bmatrix}\sqrt{\mathcal{E}_1} \\ 0\end{bmatrix} \\
&= \alpha_{21}\sqrt{\mathcal{E}_1} + \sqrt{\mathcal{E}_2 - \alpha_{21}^2}\cdot 0 \\
&= \alpha_{21}\sqrt{\mathcal{E}_1} \\
&\implies \langle s_2(t), s_1(t)\rangle = \langle \mathbf{s_2}, \mathbf{s_1}\rangle
\end{aligned}
$$

2.4 Given the signal $s_i(t)$, we can find hte energy $\mathcal{E}_i$ for $i = 1, 2$ as follows:

$$
\begin{aligned}
\mathcal{E}_i &= \int_{-\infty}^{\infty} s_i(t)s_i(t)dt \\
&= \int_{-\infty}^{\infty}\left(\sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_i t)\right)^2 dt \\
&= \int_{0}^{T}\left(\sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_i t)\right)^2 dt \\
&= \frac{2\mathcal{E}_b}{T}\int_0^T \cos^2(2\pi f_i t)\,dt \\
&= \frac{2\mathcal{E}_b}{T}\int_0^T \frac{1+\cos(4\pi f_i t)}{2}dt \\
&= \frac{2\mathcal{E}_b}{T}\cdot\frac{1}{2T} \\
&= \mathcal{E}_b
\end{aligned}
$$

We see that the final result is independent of f_i, so we have shown that $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_b$.

2.5 We can find the two orthonormal basis vectors $\Psi_1(t)$ and $\Psi_2(t)$ as follows:

$$
\begin{aligned}
\Psi_1(t) &= \frac{s_1(t)}{\|s_1(t)\|} = \frac{s_1(t)}{\sqrt{\mathcal{E}_b}} = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t) \\
\Psi_2(t) &= \frac{s_2(t)}{\|s_2(t)\|} = \frac{s_2(t)}{\sqrt{\mathcal{E}_b}} = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)
\end{aligned}
$$

2.6 To express P_e in terms of the SNR per bit, we need to find the distance d_{12} between the two signals $s_1(t)$ and $s_2(t)$:

$$\begin{aligned} d_{12}^2 &= \|s_2(t) - s_1(t)\|^2 \\ &= \int_{-\infty}^{\infty} \left(s_2(t) - s_1(t)\right)^2 dt \\ &= \sum_{i=1}^{2} (s_{2i} - s_{1i})^2 \\ &= \left(0 - \sqrt{\mathcal{E}_b}\right)^2 + \left(\sqrt{\mathcal{E}_b} - 0\right)^2 \\ &= 2\mathcal{E}_b \\ \implies d_{12} &= \sqrt{2\mathcal{E}_b} \end{aligned}$$

Therefore, our P_e can be expressed as:

$$P_e = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) = Q\left(\frac{\sqrt{2\mathcal{E}_b}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{\text{SNR}}{2}}\right)$$

3 Problem 3

3.1 a. We can find the correlation coefficient, ρ, as follows:

$$\begin{aligned} \rho &= \frac{1}{\sqrt{\mathcal{E}_1\mathcal{E}_2}} \int_{-\infty}^{\infty} s_1(t)s_2(t)dt \\ &= \frac{1}{\sqrt{\mathcal{E}_1\mathcal{E}_2}} \int_0^T \left(\sqrt{\frac{2\mathcal{E}_b}{T}} \cos(2\pi \left[f_c + \frac{\Delta f}{2}\right] t)\right) \left(\sqrt{\frac{2\mathcal{E}_b}{T}} \cos(2\pi \left[f_c - \frac{\Delta f}{2}\right] t)\right) dt \end{aligned}$$

From problem 2, we know that $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_b$, so we can simplify this to:

$$\rho = \frac{2}{T} \int_0^T \cos\left(2\pi \left[f_c + \frac{\Delta f}{2}\right] t\right) \cos\left(2\pi \left[f_c - \frac{\Delta f}{2}\right] t\right) dt$$

We can use the cosine product identity:

$$\begin{aligned} \cos(a)\cos(b) &= \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right) \\ a + b = 2\pi \left[f_c + \frac{\Delta f}{2}\right] t + 2\pi \left[f_c - \frac{\Delta f}{2}\right] t &= 4\pi f_c t \\ a - b = 2\pi \left[f_c + \frac{\Delta f}{2}\right] t - 2\pi \left[f_c - \frac{\Delta f}{2}\right] t &= 2\pi \Delta f t \end{aligned}$$

We can now substitute this into our equation:

$$
\begin{aligned}
\rho &= \frac{2}{T}\int_0^T \frac{1}{2}\left(\cos(4\pi f_c t) + \cos(2\pi \Delta f t)\right) dt \\
&= \frac{1}{T}\left[\frac{\sin(4\pi f_c t)}{4\pi f_c} + \frac{\sin(2\pi\Delta f t)}{2\pi \Delta f}\right]\Bigg|_0^T \\
&= \frac{1}{T}\left[\frac{\sin(4\pi f_c T)}{4\pi f_c} + \frac{\sin(2\pi\Delta f T)}{2\pi \Delta f}\right]
\end{aligned}
$$

Since $f_c \gg 1$, the first term with $4\pi f_c$ in the denominator will be negligible, and thus our approximation is:

$$
\begin{aligned}
\rho &\approx \frac{1}{T}\cdot\frac{\sin(2\pi\Delta f T)}{2\pi\Delta f} \\
&\approx \operatorname{sinc}(2\Delta f T)
\end{aligned}
$$

b. The sketch of this is as follows:

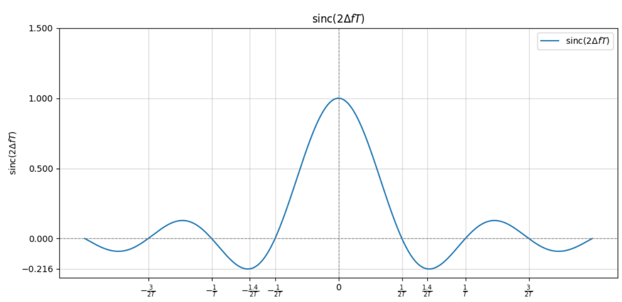

Figure 1: Plot of $\rho \approx \operatorname{sinc}(2\Delta f T)$

3.2 The minimum value in which $s_1(t)$ and $s_2(t)$ can be by setting the correlation coefficient to 0:

$$
\begin{aligned}
\rho &\approx \operatorname{sinc}(2\Delta f T) = 0 \\
\implies \Delta f &= \frac{1}{2T}
\end{aligned}
$$

3.3 We can find the probability of error by first finding the distance between the two signals:

$$
\begin{aligned}
d_{12}^2 &= \|s_2(t) - s_1(t)\|^2 \\
&= \int_{-\infty}^{\infty} \left(s_2(t) - s_1(t)\right)^2 \, dt \\
&= \int_0^T s_1^2(t) \, + dt \int_0^T s_2^2(t)\, dt - \int_0^T 2 s_1(t) s_2(t)\, dt \\
&= \mathcal{E}_1 + \mathcal{E}_2 - 2\int_0^T s_1(t)s_2(t)\, dt
\end{aligned}
$$

Rearranging the equation for the correlation coefficient:

$$\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho = \mathcal{E}_b\rho = \int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt$$

We can now use this relationship:

$$\begin{aligned} d_{12}^2 &= \mathcal{E}_1 + \mathcal{E}_2 - 2\mathcal{E}_b\rho \\ &= 2\mathcal{E}_b - 2\mathcal{E}_b\rho \\ d_{12} &= \sqrt{2\mathcal{E}_b(1-\rho)} \end{aligned}$$

We can now find the probability of error given that the power spectral density of the noise is $\frac{N_0}{2}$:

$$\begin{aligned} P_e(\rho) &= Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) \\ &= Q\left(\sqrt{\frac{\mathcal{E}_b(1-\rho)}{N_0}}\right) \end{aligned}$$

3.4 Since the Q function is a monotonic decreasing function, we can find the minimum value of P_e by finding the minimum value of ρ. This is because the minimum value of ρ will give us the maximum input for the Q function and will equate to the minimum value of P_e.

We found that the minimum value of ρ is:

$$\begin{aligned} \rho &= \operatorname{sinc}(2\Delta f T) = 0 \\ \implies \Delta f &= \frac{1}{2T} \\ \implies P_e &= Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) \end{aligned}$$

Picking Δf to be $\frac{1}{2T}$ will give us the minimum value of P_e.

3.5 With this choice of Δf, the SNR needed to achieve the same probability of error as the case for binary antipodal is:

$$\begin{aligned} P_{e,\text{binary antipodal}} &= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\text{SNR}}\right) \\ P_{e,\text{binary FSK}} &= Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{1}{2}\text{SNR}}\right) \end{aligned}$$

Thus, we need double the SNR to achieve the same probability of error as the binary antipodal case. This gives us a 3 dB difference. Meaning, if we double the energy per bit, we can achieve the same probability of error as the binary antipodal case.