# Assignment 1

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# 1 Problem 1

1. Find the Euler representation of the complex numbers $z_5, z_6$ and $z_7$.

- $z_5 = \frac{z_1}{z_2^*}$ First, we convert $z_1$ and $z_2$ to Euler form:

$$
\begin{aligned}
z_1 &= \frac{5}{2} + \frac{5\sqrt{3}}{2}j \\
|z_1| &= \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} = 5 \\
\theta_1 &= \tan^{-1}\left(\frac{5\sqrt{3}/2}{5/2}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \\
\implies z_1 &= 5e^{j\frac{\pi}{3}} \\
z_2^* &= \sqrt{3} + j \\
|z_2^*| &= \sqrt{\sqrt{3}^2 + 1^2} = 2 \\
\theta_2 &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \\
\implies z_2^* &= 2e^{j\frac{\pi}{6}}
\end{aligned}
$$

Using the Euler representation of $z_1$ and $z_2^*$, we can find the Euler representation of $z_5$:

$$
\begin{aligned}
z_5 &= \frac{z_1}{z_2^*} = \frac{5e^{j\frac{\pi}{3}}}{2e^{j\frac{\pi}{6}}} \\
&= \frac{5}{2}e^{j\left(\frac{\pi}{3} - \frac{\pi}{6}\right)} \\
&= \frac{5}{2}e^{j\frac{\pi}{6}}
\end{aligned}
$$

- $z_6 = \frac{z_3^*}{z_4}$

$$z_6 = \frac{1 + e^{jn\varphi}}{1 - e^{j\varphi}}$$

The magnitude of $z_6$:

$$|z_6| = \left|\frac{1 + e^{jn\varphi}}{1 - e^{j\varphi}}\right| = \frac{\left|1 + e^{jn\varphi}\right|}{\left|1 - e^{j\varphi}\right|}$$

And the angle is:

$$\arg(z_6) = \arg(1 + e^{jn\varphi}) - \arg(1 - e^{j\varphi})$$

Solving for the numerator first:

$$\begin{aligned}\left|1+e^{jn\varphi}\right| &= |1+cos(n\varphi)+j\sin(n\varphi)| \\ &= \sqrt{(1+\cos(n\varphi))^2+\sin^2(n\varphi)} \\ &= \sqrt{(1+2cos(n\varphi)+\cos^2(n\varphi))+\sin^2(n\varphi)} \\ &= \sqrt{2+2\cos(n\varphi)}\end{aligned}$$

The angle of the numerator is:

$$\begin{aligned}\arg(1+e^{jn\varphi}) &= \arg(1+cos(n\varphi)+j\sin(n\varphi)) \\ &= \tan^{-1}\left(\frac{\sin(n\varphi)}{1+\cos(n\varphi)}\right)\end{aligned}$$

Solving for the magnitude of the denominator:

$$\begin{aligned}\left|1-e^{j\varphi}\right| &= |1-\cos(\varphi)-j\sin(\varphi)| \\ &= \sqrt{(1-\cos(\varphi))^2+\sin^2(\varphi)} \\ &= \sqrt{(1-2\cos(\varphi)+\cos^2(\varphi))+\sin^2(\varphi)} \\ &= \sqrt{2-2\cos(\varphi)}\end{aligned}$$

The angle of the denominator is:

$$\begin{aligned}\arg(1-e^{j\varphi}) &= \arg(1-\cos(\varphi)-j\sin(\varphi)) \\ &= \tan^{-1}\left(\frac{-\sin(\varphi)}{1-\cos(\varphi)}\right)\end{aligned}$$

The overall magnitude is:

$$\begin{aligned}|z_6| &= \frac{\sqrt{2+2\cos(n\varphi)}}{\sqrt{2-2\cos(\varphi)}} \\ &= \sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}\end{aligned}$$

And the angle is:

$$\begin{aligned}\theta_{z_6} = \arg(z_6) &= \tan^{-1}\left(\frac{\sin(n\varphi)}{1+\cos(n\varphi)}\right) - \tan^{-1}\left(\frac{-\sin(\varphi)}{1-\cos(\varphi)}\right) \\ &= \tan^{-1}\left(\frac{\sin(n\varphi)}{1+\cos(n\varphi)}\right) + \tan^{-1}\left(\frac{\sin(\varphi)}{1-\cos(\varphi)}\right)\end{aligned}$$

To simplify the argument, we can use the following identities:

$$\begin{aligned}
\frac{\sin(x)}{1+\cos(x)} &= \tan\left(\frac{x}{2}\right) \\
\frac{\sin(x)}{1-\cos(x)} &= \cot\left(\frac{x}{2}\right) \\
\cot(x) &= \tan\left(\frac{\pi}{2}-x\right)
\end{aligned}$$

The first term becomes:

$$\tan^{-1}\left(\frac{\sin(n\varphi)}{1+cos(n\varphi)}\right) = \tan^{-1}\left(\tan\left(\frac{n\varphi}{2}\right)\right) = \frac{n\varphi}{2}$$

The second term becomes:

$$\begin{aligned}
\tan^{-1}\left(\frac{\sin(\varphi)}{1-\cos\varphi}\right) &= \tan^{-1}\left(\cot\left(\frac{\varphi}{2}\right)\right) \\
&= \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\varphi}{2}\right)\right) \\
&= \frac{\pi}{2}-\frac{\varphi}{2}
\end{aligned}$$

The simplified argument is:

$$\theta_{z_6} = \frac{n\varphi}{2}+\frac{\pi}{2}-\frac{\varphi}{2} = \frac{(n-1)\varphi}{2}+\frac{\pi}{2}$$

In polar form, $z_6$ is:

$$z_6 = \sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}e^{j\theta_{z_6}}$$

- $z_7 = z_5 z_6^*$

$$\begin{aligned}
z_7 &= \frac{5}{2}e^{j\frac{\pi}{6}} \cdot \sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}e^{-j\theta_{z_6}} \\
&= \frac{5}{2}\sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}e^{j\left(\frac{\pi}{6}-\frac{(n-1)\varphi}{2}-\frac{\pi}{2}\right)} \\
&= \frac{5}{2}\sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}e^{-j\left(\frac{\pi}{3}+\frac{(n-1)\varphi}{2}\right)}
\end{aligned}$$

2. Find the real and imaginary parts of $z_7^*$. We can convert from Euler form to rectangular form to easily find the real and imaginary parts of $z_7^*$:

$$
\begin{aligned}
z_7^* &= \frac{5}{2}\sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}e^{j\left(\frac{\pi}{3}+\frac{(n-1)\varphi}{2}\right)} \\
&= \frac{5}{2}\sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}\left(\cos\left(\frac{\pi}{3}+\frac{(n-1)\varphi}{2}\right)+j\sin\left(\frac{\pi}{3}+\frac{(n-1)\varphi}{2}\right)\right)
\end{aligned}
$$

This gives us the result:

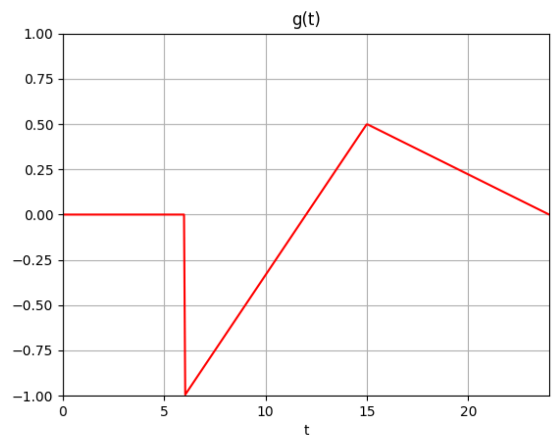
$$
\begin{aligned}
\mathrm{Re}(z_7^*) &= \frac{5}{2}\sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}\cos\left(\frac{\pi}{3}+\frac{(n-1)\varphi}{2}\right) \\
\mathrm{Im}(z_7^*) &= \frac{5}{2}\sqrt{\frac{1+\cos(n\varphi)}{1-\cos(\varphi)}}\sin\left(\frac{\pi}{3}+\frac{(n-1)\varphi}{2}\right)
\end{aligned}
$$

# 2 Problem 2

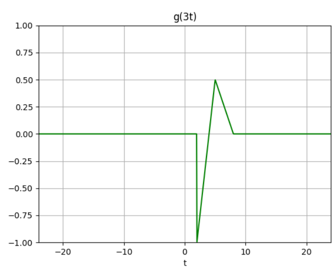
- $g_1(t) = g(-t)$
- $g_2(t) = g_1(t-1) + g(t-1) = g(-t+1) + g(t-1)$
- $g_3(t) = g_1(t+1) + g(t-1) = g(-t-1) + g(t-1)$
- $g_4(t) = g_1(t+1/2) + g(t-1/2) = g(-t-1/2) + g(t-1/2)$
- $g_5(t) = \frac{3}{2}g(\frac{t}{2}-1)$

# 3 Problem 3

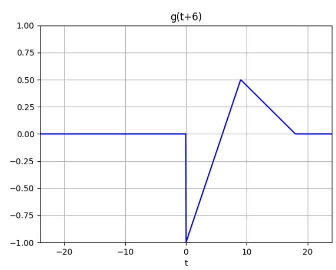
Figure 1 shows the original signal $g(t)$ and the three transformed signals $g_1(t), g_2(t)$ and $g_3(t)$. The code to generate the signals is shown in Code Snippet 1



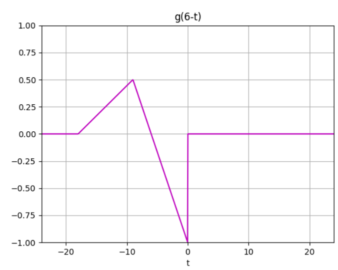

(a) Original Signal $g(t)$




(b) $g_1(t) = g(3t)$




(c) $g_2(t) = g(t + 6)$




(d) $g_3(t) = g(6 - t)$

Figure 1: Graphs of the original signal and the transformed signals

```python
import numpy as np
def g(t):
    if t > 6 and t <= 15:
        return 1/6 * t - 2
    elif t > 15 and t <= 24:
        return -1/18 * t + 4/3
    else:
        return 0
t = np.linspace(-30, 30, 1000)
g0 = [g(i) for i in t]
g1 = [g(3*i) for i in t]
g2 = [g(i+6) for i in t]
g3 = [g(6-i) for i in t]
```

Code Snippet 1: Python code to generate the signals

# 4 Problem 4

Since the signal is periodic, we can find the average power of the entire signal by integrating over 1 period and dividing by the period. The average power is given by:

$$P_g(t) = \frac{1}{T} \int_T |g(t)|^2 dt$$

The period of this signal is $T = 4$ and we can integrate from -2 to 2.

$$\begin{aligned} P_g(t) &= \frac{1}{4} \int_{-2}^{2} |g(t)|^2 dt \\ &= \frac{1}{4} \int_{-2}^{2} (t^3)^2 dt = \frac{1}{4} \int_{-2}^{2} t^6 dt \\ &= \frac{1}{4} \left[ \frac{t^7}{7} \right]_{-2}^{2} = \frac{1}{4} \left[ \frac{128}{7} - \frac{-128}{7} \right] \\ &= \frac{1}{4} \left[ \frac{256}{7} \right] = \boxed{\frac{64}{7}} \end{aligned}$$

# 5 Problem 5

5.1 Show that $\mathcal{H}[.]$ is linear. To show that $\mathcal{H}[.]$ is linear, we need to show that it satisfies both the additivity and homogeneity properties.

Additivity:

Let $x_1(t)$ and $x_2(t)$ be two signals. Then, with $h(t)$ a fixed signal, we have:

$$\begin{aligned} x_1(t) \to y_1(t) &= \mathcal{H}[x_1(t)] = x_1(t)h(t) \\ x_2(t) \to y_2(t) &= \mathcal{H}[x_2(t)] = x_2(t)h(t) \\ (x_1(t) + x_2(t)) \to y_3(t) &= \mathcal{H}[x_1(t) + x_2(t)] = (x_1(t) + x_2(t))h(t) \\ &= x_1(t)h(t) + x_2(t)h(t) = y_1(t) + y_2(t) \end{aligned}$$

Thus $\mathcal{H}[.]$ satisfies the additivity property.

Homogeneity:

Let $x(t)$ be a signal and $a$ be a scalar. Then, with $h(t)$ a fixed signal, we have:

$$\begin{aligned} x(t) \to y_1(t) &= \mathcal{H}[x(t)] = x(t)h(t) \\ ax(t) \to y_2(t) &= \mathcal{H}[ax(t)] = ax(t)h(t) \\ &= a(x(t)h(t)) = ay_1(t) \end{aligned}$$

Thus $\mathcal{H}[.]$ satisfies the homogeneity property and is linear.

5.2 Show that if $\mathcal{H}[.]$ is time-invariant, then h(t) must be a constant, i.e. $h(t) = c, \forall t, c \in \mathbb{R}$.

A system is time-invariant if a shift in the input signal results in a corresponding shift in the output signal. Let $x(t)$ be a signal.

$$\begin{aligned} x(t-t_0) \to y_1(t) = \mathcal{H}[x(t-t_0)] = x(t-t_0)h(t) \\ y_2(t-t_0) = x(t-t_0)h(t-t_0) \end{aligned}$$

Then, $y_2(t-t_0) = y_1(t)$ if, and only if, $h(t) = c$ for some constant $c$. If $h(t) = c$, then $y_1(t) = cx(t-t_0)$ and $y_2(t-t_0) = cx(t-t_0)$.

5.3 Suppose that the switch is closed for $-1/2 \leq t \leq 1/2$. Using 5.1, show that this is a linear system.

The new system is given by:

$$\mathcal{H}[x(t)] = \begin{cases} x(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Let $x_1(t)$ and $x_2(t)$ be two signals. Then we have:

$$\begin{aligned} \mathcal{H}[x_1(t) + x_2(t)] &= \begin{cases} x_1(t) + x_2(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x_1(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} x_2(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \mathcal{H}[x_1(t)] + \mathcal{H}[x_2(t)] \end{aligned}$$

And:

$$\begin{aligned} \mathcal{H}[ax(t)] &= \begin{cases} ax(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= a \begin{cases} x(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= a\mathcal{H}[x(t)] \end{aligned}$$

Thus the system satisfies the additivity and homogeneity properties and is linear.

5.4 Suppose that the switch is turned on and off at various time intervals. Is the system still linear any why? Is it time-invariant and why?

We can represent the new system as:

$$\mathcal{H}[x(t)] = \begin{cases} x(t) & \text{if } t \in \bigcup_{i=1}^{n} [a_i, b_i] \\ 0 & \text{otherwise} \end{cases}$$

We can show that it is linear as it follows both additivity and homogeneity:

$$\begin{aligned} \mathcal{H}[ax_1(t) + bx_2(t)] &= \begin{cases} ax_1(t) + bx_2(t) & \text{if } t \in \bigcup_{i=1}^{n} [a_i, b_i] \\ 0 & \text{otherwise} \end{cases} \\ &= a \begin{cases} x_1(t) & \text{if } t \in \bigcup_{i=1}^{n} [a_i, b_i] \\ 0 & \text{otherwise} \end{cases} + b \begin{cases} x_2(t) & \text{if } t \in \bigcup_{i=1}^{n} [a_i, b_i] \\ 0 & \text{otherwise} \end{cases} \\ &= a\mathcal{H}[x_1(t)] + b\mathcal{H}[x_2(t)] \end{aligned}$$

It is not time-invariant as the output signal will change depending on the time intervals the switch is on or off. A delay in the input signal will not result in a corresponding delay in the output signal.

$$x(t-t_0) \to y_1(t) = \mathcal{H}[x(t-t_0)] = \begin{cases} x(t-t_0) & \text{if } t \in \bigcup_{i=1}^{n}[a_i, b_i] \\ 0 & \text{otherwise} \end{cases}$$
$$y_2(t-t_0) = \begin{cases} x(t-t_0) & \text{if } (t-t_0) \in \bigcup_{i=1}^{n}[a_i, b_i] \\ 0 & \text{otherwise} \end{cases}$$

We see that adding a delay in the output also adds a delay to the time-intervals, unlike the delayed input signal, where it is on and off at fixed intervals.

For example, if the switch is on for $[2,3] \cup [5,6]$, a delay in the input will result in the portion of the signal being on between these two values, where as a delay in the output will result in the signal being on for $[2+t_0, 3+t_0] \cup [5+t_0, 6+t_0]$.

5.5 Suppose that the switch is turned on at $t = 0$ and stays on forever. Is the system now linear and why? Is it time-invariant and why?

We can define the new system as:

$$y(t) = \mathcal{H}[x(t)] = u(t)x(t)$$
$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The system is linear as it satisfies both additivity and homogeneity:

$$\begin{aligned} y(ax_1(t) + bx_2(t)) = \mathcal{H}[ax_1(t) + bx_2(t)] &= u(t)(ax_1(t) + bx_2(t)) \\ &= au(t)x_1(t) + bu(t)x_2(t) \\ &= a\mathcal{H}[x_1(t)] + b\mathcal{H}[x_2(t)] \\ &= ay_1(t) + by_2(t) \end{aligned}$$

It is not time-invariant and we can show this by considering a delay in the input signal:

$$\begin{aligned} x(t-t_0) \to y_1(t) &= u(t)x(t-t_0) \\ y_2(t-t_0) &= u(t-t_0)x(t-t_0) \\ &\neq y_1(t) \end{aligned}$$

Therefore, since a delay in the input does not lead to the same result as a delay in the output, the system is not time-invariant.