Assignment 2

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1 Problem 1

Given that it is an LTI system, the responses to the two new inputs can be derived from the given response.

1.1 We can express $x_2(t)$ as:

$$\begin{aligned} x_2(t) &= x_1(t) - x(t-2) \\ \implies y_2(t) &= y_1(t) - y(t-2) \\ &= 2\Lambda(t-1) - 2\Lambda(t-2) \end{aligned}$$

1.2 We can express $x_3(t)$ as:

$$\begin{aligned} x_3(t) &= x(t+1) + x(t) \\ \implies y_3(t) &= y(t+1) + y(t) \\ &= 2\Lambda(t) + 2\Lambda(t-1) \end{aligned}$$

The plots of the inputs and outputs can be seen in Figure 1.

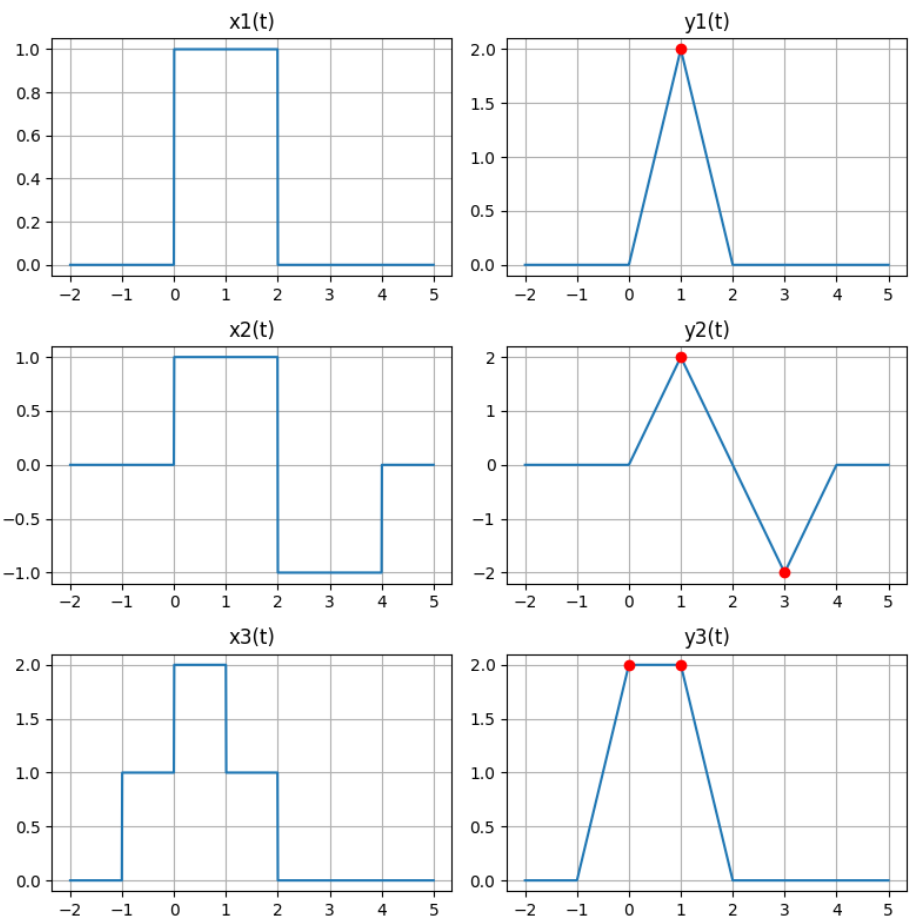

Figure 1: Plots of the inputs and outputs

2 Problem 2

We can express $h(t)$ as a sinc function:

$$\begin{aligned}
\text{sinc}(t) &= \frac{\sin(\pi t)}{\pi t} \\
\implies \frac{\sin(4[t-1])}{\pi(t-1)} &= \frac{\sin(\pi \frac{4[t-1]}{\pi})}{\pi(t-1)} \\
&= \frac{\frac{4}{\pi}\sin(\pi \frac{4[t-1]}{\pi})}{\pi \frac{4}{\pi}(t-1)} \\
&= \frac{4}{\pi}\text{sinc}\left(\frac{4[t-1]}{\pi}\right)
\end{aligned}$$

In frequency domain, using the tables, we find that the impulse response of a sinc function is a rectangle function. We also apply the time scaling property and time shifting property of the fourier transform:

$$\begin{aligned}
\text{sinc(t)} &\xrightarrow{\mathcal{F}} \text{rect}(f) \\
\text{sinc}(a(t-t_0)) &\xrightarrow{\mathcal{F}} \frac{1}{|a|}\text{rect}(f/a) \times e^{-j2\pi f t_0}
\end{aligned}$$

We have that $a = 4/\pi$ and $t_0 = 1$. Thus, the fourier transform of $h(t)$ is:

$$\begin{aligned}
H(f) &= \frac{4}{\pi} \times \frac{\pi}{4}\text{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f} \\
&= \text{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f}
\end{aligned}$$

2.1 We can rewrite $x_1(t)$ as follows:

$$x_1(t) = \frac{\sin(4[t-1])}{\pi(t-1)} = \frac{4}{\pi}\text{sinc}(4[t-1])$$

Similarly the fourier transform of $x_1(t)$ is:

$$X_1(f) = \frac{\pi}{4}\text{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f}$$

The output of the system is given by:

$$\begin{aligned}
Y_1(f) &= H(f) \cdot X_1(f) \\
&= \left(\text{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f}\right) \times \left(\frac{\pi}{4}\text{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f}\right) \\
&= \frac{\pi}{4}\text{rect}\left(\frac{\pi f}{4}\right) \times e^{-j2\pi f} \times e^{-j2\pi f}
\end{aligned}$$

Taking the inverse fourier transform of the rect function, with the time scaling and frequency shifting properties

$$\mathcal{F}^{-1}\left\{\text{rect}\left(\frac{f}{a}\right)\cdot e^{-j2\pi f\times b}\right\} = |a|\text{sinc}(a(t-b))$$
$$\text{Where } a = 4/\pi, b = 1$$
$$\implies \text{rect}\left(\frac{\pi f}{4}\right)\times e^{-j2\pi f}\times e^{-j2\pi f} = \frac{4}{\pi}\text{sinc}\left(\frac{4[t-2]}{\pi}\right)$$

Thus, the output of the system is:

$$\begin{aligned} y_1(t) &= \frac{\pi}{4}\times\frac{4}{\pi}\text{sinc}\left(\frac{4[t-2]}{\pi}\right) \\ &= \boxed{\text{sinc}\left(\frac{4[t-2]}{\pi}\right)} \end{aligned}$$

2.2 We can rewrite $x_2(t)$ as follows:

$$\begin{aligned} x_2(t) &= \left(\frac{\sin(2t)}{\pi t}\right)^2 \\ &= \left(\frac{\frac{2}{\pi}\sin\left(\pi\frac{2t}{\pi}\right)}{\pi\times\frac{2t}{\pi}}\right)^2 \\ &= \frac{4}{\pi^2}\text{sinc}^2\left(\frac{2t}{\pi}\right) \\ &= \frac{4}{\pi^2}\text{sinc}\left(\frac{2t}{\pi}\right)\times\text{sinc}\left(\frac{2t}{\pi}\right) \end{aligned}$$

Since multiplication in the time domain is convolution in the frequency domain, we can find the fourier transform of $x_2(t)$ as follows:

$$\text{sinc}\left(\frac{2t}{\pi}\right)\xrightarrow{\mathcal{F}}\frac{\pi}{2}\text{rect}\left(\frac{\pi f}{2}\right)$$

The convolution of two rectangular functions of the same width is a triangle function. Thus, the fourier transform of $x_2(t)$ is:

$$\begin{aligned} X_2(f) &= \frac{4}{\pi^2}\left(\frac{\pi}{2}\text{rect}\left(\frac{\pi f}{2}\right) * \frac{\pi}{2}\text{rect}\left(\frac{\pi f}{2}\right)\right) \\ &= \text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right) \end{aligned}$$

The output in the frequency domain is:

$$\begin{aligned} Y_2(f) &= H(f)\cdot X_2(f) \\ &= \left(\frac{\pi}{4}\text{rect}\left(\frac{\pi f}{4}\right)\times e^{-j2\pi f}\right)\times\left(\text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right)\right) \end{aligned}$$

Since the rectangle function is 1 for $|f| \leq 1/2$, mutliplying by the rectangular function on the outside is essentially mutliplying by 1. Thus, the output in the frequency domain is:

$$Y_2(f) = \frac{\pi}{4}\text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right)$$

The convolution in the frequency domain is equivalent to multiplication in the time domain. Thus, the output in the time domain is:

$$\begin{aligned} y_2(t) &= \mathcal{F}^{-1}\left(\frac{\pi}{4}\text{rect}\left(\frac{\pi f}{2}\right) * \text{rect}\left(\frac{\pi f}{2}\right)\right) \\ &= \frac{\pi}{4}\mathcal{F}^{-1}\left(\text{rect}\left(\frac{\pi f}{2}\right)\right) \times \mathcal{F}^{-1}\left(\text{rect}\left(\frac{\pi f}{2}\right)\right) \\ &= \frac{\pi}{4}\left(\frac{2}{\pi}\text{sinc}\left(\frac{2t}{\pi}\right)\right) \times \left(\frac{2}{\pi}\text{sinc}\left(\frac{2t}{\pi}\right)\right) \\ &= \boxed{\frac{1}{\pi}\left(\text{sinc}\left(\frac{2t}{\pi}\right)\right)^2} \end{aligned}$$

3 Problem 3

3.1 We can find the fourier series coefficients as follows:

$$\begin{aligned} g_n &= \frac{1}{T}\int_0^T g(t)e^{-j2\pi nt/T}dt \\ &= \frac{1}{2}\int_0^2 t^2 e^{-j\pi nt}dt \end{aligned}$$

We can solve this via integration of parts:

$$\begin{aligned} \int x^2 e^{-ax}dx &= \frac{x^2e^{-ax}}{-a} + \frac{2}{a}\int xe^{-ax}dx \\ &= \frac{x^2e^{-ax}}{-a} + \frac{2}{a}\left(\frac{xe^{-ax}}{-a} + \frac{1}{a}\int e^{-ax}dx\right) \\ &= \frac{x^2e^{-ax}}{-a} + \frac{2xe^{-ax}}{-a^2} - \frac{2}{a^2}e^{-ax} + C \\ &= -\frac{x^2e^{-ax}}{a} - \frac{2xe^{-ax}}{a^2} - \frac{2}{a^2}e^{-ax} + C \end{aligned}$$

We have that $a = j\pi n$:

$$g_n = -\frac{t^2e^{-j\pi nt}}{j\pi n} - \frac{2te^{-j\pi nt}}{(j\pi n)^2} - \frac{2e^{-j\pi nt}}{(j\pi n)^2}\Bigg|_0^2$$

3.2 We can equate the series at $t = 0$, to show the idenity:

$$\begin{aligned} g(0) &= \sum_{n=-\infty}^{\infty} g_n e^{j2\pi n0} \\ 2 &= \frac{4}{3} + \sum_{n=-\infty}^{-1} g_n + \sum_{n=1}^{\infty} g_n \end{aligned}$$

Since this is a real signal, we have that $g_n = g^*_{-n}$. Where, g_n is:

$$g_n = \frac{2(1 + j\pi n)}{\pi^2 n^2} = \frac{2}{\pi^2 n^2} + \frac{2j}{\pi n}$$

$$\begin{aligned} 2 - \frac{4}{3} &= \sum_{n=1}^{\infty} g_n + \sum_{n=1}^{\infty} g^*_{-n} \\ \frac{2}{3} &= \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} + \frac{2j}{\pi n} \right) + \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} - \frac{2j}{\pi n} \right) \\ \frac{2}{3} &= 2 \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \\ \frac{1}{3} &= \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \\ \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

And thus, the identity is proven.

3.3 We now equate at another point, $t = 1$:

$$g(1) = 1 = \sum_{n=-\infty}^{\infty} g_n e^{j\pi n}$$

We recognize that the exponential can be simplified:

$$\begin{aligned} e^{j\pi n} &= \cos(\pi n) + j \sin(\pi n) \\ &= (-1)^n \end{aligned}$$

Thus, the sum is:

$$\sum_{n=-\infty}^{\infty} g_n(-1)^n = 1$$

$$\frac{4}{3} + \sum_{n=1}^{\infty} g_n(-1)^n + \sum_{n=1}^{\infty} g_{-n}^*(-1)^n = 1$$

$$\frac{4}{3} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi^2 n^2} + \frac{2j}{\pi n}\right)(-1)^n + \sum_{n=1}^{\infty}\left(\frac{2}{\pi^2 n^2} - \frac{2j}{\pi n}\right)(-1)^n = 1$$

$$\sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}(-1)^n = 1 - \frac{4}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2} = -\frac{1}{12}$$

$$\sum_{n} = 1^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

And thus, the identity is proven.

4 Problem 4

The signal $x(t)$ whose fourier transform in the graph will be a sinc^2 function, modulated by a cosine function at a frequency, $f = 4$.

Since the plot is a triangle function, which is the convolution of two rectangular functions, we can find the original signal as follows:

$$\begin{aligned}\mathcal{F}^{-1}(X(f)) &= \mathcal{F}^{-1}\left(\text{rect}\left(\frac{f}{2}\right) * \text{rect}\left(\frac{f}{2}\right)\right) \\ &= \mathcal{F}^{-1}\left(\text{rect}\left(\frac{f}{2}\right)\right) \times \mathcal{F}^{-1}\left(\text{rect}\left(\frac{f}{2}\right)\right)\end{aligned}$$

We know:

$$\mathcal{F}(2\text{sinc}(2t)) = \text{rect}\left(\frac{f}{2}\right)$$

Thus we have that:

$$\implies 4\text{sinc}^2(2t)$$

This signal is modulated by a cosine function and the plotted fourier transform has two peaks, centered at 4 and -4. Thus, the cosine function is at a frequency of 4, and the amplitude will be scaled by 2 as modulation halves the amplitude.

Therefore, we have that the original signal is:

$$x(t) = 8\text{sinc}^2(2t) \times \cos(4\pi t)$$

5 Problem 5

5.1 We can show $g(t)$ is periodic with period T as follows:

$$g(t) = \sum_{k=-\infty}^{\infty} x(t - kT) \stackrel{?}{=} g(t + T)$$

$$\begin{aligned}
g(t+T) &= \sum_{k=-\infty}^{\infty} x(t + T - kT) \\
&= \sum_{k=-\infty}^{\infty} x(t + (1-k)T) \\
\text{Let } j &= 1 - k \\
&= \sum_{j=-\infty}^{\infty} x(t + jT) \\
&= g(t)
\end{aligned}$$

5.2 Let $x(t) = t^2$, then the plot of $g(t)$ over the interval $[-2T - \frac{\tau}{2}, 2T + \frac{\tau}{2}]$ is shown in Figure 2.

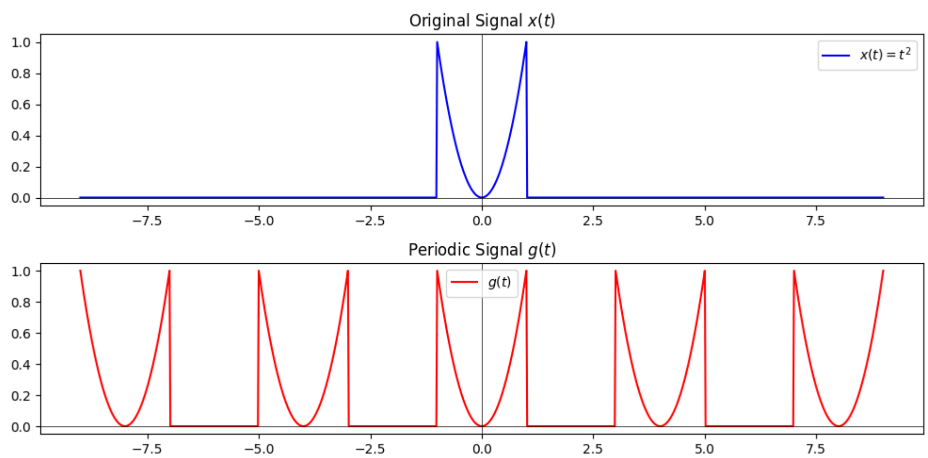

Figure 2: Plot of $x(t)$ and $g(t)$, The selected parameters are $T = 4$ and $\tau = 2$

5.3 The fourier series coefficients of $g(t)$ are:

$$\begin{aligned}
g_n &= \frac{1}{T}\int_{-T/2}^{T/2} g(t)e^{-j2\pi nt/T}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=-\infty}^{\infty} x(t-kT)e^{-j2\pi nt/T}dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\tau/2}^{\tau/2} x(t-kT)e^{-j2\pi nt/T}dt, \quad \text{Let } u = t-kT \\
&\text{The bounds can be set to } -\tau/2, \tau/2 \text{ as } x(t) \text{ is 0 outside of this range} \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{\tau/2}^{-\tau/2} x(u)e^{-j2\pi n(u+kT)/T}du \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{-j2\pi nk}\int_{\tau/2}^{-\tau/2} x(u)e^{-j2\pi nu/T}du \\
&\text{We know that for any integer } n, e^{-j2\pi nk} = 1 \\
&= \frac{1}{T}\int_{\tau/2}^{-\tau/2} x(u)e^{-j2\pi(\frac{n}{T})u}du \\
&= \frac{1}{T}X(\frac{n}{T})
\end{aligned}$$

6 Problem 6

6.1 Since $h(t)$ is an odd function, the fourier transform will be imaginary, and will have a phase of $\pm\frac{\pi}{2}$.

6.2 Evaluating the following integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} G(f)\cos(\pi f)df &= \int_{-\infty}^{\infty} G(f)\left(\frac{e^{j\pi f}+e^{-j\pi f}}{2}\right)df \\
&= \frac{1}{2}\int_{-\infty}^{\infty} G(f)e^{j\pi f}df + \frac{1}{2}\int_{-\infty}^{\infty} G(f)e^{-j\pi f}df \\
&= \frac{1}{2}\left(g(t-1/2)+g(t+1/2)\right)
\end{aligned}$$

6.3 Evaluating the following integral:

$$\int_{-\infty}^{\infty} H(f)e^{j4\pi f}df = h(t-2)$$

6.4 The plot of the odd and even parts of $g(x)$ are shown in Figure 3.

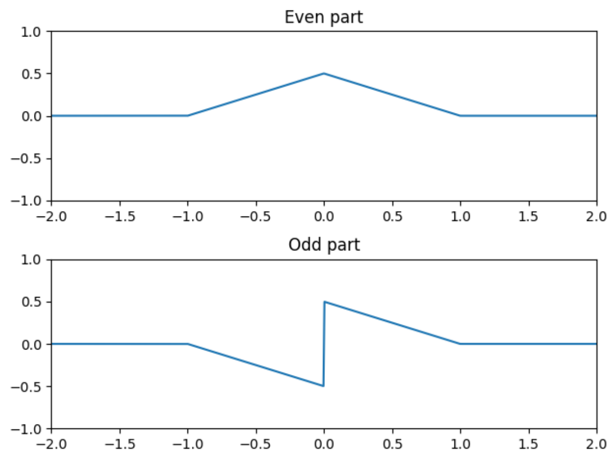

Figure 3: Plot of the odd and even parts of $g(x)$

6.5 The real part of $G(f)$ can be found from the even part of $g(t)$, $g_e(t)$:

$$
\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} (g_e(t) + g_o(t))e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} g_e(t)e^{-j2\pi ft}dt + \int_{-\infty}^{\infty} g_o(t)e^{-j2\pi ft}dt
\end{aligned}
$$

Since we are only looking for the real part of the fourier transform,

$$
\begin{aligned}
\mathrm{Re}(G(f)) &= \int_{-\infty}^{\infty} g_e(t)e^{-j2\pi ft}dt \\
&= \mathcal{F}\{g_e(t)\}
\end{aligned}
$$

6.6 In Figure 3 we see that the even part of $g(t)$ is a triangle function, which has a fourier transform of a sinc^2 function. Thus, the real part of $G(f)$ is a sinc^2 function.

$$
\mathrm{Re}(G(f)) = \frac{1}{2}\mathrm{sinc}^2(f)
$$

We can find the imaginary part of $G(f)$ as follows:

$$\begin{aligned}
\operatorname{Im}(G(f)) &= \int_{-\infty}^{\infty} g_o(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} g_o(t)(\cos(2\pi ft) - j\sin(2\pi ft))dt \\
&= \int_{-\infty}^{\infty} g_o(t)\cos(2\pi ft)dt - j\int_{-\infty}^{\infty} g_o(t)\sin(2\pi ft)dt \\
&= -j\int_{-\infty}^{\infty} g_o(t)\sin(2\pi ft)dt \\
&= -j\int_{-1}^{0}(-1/2 - t)\sin(2\pi ft)dt - j\int_{0}^{1}(1/2 - t)\sin(2\pi ft)dt
\end{aligned}$$

The first integral simplifies to:

$$\begin{aligned}
\int_{-1}^{0}\left(\frac{-1}{2} - t\right)\sin(2\pi ft)dt &= \frac{1}{2}\int_{-1}^{0}\sin(2\pi ft)dt + \int_{-1}^{0} t\sin(2\pi ft)dt \\
&= \frac{1}{2}\left(\frac{-\cos(2\pi ft)}{2\pi f}\right)\Bigg|_{-1}^{0} + \left(\frac{-t\cos(2\pi ft)}{2\pi f}\right)\Bigg|_{-1}^{0} - \int_{-1}^{0}\frac{-\cos(2\pi ft)}{2\pi f}dt \\
&= \frac{1}{4\pi f}\left(-1-(-1)\right) - \frac{1}{2\pi f} + \frac{1}{2\pi f}\left(\sin(2\pi ft)\right)\Bigg|_{-1}^{0} \\
&= -\frac{1}{2\pi f}
\end{aligned}$$

The second integral simplfies to:

$$\begin{aligned}
\int_{0}^{1}\left(\frac{1}{2} - t\right)\sin(2\pi ft)dt &= \frac{1}{2}\int_{0}^{1}\sin(2\pi ft)dt - \int_{0}^{1} t\sin(2\pi ft)dt \\
&= \frac{-t\cos(2\pi ft)}{2\pi f}\Bigg|_{0}^{1} + \int_{0}^{1}\frac{\cos(2\pi ft)}{2\pi f}dt \\
&= -\frac{1}{2\pi f}
\end{aligned}$$

Putting it together:

$$-j\left(-\frac{1}{2\pi f}\right) - j\left(-\frac{1}{2\pi f}\right) = \frac{1}{\pi f}$$