# Assignment 3

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# 1 Problem 1

1.1 We can show the following:

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\,dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\,df$$

Starting with the left-hand side:

$$\begin{aligned}
\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\,dt &= \int_{-\infty}^{\infty} g_1(t)\left(\int_{-\infty}^{\infty} G_2^*(f)e^{-j2\pi ft}\,df\right)\,dt \\
&= \int_{-\infty}^{\infty} G_2^*(f)\left(\int_{-\infty}^{\infty} g_1(t)e^{j2\pi ft}\,dt\right)\,df \\
&= \int_{-\infty}^{\infty} G_2^*(f)G_1(f)\,df \\
&= \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\,df
\end{aligned}$$

And thus, we have shown that the left-hand side is equal to the right-hand side.

1.2 *Explain how we can obtain Parseval's Theorem from (1).* To find Parseval's Theorem, we can set $g_1(t) = g_2(t) = g(t)$, which implies that $G_1(f) = G_2(f) = G(f)$. Substituting these values into (1.1), we get:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(t)g^*(t)\,dt &= \int_{-\infty}^{\infty} G(f)G^*(f)\,df \\
\int_{-\infty}^{\infty} |g(t)|^2\,dt &= \int_{-\infty}^{\infty} |G(f)|^2\,df
\end{aligned}$$

1.3 Using Parseval's Theorem, show that for any $k > 0$ we have:

$$\int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt = \frac{1}{k}$$

Here we have:

$$g(t) = \text{sinc}(kt), \quad G(f) = \frac{1}{k}\text{rect}\left(\frac{f}{k}\right)$$

The conjugate of a rect function is itself, so we have:

$$\begin{aligned}
\int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt &= \int_{-\infty}^{\infty} \frac{1}{k}\text{rect}\left(\frac{f}{k}\right)\frac{1}{k}\text{rect}\left(\frac{f}{k}\right)\,df \\
&= \frac{1}{k^2}\int_{-k/2}^{k/2} \text{rect}\left(\frac{f}{k}\right)\,df \\
&= \frac{1}{k^2}\int_{-k/2}^{k/2} 1\,df \\
&= \frac{1}{k^2}\left[f\right]_{-k/2}^{k/2} \\
&= \frac{1}{k^2}\left(\frac{k}{2} - \left(-\frac{k}{2}\right)\right) \\
&= \frac{1}{k}
\end{aligned}$$

And we have shown that $\int_{-\infty}^{\infty} \operatorname{sinc}^2(kt)\,dt = \frac{1}{k}$.

## 2 Problem 2

2.1 Show that:

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)y^*(\tau - t)\,d\tau = \int_{-\infty}^{\infty} y^*(\tau)x(\tau + t)\,d\tau$$

Since we know the correlation function is defined as:

$$\begin{aligned}
r_{xy}(t) &= x(t) * y^*(-t) \\
&= \int_{-\infty}^{\infty} x(\tau)y^*(\tau - t)\,d\tau, \quad \text{Let } u = \tau - t,\, du = d\tau \\
&= \int_{-\infty}^{\infty} x(u + t)y^*(u)\,du \\
&= \int_{-\infty}^{\infty} y^*(u)x(u + t)\,du, \quad \text{Let } \tau = u,\, d\tau = du \\
&= \int_{-\infty}^{\infty} y^*(\tau)x(\tau + t)\,d\tau
\end{aligned}$$

And we have shown as required.

2.2 Show that $r_{xy}(t) = r_{yx}(-t)^*$:

$$\begin{aligned}
r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)y^*(\tau - t)\,d\tau &= \int_{-\infty}^{\infty} y^*(\tau)x(\tau + t)\,d\tau \\
&= \left(\int_{-\infty}^{\infty} y(\tau)x^*(\tau + t)\,d\tau\right)^* \\
&= (r_{yx}(-t))^*
\end{aligned}$$

2.3 If $y(t) = x(t + T)$, we can express $r_{xy}(t)$ and $r_{yy}(t)$ in terms of $r_{xx}(t)$:

First, $r_{xy}(t)$:

$$\begin{aligned}
r_{xy}(t) &= x(t) * y^*(-t) = x(t) * x^*(T - t) \\
&= \int_{-\infty}^{\infty} x(\tau)x^*(\tau - t + T)\,d\tau \\
&= r_{xx}(t - T)
\end{aligned}$$

Now, $r_{yy}(t)$:

$$\begin{aligned}
r_{yy}(t) &= y(t) * y^*(-t) = x(t + T) * x^*(T - t) \\
&= \int_{-\infty}^{\infty} x(\tau + T)x^*(\tau - t + T)\,d\tau, \quad \text{Let } \tau' = \tau + T,\, d\tau' = d\tau \\
&= \int_{-\infty}^{\infty} x(\tau')x^*(\tau' - t)\,d\tau' \\
&= r_{xx}(t)
\end{aligned}$$

2.4 What is the relationship between the cross-ESD's $\Psi_{xy}(f)$ and $\Psi_{yx}(f)$?

Given that $\Psi_{xy}(f) = \mathcal{F}\{r_{xy}(t)\}$ and $\Psi_{yx}(f) = \mathcal{F}\{r_{yx}(t)\}$. And from above we know that $r_{xy}(t) = r_{yx}(-t)^*$. We have:

$$\begin{aligned}
\Psi_{xy}(f) = \mathcal{F}\{r_{xy}(t)\} &= \mathcal{F}\{r_{yx}(-t)^*\} \\
&= \int_{-\infty}^{\infty} r_{yx}(-t)^* e^{-j2\pi ft}\, dt \\
&= \int_{-\infty}^{\infty} \left(r_{yx}(-t)e^{j2\pi ft}\right)^*\, dt \\
&= \left(\int_{-\infty}^{\infty} r_{yx}(-t)e^{j2\pi ft}\, dt\right)^* \\
&= \left(\mathcal{F}\{r_{yx}(t)\}\right)^* \\
\Psi_{xy}(f) &= \Psi_{yx}(f)^*
\end{aligned}$$

2.5 We can find an expression of $\Psi_{xy}(f)$ in terms of $X(f)$ and $Y(f)$ as follows:

$$\begin{aligned}
\Psi_{xy}(f) = \mathcal{F}\{r_{xy}(t)\} &= \mathcal{F}\{x(t) * y^*(-t)\} \\
&= \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{y^*(-t)\} \\
&= X(f)Y^*(-f)
\end{aligned}$$

2.6 We can show that the ESD is real and positive for every $f$ as follows:

$$\begin{aligned}
\Psi_{xx}(f) = \mathcal{F}\{r_{xx}(t)\} &= \mathcal{F}\{x(t) * x^*(-t)\} \\
&= \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{x^*(-t)\} \\
&= X(f)X^*(f) \\
&= |X(f)|^2
\end{aligned}$$

Since the magnitude squared of a complex number is always real and positive, we have shown that the ESD is real and positive for every $f$.

2.7 We can find the expressions of $\Psi_{xy}(f)$ and $\Psi_{yy}(f)$ in terms of $\Psi_{xx}(f)$ and $H(f)$ as follows.

First for $\Psi_{xy}(f)$, we can use the result from 2.5:

$$\Psi_{xy}(f) = X(f)Y^*(-f)$$

And we know that $y(t) = h(t) * x(t)$, so we have for $Y^*(f)$:

$$\begin{aligned}
Y^*(f) &= H^*(f)X^*(f) \\
Y^*(-f) &= H^*(-f)X^*(-f)
\end{aligned}$$

Substituting this into $\Psi_{xy}(f)$, we get:

$$\begin{aligned}
\Psi_{xy}(f) &= X(f)H^*(-f)X^*(-f) \\
&= |X(f)|^2 H^*(-f) \\
&= \Psi_{xx}(f)H^*(-f)
\end{aligned}$$

Similarly, for $\Psi_{yy}(f)$:

$$\begin{aligned}\Psi_{yy}(f) &= Y(f)Y^*(-f) \\ &= H(f)X(f)H^*(-f)X^*(-f) \\ &= |H(f)|^2|X(f)|^2 \\ &= |H(f)|^2\Psi_{xx}(f)\end{aligned}$$

2.8 We can deduce expressions for $r_{xy}(t)$ and $r_{yy}(t)$ in terms of $h(t), r_{hh}(t)$ and $r_{xx}(t)$. Starting with $r_{xy}(t)$:

$$\begin{aligned}r_{xy}(t) &= x(t) * y^*(-t) \\ y^*(-t) &= h^*(-t) * x^*(-t) \\ \implies r_{xy}(t) &= x(t) * h^*(-t) * x^*(-t) \\ &= x(t) * x^*(-t) * h^*(-t) \\ &= r_{xx}(t) * h^*(-t)\end{aligned}$$

Similarly for $r_{yy}(t)$:

$$\begin{aligned}r_{yy}(t) &= y(t) * y^*(-t) \\ &= h(t) * x(t) * h^*(-t) * x^*(-t) \\ &= h(t) * h^*(-t) * x(t) * x^*(-t) \\ &= r_{hh}(t) * r_{xx}(t)\end{aligned}$$

# 3 Problem 3

3.1 We have the following:

$$\begin{aligned}x(t) &= m(t)\cos^3(2\pi f_c t) \\ &= B\text{sinc}^2(Bt)\cos^3(2\pi f_c t)\end{aligned}$$

The fourier transform of $x(t)$ is given by:

$$X(f) = \mathcal{F}\{x(t)\} = \mathcal{F}\{B\text{sinc}^2(Bt)\cos^3(2\pi f_c t)\}$$

We can first find the fourier transform of the message signal, and then use the modulation property 3 times to find the fourier transform of the modulated signal. We know that the fourier transform of a $\text{sinc}^2$ function is a triangle function, and by applying the time-scaling property, we have the following:

$$\begin{aligned}M(f) &= \mathcal{F}\{B\text{sinc}^2(Bt)\} \\ M(f) &= B\left(\frac{1}{B}\Lambda\left(\frac{f}{B}\right)\right) \\ &= \Lambda\left(\frac{f}{B}\right)\end{aligned}$$

We can represent the modulated signal as follows:

$$x(t) = ((m(t)\cos(2\pi f_c t))\cos(2\pi f_c t))\cos(2\pi f_c t)$$

We have:

$$
\begin{aligned}
X(f) &= \left(\left(\Lambda\left(\frac{f}{B}\right) * \frac{1}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right]\right) * \frac{1}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right]\right) * \frac{1}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right] \\
&= \frac{1}{8}\left(\left(\Lambda\left(\frac{f-f_c}{B}\right)+\Lambda\left(\frac{f-f_c}{B}\right)\right) * \left[\delta(f-f_c)+\delta(f+f_c)\right]\right) * \left[\delta(f-f_c)+\delta(f+f_c)\right] \\
&= \frac{1}{8}\left(\Lambda\left(\frac{f-2f_c}{B}\right)+2\Lambda\left(\frac{f}{B}\right)+\Lambda\left(\frac{f+2f_c}{B}\right)\right) * \left[\delta(f-f_c)+\delta(f+f_c)\right] \\
&= \frac{1}{8}\left(\Lambda\left(\frac{f-3f_c}{B}\right)+2\Lambda\left(\frac{f-f_c}{B}\right)+\Lambda\left(\frac{f+f_c}{B}\right)+\Lambda\left(\frac{f-f_c}{B}\right)+2\Lambda\left(\frac{f+f_c}{B}\right)+\Lambda\left(\frac{f+3f_c}{B}\right)\right) \\
&= \frac{1}{8}\left(\Lambda\left(\frac{f-3f_c}{B}\right)+3\Lambda\left(\frac{f-f_c}{B}\right)+3\Lambda\left(\frac{f+f_c}{B}\right)+\Lambda\left(\frac{f+3f_c}{B}\right)\right)
\end{aligned}
$$

The sketch of this signal can be seen in Figure 1

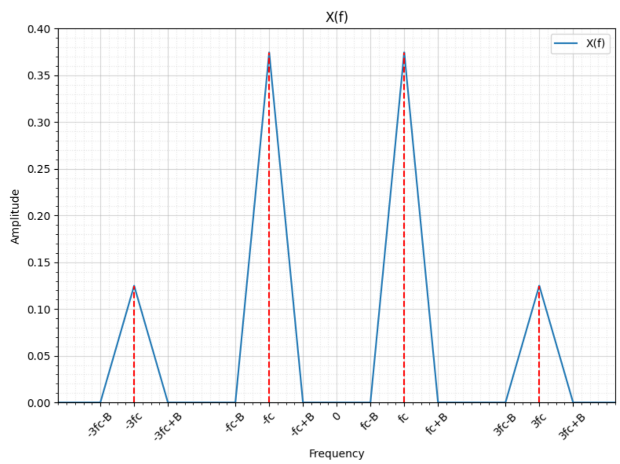


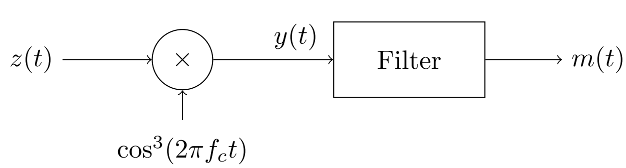

Figure 1: Sketch of $X(f)$

3.2 A suitable filter to generate hte modulated signal $z(t)$ will be a low pass filter that depends on the carrier frequency $f_c$ and the bandwidth of the message signal $B$. The cutoff frequency of the filter should be $f_c + B$. The sketch of the filter can be seen in Figure 2

3.3 The minimum usable value for the carrier frequency $f_c$ is $B$. If we pick a smaller value than the bandwidth of the signal, we will not be able to recover the original signal as amplitude modulating the signal will cause overlapping of the sidebands.

3.4 To design a receiver for the modulated signal, $z(t) = km(t)\cos(2\pi f_c t)$, and utilizing the same carrier generator of $\cos^3(2\pi f_c t)$, we can multiply the received signal by $\cos^3(2\pi f_c t)$ and pass it through a low pass filter with a cutoff frequency of $B$ to recover the message signal $m(t)$.

The block diagram of the receiver is as follows:



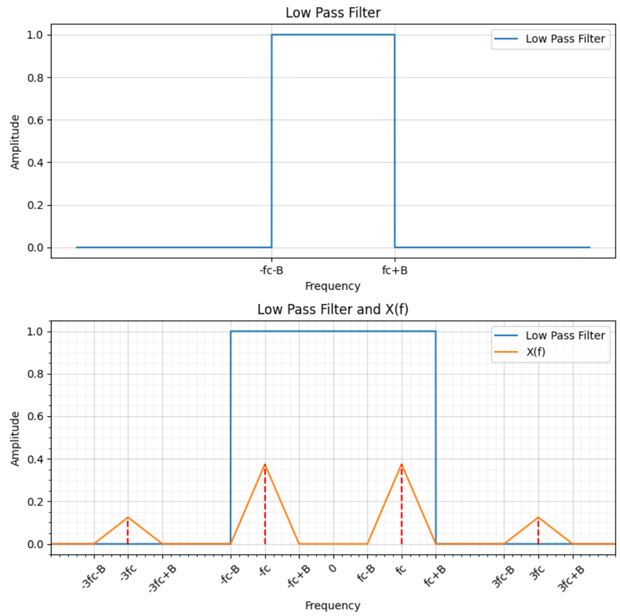



Figure 2: Sketch of the filter

We can recover the signal $m(t)$ by multiplying the received signal $z(t)$ by $\cos^3(2\pi f_c t)$ and passing it through a low pass filter with a cutoff frequency of $B$.

$$\begin{aligned} Y(f) = \mathcal{F}\{y(t)\} &= \mathcal{F}\{z(t)\cos^3(2\pi f_c t)\} \\ &= \mathcal{F}\{km(t)\cos^4(2\pi f_c t)\} \end{aligned}$$

We know from 3.1 that $x(t) = m(t)\cos^3(2\pi f_c t)$, so we can write $y(t)$ as:

$$y(t) = km(t)\cos^4(2\pi f_c t) = kx(t)\cos(2\pi f_c t)$$

We can now modulate $X(f)$, which we found in 3.1, by $\cos(2\pi f_c t)$ to get $Y(f)$:

$$\begin{aligned} Y(f) &= kX(f) * \frac{1}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right] \\ &= \frac{k}{16}\left(\Lambda\left(\frac{f-3f_c}{B}\right) + 3\Lambda\left(\frac{f-f_c}{B}\right) + 3\Lambda\left(\frac{f+f_c}{B}\right) + \Lambda\left(\frac{f+3f_c}{B}\right) * \left[\delta(f-f_c)+\delta(f+f_c)\right]\right) \\ &= \frac{k}{16}\left(\Lambda\left(\frac{f-4f_c}{B}\right) + 3\Lambda\left(\frac{f-2f_c}{B}\right) + 3\Lambda\left(\frac{f}{B}\right) + \Lambda\left(\frac{f+2f_c}{B}\right)\right. \\ &\left. + \Lambda\left(\frac{f-2f_c}{B}\right) + 3\Lambda\left(\frac{f}{B}\right) + 3\Lambda\left(\frac{f+2f_c}{B}\right) + \Lambda\left(\frac{f+4f_c}{B}\right)\right) \\ &= \frac{k}{16}\left(\Lambda\left(\frac{f-4f_c}{B}\right) + 4\Lambda\left(\frac{f-2f_c}{B}\right) + 6\Lambda\left(\frac{f}{B}\right) + 4\Lambda\left(\frac{f+2f_c}{B}\right) + \Lambda\left(\frac{f+4f_c}{B}\right)\right) \end{aligned}$$

When we pass this through a filter with a cutoff frequency of $B$, we will get the message signal $m(t)$, scaled by a constant.

$$
\begin{aligned}
M(f) = Y(f)H(f) &= Y(f) \times \Pi\left(\frac{f}{B}\right) \\
&= \frac{k}{16}\Lambda\left(\frac{f}{B}\right)
\end{aligned}
$$

$$
\frac{k}{16}\Lambda\left(\frac{f}{B}\right) \xrightarrow{\mathcal{F}^{-1}} \frac{kB}{16}\operatorname{sinc}^2(Bt) = \frac{k}{16}m(t)
$$