Assignment 1: Probability Assignment

Rajdeep Gill 7934493

ECE 4260 A01

January 24, 2025

1 Problem 1

1. Find the Euler representation of the complex numbers z_5, z_6 and z_7.

   - $z_5 = \frac{z_1}{z_2^*}$

$$
\begin{aligned}
z_5 &= \frac{z_1}{z_2^*} \\
&= \frac{\frac{5}{2} + \frac{5\sqrt{3}}{2}j}{\sqrt{3}+j} \\
&= \frac{5+5\sqrt{3}j}{2} \cdot \frac{1}{\sqrt{3}+j} \cdot \frac{\sqrt{3}-j}{\sqrt{3}-j} \\
&= \frac{(5+5\sqrt{3}j)\cdot(\sqrt{3}-j)}{2(\sqrt{3}+j)(\sqrt{3}-j)} \\
&= \frac{5\sqrt{3}-5j+15-5j}{2(3+1)} \\
&= \frac{15+5\sqrt{3}-10j}{8}
\end{aligned}
$$

To convert to Euler form, we need to find the magnitude and angle of the complex number.

$$
\begin{aligned}
|z_5| &= \frac{\sqrt{(15+5\sqrt{3})^2+(-10)^2}}{8^2} \\
&= \frac{\sqrt{225+150\sqrt{3}+75+100}}{64} \\
&= \frac{\sqrt{500+150\sqrt{3}}}{64}
\end{aligned}
$$

The angle is given by:

$$
\begin{aligned}
\theta &= \tan^{-1}\left(\frac{-10}{15+5\sqrt{3}}\right) \\
&= \tan^{-1}\left(\frac{-10}{15+5\sqrt{3}}\right)
\end{aligned}
$$

Therefore, the Euler representation of z_5 is:

$$
\begin{aligned}
z_5 &= ae^{j\theta} \\
&= \frac{\sqrt{500+150\sqrt{3}}}{64} e^{j\tan^{-1}\left(\frac{-10}{15+5\sqrt{3}}\right)}
\end{aligned}
$$

- $z_6 = \frac{z_3^*}{z_4}$

$$
\begin{aligned}
z_6 &= \frac{z_3^*}{z_4} \\
z_6 &= \frac{1+e^{jn1\varphi}}{1-e^{j\varphi}} \\
&= \frac{1+e^{j\pi\varphi}}{1-e^{j\varphi}} \cdot \frac{1-e^{-j\varphi}}{1-e^{-j\varphi}}
\end{aligned}
$$

2. Find the real and imaginary parts of z_7^*.

2 Problem 2

- $g_1(t) = g(-t)$
- $g_2(t) = g_1(t-1) + g(t-1) = g(-t+1) + g(t-1)$
- $g_3(t) = g_1(t+1) + g(t-1) = g(-t-1) + g(t-1)$
- $g_4(t) = g_1(t+1/2) + g(t-1/2) = g(-t-1/2) + g(t-1/2)$
- $g_5(t) = \frac{3}{2}g(\frac{t}{2}-1)$

3 Problem 3

Sketching

4 Problem 4

Since the signal is periodic, we can find the average power of the entire signal by integrating over 1 period and dividing by the period. The average power is given by:

$$P_g(t) = \frac{1}{T}\int_T |g(t)|^2 dt$$

The period of this signal is $T = 4$ and we can integrate from -2 to 2.

$$
\begin{aligned}
P_g(t) &= \frac{1}{4}\int_{-2}^{2} |g(t)|^2 dt \\
&= \frac{1}{4}\int_{-2}^{2} (t^3)^2 dt \\
&= \frac{1}{4}\int_{-2}^{2} t^6 dt \\
&= \frac{1}{4}\left[\frac{t^7}{7}\right]_{-2}^{2} \\
&= \frac{1}{4}\left[\frac{128}{7} - \frac{-128}{7}\right] \\
&= \frac{1}{4}\left[\frac{256}{7}\right] \\
&= \boxed{\frac{64}{7}}
\end{aligned}
$$

Therefore, the average power of the signal is $\frac{64}{7}$.

5 Problem 5

5.1 Show that $\mathcal{H}[.]$ is linear. To show that $\mathcal{H}[.]$ is linear, we need to show that it satisfies both the additivity and homogeneity properties.

Additivity:

Let $x_1(t)$ and $x_2(t)$ be two signals. Then, with $h(t)$ a fixed signal, we have:

$$
\begin{aligned}
x_1(t) \to y_1(t) &= \mathcal{H}[x_1(t)] = x_1(t)h(t) \\
x_2(t) \to y_2(t) &= \mathcal{H}[x_2(t)] = x_2(t)h(t) \\
(x_1(t) + x_2(t)) \to y_3(t) &= \mathcal{H}[x_1(t) + x_2(t)] = (x_1(t) + x_2(t))h(t) \\
&= x1(t)h(t) + x2(t)h(t) = y1(t) + y2(t)
\end{aligned}
$$

Thus $\mathcal{H}[.]$ satisfies the additivity property.

Homogeneity:

Let $x(t)$ be a signal and a be a scalar. Then, with $h(t)$ a fixed signal, we have:

$$
\begin{aligned}
x(t) \to y_1(t) &= \mathcal{H}[x(t)] = x(t)h(t) \\
ax(t) \to y_2(t) &= \mathcal{H}[ax(t)] = ax(t)h(t) \\
&= a(x(t)h(t)) = ay_1(t)
\end{aligned}
$$

Thus $\mathcal{H}[.]$ satisfies the homogeneity property and is linear.

5.2 Show that if $\mathcal{H}[.]$ is time-invariant, then h(t) must be a constant, i.e. $h(t) = c, \forall t, c \in \mathbb{R}$.

A system is time-invariant if a shift in the input signal results in a corresponding shift in the output signal. Let $x(t)$ be a signal.

$$\begin{aligned} x(t-t_0) \rightarrow y_1(t) = \mathcal{H}[x(t-t_0)] = x(t-t_0)h(t) \\ y_2(t-t_0) = x(t-t_0)h(t-t_0) \end{aligned}$$

Then, $y_2(t-t_0) = y_1(t)$ if, and only if, $h(t) = c$ for some constant c.

If $h(t) = c$, then $y_1(t) = cx(t-t_0)$ and $y_2(t-t_0) = cx(t-t_0)$. Therefore, the system is time-invariant if $h(t) = c$.

5.3 Suppose that the switch is closed for $-1/2 \leq t \leq 1/2$. Using 5.1, show that this is a linear system.

The new system is given by:

$$\mathcal{H}[x(t)] = \begin{cases} x(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Let $x_1(t)$ and $x_2(t)$ be two signals. Then we have:

$$\begin{aligned} \mathcal{H}[x_1(t) + x_2(t)] &= \begin{cases} x_1(t) + x_2(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x_1(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} x_2(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \mathcal{H}[x_1(t)] + \mathcal{H}[x_2(t)] \end{aligned}$$

And:

$$\begin{aligned} \mathcal{H}[ax(t)] &= \begin{cases} ax(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= a \begin{cases} x(t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= a\mathcal{H}[x(t)] \end{aligned}$$

Thus the system satisfies the additivity and homogeneity properties and is linear.

5.4 Suppose that the switch is turned on and off at various time intervals. Is the system still linear any why? Is it time-invariant and why?

We can represent the new system as:

$$\mathcal{H}[x(t)] = \begin{cases} x(t) & \text{if } t \in \bigcup_{i=1}^{n} [a_i, b_i] \\ 0 & \text{otherwise} \end{cases}$$

We can show that it is linear as it follows both additivity and homogeneity:

$$\begin{aligned}
\mathcal{H}[ax_1(t)+bx_2(t)] &= \begin{cases} ax_1(t)+bx_2(t) & \text{if } t \in \bigcup_{i=1}^{n}[a_i,b_i] \\ 0 & \text{otherwise} \end{cases} \\
&= a\begin{cases} x_1(t) & \text{if } t \in \bigcup_{i=1}^{n}[a_i,b_i] \\ 0 & \text{otherwise} \end{cases} + b\begin{cases} x_2(t) & \text{if } t \in \bigcup_{i=1}^{n}[a_i,b_i] \\ 0 & \text{otherwise} \end{cases} \\
&= a\mathcal{H}[x_1(t)] + b\mathcal{H}[x_2(t)]
\end{aligned}$$

It is not time-invariant as the output signal will change depending on the time intervals the switch is on or off. A delay in the input signal will not result in a corresponding delay in the output signal.

5.5 Suppose that the switch is turned on at $t = 0$ and stays on forever. Is the system now linear and why? Is it time-invariant and why?

We can define the new system as:

$$\begin{aligned}
y(t) = \mathcal{H}[x(t)] &= u(t)x(t) \\
u(t) &= \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The system is linear as it satisfies both additivity and homogeneity:

$$\begin{aligned}
y(ax_1(t)+bx_2(t)) = \mathcal{H}[ax_1(t)+bx_2(t)] &= u(t)(ax_1(t)+bx_2(t)) \\
&= au(t)x_1(t) + bu(t)x_2(t) \\
&= a\mathcal{H}[x_1(t)] + b\mathcal{H}[x_2(t)] \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

It is not time-invariant and we can show this by considering a delay in the input signal:

$$\begin{aligned}
x(t-t_0) \rightarrow y_1(t) &= u(t)x(t-t_0) \\
y_2(t-t_0) &= u(t-t_0)x(t-t_0) \\
&\neq y_1(t)
\end{aligned}$$

Therefore, since a delay in the input does not lead to the same result as a delay in the output, the system is not time-invariant.