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ECE–4260 Communication Systems

Assignment#5 (Winter 2025)

Remarks:

- This assignment is due on **April 8th, 2025**. Please scan your answers and upload them to UM Learn. Do not e-mail pdf documents.
- **Make sure to compile a single PDF document that contains all your answers, i.e., do not submit separate pages.**
- **Presenting clean and well-justified answers helps you get full marks for questions.**
- The assignments submitted after the due date will not be marked.
- Clearly show the solution method. Marks are awarded for the method and not the final answer itself.

1. **Problem 1** (**30 points**)

   Preliminaries: Probability Theory — You must read **Appendix A** in pages 7 and 8 at the end of this Assignment, before doing this Problem. You will be referred to it repeatedly in many questions.
   Problem Statement: Take a natural number $n \geq 2$ and let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed ($i.i.d$) random variables. By "identically distributed", it is meant that all random variables are distributed accoording to the same pdf, i.e., for $i = 1, 2, \ldots, n$ we have:

$$X_i \;\sim\; f_X(x_i), \tag{1}$$

   for some common pdf $f_X(.)$ that does not depend on i. This does not mean, however, that the n random variables are identical. This means that if you run a probabilistic experiment, each random variable X_i will take its own realization x_i (that is different from the realizations taken by the other RVs). Being *identically distributed* simply means that all realizations $x_1, x_2, \ldots, x_n$ that you observe are sampled from the same common distribution $f_X(.)$ in (1).
   Next, consider the following random variable, S_n, with *unknown* pdf $f_{S_n}(s)$:

$$S_n = \frac{1}{\sqrt{n}}\big(X_1 + X_2 + \ldots + X_n\big) \;\sim\; f_{S_n}(s). \tag{2}$$

   The goal of this problem is to show the *Central limit theorem (CLT)*, which is widely used in science and engineering, using standard tools of Fourier transforms we studied in this course. The CLT is stated informally as follows:

   Theorem 1: The RV S_n in (2) tends to a Gaussian RV as n tends to $+\infty$, for any common pdf $f_X(.)$ in (1) for the X_i's.

   To prove the CLT theorem, we first consider the unnormalized version of S_n, i.e.:

$$Z_n = X_1 + X_2 + \ldots + X_n, \tag{3}$$

   whose *unknown* pdf is denoted as $f_{Z_n}(z_n)$.

   1.1 Let U be any random variable with pdf $f_U(u)$ whose Fourier transform is $F_U(f)$, i.e., $f_U(u) \longleftrightarrow F_U(f)$. Using (44), show that (**2.5 points**):

$$\mathbb{E}\left\{e^{-j2\pi f U}\right\} \;=\; F_U(f). \tag{4}$$

1.2 Let $F_{Z_n}(f)$ be the Fourier transform of $f_{Z_n}(z_n)$ and $F_X(f)$ be the Fourier transform of the common pdf $f_X(x)$ in (1), i.e.:

$$f_{Z_n}(z_n) \longleftrightarrow F_{Z_n}(f) \tag{5}$$
$$f_X(x) \longleftrightarrow F_X(f) \tag{6}$$

By refering to Appendix A, show that (**5 points**):

$$F_{Z_2}(f) = \Big(F_X(f)\Big)^2. \tag{7}$$

Hint: Recall the fact the two RVs X_1 and X_2 are independent and refer to **Definition 1**.

1.3 Deduce by mathematical induction that we have (**2.5 points**):

$$F_{Z_n}(f) = \Big(F_X(f)\Big)^n. \tag{8}$$

Hint: Observe that $Z_n = Z_{n-1} + X_n$.

1.4 Using (44) , show that (**2.5 points**):

$$F_{S_n}(f) = F_{Z_n}\big(f/\sqrt{n}\big). \tag{9}$$

By Injecting (8) in (9), we conclude that :

$$\boxed{F_{S_n}(f) = \Big(F_X\big(f/\sqrt{n}\big)\Big)^n} \tag{10}$$

Now, from the basic definition of the Fourier transform we have:

$$F_X\big(f/\sqrt{n}\big) = \int_{-\infty}^{+\infty} f_X(x)e^{-j2\pi(f/\sqrt{n})x}dx \tag{11}$$

As we are tending n to $+\infty$ in the Central limit theorem (cf. **Theorem 1**), we have in mind that $2\pi(f/\sqrt{n})x$ can be made as small as we want for every x. Therefore, we will use the order-2 Taylor series expansion of $e^{-j2\pi(f/\sqrt{n})x}$ around 0. In this context, recall that the order-2 Taylor series expansion of e^t around zero is:

$$e^t = 1 + t + t^2/2 + o(t^2), \tag{12}$$

where $o(t^2)$ encompasses all the higher-order terms, which can be reasonably neglected as t becomes vanishingly small.
Also, without loss of generality, we will assume in the sequel that each RV X_i involved in (2) has zero mean and unit variance. Since from (1), we are assuming $X_i \sim f_X(x_i)\ \forall\ i$, this means that:

$$\mathbb{E}\{X_i\} = \int_{-\infty}^{+\infty} x f_X(x)dx = 0 \quad \text{and} \quad \sigma^2_{X_i} = \int_{-\infty}^{+\infty} x^2 f_X(x)dx = 1. \tag{13}$$

1.5 Show that the order-2 Taylor series expansion of $e^{-j2\pi(f/\sqrt{n})x}$ is given by (**5 points**):

$$e^{-j2\pi(f/\sqrt{n})x} \approx 1 - \frac{j2\pi f}{\sqrt{n}}x - \frac{2\pi^2 f^2}{n}x^2 \qquad \text{(for large enough } n\text{)}. \tag{14}$$

1.6 Deduce that under the assumption in (13) it follows that (**2.5 points**):

$$F_X\big(f/\sqrt{n}\big) \approx 1 - \frac{2\pi^2 f^2}{n} \qquad \text{(for large enough } n\text{)}. \tag{15}$$

1.7 Using the following well-known result from calculus:

$$\Big(1 - \frac{\alpha}{n}\Big)^n \approx e^{-\alpha} \quad \text{as} \quad n \longrightarrow +\infty, \tag{16}$$

and by recalling (10) show that (**5 points**):

$$F_{S_n}(f) \approx e^{-2\pi^2 f^2} \qquad \text{as} \quad n \longrightarrow +\infty, \tag{17}$$

1.8 By inverse Fourier transform, deduce that (**5 points**):

$$\boxed{f_{S_n}(s) \;\approx\; \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}} \qquad \text{as} \;\; n \longrightarrow +\infty.} \tag{18}$$

Hint: Recall that in class we showed that $e^{-\pi s^2} \longleftrightarrow e^{-\pi f^2}$. (Cf. course notes).

The result you established in (18) is exactly what is stated by the CLT theorem which says that S_n tends to Gaussian random variable as n tends to $+\infty$. The limiting Gaussian pdf happens to be of zero mean and unit variance just because of the assumption we made in (13) about the common pdf $f_X(x)$ of all the X_i's giving rise to S_n in (2). This result can be easily generalized to non-zero mean and non-unit variance by modifying the steps above. A MATLAB validation of the CLT theorem is provided in the next page. There, the i.i.d random variables follow a common binary distribution given by:

$$f_X(x) = \begin{cases} 1/2 & \text{if} \quad x = +1, \\ 1/2 & \text{if} \quad x = -1, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

And the plot is shown in Fig. 1 (cf. next page). **You can run the code on your computer**!

```
function [S_n] = histogram_faouzi

n = 100000;                     % Choosing a large value for n
k = 30000;                      % since S_n itself is a random variable, we need
                                % to generate a large number of its realizations
                                % (here we will generate k of them)to plot its
                                % histogram and see if it looks like a Gaussian

X = 2*randi([0, 1], n,k)-1;     % This matrix (n rows, k columns) contains
                                % n*k realizations of a binary random variable
                                % X which takes values in {-1,+1} with
                                % probabilities 1/2 and 1/2.You can check that
                                % its mean is 0 and its variance is 1

S_n = sum(X,1)/sqrt(n);         % By summing over all rows of X separately, you
                                % get k realizations of the random variable S_n
                                % (one realization for each sum).

nbins = 50;                     % number of bins used to plot the histogram of
                                % the random variable S_n
figure(1)
histogram(S_n,nbins,'Normalization','pdf')    % plot the histogram of the
hold on                                       % random variable S_n

t = -6:0.01:6;                             % plot the limiting Gaussian
plot(t,1/sqrt(2*pi)*exp(-t.^2/2), 'r')    % distribution (mean 0, variance 1)
```

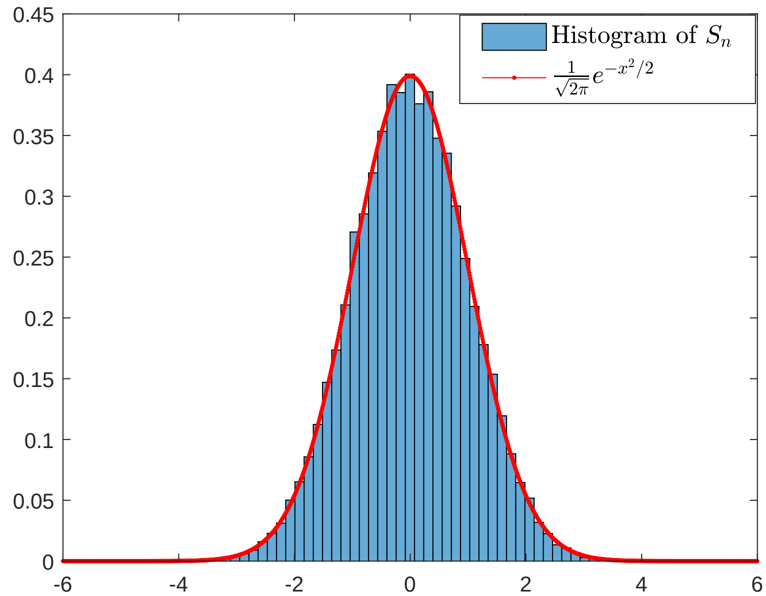

Figure 1: Plot of the histogram of S_n and the limiting Gaussian distribution in eq. (18).

2. **Problem 2 (30 points)**

Preliminaries: For ordinary vectors in $\mathbf{R}^2$, we are familiar with the notions of *inner product*, *norm*, *distance*, and *projection* which are defined as follows. Let $\mathbf{u} = [u_1,\, u_2]^\mathsf{T}$, and $\mathbf{v} = [v_1,\, v_2]^\mathsf{T}$ be two arbitrary vectors in $\mathbb{R}^2$, where the operator $(.)^\mathsf{T}$ stands for transposition. Then, their *inner product* is defined as follows:

$$\langle \mathbf{u}, \mathbf{v}\rangle \;\triangleq\; \mathbf{u}^\mathsf{T}\mathbf{v} \;=\; [u_1,\, u_2]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \;=\; u_1v_1 + u_2v_2. \tag{20}$$

The *norm* of a vector $\mathbf{u} = [u_1,\, u_2]^\mathsf{T}$ is thus given by:

$$\|\mathbf{u}\| \;=\; \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle} \;=\; \sqrt{u_1^2 + u_2^2}, \tag{21}$$

and the distance between vectors $\mathbf{u}$ and $\mathbf{v}$ is given by:

$$d_{\mathbf{uv}} \;=\; \|\mathbf{u} - \mathbf{u}\|. \tag{22}$$

All of these standard notions generalize to signals by defining the inner product as follows for any two signals/-functions $s_1(t)$ and $s_2(t)$:

$$\big\langle s_1(t), s_2(t)\big\rangle \;\triangleq\; \int_{-\infty}^{+\infty} s_1(t)s_2(t)dt. \tag{23}$$

With this inner product, we can define the notion of orthogonality of signals as follows:

$$s_1(t) \text{ and } s_1(t) \text{ are orthogonal (denoted } s_1(t) \perp s_2(t)\big) \quad\Longleftrightarrow\quad \big\langle s_1(t), s_2(t)\big\rangle = 0. \tag{24}$$

similar to ordinary vectors, we can also extend the notion of "norm" to signals as follow:

$$\|s(t)\| \;=\; \sqrt{\big\langle s(t), s(t)\big\rangle} \;=\; \left(\int_{-\infty}^{+\infty} s(t)^2 dt\right)^{\frac{1}{2}} \;=\; \sqrt{\mathcal{E}}. \tag{25}$$

where $\mathcal{E}$ is the energy of the signal $s(t)$. The distance, d_{12}, between any two signals/functions $s_1(t)$ and $s_2(t)$ is also the norm of their difference, i.e.:

$$d_{12} \;=\; \|s_1(t) - s_2(t)\|. \tag{26}$$

Problem Statement:

• **PART I**: Consider a binary communication system that conveys bits the 0 and 1 by transmitting $s_1(t)$ and $s_2(t)$, respectively, with energies $\mathcal{E}_1$ and $\mathcal{E}_2$. Consider the following two signals:

$$\Psi_1(t) = \frac{s_1(t)}{\|s_1(t)\|}, \tag{27}$$

$$\Psi_2(t) = \frac{d_2(t)}{\|d_2(t)\|}, \tag{28}$$

where

$$d_2(t) = s_2(t) - \alpha_{21}\Psi_1(t), \tag{29}$$

$$\alpha_{21} = \big\langle s_2(t), \Psi_1(t)\big\rangle. \tag{30}$$

2.1 Show that $\Psi_1(t) \perp \Psi_2(t)$ and $\|\Psi_1(t)\| = \|\Psi_2(t)\| = 1$. (**5 points**).

This shows that $\Psi_1(t)$ and $\Psi_2(t)$ are two orthornormal basis vectors for the signal space $\big\{s_1(t),\, s_2(t)\big\}$.

2.2 By expressing the two signals $s_1(t)$ and $s_2(t)$ in terms of $\Psi_1(t)$ and $\Psi_2(t)$, show that their resulting geometric vector representation is given by (**5 points**):

$$s_1(t) \longleftrightarrow \mathbf{s}_1 = \begin{bmatrix} \sqrt{\mathcal{E}_1} \\ \\ 0 \end{bmatrix} \quad \text{and} \quad s_2(t) \longleftrightarrow \mathbf{s}_2 = \begin{bmatrix} \alpha_{21} \\ \\ \sqrt{\mathcal{E}_2 - \alpha_{21}^2} \end{bmatrix}. \tag{31}$$

2.3 Show the following identities:

(a) $\|s_1(t)\|^2 = \|\mathbf{s}_1\|^2$ (**1 point**).

(b) $\|s_2(t)\|^2 = \|\mathbf{s}_2\|^2$ (**1 point**).

(c) $\|s_2(t) - s_2(t)\|^2 = \|\mathbf{s}_2 - \mathbf{s}_1\|^2$ (**2 points**).

(d) $\big\langle s_2(t), s_1(t)\big\rangle = \langle \mathbf{s}_2, \mathbf{s}_1\rangle$ (**1 point**).

> The identities established in Q.2.3 show that once the geometric representation of signals has been found, then the signals' energies (i.e., squared norms), their distance and inner product can be easily found by standard vector manipulations (using the associated geometric vector representations of course. **Do not forget This!**

• **PART II**: Now assume that two signals transmitted by the binary communication system of PART I are explicitly given by:

$$s_i(t) = \begin{cases} \sqrt{\frac{2\mathcal{E}_b}{T}} \cos\left(2\pi f_i t\right) & 0 \leq t \leq T, \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

where for $i = 1, 2$, $f_i = \frac{\ell + i}{T}$ with $\ell > 0$ being some fixed positive integer. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the respective energies of the signals $s_1(t)$ and $s_2(t)$ given in (32). The associated orthonormal basis vectors/functions are again denoted as $\Psi_1(t)$ and $\Psi_2(t)$. If $s_i(t)$ is transmitted, the corresponding received signal, denoted as $r(t)$, is given by:

$$r(t) = s_i(t) + w(t), \tag{33}$$

wherein $w(t)$ is a zero-mean white Gaussian process with auto-correlation function given by

$$\mathbb{E}\big\{w(t)w(\tau)\big\} = \frac{N_0}{2}\delta(t - \tau). \tag{34}$$

2.4 Show that $\mathcal{E}_i = \mathcal{E}_b$, for $i = 1, 2$. (**5 points**).

2.5 Find the two orthornormal basis vectors/functions $\Psi_1(t)$ and $\Psi_2(t)$. (**5 points**).

2.6 The error probability for a general binary communication system is given by (**5 points**):

$$P_e = Q(d_{12}/\sqrt{2N_0}), \qquad \text{with} \qquad Q(x) \;=\; \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-t^2/2}dt. \tag{35}$$

Express P_e in terms of the SNR per bit $\mathcal{E}_b/N_0$

3. **Problem 3** (**40 points**)

Preliminaries: The correlation coefficient, ρ, of any two signals $f(t)$ and $g(t)$ is defined as follows:

$$\rho \;\triangleq\; \frac{1}{\sqrt{\mathcal{E}_f\mathcal{E}_g}}\int_{-\infty}^{+\infty} f(t)g(t)dt \tag{36}$$

where $\mathcal{E}_f$ and $\mathcal{E}_g$ are the energies of the signals $f(t)$ and $g(t)$, respectively.

The normalized sinc function is also defined for all $t \in \mathbb{R}^*$ as follows:

$$\operatorname{sinc}(t) \;=\; \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases} \tag{37}$$

The sinc function is plotted in Fig. 2 for $-5 \leq t \leq 5$.

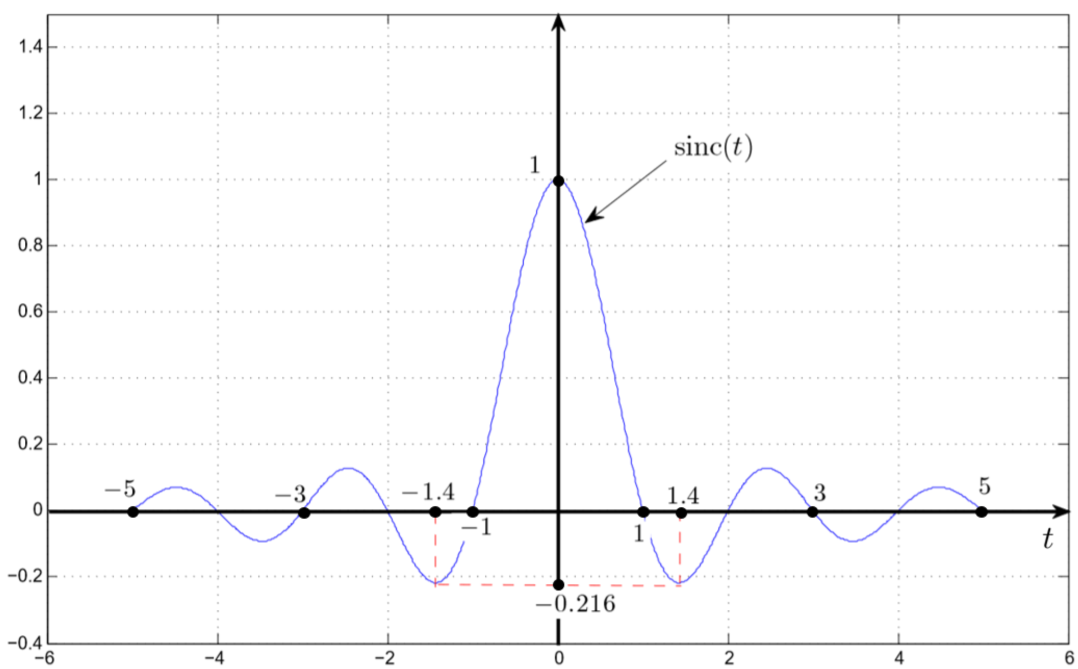

Figure 2: Plot of the sinc function defined in eq. (37).

Problem Statement: A frequency-shift keying (FSK) system, transmits either $s_1(t)$ or $s_2(t)$ to convey either 0 or 1:

$$s_1(t) \;=\; A\cos\left(2\pi\left[f_c + \frac{\Delta f}{2}\right]t\right) \qquad 0 \leq t \leq T, \tag{38}$$

$$s_2(t) \;=\; A\cos\left(2\pi\left[f_c - \frac{\Delta f}{2}\right]t\right) \qquad 0 \leq t \leq T, \tag{39}$$

in which $A = \sqrt{2\mathcal{E}_b/T}$, $f_c \gg 1$, and $\Delta f > 0$ with $\Delta f \ll f_c$. We also assume that the two signals are equally likely. The received signal is $r(t) = s_m(t) + n(t)$, $m = 1, 2$, with the additive noise, $n(t)$, being well-modeled by a zero-mean Gaussian process with power spectral density $S_N(f) = \frac{N_0}{2}$, $\forall f$.

3.1 a) Show that the correlation coefficient, ρ, of the two signals $s_1(t)$ or $s_2(t)$ is approximately given by (**8 points**)

$$\rho \approx \text{sinc}\,(2\Delta f T) \tag{40}$$

b) Use Fig. 2 to sketch the plot of the approximate expression of ρ in (40) as function of Δf. In your plot, you must also show the values of the associated key points (along the $x-$ and $y-$ axes) as it was done in Fig. 2 for the pure sinc function. (**8 points**).

3.2 What is the minimum value of frequency shift Δf for which the two signals $s_1(t)$ or $s_2(t)$ are orthogonal? (**8 points**).

3.3 Express the average probability of error, P_e, of the underlying binary FSK system as function of ρ (**8 points**).
Hint: Recall the expression of P_e as function of the distance, d_{12}, between the two signals $s_1(t)$ and $s_2(t)$.

3.4 What is the value (as function of T) of Δf that minimizes P_e. (**8 points**).

3.5 For the value of Δf found in 2.4, determine the increase in the SNR per bit, $\mathcal{E}_b/N_0$, required so that this binary FSK system has the same average probability of error as a binary antipodal system (**8 points**).

Appendix—A: Probability Theory (Preliminaries for Problem 1)

• Single random variable:
In probability theory, a continuous random variable X is completely described by its probability density function $f_X(x)$, where $-\infty < x < +\infty$ stands for all possible values that X can take. (Note here the distinction in notation between capital X for the random variable itself and the small x as a generic variable for the possible values that X can take, each with a different probability). When the pdf of X is $f_X(x)$, we use

$$X \sim f_X(x) \tag{41}$$

as a shorthand notation for saying "the RV X is distributed according to the pdf $f_X(x)$". To be a valid pdf, $f_X(x)$ must satisfy two necessary conditions, positivity and normalization:

$$f_X(x) \geq 0, \ \ \forall x \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{+\infty} f_X(x)dx = 1. \tag{42}$$

For a given (possibly complex-valued) function:

$$\begin{aligned} g \ : \ & \mathbb{R} \longrightarrow \mathbb{C} \\ & u \longrightarrow g(u) \end{aligned} \tag{43}$$

it can be shown that:

$$\mathbb{E}\{g(X)\} = \int_{-\infty}^{+\infty} g(x)f_X(x)dx. \tag{44}$$

The expected value, $m_X \triangleq \mathbb{E}\{X\}$, of the RV X is obtained as a special case of (44). Indeed, when the function $g(.)$ in (44) is given by $g(u) = u$, $\forall\ u \in \mathbb{R}$, we have $g(X) = X$ and $g(x) = x\ \forall\ x \in \mathbb{R}$, thereby leading to:

$$\mathbb{E}\{X\} = \mathbb{E}\{g(X)\}, \tag{45}$$

$$= \int_{-\infty}^{+\infty} g(x)f_X(x)dx \tag{46}$$

$$= \int_{-\infty}^{+\infty} xf_X(x)dx \tag{47}$$

The variance $\sigma_X^2 \triangleq \mathbb{E}\{(X-m_X)^2\}$, of the RV X is also obtained as a special case of (44). Indeed, when the function $g(.)$ in (44) is given by $g(u) = (u-m_X)^2, \ \forall\, u \in \mathbb{R}$, we have $g(X) = (X-m_X)^2$ and $g(x) = (x-m_X)^2 \ \forall\, x \in \mathbb{R}$, thereby leading to:

$$\begin{aligned} \mathbb{E}\{(X-m_X)^2\} &= \mathbb{E}\{g(X)\}, && (48)\\ &= \int_{-\infty}^{+\infty} g(x) f_X(x) dx && (49)\\ &= \int_{-\infty}^{+\infty} (x-m_X)^2 f_X(x) dx && (50) \end{aligned}$$

Now, the (possibly complex) quantity $Y = g(X)$ is itself a random variable (since X is random) and hence can be described by its own pdf $f_Y(y)$. Obviously, if the RV X takes the value x, then the RV Y takes the value $y = g(x)$. The expected value Y can be obtained from (47) using the pdf $f_Y(y)$ as follows:

$$\mathbb{E}\{Y\} = \int_{-\infty}^{+\infty} y f_Y(y) dy. \qquad (51)$$

Note, however, that this requires one to find the pdf $f_Y(y)$ of the *new* RV Y before being able to compute the integral in (51). Alternatively, $\mathbb{E}\{Y\}$ can be directly obtained from (44) using the *available* pdf $f_X(x)$ of the RV X as follows:

$$\mathbb{E}\{Y\} = \mathbb{E}\{g(X)\} = \int_{-\infty}^{+\infty} g(x) f_X(x) dx. \qquad (52)$$

• **Two random variables:**
If you have two random variables X_1 and X_2, then you can completely describe them by their joint pdf $f_{X_1,X_2}(x_1,x_2)$ which must also satisfy the positivity and normalization properties:

$$f_{X_1,X_2}(x_1,x_2) \geq 0, \ \ \forall x_1, x_2 \in \mathbb{R} \qquad \text{and} \qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X_1,X_2}(x_1,x_2) dx_1 dx_2 = 1. \qquad (53)$$

If you have $f_{X_1,X_2}(x_1,x_2)$, then you know everything about the two RVs X_1 and X_2 including their individual pdfs $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$, which can be found as follows:

$$f_{X_1}(x_1) = \int_{-\infty}^{+\infty} f_{X_1,X_2}(x_1,x_2) dx_2 \qquad \text{and} \qquad f_{X_2}(x_2) = \int_{-\infty}^{+\infty} f_{X_1,X_2}(x_1,x_2) dx_1 \qquad (54)$$

Definition 1: The two RVs X_1 and X_2 are said to be independent if their joint pdf $f_{X_1,X_2}(x_1,x_2)$ factors as the product of their individual pdfs $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$, in (54) i.e.:

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2), \quad \forall\, x_1, x_2 \in \mathbb{R}. \qquad (55)$$

For a known (possibly complex-valued) function $h(.,.)$:

$$\begin{aligned} h \ : \ \mathbb{R}\times\mathbb{R} &\longrightarrow \mathbb{C}\\ (u,v) &\longrightarrow h(u,v) \end{aligned} \qquad (56)$$

the quantity $Z = h(X_1, X_2)$ is another random variable which takes the value $z = h(x_1,x_2)$ whenever the RVs X_1 and X_2 take the values x_1 and x_2, respectively. The expected value, $\mathbb{E}\{\mathbb{Z}\}$, of the RV Z can be computed by first finding its pdf $f_Z(z)$ and then using (47):

$$\mathbb{E}\{Z\} = \int_{-\infty}^{+\infty} z f_Z(z) dz, \qquad (57)$$

Alternatively, $\mathbb{E}\{Z\}$ can be found directly using the *available* joint pdf $f_{X_1,X_2}(x_1,x_2)$, without the need to find $f_Z(z)$, as follows:

$$\mathbb{E}\{Z\} = \mathbb{E}\{h(X_1,X_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x_1,x_2) f_{X_1,X_2}(x_1,x_2) dx_1 dx_2 \qquad (58)$$