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ECE–4260 Communication Systems

Assignment#4 (Winter 2025)

Remarks:

- This assignment is due on **March 11th, 2025**. Please scan your answers and upload them to UM Learn. Do not e-mail pdf documents.
- **Make sure to compile a single PDF document that contains all your answers, i.e., do not submit separate pages and make sure to number your pages.**
- **Presenting clean and well-justified answers helps you get full marks for questions.**
- Assignments submitted after the due date will not be marked.
- Clearly show the solution method. Marks are awarded for the method and not the final answer itself.

1. **Problem 1 (50 points)**: Validation of the Carson's rule for bandwidth analysis of FM signals

For an FM modulated signal with (angular) carrier frequency is denoted as $\omega_c = 2\pi f_c$ and amplitude $A > 0$:

$$x_{\mathrm{FM}}(t) \,=\, A\cos\left(\omega_c t + k_f \int_{-\infty}^{t} m(\tau)d\tau\right), \tag{1}$$

and a generic message signal $m(t)$ of bandwidth B Hz, it was shown in the notes that its spectral (or bandwidth) analysis requires the use of a staircase signal approximation which lead us to the so-called Carson's rule. The latter provides an estimate of the required bandwidth, B_{FM}, of the FM-modulated signal:

$$B_{\mathrm{FM}} \,=\, 2(\Delta f + B), \tag{2a}$$
$$\,=\, 2(\beta + 1)B, \tag{2b}$$

wherein $\Delta f = k_f m_p/2\pi$ is the maximum frequency deviation with $m_p = \max|m(t)|$ and $\beta \triangleq \Delta f/B$ is the so-called frequency deviation ratio.

In this problem, we consider the special case when the message signal $m(t)$ is a sinusoid:

$$m(t) \,=\, \alpha\cos(\omega_m t), \tag{3}$$

for which a precise spectral analysis is possible (i.e., no staircase signal approximation is required). Here $\alpha > 0$ is some nonnegative real-valued amplitude and $\omega_m = 2\pi f_m$ is the angular frequency of the sinusoid, and the bandwidth of $m(t)$ is $B_m = \omega_m/2\pi$. We will use this special case along with its precice spectral analysis to verify the FM bandwidth approximation predicted by the general Carson's rule in (2).

Part I: To do so, we start by establishing some algebraic properties of the so-called Bessel function of the first kind and the nth order, $J_n(.)$, which is defined as (will be encountered later on in this problem):

$$J_n(y) \,=\, \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(y\sin x - nx)}dx, \quad y \in \mathbb{R},\ n \in \mathbb{Z}. \tag{4}$$

1.1) Show that if a function $g(x)$ is periodic with period T, then we have (**5 points**):

$$\int_0^T g(x)dx = \int_{-T/2}^{T/2} g(x)dx. \tag{5}$$

Hint: Recall the fact that $\int_a^c g(x)dx = \int_a^b g(x)dx + \int_b^c g(x)dx$.

1.2) Deduce that $J_n(y)$ is also given by (**5 points**):

$$J_n(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{j(y\sin x - nx)}dx, \quad y \in \mathbb{R}. \tag{6}$$

1.3) Deduce from (6) the following relationship (**5 points**):

$$J_{-n}(y) = (-1)^n J_n(y), \tag{7}$$

1.4) Show that for all $n \in \mathbb{Z}$ the function $J_n(y)$ is real-valued. In other words, show that (**5 points**):

$$J_n(y) \in \mathbb{R} \quad \text{for all} \quad y \in \mathbb{R} \tag{8}$$

Hint: For any complex number z, $z \in \mathbb{R}$ if and only if $z^* = z$.

Part II: Now back to the precise spectral analysis of our FM-modulated signal when the message signal is given by (3). In this part, $\Re\{.\}$ returns the real part of any complex number and we let:

$$a(t) = \int_{-\infty}^{t} m(\tau)d\tau, \tag{9}$$

1.5) Show that the deviation ratio is given by (**5 points**):

$$\beta = \frac{k_f \alpha}{\omega_m}. \tag{10}$$

1.6) With the assumtion that initially $a(-\infty) = 0$, show that the FM-modulated signal corresponding to $m(t)$ in (3) is given by (**5 points**):

$$x_{\text{FM}}(t) = A\,\Re\left\{z(t)e^{j\omega_c t}\right\}, \tag{11}$$

in which $z(t)$ is given by:

$$z(t) = e^{j\beta\sin(\omega_m t)} \tag{12}$$

1.7) Show that the signal $z(t)$ is periodic with period $2\pi/\omega_m$. (**5 points**).

The Fourier series expansion of $z(t)$ is thus given by

$$z(t) = \sum_{n=-\infty}^{+\infty} z_n e^{jn\omega_m t}, \tag{13}$$

with

$$z_n = \frac{\omega_m}{2\pi}\int_{-\pi/\omega_m}^{\pi/\omega_m} z(t)e^{-jn\omega_m t}dt. \tag{14}$$

1.8) Show that the nth Fourier series coefficient z_n is given by (**5 points**):

$$z_n = J_n(\beta). \tag{15}$$

☞ The general algebraic properties of the Bessel function we already established in Part I show that the Fourier series coefficients z_n's are real-valued and also $z_{-n} = (-1)^n z_n$.

1.9) Show that the FM-modulated signal corresponding to the message signal in (3) is given by (**5 points**):

$$x_{\mathrm{FM}}(t) = A \sum_{n=-\infty}^{+\infty} J_n(\beta) \cos\left(2\pi[f_c + n f_m]t\right). \tag{16}$$

The plots of $J_n(\beta)$ as a function of n for various values of β are depicted in Fig. 1. There the plots of $J_n(\beta)$ are depicted for for positive values of n only since you can deduce the values for negative n from the symmetry property shown in (7).

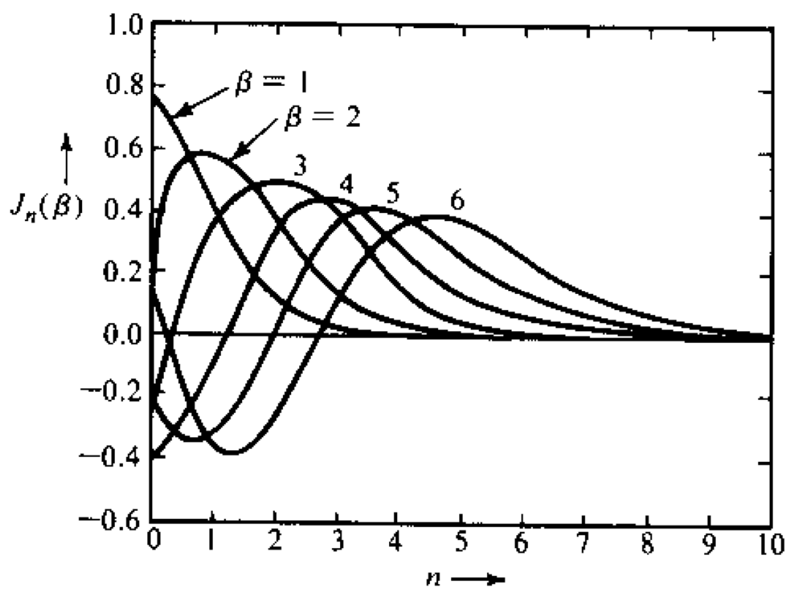

Figure 1: Variations of $J_n(\beta)$ as a fucntion of n for various values of β.

1.10) Suppose $A = 2$, $f_m = 300$ KHz, $k_f = 2\pi \times 10^5$, and $\alpha = 6$. Find and sketch the Fourier transform, $X_{\mathrm{FM}}(f)$ of the modulated signal $x_{\mathrm{FM}}(t)$. (**5 points**).

Carson's rule verification: From the plots of $J_n(\beta)$ in Fig. 1, it can be seen that for a given β, $J_n(\beta)$ decreases with n, and there are only a finite number of significant spectral lines (or harmonics). It can be seen from Fig. 1 that $J_n(\beta)$ is negligible for $n > \beta + 1$. Hence the number of significant harmonics (or sideband impulses) is $\beta + 1$. The bandwidth of the the FM-modulated signal, $x_{\mathrm{FM}}(t)$, is thus given by:

$$\begin{aligned} B_{\mathrm{FM}} &= 2(\beta+1) f_m \\ &= 2(\Delta f + B). \end{aligned}$$

which is in line with the general Carson's rule in (2).

2. **Problem 2 (25 points)**: How can you build a spectrum analyzer?

In this problem, you will understand how you can build a real-time spectrum analyzer based on the concept of frequency modulation. The spectrum analyzer equipment which you have used extensively in the labs plots the magnitude spectrum of any input real-valued signal $g(t)$.

Part I: Consider a band-pass real-valued signal $g(t)$ whose Fourier Transform $G(f)$ is depicted in. Fig. 2.

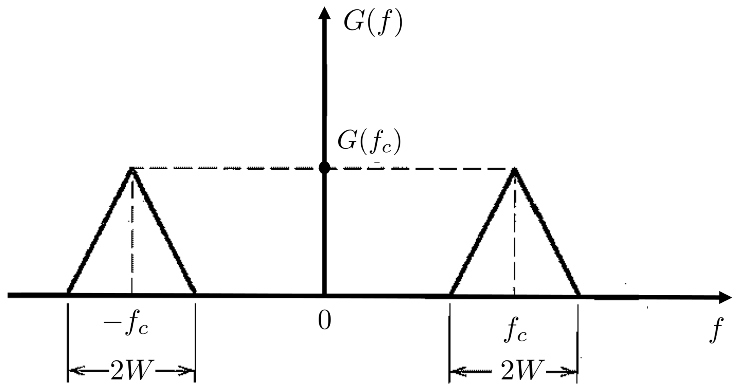

Figure 2: Fourier transform of a band-pass real-valued signal $g(t)$.

Now, consider the following two complex-valued signals:

$$g_+(t) \;=\; g(t) + j\,\widehat{g}(t), \tag{17}$$

$$\widetilde{g}(t) \;=\; g_+(t)e^{-j2\pi f_c t} \tag{18}$$

wherein $\widehat{g}(t)$ is given by ($*$ stands for convolution):

$$\widehat{g}(t) \;=\; \frac{1}{\pi t} * g(t). \tag{19}$$

2.1) Determine and sketch $G_+(f)$, the Fourier transform of $g_+(t)$. (**5 points**):

2.2) Determine and sketch $\widetilde{G}(f)$, the Fourier transform of $\widetilde{g}(t)$ (**5 points**).

2.3) Show that we have (**5 points**):

$$g(t) \;=\; \Re\left\{\widetilde{g}(t)e^{j2\pi f_c t}\right\} \tag{20}$$

The signal $\widetilde{g}(t)$ is called the complex envelope of the *pass-band* real-valued signal $g(t)$ or its *baseband*-equivalent representation. Now, consider an LTI system (e.g., filter, channel, etc.) with real-valued impulse response $h(t)$. Assume that the system is also *band-pass*, i.e., its symmetric frequency response $H(f)$ is non-zero around $\pm f_c$. Then using the exact same procedure described above (for band-pass signals) we can also define the low-pass equivalent system with *complex* impulse response $\widetilde{h}(t)$ which satisfies:

$$h(t) \;=\; \Re\left\{\widetilde{h}(t)e^{j2\pi f_c t}\right\} \tag{21}$$

Using baseband-equivalent representations, we can easily find the pass-band output, $y(t)$, of a pass-band LTI system to a pass-band input signal, $x(t)$, by simply convolving the (baseband) complex impulse response $\widetilde{h}(t)$ with the (baseband) complex envelope of the input signal $x(t)$. In other words, by letting:

$$x(t) \;=\; \Re\left\{\widetilde{x}(t)e^{j2\pi f_c t}\right\} \tag{22}$$

$$h(t) \;=\; \Re\left\{\widetilde{h}(t)e^{j2\pi f_c t}\right\} \tag{23}$$

$$\tag{24}$$

then the pass-band output $y(t)$ is obtained straightforwardly as:

$$y(t) = \Re\left\{\widetilde{y}(t)e^{j2\pi f_c t}\right\}, \tag{25}$$

with

$$\widetilde{y}(t) = \widetilde{h}(t) * \widetilde{x}(t) \tag{26}$$

Part II: Consider the block diagram shown Fig. 3 with real-valued input signal $g(t)$ and output signal $z(t)$.

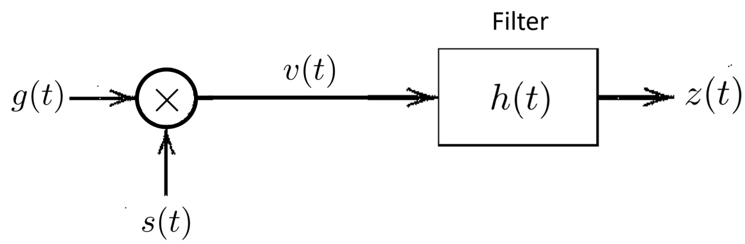

Figure 3: Block diagram of a spectrum analyzer working on the principle of frequency modulation.

The signals $s(t)$ and the filter impulse response $h(t)$ are under our control and are chosen carefully so that the output signal $z(t)$ is the magnitude spectrum of $g(t)$, which a spectrum analyzer would plot for you. One possible choice that does the job as we will now show is to use the following frequency-modulated signals:

$$s(t) = \cos\left(2\pi f_c t - \pi k t^2\right), \tag{27}$$
$$h(t) = \cos\left(2\pi f_c t + \pi k t^2\right), \tag{28}$$
$$\tag{29}$$

where k is a constant.

2.4) Show that the complex envelopes $\widetilde{v}(t)$ and $\widetilde{h}(t)$ of $v(t)$ and $h(t)$ are given by (**5 points**):

$$\widetilde{v}(t) = g(t)e^{-j\pi k t^2}, \tag{30}$$
$$\widetilde{h}(t) = e^{j\pi k t^2}. \tag{31}$$
$$\tag{32}$$

2.5) Deduce that the complex envelope of $z(t)$ is given by (**5 points**):

$$\widetilde{z}(t) = e^{j\pi k t^2}G(kt), \tag{33}$$
$$\tag{34}$$

where $G(f)$ is the Fourier transform of $g(t)$.

This shows that $|\widetilde{z}(t)| = |G(kt)|$ which means that the envelope of the filter output is equal to the magnitude spectrum of the Fourier transform of the input signal $g(t)$, with kt playing the role of frequency f. In other words, by plotting $|\widetilde{z}(t)|$ you are basically plotting the magnitude spectrum of $g(t)$ and that is how a spectrum analyzer works. FM signals are not just for radio stations!

3. **Problem 3 (25 points)**

Consider the periodic message signal $m(t)$ shown in Fig. 4 with fundamental period T_0 (i.e., with fundamental angular frequency $\omega_0 = \frac{2\pi}{T_0}$). The Fourier series expansion of $m(t)$ is written as:

$$m(t) \;=\; \sum_{-\infty}^{+\infty} m_n e^{jn\omega_0 t} \tag{35}$$

Assume that $m(t)$ is angle-modulated (either FM or PM) to produce $x(t) = A\cos(\omega_c t + \phi(t))$ whose bandwidth, B_x, can be estimated using the Carson's rule already studied in Problem 1:

$$B_x \;=\; 2(\Delta f + B_m), \tag{36}$$

where B_m is the essential bandwidth of the message signal $m(t)$ and Δf is the maximum instantaneous frequency deviation from the carrier frequency $f_c = \omega_c/2\pi$.

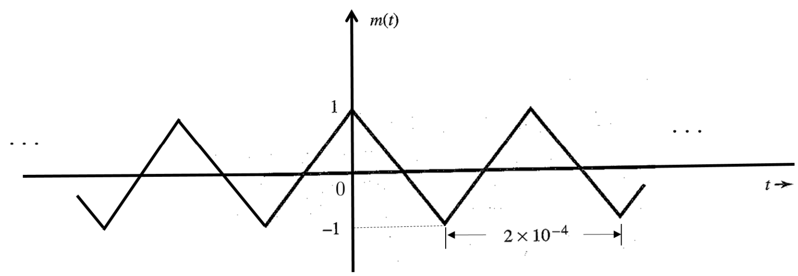

Figure 4: Plot of the periodic message signal $m(t)$.

Now, let $g(t)$ be any periodic signal with fundamental period T_0 (i.e., with fundamental angular frequency $\omega_0 = \frac{2\pi}{T_0}$) and let $\dot{g}(t) = \frac{dg(t)}{dt}$ which is also periodic with period T_0. The Fourier series expansion of $g(t)$ and $\dot{g}(t)$ are given by:

$$g(t) \;=\; \sum_{-\infty}^{+\infty} g_n e^{jn\omega_0 t} \qquad \text{and} \qquad \dot{g}(t) \;=\; \sum_{-\infty}^{+\infty} \dot{g}_n e^{jn\omega_0 t} \tag{37}$$

where

$$g_n \;=\; \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{+\pi/\omega_0} g(t)e^{-jn\omega_0 t}dt \qquad \text{and} \qquad \dot{g}_n \;=\; \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{+\pi/\omega_0} \dot{g}(t)e^{-jn\omega_0 t}dt. \tag{38}$$

3.1) Show that the Fourier series coefficients of $g(t)$ and $\dot{g}(t)$ are related as (**5 points**):

$$\dot{g}_n \;=\; jn\omega_0 g_n \tag{39}$$

3.2) In light of (39), show that the Fourier series coefficients of $m(t)$ are given by (**5 points**):

$$m_n \;=\; \begin{cases} \dfrac{4}{\pi^2 n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{40}$$

3.3) Find the power P_m of $m(t)$ using direct integration. (**2.5 points**).

3.4) Find P_m again using Parseval's theorem for periodic signals. (**2.5 points**).

Note: we have $\sum_{k=0}^{+\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$.

3.5) Assume the essential bandwidth of the periodic signal $m(t)$ as the frequency of its third harmonic. Estimate the bandwidth B_{FM} and B_{PM} of the frequency-modulated and phase-modulated signals associated with the message signal $m(t)$. Take $k_f = 2\pi \times 10^5$ and $k_p = 5\pi$. (**5 points**).

3.6) Assume that the amplitude of $m(t)$ is doubled to produce another message signal $m'(t) = 2m(t)$. Find B'_{FM} and B'_{PM} associated with $m'(t)$ for the same values of k_f and k_p given in the previous question. (**5 points**).