Assignment 3

Rajdeep Gill 7934493

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1 Problem 1

1.1 We can show the following:

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\,dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\,df$$

Starting with the left-hand side:

$$\begin{aligned}
\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\,dt &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} G_1(f)e^{j2\pi ft}\,df\right)\left(\int_{-\infty}^{\infty} G_2^*(f)e^{-j2\pi ft}\,df\right)dt \\
&= \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\left(\int_{-\infty}^{\infty} e^{j2\pi ft}e^{-j2\pi ft}\,dt\right)df \\
&= \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\left(\int_{-\infty}^{\infty} e^{(j2\pi ft-j2\pi ft)}\,dt\right)df \\
&= \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\left(\int_{-\infty}^{\infty} e^{0}\,dt\right)df \\
&= \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\left(\int_{-\infty}^{\infty} 1\,dt\right)df \\
&= \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\,df
\end{aligned}$$

And thus, we have shown that the left-hand side is equal to the right-hand side.

1.2 *Explain how we can obtain Parseval's Theorem from (1).* To find Parseval's Theorem, we can set $g_1(t) = g_2(t) = g(t)$, which implies that $G_1(f) = G_2(f) = G(f)$. Substituting these values into (1.1), we get:

$$\int_{-\infty}^{\infty} g(t)g^*(t)\,dt = \int_{-\infty}^{\infty} G(f)G^*(f)\,df$$

1.3 Using Parseval's Theorem, show that for any $k > 0$ we have:

$$\int_{-\infty}^{\infty} \mathrm{sinc}^2(kt)\,dt = \frac{1}{k}$$

Here we have:

$$g(t) = \mathrm{sinc}(kt), \quad G(f) = \frac{1}{k}\mathrm{rect}\left(\frac{f}{k}\right)$$

The conjugate of a rect function is itself, so we have:

$$\begin{aligned}
\int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt &= \int_{-\infty}^{\infty} \frac{1}{k}\text{rect}\left(\frac{f}{k}\right)\frac{1}{k}\text{rect}\left(\frac{f}{k}\right)\,df \\
&= \frac{1}{k^2}\int_{-k/2}^{k/2} \text{rect}\left(\frac{f}{k}\right)\,df \\
&= \frac{1}{k^2}\int_{-k/2}^{k/2} 1\,df \\
&= \frac{1}{k^2}\left[f\right]_{-k/2}^{k/2} \\
&= \frac{1}{k^2}\left(\frac{k}{2} - \left(-\frac{k}{2}\right)\right) \\
&= \frac{1}{k}
\end{aligned}$$

And we have shown that $\int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt = \frac{1}{k}$.

2 Problem 2

2.1 Show that:

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)y^*(\tau - t)\,d\tau = \int_{-\infty}^{\infty} y^*(\tau)x(\tau + t)\,d\tau$$

Since we know the correlation function is defined as:

$$\begin{aligned}
r_{xy}(t) &= x(t) * y^*(-t) \\
&= \int_{-\infty}^{\infty} x(\tau)y^*(\tau - t)\,d\tau, \quad \text{Let } u = \tau - t,\ du = d\tau \\
&= \int_{-\infty}^{\infty} x(u + t)y^*(u)\,du \\
&= \int_{-\infty}^{\infty} y^*(u)x(u + t)\,du, \quad \text{Let } \tau = u,\ d\tau = du \\
&= \int_{-\infty}^{\infty} y^*(\tau)x(\tau + t)\,d\tau
\end{aligned}$$

And we have shown as required.

2.2 Show that $r_{xy}(t) = r_{yx}(-t)^*$:

$$\begin{aligned}
r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)y^*(\tau - t)\,d\tau &= \int_{-\infty}^{\infty} y^*(\tau)x(\tau + t)\,d\tau \\
&= \left(\int_{-\infty}^{\infty} y(\tau)x^*(\tau + t)\,d\tau\right)^* \\
&= (r_{yx}(-t))^*
\end{aligned}$$

2.3 If $y(t) = x(t + T)$, we can express $r_{xy}(t)$ and $r_{yy}(t)$ in terms of $r_{xx}(t)$:

First, $r_{xy}(t)$:

$$\begin{aligned}
r_{xy}(t) &= x(t) * y^*(-t) = x(t) * x^*(T-t) \\
&= \int_{-\infty}^{\infty} x(\tau)x^*(\tau - t + T)\,d\tau \\
&= r_{xx}(t-T)
\end{aligned}$$

Now, $r_{yy}(t)$:

$$\begin{aligned}
r_{yy}(t) &= y(t) * y^*(-t) = x(t+T) * x^*(T-t) \\
&= \int_{-\infty}^{\infty} x(\tau+T)x^*(\tau - t + T)\,d\tau, \quad \text{Let } \tau' = \tau + T,\, d\tau' = d\tau \\
&= \int_{-\infty}^{\infty} x(\tau')x^*(\tau' - t)\,d\tau' \\
&= r_{xx}(t)
\end{aligned}$$

2.4 What is the relationship between the cross-ESD's $\Psi_{xy}(f)$ and $\Psi_{yx}(f)$?

Given that $\Psi_{xy}(f) = \mathcal{F}\{r_{xy}(t)\}$ and $\Psi_{yx}(f) = \mathcal{F}\{r_{yx}(t)\}$. And from above we know that $r_{xy}(t) = r_{yx}(-t)^*$. We have:

$$\begin{aligned}
\Psi_{xy}(f) = \mathcal{F}\{r_{xy}(t)\} &= \mathcal{F}\{r_{yx}(-t)^*\} \\
&= \int_{-\infty}^{\infty} r_{yx}(-t)^* e^{-j2\pi ft}\,dt \\
&= \int_{-\infty}^{\infty} \left(r_{yx}(-t)e^{j2\pi ft}\right)^*\,dt \\
&= \left(\int_{-\infty}^{\infty} r_{yx}(-t)e^{j2\pi ft}\,dt\right)^* \\
&= \left(\mathcal{F}\{r_{yx}(t)\}\right)^* \\
\Psi_{xy}(f) &= \Psi_{yx}(f)^*
\end{aligned}$$

2.5 We can find an expression of $\Psi_{xy}(f)$ in terms of $X(f)$ and $Y(f)$ as follows:

$$\begin{aligned}
\Psi_{xy}(f) = \mathcal{F}\{r_{xy}(t)\} &= \mathcal{F}\{x(t) * y^*(-t)\} \\
&= \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{y^*(-t)\} \\
&= X(f)Y^*(-f)
\end{aligned}$$

2.6 We can show that the ESD is real and positive for every f as follows:

$$\begin{aligned}
\Psi_{xx}(f) = \mathcal{F}\{r_{xx}(t)\} &= \mathcal{F}\{x(t) * x^*(-t)\} \\
&= \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{x^*(-t)\} \\
&= X(f)X^*(f) \\
&= |X(f)|^2
\end{aligned}$$

Since the magnitude squared of a complex number is always real and positive, we have shown that the ESD is real and positive for every f.

2.7 We can find the expressions of $\Psi_{xy}(f)$ and $\Psi_{yy}(f)$ in terms of $\Psi_{xx}(f)$ and $H(f)$ as follows.

First, $\Psi_{xy}(f)$. We know that $y(t) = h(t) * x(t)$, so we have:

$$\Psi_{xy}(f) = \mathcal{F}\{r_{xy}(t)\} = \mathcal{F}\{x(t) * y^*(-t)\} = \mathcal{F}\{x(t) * h^*(-t) * x^*(-t)\} \tag{1}$$