Lab 5

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ECE 4830 B03

March 24, 2025

Contents

1 Audio Filtering

To filter the complete song and eliminate the complete tone using a pole-zero plot, we first define $H(z)$. We note the general of the frequency response from pole-zero positions as:

$$H(z) = b_0 \cdot \frac{(z - z_1)(z - z_2) \cdots (z - z_N)}{(z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_N)}$$

We require no amplification and set $b_0 = 1$, and knowing that the tone is at a specific frequency f_0, we set $z = e^{j\omega_0}$. This simplifies our notch filter to include only two poles and two zeroes (complex conjugate pairs) as:

$$H(z) = \frac{(z - e^{j\omega_0})(z - e^{-j\omega_0})}{(z - re^{j\omega_0})(z - re^{-j\omega_0})} \tag{1}$$

We can simplify the numerator to be in terms of z^{-1}:

$$\begin{aligned}\left(z - e^{j\omega_0}\right)\left(z - e^{-j\omega_0}\right) &= z^2 - \left(e^{j\omega_0} + e^{-j\omega_0}\right)z + 1 \\ &= z^2 - 2\cos(\omega_0)z + 1 \\ &= z^2\left(1 - 2\cos(\omega_0)z^{-1} + z^{-2}\right)\end{aligned}$$

Similarly, the denominator becomes:

$$(z - re^{j\omega_0})(z - re^{-j\omega_0}) \implies z^2\left(1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}\right)$$

Putting the above two into equation (1), we get:

$$H(z) = \frac{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}} \tag{2}$$

We note that $f_0 = 2000$ Hz from Lab 4 (the frequency of the tone to be eliminated), and since we want our filter to be stable, we want $r < 1$. Setting $r = 0.98$ will satisfy this requirement while giving us a narrow, focused supression of ω_0.

With the frequency response from pole-zero positions $H(z)$ defined, we now work toward converting it into a difference equation. We recall:

$$H(z) = \frac{Y(z)}{X(z)} \implies Y(z) = H(z) \cdot X(z)$$

Applying this relationship to equation (2), we are left with:

$$Y(z)(1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}) = X(z)(1 - 2\cos(\omega_0)z^{-1} + z^{-2}) \tag{3}$$

Applying the inverse Z-transform to equation (3), we get the following difference equation:

$$y[n] = x[n] - 2\cos(\omega_0)x[n-1] + x[n-2] + 2r\cos(\omega_0)y[n-1] - r^2 y[n-2] \tag{4}$$

With our equations prepared, we write MATLAB code to set out variables and then compute the desired filters:

```
% Target tone to remove
f0 = 2000; % Frequency to eliminate
omega0 = 2*pi*f0/Fs;
r = 0.98; % stable and focused

% Coefficients from H(z)
b = [1, -2*cos(omega0), 1];         % numerator of equation (2)
a = [1, -2*r*cos(omega0), r^2];     % denominator of equation (2)

% --- Built-in filtering ---
y_builtin = filter(b, a, x);

% --- Manual filtering using a for-loop ---
y_manual = zeros(size(x));
for n = 3:length(x)
    y_manual(n) = b(1)*x(n) + b(2)*x(n-1) + b(3)*x(n-2) ...
                - a(2)*y_manual(n-1) - a(3)*y_manual(n-2);
end
```

Code Snippet 1: MATLAB code to filter the audio signal.

Listening to the generated audio confirmed the validity of our filters, effectively removing the single sinusoidal tone using a pole-zero plot.

Found below is an FFT plot of all three waveforms (before filtering, after built-in filtering, and after manual filtering) that curtly showcases the removal of the 2000 Hz tone from both filters:

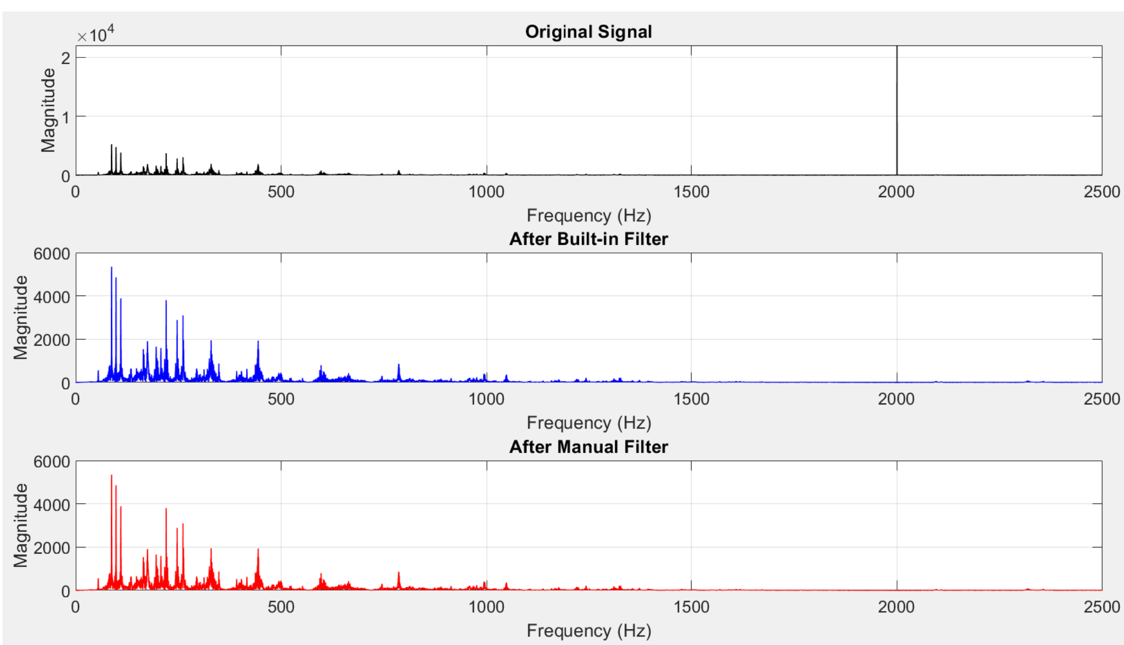

Figure 1: Fast Fourier Transform for Before and after filtering.

2 Problems

P1.1 The function $x[n]$ is defined as:

$$x[n] = \begin{cases} 3 & \text{for } n = 3 \\ 1 & \text{for } n = 5 \\ -1 & \text{for } n = 7 \\ -3 & \text{for } n > 8 \\ 0 & \text{otherwise} \end{cases}$$

We can determine the unilateral Z-transform of $x[n]$ by using the following formula:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Substituting the values of $x[n]$ into the formula, we get:

$$\begin{aligned} X(z) &= 3z^{-3} + z^{-5} - z^{-7} - 3\sum_{n=9}^{\infty} z^{-n} \\ &= 3z^{-3} + z^{-5} - z^{-7} - 3\left(\frac{z^{-9}}{1 - z^{-1}}\right) \end{aligned}$$

P1.2 We can use the definition of the z-transform and then determine its region of convergence (ROC).

(a) The z-transform of $x[n] = \gamma^n \cos(\pi n)u(n)$ is:

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \gamma^n \cos(\pi n) z^{-n} \\ &= \sum_{n=0}^{\infty} \gamma^n (-1)^n z^{-n} \\ &= \sum_{n=0}^{\infty} (-\gamma z^{-1})^n \\ &= \frac{1}{1 + \gamma z^{-1}} \end{aligned}$$

The ROC is $|z| > |\gamma|$. This is because for this summation to converge, the ratio $|\gamma z^{-1}|$ must be less than 1.

$$|\gamma z^{-1}| < 1 \implies |z| > |\gamma|$$

(b) The z-transform of $x[n] = \left[2^{n-1} - (-2)^{n-1}\right] u(n)$ is given by:

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} \left[2^{n-1} - (-2)^{n-1}\right] z^{-n} \\
&= \sum_{n=0}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} (-2)^{n-1} z^{-n} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^{n} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{-2}{z}\right)^{n} \\
&= \frac{1}{2} \left(\frac{1}{1 - \frac{2}{z}}\right) + \frac{1}{2} \left(\frac{1}{1 + \frac{2}{z}}\right) \\
&= \frac{z}{z-2} + \frac{z}{z+2} \\
&= \frac{2z^2}{z^2 - 4}
\end{aligned}$$

The radius of convergence will be when both the summations converge, that is:

$$\left|\frac{2}{|z|}\right| < 1 \implies |z| > 2$$

Therefore, the ROC is $|z| > 2$.

P1.3 We can find the inverse z-transform of the following functions.

(a) The inverse z-transform of $X(z) = \frac{(z-1)^2}{z^3}$ is:

$$\begin{aligned}
X(z) &= \frac{(z-1)^2}{z^3} \\
&= \frac{z^2 - 2z + 1}{z^3} \\
&= z^{-1} - 2z^{-2} + z^{-3}
\end{aligned}$$

We use the common z-transform pairs to find the inverse z-transform:

$$z^{-n_0} \longleftrightarrow \delta(n - n_0)$$

Therefore, the inverse z-transform of $X(z)$ is:

$$x[n] = \delta[n-1] - 2\delta[n-2] + \delta[n-3]$$

(b) The inverse z-transform of $X(z) = \frac{z-4}{z^2-5z+6}$ can be found first by doing a partial fraction decomposition:

$$\begin{aligned}
X(z) &= \frac{z-4}{z^2 - 5z + 6} = \frac{z-4}{(z-2)(z-3)} \\
&= \frac{A}{z-2} + \frac{B}{z-3}
\end{aligned}$$

We can then use the cover-up method to find the values of A and B:

$$A : \text{Let z} = 2 \implies A = \frac{2-4}{2-3} = 2$$
$$B : \text{Let z} = 3 \implies B = \frac{3-4}{3-2} = -1$$

We now have:

$$X(z) = \frac{2}{z-2} - \frac{1}{z-3}$$

We use the following transform pair:

$$\gamma^{n-1}u[n] \longleftrightarrow \frac{1}{z-\gamma}$$

Therefore, the inverse z-transform of $X(z)$ is:

$$\begin{aligned} x[n] &= 2(2^{n-1})u[n] - (3^{n-1})u[n] \\ &= (2(2^{n-1}) - 3^{n-1})u[n] \end{aligned}$$