Linear Space

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1 Introduction

In defning linear space we defne the axims the elements of the space must satisfy. We do not defne the elements or the operations on them. example: $\mathbb{R}^n, \mathbb{R}, \mathbb{C}$, polynomials, etc.

2 Defnition

let V be a non-empty set of objects called elements. Then V is called a linear space if the following axioms are satisfed:

1. V is closed under addition. For all $x, y \in V$, there exist an element $x + y \in V$ called the sum of x and y.
2. V is closed under scalar multiplication. For all $x \in V$ and all scalars $c \in \mathbb{R}$, there exists an element $cx \in V$ called the scalar product of c and x.
3. Addition is commutative. For all $x, y \in V$, $x + y = y + x$.
4. Addition is associative. For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$.
5. V there exists an element zero O such that for all $x \in V$, $x + O = x$.
6. For each $x \in V$, there exists an element $-x$ such that $x + (-x) = O$.
7. Scalar multiplication is associative. For all $x \in V$ and all scalars $a, b \in \mathbb{R}$, $(ab)x = a(bx)$.
8. Scalar multiplication distributes over vector addition. For all $x, y \in V$ and all scalars $a \in \mathbb{R}$, $a(x + y) = ax + ay$.
9. Scalar multiplication distributes over scalar addition. For all $x \in V$ and all scalars $a, b \in \mathbb{R}$, $(a + b)x = ax + bx$.
10. 1 is the identity of multiplication. For all $x \in V$, $1x = x$.

The vector space or linear space which satify the above for all real number are called real vector space or real linear space. The vector space or linear space which satify the above for all complex number are called complex vector space or complex linear space.

3 Theorem

Theorem 1: The zero vector is unique.

Proof: Let O_1 and O_2 be two zero vectors in V.
Then $O_1 + O_2 = O_1$ and
$O_1 + O_2 = O_2$.
Thus $O_1 = O_2$.

Theorem 2: Uniqueness of additive inverse.

Proof: Let x be an element of V and let y_1 and y_2 be two additive inverses of x.
Then $x + y_1 = O$ and $x + y_2 = O$.
$y_1 = y_1 + O = y_1 + (x + y_2) = (y_1 + x) + y_2 = O + y_2 = y_2$.
hence $y_1 = y_2$.

note that -x = $(-1)x$ is the additive inverse of x. follows from uniqueness of distributive law.

4 Properties of Linear Space

Theorem 3: Let $x, y \in V$ and $a, b \in \mathbb{R}$. Then:

1. $0 \cdot x = O$
2. $a \cdot O = O$
3. $(-a) \cdot x = -(a \cdot x) = a \cdot (-x)$
4. If $a \cdot x = O$, then either $a = 0$ or $x = O$
5. If $a \cdot x = a \cdot y$ and $a \neq 0$, then $x = y$
6. If $a \cdot x = b \cdot x$ and $x \neq O$, then $a = b$
7. $-(x + y) = (-x) + (-y)$
8. $x + x = 2x$, $x + x + x = 3x$, in general, $nx = x + x + \cdots + x$ (n times)

Proof

(1) $0 \cdot x = O$

$$2(0 \cdot x) = (2 \cdot 0) \cdot x = 0 \cdot x$$

Now, add $-1(0 \cdot x$