Intro to AI Assignment 3 - Probabilistic Reasoning

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April 16, 2025

§1 Problem 1

Part a

The joint probability for the events A, B, C, D, E is defined by the chain rule as:

$$P(A, B, C, D, E) = P(A) \cdot P(B) \cdot P(C) \cdot P(D|A, B) \cdot P(E|B, C).$$

For the specific case where all events are true (T), the joint probability is:

$$P(A = T, B = T, C = T, D = T, E = T) = P(A = T) \cdot P(B = T) \cdot P(C = T) \cdot P(D = T|A = T, B = T) \cdot P$$

Substituting the provided numerical values:

$$\begin{aligned} P(A = T, B = T, C = T, D = T, E = T) &= 0.2 \cdot 0.5 \cdot 0.8 \cdot 0.1 \cdot 0.3 \\ &= 0.0024 \end{aligned}$$

Thus, the likelihood of the combined event where A, B, C, D, and E are all true is 0.0024.

Part b

Now, let's analyze the scenario where all these events are false (F). The joint probability is:

$$P(A = F, B = F, C = F, D = F, E = F) = P(A = F) \cdot P(B = F) \cdot P(C = F) \cdot P(D = F|A = F, B = F) \cdot$$

Inserting the given probabilities for this scenario:

$$\begin{aligned} P(A = F, B = F, C = F, D = F, E = F) &= 0.8 \cdot 0.5 \cdot 0.2 \cdot 0.1 \cdot 0.8 \\ &= 0.0064 \end{aligned}$$

Consequently, the probability of the joint event where A, B, C, D, and E are all false is 0.0064.

Part c

We aim to determine the conditional probability $P(\neg A|B,C,D,E)$. Employing Bayes' theorem, this can be expressed as proportional to the joint probability $P(\neg A,B,C,D,E)$:

$$P(\neg A|B,C,D,E) \propto P(\neg A,B,C,D,E).$$

Let the normalization factor be α, defined as:

$$\alpha = \frac{1}{P(A,B,C,D,E) + P(\neg A,B,C,D,E)}.$$

We are given the calculation for α:

$$\alpha = \frac{1}{(0.2 \cdot 0.5 \cdot 0.8 \cdot 0.1 \cdot 0.3) + (0.8 \cdot 0.5 \cdot 0.8 \cdot 0.6 \cdot 0.3)}.$$

$$\alpha = \frac{1}{0.0024 + 0.0576} = \frac{1}{0.06} = \frac{50}{3}.$$

Now, we can compute the conditional probability $P(\neg A|B,C,D,E)$:

$$P(\neg A|B,C,D,E) = \alpha \cdot P(\neg A,B,C,D,E).$$

$$P(\neg A|B,C,D,E) = \frac{50}{3} \cdot 0.0576.$$
$$P(\neg A|B,C,D,E) = 0.96.$$

Therefore, the conditional probability $P(\neg A|B,C,D,E)$ is 0.96.

Problem 2

Part a

Our goal is to determine the conditional probability $P(Burglary|JohnsCalls = \text{true}, MaryCalls = \text{true})$. We are given the formulation:

$$P(B|J,M) = \alpha \cdot P(B) \sum_E P(E) \sum_A P(A|B,E) \cdot P(J|A) \cdot P(M|A)$$

where $\alpha = \dfrac{1}{P(J,M)}$.

Following the provided steps and correcting the final calculation:

$$\begin{aligned}
P(B|J,M) = \alpha \cdot \begin{pmatrix} 0.00059224259 \\ 0.0014918576 \end{pmatrix} \\
&\text{(where the top element corresponds to } B = T \text{ and the bottom to } B = F\text{)} \\
&\left(\text{and } \alpha = \frac{1}{0.0020853609}\right) \\
&= \frac{1}{0.0020853609} \cdot \begin{pmatrix} 0.00059224259 \\ 0.0014918576 \end{pmatrix} \\
&= \begin{pmatrix} \frac{0.00059224259}{0.0020853609} \\ \frac{0.0014918576}{0.0020853609} \end{pmatrix} \\
&= \begin{pmatrix} 0.284 \\ 0.716 \end{pmatrix}
\end{aligned}$$

Here, 0.284 signifies the probability of a burglary occurring given that John and Mary call, while 0.716 represents the probability of no burglary under the same conditions.

Part b

What is the computational cost of determining $P(X_1|X_n = \text{true})$ via enumeration? What is the cost using variable elimination?

Complexity via Enumeration

To compute $P(X_1|X_n = \text{true})$ by enumeration, we initially assess two binary trees for each state of X_1. Each of these trees possesses a depth of $n-2$. Consequently, the aggregate computational effort for enumeration amounts to $\mathbf{O(2^n)}$.

Complexity via Variable Elimination

Moving on to variable elimination, the size of the factors will not exceed two variables. For instance, when computing $P(X_1|X_n = \text{true})$:

$$\begin{aligned}
P(X_1|X_n = \text{true}) &= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} P(x_{n-1}|x_{n-2}) P(X_n = \text{true}|x_{n-1}) \\
&= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} f_{X_{n-1}}(x_{n-1}, x_{n-2}) f_{X_n}(x_{n-1}) \\
&= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) f_{\frac{x_{n-2}}{X_{n-1} \cdot X_n}}
\end{aligned}$$

As evident, this mirrors a problem with $n-1$ variables rather than n. Hence, the computational work remains constant, independent of n, and the overall complexity is $\mathbf{O(n)}$.

§2 Problem 3

§3 Problem 4

We can model the system as a hidden Markov model. We can model X_t as a Markov chain with the states $\{A, B, C, D, E, F\}$ and transition matrix:

$$\begin{pmatrix}
0.2 & 0.8 & 0 & 0 & 0 & 0 \\
0 & 0.2 & 0.8 & 0 & 0 & 0 \\
0 & 0 & 0.2 & 0.8 & 0 & 0 \\
0 & 0 & 0 & 0.2 & 0.8 & 0 \\
0 & 0 & 0 & 0 & 0.2 & 0.8 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

In addition, we have the observation matrices for hot and cold:

§3.1 Part 1

We know that the rover starts at state A with probability 1, so $P(X_1 = A) = 1$. The initial state vector is therefore $[1, 0, 0, 0, 0, 0]^T$.

§4 Problem 5