Temporal Models Example Problems
Intro to AI - Spring 2019

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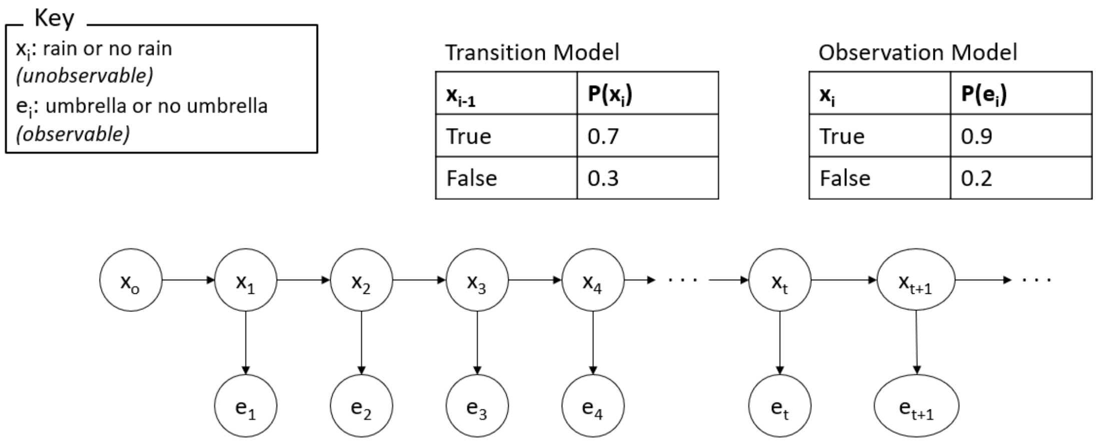
Temporal State Estimation

We will use the umbrella and rain example from lecture:



Transition Model

x_{i-1}	$P(x_i)$
True	0.7
False	0.3

Observation Model

x_i	$P(e_i)$
True	0.9
False	0.2

Figure 1: Dynamic Bayesian Network - Umbrella example

Let us answer the following:

1. Filtering: $P(x_2|e_1 = t, e_2 = t)$
2. Prediction: $P(x_2|e_1 = t)$ and $P(x_3|e_1 = t)$
3. Smoothing: $P(x_1|e_1 = t, e_2 = t, e_3 = f)$

Given:

1. Transition Model: shown above
2. Observation Model: shown above
3. Initial Probability: $P(x_0 = t) = 0.5$

Filtering

For the first two days, we observe the director coming in with an umbrella, that is, $e_1 = t$ and $e_2 = t$. Given this evidence, we can update our belief on day two via filtering.

$$P(x_2|e_1 = t, e_2 = t)$$

$$= \alpha_2 P(e_2|x_2) \sum_{x_1} P(x_2|x_1)P(x_1|e_1)$$

But $P(x_1|e_1)$ is not known, and is in fact another filtering problem. Thus, we recursively expand this.

$$= \alpha_2 P(e_2|x_2) \sum_{x_t} P(x_2|x_1)[\alpha_1 P(e_1|x_1) \sum_{x_0} P(x_1|x_0)P(x_0)]$$

We can stop here because $P(x_0)$ is known, and all other terms are given either by the transition model or observation model. Thus, the problem is fully specified.

The computation is done as follows, for which $< value1, value2 >$ keeps track of both $x_i = t$ and $x_i = f$ simultaneously, to make standardization clearer.

$$\begin{aligned}
P(x_0) &=< P(x_0 = t), P(x_0 = f) > \\
&=< 0.5, 0.5 >
\end{aligned}$$

$$\begin{aligned}
P(x_1) &= \sum_{x_0} P(x_1|x_0)P(x_0) \\
&=< 0.7, 0.3 > \times(0.5)+ < 0.3, 0.7 > \times(0.5) \\
&=< 0.5, 0.5 >
\end{aligned}$$

$$\begin{aligned}
P(x_1|e_1 = t) &= \alpha_1 P(e_1 = t|x_1)P(x_1) \\
&= \alpha_1 < 0.9, 0, 2 > \times < 0.5, 0.5 > \\
&= \alpha_1 < 0.45, 0.1 > \\
&=< 0.818, 0.182 >
\end{aligned}$$

$$\begin{aligned}
P(x_2|e_1 = t) &= \sum_{x_1} P(x_2|x_1)P(x_1|e_1 = t) \\
&=< 0.7, 0.3 > \times(0.818)+ < 0.3, 0.7 > \times(0.182) \\
&=< 0.627, 0.373 >
\end{aligned}$$

$$\begin{aligned}
P(x_2|e_1 = t, e_2 = t) &= \alpha_2 P(e_2|x_2)P(x_2|e_1) \\
&= \alpha_2 < 0.9, 0.2 > \times < 0.627, 0.373 > \\
&= \alpha_2 < 0.565, 0.075 > \\
&=< 0.883, 0.117 >
\end{aligned}$$

Prediction

On the day 1, we observe the director coming in with an umbrella, i.e. $e_1 = t$ (here, assume that we do not know the evidence of day 2). Given this evidence, we would like to predict whether it would rain on day 2.

$$P(x_2|e_1 = t) = \sum_{x_1} P(x_2|x_1)P(x_1|e_1)$$

Again, $P(x_1|e_1)$ is not known. Thus, we recursively expand this.

$$\begin{aligned} P(x_2|e_1 = t) &= \sum_{x_1} P(x_2|x_1)P(x_1|e_1) \\ &= \sum_{x_1} P(x_2|x_1)[\alpha P(e_1|x_1) \sum_{x_0} P(x_1|x_0)P(x_0)] \end{aligned}$$

We stop here since $P(x_0)$ is known, and all other terms are given either by the transition model or the observation model. Thus, the problem is fully specified.

The computation is done as in the filtering question.

$$P(x_0) =< P(x_0 = t), P(x_0 = f) >=< 0.5, 0.5 >$$

$$\begin{aligned} P(x_1) &= \sum_{x_0} P(x_1|x_0)P(x_0) \\ &=< 0.7, 0.3 > \times 0.5+ < 0.3, 0.7 > \times 0.5 \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} P(x_1|e_1 = t) &= \alpha P(e_1 = t|x_1)P(x_1) \\ &= \alpha < 0.9, 0.2 > \times < 0.5, 0.5 > \\ &= \alpha < 0.45, 0.1 > \\ &=< 0.818, 0.182 > \end{aligned}$$

$$\begin{aligned} P(x_2|e_1 = t) &= \sum_{x_1} P(x_2|x_1)P(x_1|e_1 = t) \\ &=< 0.7, 0.3 > \times 0.818+ < 0.3, 0.7 > \times 0.182 \\ &=< 0.627, 0.373 > \end{aligned}$$

Predicting one more time step into the future, we have:

$$\begin{aligned} P(x_3|e_1 = t) &= \sum_{x_2} P(x_3|x_2)P(x_2|e_1 = t) \\ &=< 0.7, 0.3 > \times 0.627+ < 0.3, 0.7 > \times 0.373 \\ &=< 0.551, 0.449 > \end{aligned}$$

As we can see from the above calculations, the less the evidence that is provided to us, our probability distribution $P(x_{t+k}|e_{1:t})$ converges to the stationary distribution determined by the transition model. In this case, it happens to be our prior distribution $P(x_0) =< 0.5, 0.5 >$.

Smoothing

Because $P(x_1|e_1 = t, e_2 = t, e_3 = f) = \alpha P(x_1|e_1 = t)P(e_2 = t, e_3 = f|x_1)$ and we already have $P(x_1|e_1 = t)$ from filtering, we only need to compute $P(e_2 = t, e_3 = f|x_1)$.

And since $P(e_2 = t, e_3 = f|x_1) = \sum_{x_2} P(e_2 = t|x_2)P(e_3 = f|x_2)P(x_2|x_1)$, we have to first compute $P(e_3 = f|x_2)$:

$$\begin{aligned}
P(e_3 = f|x_2 = t) &= \sum_{x_3} P(e_3 = f|x_3)P(x_3|x_2 = t) \\
&= P(e_3 = f|x_3 = t)P(x_3 = t|x_2 = t) + P(e_3 = f|x_3 = f)P(x_3 = f|x_2 = t) \\
&= 0.1 \times 0.7 + 0.8 \times 0.3 \\
&= 0.31 \\
P(e_3 = f|x_2 = f) &= \sum_{x_3} P(e_3 = f|x_3)P(x_3|x_2 = f) \\
&= P(e_3 = f|x_3 = t)P(x_3 = t|x_2 = f) + P(e_3 = f|x_3 = f)P(x_3 = f|x_2 = f) \\
&= 0.1 \times 0.3 + 0.8 \times 0.7 \\
&= 0.59
\end{aligned}$$

Therefore, we can compute $P(e_2 = t, e_3 = f|x_1)$:

$$\begin{aligned}
P(e_2 = t, e_3 = f|x_1 = t) &= \sum_{x_2} P(e_2 = t|x_2)P(e_3 = f|x_2)P(x_2|x_1 = t) \\
&= P(e_2 = t|x_2 = t)P(e_3 = f|x_2 = t)P(x_2 = t|x_1 = t)+ \\
&\quad P(e_2 = t|x_2 = f)P(e_3 = f|x_2 = f)P(x_2 = f|x_1 = t) \\
&= 0.9 \times 0.31 \times 0.7 + 0.2 \times 0.59 \times 0.3 \\
&= 0.2307 \\
P(e_2 = t, e_3 = f|x_1 = f) &= \sum_{x_2} P(e_2 = t|x_2)P(e_3 = f|x_2)P(x_2|x_1 = f) \\
&= P(e_2 = t|x_2 = t)P(e_3 = f|x_2 = t)P(x_2 = t|x_1 = f)+ \\
&\quad P(e_2 = t|x_2 = f)P(e_3 = f|x_2 = f)P(x_2 = f|x_1 = f) \\
&= 0.9 \times 0.31 \times 0.3 + 0.2 \times 0.59 \times 0.7 \\
&= 0.1663
\end{aligned}$$

Finally, we can get $P(x_1|e_1 = t, e_2 = t, e_3 = f)$:

$$\begin{aligned}
P(x_1|e_1 = t, e_2 = t, e_3 = f) &= \alpha P(x_1|e_1 = t)P(e_2 = t, e_3 = f|x_1) \\
&= \alpha < 0.818, 0.182 > \times < 0.2307, 0.1663 > \\
&= \alpha < 0.1887, 0.03076 > \\
&=< 0.86, 0.14 >
\end{aligned}$$