# Intro to AI Assignment 3 - Probabilistic Reasoning

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## §1 Problem 1

### §1.1 Part a

The joint probability for the events $A, B, C, D, E$ is defined by the chain rule as:

$$P(A, B, C, D, E) = P(A) \cdot P(B) \cdot P(C) \cdot P(D|A, B) \cdot P(E|B, C)$$

For the specific case where all events are true ($T$), the joint probability is:

$$\begin{aligned} P(A = T, B = T, C = T, D = T, E = T) &= P(A = T) \cdot P(B = T) \cdot P(C = T) \\ &\quad \cdot P(D = T|A = T, B = T) \cdot P(E = T|B = T, C = T) \end{aligned}$$

Substituting the provided numerical values:

$$\begin{aligned} P(A = T, B = T, C = T, D = T, E = T) &= 0.2 \cdot 0.5 \cdot 0.8 \cdot 0.1 \cdot 0.3 \\ &= 0.0024 \end{aligned}$$

Thus, the likelihood of the combined event where $A, B, C, D$ and $E$ are all true is $\boxed{0.0024}$.

### §1.2 Part b

Now, let's analyze the scenario where all these events are false ($F$). The joint probability is:

$$\begin{aligned} P(A = F, B = F, C = F, D = F, E = F) &= P(A = F) \cdot P(B = F) \cdot P(C = F) \\ &\quad \cdot P(D = F|A = F, B = F) \cdot P(E = F|B = F, C = F) \end{aligned}$$

Inserting the given probabilities for this scenario:

$$\begin{aligned} P(A = F, B = F, C = F, D = F, E = F) &= 0.8 \cdot 0.5 \cdot 0.2 \cdot 0.1 \cdot 0.8 \\ &= 0.0064 \end{aligned}$$

Consequently, the probability of the joint event where $A, B, C, D$, and $E$ are all false is $\boxed{0.0064}$.

## §1.3 Part c

We aim to determine the conditional probability $P(\neg A|B,C,D,E)$. Employing Bayes' theorem, this can be expressed as proportional to the joint probability $P(\neg A,B,C,D,E)$:

$$P(\neg A|B,C,D,E) \propto P(\neg A,B,C,D,E).$$

Let the normalization factor be $\alpha$, defined as:

$$\alpha = \frac{1}{P(A,B,C,D,E) + P(\neg A,B,C,D,E)}.$$

We are given the calculation for $\alpha$:

$$\alpha = \frac{1}{(0.2 \cdot 0.5 \cdot 0.8 \cdot 0.1 \cdot 0.3) + (0.8 \cdot 0.5 \cdot 0.8 \cdot 0.6 \cdot 0.3)}.$$
$$\alpha = \frac{1}{0.0024 + 0.0576} = \frac{1}{0.06} = \frac{50}{3}.$$

Now, we can compute the conditional probability $P(\neg A|B,C,D,E)$:

$$P(\neg A|B,C,D,E) = \alpha \cdot P(\neg A,B,C,D,E).$$
$$P(\neg A|B,C,D,E) = \frac{50}{3} \cdot 0.0576.$$
$$P(\neg A|B,C,D,E) = 0.96.$$

Therefore, the conditional probability $P(\neg A|B,C,D,E)$ is $\boxed{0.96}$.

# §2 Problem 2

## §2.1 Part a

Our goal is to determine the conditional probability $P(\text{Burglary}|\text{JohnsCalls} = \text{true}, \text{MaryCalls} = \text{true})$. We are given the formulation:

$$P(B|J,M) = \alpha \cdot P(B) \sum_E P(E) \sum_A P(A|B,E) \cdot P(J|A) \cdot P(M|A)$$

where $\alpha = \dfrac{1}{P(J,M)}$.

Following the provided steps and correcting the final calculation:

$$
\begin{aligned}
P(B|J,M) = \alpha \cdot \begin{pmatrix} 0.00059224259 \\ 0.0014918576 \end{pmatrix} \\
&\text{(where the top element corresponds to } B = T \text{ and the bottom to } B = F\text{)} \\
&\left(\text{and } \alpha = \frac{1}{0.0020853609}\right) \\
&= \frac{1}{0.0020853609} \cdot \begin{pmatrix} 0.00059224259 \\ 0.0014918576 \end{pmatrix} \\
&= \begin{pmatrix} \frac{0.00059224259}{0.0020853609} \\ \frac{0.0014918576}{0.0020853609} \end{pmatrix} \\
&= \begin{pmatrix} 0.284 \\ 0.716 \end{pmatrix}
\end{aligned}
$$

Here, $\boxed{0.284}$ signifies the probability of a burglary occurring given that John and Mary call, while $\boxed{0.716}$ represents the probability of no burglary under the same conditions.

## §2.2 Part b

What is the computational cost of determining $P(X_1|X_n = \text{true})$ via enumeration? What is the cost using variable elimination?

### §2.2.1 Complexity via Enumeration

To compute $P(X_1|X_n = \text{true})$ by enumeration, we initially assess two binary trees for each state of $X_1$. Each of these trees possesses a depth of $n-2$. Consequently, the aggregate computational effort for enumeration amounts to $\boxed{O(2^n)}$.

### §2.2.2 Complexity via Variable Elimination

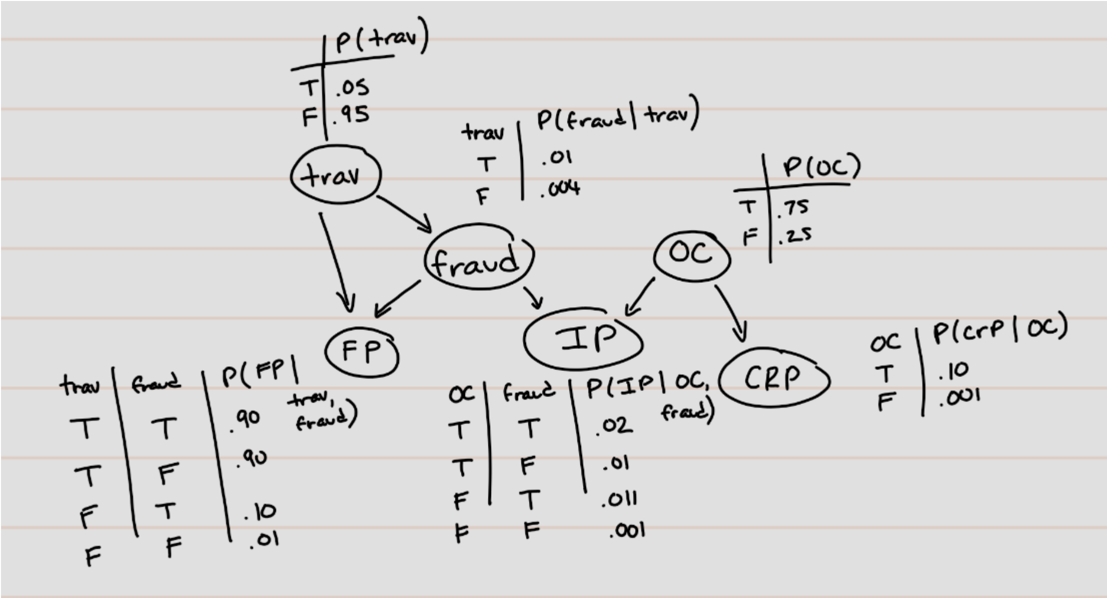
Moving on to variable elimination, the size of the factors will not exceed two variables. For instance, when computing $P(X_1|X_n = \text{true})$:

$$
\begin{aligned}
P(X_1|X_n = \text{true}) &= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} P(x_{n-1}|x_{n-2}) P(X_n = \text{true}|x_{n-1}) \\
&= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} f_{X_{n-1}}(x_{n-1}, x_{n-2}) f_{X_n}(x_{n-1}) \\
&= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) f_{\frac{x_{n-2}}{X_{n-1} \cdot X_n}}
\end{aligned}
$$

As evident, this mirrors a problem with $n-1$ variables rather than $n$. Hence, the computational work remains constant, independent of $n$, and the overall complexity is $\boxed{O(n)}$.

# §3 Problem 3

## §3.1 Part A




## §3.2 Part B

Prior probability of fraud:

$P(fraud) = P(travel) * P(fraud|travel) + P(\neg travel) * P(fraud|\neg travel)$

$= 0.5 * P(fraud|travel) + 0.95 * P(fraud|\neg travel)$

$= 0.5 * 0.01 + 0.95 * 0.004$

$= 0.0088$

$P(fraud|FP, \neg IP, CRP)$:

$P(fraud|FP, \neg IP, CRP) = *P(fraud, FP, \neg IP, CRP)$

$P(fraud, FP, \neg IP, CRP) = \sum_{trav} \alpha \sum_{OC} P(trav) * P(fraud|trav) * P(FP|fraud, trav) * P(OC) * P(\neg IP|OC, fraud) * P(CRP|OC)$

$= \sum_{OC} P(trav) * P(fraud|trav) * P(FP|fraud, trav) * P(OC) * P(\neg IP|OC, fraud) * P(CRP|OC) +$

$\sum_{OC} P(\neg trav) * P(fraud|\neg trav) * P(FP|fraud, \neg trav) * P(OC) * P(\neg IP|OC, fraud) * P(CRP|OC)$

$= P(trav) * P(fraud|trav) * P(FP|fraud, trav) * P(OC) * P(\neg IP|OC, fraud) * P(CRP|OC) +$

P(trav) * P(fraud — trav) * P(FP — fraud, trav) * P($\neg OC$) $* P(\neg IP|\neg OC, fraud) * P(CRP|\neg OC) +$

P($\neg trav) * P(fraud|\neg trav) * P(FP|fraud, \neg trav) * P(OC) * P(\neg IP|OC, fraud) * P(CRP|OC) +$

P($\neg trav$) $*$ $P(fraud|\neg trav)$ $*$ $P(FP|fraud, \neg trav)$ $*$ $P(\neg OC)$ $*$ $P(\neg IP|\neg OC, fraud)$ $*$ $P(CRP|\neg OC)$

$= 0.05 * 0.01 * .90 * .75 * (1 - 0.02) * 0.10+$

0.05 * 0.01 * .90 * .25 * (1 - 0.011) * 0.001 +

0.95 * 0.004 * .10 * .75 * (1 - 0.02) * 0.10 +

0.95 * 0.004 * .10 * .25 * (1 - 0.011) * 0.001

$= 0.000033075 + 1.112625 \times 10^{-7} + 0.00002793 + 9.3955 \times 10^{-8}$

$= 0.0000612102175$

Next, calculate alpha. We first need $P(\neg fraud, FP, \neg IP, CRP)$.

$P(\neg fraud, FP, \neg IP, CRP) = \sum_{trav} \alpha \sum_{OC} P(trav) * P(\neg fraud|trav) * P(FP|\neg fraud, trav) * P(OC) * P(\neg IP|OC, \neg fraud) * P(CRP|OC)$

$= \sum_{OC} P(trav) * P(\neg fraud|trav) * P(FP|\neg fraud, trav) * P(OC) * P(\neg IP|OC, \neg fraud) * P(CRP|OC) +$

$\sum_{OC} P(\neg trav) * P(\neg fraud|\neg trav) * P(FP|\neg fraud, \neg trav) * P(OC) * P(\neg IP|OC, \neg fraud) * P(CRP|OC)$

$= P(trav) * P(\neg fraud|trav) * P(FP|\neg fraud, trav) * P(OC) * P(\neg IP|OC, \neg fraud) * P(CRP|OC) +$

P(trav) * P($\neg fraud|trav$) $*$ $P(FP|\neg fraud, trav)$ $*$ $P(\neg OC)$ $*$ $P(\neg IP|\neg OC, \neg fraud)$ $*$ $P(CRP|\neg OC) +$

P($\neg trav$) $*$ $P(\neg fraud|\neg trav)$ $*$ $P(FP|\neg fraud, \neg trav)$ $*$ $P(OC)$ $*$ $P(\neg IP|OC, \neg fraud)$ $*$ $P(CRP|OC) +$

P($\neg trav$)$*P(\neg fraud|\neg trav) * P(FP|\neg fraud, \neg trav) * P(\neg OC) * P(\neg IP|\neg OC, \neg fraud) * P(CRP|\neg OC)$

Alpha $= 1/(P(fraud, FP, \neg IP, CRP) + P(\neg fraud, FP, \neg IP, CRP))$.

The final answer is $0.0000612102175*$ alpha

## §4 Problem 4

We can model the system as a hidden Markov model. We can model $X_t$ as a Markov chain with the states $\{A, B, C, D, E, F\}$ and transition matrix:

$$T_{i,j} = P(X_t = j|X_{t-1} = i)$$
$$T = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In addition, we have the observation matrices for hot and cold:

$$O_{\text{hot}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$O_{\text{cold}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We know that the rover starts at state A with probability 1, so $P(X_1 = A) = 1$. The initial state vector is therefore $f_1 = [1, 0, 0, 0, 0, 0]^T$.

## §4.1 Part 1

We are being asked to compute the state distribution $f_3$ given the observations $O_1 = \text{hot}$, $O_2 = \text{cold}$, and $O_3 = \text{cold}$.

$$\begin{aligned} f_3 &= \alpha O_{e_3} T f_2 \\ &= \alpha O_{e_3} T \left( \alpha O_{e_2} T f_1 \right) \\ &= \alpha O_{\text{cold}} T O_{\text{cold}} T f_1 \\ &= \alpha \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &\cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= [0, 0.2, 0.8, 0, 0, 0] \end{aligned}$$

# §5 Problem 5