

## Intro to AI Assignment 3 - Probabilistic Reasoning

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### §1 Problem 1

#### §1.1 Part a

The joint probability for the events  $A, B, C, D, E$  is defined by the chain rule as:

$$P(A, B, C, D, E) = P(A) \cdot P(B) \cdot P(C) \cdot P(D|A, B) \cdot P(E|B, C)$$

For the specific case where all events are true ( $T$ ), the joint probability is:

$$\begin{aligned} P(A = T, B = T, C = T, D = T, E = T) &= P(A = T) \cdot P(B = T) \cdot P(C = T) \\ &\quad \cdot P(D = T|A = T, B = T) \cdot P(E = T|B = T, C = T) \end{aligned}$$

Substituting the provided numerical values:

$$\begin{aligned} P(A = T, B = T, C = T, D = T, E = T) &= 0.2 \cdot 0.5 \cdot 0.8 \cdot 0.1 \cdot 0.3 \\ &= 0.0024 \end{aligned}$$

Thus, the likelihood of the combined event where  $A, B, C, D$  and  $E$  are all true is 0.0024.

#### §1.2 Part b

Now, let's analyze the scenario where all these events are false ( $F$ ). The joint probability is:

$$\begin{aligned} P(A = F, B = F, C = F, D = F, E = F) &= P(A = F) \cdot P(B = F) \cdot P(C = F) \\ &\quad \cdot P(D = F|A = F, B = F) \cdot P(E = F|B = F, C = F) \end{aligned}$$

Inserting the given probabilities for this scenario:

$$\begin{aligned} P(A = F, B = F, C = F, D = F, E = F) &= 0.8 \cdot 0.5 \cdot 0.2 \cdot 0.1 \cdot 0.8 \\ &= 0.0064 \end{aligned}$$

Consequently, the probability of the joint event where  $A, B, C, D$ , and  $E$  are all false is 0.0064.

## §1.3 Part c

We aim to determine the conditional probability  $P(\neg A|B, C, D, E)$ . Employing Bayes' theorem, this can be expressed as proportional to the joint probability  $P(\neg A, B, C, D, E)$ :

$$P(\neg A|B, C, D, E) \propto P(\neg A, B, C, D, E).$$

Let the normalization factor be  $\alpha$ , defined as:

$$\alpha = \frac{1}{P(A, B, C, D, E) + P(\neg A, B, C, D, E)}.$$

We are given the calculation for  $\alpha$ :

$$\begin{aligned} \alpha &= \frac{1}{(0.2 \cdot 0.5 \cdot 0.8 \cdot 0.1 \cdot 0.3) + (0.8 \cdot 0.5 \cdot 0.8 \cdot 0.6 \cdot 0.3)} \\ \alpha &= \frac{1}{0.0024 + 0.0576} = \frac{1}{0.06} = \frac{50}{3}. \end{aligned}$$

Now, we can compute the conditional probability  $P(\neg A|B, C, D, E)$ :

$$\begin{aligned} P(\neg A|B, C, D, E) &= \alpha \cdot P(\neg A, B, C, D, E). \\ P(\neg A|B, C, D, E) &= \frac{50}{3} \cdot 0.0576. \\ P(\neg A|B, C, D, E) &= 0.96. \end{aligned}$$

Therefore, the conditional probability  $P(\neg A|B, C, D, E)$  is 0.96.

## §2 Problem 2

### §2.1 Part a

Our goal is to determine the conditional probability  $P(\text{Burglary} | \text{JohnsCalls} = \text{true}, \text{MaryCalls} = \text{true})$ . We are given the formulation:

$$P(B|J, M) = \alpha \cdot P(B) \sum_E P(E) \sum_A P(A|B, E) \cdot P(J|A) \cdot P(M|A)$$

where  $\alpha = \frac{1}{P(J, M)}$ .

Following the provided steps and correcting the final calculation:

$$\begin{aligned}
 P(B|J, M) &= \alpha \cdot \begin{pmatrix} 0.00059224259 \\ 0.0014918576 \end{pmatrix} \\
 &\quad \text{(where the top element corresponds to } B = T \text{ and the bottom to } B = F) \\
 &\quad \left( \text{and } \alpha = \frac{1}{0.0020853609} \right) \\
 &= \frac{1}{0.0020853609} \cdot \begin{pmatrix} 0.00059224259 \\ 0.0014918576 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{0.00059224259}{0.0020853609} \\ \frac{0.0014918576}{0.0020853609} \end{pmatrix} \\
 &= \begin{pmatrix} 0.284 \\ 0.716 \end{pmatrix}
 \end{aligned}$$

Here,  $\boxed{0.284}$  signifies the probability of a burglary occurring given that John and Mary call, while  $\boxed{0.716}$  represents the probability of no burglary under the same conditions.

## §2.2 Part b

What is the computational cost of determining  $P(X_1|X_n = \text{true})$  via enumeration? What is the cost using variable elimination?

### §2.2.1 Complexity via Enumeration

To compute  $P(X_1|X_n = \text{true})$  by enumeration, we initially assess two binary trees for each state of  $X_1$ . Each of these trees possesses a depth of  $n - 2$ . Consequently, the aggregate computational effort for enumeration amounts to  $\boxed{O(2^n)}$ .

### §2.2.2 Complexity via Variable Elimination

Moving on to variable elimination, the size of the factors will not exceed two variables. For instance, when computing  $P(X_1|X_n = \text{true})$ :

$$\begin{aligned}
 P(X_1|X_n = \text{true}) &= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} P(x_{n-1}|x_{n-2}) P(X_n = \text{true}|x_{n-1}) \\
 &= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} f_{X_{n-1}}(x_{n-1}, x_{n-2}) f_{X_n}(x_{n-1}) \\
 &= \alpha \cdot P(X_1) \cdots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) f_{\frac{x_{n-2}}{X_{n-1} \cdot X_n}}
 \end{aligned}$$

As evident, this mirrors a problem with  $n - 1$  variables rather than  $n$ . Hence, the computational work remains constant, independent of  $n$ , and the overall complexity is  $\boxed{O(n)}$ .

### §3 Problem 3

### §4 Problem 4

We can model the system as a hidden Markov model. We can model  $X_t$  as a Markov chain with the states  $\{A, B, C, D, E, F\}$  and transition matrix:

$$T_{i,j} = P(X_t = j | X_{t-1} = i)$$
$$T = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In addition, we have the observation matrices for hot and cold:

$$O_{\text{hot}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$O_{\text{cold}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We know that the rover starts at state A with probability 1, so  $P(X_1 = A) = 1$ . The initial state vector is therefore  $f_1 = [1, 0, 0, 0, 0, 0]^T$ .

#### §4.1 Part 1

We are being asked to compute the state distribution  $f_3$  given the observations  $O_1 = \text{hot}$ ,  $O_2 = \text{cold}$ , and  $O_3 = \text{cold}$ .

$$\begin{aligned}f_3 &= \alpha O_{e_3} T f_2 \\&= \alpha O_{e_3} T (\alpha O_{e_2} T f_1) \\&= \alpha O_{\text{cold}} T O_{\text{cold}} T f_1 \\&= \alpha \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\&\quad \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\&= [0, 0.2, 0.8, 0, 0, 0]\end{aligned}$$

## §5 Problem 5