Course 16 :198 :440 : Introduction To Artificial Intelligence
Lecture 5

Solving Problems by Searching: Adversarial Search

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Outline

We examine the problems that arise when we make decisions in a world where other agents are also acting, possibly against us.

1. The minimax algorithm
2. Alpha-beta pruning
3. Imperfect fast decisions
4. Stochastic games
5. Partially observable games

Games

- **Multiagent environments** are environments where more than one agent is acting, simultaneously or at different times.
- **Contingency plans** are necessary to account for the unpredictability of other agents.
- Each agent has its own personal **utility function**.
- The corresponding **decision-making** problem is called a **game**.
- A game is **competitive** if the utilities of different agents are maximized in different states.
- In **zero-sum games**, the sum of the utilities of all agents is **constant**.
- Zero-sum games are purely competitive.

Games

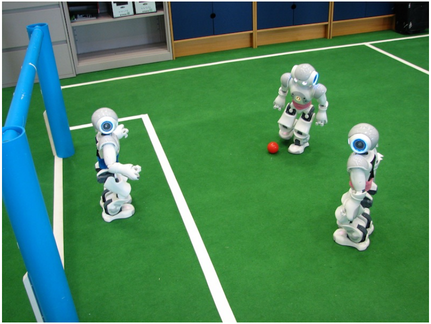
- The abstract nature of games, such as chess, makes them appealing to study in AI.
- The state of a game is easy to represent and agents typically have a small number of actions to choose from.

(a) (b)

Left : Computer chess pioneers Herbert Simon and Allen Newell (1958). Right : John McCarthy and the Kotok-McCarthy program on an IBM 7090 (1967)

Games

- The abstract nature of games, such as chess, makes them appealing to study in AI.
- The state of a game is easy to represent and agents typically have a small number of actions to choose from.
- Physical games, such as soccer, are more difficult to study due to their continuous state and action spaces.



Robot soccer
(from ri.cmu.edu)

Games

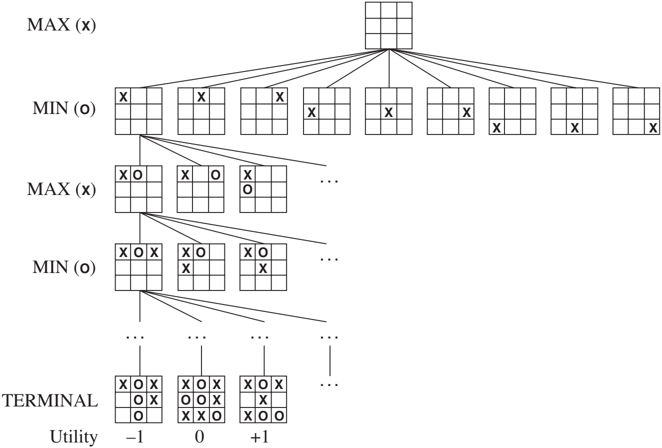
A game is described by :

- $\mathbf{S_0}$: the initial state (how is the game set up at the start ?).
- **PLAYER(s)** : Indicates which player has the move in state s (whose turn is it ?).
- **ACTIONS(s)** : Set of legal actions in state s.
- **RESULT(s, a)** : returns the next state after we play action a in state s.
- **TERMINAL-TEST(s)** : Indicates if s is a terminal state.
- **UTILITY(s, p)** : (also called objective or payoff function) defines a numerical value for a game that ends in terminal state s for player p

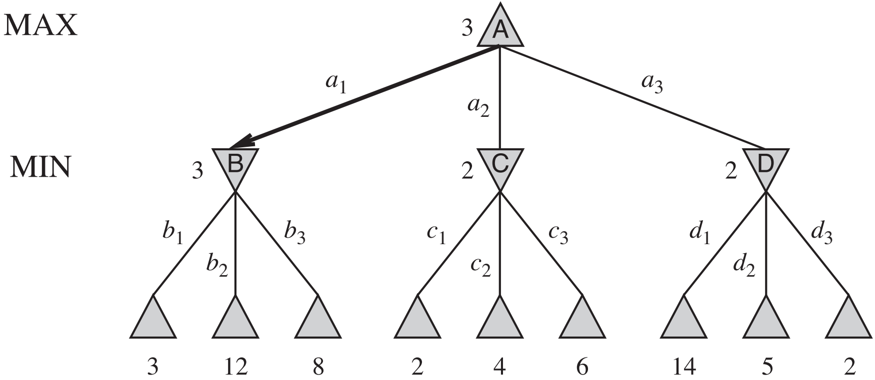
Example : tic-tac-toe

We suppose there are two players in a zero-sum game

- We call our player **MAX**, she tries to maximize our utility.
- We call our opponent **MIN**, she tries to minimize our utility (i.e. maximize her utility).



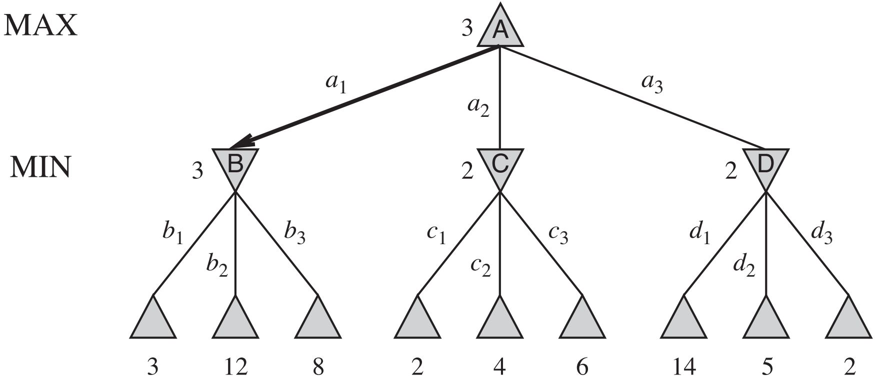
Optimal games



- $\triangle$ Indicates states where MAX should play.
- ∇ Indicates states where MIN should play.

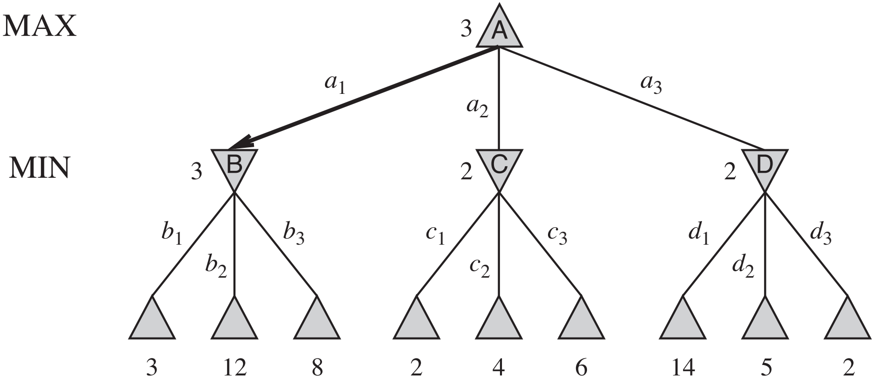
Which action among $\{a_1, a_2, a_3\}$ should MAX play ?

Minimax strategy



- We don't take any risk, we assume that MIN will play optimally.
- We look for the best action for the worst possible scenario.
- What if our opponent is not optimal ? Can we learn the opponent's behaviour ?
- What if our player is trying to fool us by behaving in a certain way ?

Minimax strategy



$$\text{MINIMAX}(s) = \begin{cases} \text{UTILITY}(s) & \text{if TERMINAL-TEST}(s) \\ \max_{a \in Actions(s)} \text{MINIMAX}(\text{RESULT}(s,a)) & \text{if PLAYER}(s) = MAX, \\ \min_{a \in Actions(s)} \text{MINIMAX}(\text{RESULT}(s,a)) & \text{if PLAYER}(s) = MIN. \end{cases}$$

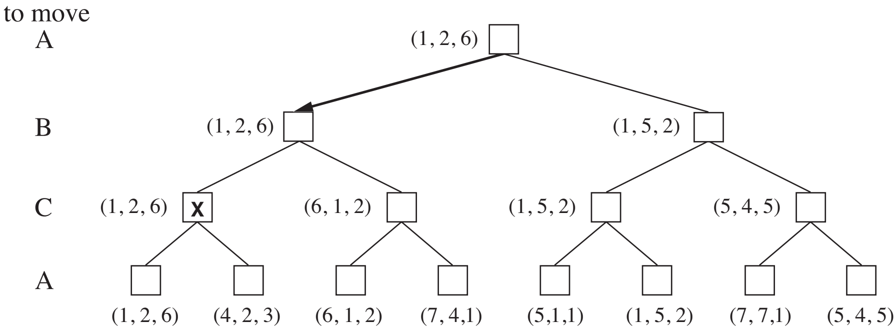
Minimax algorithm

```
function MINIMAX-DECISION(state) returns an action
  return $\arg\max_{a \in \text{ACTIONS}(s)}$ MIN-VALUE(RESULT(state, a))
```

```
function MAX-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  $v \leftarrow -\infty$
  for each a in ACTIONS(state) do
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(s, a)))
  return v
```

```
function MIN-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  $v \leftarrow \infty$
  for each a in ACTIONS(state) do
    $v \leftarrow$ MIN(v, MAX-VALUE(RESULT(s, a)))
  return v
```

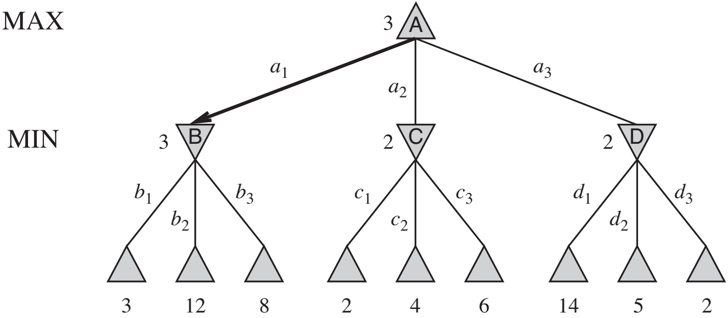
Minimax strategy in multiplayer games



- We have three players A, B, and C.
- The utilities are represented by a 3-dimensional vector (v_A, v_B, v_C).
- We apply the same principle : assume that every player is optimal.
- If the game is not zero-sum, implicit collaborations may occur.

Alpha-Beta pruning

- Time is a major issue in game search trees. Searching the complete tree takes $O(b^m)$ operations, where b is the branching factor and m is the depth of the tree (the horizon).
- Do we really need to parse the whole tree to find a minimax strategy ?

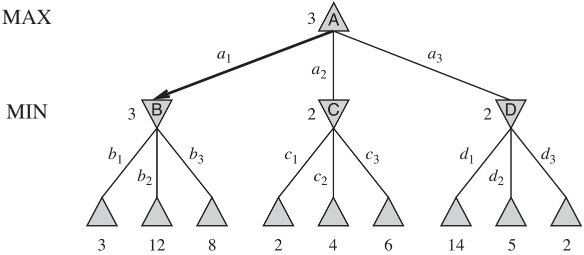


Alpha-Beta pruning

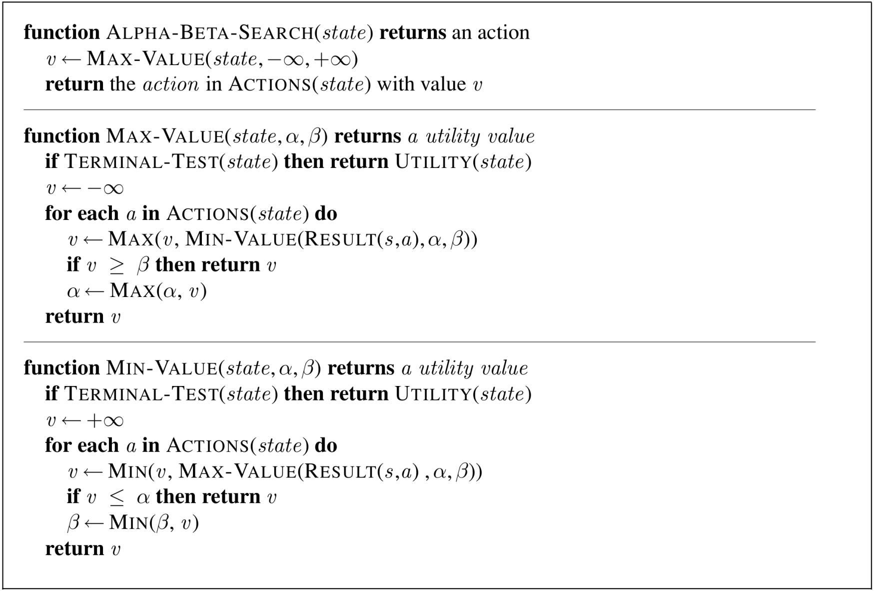
Example

- Assume that the search tree has been parsed except for actions c_2 and c_3.
- Let us denote the utilities of c_2 and c_3 by x and y respectively.

$$\begin{aligned}\text{MINIMAX(root)} &= \max(\min(3,12,8),\min(2,x,y),\min(14,5,2)) \\ &= \max(3,\min(2,x,y),2) \\ &= \max(3,z,2) \text{ where } z=\min(2,x,y)\leq 2 \\ &= 3.\end{aligned}$$

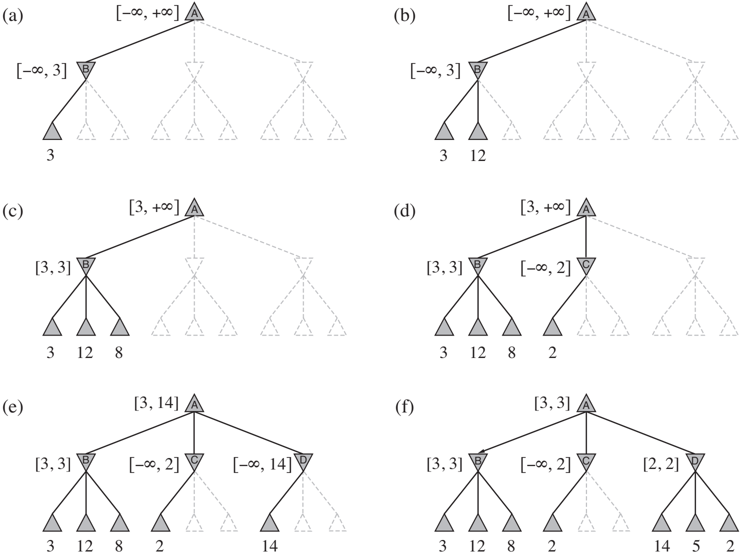


Alpha-Beta pruning



```
function ALPHA-BETA-SEARCH(state) returns an action
  $v \leftarrow$ MAX-VALUE(state, $-\infty, +\infty$)
  return the action in ACTIONS(state) with value $v$
-----------------------------------------------------------------
function MAX-VALUE(state, $\alpha, \beta$) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  $v \leftarrow -\infty$
  for each $a$ in ACTIONS(state) do
    $v \leftarrow$ MAX($v$, MIN-VALUE(RESULT($s$,$a$), $\alpha, \beta$))
    if $v \geq \beta$ then return $v$
    $\alpha \leftarrow$ MAX($\alpha$, $v$)
  return $v$
-----------------------------------------------------------------
function MIN-VALUE(state, $\alpha, \beta$) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  $v \leftarrow +\infty$
  for each $a$ in ACTIONS(state) do
    $v \leftarrow$ MIN($v$, MAX-VALUE(RESULT($s$,$a$) , $\alpha, \beta$))
    if $v \leq \alpha$ then return $v$
    $\beta \leftarrow$ MIN($\beta$, $v$)
  return $v$
```

Alpha-Beta pruning



Imperfect fast decisions

- The minimax algorithm generates the entire search tree.
- The alpha-beta algorithm allows us to prune large parts of the search tree, but its complexity is still exponential in the branching factor (number of actions).
- This is still non-practical because moves should be made very quickly.

Cutting-off the search

- A cutoff test is used to decide when to stop looking further.
- A heuristic **evaluation function** is used to estimate the utility where the search is cut off.

$$\text{H-MINIMAX}(s,d) = \begin{cases} \text{EVAL}(s) & \text{if CUTOFF-TEST}(s,d) = true, \\ \max_{a \in Actions(s)} \text{H-MINIMAX}(\text{RESULT}(s,a), d+1) & \text{if PLAYER}(s) = MAX, \\ \min_{a \in Actions(s)} \text{H-MINIMAX}(\text{RESULT}(s,a), d+1) & \text{if PLAYER}(s) = MIN. \end{cases}$$

Evaluation functions

- Human chess players have ways of judging the value of a position without imagining all the moves ahead until a check-mate.
- A good evaluation function should order the actions correctly according to their true utilities.
- Evaluation functions should be computed very quickly.

Example : evaluation functions in chess

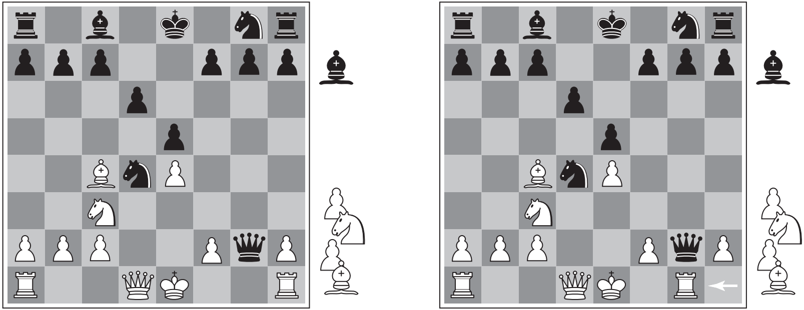
- Evaluation functions use **features** of given positions in the game.
- Example : number of pawns in the given position. If we know from experience that 72% of positions “two pawns vs one pawn” lead to a win (utility $+1$) ; 20% to a loss (0), and 8% to a draw (1/2), then the expected value of these positions is 0.76.
- Other functions such as the advantage in each piece, “good pawn structure”, and “king safety” can be used as features f_i.
- The evaluation function can be given using a weighted linear model :

$$\mathsf{EVAL}(s) = \sum_{i=1}^{n} w_i f_i(s),$$

where w_i is the importance of feature f_i.

Example : evaluation functions in chess

- Linear models assume that the features are independent, which is not always true (bishops are more efficient at endgame).
- Values of some features do not increase linearly (two knights are way more useful than one knight).

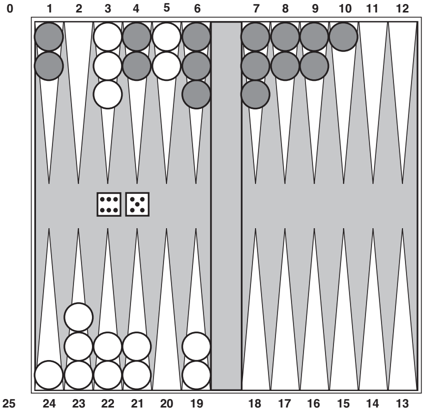


(a) White to move (b) White to move

In the two positions, the two players have the same number of pieces. The position on the right is much worse than the one on the left for Black. What if the search cutoff happens in the left ?

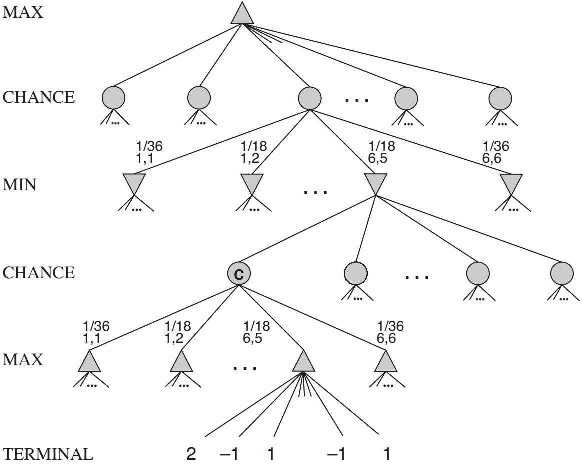
Stochastic games

- In some games, the state of the game changes randomly depending on the selected actions
- Backgammon is a typical game that combines luck and skill.



Stochastic games

We can use the same minimax strategy, but we need to take the randomness into account by computing the expected utilities.



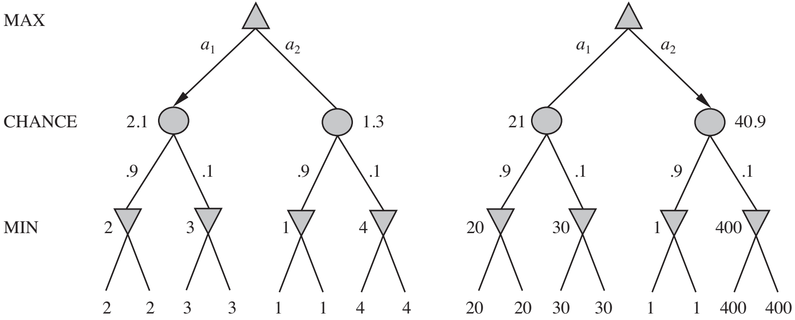
Stochastic games

We can use the same minimax strategy, but we need to take the randomness into account by computing the expected utilities.

$$\text{EXPECMINIMAX}(s) = \begin{cases} \text{UTILITY}(s) & \text{if TERMINAL-TEST}(s) = true, \\ \max_{a \in Actions(s)} \text{EXPECMINIMAX}(\text{RESULT}(s, a)) & \text{if PLAYER}(s) = MAX, \\ \min_{a \in Actions(s)} \text{EXPECMINIMAX}(\text{RESULT}(s, a)) & \text{if PLAYER}(s) = MIN \\ \sum_r P(r)\text{EXPECMINIMAX}(\text{RESULT}(s, r)) & \text{if PLAYER}(s) = CHANCE, \end{cases}$$

where r is the outcome of a chance event (e.g, roll of a single dice, in which case $r \in \{1, 2, 3, 4, 5, 6\}$, and $P(r)$ is the probability of getting outcome r.

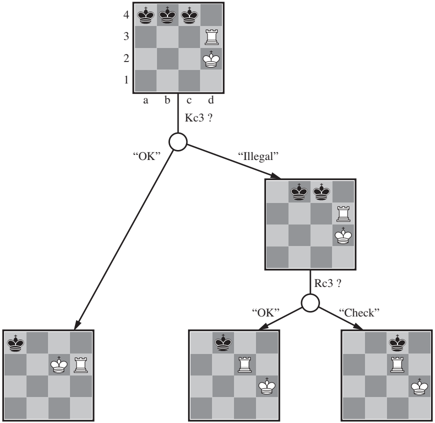
Trouble with evaluation functions in stochastic games



If the evaluation function rescales the values of the cut-off states with different factors, despite preserving their order, the order of the expected values of actions may change.

Partially observable

- In some games, the state of the game is not fully known.
- Cards and Kriegspiel are examples of such games.
- The state of the game can be tracked by remembering past actions and observations.



Game Theory

What is game theory ?

Game theory is the formal study of conflict and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, robots, firms, etc. The concepts of game theory provide a language to formulate, structure, analyze, and understand strategic scenarios

Game Theory is the formal study of strategic interaction.

Game Theory

Examples

- poker, chess, soccer, driving, dating, stock market
- advertising, setting prices, entering new markets, building a reputation
- bargaining, partnerships, job market search and screening
- designing contracts, auctions, insurance, environmental regulations
- international relations, trade agreements, electoral campaigns

Game theory plays a fundamental role in modern economics.

Game Theory

A game is a triplet (N, A, V)

- $N = \{1, 2, \ldots, n\}$: finite set of players
- A_i : set of actions (or strategies) for player i
- $A = A_1 \times A_2 \times \ldots A_n$: set of joint actions
- $A_{-i} = \prod_{j \neq i} A_j$: set of joint actions of i's opponents
- $V_i : A \to \mathbb{R}$, payoff (or value) function

All players know (N, A, V), and know that their opponents know it, and know that their opponents know that everyone knows, and so on.

Game Theory

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Mixed (or stochastic) strategies

- $\Delta(X)$: set of probability measures (or distributions) over the set X
- $\delta_i \in \Delta(A_i)$: mixed strategy of player i
- $\delta \in \Delta(A_1) \times \Delta(A_2) \times \ldots \Delta(A_n)$: mixed joint strategy
- Payoff of a mixed strategy :

$$V_i(\delta_i, a_{-i}) = \sum_{a_i \in A_i} \delta_i(a_i) V_i(a_i, a_{-i})$$

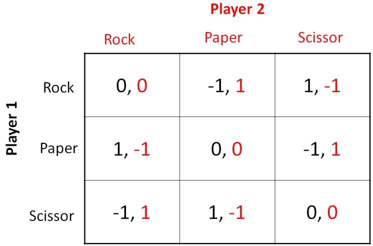
Example : Rock-Paper-Scissors

At each round of the game, the two players simultaneously choose their actions and receive payoffs accordingly.

			Player 2	
		Rock	Paper	Scissor
	Rock	0, 0	-1, 1	1, -1
Player 1	Paper	1, -1	0, 0	-1, 1
	Scissor	-1, 1	1, -1	0, 0

- $N = \{1, 2\}$: set of players
- $A_1 = A_2 = \{rock, paper, scissor\}$: sets of actions
- $A = \{(rock, rock), (rock, paper), \ldots, (scissor, scissor)\}$: set of joint actions
- Payoff function : $V_1(rock, rock) = 0$, $V_1(rock, paper) = -1$, etc.

Example : Rock-Paper-Scissors



		Player 2		
		Rock	Paper	Scissor
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissor	-1, 1	1, -1	0, 0

- Example of a mixed action (mixed strategy) δ_1 for player 1 :

$$\Big(P(a_1 = rock) = 0.75, P(a_1 = paper) = 0.15, P(a_1 = scissor) = 0.10\Big)$$

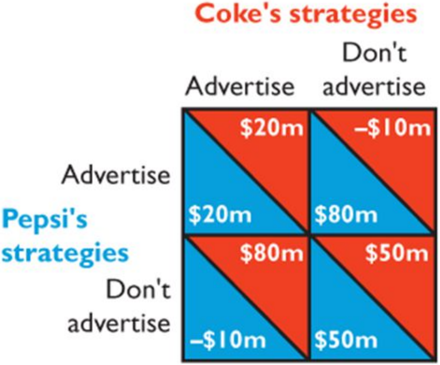
- Example of a mixed action (mixed strategy) δ_2 for player 2 :

$$\Big(P(a_2 = rock) = 0.55, P(a_2 = paper) = 0.40, P(a_2 = scissor) = 0.05\Big)$$

- (δ_1, δ_2) is a mixed joint action.

Dominated Strategies

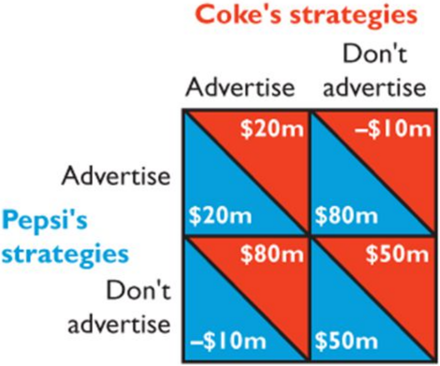
Are there actions that are obviously bad ?



		Coke's strategies	
		Advertise	Don't advertise
Pepsi's strategies	Advertise	$20m $20m	–$10m $80m
	Don't advertise	$80m –$10m	$50m $50m

Dominated Strategies

Are there actions that are obviously bad ?

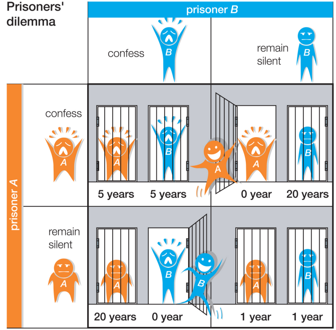


		Coke's strategies	
		Advertise	Don't advertise
Pepsi's strategies	Advertise	$20m $20m	–$10m $80m
	Don't advertise	$80m –$10m	$50m $50m

The action "Don't advertise" is clearly dominated by the action "advertise".

Dominated Strategies

Prisoners' Dilemma (PD) : Flood and Dresher (1950) and RAND corporation's investigations into game theory for possible applications to global nuclear strategy.



Dominated Strategies

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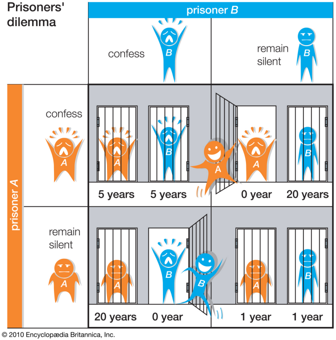


Each prisoner is better off defecting regardless of what the other does. We say "Confess" strictly dominates "Don't Confess" for each prisoner. Dilemma : the resulting outcome is (Confess, Confess), which is worse than (Don't Confess, Don't Confess) for both players.

Dominated Strategies

Definition

A strategy $a_i \in A_i$ is strictly dominated by $\delta_i \in \Delta A_i$ if

$$V_i(\delta_i, a_{-i}) > V_i(a_i, a_{-i}), \forall a_{-i} \in A_{-i}$$

Pure Strategies May Be Dominated by Mixed Strategies

	L	R
T	$3, x$	$0, x$
M	$0, x$	$3, x$
B	$1, x$	$1, x$

Figure: B is s. dominated by $1/2T + 1/2M$.

Nash Equilibrium

Often, it is not possible to find a strategy that dominates all other strategies. Nevertheless, the involved parties find a solution.
A Nash equilibrium is a joint strategy with the property that no player can benefit by deviating from his corresponding strategy.

Definition (Nash 1950)

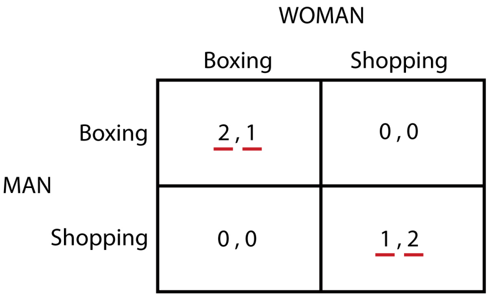
A joint strategy $\delta \in \Delta(A)$ is a Nash equilibrium if for every $i \in N$

$$V_i(\delta_i, \delta_{-i}) \geq V_i(a_i, \delta_{-i}), \forall a_i \in A_i$$

John Nash proved that every finite game has a Nash equilibrium.
Not all games have pure strategy Nash equilibria, some have only mixed strategy Nash equilibria.

Nash Equilibrium

Example : Battle of the sexes



		WOMAN	
		Boxing	Shopping
MAN	Boxing	2 , 1	0 , 0
	Shopping	0 , 0	1 , 2

In game theory, battle of the sexes (BoS) is a two-player coordination game. Imagine a couple that agreed to meet this weekend, but cannot recall if they will be shopping or attending a boxing match (and the fact that they forgot is common knowledge). The husband would prefer to go to the boxing match. The wife would rather go shopping. Both would prefer to go to the same place rather than different ones. If they cannot communicate, where should they go ? (from Wikipedia)