Course 16:198:440: Introduction To Artificial Intelligence
Lecture 12

Temporal Models

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Outline

In the previous lectures on Bayesian and Markov networks, we were concerned about problems where the values of variables are **static**. We now consider problems wherein variables **change over time**.

Example 1: gait modeling

- The joint angles of a person are random variables that change over time.
- Probabilistic modeling can be used to predict the values of the joint angles in a sequence, or to recognize a person from her gait.

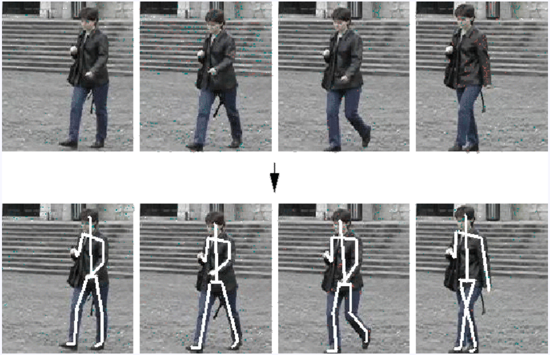
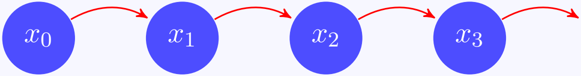
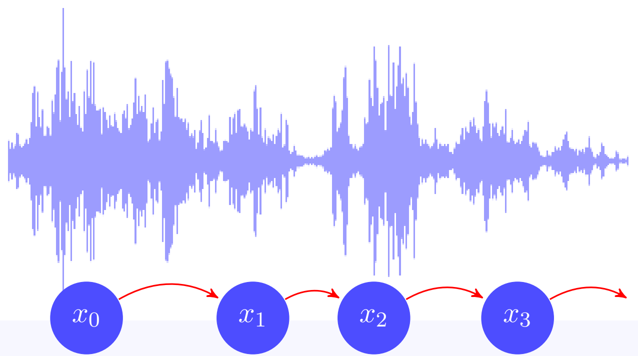


image from inf.ed.ac.uk



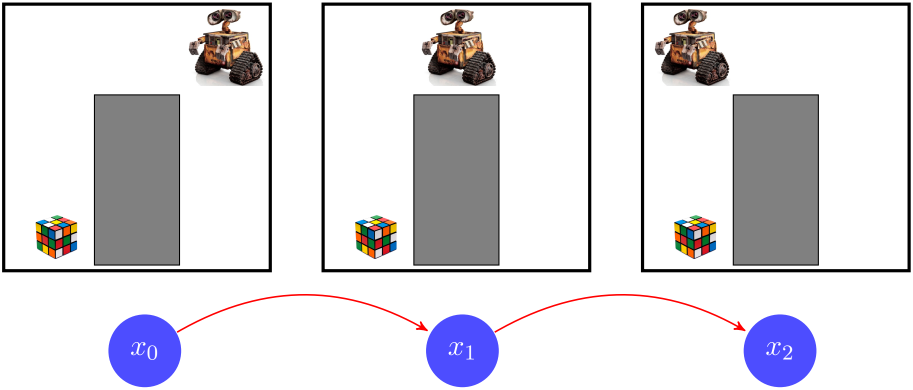
Example 2: natural language processing

- A speech is a sequence of utterances take different values over time.
- Probabilistic modeling is used for speech processing and recognition.



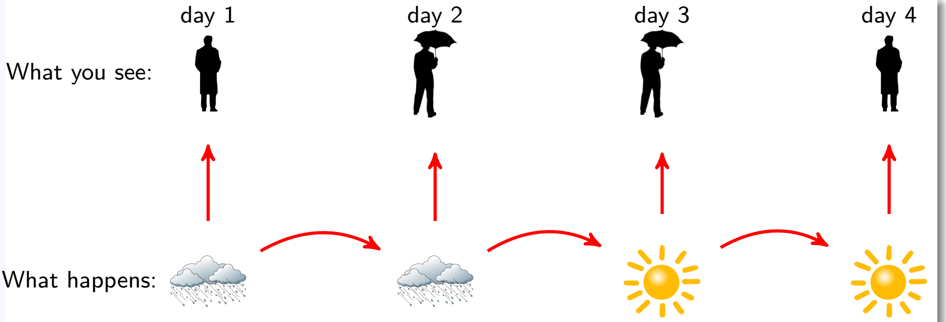
Example 3: robot navigation

The robot's exact position is a random variable that changes over time.



Toy example

- You are the security guard stationed at a secret underground installation.
- You want to know whether it's raining today,
- but your only access to the outside world occurs each morning when you see the director coming in with, or without, an umbrella.



States and observations

- In a temporal model, the world is seen as a sequence of snapshots, or *time slices*.
- A **state** is defined as a set of random variables $\mathbf{X_t}$.
- The variables in $\mathbf{X_t}$ take new values at each time t.
- The variables in $\mathbf{X_t}$ are **unobservable** (or unknown).
- The set of observed variables (**evidence**) at time t is denoted by $\mathbf{E_t}$.
- We assume that time t is discrete, $t \in \{0, 1, 2, 3, \dots\}$.
- We use $X_{a:b}$ to denote the variable X at times $t = a$ to $t = b$. $X_{a:b} = [X_a, X_{a+1}, \dots, X_b]$ and $E_{a:b} = [E_a, E_{a+1}, \dots, E_b]$

In our previous example:

- Each time step corresponds to a day.
- There is one state variable $X_t = \{\text{rain}, \text{no rain}\}$.
- There is one evidence variable $E_t = \{\text{umbrella}, \text{no umbrella}\}$.

Transition model

- The transition model specifies the probability distribution over current state values, given all the previous states: $P(X_t \mid X_{0:t-1})$.
- Problem: $X_{0:t-1}$ is unbounded in size as t increases.
- **Markov assumption**: the last state X_{t-1} has all the information from the past. This defines a first-order Markov chain.

$$P(X_t \mid X_{0:t-1}) = P(X_t \mid X_{t-1}).$$

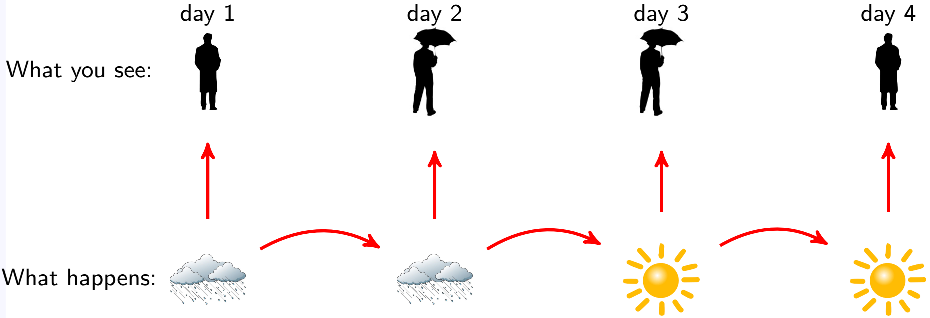
- One can also define a second-order Markov chain by assuming

$$P(X_t \mid X_{0:t-1}) = P(X_t \mid X_{t-1}, X_{t-2}).$$

- Any n-order Markov chain can be reduced to a first-order Markov chain by redefining the states.

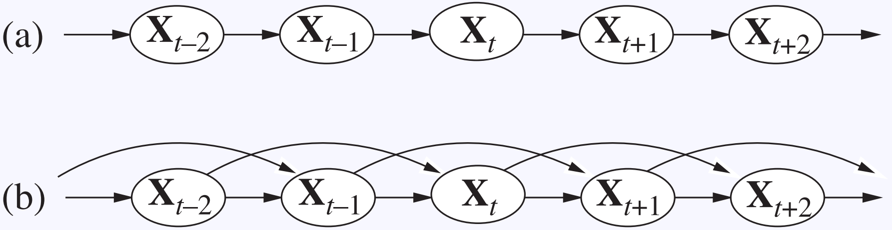
Transition model

The transition model is the probabilities:
P(it rains today | it rained yesterday),
P(it rains today | it didn't rain yesterday).



Transition model

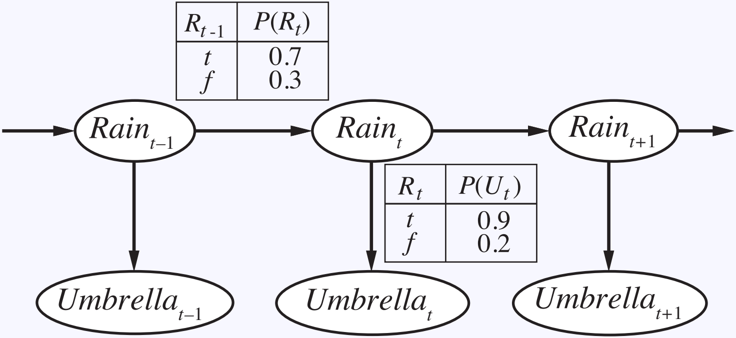
The transition model is a *dynamic* Bayesian network. Previous states are the cause of the current state.



(a) first-order Markov chain, (b) second-order Markov chain

Observation model

- The observation model specifies the probability distribution over current observations, given the current state: $P(E_t \mid X_t)$.
- $P(E_t \mid X_t)$ is also called the **sensor model**.
- The transition and observation models form together a *dynamic* Bayesian network.



Inference problems in temporal models

Inference

The joint probability of a sequence of states $X_{0:t}$ and observations $E_{1:t}$ is given by

$$P(X_{0:t}, E_{1:t}) = P(X_0) \prod_{i=1}^{t} P(X_i \mid X_{i-1}) P(E_i \mid X_i).$$

Inference problems in temporal models

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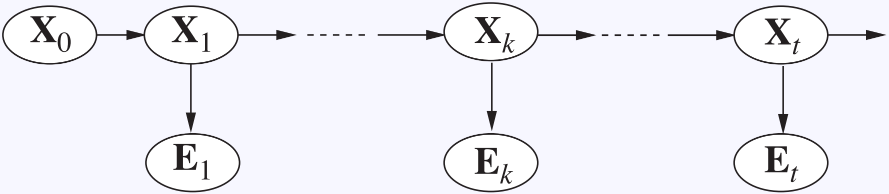
Inference problems

- **Filtering** (a.k.a state estimation): Find $P(X_t \mid e_{1:t})$.
- **Prediction**: Find $P(E_{t+1:T} \mid e_{1:t})$.
- **Smoothing**: Find $P(X_{0:t} \mid e_{1:t})$.
- **Most Likely Explanation**: Find $\arg\max_{X_{0:t}} P(X_{0:t} \mid e_{1:t})$.

Filtering (state estimation)

Given all the past and the current observations $e_{1:t} = [e_1, e_2, \ldots e_t]$, we want to compute a distribution on the current state X_t (without knowing the previous states $X_{0:t-1}$).
$P(X_t \mid e_{1:t})$ is know as the **belief state**.



Filtering (state estimation)

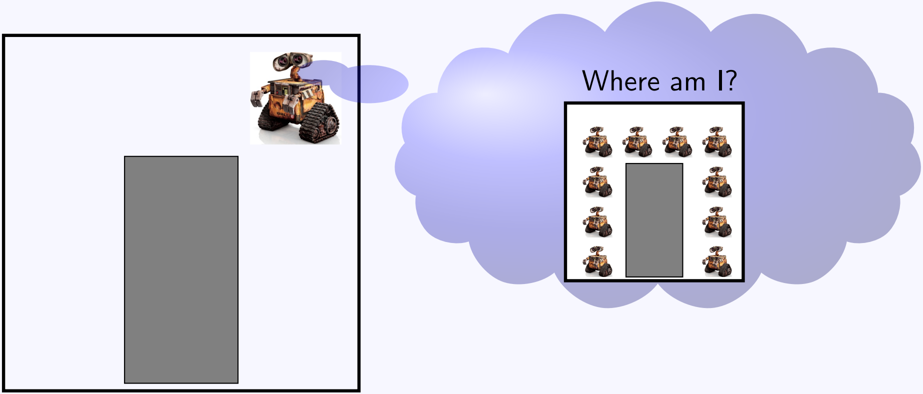


Figure: Initial belief state.

Filtering (state estimation)

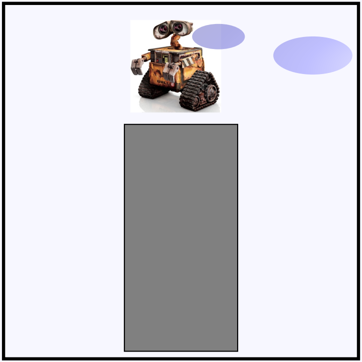


Figure: Belief state after moving left.

Filtering (state estimation)

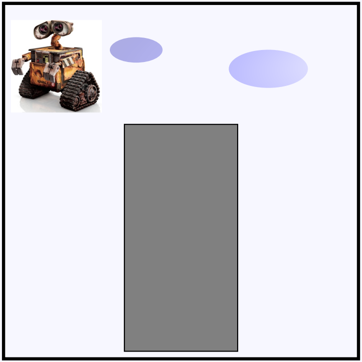


Figure: Belief state after moving left twice.

Filtering (state estimation)

$$P(X_t \mid e_{1:t}) \quad = \quad P(X_t \mid e_{1:t-1}, e_t)$$

Filtering (state estimation)

$$
\begin{aligned}
P(X_t \mid e_{1:t}) &= P(X_t \mid e_{1:t-1}, e_t) \\
&= \frac{P(e_t \mid X_t, e_{1:t-1}) P(X_t \mid e_{1:t-1})}{P(e_t | e_{1:t-1})} \text{ (Bayes' Rule)}
\end{aligned}
$$

Filtering (state estimation)

$$
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&= \frac{P(e_t \mid X_t) P(X_t \mid e_{1:t-1})}{P(e_t|e_{1:t-1})} \\
&= \frac{P(e_t \mid X_t) \sum_{x_{t-1}} P(X_t \mid x_{t-1}, e_{1:t-1}) P(x_{t-1} \mid e_{1:t-1})}{P(e_t|e_{1:t-1})}
\end{aligned}
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&= \frac{P(e_t \mid X_t) \sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1:t-1})}{P(e_t | e_{1:t-1})} \text{ (Markov)}
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Filtering (state estimation)

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&= \frac{P(e_t \mid X_t) \sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1:t-1})}{P(e_t | e_{1:t-1})} \text{ (Markov)} \\
&= \frac{P(e_t \mid X_t) \sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1:t-1})}{\sum_{x_t} P(e_t \mid x_t) \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1} \mid e_{1:t-1})}
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&= \frac{P(e_t \mid X_t) P(X_t \mid e_{1:t-1})}{P(e_t|e_{1:t-1})} \\
&= \frac{P(e_t \mid X_t) \sum_{x_{t-1}} P(X_t \mid x_{t-1}, e_{1:t-1}) P(x_{t-1} \mid e_{1:t-1})}{P(e_t|e_{1:t-1})} \\
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\end{aligned}
$$

$P(X_t \mid e_{1:t})$ can be computed recursively by starting with the prior $P(X_0)$.

Filtering (state estimation)

$$P(X_t \mid e_{1:t}) = \frac{P(e_t \mid X_t) \sum_{x_{t-1}} P(X_t \mid x_{t-1}) P(x_{t-1} \mid e_{1:t-1})}{\sum_{x_t} P(e_t \mid x_t) \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1} \mid e_{1:t-1})}$$

$P(X_t \mid e_{1:t})$ can be computed recursively by starting with the prior $P(X_0)$.

$$P(X_t \mid e_{1:t}) = \text{FORWARD}\Big(P(X_{t-1} \mid e_{1:t-1}), e_t\Big).$$

Prediction

Let's say we have received a sequence of observations $e_{1:t} = (e_1, e_2, \ldots, e_t)$ and we want to compute $P(E_{t+1:T} \mid e_{1:t})$, the probability distribution over future observations. We have

$$P(E_{t+1:T} \mid e_{1:t}) = \sum_{x_t} P(x_t \mid e_{1:t}) P(E_{t+1:T} \mid x_t),$$

where computing $P(x_t \mid e_{1:t})$ is a filtering problem.
We need then to find how to compute $P(E_{t+1:T} \mid x_t)$.

Prediction

$$\begin{aligned}
P(E_{t+1:T} \mid x_t) &= \sum_{x_{t+1}} P(x_{t+1} \mid x_t) P(E_{t+1:T} \mid x_{t+1}) \text{ (Markov property)} \\
&= \sum_{x_{t+1}} P(x_{t+1} \mid x_t) P(E_{t+1}, E_{t+2:T} \mid x_{t+1}) \\
&= \sum_{x_{t+1}} P(x_{t+1} \mid x_t) P(E_{t+1} \mid x_{t+1}) P(E_{t+2:T} \mid x_{t+1})
\end{aligned}$$

Prediction

$$
\begin{aligned}
P(E_{t+1:T} \mid x_t) &= \sum_{x_{t+1}} P(x_{t+1} \mid x_t) P(E_{t+1:T} \mid x_{t+1}) \text{ (Markov property)} \\
&= \sum_{x_{t+1}} P(x_{t+1} \mid x_t) P(E_{t+1}, E_{t+2:T} \mid x_{t+1}) \\
&= \sum_{x_{t+1}} P(x_{t+1} \mid x_t) P(E_{t+1} \mid x_{t+1}) P(E_{t+2:T} \mid x_{t+1})
\end{aligned}
$$

$P(E_{t+1:T} \mid x_t)$ can be computed recursively, starting at $t = T - 1$ and setting $P(E_{T+1:T} \mid x_T) = 1$.

Prediction

$$P(E_{t+1:T} \mid X_t) = \text{BACKWARD}\Big(P(E_{t+2:T} \mid X_{t+1}), E_{t+1}\Big)$$

$P(E_{t+1:T} \mid x_t)$ can be computed recursively, starting at $t = T - 1$ and setting $P(E_{T+1:T} \mid x_T) = 1$.

Smoothing

Let's say we have received a sequence of observations $e_{1:t} = (e_1, e_2, \ldots, e_t)$ and we want to compute $P(X_k \mid e_{1:t})$, the probability distribution over past states at time $k < t$.

$$
\begin{aligned}
P(X_k \mid e_{1:t}) &= P(X_k \mid e_{1:k}, e_{k+1:t}) \\
&= \frac{P(e_{k+1:t} \mid e_{1:k}, X_k) P(X_k \mid e_{1:k})}{P(e_{k+1:t} \mid e_{1:k})} \text{ (Bayes' Rule)} \\
&= \frac{P(e_{k+1:t} \mid X_k) P(X_k \mid e_{1:k})}{P(e_{k+1:t} \mid e_{1:k})} \\
&= \frac{P(e_{k+1:t} \mid X_k) P(X_k \mid e_{1:k})}{\sum_{x_k} P(e_{k+1:t} \mid x_k) P(x_k \mid e_{1:k})}
\end{aligned}
$$

Smoothing

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\begin{aligned}
P(X_k \mid e_{1:t}) &= P(X_k \mid e_{1:k}, e_{k+1:t}) \\
&= \frac{P(e_{k+1:t} \mid e_{1:k}, X_k) P(X_k \mid e_{1:k})}{P(e_{k+1:t} \mid e_{1:k})} \text{ (Bayes' Rule)} \\
&= \frac{P(e_{k+1:t} \mid X_k) P(X_k \mid e_{1:k})}{P(e_{k+1:t} \mid e_{1:k})} \\
&= \frac{P(e_{k+1:t} \mid X_k) P(X_k \mid e_{1:k})}{\sum_{x_k} P(e_{k+1:t} \mid x_k) P(x_k \mid e_{1:k})}
\end{aligned}
$$

Notice that computing

- $P(X_k \mid e_{1:k})$ is a filtering problem (state estimation), which can be done **forward**,
- $P(e_{k+1:t} \mid X_k)$ is a prediction problem, which can be done **backward**.

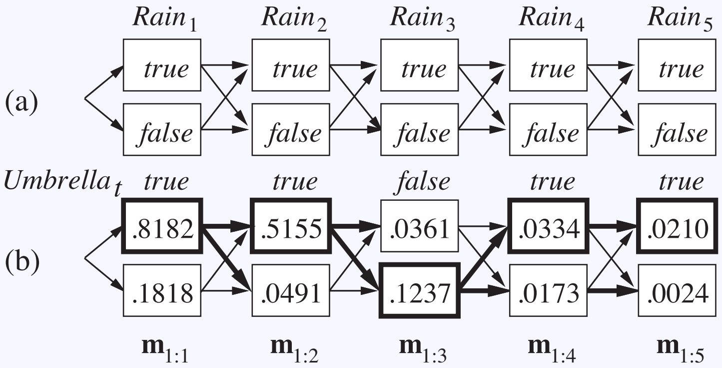
The forward-backward algorithm for smoothing

function FORWARD-BACKWARD(**ev**, $prior$) **returns** a vector of probability distributions
inputs: **ev**, a vector of evidence values for steps $1, \ldots, t$
$prior$, the prior distribution on the initial state, $\mathbf{P}(\mathbf{X}_0)$
local variables: **fv**, a vector of forward messages for steps $0, \ldots, t$
b, a representation of the backward message, initially all 1s
sv, a vector of smoothed estimates for steps $1, \ldots, t$

```
fv[0] ← prior
for i = 1 to t do
    fv[i] ← FORWARD(fv[i − 1], ev[i])
for i = t downto 1 do
    sv[i] ← NORMALIZE(fv[i] × b)
    b ← BACKWARD(b, ev[i])
return sv
```

Most Likely Explanation

Suppose that [true, true, false, true, true] is the umbrella sequence in the previous example. What is the weather sequence most likely to explain this?



Possible state sequences for $Rain_t$ can be viewed as paths through a graph of the possible states at each time step.

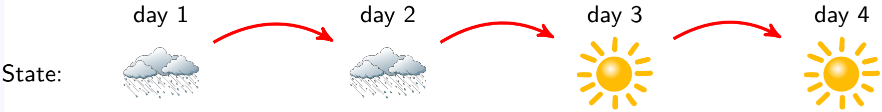
Most Likely Explanation: The Veterbi Algorithm

$$\max_{x_1 \ldots x_t} P(x_1, \ldots, x_t, X_{t+1} \mid e_{1:t+1}) =$$

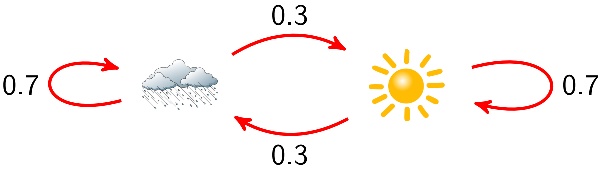
$$\frac{1}{z} P(e_{t+1} \mid X_{t+1}) \max_{x_t} \Big(P(X_{t+1} \mid x_t) max_{x_1 \ldots x_{t-1}} P(x_1, \ldots, x_{t-1}, x_t \mid e_{1:t}) \Big)$$

Markov Chain

A Markov Chain is a temporal model where the state is a single random variable that is always known.

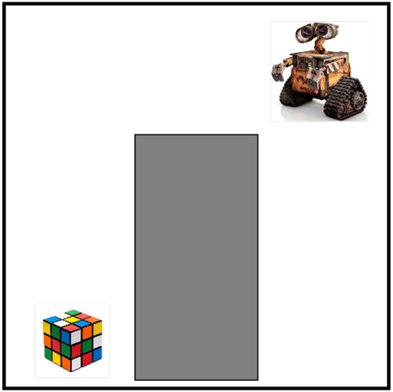


Transition Function



Weather remains the same with probability 0.7, and changes with probability 0.3.

Example of a Markov Chain: Robot searching for an object



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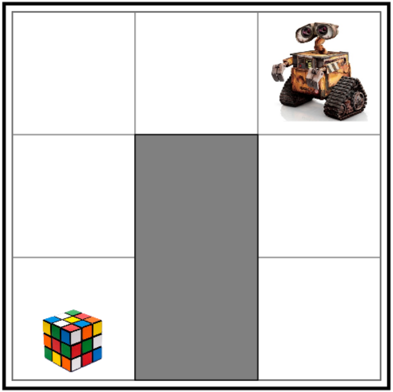
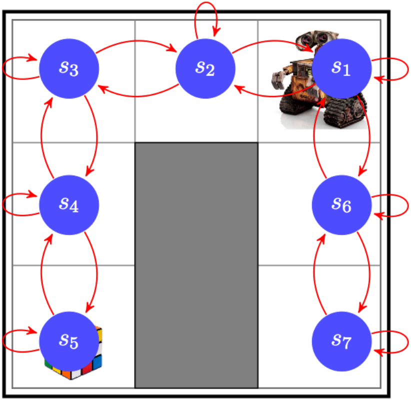


Example of a Markov Chain: Robot searching for an object

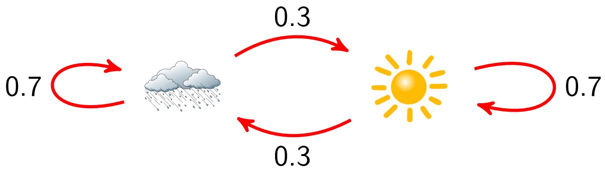


Transition Matrix

Let $\{s^i\}$ be the states of a Markov chain.
The transition function can be represented as a matrix T, where

$$T[i,j] = P(s_{t+1} = s^j \mid s_t = s^i).$$

Example



$$T = \begin{bmatrix} P(rainy \mid rainy) & P(sunny \mid rainy) \\ P(rainy \mid sunny) & P(sunny \mid sunny) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Transition Matrix

The transition matrix can be used to easily compute the state distributions in the future.
Let f_t denote the state distribution at time t, i.e. $f_t[i] = P(s_t = s^i)$, then

$$\begin{aligned} f_t &= f_0 \underbrace{T \times T \times T \times \cdots \times T}_{t \text{ times}} \\ &= f_{t-1} T. \end{aligned}$$

Example

$$f_0 = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}, T = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

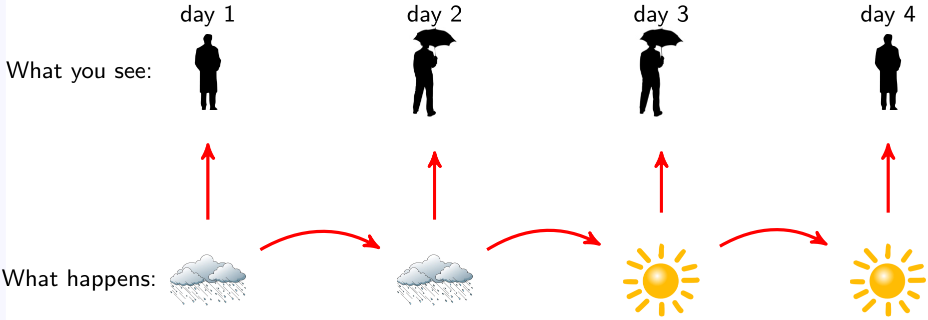
$$f_1 = f_0 T = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \times \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.62 & 0.38 \end{bmatrix},$$

$$f_2 = f_0 T T = f_1 T = \begin{bmatrix} 0.62 & 0.38 \end{bmatrix} \times \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.548 & 0.452 \end{bmatrix}.$$

Hidden Markov Model (HMM)

A Hidden Markov Model (HMM) is a temporal model where the state is a single random variable that is unknown (hidden). An observable variable is used as evidence to infer the state.

Example

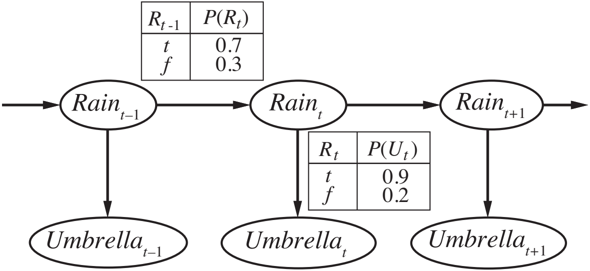


In addition to the transition matrix, we define an observation matrix O_e for each possible value e of the evidence variable E.
Observation matrix O_e has zeros everywhere except on the diagonal, where $O_e[i, i] = P(E_t = e \mid s_t = s^i)$.

$$O_e = \begin{bmatrix} P(e \mid s^0) & 0 & 0 & \dots & 0 \\ 0 & P(e \mid s^1) & 0 & \dots & 0 \\ 0 & 0 & P(e \mid s^2) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & P(e \mid s^n) \end{bmatrix}$$

Hidden Markov Model (HMM)

Example



$$O_{\text{umbrella } true} = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$O_{\text{umbrella } false} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.8 \end{bmatrix}.$$

What is the advantage of using vector and matrix notations?

- Vector and matrix notations make the calculations simple and elegant.
- Forward: Let $f_t[i] = P(s_t = s^i)$, then

$$
\begin{aligned}
f_{t+1} &= \alpha f_t T O_{e_{t+1}}, \\
\alpha &= \left(f_t T O_{e_{t+1}} \mathbf{1}\right)^{-1} \\
\mathbf{1} &= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.
\end{aligned}
$$

Hidden Markov Model (HMM)

What is the advantage of using vector and matrix notations?

- Vector and matrix notations make the calculations simple and elegant.
- Backward: Let $b_t[i] = P(e_{t+1:k} \mid s_t = s^i)$, then

$$b_t = TO_{e_{t+1}} b_{t+1}.$$

where we start from the last time-step k with $b_k = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.