CS440 Assignment 3: Probabilistic Reasoning Solutions

1 Problem 1

(a)

$$\begin{aligned} Pr(A, B, C, D, E) &= Pr(A) \times Pr(B) \times Pr(C) \times Pr(D|A, B) \times Pr(E|B, C) \\ &= 0.2 \times 0.5 \times 0.8 \times 0.1 \times 0.3 \\ &= 0.0024. \end{aligned}$$

(b)

$$\begin{aligned} Pr(\neg A, \neg B, \neg C, \neg D, \neg E) &= Pr(\neg A) \times Pr(\neg B) \times Pr(\neg C) \times Pr(\neg D|\neg A, \neg B) \times Pr(\neg E|\neg B, \neg C) \\ &= 0.8 \times 0.5 \times 0.2 \times 0.1 \times 0.8 \\ &= 0.0064. \end{aligned}$$

(c)

$$\begin{aligned} Pr(\neg A|B, C, D, E) &= \frac{Pr(\neg A, B, C, D, E)}{Pr(A, B, C, D, E) + Pr(\neg A, B, C, D, E)} \\ &= \frac{Pr(\neg A) \times Pr(B) \times Pr(C) \times Pr(D|\neg A, B) \times Pr(E|B, C)}{0.0024 + Pr(\neg A) \times Pr(B) \times Pr(C) \times Pr(D|\neg A, B) \times Pr(E|B, C)} \\ &= \frac{0.8 \times 0.5 \times 0.8 \times 0.6 \times 0.3}{0.0024 + 0.8 \times 0.5 \times 0.8 \times 0.6 \times 0.3} \\ &= \frac{0.0576}{0.0024 + 0.0576} \\ &= 0.96. \end{aligned}$$

2 Problem 2

a)
$$
\begin{aligned}
Pr(B|JC=t, MC=t) &= \alpha Pr(B, JC=t, MC=t) \\
&= \alpha \sum_E \sum_A Pr(B, E, A, JC=t, MC=t) \\
&= \alpha \sum_E \sum_A Pr(B) Pr(E) Pr(A|B,E) Pr(JC=t|A) Pr(MC=t|A) \\
&= \alpha Pr(B) \sum_E Pr(E) \sum_A Pr(A|B,E) Pr(JC=t|A) Pr(MC=t|A) \\
&= \alpha \times 0.001 \times [0.002 \times \{0.95 \times 0.9 \times 0.7 + 0.05 \times 0.05 \times 0.01\} + \\
&0.998 \times \{0.94 \times 0.9 \times 0.7 + 0.06 \times 0.05 \times 0.01\}] \\
&= \alpha \times 0.001 \times [0.002 \times 0.5985 + 0.998 \times 0.5922] \\
&= \alpha \times 0.001 \times 0.5922 \\
&= 0.0005922\alpha
\end{aligned}
$$

$$
\begin{aligned}
Pr(\neg B|JC=t, MC=t) &= \alpha Pr(\neg B, JC=t, MC=t) \\
&= \alpha \sum_E \sum_A Pr(\neg B, E, A, JC=t, MC=t) \\
&= \alpha \sum_E \sum_A Pr(\neg B) Pr(E) Pr(A|B,E) Pr(JC=t|A) Pr(MC=t|A) \\
&= \alpha Pr(\neg B) \sum_E Pr(E) \sum_A Pr(A|\neg B,E) Pr(JC=t|A) Pr(MC=t|A) \\
&= \alpha \times 0.999 \times [0.002 \times \{0.29 \times 0.9 \times 0.7 + 0.71 \times 0.05 \times 0.01\} + \\
&0.998 \times \{0.001 \times 0.9 \times 0.7 + 0.999 \times 0.05 \times 0.01\}] \\
&= \alpha \times 0.999 \times [0.002 \times 0.1831 + 0.998 \times 0.0011] \\
&= \alpha \times 0.999 \times 0.0015 \\
&= 0.0014985\alpha
\end{aligned}
$$

Thus, $P(B|JC=t, MC=t) = 478.3087 \times 0.0005922 = 0.2833.$

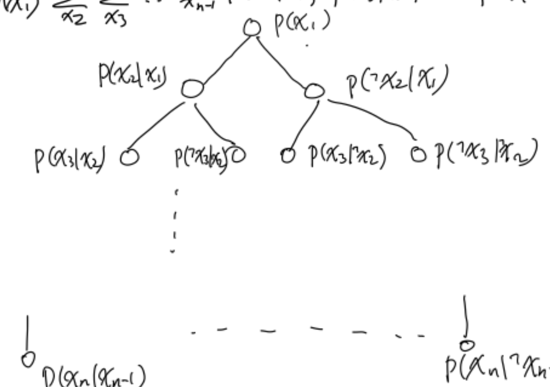
$$\alpha = \frac{1}{0.0005922 + 0.0014985} = 478.3087$$

b) ① enumeration

$$P(x_1 | x_n = true) = \alpha \sum_{x_2} \sum_{x_3} \cdots \sum_{x_{n-1}} P(x_1 | x_2, x_3, \cdots x_{n-1}, x_n = true)$$

$$= \alpha \sum_{x_2} \sum_{x_3} \cdots \sum_{x_{n-1}} P(x_1) \cdot P(x_2 | x_1) \cdot P(x_3 | x_2) \cdots P(x_n = true | x_{n-1})$$

$$= \alpha P(x_1) \sum_{x_2} \sum_{x_3} \cdots \sum_{x_{n-1}} P(x_2 | x_1) \cdot P(x_3 | x_2) \cdots P(x_n = true | x_{n-1})$$



DFS can be used to parse the tree, and the time complexity is $O(2^n)$

② Variable elimination

$$P(x_1 | x_n = true) = \alpha P(x_1) \sum_{x_2} \sum_{x_3} \cdots \sum_{x_{n-1}} \underbrace{P(x_2 | x_1)}_{\text{compute twice}} \cdots \underbrace{P(x_{n-1} | x_{n-2})}_{\text{comput } 2^{n-2} \text{ times}} \cdot P(x_n = true | x_{n-1})$$

Initial factors: $P(x_1)$ $P(x_2|x_1)$ ... $P(x_n|x_{n-1})$

Known values are selected $P(x_1)$ $P(x_2|x_1)$ ... $P(x_n = true | x_{n-1})$

$P(x_1) \times P(x_2|x_1) \to P(x_1, x_2) = P(x_1) \cdot P(x_2|x_1)$ $\forall x_2, x_1$ new factor

new factors are: $P(x_1, x_2)$, $P(x_3|x_2)$ ... $P(x_n|x_{n-1})$

$P(x_1, x_2) \times P(x_3|x_2) \to P(x_1, x_2, x_3) = P(x_1, x_2) \cdot P(x_3|x_2)$ $\forall x_1, x_2, x_3$ new factor

new factors are $P(x_1, x_2, x_3)$, $P(x_4|x_3)$ ... $P(x_n|x_{n-1})$

⋮

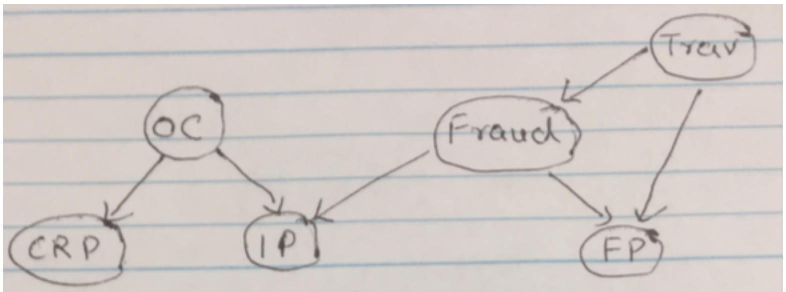
Finally we have new factor: $P(x_1, \ldots x_n)$ through n-1 steps

$P(x_1 | x_n = true)$ comes from $P(x_1, x_n = true)$ which is summing out variable $x_2 \cdots x_{n-1}$ from $P(x_1, \ldots x_n)$

the time complexity is $O(n)$

Problem 3

a)



P(Trav) = 0.05
P(OC) = 0.75

Trav	P(Fraud)
t	0.01
f	0.004

Fraud	Travel	P(FP)
t	t	0.9
t	f	0.1
f	t	0.9
f	f	0.01

Fraud	OC	P(IP)
t	t	0.02
t	f	0.011
f	t	0.01
f	f	0.001

OC	P(CRP)
t	0.1
f	0.001

3 Problem 3

b)

$$
\begin{aligned}
P(Fraud) &= P(Fraud|Trav)P(Trav) + P(Fraud|\neg Trav)P(\neg Trav) \\
&= 0.01 \times 0.05 + 0.004 \times 0.95 \\
&= 0.0005 + 0.0038 \\
&= 0.0043.
\end{aligned}
$$

$$
\begin{aligned}
P(Fraud|FP, \neg IP, CRP) &= \alpha P(Fraud, FP, \neg IP, CRP) \\
&= \alpha \sum_{Trav} \sum_{OC} P(Fraud, FP, \neg IP, CRP, Trav, OC) \\
&= \alpha \sum_{Trav} P(Trav)P(Fraud|Trav)P(FP|Trav, Fraud) \times \\
&(\sum_{OC} P(OC)P(CRP|OC)P(\neg IP|Fraud, OC)) \\
&= \alpha \times [0.05 \times 0.01 \times 0.9 \times \{0.75 \times 0.1 \times 0.98 + 0.25 \times 0.001 \times 0.989\} + \\
&0.95 \times 0.004 \times 0.1\{0.75 \times 0.1 \times 0.98 + 0.25 \times 0.001 \times 0.989\}] \\
&= \alpha \times [0.00045 \times (0.0735 + 0.00024725) + 0.00038 \times (0.0735 + 0.00024725)] \\
&= \alpha \times [0.00003318626 + 0.000028029413] \\
&= 0.00006121
\end{aligned}
$$

$$
\begin{aligned}
P(\neg Fraud|FP, \neg IP, CRP) &= \alpha P(\neg Fraud, FP, \neg IP, CRP) \\
&= \alpha \sum_{Trav} \sum_{OC} P(\neg Fraud, FP, \neg IP, CRP, Trav, OC) \\
&= \alpha \sum_{Trav} P(Trav)P(\neg Fraud|Trav)P(FP|Trav, \neg Fraud) \times \\
&(\sum_{OC} P(OC)P(CRP|OC)P(\neg IP|\neg Fraud, OC)) \\
&= \alpha \times [0.05 \times 0.99 \times 0.9 \times \{0.75 \times 0.1 \times 0.99 + 0.25 \times 0.001 \times 0.999\} + \\
&0.95 \times 0.996 \times 0.01\{0.75 \times 0.1 \times 0.99 + 0.25 \times 0.001 \times 0.999\}] \\
&= \alpha \times [0.04455 \times (0.07425 + 0.00024975) + 0.009462 \times (0.07425 + 0.00024975)] \\
&= \alpha \times 0.0745 \times 0.054012 \\
&= 0.00402392
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \frac{1}{0.00006121 + 0.0040239} \\
&= 244.8
\end{aligned}
$$

Thus, $P(Fraud|FP, \neg IP, CRP) = 244.8 \times 0.00006121 = 1.498\%$

Problem 4

1. Filtering: Three days have passed since the rover fell into the ravine. The observations were $(E1 = hot, E2 = cold, E3 = cold)$. What is $P(X3|hot1, cold2, cold3)$, the probability distribution over the rover's position on day 3, given the observations? (This is a probability distribution over the six possible positions).

In this problem, we will represent the probability distribution of State X as:

$$P(X) = \begin{bmatrix} P(X = A) \\ P(X = B) \\ P(X = C) \\ P(X = D) \\ P(X = E) \\ P(X = F) \end{bmatrix}$$

We assume all multiplication between vectors are element wise multiplication
We will be using the equation for filtering:

$$P(X_t|e_{1:t}) = \alpha P(e_t|X_t) \sum_{x_{t-1}} P(X_t|x_{t-1}) P(x_{t-1}|e_{1:t-1}) \tag{1}$$

Since $X_1 = A$ is the given information to us, so the probability distribution of day 1 is:

$$P(X_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

For day 2, we want to find $P(X_2|e_{1:2})$, apply equation (1), we can get:

$$P(X_2|e_{1:2}) = \alpha P(e_2|X_2) \sum_{x_1} P(X_2|x_1) P(x_1|e_1) \tag{3}$$

when we sum over x_1, from (2), we know that when $x_1 = A$, $P(x_1|e_1) = 1$, and when $x_1 = B, C, D, E, F$, $P(x_1|e_1) = 0$. So we only consider the situation where $x_1 = A$. When $x_1 = A$:

$$P(X_2|x_1 = A) = \begin{bmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

Therefore,

$$\sum_{x_1} P(X_2|x_1)P(x_1|e_1) = \begin{bmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot 1 + 0 + 0 + 0 + 0 + 0 = \begin{bmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

Since we assume the sensor never fails, and $e_2 = cold$ so:

$$\begin{aligned} P(e_2 = cold|X_2 = A) = 0 \\ P(e_2 = cold|X_2 = B) = 1 \\ P(e_2 = cold|X_2 = C) = 1 \\ P(e_2 = cold|X_2 = D) = 0 \\ P(e_2 = cold|X_2 = E) = 1 \\ P(e_2 = cold|X_2 = F) = 1 \end{aligned}$$

Therefore,

$$P(e_2 = cold|X_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{6}$$

Based on (3), (5) and (6), we can get:

$$P(X_2|e_{1:2}) = \alpha \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

For day 3, we want to find $P(X_3|e_{1:3})$, apply equation (1), we can get:

$$P(X_3|e_{1:3}) = \alpha P(e_3|X_3) \sum_{x_2} P(X_3|x_2)P(x_2|e_{1:2}) \tag{8}$$

when we sum over x_2, from (8), we know that when $x_2 = B$, $P(x_2|e_{1:2}) = 1$, and when $x_2 = A, C, D, E, F$, $P(x_2|e_{1:2}) = 0$. So we only consider the situation where $x_2 = B$. When $x_2 = B$:

$$P(X_3|x_2 = B) = \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

Therefore,

$$\sum_{x_2} P(X_3|x_2)P(x_2|e_{1:2}) = 0 + \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot 1 + 0 + 0 + 0 + 0 = \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

Since we assume the sensor never fails, and $e_3 = cold$ so:

$$P(e_3 = cold|X_3 = A) = 0$$
$$P(e_3 = cold|X_3 = B) = 1$$
$$P(e_3 = cold|X_3 = C) = 1$$
$$P(e_3 = cold|X_3 = D) = 0$$
$$P(e_3 = cold|X_3 = E) = 1$$
$$P(e_3 = cold|X_3 = F) = 1$$

Therefore,

$$P(e_3 = cold|X_3) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{11}$$

Based on (8), (10) and (11), we can get:

$$P(X_3|e_{1:3}) = \alpha \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

So the probability distribution on day 3 is $\begin{bmatrix} P(X = A) = 0 \\ P(X = B) = 0.2 \\ P(X = C) = 0.8 \\ P(X = D) = 0 \\ P(X = E) = 0 \\ P(X = F) = 0 \end{bmatrix}$

2. Smoothing: What is $P(X_2|hot_1, cold_2, cold_3)$, the probability distribution over the rover's position on day 3, given the observations? (This is a probability distribution over the six possible positions).

In this problem, we will represent the probability distribution of of state X as:

$$P(X) = \begin{bmatrix} P(X = A) \\ P(X = B) \\ P(X = C) \\ P(X = D) \\ P(X = E) \\ P(X = F) \end{bmatrix}$$

Here, all multiplications between vectors are element wise multiplications.
We will be using the equation for filtering:

$$P(X_t|e_{1:t}) = \alpha P(e_t|X_t) \sum_{x_{t-1}} P(X_t|x_{t-1})P(x_{t-1}|e_{1:t-1}) \tag{13}$$

the equation for prediction:

$$P(E_{t+1:T}|x_t) = \sum_{x_{t+1}} P(x_{t+1}|x_t)P(E_{t+1}|x_{t+1})P(E_{t+2:T}|x_{t+1}) \tag{14}$$

And the equation for Smoothing:

$$P(X_k|e_{1:t}) = \alpha P(e_{k+1:t}|X_k)P(X_k|e_{1:k}) \tag{15}$$

We want to compute:

$$P(X_2|e_{1:3}) = \alpha P(e_{3:3}|X_2)P(X_2|e_{1:2}) \tag{16}$$

where $e_{1:3} = \{hot, cold, cold\}$
To compute $P(e_{3:3}|X_2)$ over all possible x_3, we apply (14)

$$P(E_{3:3} = cold|X_2) = \sum_{x_3} P(x_3|X_2)P(E_3 = cold|x_3)P(E_{4:3}|x_3) \tag{17}$$

since $t = T - 1 = 3 - 1 = 2$, so we set $P(E_{4:3}|x_3) = 1$
since we know that $e_3 = cold$, so $P(E_3 = cold|x_3) = 1$ when $x_3 = B, C, E, F$,

and $P(E_3 = cold|x_3) = 0$ when $x_3 = A, D$

$$P(E_{3:3} = cold|x_2) = \sum_{x_3} P(x_3|x_2)P(E_3 = cold|x_3) \cdot 1 =$$

$$\begin{bmatrix} P(x_3 = B|x_2 = A) + P(x_3 = C|x_2 = A) + P(x_3 = E|x_2 = A) + P(x_3 = F|x_2 = A) \\ P(x_3 = B|x_2 = B) + P(x_3 = C|x_2 = B) + P(x_3 = E|x_2 = B) + P(x_3 = F|x_2 = B) \\ P(x_3 = B|x_2 = C) + P(x_3 = C|x_2 = C) + P(x_3 = E|x_2 = C) + P(x_3 = F|x_2 = C) \\ P(x_3 = B|x_2 = D) + P(x_3 = C|x_2 = D) + P(x_3 = E|x_2 = D) + P(x_3 = F|x_2 = D) \\ P(x_3 = B|x_2 = E) + P(x_3 = C|x_2 = E) + P(x_3 = E|x_2 = E) + P(x_3 = F|x_2 = E) \\ P(x_3 = B|x_2 = F) + P(x_3 = C|x_2 = F) + P(x_3 = E|x_2 = F) + P(x_3 = F|x_2 = F) \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 + 0 + 0 + 0 \\ 0.2 + 0.8 + 0 + 0 \\ 0 + 0.2 + 0 + 0 \\ 0 + 0 + 0.8 + 0 \\ 0 + 0 + 0.2 + 0.8 \\ 0 + 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} P(E_{3:3} = cold|x_2 = A) = 0.8 \\ P(E_{3:3} = cold|x_2 = B) = 1 \\ P(E_{3:3} = cold|x_2 = C) = 0.2 \\ P(E_{3:3} = cold|x_2 = D) = 0.8 \\ P(E_{3:3} = cold|x_2 = E) = 1 \\ P(E_{3:3} = cold|x_2 = F) = 1 \\ \end{bmatrix}$$

So we get:

$$P(E_{3:3} = cold|X_2) = \begin{bmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{bmatrix} \tag{18}$$

From (7), we know that

$$P(X_2|e_{1:2}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

Based on (16), (18), and (19), we can get:

$$P(X_2|e_{1:3}) = \alpha P(e_{3:3}|X_2)P(X_2|e_{1:2}) \tag{20}$$

$$P(X_2|e_{1:3}) = \alpha \cdot \begin{bmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

3. Most Likely Explanation: What is the most likely sequence of the rover's positions in the three days given the observations $(E1 = hot, E2 = cold, E3 = cold)$**?**

Since it is the given information to us that the rover's position in day1 is A, so:

$$P(x_1 = A|e_1 = hot) = 1$$
$$P(x_1 = B|e_1 = hot) = 0$$
$$P(x_1 = C|e_1 = hot) = 0$$
$$P(x_1 = D|e_1 = hot) = 0$$
$$P(x_1 = E|e_1 = hot) = 0$$
$$P(x_1 = F|e_1 = hot) = 0$$

For $x_2 = A$:

$P(x_1 = A|e_1 = hot) \cdot P(x_2 = A|x_1 = A) \cdot P(e_2 = cold|x_2 = A) = 1 * 0.2 * 0 = 0$
$P(x_1 = B|e_1 = hot) \cdot P(x_2 = A|x_1 = B) \cdot P(e_2 = cold|x_2 = A) = 0$
$P(x_1 = C|e_1 = hot) \cdot P(x_2 = A|x_1 = C) \cdot P(e_2 = cold|x_2 = A) = 0$
$P(x_1 = D|e_1 = hot) \cdot P(x_2 = A|x_1 = D) \cdot P(e_2 = cold|x_2 = A) = 0$
$P(x_1 = E|e_1 = hot) \cdot P(x_2 = A|x_1 = E) \cdot P(e_2 = cold|x_2 = A) = 0$
$P(x_1 = F|e_1 = hot) \cdot P(x_2 = A|x_1 = F) \cdot P(e_2 = cold|x_2 = A) = 0$
$Max = 0$

This result is from $x_1 = A$

For $x_2 = B$:

$P(x_1 = A|e_1 = hot) \cdot P(x_2 = B|x_1 = A) \cdot P(e_2 = cold|x_2 = B) = 1 * 0.8 * 1 = 0.8$
$P(x_1 = B|e_1 = hot) \cdot P(x_2 = B|x_1 = B) \cdot P(e_2 = cold|x_2 = B) = 0$
$P(x_1 = C|e_1 = hot) \cdot P(x_2 = B|x_1 = C) \cdot P(e_2 = cold|x_2 = B) = 0$
$P(x_1 = D|e_1 = hot) \cdot P(x_2 = B|x_1 = D) \cdot P(e_2 = cold|x_2 = B) = 0$
$P(x_1 = E|e_1 = hot) \cdot P(x_2 = B|x_1 = E) \cdot P(e_2 = cold|x_2 = B) = 0$
$P(x_1 = F|e_1 = hot) \cdot P(x_2 = B|x_1 = F) \cdot P(e_2 = cold|x_2 = B) = 0$
$Max = 0.8$

This result is from $x_1 = A$

For $x_2 = C$:

$$P(x_1 = A|e_1 = hot) \cdot P(x_2 = C|x_1 = A) \cdot P(e_2 = cold|x_2 = C) = 1 * 0 * 1 = 0$$
$$P(x_1 = B|e_1 = hot) \cdot P(x_2 = C|x_1 = B) \cdot P(e_2 = cold|x_2 = C) = 0$$
$$P(x_1 = C|e_1 = hot) \cdot P(x_2 = C|x_1 = C) \cdot P(e_2 = cold|x_2 = C) = 0$$
$$P(x_1 = D|e_1 = hot) \cdot P(x_2 = C|x_1 = D) \cdot P(e_2 = cold|x_2 = C) = 0$$
$$P(x_1 = E|e_1 = hot) \cdot P(x_2 = C|x_1 = E) \cdot P(e_2 = cold|x_2 = C) = 0$$
$$P(x_1 = F|e_1 = hot) \cdot P(x_2 = C|x_1 = F) \cdot P(e_2 = cold|x_2 = C) = 0$$
$$Max = 0$$

This result is from $x_1 = A$

For $x_2 = D$:

$$P(x_1 = A|e_1 = hot) \cdot P(x_2 = D|x_1 = A) \cdot P(e_2 = cold|x_2 = D) = 1 * 0 * 0 = 0$$
$$P(x_1 = B|e_1 = hot) \cdot P(x_2 = D|x_1 = B) \cdot P(e_2 = cold|x_2 = D) = 0$$
$$P(x_1 = C|e_1 = hot) \cdot P(x_2 = D|x_1 = C) \cdot P(e_2 = cold|x_2 = D) = 0$$
$$P(x_1 = D|e_1 = hot) \cdot P(x_2 = D|x_1 = D) \cdot P(e_2 = cold|x_2 = D) = 0$$
$$P(x_1 = E|e_1 = hot) \cdot P(x_2 = D|x_1 = E) \cdot P(e_2 = cold|x_2 = D) = 0$$
$$P(x_1 = F|e_1 = hot) \cdot P(x_2 = D|x_1 = F) \cdot P(e_2 = cold|x_2 = D) = 0$$
$$Max = 0$$

This result is from $x_1 = A$

For $x_2 = E$:

$$P(x_1 = A|e_1 = hot) \cdot P(x_2 = E|x_1 = A) \cdot P(e_2 = cold|x_2 = E) = 1 * 0 * 1 = 0$$
$$P(x_1 = B|e_1 = hot) \cdot P(x_2 = E|x_1 = B) \cdot P(e_2 = cold|x_2 = E) = 0$$
$$P(x_1 = C|e_1 = hot) \cdot P(x_2 = E|x_1 = C) \cdot P(e_2 = cold|x_2 = E) = 0$$
$$P(x_1 = D|e_1 = hot) \cdot P(x_2 = E|x_1 = D) \cdot P(e_2 = cold|x_2 = E) = 0$$
$$P(x_1 = E|e_1 = hot) \cdot P(x_2 = E|x_1 = E) \cdot P(e_2 = cold|x_2 = E) = 0$$
$$P(x_1 = F|e_1 = hot) \cdot P(x_2 = E|x_1 = F) \cdot P(e_2 = cold|x_2 = E) = 0$$
$$Max = 0$$

This result is from $x_1 = A$

For $x_2 = E$:

$$
\begin{aligned}
&P(x_1 = A|e_1 = hot) \cdot P(x_2 = F|x_1 = A) \cdot P(e_2 = cold|x_2 = F) = 1 * 0 * 1 = 0 \\
&P(x_1 = B|e_1 = hot) \cdot P(x_2 = F|x_1 = B) \cdot P(e_2 = cold|x_2 = F) = 0 \\
&P(x_1 = C|e_1 = hot) \cdot P(x_2 = F|x_1 = C) \cdot P(e_2 = cold|x_2 = F) = 0 \\
&P(x_1 = D|e_1 = hot) \cdot P(x_2 = F|x_1 = D) \cdot P(e_2 = cold|x_2 = F) = 0 \\
&P(x_1 = E|e_1 = hot) \cdot P(x_2 = F|x_1 = E) \cdot P(e_2 = cold|x_2 = F) = 0 \\
&P(x_1 = F|e_1 = hot) \cdot P(x_2 = F|x_1 = F) \cdot P(e_2 = cold|x_2 = F) = 0 \\
&Max = 0
\end{aligned}
$$

This result is from $x_1 = A$

Normalize all the Max terms, we have $P(x_2 = B) = 1$ and $P(x_2 = k) = 0$ for $k = A, C, D, E, F$.

For $x_3 = A$:

$$
\begin{aligned}
&P(x_2 = A) \cdot P(x_3 = A|x_2 = A) \cdot P(e_3 = cold|x_3 = A) = 0 \\
&P(x_2 = B) \cdot P(x_3 = A|x_2 = B) \cdot P(e_3 = cold|x_3 = A) = 0.5 * 0 * 0 = 0 \\
&P(x_2 = C) \cdot P(x_3 = A|x_2 = C) \cdot P(e_3 = cold|x_3 = A) = 0 \\
&P(x_2 = D) \cdot P(x_3 = A|x_2 = D) \cdot P(e_3 = cold|x_3 = A) = 0 \\
&P(x_2 = E) \cdot P(x_3 = A|x_2 = E) \cdot P(e_3 = cold|x_3 = A) = 0 \\
&P(x_2 = F) \cdot P(x_3 = A|x_2 = F) \cdot P(e_3 = cold|x_3 = A) = 0 \\
&Max = 0
\end{aligned}
$$

This result is from $x_2 = B$

For $x_3 = B$:

$$
\begin{aligned}
&P(x_2 = A) \cdot P(x_3 = B|x_2 = A) \cdot P(e_3 = cold|x_3 = B) = 0 \\
&P(x_2 = B) \cdot P(x_3 = B|x_2 = B) \cdot P(e_3 = cold|x_3 = B) = 0.8 * 0.2 * 1 = 0.16 \\
&P(x_2 = C) \cdot P(x_3 = B|x_2 = C) \cdot P(e_3 = cold|x_3 = B) = 0 \\
&P(x_2 = D) \cdot P(x_3 = B|x_2 = D) \cdot P(e_3 = cold|x_3 = B) = 0 \\
&P(x_2 = E) \cdot P(x_3 = B|x_2 = E) \cdot P(e_3 = cold|x_3 = B) = 0 \\
&P(x_2 = F) \cdot P(x_3 = B|x_2 = F) \cdot P(e_3 = cold|x_3 = B) = 0 \\
&Max = 0.16
\end{aligned}
$$

This result is from $x_2 = B$

For $x_3 = C$:

$$
\begin{aligned}
&P(x_2 = A) \cdot P(x_3 = C|x_2 = A) \cdot P(e_3 = cold|x_3 = C) = 0\\
&P(x_2 = B) \cdot P(x_3 = C|x_2 = B) \cdot P(e_3 = cold|x_3 = C) = 0.8 * 0.8 * 1 = 0.64\\
&P(x_2 = C) \cdot P(x_3 = C|x_2 = C) \cdot P(e_3 = cold|x_3 = C) = 0\\
&P(x_2 = D) \cdot P(x_3 = C|x_2 = D) \cdot P(e_3 = cold|x_3 = C) = 0\\
&P(x_2 = E) \cdot P(x_3 = C|x_2 = E) \cdot P(e_3 = cold|x_3 = C) = 0\\
&P(x_2 = F) \cdot P(x_3 = C|x_2 = F) \cdot P(e_3 = cold|x_3 = C) = 0\\
&Max = 0.64
\end{aligned}
$$

This result is from $x_2 = B$

For $x_3 = D$:

$$
\begin{aligned}
&P(x_2 = A) \cdot P(x_3 = D|x_2 = A) \cdot P(e_3 = cold|x_3 = D) = 0\\
&P(x_2 = B) \cdot P(x_3 = D|x_2 = B) \cdot P(e_3 = cold|x_3 = D) = 0.8 * 0 * 0 = 0\\
&P(x_2 = C) \cdot P(x_3 = D|x_2 = C) \cdot P(e_3 = cold|x_3 = D) = 0\\
&P(x_2 = D) \cdot P(x_3 = D|x_2 = D) \cdot P(e_3 = cold|x_3 = D) = 0\\
&P(x_2 = E) \cdot P(x_3 = D|x_2 = E) \cdot P(e_3 = cold|x_3 = D) = 0\\
&P(x_2 = F) \cdot P(x_3 = D|x_2 = F) \cdot P(e_3 = cold|x_3 = D) = 0\\
&Max = 0
\end{aligned}
$$

This result is from $x_2 = B$

For $x_3 = E$:

$$
\begin{aligned}
&P(x_2 = A) \cdot P(x_3 = E|x_2 = A) \cdot P(e_3 = cold|x_3 = E) = 0\\
&P(x_2 = B) \cdot P(x_3 = E|x_2 = B) \cdot P(e_3 = cold|x_3 = E) = 0.8 * 0 * 1 = 0\\
&P(x_2 = C) \cdot P(x_3 = E|x_2 = C) \cdot P(e_3 = cold|x_3 = E) = 0\\
&P(x_2 = D) \cdot P(x_3 = E|x_2 = D) \cdot P(e_3 = cold|x_3 = E) = 0\\
&P(x_2 = E) \cdot P(x_3 = E|x_2 = E) \cdot P(e_3 = cold|x_3 = E) = 0\\
&P(x_2 = F) \cdot P(x_3 = E|x_2 = F) \cdot P(e_3 = cold|x_3 = E) = 0\\
&Max = 0
\end{aligned}
$$

This result is from $x_2 = B$

For $x_3 = F$:

$$
\begin{aligned}
&P(x_2 = A) \cdot P(x_3 = F|x_2 = A) \cdot P(e_3 = cold|x_3 = F) = 0\\
&P(x_2 = B) \cdot P(x_3 = F|x_2 = B) \cdot P(e_3 = cold|x_3 = F) = 0.8 * 0 * 1 = 0\\
&P(x_2 = C) \cdot P(x_3 = F|x_2 = C) \cdot P(e_3 = cold|x_3 = F) = 0\\
&P(x_2 = D) \cdot P(x_3 = F|x_2 = D) \cdot P(e_3 = cold|x_3 = F) = 0\\
&P(x_2 = E) \cdot P(x_3 = F|x_2 = E) \cdot P(e_3 = cold|x_3 = F) = 0\\
&P(x_2 = F) \cdot P(x_3 = F|x_2 = F) \cdot P(e_3 = cold|x_3 = F) = 0\\
&Max = 0
\end{aligned}
$$

This result is from $x_2 = B$

Normalize all the Max terms, we have $P(x_3 = B) = 0.2$, $P(x_3 = C) = 0.8$ and $P(x_3 = k) = 0$ for $k = A, D, E, F$.

From the above computations, we can generate the following graph as figure 3 shows

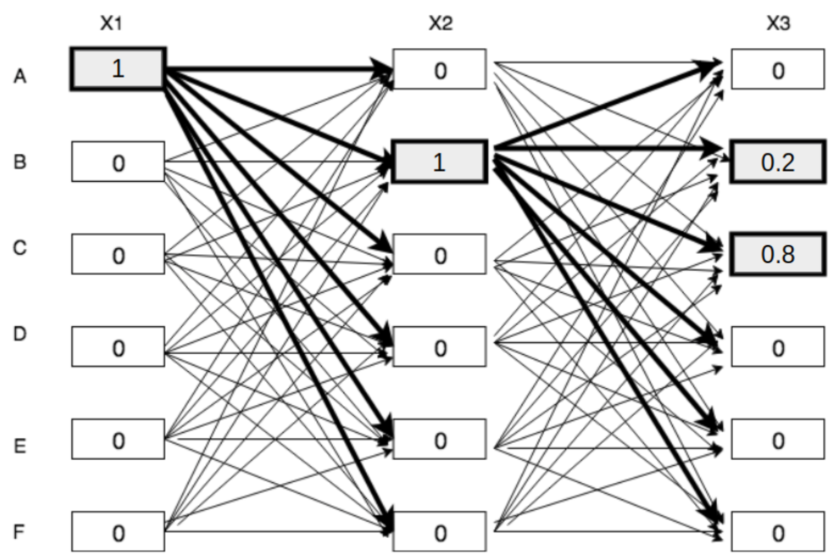


Figure 1: graph of the possible states at each time step

In conclusion,

$$x_1 = A, x_2 = B, x_3 = C$$

is the most likely sequence.

4. Prediction: What is $P(hot4, hot5, cold6|hot1, cold2, cold3)$, the probability of observing hot4 and hot5 and cold6 in days 4,5,6 respectively, given the previous observations in days 1,2, and 3? (This is a single value, not a distribution).

To solve this question, we are going to use the equation for Prediction:

$$P(E_{t+1:T}|e_{1:t}) = \sum_{x_t} P(x_t|e_{1:t})P(E_{t+1:T}|x_t) \tag{22}$$

from question 1 (equation 12), we can get that:

$$\begin{bmatrix} P(x_3 = A|e_{1:3}) = 0 \\ P(x_3 = B|e_{1:3}) = 0.2 \\ P(x_3 = C|e_{1:3}) = 0.8 \\ P(x_3 = D|e_{1:3}) = 0 \\ P(x_3 = E|e_{1:3}) = 0 \\ P(x_3 = F|e_{1:3}) = 0 \end{bmatrix}$$

$$P(E_{t+1:T}|x_t) = \sum_{x_{t+1}} P(x_{t+1}|x_t)P(E_{t+1}|x_{t+1})P(E_{t+2:T}|x_{t+1}) \tag{23}$$

we want to compute $P(E_{4:6}|X_3)$, we start with computing $P(E_{6:6} = cold|X_5)$ and we set $P(E_{7:6}|X_6) = 1$

$$P(E_{6:6} = cold|x_5) = \sum_{x_6} P(x_6|x_5)P(E_6 = cold|x_6)P(E_{7:6}|x_6)$$

$$= \begin{bmatrix} \sum_{x_6} P(x_6|x_5 = A)P(E_6 = cold|x_6)P(E_{7:6}|x_6) \\ \sum_{x_6} P(x_6|x_5 = B)P(E_6 = cold|x_6)P(E_{7:6}|x_6) \\ \sum_{x_6} P(x_6|x_5 = C)P(E_6 = cold|x_6)P(E_{7:6}|x_6) \\ \sum_{x_6} P(x_6|x_5 = D)P(E_6 = cold|x_6)P(E_{7:6}|x_6) \\ \sum_{x_6} P(x_6|x_5 = E)P(E_6 = cold|x_6)P(E_{7:6}|x_6) \\ \sum_{x_6} P(x_6|x_5 = F)P(E_6 = cold|x_6)P(E_{7:6}|x_6) \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 0.8 + 0 + 0 + 0 + 0 \\ 0 + 0.2 + 0.8 + 0 + 0 + 0 \\ 0 + 0 + 0.2 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0.8 + 0 \\ 0 + 0 + 0 + 0 + 0.2 + 0.8 \\ 0 + 0 + 0 + 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 \\ 1 \\ 0.2 \\ 0.8 \\ 1 \\ 1 \end{bmatrix}$$

Now we can compute for $t = 4$, $P(E_{5:6}|X_4)$

$$P(E_{5:6}|x_4) = \sum_{x_5} P(x_5|x_4)P(E_5 = hot|x_5)P(E_{6:6}|x_5)$$

$$= \begin{bmatrix} \sum_{x_5} P(x_5|x_4 = A)P(E_5 = hot|x_5)P(E_{6:6}|x_5) \\ \sum_{x_5} P(x_5|x_4 = B)P(E_5 = hot|x_5)P(E_{6:6}|x_5) \\ \sum_{x_5} P(x_5|x_4 = C)P(E_5 = hot|x_5)P(E_{6:6}|x_5) \\ \sum_{x_5} P(x_5|x_4 = D)P(E_5 = hot|x_5)P(E_{6:6}|x_5) \\ \sum_{x_5} P(x_5|x_4 = E)P(E_5 = hot|x_5)P(E_{6:6}|x_5) \\ \sum_{x_5} P(x_5|x_4 = F)P(E_5 = hot|x_5)P(E_{6:6}|x_5) \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 * 0.8 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0.8 * 0.8 + 0 + 0 \\ 0 + 0 + 0 + 0.2 * 0.8 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.16 \\ 0 \\ 0.64 \\ 0.16 \\ 0 \\ 0 \end{bmatrix}$$

Now we can compute t=3, $P(E_{4:6}|X_3)$

$$
\begin{aligned}
P(E_{4:6}|x_3) &= \sum_{x_4} P(x_4|x_3)P(E_4 = hot|x_4)P(E_{5:6}|x_4) \\
&= \begin{bmatrix} \sum_{x_4} P(x_4|x_3 = A)P(E_4 = hot|x_4)P(E_{5:6}|x_4) \\ \sum_{x_4} P(x_4|x_3 = B)P(E_4 = hot|x_4)P(E_{5:6}|x_4) \\ \sum_{x_4} P(x_4|x_3 = C)P(E_4 = hot|x_4)P(E_{5:6}|x_4) \\ \sum_{x_4} P(x_4|x_3 = D)P(E_4 = hot|x_4)P(E_{5:6}|x_4) \\ \sum_{x_4} P(x_4|x_3 = E)P(E_4 = hot|x_4)P(E_{5:6}|x_4) \\ \sum_{x_4} P(x_4|x_3 = F)P(E_4 = hot|x_4)P(E_{5:6}|x_4) \end{bmatrix} \\
&= \begin{bmatrix} 0.2 * 0.16 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0.8 * 0.16 + 0 + 0 \\ 0 + 0 + 0 + 0.2 * 0.16 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.032 \\ 0 \\ 0.128 \\ 0.032 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

So we get:

$$
P(E_{4:6}|x_3) = \begin{bmatrix} 0.032 \\ 0 \\ 0.128 \\ 0.032 \\ 0 \\ 0 \end{bmatrix} \tag{24}
$$

Based on equation (22),(23),(24) we can get the following:

$$
\begin{aligned}
P(E_{4:6}|e_{1:3}) &= \sum_{x_3} P(x_3|e_{1:3})P(E_{4:6}|x_3) \\
&= 0 * 0.032 + 0.2 * 0 + 0.8 * 0.128 + 0 * 0.032 + 0 * 0 + 0 * 0 \\
&= 0.1024
\end{aligned}
$$

5. Prediction: You decide to attempt to rescue the rover on day 4. However, the transmission of E4 seems to have been corrupted, and so it is not observed. What is the rover's position distribution for day 4 given the same evidence, $P(X4|hot1, cold2, cold3)$?
The same thing happens again on day 5. What is the rover's position distribution for day 5 given the same evidence, $P(X5|hot1, cold2, cold3)$**?**

To compute position distribution of $P(X_4|hot_1, cold_2, cold_3)$, we are going to first find the position distribution of $P(X_3|e_{1:3})$, and then apply the transition model to find the position distribution on day 4.

From the first question, we know that

$$P(X_3|e_{1:3}) = \begin{bmatrix} P(X_3 = A) = 0 \\ P(X_3 = B) = 0.2 \\ P(X_3 = C) = 0.8 \\ P(X_3 = D) = 0 \\ P(X_3 = E) = 0 \\ P(X_3 = F) = 0 \end{bmatrix} \tag{25}$$

$$P(X_4|hot_1, cold_2, cold_3) = \sum_{x_3} P(X_4|x_3)P(x_3|hot1, cold2, cold3)$$

$$= \begin{bmatrix} \sum_{x_3} P(X_4 = A|x_3)P(x_3|e_{1:3}) \\ \sum_{x_3} P(X_4 = B|x_3)P(x_3|e_{1:3}) \\ \sum_{x_3} P(X_4 = C|x_3)P(x_3|e_{1:3}) \\ \sum_{x_3} P(X_4 = D|x_3)P(x_3|e_{1:3}) \\ \sum_{x_3} P(X_4 = E|x_3)P(x_3|e_{1:3}) \\ \sum_{x_3} P(X_4 = F|x_3)P(x_3|e_{1:3}) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+0+0+0 \\ 0+0.2*0.2+0+0+0+0 \\ 0+0.2*0.8+0.8*0.2+0+0+0 \\ 0+0+0.8*0.8+0+0+0 \\ 0+0+0+0+0+0 \\ 0+0+0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.04 \\ 0.32 \\ 0.64 \\ 0 \\ 0 \end{bmatrix}$$

To compute $P(X_5|hot1, cold2, cold3)$, we can use the same technique for day 4

$$P(X_5|hot_1, cold_2, cold_3) = \sum_{x_4} P(X_5|x_4)P(x_4|hot1, cold2, cold3)$$

$$= \begin{bmatrix} \sum_{x_4} P(X_5 = A|x_4)P(x_4|e_{1:3}) \\ \sum_{x_4} P(X_5 = B|x_4)P(x_4|e_{1:3}) \\ \sum_{x_4} P(X_5 = C|x_4)P(x_4|e_{1:3}) \\ \sum_{x_4} P(X_5 = D|x_4)P(x_4|e_{1:3}) \\ \sum_{x_4} P(X_5 = E|x_4)P(x_4|e_{1:3}) \\ \sum_{x_4} P(X_5 = F|x_4)P(x_4|e_{1:3}) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+0+0+0 \\ 0+0.2*0.04+0+0+0+0 \\ 0+0.8*0.04+0.2*0.32+0+0+0 \\ 0+0+0.8*0.32+0.2*0.64+0+0 \\ 0+0+0+0.8*0.64+0+0 \\ 0+0+0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.008 \\ 0.096 \\ 0.384 \\ 0.512 \\ 0 \end{bmatrix}$$

Therefore:

$$P(X_4|hot_1, cold_2, cold_3) = \begin{bmatrix} 0 \\ 0.04 \\ 0.32 \\ 0.64 \\ 0 \\ 0 \end{bmatrix}$$

$$P(X_5|hot_1, cold_2, cold_3) = \begin{bmatrix} 0 \\ 0.008 \\ 0.096 \\ 0.384 \\ 0.512 \\ 0 \end{bmatrix}$$

3. Prediction: What is $P(hot_4|hot_1, cold_2, cold_3)$, the probability of observing hot4 in day 4, given the previous observations in days 1,2, and 3? (This is a single value, not a distribution).

To solve this question, we are going to use the equation for Prediction:

$$P(E_{t+1:T}|e_{1:t}) = \sum_{X_t} P(X_t|e_{1:t})P(E_{t+1:T}|X_t) \tag{22}$$

From (12), we can get that:

$$\begin{bmatrix} P(X_3 = A|e_{1:3}) = 0 \\ P(X_3 = B|e_{1:3}) = 0.2 \\ P(X_3 = C|e_{1:3}) = 0.8 \\ P(X_3 = D|e_{1:3}) = 0 \\ P(X_3 = E|e_{1:3}) = 0 \\ P(X_3 = F|e_{1:3}) = 0 \end{bmatrix}$$

$$P(E_{t+1:T}|X_t) = \sum_{X_{t+1}} P(X_{t+1}|X_t)P(E_{t+1}|X_{t+1})P(E_{t+2:T}|X_{t+1}) \tag{23}$$

From (12), we know that:

$$P(X_3|hot_1, cold_2, cold_3) = \begin{bmatrix} 0 \\ 0.2 \\ 0.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying the transition model to find the position distribution on day 4, we have:

$$P(X_4|hot_1, cold_2, cold_3) = \sum_{X_3} P(X_4|X_3)P(X_3|hot1, cold2, cold3)$$

$$= \begin{bmatrix} \sum_{X_3} P(X_4 = A|X_3)P(X_3|e_{1:3}) \\ \sum_{X_3} P(X_4 = B|X_3)P(X_3|e_{1:3}) \\ \sum_{X_3} P(X_4 = C|X_3)P(X_3|e_{1:3}) \\ \sum_{X_3} P(X_4 = D|X_3)P(X_3|e_{1:3}) \\ \sum_{X_3} P(X_4 = E|X_3)P(X_3|e_{1:3}) \\ \sum_{X_3} P(X_4 = F|X_3)P(X_3|e_{1:3}) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+0+0+0 \\ 0+0.2*0.2+0+0+0+0 \\ 0+0.8*0.2+0.2*0.8+0+0+0 \\ 0+0+0.8*0.8+0+0+0 \\ 0+0+0+0+0+0 \\ 0+0+0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.04 \\ 0.32 \\ 0.64 \\ 0 \\ 0 \end{bmatrix}$$

We can observe hot_4 prediction only at $X_4 = A$ or $X_4 = D$. So, $P(hot_4|hot_1, cold_2, cold_3 = 0 \times 1 + 0.64 \times 1 = 0.64$.

Problem 5

(a)

$$V^\pi(s) = R(s,a) + \gamma \sum_{s'} T(s,a,s') V^\pi(s')$$

(b)

State	V_0	V_1	V_2
A	0	0	$0 + 1 \times 0 = 0$
B	0	5	$5 + 1 \times 5 = 10$

(c)

State	Action 1	Action 2
A	$0 + 1 \times 0 = 0$	$-1 + 1 \times (0.5 \times 0 + 0.5 \times 10) = 4$
B	$5 + 1 \times 10 = 15$	$0 + 1 \times 10 = 10$

New policy: $\pi_{new}(A) = 2, \pi_{new}(B) = 1$.