Math 291: Challenge Problems 1
Fall 2021

Due September 20, 2021

This set of problems focuses on orthogonal/orthonormal vectors, the Gram-Schmidt orthogalization algorithm, and distance problems (equivalently, least squares problems).

1. Let W be the subspace of $\mathbb{R}^3$ spanned by the vector $\mathbf{u} = (1, 2, 1)$.

1.a. (1 point) Let $\mathbf{v} = (1, 1, 1)$. Find the parallel component $\mathbf{w}$ of $\mathbf{v}$ onto W.

1.b. (2 point) Define $W^\perp$ to be the set of vectors in $\mathbb{R}^3$ orthogonal to $\mathbf{u}$. Suppose $\mathbf{x} = (x_1, x_2, x_3)$ is in $W^\perp$. Write down the equation satisfied by x_1, x_2, x_3. Define $\mathbf{v}^\perp = \mathbf{v} - \mathbf{w}$ where $\mathbf{v}$ and $\mathbf{w}$ are from the previous part. Verify that $\mathbf{v}^\perp$ is a vector in $W^\perp$.

1.c. (4 points) Parametrize the set of vectors in $W^\perp$ in the form of a linear combination of vectors. How many independent variables do you need?

1.d. (8 points) Find the point(s) in $W^\perp$ closest to $\mathbf{v}$. Justify your answer.

2. Let W be the span in $\mathbb{R}^4$ of the following three vectors

$$\mathbf{v}_1 = (1, 1, -1, -1), \quad \mathbf{v}_2 = (2, 0, -2, 0), \quad \mathbf{v}_3 = (4, -2, -2, 0). \tag{0.1}$$

2.a. (8 points) Apply the Gram-Schmidt process to the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ given above to obtain an orthonormal set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Provide adequate details on the construction of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Express $\mathbf{v}_2$ and $\mathbf{v}_3$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

2.b. (9 points) Take $\mathbf{v}_4 = (6, 4, 2, 0)$. Find the point(s) in W closest to $\mathbf{v}_4$. Provide adequate explanations for your procedures and justify them. You may use the computations from the previous questions.

2.c. (8 points) Find the distance between the plane V spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$ and the line $L = \{\mathbf{v}_4 + t\mathbf{v}_3 : \ t \in \mathbb{R}\}$. Identify the points on V and L which attain this distance. Again, justify your procedures and answers.