Math 291H Homework #2

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Honors Pledge Statement: “The writeup of this submission is my own work alone.”

Problem 1.19

a

$$\begin{aligned}
(1,1,0)+t\,(1,-1,2) &= (2,0,2)+s\,(-1,1,0) \\
(t,-t,2t)-(-s,s,0) &= (1,-1,2) \\
(t+s,-t-s,2t) &= (1,-1,2)
\end{aligned}$$

This implies that $t = 1$, which means that $s = 0$. Resubstituting, we find the point of intersection to be $\boxed{(2,0,2)}$.

b

Let $\boldsymbol{v}_1 = (1,-1,2)$ and $\boldsymbol{v}_2 = (-1,1,0)$. Both these vectors are in the plane. We next find the normal vector $\boldsymbol{n}$.

$$\boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2 = (-2,-2,0)$$

The general form of a plane is $\boldsymbol{n} \cdot \boldsymbol{x} = d$. Substituting $\boldsymbol{x} = (1,1,0)$, we see that $d = -4$. The equation for the plane is therefore

$$-2x + -2y = -4$$

We can simplify this a little to get

$$\boxed{x + y = 2}$$

Problem 1.20

Defining $\boldsymbol{n} := (2, -1, 3)$ and $\boldsymbol{x}_0 := (2, 0, 0)$, we can write the given equation as

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) = 0$$

We normalize $\boldsymbol{n}$, getting

$$\boldsymbol{u} = \frac{1}{\sqrt{14}} (2, -1, 3)$$

The minimal distance between this plane and $\boldsymbol{p}$ is then

$$|(\boldsymbol{x}_0 - \boldsymbol{p}) \cdot \boldsymbol{u}| = \frac{1}{\sqrt{14}} |(2, 3, 0) \cdot (2, -1, 3)| = \boxed{\frac{1}{\sqrt{14}}}$$

Problem 1.24

Let $k \in \mathbb{R}$, so that

$$h_{\boldsymbol{u}}(\boldsymbol{x}) = k\boldsymbol{e}_1$$

Since $\|h_{\boldsymbol{u}}(\boldsymbol{x})\| = \|\boldsymbol{x}\|$, $\boxed{k = \pm 7}$.
By construction of the Householder reflection,

$$\begin{aligned}
\boldsymbol{u} &= \pm \frac{\boldsymbol{x} - k\boldsymbol{e}_1}{\|\boldsymbol{x} - k\boldsymbol{e}_1\|} \\
&= \pm \frac{(5 \pm 7, 2, 4, 2)}{\|(5 \pm 7, 2, 4, 2)\|} \\
&= \boxed{\frac{(6,1,2,1)}{\sqrt{42}}, -\frac{(6,1,2,1)}{\sqrt{42}}, \frac{(-1,1,2,1)}{\sqrt{10}}, -\frac{(-1,1,2,1)}{\sqrt{10}}}
\end{aligned}$$

(In the first two lines, the outer $\pm$ signs are independent of the inner ones.)

Problem 1.27

We define the orthonormal basis

$$\begin{aligned}
\boldsymbol{u}_1 &= \frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|} \\
&= \frac{1}{\sqrt{14}}(-3,-2,1) \\
\boldsymbol{u}_2 &= \frac{1}{\|\boldsymbol{v}_1\|}\boldsymbol{v}_1 \times \boldsymbol{u}_1 \\
&= \frac{1}{\sqrt{10}}(-1,0,-3) \\
\boldsymbol{u}_3 &= \boldsymbol{u}_1 \times \boldsymbol{u}_2 \\
&= \frac{1}{\sqrt{35}}(3,-5,-1)
\end{aligned}$$

We can then define

$$\boldsymbol{b} = \boldsymbol{x}_1 - \boldsymbol{x}_2 = (0,3,1)$$

We then have

$$t = \frac{\boldsymbol{b} \cdot \boldsymbol{u}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{u}_2} = \frac{-5}{10} = -\frac{1}{2}$$

The point on the first line that corresponds to this t-value is

$$\boxed{\left(\frac{3}{2}, -\frac{1}{2}, -\frac{5}{2}\right)}$$

Similarly,

$$s = \frac{t\,(\boldsymbol{v}_2 - \boldsymbol{b}) \cdot \boldsymbol{u}_3}{\boldsymbol{v}_1 \cdot \boldsymbol{u}_3} = \frac{-\frac{1}{2}(-1,0,2)\cdot(3,-5,-1)}{(1,-4,-2)\cdot(3,-5,-1)} = \frac{\frac{5}{2}}{25} = \frac{1}{10}$$

$$\boxed{\left(\frac{33}{10}, \frac{3}{2}, \frac{9}{10}\right)}$$

The distance between these points is

$$|\boldsymbol{b} \cdot \boldsymbol{u}_1| = \boxed{\frac{5}{\sqrt{14}}}$$

Problem 1.29

a

It's easy to see that $\|\boldsymbol{u}_i\| = 1$. In addition, we can compute

$$
\begin{aligned}
\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 &= \frac{1}{9}\left(1 \cdot 2 + 2 \cdot 1 - 2 \cdot 2\right) = 0 \\
\boldsymbol{u}_1 \cdot \boldsymbol{u}_3 &= \frac{1}{9}\left(1 \cdot 2 + 2 \cdot (-2) - 2 \cdot (-1)\right) = 0 \\
\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 &= \frac{1}{9}\left(2 \cdot 2 + 1 \cdot (-2) + 2 \cdot (-1)\right) = 0
\end{aligned}
$$

We see that

$$\boldsymbol{u}_1 \times \boldsymbol{u}_2 = \frac{1}{3}\left(2, -2, -1\right) = \boldsymbol{u}_3$$

Therefore, this is a right-handed orthonormal basis.

b

$$\boldsymbol{u} = \frac{\boldsymbol{u}_1 - \boldsymbol{e}_1}{\|\boldsymbol{u}_1 - \boldsymbol{e}_1\|} = \frac{\frac{1}{3}\left(-2, 2, -2\right)}{\frac{1}{3}\left\|(-2, 2, -2)\right\|} = \boxed{\frac{1}{\sqrt{3}}\left(-1, 1, -1\right)}$$

c

$$
\begin{aligned}
h_{\boldsymbol{u}}\left(\boldsymbol{u}_2\right) &= \boldsymbol{u}_2 - 2\left(\boldsymbol{u}_2 \cdot \boldsymbol{u}\right)\boldsymbol{u} \\
&= \frac{1}{3}\left(2, 1, 2\right) - \frac{2}{9}\left(-3\right)\left(-1, 1, -1\right) \\
&= \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) + \left(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right) \\
&= \boxed{(0, 1, 0) = \boldsymbol{e}_2} \\
h_{\boldsymbol{u}}\left(\boldsymbol{u}_3\right) &= \boldsymbol{u}_3 - 2\left(\boldsymbol{u}_3 \cdot \boldsymbol{u}\right)\boldsymbol{u} \\
&= \frac{1}{3}\left(2, -2, -1\right) - \frac{2}{9}\left(-3\right)\left(-1, 1, -1\right) \\
&= \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right) + \left(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right) \\
&= \boxed{(0, 0, 1) = \boldsymbol{e}_1}
\end{aligned}
$$

Problem 1.32

a

Define $V := V_1 \cap V_2$. Suppose we have vectors $\boldsymbol{r}_1, \boldsymbol{r}_2 \in V$ and scalars $a, b \in \mathbb{R}$. By definition, $\boldsymbol{r}_1 \in V_1, V_2$ and $\boldsymbol{r}_2 \in V_1, V_2$. By definition of a subspace, $a\boldsymbol{r}_1 \in V_1, V_2$ and $b\boldsymbol{r}_1 \in V_1, V_2$. Since these vectors are in both subspaces, we can say that

$$a\boldsymbol{r}_1 + b\boldsymbol{r}_2 \in V_1, V_2$$

This is what we wanted to show.

b

If we scale up $\boldsymbol{z}$ by a factor of α, we get

$$\alpha\boldsymbol{z} = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$$

Since $\alpha\boldsymbol{x} \in V_1$ and $\alpha\boldsymbol{x} \in V_2$ by the definition of a subspace, $\alpha\boldsymbol{z} \in V_1 + V_2$ by the construction of $V_1 + V_2$. Similarly, let

$$\boldsymbol{z'} = \boldsymbol{z}_1 + \boldsymbol{z}_2 = \boldsymbol{x}_1 + \boldsymbol{y}_1 + \boldsymbol{x}_2 + \boldsymbol{y}_2 = (\boldsymbol{x}_1 + \boldsymbol{x}_1) + (\boldsymbol{y}_1 + \boldsymbol{y}_2)$$

By the definition of a subspace, $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in V_1$. Similarly, $\boldsymbol{y}_1 + \boldsymbol{y}_2 \in V_2$. Therefore, by construction, $\boldsymbol{z}_1 + \boldsymbol{z}_2 \in V_1 + V_2$.

Problem 1.33

Define $V_3 := V_1 + V_2$ and $V := V_1 \cap V_2$. We then define an orthonormal basis for V.

$$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$$

Therefore, $\dim(V) = p$. We can extend $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ to find orthonormal bases for V_1 and V_2. By definition, $V \subseteq V_1$ and $V \subseteq V_2$. Therefore, $\operatorname{span} V \subseteq \operatorname{span} V_1$ and $\operatorname{span} V \subseteq \operatorname{span} V_2$. Similarly, $\operatorname{span} V_1 \subseteq \operatorname{span} V_3$ and $\operatorname{span} V_2 \subseteq V_3$. We can therefore write and define

$$\begin{aligned}
S &:= \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q\} \\
T &:= \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_r\} \\
V_1 &= \operatorname{span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q\} \\
V_2 &= \operatorname{span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_r\}
\end{aligned}$$

It's easy to see that $V_1 \subseteq \operatorname{span}(S \cup T)$ and $V_2 \subseteq \operatorname{span}(S \cup T)$. Therefore, $V_3 \subseteq \operatorname{span}(S \cup T)$. In addition, $(S \cup T) \subseteq V_3$, so $\operatorname{span}(S \cup T) \subseteq V_3$. Therefore,

$$\begin{aligned}
V_3 &= \operatorname{span}(S \cup T) \\
&= \operatorname{span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_r\} \\
\dim V_3 &= p + q + r
\end{aligned}$$

We can now verify that

$$\begin{gathered}
\dim V_3 + \dim V = \dim V_1 + \dim V_2 \\
(p + q + r) + (p) = (p + q) + (p + r)
\end{gathered}$$