Math 291: Honors Calculus III

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- Class taught by Ian Jauslin (assistant professor)
- Recitations taught by Yael Davidov (5th year grad student)
- Textbook is *Multivariable Calculus* by Eric Carlen [1]
  - Given to us
  - Author taught class for several years
    - Last taught 2019-2020 school year
  - Last updated August 2019
  - Has a lot of errors
- Additional resources
  - *Vector Calculus, Linear Algebra, and Differential Forms* by Hubbard and Hubbard [2]

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§1 Lecture 1: September 1

§1.1 Introduction

Course is about calculus - multivariable calculus.

Single variable calculus is assumed.

The course is very dense and has a lot of material. As such, some nonessential/easy proofs/examples will be relegated to the textbook or recitations. In addition, there's a lot of material and this class is mostly for masochists.

§1.2 Why 3D Calc

The world is 3D. We need to generalize 1D calc for it to be applicable to a lot of things. We're gonna be working in an arbitrary number of dimensions. 3 dimensions isn't enough - we often deal with rotation, which adds 3 more angular parameters in addition to 3 for position. But we don't stop here - we continue onwards to n dimensions.

We focus less on numbers and more on the concepts - this lets us learn linear algebra, which isn't usually introduced in a typical calculus class.

§1.3 Vector Variables and Cartesian Coordinates

This section follows 1.1.2 of [1].

Definition 1.1: Functions

A function takes an ordered collection of numbers and maps it to another ordered collection of numbers.

Definition 1.2: Cartesian coordinates

Tells you where a particle is using a base point (the origin O) and a set of base vectors: $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$. We can follow each base vector a certain distance from the base point to get coordinates that specify direction. We can extend this to an arbitrary number of dimensions by increasing the number of base vectors.

The Pythagorean theorem in multiple dimensions can be used to find distance between these two points.

You can specify a geometric object by defining it as the set of points that satisfy an equation.

Example 1.1

The unit circle is defined as the set of points in 2D that satisfy $x^2 + y^2 = 1$.

§1.4 Parametrization

Definition 1.3: Parametrization

It maps complicated geometric objects to simpler objects (usually intervals) using analytic functions.

Example 1.2

$(x = \cos\theta, y = \sin\theta)$ where $\theta \in [-\pi, \pi]$ is a map that maps θ to a pair of coordinates.

$$X(\theta) = (\cos\theta, \sin\theta)$$

This is called a coordinate function.

Example 1.3

The inverse of this map:

$$\theta := \begin{cases} \arccos x & y \geq 0 \\ -\arccos x & y < 0 \end{cases}$$

The inverse cosine is only defined over the interval $\theta \in (-\pi, \pi]$.

Example 1.4

$$x^2 + y^2 = r^2$$

We assume that $r > 0$. With the scaling variable r we can transform the parametrization of the unit circle (just above) into an appropriate parametrization for a circle of an arbitray radius.

$$\theta \mapsto (r\cos\theta, r\sin\theta)$$

We can also define the inverse: the coordinate function.

$$(x, y) \mapsto \theta := \begin{cases} \arccos\left(x/\sqrt{x^2 + y^2}\right) & y \geq 0 \\ -\arccos\left(x/\sqrt{x^2 + y^2}\right) & y < 0 \end{cases}$$

Example 1.5

We parametrize the unit sphere.

$$x^2 + y^2 + z^2 = 1$$

We need two variables since its a circle (which is 1D - a line shaped into a circle) rotated.

$$z = \cos\phi, \ \phi \in [0, \pi]$$

This implies

$$x^2 + y^2 = \sin^2\phi$$

x and y lie on a circle of radius $\sin\phi$.
It's pretty easy to parametrize this

$$x = \sin\phi\cos\theta$$

$$y = \sin\phi\sin\theta$$

$$\theta \in [-\pi, \pi]$$

Here, ϕ is the angle between the point on the sphere and the z-axis. At the North and South poles, θ doesn't matter anymore. The parametrization function only holds on $\phi \neq 0, \pi$.

Parametrization can be used to simplify all sorts of computationally tedious problems.

Example 1.6

We compute the tangent plane to a sphere.
In 1D calculus, the tangent line to curve is the best local approximation of the curve. For example, at a point x_0, $y = \cos(x_0 + t) \approx \cos(x_0) - t\sin(x_0)$.

To make this easier suppose we have a fixed point: $(x_0, y_0, z_0) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ or $\phi = \frac{\pi}{4}, \theta = \frac{\pi}{4}$.

We can parametrize the shift in the angles with $\phi = \frac{\pi}{4} + s, \theta = \frac{\pi}{4} + t$. For $s, t \ll 1$ (very small changes), we can write

$$\begin{aligned}
x &= \sin\left(\frac{\pi}{4} + s\right)\cos\left(\frac{\pi}{4} + t\right) \\
&\approx \left(\sin\left(\frac{\pi}{4}\right) + s\cos\left(\frac{\pi}{4}\right)\right)\left(\cos\left(\frac{\pi}{4}\right) - t\sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{1}{2}(1+s)(1-t) \\
y &= \sin\left(\frac{\pi}{4} + s\right)\sin\left(\frac{\pi}{4} + t\right) \\
&\approx \left(\sin\left(\frac{\pi}{4}\right) + s\cos\left(\frac{\pi}{4}\right)\right)\left(\sin\left(\frac{\pi}{4}\right) + t\cos\left(\frac{\pi}{4}\right)\right) \\
&= \frac{1}{2}(1+s)(1+t) \\
z &= \cos\left(\frac{\pi}{4} + s\right) \\
&\approx \frac{1}{\sqrt{2}} - \frac{s}{\sqrt{2}}
\end{aligned}$$

Note: taking a Taylor expansion and taking a derivative have a lot of things in common.

§1.5 Vector Space $\mathbb{R}^n$

Definition 1.4: Vectors

A vector is an ordered collection of numbers in $\mathbb{R}$.

$$\boldsymbol{x} \in \mathbb{R}^m : m \in \mathbb{N}_* : \boldsymbol{x} \equiv (x_1, \dots, x_m)$$

This means $\mathbb{R}^n$ is a vector space.

Definition 1.5: Vector Space

You can multiply a member of the space by a scalar and still be in the vector space. You can also add members of the same vector space and the resultant vector will still be in the vector space.
Distributive property: if you multiply by a scalar α the resultant vector will still be in the vector space.
Commutative property: the order in which you add vectors of a vector space doesn't matter; the sum will still be the same member of the vector space.
Associative property: the order in which you group addition doesn't matter.

Definition 1.6: Span

$\text{span}(V)$
The span of a subspace is all possible linear combinations of vectors in V - also a vector space.

Theorem 1.1: Span Identity Theorem

Let $W = \text{span}(V)$, where $V \subset \mathbb{R}^n$. Then $\text{span}(W) = W$.

Proof. Since $W = \text{span}\, V$, $V \subseteq W$. Therefore, $\text{span}\, V \subseteq \text{span}\, W$. Pick indices $i, j, k \in \mathbb{N}$ such that $i + j = k$ and scalars $s_1, \cdots, s_k \in \mathbb{R}$. In addition, we also choose vectors $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$ and $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k \in V$, where

$$\boldsymbol{w}_1 = \sum_{n=0}^{i} s_n \boldsymbol{v}_n$$
$$\boldsymbol{w}_2 = \sum_{n=i+1}^{k} s_n \boldsymbol{v}_n$$

■

It's sometimes useful to note that bases are the shortest way to write a span.

Theorem 1.2: Standard Bases

$\text{span}(e_1, \cdots, e_n) = \mathbb{R}^m$
All vectors in $\mathbb{R}^m$ are linear combinations of the standard bases

Proof. Pick some scalars $(t_1, \cdots t_m) \in \mathbb{R}$. It's easy to see that for a sufficient choice for t_i, we can create any vector $\boldsymbol{x} \in \mathbb{R}^m$.

$$\boldsymbol{x} = \sum t_i \boldsymbol{e}_i$$

■

§2 Recitation 1: September 2

Example 2.1

Suppose $V = \{(0,0,0)\}$. Then $\operatorname{span} V$ only has one element since you can't make anything but $\mathbf{0}$ out of the zero-vector.

Example 2.2

Let $V = \{(1,0,0)\}$.

$$\operatorname{span} V = \{(t,0,0) : t \in \mathbb{R}\}$$

Example 2.3

Let $\boldsymbol{v}_1 = (1,2,-3)$ and $\boldsymbol{v}_2 = (1,-2,1)$. Can we find an equation that is satisfied by a vector $\boldsymbol{v} = (x,y,z)$ if and only if $\boldsymbol{v} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$?
Letting $s,t \in \mathbb{R}$,

$$\begin{aligned}
&\boldsymbol{v} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \\
&\boldsymbol{v} = s\boldsymbol{v}_1 + t\boldsymbol{v}_2 \\
&\quad = s(1,2,-3) + t(1,-2,1) \\
&\quad = (s+t, 2s-2t, -3s+t) \\
&x = s+t \\
&y = 2s-2t \\
&z = -3s+t
\end{aligned}$$

We solve the equations for x and y to see that

$$\begin{aligned}
s &= \frac{1}{4}(2x+y) \\
t &= \frac{1}{4}(2x+y)
\end{aligned}$$

Resubstituting into the equation for z, we see that

$$\begin{aligned}
z &= -x-y \\
x+y+z &= 0
\end{aligned}$$

So $\boldsymbol{v} = (x,y,z) \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \leftrightarrow z+x+y = 0$

Example 2.4

Let $\boldsymbol{v}_3 = (-2, 1, 1)$. Can you describe $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$? (vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ from previous example)?
Let $r, s, t \in \mathbb{R}$. Suppose $\boldsymbol{u} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Then,

$$\begin{aligned} \boldsymbol{u} &= r\boldsymbol{v}_1 + s\boldsymbol{v}_2 + t\boldsymbol{v}_3 \\ (x, y, z) &= (r + s - 2t, 2r - 2s + t, -3r + s + t) \end{aligned}$$

Adding all the equations together, we see that $x + y + z = 0$, once again. This means that the span hasn't changed.

Example 2.5

Let $\boldsymbol{u} = (1, 2), \boldsymbol{v} = (-2, -4), \boldsymbol{w} = (7, 14)$ be three vectors in $\mathbb{R}^2$. Show that $\boldsymbol{w} \in \operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$.
We can write $\boldsymbol{w} = s\boldsymbol{u} + t\boldsymbol{v}$. There are many such pairs of (s, t), which are computationally trivial to solve for.
Another, quicker, approach is to see that $\boldsymbol{u} \in \operatorname{span}\boldsymbol{v}$ and vice versa. Therefore, $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \operatorname{span}\boldsymbol{v}$.

Example 2.6

Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be orthogonal unit vectors. Define

$$\boldsymbol{v} := a\boldsymbol{u}_1 + b\boldsymbol{u}_2$$

Find $\boldsymbol{v} \cdot \boldsymbol{u}_1$.

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{u}_1 &= a\boldsymbol{u}_1 \cdot \boldsymbol{u}_1 + b\boldsymbol{u}_2 \cdot \boldsymbol{u}_1 \\ &= a + b \cdot 0 \\ &= a \end{aligned}$$

Definition 2.1: Set Equality

If you want to prove that $A = B$, then it is sufficient to prove that both $A \subseteq B$ and $B \subseteq A$.

§3 Lecture 2: September 8

§3.1 Dot Products

This section follows 1.1.5 of [1].

The fundamental property of geometry is distance.

Definition 3.1: Metric

A metric ρ takes a pair of points and returns a non-negative real number.
Properties:

$$\begin{aligned}\rho(x,y) &= \rho(y,x)\\ \rho(x,y) &= 0 \leftrightarrow x = y\\ \rho(x,z) &\le \rho(x,y) + \rho(y,z)\end{aligned}$$

The last property is satisfied by the Euclidean norm using the triangular inequality.

Definition 3.2: Euclidean Norm

[1] calls this Euclidean distance.
Suppose we have a vector $\boldsymbol{x} \in \mathbb{R}^n$. We define the Euclidean norm of that vector $\|\boldsymbol{x}\|$ by

$$\|\boldsymbol{x}\| = \sqrt{\sum_{j=1}^{n} x_i^2}$$

Example 3.1

The norm in $\mathbb{R}^2$ is

$$\sqrt{x^2 + y^2}$$

Example 3.2

The norm in $\mathbb{R}^3$ is

$$\sqrt{x^2 + y^2 + z^2}$$

The norm of the difference between two vectors is their Euclidean distance.

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \rho_E(x,y)$$

Some properties of the Euclidean distance:

Another cool property is that for any $t \in \mathbb{R}$,

$$\|t\boldsymbol{x}\| = |t|\,\|\boldsymbol{x}\|$$

Definition 3.3: Unit(ary) vector

You can obtain a unit vector by dividing a vector by its norm. Let $\boldsymbol{x} \in \mathbb{R}^n$. The unit vector $\boldsymbol{u}$ is

$$\boldsymbol{u} := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

By definition (and it's also pretty easy to see this), $\|\boldsymbol{u}\| = 1$ and $\boldsymbol{u}$ and $\boldsymbol{x}$ point in the same direction.

Definition 3.4: Dot Product AKA Scalar Product

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.
The dot product of these two vectors is defined as

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{i=n} x_i y_i$$

The norm is the square root of vector dotted with itself:

$$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$$

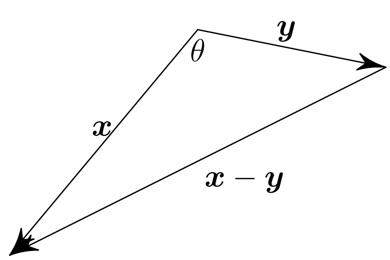
This is analogous to absolute value:

$$|x| = \sqrt{x \cdot x}$$

The dot product is distrbutive:

$$\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}$$

$$\begin{aligned}
\|\boldsymbol{x} - \boldsymbol{y}\|^2 &= (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y} - 2\boldsymbol{x} \cdot \boldsymbol{y} \\
&= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y}
\end{aligned}$$



This bears a striking similarity to the law of cosines. We see in the triangle above that

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\|\boldsymbol{x}\|\,\|\boldsymbol{x}\|\cos\theta$$

This is how you see $\boldsymbol{x} \cdot \boldsymbol{y} = \|x\|\,\|y\|\cos\theta$.

Definition 3.5: Angle Between Vectors

The angle θ is defined using the dot product:

$$\theta = \arccos \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|}$$

Note: we need to prove that this quotient is in the domain of the inverse cosine function, which we do using the Cauchy-Schwarz Inequality (below).

Theorem 3.1: Cauchy-Schwarz Inequality

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$$

Proof. Define two unitary versions of x and y, $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

$$\boldsymbol{a} := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$
$$\boldsymbol{b} := \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}$$

We have

$$\begin{aligned} \|\boldsymbol{a} - \boldsymbol{b}\|^2 &= \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} \\ &= 2\,(1 - \boldsymbol{a} \cdot \boldsymbol{b}) \end{aligned}$$

Since $\|\boldsymbol{a} - \boldsymbol{b}\|^2 \geq 0$, that means $\boldsymbol{a} \cdot \boldsymbol{b} \leq 1$. Similarly,

$$\|\boldsymbol{a} + \boldsymbol{b}\|^2 = 2\,(1 + \boldsymbol{a} \cdot \boldsymbol{b})$$

Therefore, $\boldsymbol{a} \cdot \boldsymbol{b} \geq -1$. Putting these two inequalities together,

$$-1 \leq \boldsymbol{a} \cdot \boldsymbol{b} \leq 1$$

By the definition of $\boldsymbol{a}, \boldsymbol{b}$, this proves the inequality. ∎

Theorem 3.2: Triangle Inequality

For any three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$,

$$\|\boldsymbol{x}-\boldsymbol{z}\| \geq \|\boldsymbol{x}-\boldsymbol{y}\| + \|\boldsymbol{y}-\boldsymbol{z}\|$$

Proof. Define:

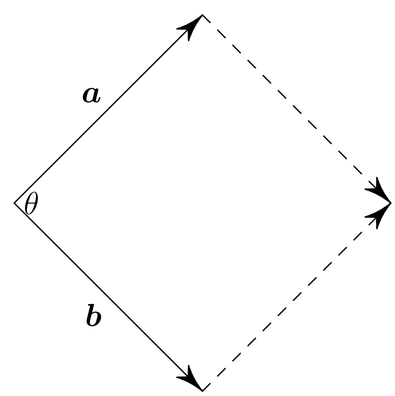
$$\begin{aligned} \boldsymbol{a} &:= \boldsymbol{x}-\boldsymbol{y} \\ \boldsymbol{b} &:= \boldsymbol{z}-\boldsymbol{y} \end{aligned}$$

We then have

$$\begin{aligned} \|\boldsymbol{x}-\boldsymbol{z}\|^2 &= \|\boldsymbol{a}-\boldsymbol{b}\|^2 \\ &= \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} \\ \text{(Cauchy-Schwartz)} &\leq \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} \\ &= (\|\boldsymbol{a}\| + \|\boldsymbol{b}\|)^2 \\ \|\boldsymbol{x}-\boldsymbol{z}\| &\leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\| \end{aligned}$$

Using the definition of $\boldsymbol{a}, \boldsymbol{b}$ finishes the proof. ■

§3.2 Cross Products in $\mathbb{R}^3$



The area of parallelogram is (from basic geometry)

$$A = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \sin\theta$$

$$\begin{aligned}
A^2 &= \|\boldsymbol{a}\|^2 \|\boldsymbol{b}\|^2 \sin^2\theta \\
&= \|\boldsymbol{a}\|^2 \|\boldsymbol{b}\|^2 \left(1 - \cos^2\theta\right) \\
&= \|\boldsymbol{a}\|^2 \|\boldsymbol{b}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2
\end{aligned}$$

Splitting $\boldsymbol{a}$ and $\boldsymbol{b}$ into components, we see that

$$A^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

We define a useful operation to get this quantity, which looks like the norm of a vector: the cross product.

Definition 3.6: Cross Product

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2b_3 - a_3b_2)\,\boldsymbol{e}_1 + (a_3b_1 - a_1b_3)\,\boldsymbol{e}_2 + (a_1b_2 - a_2b_1)\,\boldsymbol{e}_3$$

A more convenient way to remember this is

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

By construction,

$$A = \|\boldsymbol{a} \times \boldsymbol{b}\|$$

It's easy to see with some algebra that the cross product is anticommutative, so

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$$

Example 3.3

It's easy to see that

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

It follows that for all $t \in \mathbb{R}$

$$\boldsymbol{a} \times (t\boldsymbol{a}) = 0$$

This means that when parallel or anti-parallel vectors are crossed, the result is always 0.

Theorem 3.3:

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}$$

Note the cylic permutations of $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. If acylic permutations are introduced, then there's a negative sign added.

Proof. Trivial by algebraic expansion. ■

Example 3.4

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = (\boldsymbol{a} \times \boldsymbol{a}) \cdot \boldsymbol{b} = 0$$

This means $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$.

Cross products are less relevant in higher dimensions:

- In 4D, the cross product is a plane
- In 5D, the cross product is a sphere

Theorem 3.4: Lagrange's Identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c}$$

Proof. ■

§3.3 Lines and Planes in $\mathbb{R}^3$

§3.3.1 Lines

Definition 3.7: Line

A line is defined as the set of points $\boldsymbol{x}$ that satisfy

$$\boldsymbol{a} \times (\boldsymbol{x} - \boldsymbol{x}_0) = 0$$

Parametrization:

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + t\boldsymbol{a}$$

It's easy to see that this satisfies the definition of a line.

§3.3.2 Planes

(insert airplane joke here)

Definition 3.8: Planes

A plane is the set of points $\boldsymbol{x}$ such that

$$(\boldsymbol{x} - \boldsymbol{x}_0) \cdot \boldsymbol{a} = 0$$

Parametrization:

$$\boldsymbol{x}(s, t) = \boldsymbol{x}_0 + s\boldsymbol{v}_0 + t\boldsymbol{v}_1$$

Here, $\boldsymbol{v}_0, \boldsymbol{v}_1$ are vectors in the plane that they help define.

§3.4 Orthonormal Bases

Definition 3.9: Orthonormal Bases

Define the unit vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$. We define an orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ by assigning the unit vectors the following properties:

$$\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0 \, \forall i \neq j$$
$$\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 1 \, \forall i = j$$

For the following two definitions, we work with the orthonormal basis

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$$

Definition 3.10: Right-Handed Orthonormal Bases

$$\boldsymbol{u}_1 \times \boldsymbol{u}_2 = \boldsymbol{u}_3$$

Definition 3.11: Left-Handed Orthonormal Bases

$$\boldsymbol{u}_1 \times \boldsymbol{u}_2 = -\boldsymbol{u}_3$$

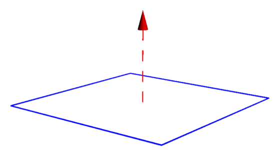
§4 Recitation 2: September 9

§4.1 Lines and Planes

§4.1.1 Planes

$$\boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) = 0$$

If you take a vector lying in the plane, it has to be perpendicular to $\boldsymbol{a}$. (draw picture)



If you take two vectors that aren't parallel, their span is the plane that they are in. This naive plane will go through the origin. However, you also need planes that don't go through the origin. Therefore, you can use a constant term $\boldsymbol{x}_0$ to also write the plane as

$$\boldsymbol{x}(s,t) = s\boldsymbol{v}_1 + t\boldsymbol{v}_2 + \boldsymbol{x}_0$$

§4.1.2 Lines

$$\boldsymbol{a} \times (\boldsymbol{x} - \boldsymbol{x}_0) = 0$$

All vectors on the line must be parallel or anti-parallel to $\boldsymbol{a}$.

§4.2 Orthonormal Bases

Example 4.1

(Example 1.11 of [1])
Given a nonzero vector $\boldsymbol{v} \in \mathbb{R}^3$, how do we find an orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ in which $\boldsymbol{u}_3$ is a positive multiple of $\boldsymbol{v}$?
The general procedure is as follows: $\boldsymbol{u}_3 = \|\boldsymbol{v}\|^{-1}\boldsymbol{v}$ is the only unit vector that is a positive mutliple of $\boldsymbol{v}$. To pick another vector $\boldsymbol{u}_1$, we must find a vector orthogonal to $\boldsymbol{v}$. Let

$$\boldsymbol{w} := \boldsymbol{e}_3 \times \boldsymbol{v} = (-v_2, v_1, 0)$$

Then, we can normalize this vector to find $\boldsymbol{u}_1$. Finally, we use the cross product to find $\boldsymbol{u}_2 = \boldsymbol{u}_3 \times \boldsymbol{u}_1$. This basis will be right-handed.

§4.3 Worked Examples

Example 4.2

Example 1.12 of [1].
Consider the plane passing through $\boldsymbol{x}_0$ that is orthogonal to the vector $\boldsymbol{a}$. For the sake of concreteness, let

$$\begin{aligned}\boldsymbol{a} &= (1,2,3)\\ \boldsymbol{x}_0 &= (3,2,1)\end{aligned}$$

By definition, the plane is the set of points $\boldsymbol{x}$ that satisfy

$$\boldsymbol{a}\cdot(\boldsymbol{x}-\boldsymbol{x}_0) = 0$$

Take dot product and simplify to get

$$x + 2y + 3z = 10$$

Example 4.3

Example 1.13 of [1].
Suppose we have the equation

$$\boldsymbol{a}\cdot\boldsymbol{x} = d$$

Let $\boldsymbol{a} = (1,2,1)$ and $d = 10$.
We can take this dot product fairly easily, getting

$$x + 2y + z = 10 \iff z = 10 - 2y - x$$

We can also define the plane as the set of points $\boldsymbol{x}$ that satisfy this relation. We express this as

$$\begin{aligned}\boldsymbol{x} &= (x, y, 10-2y-x)\\ &= (0,0,10) + x\,(1,0,-1) + y\,(0,1,-2)\end{aligned}$$

We'll be parametrizing planes a lot when we take triple surface integrals and other spicy objects later in the class.

Example 4.4

Are these points on the same plane?

$$\begin{aligned}
\boldsymbol{p}_1 &= (1, 2, 3) \\
\boldsymbol{p}_2 &= (3, 2, 1) \\
\boldsymbol{p}_3 &= (1, 3, 2) \\
\boldsymbol{p}_4 &= (4, -1, 3)
\end{aligned}$$

We immediately notice that $x + y + z = 6$ for all 4 points. Therefore the answer is yes!
Another, slower, way to do this is to find two vectors that lie in the plane, find their normal vector, and construct a plane from it.

§5 Lecture 3: September 13

We covered (partially in almost all cases) 1.1.6, 1.1.7, 1.1.8, 1.2.3 of [1].

§5.1 Parallel and Orthogonal Components

1.1.6 of [1].

In the book, the section almost feels like a small comment, but is in reality really important for a lot of proofs.

Let $\boldsymbol{x} \in \mathbb{R}^m$. Fix a vector $\boldsymbol{a} \in \mathbb{R}^m$.

We want to find t such that

$$\boldsymbol{x} = t\boldsymbol{a} + (\boldsymbol{x} - t\boldsymbol{a})$$

This is of course, a tautology, but the point is that we want a component that goes in the direction of $\boldsymbol{a}$ and a component that is orthogonal to it. Taking the dot product with respect to $\boldsymbol{a}$, we will later find it easy to see that this holds true when $t = \dfrac{\boldsymbol{x} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}$.

$$\boldsymbol{x} = \frac{\boldsymbol{x} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} + \left(\boldsymbol{x} - \frac{\boldsymbol{x} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}\right)$$

Looking at the two distinct components of the RHS of this equation, we can define the parallel and orthogonal components of a vector.

Definition 5.1: Parallel and Orthogonal Components

Given vectors $\boldsymbol{x}, \boldsymbol{a} \in \mathbb{R}^n$, the parallel and orthogonal components with respect to $\boldsymbol{a}$ are, respectively,

$$\boldsymbol{x}_{\|} = \frac{\boldsymbol{x} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$
$$\boldsymbol{x}_{\perp} = \boldsymbol{x} - \frac{\boldsymbol{x} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

We can define $\boldsymbol{u} := \dfrac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2}$.

$$\boldsymbol{x}_{\|} = (\boldsymbol{x} \cdot \boldsymbol{u})\,\boldsymbol{u}$$
$$\boldsymbol{x}_{\perp} = \boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{u})\,\boldsymbol{u}$$

Note that introducing a system of coordinates involving $\boldsymbol{a}$ is useful in a lot of problems.

§5.2 Orthonormal Subsets

Definition 5.2: Orthonormal Subsets

An orthonormal subset $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\} \in \mathbb{R}^n$ satisfies the following properties:

$$\begin{aligned} \boldsymbol{u}_i &\in \mathbb{R}^n \\ \|\boldsymbol{u}_i\| &= 1 \\ \forall i, j : i \neq j,\ \boldsymbol{u}_i \cdot \boldsymbol{u}_j &= 0 \end{aligned}$$

Definition 5.3: Orthonormal Bases

An orthonormal basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\} \in \mathbb{R}^n$ is an orthonormal subset that satisfies $m = n$. In other words, the length of the set is equal to the number of dimensions.

Theorem 5.1: Fundamental Theorem on Orthonormal Bases

Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} \in \mathbb{R}^m$ be an orthonormal basis of $\mathbb{R}^m$.

$$\forall \boldsymbol{x} \in \mathbb{R}^m, \ \boldsymbol{x} = \sum_{j=1}^{n} (\boldsymbol{x} \cdot \boldsymbol{u}_j)\, \boldsymbol{u}_j$$

The norm can also be computed from the orthonormal basis.

$$\|\boldsymbol{x}\|^2 = \sum_{j=1}^{n} (\boldsymbol{x} \cdot \boldsymbol{u}_j)^2$$

Proof. We prove this theorem by contradiction. Something about the way this statement is written screams for taking a dot product. Define

$$\begin{aligned}
\boldsymbol{z} &:= \boldsymbol{x} - \sum_{j} (\boldsymbol{x} \cdot \boldsymbol{u}_j)\, \boldsymbol{u}_j \\
\boldsymbol{z} \cdot \boldsymbol{u}_i &= \boldsymbol{x} \cdot \boldsymbol{u}_i - \sum_{j} (\boldsymbol{x} \cdot \boldsymbol{u}_j)\, (\boldsymbol{u}_j \cdot \boldsymbol{u}_i) \\
&= \boldsymbol{x} \cdot \boldsymbol{u}_i - \boldsymbol{x} \cdot \boldsymbol{u}_i \\
&= 0 \\
\boldsymbol{z} &\perp \boldsymbol{u}_i \ \forall i
\end{aligned}$$

But if $\boldsymbol{z} \neq 0$,

$$\frac{\boldsymbol{z}}{\|\boldsymbol{z}\|} = \boldsymbol{u}_{n+1}$$

This is a direct contradiction of Lemma 1, so $\boldsymbol{z} = 0$ is true. ■

Remark 5.1

A lemma is a statement that is either less important or used to prove a theorem.

Lemma 5.1:

In $\mathbb{R}^n$, there is no orthonormal subset of vectors with greater than n elements.

Proof. We use induction on n.
We start at $n = 1$. In $n = 1$, there are two unit vectors: $\{1, -1\}$, which isn't an orthonormal set.
Assume that the statement is true for $\mathbb{R}^{n-1}$. We proceed by contradiction. Suppose we can find an orthonormal set

$$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{n+1}\} \in \mathbb{R}^n$$

Then, define another orthonormal set

$$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1}\} \in \mathbb{R}^n, \ \boldsymbol{v}_{n+1} = \boldsymbol{e}_n$$

If $\boldsymbol{u}_{n+1} = \boldsymbol{e}_n$, then $\boldsymbol{v}_i = \boldsymbol{u}_i$. If $\boldsymbol{u}_{n+1} \neq \boldsymbol{e}_n$, we can take the Householder reflection of $\boldsymbol{u}_i$. Lemma 3 guarentees the existence of a unit vector $\boldsymbol{a}$ such that

$$h_{\boldsymbol{a}}\left(\boldsymbol{u}_{n+1}\right) = \boldsymbol{e}_n$$

In both cases, we have an orthonormal set with

$$\forall j \leq n: \ \boldsymbol{v}_j \cdot \boldsymbol{e} = 0$$

But if we remove the last component, that forms an orthonormal set (by Lemma 2) in $\mathbb{R}^{n-1}$. This is a contradiction, so we can conclude the proof by induction. ■

Example 5.1

The most trivial example uses the fundamental basis vectors $\boldsymbol{e}_i$ to express a general vector $\boldsymbol{x}$.

$$\boldsymbol{x} = \sum^{n} \boldsymbol{x}_j \boldsymbol{e}_j$$

This is fairly easy to see given a concrete $\boldsymbol{x}$.

Definition 5.4: Householder Reflections

We define a reflection about a symmetrical object. Let $\boldsymbol{u}$ be a unit vector orthogonal to this object. Let

$$\boldsymbol{x} = \boldsymbol{x}_{\perp} + \boldsymbol{x}_{\parallel}$$

Here, the parallel and perpendicular components are taken with respect to $\boldsymbol{u}$. The Householder reflection of a vector is the transformation

$$\begin{aligned} \boldsymbol{x}_{\parallel} &\to -\boldsymbol{x}_{\parallel} \\ \boldsymbol{x}_{\perp} &\to \boldsymbol{x}_{\perp} \end{aligned}$$

In other words,

$$h_{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{x}_{\perp} - \boldsymbol{x}_{\parallel}$$

Lemma 5.2: Dot Product Under Householder Reflections

$$h_{\boldsymbol{u}}(\boldsymbol{x}) \cdot h_{\boldsymbol{u}}(\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$$

Proof.

$$\begin{aligned} h_{\boldsymbol{u}}(\boldsymbol{x}) \cdot h_{\boldsymbol{u}}(\boldsymbol{y}) &= \left(\boldsymbol{x}_{\perp} - \boldsymbol{x}_{\parallel}\right) \cdot \left(\boldsymbol{y}_{\perp} - \boldsymbol{y}_{\parallel}\right) \\ &= \boldsymbol{x}_{\perp} \cdot \boldsymbol{y}_{\perp} + \boldsymbol{x}_{\parallel}\boldsymbol{y}_{\parallel} \\ \boldsymbol{x} \cdot \boldsymbol{y} &= \left(\boldsymbol{x}_{\perp} - \boldsymbol{x}_{\parallel}\right) \cdot \left(\boldsymbol{y}_{\perp} - \boldsymbol{y}_{\parallel}\right) \\ &= \boldsymbol{x}_{\perp} \cdot \boldsymbol{y}_{\perp} + \boldsymbol{x}_{\parallel}\boldsymbol{y}_{\parallel} \end{aligned}$$

Here, the cross terms dissappear by orthogonality. ■

Its easy to see that

$$\|h_{\boldsymbol{u}}(\boldsymbol{x})\| = \|\boldsymbol{x}\|$$

Lemma 5.3:

Suppose we have two vectors $\boldsymbol{x}, \boldsymbol{y}$ such that

$$\|\boldsymbol{x}\| = \|\boldsymbol{y}\|$$

If $\boldsymbol{x} \neq \boldsymbol{y}$, then $\exists \boldsymbol{u}$ such that

$$\boldsymbol{y} = h_{\boldsymbol{u}}(\boldsymbol{x})$$

In fact,

$$\boldsymbol{u} = \frac{\boldsymbol{x} - \boldsymbol{y}}{\|\boldsymbol{x} - \boldsymbol{y}\|}$$

Proof.

$$\begin{aligned}
h_{\boldsymbol{u}}(x) &= \boldsymbol{x}_{\perp} - \boldsymbol{x}_{\|} \\
&= \boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{u})\,\boldsymbol{u} - (\boldsymbol{x} \cdot \boldsymbol{u})\,\boldsymbol{u} \\
&= \boldsymbol{x} - 2\,(\boldsymbol{x} \cdot \boldsymbol{u})\,\boldsymbol{u} \\
&= \boldsymbol{x} - \frac{2}{\|\boldsymbol{x} - \boldsymbol{y}\|^2}\,(\boldsymbol{x} \cdot (\boldsymbol{x} - \boldsymbol{y}))\,(\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{x} - \frac{2}{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y}}\,(\boldsymbol{x} \cdot (\boldsymbol{x} - \boldsymbol{y}))\,(\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{x} - \frac{2}{2\left(\|\boldsymbol{x}\|^2 - \boldsymbol{x} \cdot \boldsymbol{y}\right)}\,(\boldsymbol{x} \cdot (\boldsymbol{x} - \boldsymbol{y}))\,(\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{x} - \frac{1}{\left(\|\boldsymbol{x}\|^2 - \boldsymbol{x} \cdot \boldsymbol{y}\right)}\,(\boldsymbol{x} \cdot (\boldsymbol{x} - \boldsymbol{y}))\,(\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{x} - \frac{1}{\boldsymbol{x} \cdot (\boldsymbol{x} - \cdot \boldsymbol{y})}\,(\boldsymbol{x} \cdot (\boldsymbol{x} - \boldsymbol{y}))\,(\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{x} - (\boldsymbol{x} - \boldsymbol{y}) \\
&= \boldsymbol{y}
\end{aligned}$$

■

Remark 5.2

You should never state something is obvious when writing a proof when it really isn't. Definitely not on homework assignments and *definitely* not on exams.

§5.3 Distance Problems in $\mathbb{R}^3$

Theorem 5.2: Distance Between a Point and a Line

Given a line

$$L : \boldsymbol{x}(t) = \boldsymbol{x}_0 + t\boldsymbol{a}$$

and a point $\boldsymbol{p} \in \mathbb{R}^3$, the minimal distance between the two is

$$\|(\boldsymbol{x}_0 - \boldsymbol{p}) \times \boldsymbol{a}\|$$

Proof. We can rescale to constrain $\|\boldsymbol{a}\| = 1$. The distance between L and $\boldsymbol{p}$ is defined as

$$\min_t \|\boldsymbol{x}(t) - \boldsymbol{p}\|$$

$$\begin{aligned}
\|\boldsymbol{x}(t) - \boldsymbol{p}\|^2 &= \left\|(\boldsymbol{x}(t) - \boldsymbol{p})_{\|}\right\|^2 + \|(\boldsymbol{x}(t) - \boldsymbol{p})_{\perp}\|^2 \\
&= \|(\boldsymbol{x}(t) - \boldsymbol{p}) \cdot \boldsymbol{a}\|^2 + \|(\boldsymbol{x}(t) - \boldsymbol{p}) \times \boldsymbol{a}\|^2 \\
&= \|(\boldsymbol{x}_0 \cdot \boldsymbol{a} - \boldsymbol{p} \cdot \boldsymbol{a} + t)\,\boldsymbol{a}\|^2 + \|x_0 \times \boldsymbol{a} - \boldsymbol{p} \times \boldsymbol{a}\|^2
\end{aligned}$$

The first term is minimized when $t = (\boldsymbol{p} - \boldsymbol{x}_0) \cdot \boldsymbol{a}$. Therefore, the distance between the line and point is $\|(\boldsymbol{x}_0 - \boldsymbol{p}) \times \boldsymbol{a}\|$. In addition, the point on L that is closest to $\boldsymbol{p}$ is $\boldsymbol{x}_0 + ((\boldsymbol{p} - \boldsymbol{x}_0) \cdot \boldsymbol{a})\,\boldsymbol{a}$. ∎

Aside: by Lagrange's identity, for a unit vector $\boldsymbol{u}$

$$\begin{aligned}
-\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{c}) &= -\left((\boldsymbol{u} \cdot \boldsymbol{c})\,\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{u})\,\boldsymbol{c}\right) \\
&= \boldsymbol{c} - (\boldsymbol{u} \cdot \boldsymbol{c})\,\boldsymbol{u} \\
&= \boldsymbol{c}_{\perp}
\end{aligned}$$

§6 Lecture 4: September 15

§7 Recitation 3: September 16

§8 Lecture 5: September 20

§9 Lecture 6: September 22

§10 Recitation 4: September 23

§11 Lecture 7: September 27

§12 Lecture 8: September 29

§13 Recitation 5: September 30

§14 Lecture 9: October 4

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§25 Recitation 9: October 28

§26 Lecture 17: November 1

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§31 Recitation 11: November 11

§32 Lecture 21: November 15

§33 Lecture 22: November 17

§34 Recitation 12: November 18

§35 Lecture 23: November 22

§36 Lecture 24: November 29

§37 Lecture 25: December 1

§38 Recitation 13: December 2

§39 Lecture 26: December 6

§40 Lecture 27: December 8

§41 Recitation 14: December 9

§42 Lecture 28: December 13

References

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