Math 291H Challenge Problems #1

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Honors Pledge Statement: “The writeup of this submission is my own work alone.”

Problem 1

a

We need to find the unit vector in the same direction as $\boldsymbol{u}$. By construction, this unit vector will be in the span of $\boldsymbol{u}$ and will therefore be in W.

$$\boldsymbol{u'} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} = \frac{1}{\sqrt{6}}(1,2,1)$$

The parallel component $\boldsymbol{w}$ in the same direction as $\boldsymbol{u'}$ is then

$$\begin{aligned}\boldsymbol{w} &= (\boldsymbol{v} \cdot \boldsymbol{u})\,\boldsymbol{u} \\ &= \frac{1}{6}\left((1,1,1) \cdot (1,2,1)\right)(1,2,1) \\ &= \boxed{\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)}\end{aligned}$$

b

If $\boldsymbol{x}$ is in $W^{\perp}$, then

$$\begin{aligned}\boldsymbol{u} \cdot \boldsymbol{x} &= 0 \\ x_1 + 2x_2 + x_3 &= 0\end{aligned}$$

$$\begin{aligned}\boldsymbol{v}^{\perp} &= \boldsymbol{v} - \boldsymbol{w} \\ &= (1,1,1) - \left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right) \\ &= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)\end{aligned}$$

We can compute

$$\frac{1}{3} + (2)\left(-\frac{1}{3}\right) + \frac{1}{3} = 0$$

Therefore, $\boldsymbol{v}^{\perp} \in W^{\perp}$.

c

We observe that

$$x_1 = -2x_2 - x_3$$

Therefore any $\boldsymbol{x} \in W^{\perp}$ can be written

$$\begin{aligned}\boldsymbol{x} &= (-2x_2 - x_3, x_2, x_3) \\ &= x_2 (-2, 1, 0) + x_3 (-1, 0, 1)\end{aligned}$$

In terms of vectors $\boldsymbol{v}_1 = (-2, 1, 0), \boldsymbol{v}_2 = (-1, 0, 1)$, we can parametrize

$$\boldsymbol{x}(s,t) = s\boldsymbol{v}_1 + t\boldsymbol{v}_2$$

This parametrization uses a total of 2 independent variables. That makes sense since a plane is a 2 dimensional object. To check that these are the correct vectors, we can verify that $\boldsymbol{u} = \boldsymbol{v}_1 \times \boldsymbol{v}_2$.

d

We wish to find (s,t) such that the distance $\|\boldsymbol{x}(s,t) - \boldsymbol{v}\|$ is minimized. To do this, we need to define a convenient orthonormal basis so that calculations become easier. Let this basis be $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$. To make the most terms vanish, we let $\boldsymbol{u}_3 := \boldsymbol{u'}$ and $\boldsymbol{u}_1 := \dfrac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$. To find $\boldsymbol{u}_2$, we compute $\boldsymbol{w}_2 := \boldsymbol{u} \times \boldsymbol{v}_1 = (-1, -2, 5)$. We then normalize this vector to find $\boldsymbol{u}_2$.

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} := \left\{\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}, \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|}, \boldsymbol{u'}\right\} = \left\{\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right), \left(-\frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right), \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$$

$$\begin{aligned}\|\boldsymbol{x}(s,t) - \boldsymbol{v}\|^2 &= \sum_{i=1}^{i=3} |(\boldsymbol{x}(s,t) - \boldsymbol{v}) \cdot \boldsymbol{u_i}|^2 \\ &= \sum_{i=1}^{i=3} |\boldsymbol{x}(s,t) \cdot \boldsymbol{u}_i - \boldsymbol{v} \cdot \boldsymbol{u}_i|^2 \\ &= \sum_{i=1}^{i=3} |s\boldsymbol{v}_1 \cdot \boldsymbol{u}_i + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_i - \boldsymbol{v} \cdot \boldsymbol{u}_i|^2 \\ &= |s\boldsymbol{v}_1 \cdot \boldsymbol{u}_1 + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_1 - \boldsymbol{v} \cdot \boldsymbol{u}_1|^2 + |s\boldsymbol{v}_1 \cdot \boldsymbol{u}_2 + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_2 - \boldsymbol{v} \cdot \boldsymbol{u}_2|^2 + |s\boldsymbol{v}_1 \cdot \boldsymbol{u}_3 + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_3 - \boldsymbol{v} \cdot \boldsymbol{u}_3|^2 \\ &= |s\|\boldsymbol{v}_1\| + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_1 - \boldsymbol{v} \cdot \boldsymbol{u}_1|^2 + |0 + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_2 - \boldsymbol{v} \cdot \boldsymbol{u}_2|^2 + |0 + 0 - \boldsymbol{v} \cdot \boldsymbol{u}_3|^2 \\ &= |s\|\boldsymbol{v}_1\| + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_1 - \boldsymbol{v} \cdot \boldsymbol{u}_1|^2 + |t\boldsymbol{v}_2 \cdot \boldsymbol{u}_2 - \boldsymbol{v} \cdot \boldsymbol{u}_2|^2 + |\boldsymbol{v} \cdot \boldsymbol{u}_3|^2\end{aligned}$$

Here, we use orthogonality to make a lot of the expression vanish. We see that the first term depends on both s and t, the second term depends on t only, and the final term depends on neither. To minimize this sum, we first minimize the second term and solve for t

$$t\boldsymbol{v}_2 \cdot \boldsymbol{u}_2 - \boldsymbol{v} \cdot \boldsymbol{u}_2 = 0 \iff t = \frac{\boldsymbol{v} \cdot \boldsymbol{u}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{u}_2} = \frac{\boldsymbol{v} \cdot \boldsymbol{w}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{w}_2} = \frac{1}{3}$$

Now that we know the optimal value of t, we can use this to minimize the first term as well.

$$s\left\|\boldsymbol{v}_1\right\| + t\boldsymbol{v}_2 \cdot \boldsymbol{u}_1 - \boldsymbol{v} \cdot \boldsymbol{v}_1 = 0 \iff s = \frac{\boldsymbol{v} \cdot \boldsymbol{u}_1 - t\boldsymbol{v}_2 \cdot \boldsymbol{u}_1}{\left\|\boldsymbol{v}_1\right\|} = \frac{\boldsymbol{v} \cdot \boldsymbol{v}_1 - t\boldsymbol{v}_2 \cdot \boldsymbol{v}_1}{\left\|\boldsymbol{v}_1\right\|^2} = \frac{-1 - \frac{1}{3}(2)}{5} = -\frac{1}{3}$$

Finally, we can substitute these values for s and t, getting

$$\boldsymbol{x}\left(-\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3}(-2, 1, 0) + \frac{1}{3}(-1, 0, 1) = \boxed{\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)}$$

Problem 2

a

We normalize $\boldsymbol{v}_1$ to get $\boldsymbol{u}_1$.

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{2}(1, 1, -1, -1)$$

We define

$$\begin{aligned}
\boldsymbol{w}_2 &:= \boldsymbol{v}_2 - (\boldsymbol{u}_1 \cdot \boldsymbol{v}_2)\,\boldsymbol{u}_1 \\
&= (2, 0, -2, 0) - \frac{1}{4}(4)(1, 1, -1, -1) \\
&= (1, -1, -1, 1)
\end{aligned}$$

Note that by construction, $\boldsymbol{w}_2$ is orthogonal to $\boldsymbol{u}_1$, since $\boldsymbol{w}_2 \cdot \boldsymbol{u}_1 = 0$. Normalizing $\boldsymbol{w}_2$ and defining the resultant vector to be $\boldsymbol{u}_2$, we find that

$$\boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} = \frac{1}{2}(1, -1, -1, 1)$$

Similarly, we can do the same for $\boldsymbol{v}_3$.

$$\begin{aligned}
\boldsymbol{w}_3 &:= \boldsymbol{v}_3 - (\boldsymbol{v}_3 \cdot \boldsymbol{u}_1)\,\boldsymbol{u}_1 - (\boldsymbol{v}_3 \cdot \boldsymbol{u}_2)\,\boldsymbol{u}_2 \\
&= (4, -2, -2, 0) - \frac{1}{4}(4)(1, 1, -1, -1) - \frac{1}{4}(8)(1, -1, -1, 1) \\
&= (4, -2, -2, 0) - (1, 1, -1, -1) - (2, -2, -2, 2) \\
&= (1, -1, 1, -1) \\
\boldsymbol{u}_3 &= \frac{\boldsymbol{w}_3}{\|\boldsymbol{w}_3\|} \\
&= \frac{1}{2}(1, -1, 1, -1)
\end{aligned}$$

Note that by construction, $\boldsymbol{w}_3 \cdot \boldsymbol{u}_i = 0,\ \forall i \in \{1, 2\}$. The orthonormal basis is then

$$\boxed{\left\{\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\right\}}$$

Noting that the second and fourth components of $\boldsymbol{v}_2$ are 0, we can save some time by and look for unit vectors whose sum also has this property. Doing so, we observe that $\boldsymbol{u}_1 + \boldsymbol{u}_2 = (1, 0, -1, 0)$. This vector is in the same direction as $\boldsymbol{v}_2$ and simply needs to be scaled up by a factor of 2 to construct $\boldsymbol{v}_2$.

$$\boxed{\boldsymbol{v}_2 = 2\boldsymbol{u}_1 + 2\boldsymbol{u}_2}$$

Unfortunately we have to do the old-fashioned approach to write $\boldsymbol{v}_3$.

$$\begin{aligned}
\boldsymbol{v}_3 &= a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3 \\
(4,-2,-2,0) &= \frac{a}{2}(1,1,-1,-1) + \frac{b}{2}(1,-1,-1,1) + \frac{c}{2}(1,-1,1,-1) \\
(8,-4,-4,0) &= (a+b+c, a-b-c, -a-b+c, -a+b-c) \\
&\begin{cases} 8 = a+b+c \\ -4 = a-b-c \\ -4 = -a-b+c \\ 0 = -a+b-c \end{cases}
\end{aligned}$$

Solving the system of equations, we find $a = 2, b = 4, c = 2$. Therefore,

$$\boxed{\boldsymbol{v}_3 = 2\boldsymbol{u}_1 + 4\boldsymbol{u}_2 + 2\boldsymbol{u}_3}$$

b

Let $r, s, t \in \mathbb{R}$ and let $\boldsymbol{x} \in W$. We can parametrize $\boldsymbol{x}$ with the fact that it can be written as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

$$\boldsymbol{x} = r\boldsymbol{v}_1 + s\boldsymbol{v}_2 + t\boldsymbol{v}_3$$

In order to make it easier for us, we decompose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ into the unit vectors from the previous part.

$$\boldsymbol{x} = 2r\boldsymbol{u}_1 + 2s\boldsymbol{u}_1 + 2s\boldsymbol{u}_2 + 2t\boldsymbol{u}_1 + 4t\boldsymbol{u}_2 + 2t\boldsymbol{u}_3 = (2r+2s+2t)\,\boldsymbol{u}_1 + (2s+4t)\,\boldsymbol{u}_2 + 2t\boldsymbol{u}_3$$

To use more convenient variables, we define the mapping

$$(a, b, c) \to (2r+2s+2t, 2s+4t, 2t)$$

We can now parametrize $\boldsymbol{x}$ as

$$\boldsymbol{x}\,(a,b,c) = a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3$$

We wish to minimize the quantity $\|\boldsymbol{x}\,(a,b,c) - \boldsymbol{v}_4\|$ by choosing an appropriate (a, b, c). Using the orthonormal basis we found in the previous part, we can compute

$$\begin{aligned}
\|\boldsymbol{x}\,(a,b,c) - \boldsymbol{v}_4\|^2 &= \sum_{i=1}^{i=3} |(a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3 - \boldsymbol{v}_4)\cdot\boldsymbol{u}_i|^2 \\
&= |a - \boldsymbol{v}_4\cdot\boldsymbol{u}_1|^2 + |b - \boldsymbol{v}_4\cdot\boldsymbol{u}_2|^2 + |c - \boldsymbol{v}_4\cdot\boldsymbol{u}_3|^2
\end{aligned}$$

Here, we have used the fact that $\boldsymbol{u}_i\cdot\boldsymbol{u}_j = \delta_{ij}$. ($\delta_{ij}$ is the Kronecker delta.) To minimize everything, we have

$$\begin{aligned}
a &= \boldsymbol{v}_4\cdot\boldsymbol{u}_1 = 4 \\
b &= \boldsymbol{v}_4\cdot\boldsymbol{u}_2 = 0 \\
c &= \boldsymbol{v}_4\cdot\boldsymbol{u}_3 = 2
\end{aligned}$$

Therefore, the point closest is

$$
\begin{aligned}
\boldsymbol{x} &= 4\left(\frac{1}{2}\right)(1,1,-1,-1) + 0\left(\frac{1}{2}\right)(1,-1,-1,1) + 2\left(\frac{1}{2}\right)(1,-1,1,-1) \\
&= (2,2,-2,-2) + (1,-1,1,-1) \\
&= \boxed{(3,1,-1,-3)}
\end{aligned}
$$

C

In order to do this problem efficiently, we continue the Gram-Schmidt process with $\boldsymbol{v}_4$ to add to our orthonormal subset.

$$
\begin{aligned}
\boldsymbol{w}_4 &= \boldsymbol{v}_4 - (\boldsymbol{v}_4 \cdot \boldsymbol{u}_1)\,\boldsymbol{u}_1 - (\boldsymbol{v}_4 \cdot \boldsymbol{u}_2)\,\boldsymbol{u}_2 - (\boldsymbol{v}_4 \cdot \boldsymbol{u}_3)\,\boldsymbol{u}_3 \\
&= (6,4,2,0) - (2,2,-2,-2) - 0\,(1,-1,-1,1) - (1,-1,1,-1) \\
&= (3,3,3,3) \\
\boldsymbol{u}_4 &= \frac{\boldsymbol{w}_4}{\|\boldsymbol{u}_4\|} \\
&= \frac{1}{2}(1,1,1,1)
\end{aligned}
$$

We can solve a system of equations to find $\boldsymbol{v}_4 = 4\boldsymbol{u}_1 + 2\boldsymbol{u}_3 + 6\boldsymbol{u}_4$.

By the Gram-Schmidt process,

$$\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$$

Suppose we have a point $\boldsymbol{x} \in V$. Then for some $r, s \in \mathbb{R}$, we can write

$$\boldsymbol{x}(r,s) = r\boldsymbol{u}_1 + s\boldsymbol{u}_2$$

Suppose we have another point $\boldsymbol{y} \in L$. We can write

$$\boldsymbol{y}(t) = \boldsymbol{v}_4 + t\boldsymbol{v}_3 = 4\boldsymbol{u}_1 + 2\boldsymbol{u}_3 + 6\boldsymbol{u}_4 + t\,(2\boldsymbol{u}_1 + 4\boldsymbol{u}_2 + 2\boldsymbol{u}_3)$$

We wish to find $\min \|\boldsymbol{x} - \boldsymbol{y}\|$. We have

$$
\begin{aligned}
\|\boldsymbol{x} - \boldsymbol{y}\| &= \|r\boldsymbol{u}_1 + s\boldsymbol{u}_2 - (4\boldsymbol{u}_1 + 2\boldsymbol{u}_3 + 6\boldsymbol{u}_4 + t\,(2\boldsymbol{u}_1 + 4\boldsymbol{u}_2 + 2\boldsymbol{u}_3))\| \\
&= \|\boldsymbol{u}_1\,(r - 4 - 2t) + \boldsymbol{u}_2\,(s - 4t) + \boldsymbol{u}_3\,(2t - 3) - 4\boldsymbol{u}_4\|
\end{aligned}
$$

We can minimize this quantity by first minimizing the $\boldsymbol{u}_1$-component.

$$2t - 3 = 0 \iff t = \frac{3}{2}$$

Similarly, we find $r = 7$ and $s = 6$ by minimizing the $\boldsymbol{u}_1$ and $\boldsymbol{u}_3$-components, respectively. We can then simply compute

$$\boxed{\min \|\boldsymbol{x} - \boldsymbol{y}\| = 4}$$

Similarly, we can find the points on V and L.

$$\boldsymbol{y} = 4\boldsymbol{u}_1 + 2\boldsymbol{u}_3 + 6\boldsymbol{u}_4 + 3\boldsymbol{u}_1 + 6\boldsymbol{u}_2 + 3\boldsymbol{u}_3 = 7\boldsymbol{u}_1 + 6\boldsymbol{u}_2 + 5\boldsymbol{u}_3 + 6\boldsymbol{u}_4 = \boxed{(12, 1, -1, 0)}$$

$$\boldsymbol{x} = 7\boldsymbol{u}_1 + 6\boldsymbol{u}_2 = \boxed{\left(\frac{13}{2}, \frac{1}{2}, -\frac{13}{2}, \frac{1}{2}\right)}$$