Math 291: Honors Calculus III

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- Class taught by Ian Jauslin (assistant professor)
- Recitations taught by Yael Davidov (5th year grad student)
- Textbook is *Multivariable Calculus* by Eric Carlen [1]
  - Given to us
  - Author taught class for several years
    - Last taught 2019-2020 school year
  - Last updated August 2019
- Additional resources
  - *Vector Calculus, Linear Algebra, and Differential Forms* by Hubbard and Hubbard [2]

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§1 Lecture 1: September 1

§1.1 Introduction

Course is about calculus - multivariable calculus.

Single variable calculus is assumed.

The course is very dense and has a lot of material. As such, some nonessential/easy proofs/examples will be relegated to the textbook or recitation.

§1.2 Why 3D Calc

The world is 3D. We need to generalize 1D calc for it to be applicable to a lot of things. We're gonna be working in an arbitrary number of dimensions. 3 dimensions isn't enough - we often deal with rotation, which adds 3 more angular parameters in addition to 3 for position. But we don't stop here - we continue onwards to n dimensions.

We focus less on numbers and more on the concepts - this lets us learn linear algebra, which isn't usually introduced in a typical calculus class.

§1.3 Vector Variables and Cartesian Coordinates

This section follows 1.1.2 of [1].

Definition 1.1: Functions

A function takes an ordered collection of numbers and maps it to another ordered collection of numbers.

Definition 1.2: Cartesian coordinates

Tells you where a particle is using a base point (the origin O) and a set of base vectors: $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$. We can follow each base vector a certain distance from the base point to get coordinates that specify direction. We can extend this to an arbitrary number of dimensions by increasing the number of base vectors.

The Pythagorean theorem in multiple dimensions can be used to find distance between these two points.

You can specify a geometric object by defining it as the set of points that satisfy an equation.

Example 1.1

The unit circle is defined as the set of points in 2D that satisfy $x^2 + y^2 = 1$.

§1.4 Parametrization

Definition 1.3: Parametrization

It maps complicated geometric objects to simpler objects (usually intervals) using analytic functions.

Example 1.2

$(x = \cos\theta, y = \sin\theta)$ where $\theta \in [-\pi, \pi]$ is a map that maps θ to a pair of coordinates.

$$X(\theta) = (\cos\theta, \sin\theta)$$

This is called a coordinate function.

Example 1.3

The inverse of this map:

$$\theta := \begin{cases} \arccos x & y \geq 0 \\ -\arccos x & y < 0 \end{cases}$$

The inverse cosine is only defined over the interval $\theta \in (-\pi, \pi]$.

Example 1.4

$$x^2 + y^2 = r^2$$

We assume that $r > 0$. With the scaling variable r we can transform the parametrization of the unit circle (just above) into an appropriate parametrization for a circle of an arbitray radius.

$$\theta \mapsto (r\cos\theta, r\sin\theta)$$

We can also define the inverse: the coordinate function.

$$(x, y) \mapsto \theta := \begin{cases} \arccos\left(x/\sqrt{x^2+y^2}\right) & y \geq 0 \\ -\arccos\left(x/\sqrt{x^2+y^2}\right) & y < 0 \end{cases}$$

Example 1.5

We parametrize the unit sphere.

$$x^2 + y^2 + z^2 = 1$$

We need two variables since its a circle (which is 1D - a line shaped into a circle) rotated.

$$z = \cos\phi,\ \phi \in [0, \pi]$$

This implies

$$x^2 + y^2 = \sin^2\phi$$

x and y lie on a circle of radius $\sin\phi$.
It's pretty easy to parametrize this

$$x = \sin\phi\cos\theta$$

$$y = \sin\phi\sin\theta$$

$$\theta \in [-\pi, \pi]$$

Here, ϕ is the angle between the point on the sphere and the z-axis. At the North and South poles, θ doesn't matter anymore. The parametrization function only holds on $\phi \neq 0, \pi$.

Parametrization can be used to simplify all sorts of computationally tedious problems.

Example 1.6

We compute the tangent plane to a sphere.
In 1D calc, the tangent line to curve is the best local approximation of the curve. For example, at a point x_0, $y = \cos(x_0 + t) \approx \cos(x_0) - t\sin(x_0)$.
To make this easier suppose we have a fixed point: $(x_0, y_0, z_0) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ or $\phi = \frac{\pi}{4}, \theta = \frac{\pi}{4}$.
We can parametrize the shift in the angles with $\phi = \pi/4 + s, \theta = \pi/4 + t$. For $s, t \ll 1$ (very small changes), we can write

$$\begin{aligned}
x &= \sin(\pi/4 + s)\cos(\pi/4 + t) \\
&\approx (\sin(\pi/4) + s\cos(\pi/4))(\cos(\pi/4) - t\sin(\pi/4)) \\
&= \frac{1}{2}(1 + s)(1 - t) \\
y &= \sin\left(\pi/4 + s\right)\sin\left(\pi/4 + t\right) \\
&\approx \left(\sin\pi/4 + s\cos\pi/4\right)\left(\sin\pi/4 + t\cos\pi/4\right) \\
&= \frac{1}{2}\left(1 + s\right)\left(1 + t\right) \\
z &= \cos\left(\pi/4 + s\right) \\
&\approx \frac{1}{\sqrt{2}} - s\frac{1}{\sqrt{2}}
\end{aligned}$$

Note: taking a Taylor expansion and taking a derivative have a lot of things in common.

§1.5 Vector Space $\mathbb{R}^n$

Definition 1.4: Vectors

A vector is an ordered collection of numbers in $\mathbb{R}$.

$$\boldsymbol{x} \in \mathbb{R}^m : m \in \mathbb{N}_* : \boldsymbol{x} \equiv (x_1, \ldots, x_m)$$

This means $\mathbb{R}^n$ is a vector space.

Definition 1.5: Vector Space

You can multiply a member of the space by a scalar and still be in the vector space. You can also add members of the same vector space and the resultant vector will still be in the vector space.
Distributive property: if you multiply by a scalar α the resultant vector will still be in the vector space.
Commutative property: the order in which you add vectors of a vector space doesn't matter; the sum will still be the same member of the vector space.
Associative property: the order in which you group addition doesn't matter.

The linear combination of members of a vector space is still in the vector space. Let V be an arbitrary subset of a vector space. $V \subset \mathbb{R}^n$

Definition 1.6: Span

$\operatorname{span}(V)$
The span of a subspace is all possible linear combinations of vectors in V - also a vector space.

Definition 1.7: Set Equality

If you want to prove that $A = B$, then it is sufficient to prove that both $A \subseteq B$ and $B \subseteq A$.

Theorem 1.1: Span Identity Theorem

Let $W = \operatorname{span}(V)$, where $V \subset \mathbb{R}^n$. Then $\operatorname{span}(W) = W$.

Proof. Since $W = \operatorname{span} V$, $V \subseteq W$. Therefore, $\operatorname{span} V \subseteq \operatorname{span} W$. Pick indices $i, j, k \in \mathbb{N}$ such that $i + j = k$ and scalars $s_1, \cdots, s_k \in \mathbb{R}$. In addition, we also choose vectors $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$ and $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k \in V$, where

$$\boldsymbol{w}_1 = \sum_{n=0}^{i} s_n \boldsymbol{v}_n$$
$$\boldsymbol{w}_2 = \sum_{n=i+1}^{k} s_n \boldsymbol{v}_n$$

We attempt to write another distinct vector $\boldsymbol{b}$ (a generalized member of $\operatorname{span} W$) that is a linear combination of these two vectors using scalars $t_1, t_2 \in \mathbb{R}$.

$$\boldsymbol{b} = t_1 \boldsymbol{w}_1 + t_2 \boldsymbol{w}_2 = \sum_{n=1}^{n=i} (t_1 s_n)\, \boldsymbol{v}_n + \sum_{n=i+1}^{n=k} (t_2 s_n)\, \boldsymbol{v}_n$$

However, we see that this is simply a linear combination of $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$. Therefore, $\boldsymbol{b} \in \operatorname{span} V = W$. Since $\boldsymbol{b}$ is really a formalized way to represent a member of $\operatorname{span} W$, we can see that. Therefore, $\operatorname{span} W \subseteq \operatorname{span} V$. However since we've already established that $\operatorname{span} V \subseteq \operatorname{span} W$, the only option is for $\operatorname{span} V = \operatorname{span} W$ or $\operatorname{span} W = W$.
This proves it for the case of 2 vectors. We need to generalize it to an arbitray number of vectors, after which we can complete the proof using induction. ■

Definition 1.8: Standard Bases

Standard bases are the vectors $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n \in \mathbb{R}^n$, where

$$\begin{aligned} \boldsymbol{e}_1 &= (1, 0, \cdots, 0) \\ &\vdots \\ \boldsymbol{e}_n &= (0, 0, \cdots, 1) \end{aligned}$$

It's sometimes useful to note that bases are the shortest way to write a span.

Theorem 1.2: Standard Bases

$\text{span}(e_1, \cdots, e_n) = \mathbb{R}^m$
All vectors in $\mathbb{R}^m$ are linear combinations of the standard bases

Proof. Pick some scalars $(t_1, \cdots t_m) \in \mathbb{R}$. It's easy to see that for a sufficient choice for t_i, we can create any vector $\boldsymbol{x} \in \mathbb{R}^m$.

$$\boldsymbol{x} = \sum t_i \boldsymbol{e}_i$$

■

§2 Recitation 1: September 2

Given a subset V, $\operatorname{span} V$ is intuitively the set of linear combinations of the vectors in V.
$\boldsymbol{x} \in \operatorname{span} V \leftrightarrow \boldsymbol{x} = \sum_i t_i \boldsymbol{v}_i$ where $t_i \in \mathbb{R}$ and $\boldsymbol{v}_i \in V$.

Example 2.1

Suppose $V = \{(0,0,0)\}$. Then $\operatorname{span} V$ only has one element since you can't make anything but $\mathbf{0}$ out of the zero-vector.

Example 2.2

Let $V = \{(1,0,0)\}$.

$$\operatorname{span} V = \{(t,0,0) : t \in \mathbb{R}\}$$

Example 2.3

Let $\boldsymbol{v}_1 = (1,2,-3)$ and $\boldsymbol{v}_2 = (1,-2,1)$. Can we find an equation that is satisfied by a vector $\boldsymbol{v} = (x,y,z)$ if and only if $\boldsymbol{v} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$?
Letting $s,t \in \mathbb{R}$,

$$\begin{aligned}
&\boldsymbol{v} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \\
&\boldsymbol{v} = s\boldsymbol{v}_1 + t\boldsymbol{v}_2 \\
&\quad = s(1,2,-3) + t(1,-2,1) \\
&\quad = (s+t, 2s-2t, -3s+t) \\
&x = s+t \\
&y = 2s-2t \\
&z = -3s+t
\end{aligned}$$

We solve the equations for x and y to see that

$$\begin{aligned}
s &= \frac{1}{4}(2x+y) \\
t &= \frac{1}{4}(2x+y)
\end{aligned}$$

Resubstituting into the equation for z, we see that

$$\begin{aligned}
z &= -x-y \\
x+y+z &= 0
\end{aligned}$$

So $\boldsymbol{v} = (x,y,z) \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \leftrightarrow z+x+y = 0$

Example 2.4

Let $\boldsymbol{v}_3 = (-2, 1, 1)$. Can you describe $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$? (vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ from previous example)?

Example 2.5

Let $\boldsymbol{u} = (1, 2), \boldsymbol{v} = (-2, -4), \boldsymbol{w} = (7, 14)$ be three vectors in $\mathbb{R}^2$. Show that $\boldsymbol{w} \in \operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$.
We can write $\boldsymbol{w} = s\boldsymbol{u} + t\boldsymbol{v}$. There are many such pairs of (s, t), which are computationally trivial to solve for.
Another, quicker, approach is to see that $\boldsymbol{u} \in \operatorname{span} \boldsymbol{v}$ and vice versa. Therefore, $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \operatorname{span} \boldsymbol{v}$.

Example 2.6

Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be orthogonal unit vectors. Define

$$\boldsymbol{v} := a\boldsymbol{u}_1 + b\boldsymbol{u}_2$$

Find $\boldsymbol{v} \cdot \boldsymbol{u}_1$.

$$\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{u}_1 &= a\boldsymbol{u}_1 \cdot \boldsymbol{u}_1 + b\boldsymbol{u}_2 \cdot \boldsymbol{u}_1 \\
&= a + b \cdot 0 \\
&= a
\end{aligned}$$

§3 Lecture 2: September 8

dot products!

§4 Recitation 2: September 9

§5 Lecture 3: September 13

§6 Lecture 4: September 15

§7 Recitation 3: September 16

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§40 Lecture 27: December 8

§41 Recitation 14: December 9

§42 Lecture 28: December 13

References

[1] Eric A. Carlen. *Multivariable Calculus*. Aug. 2019.

[2] John H. Hubbard and Barbara Burke Hubbard. *Vector Calculus, Linear Algebra, and Differential Equations: A Unified Approach*. Matrix Editions, 2015. ISBN: 9780971576681.