Math 291H Homework #1

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Honors Pledge Statement: "The writeup of this submission is my own work alone."

Problem 1.1

$$
\begin{aligned}
(3,-1) &= s\,(2,1)+t\,(1,3)\\
(3,-1) &= (2s+t, s+3t)
\end{aligned}
$$

We turn this into a system of equations.

$$
\begin{cases}
2s+t=3\\
s+3t=-1
\end{cases}
$$

Solving, we find that $\boxed{s=2, t=-1}$.

Problem 1.4

$$\|\boldsymbol{x}\| = \sqrt{4^2 + 7^2 + (-4)^2 + 1^2 + 2^2 + (-2)^2} = \boxed{\sqrt{15}}$$

$$\|\boldsymbol{y}\| = \sqrt{2^2 + 1^2 + 2^2 + 2^2 + (-1)^2 + (-1)^2} = \boxed{\sqrt{11}}$$

$$\begin{aligned}
\cos\theta &= \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|} \\
&= \frac{4 \cdot 2 + 7 \cdot 1 + (-4) \cdot 2 + 1 \cdot (-1) + (-2) \cdot (-1)}{\sqrt{15} \cdot \sqrt{11}} \\
&= \frac{8}{\sqrt{165}} \\
&= \frac{8\sqrt{165}}{165} \\
\theta &= \boxed{\arccos \frac{8\sqrt{165}}{165}}
\end{aligned}$$

Problem 1.5

$$\|\boldsymbol{x}\| = \sqrt{4^2 + 7^2 + 4^2} = \boxed{9}$$

$$\|\boldsymbol{y}\| = \sqrt{2^2 + 1^2 + 2^2} = \boxed{3}$$

$$\begin{aligned}\cos\theta &= \frac{4 \cdot 2 + 7 \cdot 1 + 4 \cdot 2}{9 \cdot 3} \\ &= \frac{23}{27} \\ \theta &= \boxed{\arccos\frac{23}{27}}\end{aligned}$$

Problem 1.7

a

$$
\begin{aligned}
\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 &= \frac{1}{81}\left(1 \cdot 8 + (-4) \cdot 4 + (-8) \cdot (-1)\right) = 0 \\
\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 &= \frac{1}{81}\left(8 \cdot 4 + 4 \cdot (-7) + (-1) \cdot 4\right) = 0 \\
\boldsymbol{u}_1 \cdot \boldsymbol{u}_3 &= \frac{1}{81}\left(1 \cdot 4 + (-4) \cdot (-7) + (-8) \cdot 4\right) = 0
\end{aligned}
$$

Therefore, $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ *is* an orthonormal basis of $\mathbb{R}^3$.

$$
\begin{aligned}
\boldsymbol{u}_1 \times \boldsymbol{u}_2 &= \frac{1}{81}\left(1, -4, -8\right) \times \left(8, 4, -1\right) \\
&= \frac{1}{81}\left(36, -63, 36\right) \\
&= \frac{1}{9}\left(4, -7, 4\right) \\
&= \boldsymbol{u}_3
\end{aligned}
$$

Since $\boldsymbol{u}_1 \times \boldsymbol{u}_2 = \boldsymbol{u}_3$, $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a *right-handed* orthonormal basis of $\mathbb{R}^3$.

b

$$
\begin{aligned}
y_1\boldsymbol{u}_1 + y_2\boldsymbol{u}_2 + y_3\boldsymbol{u}_3 &= (10, 11, -11) \\
y_1\left(1, -4, -8\right) + y_2\left(8, 4, -1\right) + y_3\left(4, -7, 4\right) &= (90, 99, -99) \\
\left(y_1 + 8y_2 + 4y_3, -4y_1 + 4y_2 - 7y_3, -8y_1 - y_2 + 4y_3\right) &= (90, 99, -99)
\end{aligned}
$$

We can solve this system to find $\boxed{y_1 = 6, y_2 = 15, y_3 = -9}$.

$$
\begin{aligned}
\|(y_1, y_2, y_3)\| &= \sqrt{6^2 + 15^2 + (-9)^2} \\
&= \boxed{3\sqrt{38}} \\
\|(10, 11, -11)\| &= \sqrt{10^2 + 11^2 + (-11)^2} \\
&= \boxed{3\sqrt{38}}
\end{aligned}
$$

Problem 1.14

a

Let $\boldsymbol{v}_1$ be the vector that passes through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Let $\boldsymbol{v}_2$ be the vector that passes through $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$. Let $\boldsymbol{v}_3$ be the vector that passes through $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. Let $\boldsymbol{v}_4$ be the vector that passes through $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$.

$$\begin{aligned}
\boldsymbol{v}_1 &= (-2, 0, -4)\\
\boldsymbol{v}_2 &= (3, -5, 1)\\
\boldsymbol{v}_3 &= (0, -1, 1)\\
\boldsymbol{v}_4 &= (-1, 1, 0)
\end{aligned}$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2$ lie in the plane P_1, their cross product $\boldsymbol{n}_1$ is perpendicular to P_1. Likewise for $\boldsymbol{v}_3, \boldsymbol{v}_4, P_2,$ and $\boldsymbol{n}_2$, respectively.

$$\begin{aligned}
\boldsymbol{n}_1 &= (20, -10, 10)\\
\boldsymbol{n}_2 &= (-1, -1, -1)
\end{aligned}$$

We substitute into the standard form equation for a plane:

$$\begin{aligned}
P_1 &: \boldsymbol{n}_1 \cdot \boldsymbol{r} + d_1 = 0\\
P_2 &: \boldsymbol{n}_2 \cdot \boldsymbol{r} + d_2 = 0
\end{aligned}$$

Substituting $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$, respectively, we find that $d_1 = -10$ and $d_2 = 3$. After simplifying,

$$\begin{aligned}
P_1 &: \boxed{2x - y + z - 1 = 0}\\
P_2 &: \boxed{x + y + z - 3 = 0}
\end{aligned}$$

b

Adding and subtracting the two equations, respectively

$$\begin{cases}
x - 2y + 2 = 0 \iff y = \dfrac{1}{2}x + 1\\
3x + 2z - 4 = 0 \iff z = -\dfrac{3}{2}x + 2
\end{cases}$$

Letting $t \in \mathbb{R}$, we can write

$$\boldsymbol{x}(t) = \left(t, \frac{1}{2}t + 1, -\frac{3}{2}t + 2\right)$$
$$\boldsymbol{x}(t) = \boxed{(0, 1, 2) + t\left(0, \frac{1}{2}, -\frac{3}{2}\right)}$$

This is of the form $\boldsymbol{x}(t) = \boldsymbol{x}_0 + t\boldsymbol{v}$. To find the distance, we must first normalize $\boldsymbol{v}$.

$$\begin{aligned}\boldsymbol{u} &= \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \\ &= \frac{\sqrt{10}}{5}\left(0, \frac{1}{2}, -\frac{3}{2}\right)\end{aligned}$$

Suppose the point on the line closest to $\boldsymbol{a}_1$ is $\boldsymbol{p}$. The shortest distance is then

$$\begin{aligned}\|\boldsymbol{p} - \boldsymbol{a}_1\|^2 &= \|\boldsymbol{x}_0 - \boldsymbol{a}_1\|^2 - \|(\boldsymbol{x}_0 - \boldsymbol{a}_1) \cdot \boldsymbol{u}\|^2 \\ &= \|(0,1,2) - (1,2,1)\|^2 - \frac{10}{25}\left|((0,1,2) - (1,2,1)) \cdot \left(0, \frac{1}{2}, -\frac{3}{2}\right)\right|^2 \\ &= \|(-1,-1,1)\|^2 - \frac{2}{5}\left|(-1,-1,1) \cdot \left(0, \frac{1}{2}, -\frac{3}{2}\right)\right| \\ &= 3 - \frac{2}{5}(2) \\ &= \frac{13}{5} \\ \|\boldsymbol{p} - \boldsymbol{a}_1\| &= \boxed{\sqrt{\frac{13}{5}}}\end{aligned}$$

C

$$\boldsymbol{x} = \boldsymbol{b}_1 + t\boldsymbol{a}$$

Suppose the line is at $\boldsymbol{b}_1$ when $t = 0$. In addition, at $t = 1$, suppose the line is at $\boldsymbol{b}_2$. Then, $\boldsymbol{a} = \boldsymbol{b}_2 - \boldsymbol{b}_1 = (0, -1, 1)$. Therefore,

$$\boldsymbol{x} = (1, 1, 0) + t(0, -1, 1)$$

The vector equation of the line is therefore

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{x} - \boldsymbol{b}_1) &= 0 \\ (0, -1, 1) \times (\boldsymbol{x} - (1, 1, 0)) &= 0 \\ (0, -1, 1) \times (x - 1, y - 1, z) &= 0 \\ (-y - z + 1, x - 1, x - 1) &= 0 \end{aligned}$$

Clearly, $x = 1$ from the $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ components of this equation. In addition, $y + z = 1$ from the $\boldsymbol{e}_1$ component. The equation for P_1 is $2x - y + z = 1$. Substituting $x = 1$, we have

$$\begin{cases} y + z = 1 \\ y - z = 1 \end{cases}$$

Solving, we find $y = 1$ and $z = 0$. Finally, the point of intersection is $\boxed{(1, 1, 0)}$.

Problem 1.15

Let the orthonormal basis be composed of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. We let

$$\boldsymbol{c} := \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{1}{\sqrt{26}}\,(1, 4, 3)$$

In addition, we define

$$\boldsymbol{w} := (-4, 1, 0)$$

Note that $\boldsymbol{w}$ and $\boldsymbol{c}$ are orthogonal. We normalize and let the resultant unit vector be $\boldsymbol{a}$.

$$\boldsymbol{a} := \frac{1}{\sqrt{17}}\,(-4, 1, 0)$$

To make $\boldsymbol{b}$ orthogonal to the other two vectors, we can compute the final vector:

$$\begin{aligned}
\boldsymbol{y} &:= \boldsymbol{v} \times \boldsymbol{w} \\
&= (-3, -12, 15) \\
\boldsymbol{b} &= \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \\
\boldsymbol{b} &= \frac{1}{\sqrt{378}}\,(-3, -12, 15)
\end{aligned}$$

Problem 1.16

Let the orthonormal basis be $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$\begin{aligned}
\boldsymbol{u}_1 &= \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \\
&= \frac{1}{\sqrt{26}}\,(1,4,3) \\
\boldsymbol{u}_2 &= \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|} \\
&= \frac{1}{\sqrt{14}}\,(3,2,1) \\
\boldsymbol{c} &= \boldsymbol{a} \times \boldsymbol{b} \\
&= (-2,8,-10) \\
\boldsymbol{u}_3 &= \frac{\boldsymbol{c}}{\|\boldsymbol{c}\|} \\
&= \frac{1}{\sqrt{168}}\,(-2,8,-10)
\end{aligned}$$

Problem 1.17

a

Let $(s,t) = (0,0), (0,1), (1,0)$ correspond to $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$, respectively. We see that $\boldsymbol{x}_0 = \boldsymbol{p}_1 = (-2,0,2)$. Further, we also see that

$$\boldsymbol{v}_1 = \boldsymbol{p}_2 - \boldsymbol{p}_1 = (3,-2,0)$$

and

$$\boldsymbol{v}_2 = \boldsymbol{p}_3 - \boldsymbol{p}_1 = (5,-1,-4)$$

$$\boldsymbol{x}(s,t) = (-2,0,2) + s(3,-2,0) + t(5,-1,-4)$$

b

Let $u = 0$ at $\boldsymbol{x}_0$, so $\boldsymbol{z}_0 = \boldsymbol{x}_0 = \boxed{(1,4,-2)}$. Letting $u = 1$ at $\boldsymbol{x}_1$ lets us see that $\boldsymbol{w} = \boldsymbol{z}_1 - \boldsymbol{z}_0 = \boxed{(-1,-7,3)}$.

c

We can compute the normal vector by

$$\begin{aligned}\boldsymbol{n} &= \boldsymbol{v}_1 \times \boldsymbol{v}_2 \\ &= (3,-2,0) \times (5,-1,-4) \\ &= (8,12,7)\end{aligned}$$

We know that $\boldsymbol{n} \cdot \boldsymbol{x} + d = 0$ is the general vector equation for the plane. Substituting $\boldsymbol{x} = \boldsymbol{p}_1$, we see that $d = 2$. We can then expand,

$$\boxed{8x + 12y + 7z + 2 = 0}$$

d

The general vector equation for a line is

$$\begin{aligned}\boldsymbol{w} \times (\boldsymbol{z} - \boldsymbol{z}_0) &= 0 \\ (-1,-7,3) \times ((x,y,z) - (1,4,-2)) &= 0 \\ (-1,-7,3) \times (x-1, y-4, z+2) &= 0 \\ (-2-3y-7z, -1+3x+z, -3+7x-y) &= 0\end{aligned}$$

We therefore have the system,

$$\begin{cases} 3y + 7z = -2 \\ 3x + z = 1 \\ 7x - y = 3 \end{cases}$$

e

Solving for y in the last equation,

$$y = 7x - 3$$

Substituting into the equation for the plane,

$$\begin{cases} 92x + 7z + 2 = 0 \\ 3x + z = 1 \end{cases}$$

Solving and resubstituting, we find $\boxed{\left(\frac{24}{71}, -\frac{24}{71}, -\frac{10}{71}\right)}$.

f

We must first normalize $\boldsymbol{w}$.

$$\begin{aligned} \boldsymbol{u} &= \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} \\ &= \frac{1}{\sqrt{59}}\,(-1, -7, 3) \end{aligned}$$

Suppose the point on the line closest to $\boldsymbol{p}_1$ is $\boldsymbol{q}$.

$$\begin{aligned} \|\boldsymbol{p}_1 - \boldsymbol{q}\| &= \|(\boldsymbol{x}_0 - \boldsymbol{p}_1) \times \boldsymbol{u}\| \\ &= \frac{1}{\sqrt{59}}\,\|((1, 4, -2) - (-1, -3, 0)) \times (-1, -7, 3)\| \\ &= \frac{1}{\sqrt{59}}\,\|(2, 7, -2) \times (-1, -7, 3)\| \\ &= \frac{1}{\sqrt{59}}\,\|(7, -4, -7)\| \\ &= \boxed{\sqrt{\frac{114}{59}}} \end{aligned}$$

g

We must first normalize $\boldsymbol{n}$.

$$\begin{aligned} \boldsymbol{u} &= \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} \\ &= \frac{1}{\sqrt{257}}\,(8, 12, 7) \end{aligned}$$

The distance is then

$$\begin{aligned}
|(\boldsymbol{x}_0 - \boldsymbol{z}_0) \cdot \boldsymbol{u}| &= \frac{1}{\sqrt{257}} \left|(-2,0,2) - (1,4,-2) \cdot (8,12,7)\right| \\
&= \frac{1}{\sqrt{257}} \left|(-3,-4,4) \cdot (8,12,7)\right| \\
&= \frac{1}{\sqrt{257}} |-44| \\
&= \boxed{\frac{44}{\sqrt{257}}}
\end{aligned}$$