Honors Pledge Statement: "The writeup of this submission is my own work alone".

## 1.1

$$(3, -1) = s\,(2, 1) + t\,(1, 3)$$
$$(3, -1) = (2s + t, s + 3t)$$

We turn this into a system of equations.

$$\begin{cases} 2s + t = 3 \\ s + 3t = -1 \end{cases}$$

Solving, we find that $\boxed{s = 2, t = -1}$.

## 1.4

$$\|\boldsymbol{x}\| = \sqrt{4^2 + 7^2 + (-4)^2 + 1^2 + 2^2 + (-2)^2} = \boxed{3\sqrt{10}}$$

$$\|\boldsymbol{y}\| = \sqrt{2^2 + 1^2 + 2^2 + 2^2 + (-1)^2 + (-1)^2} = \boxed{\sqrt{11}}$$

$$\begin{aligned} \cos\theta &= \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|} \\ &= \frac{4 \cdot 2 + 7 \cdot 1 + (-4) \cdot 2 + 1 \cdot (-1) + (-2) \cdot (-1)}{3\sqrt{10} \cdot \sqrt{11}} \\ &= \frac{8}{3\sqrt{110}} \\ &= \frac{4\sqrt{110}}{165} \\ \theta &\approx \boxed{1.314} \end{aligned}$$

## 1.5

$$\|\boldsymbol{x}\| = \sqrt{4^2 + 7^2 + 4^2} = \boxed{9}$$

$$\|\boldsymbol{y}\| = \sqrt{2^2 + 1^2 + 2^2} = \boxed{3}$$

$$\begin{aligned} \cos\theta &= \frac{4 \cdot 2 + 7 \cdot 1 + 4 \cdot 2}{9 \cdot 3} \\ &= \frac{23}{27} \\ \theta &\approx \boxed{0.551} \end{aligned}$$

## 1.7

### a

$$
\begin{aligned}
\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 &= \frac{1}{81}\left(1 \cdot 8 + (-4) \cdot 4 + (-8) \cdot (-1)\right) = 0 \\
\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 &= \frac{1}{81}\left(8 \cdot 4 + 4 \cdot (-7) + (-1) \cdot 4\right) = 0 \\
\boldsymbol{u}_1 \cdot \boldsymbol{u}_3 &= \frac{1}{81}\left(1 \cdot 4 + (-4) \cdot (-7) + (-8) \cdot 4\right) = 0
\end{aligned}
$$

Therefore, $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

$$
\begin{aligned}
\boldsymbol{u}_1 \times \boldsymbol{u}_2 &= \frac{1}{81}(1, -4, -8) \times (8, 4, -1) \\
&= \frac{1}{81}(36, -63, 36) \\
&= \frac{1}{9}(4, -7, 4) \\
&= \boldsymbol{u}_3
\end{aligned}
$$

Since $\boldsymbol{u}_1 \times \boldsymbol{u}_2 = \boldsymbol{u}_3$, $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a right-handed orthonormal basis of $\mathbb{R}^3$.

### b

$$
\begin{aligned}
y_1\boldsymbol{u}_1 + y_2\boldsymbol{u}_2 + y_3\boldsymbol{u}_3 &= (10, 11, -11) \\
y_1(1, -4, -8) + y_2(8, 4, -1) + y_3(4, -7, 4) &= (90, 99, -99) \\
(y_1 + 8y_2 + 4y_3, -4y_1 + 4y_2 - 7y_3, -8y_1 - y_2 + 4y_3) &= (90, 99, -99)
\end{aligned}
$$

We can solve this system to find $\boxed{y_1 = 6, y_2 = 15, y_3 = -9}$.

$$
\begin{aligned}
\|(y_1, y_2, y_3)\| &= \sqrt{6^2 + 15^2 + (-9)^2} \\
&= \boxed{3\sqrt{38}} \\
\|(10, 11, -11)\| &= \sqrt{10^2 + 11^2 + (-11)^2} \\
&= \boxed{3\sqrt{38}}
\end{aligned}
$$

## 1.14

### a

Let $\boldsymbol{v}_1$ be the vector that passes through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Let $\boldsymbol{v}_2$ be the vector that passes through $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$. Let $\boldsymbol{v}_3$ be the vector that passes through $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. Let $\boldsymbol{v}_4$ be the vector that passes through $\boldsymbol{b}_2$ and $\boldsymbol{b}_3$.

$$\begin{aligned}
\boldsymbol{v}_1 &= (-2, 0, -4) \\
\boldsymbol{v}_2 &= (3, -5, 1) \\
\boldsymbol{v}_3 &= (0, -1, 1) \\
\boldsymbol{v}_4 &= (-1, 1, 0)
\end{aligned}$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2$ lie in the plane $P_1$, their cross product $\boldsymbol{n}_1$ is perpendicular to $P_1$. Likewise for $\boldsymbol{v}_3, \boldsymbol{v}_4, P_2, \boldsymbol{n}_2$, respectively.

$$\begin{aligned}
\boldsymbol{n}_1 &= (20, -10, 10) \\
\boldsymbol{n}_2 &= (1, 1, -1)
\end{aligned}$$

We substitute into the standard form equation for a plane:

$$\begin{aligned}
P_1 &: \boldsymbol{n}_1 \cdot \boldsymbol{r} + d_1 = 0 \\
P_2 &: \boldsymbol{n}_2 \cdot \boldsymbol{r} + d_2 = 0
\end{aligned}$$

Substituting $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$, respectively, we find that $d_1 = -10$ and $d_2 = -2$. After simplifying $P_1$,

$$\begin{aligned}
P_1 &: \boxed{2x - y + z - 1 = 0} \\
P_2 &: \boxed{x + y - z - 2 = 0}
\end{aligned}$$

### b

Adding the equations together, we see that the lines intersect at $x = 3$. Substituting this back, we find

$$z = y - 1$$

Putting these results back into vector notation,

$$\boldsymbol{r} = (3, y, y - 1) = (3, 0, -1) + y\,(1, 1, 1)$$

Parametrizing using $t \in \mathbb{R}$,

$$\boxed{\boldsymbol{r}(t) = (3,0,-1) + t(1,1,1)\,,\ t \in \mathbb{R}}$$

The distance between this line and $\boldsymbol{a}_1$ is

$$\begin{aligned}
\sqrt{\|(3,0,-1)-(1,2,1)\|^2 - \left\|((3,0,-1)-(1,2,1)) \times \frac{1}{\sqrt{3}}(1,1,1)\right\|^2} &= \sqrt{\|(2,-2,-2)\|^2 - \frac{1}{3}\|(3,-2,-2)\times(1,1,1)\|^2} \\
&= \sqrt{\left(2\sqrt{3}\right)^2 - \frac{1}{3}\|(0,5,5)\|^2} \\
&= \sqrt{12 - \frac{1}{3}(50)} \\
&= \sqrt{12 - \frac{50}{3}}
\end{aligned}$$

Something went wrong here...

### c

$$\boldsymbol{r} = \boldsymbol{b}_1 + t\boldsymbol{a}$$

Suppose the line is at $\boldsymbol{b}_1$ when $t = 0$. In addition, at $t = 1$, suppose the line is at $\boldsymbol{b}_2$. Then, $\boldsymbol{a} = \boldsymbol{b}_2 - \boldsymbol{b}_1 = (0,-1,1)$. Therefore,

$$\boldsymbol{r} = (1,1,0) + t(0,-1,1)$$

Recall that a line in $\mathbb{R}^3$ is simply the intersection of two nonparallel planes.

## 1.15

Let the orthonormal basis be composed of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. We let

$$\boldsymbol{c} := \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{1}{\sqrt{26}}(1,4,3)$$

In addition, we define

$$\boldsymbol{w} := (-4,1,0)$$

Note that $\boldsymbol{w}$ and $\boldsymbol{c}$ are orthogonal. We normalize and let that be $\boldsymbol{a}$.

$$\boldsymbol{a} := \frac{1}{\sqrt{17}}(-4,1,0)$$

To make $\boldsymbol{b}$ orthogonal to the other two vectors, we can compute the final vector:

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{v} \times \boldsymbol{w} \\
&= (-3, -12, 15) \\
\boldsymbol{b} &= \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \\
\boldsymbol{b} &= \frac{1}{\sqrt{378}} (-3, -12, 15)
\end{aligned}
$$

# 1.16

Let the orthonormal basis be $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$
\begin{aligned}
\boldsymbol{u}_1 &= \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \\
&= \frac{1}{\sqrt{26}} (1, 4, 3) \\
\boldsymbol{u}_2 &= \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|} \\
&= \frac{1}{\sqrt{14}} (3, 2, 1) \\
\boldsymbol{c} &= \boldsymbol{a} \times \boldsymbol{b} \\
&= (-2, 8, -10) \\
\boldsymbol{u}_3 &= \frac{\boldsymbol{c}}{\|\boldsymbol{c}\|} \\
&= \frac{1}{\sqrt{168}} (-2, 8, -10)
\end{aligned}
$$

# 1.17

## a

Let $(s, t) = (0, 0), (0, 1), (1, 0)$ correspond to $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$, respectively. We see that $\boldsymbol{x}_0 = \boldsymbol{p}_1 = (-2, 0, 2)$. Further, we also see that

$$\boldsymbol{v}_1 = \boldsymbol{p}_2 - \boldsymbol{p}_1 = (3, -2, 0)$$

and

$$\boldsymbol{v}_2 = \boldsymbol{p}_3 - \boldsymbol{p}_1 = (5, -1, -4)$$

$$\boldsymbol{x}(s, t) = (-2, 0, 2) + s (3, -2, 0) + t (5, -1, -4)$$

## b

Let $t = 0$ at $\boldsymbol{x}_0$, so $\boldsymbol{z}_0 = \boldsymbol{x}_0$.