$\mathcal{P}$1

Let $A = \{1, 2\}, B = \{x, y\}$. List all functions from $A \to B$ that are:

$\mathcal{P}$1.a

injections;

By the definition of an injection, each element of B may have at most 1 element of A that maps to it. The possible functions are then,

$$\{(1, x), (2, y)\}$$
$$\{(2, x), (1, y)\}$$

$\mathcal{P}$1.b

surjections;

By the definition of a surjection, each element of B must be mapped to by an element of A at least once. The possible functions are then,

$$\{(1, x), (2, y)\}$$
$$\{(2, x), (1, y)\}$$

$\mathcal{P}$1.c

bijections;

A function is only a bijection if it is both an injection and a surjection. The possible functions are then,

$$\{(1, x), (2, y)\}$$
$$\{(2, x), (1, y)\}$$

$\mathcal{P}$1.d

none of the above.

The other functions that exist to map these two sets are

$$\{(1, x), (2, x)\}$$
$$\{(1, y), (2, y)\}$$

$\mathcal{P}$2

$\mathcal{P}$2.a

Supply a proof for Proposition 7.8.

Proposition 7.8. If $f : A \to B$ and $g : B \to C$ are both injective, then their composition $g \circ f : A \to C$ is injective.

Consider two arbitrary elements $x, y \in A$ such that $g(f(x)) = g(f(y))$. Since g is an injection, this means $f(x) = f(y)$. Similarly, since f is also an injection, this means $x = y$. This means that $g \circ f$ is also an injection.

$\mathcal{P}$2.b

Suppose $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective, is it necessarily the case that f is injective? Is it necessarily the case that g is injective? Prove or disprove your claims.

We show that f is necessarily injective. Pick some arbitrary $x, y \in A$ such that $f(x) = f(y)$. This is equivalent to $g(f(x)) = g(f(x))$. Since $g \circ f$ is injective, this implies that $x = y$. Therefore, f is injective.

Note that g doesn't need to be injective we can't go from $g(x) = g(y)$ to $g(f(x)) = g(f(y))$ without changing sets.

$\mathcal{P}$2.c

Repeat part (b) but replace injective with surjective.

We show that g is necessarily surjective. Choose $z \in C$. We know that there exists $x \in A$ such that $g \circ f(x) = z$ since $g \circ f$ is surjective. Define $y := f(x)$. Then there exists a $y \in B$ such that $g(y) = z$, so g is surjective.

We can't show the same for f since it is already inside another function, so we can't make the same argument.

$\mathcal{P}$3

Find a function from $\mathbb{R}$ to $\mathbb{R}$, and supply a proof of your claim, that is:

$\mathcal{P}$3.a

an injection but not a surjection;

Consider $f(x) = e^x$. It is an injection because each x-value results in a different output. It is not a surjection because non-positive real numbers aren't achieved no matter what the value of x is.

$\mathcal{P}$3.b

a surjection but not an injection;

Consider $f(x) = x \sin x$. This function is a surjection because all real numbers are achieved given a suitable real number x. However, this function is not an injection because there are $f(x)$-values that are produced by more than one x-value.

$\mathcal{P}$3.c

a bijection;

Consider $f(x) = x$. This function is an injection because each value of $f(x)$ is produced by a distinct x-value. Similarly, this function is a surjection because all real numbers are achieved given a suitable real number x. Since the function is both an injection and a surjection, it is a bijection.

$\mathcal{P}$3.d

neither a surjection nor an injection.

Consider $f(x) = x^2$. This function is not an injection because multiple x values can map to each $f(x)$-value. This function is not a surjection because the negative real numbers cannot be achieved for any real x.

$\mathcal{P}$4

Let $f : A \to B$, and let $G_1, G_2 \subseteq A$, and let $H_1, H_2 \subseteq B$.

$\mathcal{P}$4.a

Is it true that $f^{-1}(f(G_1)) = G_1$. If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.

The image of G_1, $f(G_1)$ will be some subset of B, which we denote by B_1. Unfortunately, we don't know whether or not the function is injective, so the pre-image $f^{-1}(B_1)$ may contain extra elements. For a counterexample, consider the function $f\left(x^2\right)$ over the subset $[-2, -1]$. The final set that is returned will be $[-2, -1] \cup [1, 2]$.

The correct relation is $f^{-1}(f(G_1)) \subseteq G_1$.

$\mathcal{P}$4.b

Is it true that $f\left(f^{-1}(H_1)\right) = H_1$. If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.

Let the pre-image of H_1 be A_1. Since f is a function, by definition, each element of A_1 is mapped to its appropriate element of H_1. Therefore, $f(A_1) = H_1$, proving this statement to be true.

$\mathcal{P}$4.c

As a counterexample, we choose $f(x) = x^2$ and the intervals $[-1, 0]$ and $[0, 1]$.