Foreword

Mathematical rigor is not intuitive. The human brain makes leaps and strides constantly, and must do so to get anywhere. But in order to do mathematics properly, we must first slow this process to a halt, analyze everything atomically, and slowly build the intuition to do things right.

Math 300 should only serve to give you an idea of what this looks like. You must take additional math courses such as 311 (analysis) and 351 (algebra) to learn how to write proofs. Further courses will serve to strengthen your ability to rigorously analyze statements and determine potential inaccuracies.

This course is difficult and I ask that you spend the time to learn it properly, as it will help you greatly wherever you go.

Formally Speaking

Definition 1.1. *A* ***proposition*** *is a statement which is either true or false.*
Examples: I am a dog. Sometimes there are clouds in the sky. The Riemann Hypothesis is true.
Nonexamples: Blue. Apricot. Code.pdf.

Note 1.2. *Whenever you see a definition, it is usually helpful to construct both a nontrivial example and a nontrivial nonexample.*

Definition 1.3. *A* ***boolean function*** *is a proposition whose truthiness may depend on its inputs.*
Example: D(x) := x is a dog. L(x) := x loves dogs. F(x, y) := x considers y a friend.

Note 1.4. *While not necessary, explicitly writing := shows that we are defining something.*

Definition 1.5. *Conjunction: Logical and, written as* $\land$ `(\land)`. *Disjunction: Logical or, written as* $\lor$ `(\lor)`.

Note 1.6. *I will periodically introduce LaTeX notation by including the LaTeX command next to the symbol.*

Note 1.7. *Disjunction does not work like it does in English. In English, "or" tends to mean exactly one of the presented options (ex. "get me a spoon or a fork"). In logic, it means at least one of the presented options.*

Conjunction and disjunction alone are fairly powerful. I can define some simple boolean functions, and make more complex boolean functions by composing them. For example, using the functions defined above, I could write something like $G(x, y) := L(x) \land L(y) \land F(x, y) \land F(y, x)$. Intuitively, G is a function that tells me if two people are dog lovers and also friends.

However, I might want to make statements that are true of anyone, or more specifically, of any element of a set. For now, we will say a set is simply a collection of elements and leave it at that.

Definition 1.8. $\forall$ *is the* ***universal quantifier****. There is only one way to use it:* ***For all*** *elements x in a set X, $F(x)$ is true. The shorthand is simply* $\forall x \in X, F(x)$. `(\forall x \in X, F(x))`

Note 1.9. *In general, when trying to make an English statement rigorous, you will have to think (perhaps quite a while) on which variables and functions you might need in order to write it.*

Let's say we have the English statement: Bob considers everyone in his class a friend. We are going to have to make use of the universal quantifier, and there is only one way to use it. We'll define $x :=$ Bob, C as Bob's class, and use the function F from before.

Then our rigorous statement is simply $\forall y \in C, F(x, y)$.

Definition 1.10. $\exists$ *is the* ***existential quantifier****. There is only one way to use it: There is* ***at least one*** *x in a set X, $F(x)$ is true. The shorthand is simply* $\exists x \in X, F(x)$. `(\exists)`

We'll take another English statement: Among all animals on Earth, at least one is a dog. Let's define A as the set of all animals on Earth, and then our statement is simply $\exists x \in A : D(x)$.

Note 1.11. *The symbol : should be read as "such that."*

I will point out now that a boolean function itself can make use of quantifiers. Did someone say recursion?

There is a person in the class who considers everyone to be a friend : $\exists x \in C : \forall y \in CF(x, y)$.
There is a person in the class who everyone considers to be a friend : $\exists x \in C : \forall y \in CF(y, x)$.
Everyone has at least one person they consider a friend : $\forall x \in C : \exists y \in CF(x, y)$.
Everyone considers everyone a friend: $\forall x, y \in C, F(x, y)$.
Everyone has at least *two* people who they consider to be friends : $\forall x \in C, \exists y, z \in C : F(x, y) \wedge F(x, z) \wedge y \neq z$. (`\neq`)

Note 1.12. *I suppose I lied earlier when I said there is only one way. It is kosher to combine adjacent quantifiers, so long as they both use the same set, as done above with two variables both being in C.*

Definition 1.13. ***Negation** flips the truthiness of a proposition. It is written as $\neg P$* (`\lnot`).

To negate a general proposition, we have to figure out how negation works with quantifiers. It is the case that $\neg\forall x \in X, F(x)$ is the same as $\exists x \in X : \neg F(x)$ (Why?). Similarly, $\neg\exists x \in X, F(x)$ is the same as $\forall x \in X : \neg F(x)$.

This means that we have a general rule for negating statements with chained quantifiers.

To negate a disjunction, we have $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$. Similarly, $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.

Note 1.14. *The symbol $\equiv$* (`\equiv`) *is used to represent logical equivalence of propositions.*

The previous two properties are known as DeMorgan's laws. You may have caught onto the general rule for remembering how to negate things: Just flip everything you see. I will present some examples of negations to make sure it is clear.

$\neg(\exists x \in C : \forall y \in C, F(x, y)) \equiv \forall x \in C, \exists y \in C : \neg F(x, y)$
$\neg(\forall x \in C, \exists y \in C : F(x, y)) \equiv \exists x \in C : \forall y \in C, \neg F(x, y)$
$\neg(\forall x \in C, \exists y, z \in C : F(x, y) \wedge F(x, z) \wedge y \neq z) \equiv \exists x \in C : \forall y, z \in C, \neg F(x, y) \vee \neg F(x, z) \vee y = z$

The last tool we need to learn is the implication. It turns out that it can already be constructed with what we have, but it is a lot easier to do math with it defined.

Definition 1.15. *An implication $P \Rightarrow Q$* (`P \Rightarrow Q`) *is true as long as Q is true whenever P is true.*

Note 1.16. *Please always use a **double** arrow for implication. A single arrow has other uses.*

The first thing to note in this definition is that an implication is *false* precisely if P is true yet Q is false. Thus we have $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$. Somewhat counterintuitively, negating both sides yields $P \Rightarrow Q \equiv \neg P \vee Q$.

To break it down: suppose we are considering if P implies Q. There are two ways to make this statement check out. The first is if P is simply never true. If P is never true, then we can flee from the consequences of saying $P \Rightarrow Q$ because the condition for this implication will simply never be true. Similarly, if Q is always true, then my implication is also correct. Q follows regardless, so the conclusion of Q is always correct.

Another quality of life upgrade to our toolkit is "if and only if", sometimes abbreviated in text as iff. $P \Leftrightarrow Q$ is the same as saying $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. This is quite similar to logical equivalence, but in practice it makes a lot more sense to think of it this way.

Definition 1.17. *P is **sufficient** for Q if P being true means Q must be true - in other words, $P \Rightarrow Q$. P is **necessary** for Q if Q cannot be true without P - in other words, $Q \Rightarrow P$.*

Note 1.18. *As a result, P being a necessary and sufficient condition for Q is equivalent to simply $P \Leftrightarrow Q$.*

Homework

Homework must be completed in LaTeX. You can get started in Overleaf, and you can use this document as a guide to figure out how to format the homework.

Note: † `(\dagger)` (read as "dagger") problems are typically listed in math textbooks as challenge problems. Particularly difficult problems may even be listed as ††.

1. Convert the following English sentences to rigorous logical statements, as shown above. Remember that the only acceptable format is a chain of quantifiers, followed by a proposition using (defined) variables, combined with $\neg, \vee, \wedge, \Rightarrow$ and $\Leftrightarrow$*. You will have to define some variables and sets before you begin writing the actual statement. Note: The statements will get more and more subjectively difficult to make rigorous. Try your best.

   *Actually, statements can be more complex. But it is possible to answer all the questions using precisely the format I have given.

   (a) Everyone loves dogs.

   (b) Everyone either loves at least one dog, or is a dog themself.

   (c) There is no largest integer.

   (d) Everyone needs rest, exercise and a good diet to be healthy.

   (e) There is no largest integer (In this version, you may only use the logical symbols defined in the notes, the set of integers $\mathbb{Z}$ `(\mathbb{Z})`, and the symbol $>$.

   (f) x is a prime number. (The only boolean function you may define is $D(x, y) := x$ divides y.)
   [Hint: An integer x is prime when its only positive divisors are x and 1, and $x > 1$.]

2. Negate each of the statements you have written above.

3. Visit `http://www.personal.psu.edu/tcr2/311w/logicQuickRef.pdf`. Use it to follow along with the discussion below. There are a few exercises scattered in the discussion; do them.

   - Idempotence: The $\Leftrightarrow$ might throw you off, but it's just saying p or p is the same as p. Same with p and p.
   - Law of Excluded Middle: Believe it or not, there are "LEM" (for short) deniers. I met one in my CS 538 class. In any case, it is saying that a statement cannot be both true and false.
   - Consistency: This is the same as LEM! Exercise: Use DeMorgan's law on both sides of LEM to receive this.
   - Double Negative: Duh.
   - Commutativity: Just a sanity check, but yes, all these things are commutative. If you don't remember what that means, it just says that the operator is such that you can flip the two operands and the result is still the same.
   - Associativity: Another sanity check. This says that if you have a bunch of and's chained in a row, the order in which you evaluate adjacent pairs of and's doesn't matter. This allows us to remove some parentheses.
   - DeMorgan: Consistent with prior notes.
   - Distribution: I recommend that you check these carefully. Distribution can sometimes help simplify some logical statements. The second two are redundant as these operations are all commutative. So just look at the first two and notice how or splits over and, and how and splits over or.
   - Definition: These are written very poorly. Ignore the and in the middle and then just note that it is saying p and T is precisely just p, and so on and so forth
   - Absorption: Useful trick to remember. Exercise: Use distributivity, along with other logical identities listed above, to prove that $(p \wedge (p \vee q))$ and $(p \vee (p \wedge q))$ are logically equivalent.

- Contrapositive: You may have heard this before. It is an important one that we will come back to in great detail later. Exercise: Explain why these two are the same.
- Other Logical Identities: Just skim these. They are not extremely important.
- Modus Ponens: Another important one we will come back to later. An **argument** has the structure of a set of hypotheses and a conclusion, and posits that the hypotheses implies the conclusion. An argument is **valid** if the implication itself is true. An argument is **sound** if it is valid, and all the hypotheses are true as well. Modus ponens simply says that if an argument is sound, then the conclusion is true.
- Modus Tollens: If you replace the implication with the contrapositive, you find that this is just a reworded modus ponens. Nothing special happening here. Exercise: Rewrite modus tollens to look like modus ponens.
- Common implication tautologies: Just skim these.

4. We've remarked that disjunction in mathematical reasoning acts differently from plain English. Let the symbol $\otimes$ denote $P \otimes Q$ is true only if either P or Q is true, but not both. Write an equivalent expression for $P \otimes Q$ in terms of $P, Q, \wedge, \vee, \neg$, (and parentheses), and justify your answer.

5. In this exercise, we will show that swapping the order of quantifiers changes the meaning of the proposition. Translate each logical statement into English, and determine their truth values. Justify your answer.

   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : (x + y = 0)$

   (b) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} \, (x + y = 0)$

6. Observe that a logical sentence using the existential quantifier, $\exists x \in X : P(x)$ only tells us if there is an $x \in X$ such that $P(x)$ is true. Consider the quantifier $\exists!$, which is given as follows:

   $\exists! x \in X : P(x)$ is true if and only if there is a **unique** element $x \in X$ such that $P(x)$ is true.

   Write down an equivalent expression $\exists! x \in X : P(x)$ using just $\forall, \exists, \wedge, \vee, \implies, \neg, =$, and justify your answer.