Relatively Innocuous Title, Pt. 2

Definition 7.1. *A **function** f from sets A to B, is a relation such that $\forall a, \exists! b$ such that afb. It is written as $f : A \to B$* (`f : A \to B`). *A is referred to as the **domain** of f, and B is referred to as the **codomain** of f.*

Note 7.2. *We of course write this normally as just $f(a) = b$. Also, recall that $\exists!$ means "there exists a **unique**..."*

Example: $A = B = \mathbb{N}$. $f = \{(a, a) \mid a \in \mathbb{N}\}$. This function is known as the **identity function**.

Nonexample: $A = B = \mathbb{Z}$. $f = \sim_3$, as defined in last lecture.

Definition 7.3. *A function $f : A \to B$ is **injective**, or **one-to-one**, if $\forall x, y \in A, f(x) = f(y) \Rightarrow x = y$. Alternatively, every element in B has at most one element in A "mapping" to it.*

Example: $f : \mathbb{N} \to \mathbb{N}, f(x) = x$. Suppose that $x, y \in \mathbb{N}$, and $f(x) = f(y)$. $x = f(x) = f(y) = y$, so $x = y$.

Nonexample: $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin x$. This function is not injective, because $f(0) = f(2\pi)$ and $0 \neq 2\pi$.

Definition 7.4. *A function $f : A \to B$ is **surjective**, or **onto** B, if for any y in B, there is some x in A such that $f(x) = y$. In other words, all of B is "covered" by elements from A.*

Example: $f : \mathbb{R} \to [1, 1], f(x) = 1$. This is a surjection, because $[1, 1] = \{1\}$, and $f(0) = 1$.

Nonexample: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$. We notice that there is no element in $\mathbb{R}$ such that $x^2 = -1$.

Definition 7.5. *A function is a **bijection**, or a **one-to-one correspondence**, if it is both a surjection and an injection.*

Example: $f : \mathbb{R} \to \mathbb{R}, f(x) = -x^3$.

Nonexample: $f : \mathbb{R} \to \mathbb{R}, f = \arctan x$. f is not a surjection.

Definition 7.6. *Suppose $f : A \to B$ and $g : B \to C$ are functions. We define their **composition** to be $g \circ f : A \to C$ given by $(g \circ f)(x) = g(f(x))$.*

Proposition 7.7. *Suppose $f : A \to B$ and $g : B \to C$ are both surjective. Then their composition $g \circ f : A \to C$ is surjective.*

Proof. Suppose $z \in C$. We will need to exhibit an $x \in A$ such that $(g \circ f)(x) = z$. Since g is surjective, there exists some $y \in B$ such that $g(y) = z$. But since f is surjective, there exists some $x \in A$ such that $f(x) = y$. Then clearly we now have $(g \circ f)(x) = g(f(x)) = g(y) = z$, and so we have found the desired x. $\square$

Proposition 7.8. *If $f : A \to B$ and $g : B \to C$ are both injective, then their composition $g \circ f : A \to C$ is injective.*

Proof. Exercise. $\square$

Corollary 7.9. *If $f : A \to B$ and $g : B \to C$ are both bijective, then their composition $g \circ f : A \to C$ is bijective.*

Armed to the Teeth

The basic idea is that you can "overload" operations that work on certain types to work on sets of those types. This will work for images and pre-images, but also operators that act on any domain.

Definition 7.10. *Let $f : A \to B$. Then the* ***image*** *of $X \subseteq A$ is $\{f(x) \mid x \in X\}$. You can write this as $f(X)$.*

Example: Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$. Then the image of the positive reals is the positive reals.

Definition 7.11. *Let $f : A \to B$. The* ***image of*** *f, or the* ***range*** *of f, is the image of A itself. It is written as $f(A)$ or $\mathrm{Im}(f)$ (`\mathrm{Im}(f)`).*

Example: In the function above, the image of f is the non-negative reals.

Example: A function is surjective iff it its image is equal to its codomain.

Definition 7.12. *Let $f : A \to B$. Then the* ***pre-image*** *of $Y \subseteq B$ is $\{x \mid x \in A, f(x) \in Y\}$. You can write this as $f^{-1}(Y)$.*

Example: For an injection, the pre-image of any element is either a singleton set (a set with size 1) or the empty set.

Definition 7.13. *Let $\cdot$ be any binary operator on the set A. Then for any $a \in A, X \subseteq A$, we define $a \cdot X = \{a \cdot x \mid x \in X\}$.*

Example: $\mathbb{Z} + 0.5$ is the set $\{\dots, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, \dots\}$

Definition 7.14. *Let $\cdot$ be any binary operator on the set A. Then for any $X, Y \subseteq A$, we define $X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}$.*

Example: The set $(3 \cdot \mathbb{Z}) + \{0, 1, 2\}$ is $\mathbb{Z}$.

Definition 7.15. *Let $f : A \to B$, and $g : B \to A$. We say that g is a* ***left inverse*** *of f if $g \circ f = \mathrm{id}_A$ the identity function on A, and g is a* ***right inverse*** *of f if $f \circ g = \mathrm{id}_B$ the identity function on B.*

Example: $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 1$. Then the function $g : \mathbb{N} \to \mathbb{N}$ given by

$$g(n) = \begin{cases} 2, & n = 1 \\ n - 1, & n > 1 \end{cases}.$$

Then $g \circ f = \mathrm{id}_{\mathbb{N}}$, so g is a left inverse for f. However, g is not a right inverse for f, since $(f \circ g)(1) = f(2) = 3$, and so it disagrees with $\mathrm{id}_{\mathbb{N}}$ at $n = 1$. This shows that f is a right inverse of g also, but f is not a left inverse of g.

Proposition 7.16. *A function $f : A \to B$ is one-to-one if and only if there exists a left inverse.*

Proof. ($\Rightarrow$) Suppose $f : A \to B$ is one-to-one. Then for each $y \in f(A) \subseteq B$, there exists a unique $x \in A$ such that $f(x) = y$. This allows us to define a function $g : f(A) \to A$ using that rule: send each $y \in f(A)$ to its unique preimage in A. We will now extend this function g to B. Let $x_0 \in A$ be arbitrary; now define $g^* : B \to A$ by

$$g^*(y) = \begin{cases} g(y), & y \in f(A) \\ x_0, & y \notin f(A) \end{cases}.$$

We now need to show that this g^* satisfies the desired properties. To this end, let $x \in A$; then we have $(g^* \circ f)(x) = g^*(f(x)) = x$ since $f(x) \in f(A)$. Thus $g^* \circ f \equiv \mathrm{id}_A$, and so we have verified the existence of the left inverse.

($\Leftarrow$) Suppose f has a left inverse g, and suppose there are $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. Then applying g to both sides, we have $(g \circ f)(x_1) = (g \circ f)(x_2)$, but since g is a left inverse for f, $g \circ f \equiv \mathrm{id}_A$, and so this implies $x_1 = x_2$. Hence f is injective. $\square$

Proposition 7.17. *A function $f : A \to B$ is surjective if and only if there exists a right inverse.*

We will omit the proof of this proposition: the backwards direction is easy; the forwards direction requires the Axiom of Choice!

Definition 7.18. *Let $f : A \to B$ be a function. We will define the* ***inverse*** *of a function to be $f^{-1} : B \to \mathcal{P}(A)$, where $f^{-1}(y) := f^{-1}(\{y\})$ in the sense of preimages defined above. We will say f is* ***invertible*** *if $|f^{-1}(y)| = 1$ for each $y \in B$.*

Note that if a function f is invertible, then its inverse f^{-1} is both a left and right inverse.

Theorem 7.19. *A function $f : A \to B$ is invertible if and only if it is a bijection.*

Proof. Omitted. Proceed as in the proof of Proposition 7.16. □

Recall that we said a function $f : A \to B$ is a relation; that means that $f \subseteq A \times B$ by the definition of a relation! That is, we say $f(x) = y$ if and only if $(x, y) \in f$. That's right: everything is a set! This gives us a neat way to think about when functions are equal, function composition, and inverse functions. In fact, the notions such as domain, codomain, image, composition, and inverses may be defined for relations, not just functions! Two functions f and g are equal if and only if $f \subseteq g$ and $g \subseteq f$. The inverse of a function would be $f^{-1} := \{(y, x) \in B \times A \mid (x, y) \in f\}$. Finally, we would define the composition as $g \circ f := \{(x, z) \in A \times C \mid (x, y) \in f \text{ and } (y, z) \in g\}$.

Homework

1. Let $A = \{1, 2\}, B = \{x, y\}$. List all functions from $A \to B$ that are:
   (a) injections;
   (b) surjections;
   (c) bijections;
   (d) none of the above.
2. (a) Supply a proof for Proposition 7.8.
   (b) Suppose $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective, is it necessarily the case that f is injective? Is it necessarily the case that g is injective? Prove or disprove your claims.
   (c) Repeat part (b) but replace injective with surjective.
3. Find a function from $\mathbb{R}$ to $\mathbb{R}$, and supply a proof of your claim, that is:
   (a) an injection but not a surjection;
   (b) a surjection but not an injection;
   (c) a bijection;
   (d) neither a surjection nor an injection.
4. Let $f : A \to B$, and let $G_1, G_2 \subseteq A$, and let $H_1, H_2 \subseteq B$.
   (a) Is it true that $f^{-1}(f(G_1)) = G_1$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.
   (b) Is it true that $f(f^{-1}(H_1)) = H_1$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.
   (c) Is it true that $f(G_1 \cap G_2) = f(G_1) \cap f(G_2)$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.
   (d) Is it true that $f^{-1}(H_1 \cap H_2) = f^{-1}(H_1) \cap f^{-1}(H_2)$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.

Additional Exercises for Extra Practice

5. Let $G = (V, E)$ be a simple graph. Now, suppose we are interested in knowing how many possible **walks** there are from u to v, of a particular length l. A walk is much like a path, but you are allowed to repeat edges/vertices.

   Recall that E is simply a relation on V. Define the **adjacency matrix** of a graph to be the $|V| \times |V|$ matrix A, where A_{ij} is 1 when $(i, j) \in E$, and 0 otherwise.

   (a) † Prove that the number of walks of length l from u to v is simply $(A^l)_{uv}$.

   (b) *(For the computer scientist's fun)* Show an algorithm in $\mathcal{O}(|V|^3 \log_2(l))$ to calculate A^l.

   (c) †† You are gaming with Joe[1]. In the game, there are many levels. Levels can be played any number of times, including 0. If Joe plays level x, this is considered to be **sussy**, and he will choose the next level to play from a set $S(x)$. If you play a level x, this is considered to be **drippy**, and you will choose the next level to play from a set $D(x)$.

   You will take turns playing the game, but not in the traditional way. The order of your moves is determined as follows. There are n *stages* in the game. In the first stage, you will play the first level, then Joe will play the second level.

   Now, in general on the i^{th} stage, take the sequence of turns played across all previous stages, swap yourself and Joe, and play in that order. For example, the first few stages would look like

   You, Joe,

   Joe, You,

   Joe, You, You, Joe,

   Joe, You, You, Joe, You, Joe, Joe, You.

   The next turn would be Joe's, as you played first in the last sixteen moves.

   Given a set of levels L, where for each x in L, $S(x), D(x) \subseteq L$, find the number of ways you and Joe can play through n stages of the game without running into any dead ends, assuming you begin by playing level 1. A dead end would be considered if it is your turn on level x but $D(x)$ is empty, or it is Joe's turn but $S(x)$ is empty. Your algorithm must work in $\mathcal{O}(n|L|^3)$.

[1] This problem comes from a practice contest I did in competitive programming, albeit phrased a little differently. I will link the problem in the solutions, as not to spoil anything.