It Never Ends: Introductory Real Analysis

You might recall from one of the earlier lectures the following proposition:

Proposition 10.1. *There is no rational number r such that $r^2 = 2$.*

This proposition tells us that the rational number has holes! Recall that you proved that there is no smallest positive rational number. Similarly, you can show that $A = \{r \in \mathbb{Q} \mid r^2 < 2\}$ has no largest element, and so you can get *as close to $\sqrt{2}$ as you want, but no rational will ever reach it.* We will want to eventually get around this. Nevertheless, $\mathbb{Q}$ is a nice set: in fact, it is an **ordered field**, meaning that it is a field such that we can define a total order on it. If you don't know what a field is, don't worry, it's not that important, but we'll write it here for the sake of completeness.

Definition 10.2. *A set $\mathbb{F}$ equipped with operations $+$ and $\cdot$ is called a **field** if it satisfies the following properties:*

(i) (Closure under Addition) If $x, y \in \mathbb{F}$, then $x + y \in \mathbb{F}$.

(ii) (Commutativity of Addition) If $x, y \in \mathbb{F}$, then $x + y = y + x$.

(iii) (Associativity of Addition) If $x, y, z \in \mathbb{F}$, then $(x + y) + z = x + (y + z)$.

(iv) (Existence of Additive Identity) There exists an element 0 such that $0 + x = x$ for all $x \in \mathbb{F}$.

(v) (Existence of Additive Inverse) For all $x \in \mathbb{F}$, there exists $-x \in \mathbb{F}$ such that $x + (-x) = 0$.

(vi) (Closure under Multiplication) If $x, y \in \mathbb{F}$, then $x \cdot y \in \mathbb{F}$.

(vii) (Commutativity of Multiplication) If $x, y \in \mathbb{F}$, then $x \cdot y = y \cdot x$.

(viii) (Associativity of Multiplication) If $x, y, z \in \mathbb{F}$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(ix) (Existence of Multiplicative Identity) There exists an element $1 \in \mathbb{F}$ such that for all $x \in \mathbb{F}$, $1 \cdot x = x$.

(x) (Existence of Multiplicative Inverse) If $x \in \mathbb{F}$ with $x \neq 0$, then there exists an element $1/x \in \mathbb{F}$ such that $x \cdot (1/x) = 1$.

(xi) (Distributivity) For all $x, y, z \in \mathbb{F}$, $x(y + z) = (x \cdot y) + (x \cdot z)$.

In short, it is a set that has the usual operations addition, subtraction, multiplication, and division, and they all work as we would expect them to.

Proposition 10.3. *$\mathbb{Q}$ is a field.*

Proof. Exercise. ■

But now it is abundantly clear that $\mathbb{Q}$ and $\mathbb{R}$ are both fields: how do we differentiate the two? What is the defining property of $\mathbb{R}$ that $\mathbb{Q}$ does not satisfy? To do this, we will recall from our study of relations and order relations the following definition:

Definition 10.4. *If $(S, \leq)$ is a partially ordered set, and $E \subseteq S$. If there exists a $\beta \in S$ such that for all $x \in E$, $x \leq \beta$, then we say that β is an **upper bound** for E, and E is **bounded above**. Further, if for all other upper bound α for E, we have $\beta \leq \alpha$, then we say that β is a **least upper bound** of E, or **supremum** of E, and we write $\sup E$.*

We may similarly define lower bound, bounded below, greatest lower bound, and infimum.

Remark. The B-man uses $\mathrm{LUB}(E)$ instead of $\sup E$.

Example 10.5. *If we let $S = \mathbb{Q}$ and consider A as a subset of $\mathbb{Q}$, then we have shown that $\sup A$ does not exist!*

Now we are ready to state the defining property of $\mathbb{R}$.

Axiom of Completeness. *Every nonempty subset of $\mathbb{R}$ that has an upper bound has a least upper bound.*

Now we say that $\mathbb{R}$ is an ordered field that satisfies the Axiom of Completeness. To show that this set exists is nontrivial and outside the scope of this lecture. The two main ways (among others!) to construct $\mathbb{R}$ are to either use Dedekind cuts, or to use equivalence relations of Cauchy sequences. Now we will show a few consequences of the Axiom of Completeness, and move onto other topics.

Proposition 10.6 (Archimedean Property)**.** *For any $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > x$.*

Proof. Towards a contradiction suppose otherwise. Then there exists an $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $n \leq x$. Then x is an upper bound for $\mathbb{N}$, and the Axiom of Completeness tells us that there exists an $\alpha = \sup \mathbb{N}$. Then $\alpha - 1$ is not an upper bound, since if that were the case, then α would no longer be the least upper bound. That means there is some $m \in \mathbb{N}$ such that $m > \alpha - 1$. But this implies that $\alpha < m + 1$, and $m + 1 \in \mathbb{N}$, a contradiction. ■

Corollary 10.7. *For all $x > 0$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < y$.*

Proof. Exercise. ■

Theorem 10.8 (Existence of $\sqrt{2}$)**.** *There exists an $s \in \mathbb{R}$ such that $s^2 = 2$.*

Proof. Consider the set A as defined right after Proposition 10.1. This set is clearly bounded above, so $\alpha = \sup A$ exists. Then it is the case that one of $\alpha^2 < 2$, $\alpha^2 = 2$, or $\alpha^2 > 2$ is true. Suppose for a contradiction that $\alpha^2 < 2$. Then $2 - \alpha^2 > 0$, and hence

$$\frac{2 - \alpha^2}{2\alpha + 1} > 0.$$

Hence using Corollary 10.7, there exists an $n_0 \in \mathbb{N}$ such that

$$0 < \frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}.$$

Rearranging yields

$$\frac{2\alpha + 1}{n_0} < 2 - \alpha^2,$$

and so

$$\begin{aligned} \left(\alpha + \frac{1}{n_0}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n} < \alpha^2 + (2 - \alpha) = 2. \end{aligned}$$

This implies that $\alpha + \frac{1}{n_0} \in A$. But α is an upper bound for T, a contradiction. The case $\alpha^2 > 2$ is similar; hence the only possibility remaining is $\alpha^2 = 2$. ■

This concludes the discussion about the Axiom of Completeness. Onto more things!

Sequences, Sequences, Sequences

Recall that a sequence $(x_n)_{n=1}^{\infty} = (x_n)_{n \in \mathbb{N}}$ is a bijection from $\mathbb{N}$ to S, $n \mapsto x_n$. Often we'll abuse notation and say that a sequence of elements in S is $(x_n)_{n \in \mathbb{N}} \subseteq S$. Later we'll even drop the indices and understand that (x_n) is a sequence. Armed with the new Axiom of Completeness for $\mathbb{R}$, we can now discuss when a sequence of real numbers converges (for a sequence of rational numbers that doesn't converge, see the homework below).

Definition 10.9. *Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence of real numbers. We say that (x_n)* ***converges*** *to $x \in \mathbb{R}$ if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, $|x_n - x| < \epsilon$. Whenever this happens, we write $\lim_{n\to\infty} x_n = x$* (`\lim_{n\to\infty} x_n=x`)*, or $x_n \to x$* (`x\to x`)*, or even $x \xrightarrow[n\to\infty]{} x$.* (`x_n\rightarrow[n\to\infty]{}x`)

To scare you, let's try writing that as a logical proposition:

$$\lim_{n\to\infty} x_n = x \iff \forall\epsilon > 0, \exists N \in \mathbb{N},\ \forall n \in \mathbb{N},\ (n \geq N \implies |x_n - x| < \epsilon).$$

This may be a little daunting, so let's think of an example.

Example 10.10. $\lim_{n\to\infty} \frac{1}{n} = 0$.

In the definition of a limit, there are multiple quantifiers to unravel. There is a universal quantifier, and then an existential quantifier, and another universal quantifier. So one begins with letting $\epsilon > 0$ be arbitrary, and then proceeding with a choice of N. Finally, one would show that for this particular choice of N, one has x_n is within ϵ of N. Let's look at an example.

Proof. Let $\epsilon > 0$ be given. Then by Corollary 10.7, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if $n \geq N$, observe:

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Therefore $\frac{1}{n} \xrightarrow[n\to\infty]{} 0$. ∎

Now to prove that a sequence does NOT converge to an $x \in \mathbb{R}$, then we must come up with an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists some $n \geq N$ such that $|x_n - x| \geq \epsilon$. To prove that a sequence does NOT converge to any x, then we would usually use a proof by contradiction: suppose it DOES converge to some x, and derive a contradiction. You will have more practice with this in the homework below.

Proposition 10.11. *If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences such that $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.*

Proof. Let $\epsilon > 0$ be arbitrary. Then since (x_n) converges to x, there exists an $N_x \in \mathbb{N}$ such that if $n \geq N_x$, then $|x_n - x| < \frac{\epsilon}{2}$. Similarly, since (y_n) converges to y, there exists an $N_y \in \mathbb{N}$ such that if $n \geq N_y$, then $|y_n - y| < \frac{\epsilon}{2}$. Now set $N := \max\{N_x, N_y\}$. If $n \geq N$, we have, using the Triangle Inequality,

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $(x_n + y_n)_{n\in\mathbb{N}}$ converges to $x + y$. ∎

Proposition 10.12. *Limits of sequences are unique: if $x_n \to x$ and $x_n \to y$, then necessarily $x = y$.*

Proof. Towards a contradiction assume that $x \neq y$, and set $\epsilon = \frac{|x-y|}{2}$. Then by passing to the maximum if necessary as done in the previous proposition, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, $|x_n - x| < \frac{\epsilon}{2}$ and $|x_n - y| < \frac{\epsilon}{2}$. Then if $n \geq N$, we have by the Triangle Inequality,

$$|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon = \frac{|x-y|}{2}.$$

This is a contradiction, and so $x = y$. ∎

Homework

1. (a) Prove Corollary 10.7.
   (b) Complete the Proof of Theorem 10.8 by analyzing the case $\alpha^2 > 2$.
2. **(Lemma 1.3.8).** Suppose $A \subseteq \mathbb{R}$ and s is an upper bound for A. Prove that $s = \sup A$ if and only if for all $\epsilon > 0$, there exists an $a \in A$ such that $s - \epsilon < a$.
3. Find, and supply a proof, the supremum of the following sets:
   (a) $[0, 1]$
   (b) For $a, b \in \mathbb{R}$ with $a < b$, $\mathbb{Q} \cap [a, b]$
   (c) For $A \subseteq \mathbb{R}$ with $\sup A = s$, $\alpha + A \coloneqq \{\alpha + x \mid x \in A\}$.
4. Find the limit of the following sequences, and supply a proof, or prove that they do not exist:
   (a) $\lim_{n\to\infty} \frac{\sin n}{n}$.
   (b) $\lim_{n\to\infty} \frac{2n+3}{n+1}$.
   (c) $\lim_{n\to\infty} \frac{n+2}{3n+4}$.
   (d) $\lim_{n\to\infty} \sin\left(\frac{n\pi}{2}\right)$.
5. Prove or disprove the following:
   (a) If (x_n) converges to x, then $|x_n| \xrightarrow[n\to\infty]{} |x|$.
   (b) If (x_n) is a sequence such that $|x_n|$ converges, then (x_n) converges.
   (c) If (x_n) converges to x, then (cx_n) converges to cx.
   (d) There exists a sequence with an infinite number of ones that does not converge to one.
   (e) There exists a sequence with an infinite number of ones that converges to a limit not equal to one.
   (f) If (x_n) converges to x, then the sequence is **bounded**, that is, there exists some $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.
6. Give an example of a sequence $(q_n)_{n\in\mathbb{N}}$ that converges in the reals but does not converge in the rationals.
7. Prove or disprove: if a sequence $(x_n)_{n\in\mathbb{N}}$ has the property that $|x_{n+1} - x_n| \xrightarrow[n\to\infty]{} 0$, then (x_n) converges.
8. **(Monotone Convergence Theorem).** † We say that a sequence **increases monotonically** if for all n, $x_n \leq x_{n+1}$. Prove that every bounded monotonically increasing sequence converges.