How to Put Things Together

Definition 2.1. *A* ***set*** *is an unordered collection of items, where each item appears at most once.*

Note 2.2. *This is a lie. Consider the set* $A = \{X \mid X \notin X\}$ `(A = \{X \mid X \not \in X\})`*. If* $A \notin A$*, then by definition of* A*,* $A \in A$*. This is a contradiction. So, what should we conclude?*

Naïve set theory runs into some issues. The modern day version of set-theory uses axioms referred to as ZF, and with these axioms, things like A do not meet the criteria to be a set. We will not concern ourselves with the fine details. I do recommend glancing over the foundation of ZF at some later point in time, though.

Here are some examples of common sets: $\mathbb{Z}$ `(\mathbb{Z})` (the integers), $\mathbb{N}$ (the natural numbers, which are *positive* integers), $\mathbb{Q}$ (rational numbers, the quotient of two integers where the latter is nonzero), $\mathbb{R}$ (the real numbers, whose definition we will ignore for now), $\mathbb{C}$ (the complex numbers, obtained by adjoining the imaginary number i to $\mathbb{R}$), and $\mathbb{H}$ (the quaternions, obtained by adjoining two more elements to $\mathbb{C}$).

Definition 2.3. *For two sets* A *and* B*,* A *is a* ***subset*** *of* B *when* $\forall a \in A, a \in B$*. We write this as* $A \subseteq B$ `(A \subseteq B)`*.*

Definition 2.4. *Two sets* A *and* B *are set to be* ***equal*** *when* $A \subseteq B \wedge B \subseteq A$*.*

Note 2.5. *We use* $\subseteq$ *when it is possible that the two sets are in fact equal (similar to saying* $\leq$ `(\leq)`*). If we want to distinguish the fact that the two sets are definitely* ***not*** *equal, then we can write* $\subset$ `(\subset)` *instead.*

Here are some nice examples of sets: $\{1, 2\}$, $\{\text{code.pdf}, 15, \texttt{\$\$\$\$\$\$\$}\}$. Here are some less nice examples of sets: $\{1, \{2\}\}, \{\}, \{\{\}\}$. Here are some even less nice examples of sets: $\{\{\{\{\}, \{\{\}\}\}\}, \{\}\}, \{A \mid A \subseteq \{\{\}\}\}$.

Sets can be very ugly. If someone says "here is a set", and tells you nothing else about it, be very afraid as to what might be inside.

Note 2.6. *I once had a nightmare about* $\mathbb{R}/\mathbb{Q}$ *(read as "R mod Q"). It is a very scary set. Thankfully I do not think of myself as a mathematician anymore, so it has left me alone.*

Definition 2.7. *The* ***powerset*** *of a set* A *is written as* $\mathcal{P}(A)$ `(\mathcal{P}(A))`*. It is the set which contains every subset of* A*.*

Definition 2.8. *The* ***empty set*** *is the set which contains nothing. It is written as* $\varnothing$ `(\varnothing)`*.*

Computer science has a very nice way of thinking about powersets. Consider a set A of size n, and consider some $B \subseteq A$. We will construct a bitstring of size n that represents B. Arbitrarily order the elements of A; then, for each element in A, if it appears in B, add a 1 to the bitstring. Otherwise, add a 0.

For example, if $A = \{1, 2, 3, 4\}$, and $B = \{1, 3\}$, our bitstring will read as 1010. This is because the first element is present in B, the second is not, the third is, and the fourth is not.

Notice that we can map from a bitstring to a subset of A. If I gave you a bitstring 0010, that would correspond to $\{3\}$, a subset of A.

Finally, note that we can loop over all bitstrings of size n, and if we map each of these to a subset, we can loop over all subsets. Since we looped over all subsets, we looped over every element in the powerset of A.

Definition 2.9. *The* ***union*** *of two sets* A *and* B *is the set* $\{x \mid x \in A \vee x \in B\}$*. It is written as* $A \cup B$ `(A \cup B)`*.*

Definition 2.10. *The* ***intersection*** *of two sets* A *and* B *is the set* $\{x \mid x \in A \wedge x \in B\}$*. It is written as* $A \cap B$ `(A \cap B)`*.*

Let A and B be any two sets. We will prove the following: $(A \cap B) \subseteq A \subseteq (A \cup B)$. We can split this into two things two prove: $(A \cap B) \subseteq A$ and $A \subseteq (A \cup B)$.

To show $(A \cap B) \subseteq A$, we have to prove $\forall x \in (A \cap B), x \in A$. This is by Definition 0.3. While I will not go heavily into the various proof techniques yet, I will use something known as *universal instantiation*. When trying to prove something of the form $\forall x \in \mathcal{X}, F(x)$, I am allowed to say basically "Let x be some arbitrary element of $\mathcal{X}$. Now, abracadabra, $F(x)$ is true!"

We will do just that. Let $x \in (A \cap B)$. By the definition of intersection, we know that $x \in A \wedge x \in B$. Thus it trivially follows that $x \in A$, and I am done. We started from $x \in (A \cap B)$ and concluded with $x \in A$.

Now for the next part, $A \subseteq (A \cup B)$. Let $x \in A$. Then it clear that $x \in A \vee x \in B$. Therefore $x \in (A \cup B)$.

Now that I have proven $\forall x \in (A \cap B), x \in A$ and $\forall x \in A, x \in (A \cup B)$, I know that $(A \cap B) \subseteq A$ and $A \subseteq (A \cup B)$. Therefore $(A \cap B) \subseteq A \subseteq (A \cup B)$ and I am done.

For another example, we will prove that $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$. Suppose $X \in \mathcal{P}(A)$. Then $X \subseteq A$. Therefore $X \subseteq (A \cup B)$ (we just proved this!). From there it follows that $X \in \mathcal{P}(A \cup B)$.

We have shown that for arbitrary $X \in \mathcal{P}(A), X \in \mathcal{P}(A \cup B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$.

Definition 2.11. *The **complement** of a set $A \subseteq B$ is the set of all elements of B which are not in A. Generally, the set B is the largest set that we are considering at the time. It is written as A^C* `(A^C)`.

Definition 2.12. $[n] = \{1, \ldots, n\}$.

We'll again return to our example of bitstrings. Take $B = [4]$. Then for $A = \{2, 4\}$, the corresponding bitstring is 0101. The fun thing is now that the bitstring of A^C is precisely 1010. Do you see why?

There are more DeMorgan's laws, this time for sets. Suppose that there is some universal set U, and $A, B \subseteq U$. Then $(A \cup B)^C = A^C \cap B^C$ and $(A \cap B)^C = A^C \cup B^C$.

We can extend the concept of unioning to any finite number of sets being unioned together. In fact, we can even union an *infinite* number of sets together.

Definition 2.13. *An **indexing set** is any set which we use to index some collection.*

Example: For a series $a_i = 2^i$, our indexing set might be the natural numbers, $\mathbb{N}$.

Definition 2.14. *An **indexed union** is the set $\{a \in U \mid \exists i \in I : a \in A_i\}$, where I is the indexing set of A_i.*

Note 2.15. *Most often, indexing sets are simply just $\mathbb{N}$. However, they can sometimes be more interesting, like $\mathbb{Q}$ or even $\mathbb{R}$!*

Define $A_i = \{2 \cdot i + 1\}$, for $i \in \mathbb{N}$. What is the union of all A_i when $I = \mathbb{N}$?

What about when $I = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$?

Definition 2.16. *An **indexed intersection** is the set $\{a \in U \mid \forall i \in I : a \in A_i\}$, where I is the indexing set of A_i.*

Definition 2.17. *The set difference of two sets A and B is every element of A that does not appear in B. It is written as $A \setminus B$* `(A \setminus B)`*. Formally, you can describe it as $\{a \in A \mid a \notin B\}$.*

Example: Let $A = [4]$, and $B = \{3, 4, 5, 6\}$. Then $A \setminus B = [2]$, and $B \setminus A = \{5, 6\}$.

Definition 2.18. *The **Cartesian product** of two sets A and B is $\{(a, b) \mid a \in A \wedge b \in B\}$. It is written as $A \times B$* `(A \times B)`.

Note 2.19. *The Cartesian product is simply all pairs between A and B. But what if I wanted all between three sets? Would I end up with elements like $((a, b), c)$ if I did $(A \times B) \times C$? Thankfully not, as by convention we **flatten** tuples to all look like $(a_1, ..., a_n)$.*

Example: $\mathbb{R}$ is the real numbers. You are familiar with $\mathbb{R}^2$, which is the 2D Cartesian plane. It can (and should be) thought of as the Cartesian product of $\mathbb{R}$ and itself.

Definition 2.20. *The **cardinality** of a set is the number of elements that it has. It is written as $|A|$* `(|A|)`.

Example: The cardinality of the empty set is 0. If the cardinality of a set A is n, then the cardinality of $\mathcal{P}(A)$ is 2^n.

What happens with infinite sets? We will not explore infinite cardinals now, but I would like to mention **Cantor's Theorem**, which has important consequences in mathematics.

Theorem 2.21. ***Cantor's Theorem:*** $|\mathcal{P}(A)| > |A|$.

For *finite* sets, this is pretty clear ($\forall n \geq 0, 2^n > n$) `(\forall n \geq 0, 2^n > n)`. However, it's significantly more nontrivial to prove for infinite sets.

Homework

1. An **open interval**, written as (a, b) for $a, b \in \mathbb{R}$, is all real numbers strictly between a and b. If either the first or second parend is replaced with a square bracket, that means that endpoint is included in the interval as well.

   (a) What set (that we know) is $\bigcup_{x \in \mathbb{Z}} \{x, x+1, x+2\}$?

   Note 2.22. *The notation above is shorthand indexed union over* $\mathbb{Z}$.

   (b) What set (that we know) is $\bigcup_{n \in \mathbb{N}} (-n, n)$? [Hint: remember that for the definition of indexed union, $x \in$`<indexed union set>` $\Leftrightarrow \exists i \in I : x \in A_i$]

   (c) What set is $\bigcap_{n \in \mathbb{N}} (-n, n)$? Use formal set builder notation to describe it.

   (d) What set is $\bigcup_{n=2}^{\infty} [0, 1 - 1/n)$ `(\bigcup_{n = 2} ^\infty [0, 1 - 1/n))`?

   (e) What set is $\bigcup_{x \in \mathbb{Z}} \left(\bigcup_{n=2}^{\infty} [x, x + 1 - 1/n) \right)$? Express your answer as simply as possible.

2. For the sets in problem 1, find the complement of each set with respect to $\mathbb{R}$. (Recall that the complement of A with respect to B is every element in B that is not in A).

3. (a) Prove that the cardinality of $|A \times B|$ is $|A| \cdot |B|$, assuming that A and B are finite. It does not have to be particularly rigorous.

   (b) What is the cardinality of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))))$? What about if I apply the powerset function to the empty set n times in general?

4. (a) Prove that $(A \setminus B) \subseteq A$.

   (b) Prove the first DeMorgan's law for sets that I listed above:

$$(A \cup B)^C = A^C \cap B^C.$$

5. Define the **symmetric difference** of two sets A, B as follows:

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

   Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Additional Problems for Practice

6. We can perform unions and intersections without an indexing set, as well. For example, we can write

$$\bigcup_{X \in \mathcal{P}(A)} X = A.$$

   (You can prove this to yourself if you'd like). Now let's define the following set:

$$\mathcal{T}_A := \{X \subseteq A \mid |X| = 2\}.$$

   We say this set is *parameterized* by a set A. Now, for which A is the union

$$\bigcup_{X \in \mathcal{T}_A} X = A?$$

7. †† Prove the following: $\forall \varepsilon > 0, \bigcup_{q \in \mathbb{Q}} (q - \varepsilon, q + \varepsilon) = \mathbb{R}$. Remember that set equality requires you to prove both directions ($A \subseteq B \wedge B \subseteq A$). Please do not spend too much time on the question, it is more of a bonus.