Foreword

Congrats! We are at the end of the core material.

Top Ten Amazing Card Tricks to Impress Your Mom - Honorable Mentions

I lied to you previously - summarily, cardinal numbers do not **necessarily** work like natural numbers.

Definition 9.1. *The cardinality of a set, A, is a* ***cardinal number****. It is written as $|A|$.*

Recalling the definition of cardinality relations with injections and surjections, we can see that there is an ordering on cardinal numbers. In fact, we have by Cantor's Theorem,

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots .$$

In fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$—you can prove this using either binary expansions, or indicator functions. Moreover, some of the properties of "orderings" act like what we expect:

Proposition 9.2. $|A| \leq |B|$ *if and only if* $|B| \geq |A|$.

Proof. Revisit the proofs of Propositions 7.16 and 7.17 and try to apply the ideas here. □

We also happen to represent some cardinal numbers as what *look* like natural numbers, but aren't really natural numbers.

It turns out that cardinal numbers that represent finite sets work exactly like natural numbers - we can order them like natural numbers, and do arithmetic with them. But the first of these properties (ordering) is nontrivial, and is also basically what the pigeonhole principle says. Doing arithmetic with them also involves having to mess around with some definitions, and is similarly nontrivial.

Thus in the previous homework, since you are basically proving that cardinal numbers work like natural numbers, you cannot assume they work like natural numbers. That would be circular.

Furthermore, we cannot assume directly that $|A| \leq |B|$ and $|A| \geq |B|$ implies $|A| = |B|$. Actually,

Theorem 9.3. ***Cantor-Schröder-Bernstein Theorem****. If there is an injection and surjection from A to B, then there is a bijection from A to B.*

Surprising that this is not trivially true, but this is how the definitions we picked follow through.

Step by Step by Step...

Definition 9.4. *Suppose $F : \mathbb{N} \to \{\mathrm{T}, \mathrm{F}\}$ is a boolean function and we are trying to prove it is true everywhere. We may alternatively do induction on G, where $G(n) := \forall m \in \mathbb{N} : m \leq n, F(m)$. This is known as* ***strong induction****.*

The template for strong induction does not differ much from the tempalte for weak induction (i.e., the normal version of induction). The only difference is that instead of assuming $F(n)$ and proving $F(n+1)$, you may instead assume $F(1), F(2), ..., F(n)$ and then try to prove $F(n+1)$.

In a nutshell, this differs from weak induction in the sense that sometimes you may have to refer to the fact that a domino before the one directly prior to you fell over.

Example: Prove that for any value of 8 cents or higher can be achieved using only 3 and 5 cent coins.

Proof. We will proceed by strong induction. For 8 cents, we can pick both types of coins. For 9 cents, we can pick 3 of the 3 cent coins. For 10 cents, we can pick 2 of the 5 cent coins.

Suppose that for any $n \geq 10$, we can create any value strictly between 8 and n using 3 and 5 cent coins. Then we can create $n-2$ cents using some combination of coins, so we can create $n+1$ cents by adding another 3 cent coin to the pile. $\square$

Example: Prove that a tree (V, E) has $|V|-1$ edges.

Proof. We will proceed with strong induction. A tree with 1 vertex necessarily has 0 edges.

Let $n \geq 1$, and suppose that for any tree with m vertices, for $1 \leq m \leq n$, it has $m-1$ edges. Let G be an arbitrary tree with $n+1$ vertices. Select an arbitrary vertex v, and let k be its degree.

Let G' be the graph obtained after deleting v from G and all its edges connected to v. Let $u_1, ..., u_k$ be its neighbors. Every vertex remaining must be connected to at least one u_i, otherwise it would not have been connected to v in G. No two vertices connected to distinct u_i's may be connected, otherwise there would be a cycle.

Therefore G' is partitioned into precisely k connected components, all of which must be acyclic due to acyclicity of G. If s_i is the number of vertices in the i^{th} component, the number of edges in G must be

$$k + \sum_{i=1}^{k}(s_i - 1) = k - k + \sum_{i=1}^{k} s_i = n.$$

$\square$

Much easier with strong induction, no? No lemmas needed.

Example (300H final, Spring 2021): Recall that, having written the Fibonacci numbers as $(f_n)_{n=1}^{\infty}$, where

$$f_n = \begin{cases} 1, & n = 1 \text{ or } 2 \\ f_{n-1} + f_{n-2}, & n > 2. \end{cases}$$

Prove that for all $n \in \mathbb{N}$, $f_n < 1.7^n$.

Proof. We will proceed using strong induction. Note that for $n = 1$, $f_1 = 1 < 1.7$, and for $n = 2$, $f_2 = 1 < 1.7^2 = 2.89$. Now suppose we have the result for $1, 2, \ldots, n$ for $n > 2$; we will now show the result for $n+1$. Observe:

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} \\ &< 1.7^n + 1.7^{n-1} \\ &= 1.7^{n-1}(1.7 + 1) \\ &< 1.7^{n-1}(2.89) \\ &= 1.7^{n+1}. \end{aligned}$$

Therefore the result follows by strong induction. $\square$

Example (Binet's Formula): Denote φ, ψ to be the solutions to the quadratic equation $x^2 - x - 1 = 0$, where $\varphi > 0$ and $\psi < 0$ (φ is known as the **golden ratio**). Show that for all $n \in \mathbb{N}$,

$$f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}.$$

Proof. Let's do this together in lecture! $\square$

Muuuuuuuuuch easier with strong induction...

Step by Step by Step $\cdots$

by Step by Step $\cdots$

by Step $\cdots$

$\vdots$

This stuff is really strong, okay? Use it carefully.

Definition 9.5. *An element a in a partially ordered set (written normally as a pair $(A, \leq)$) is known as a **minimum** when for any other element $b \neq a$, $a \not\leq b$.*

Definition 9.6. *A partially ordered set, is **well-founded** when for any non-empty subset of A, there is a minimum element in the subset.*

Note 9.7. *A well-founded partial ordering really just means there are no infinitely descending chains elements, like in $(\mathbb{Z}, \leq)$.*

Definition 9.8. *Let M be the set of minimums in a partial ordering P. If we prove $F(m)$ for any $m \in M$, and $F(a) \Rightarrow F(b)$ for any $a \leq b$, then $F(x)$ is true for all $x \in P$. This is known as **stronger induction**.*

The idea is basically this: You knock over all the dominoes which are minimums. Every other domino which is not a minimum must be $\geq$ at least one other minimum, otherwise it would be a minimum itself. Thus all dominoes will fall over.

Figure 1: Various types of induction.

Homework

1. Show that every natural number can be written as the sum of *distinct* powers of two.

2. Prove that every natural greater than 23 can be written as $5x + 7y$, for $x, y \geq 0$.

3. † Prove that every natural number can be written as the sum of distinct Fibonacci numbers.

4. Prove the existence part of the Fundamental Theorem of Arithmetic, i.e., that every number can be written as the product of primes.

5. Consider a graph G with n vertices and exactly k connected components. We would like to maximize the number of edges in G.

   (a) What is the largest number of edges a component with x vertices can have?

   (b) Conjecture the largest number of edges that a graph with n vertices and exactly k connected components can contain.

   (c) First, a definition:

   Definition 9.9. *An* ***integer partition*** *of the number n and length k is a nonincreasing finite sequence of length k whose entries sum to n.*

   How could one represent the graph G as an integer partition of n and length k?

   (d) †† (Bonus) Conjecture a partial ordering that one could do stronger induction on to prove the result.

   (e) †† (Bonus) Prove the base case and inductive step on your partial ordering.