$\mathcal{P}$1

$\mathcal{P}$1.a $\bigcup_{x \in \mathbb{Z}} \{x, x+1, x+2\}$

The resultant set will be $\mathbb{Z}$ because of the range of the indexing set, which is also $\mathbb{Z}$.

$\mathcal{P}$1.b $\bigcup_{n \in \mathbb{N}} (-n, n)$

As n approaches infinity, the interval described also expands to infinity. The union of all of these intervals is therefore $\mathbb{R}$.

$\mathcal{P}$1.c $\bigcap_{n \in \mathbb{N}} (-n, n)$

As n increases, the width of each interval also increases, but all of them are centered at 0. The smallest interval is the original interval: $(-1, 1)$.

$\mathcal{P}$1.d $\bigcup_{n=2}^{\infty} [0, 1 - 1/n)$

As the value of n increases, the right endpoint gets closer and closer to 1, but never reaches. Indeed,

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right) = 1$$

This means that the union of all of these intervals also won't include 1. The answer is therefore $[0, 1)$.

$\mathcal{P}$1.e $\bigcup_{x \in \mathbb{Z}} \left(\bigcup_{n=2}^{\infty} [x, x + 1 - 1/n) \right)$

Using the results of the last subproblem, the inner union is $[x, x+1)$. The outer union then takes this interval and repeats it across all integers x. This gives $\mathbb{R}$ as the result.

$\mathcal{P}$2

$\mathcal{P}$2.a $\left(\bigcup_{x \in \mathbb{Z}} \{x, x+1, x+2\} \right)^C$

Taking the complement results in removing the integers from the reals, or in shorter form, $\mathbb{R} \backslash \mathbb{Z}$.

$\mathcal{P}$2.b $\left(\bigcup_{n \in \mathbb{N}} (-n, n) \right)^C$

The complement of $\mathbb{R}$ with respect to $\mathbb{R}$ is the empty set, $\{\}$.

$\mathcal{P}$2.c $\left(\bigcup_{n=2}^{\infty}[0,1-1/n)\right)^C$

The complement is the result of simply removing $[0,1)$ from the reals. Doing so yields $pars{-\infty,0} \cup [1,\infty)$.

$\mathcal{P}$2.d $\left(\bigcup_{x\in\mathbb{Z}}\left(\bigcup_{n=2}^{\infty}[x,x+1-1/n)\right)\right)^C$

The complement of the reals is just the empty set, $\{\}$.

$\mathcal{P}$3

$\mathcal{P}$3.a $|A\times B| = |A|\cdot|B|$

We can use the A as the outer indexing collection and B as the inner indexing collection. To carry out the cartesian product, we traverse through the entirety of B $|A|$ times. This results in a set with cardinality $|A|\cdot|B|$.

$\mathcal{P}$3.b $|\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\varnothing\right)\right)\right)\right)|$

We won't take each powerset function, since that would be very cumbersome. The initial set is the empty set, therefore it has 0 elements. For an initial set with cardinality n, the powerset produces a set with cardinality 2^n. This makes sense because for each element, there are two choices: include them in the subset or don't. We can then compute,

$$
\begin{aligned}
|\mathcal{P}\left(\varnothing\right)| &= 1 \\
|\mathcal{P}\left(\mathcal{P}\left(\varnothing\right)\right)| &= 2 \\
|\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\varnothing\right)\right)\right)| &= 4 \\
|\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\varnothing\right)\right)\right)\right)| &= 16
\end{aligned}
$$

In general, define $\mathcal{P}_n\left(\varnothing\right)$ as the set produced by applying the powerset n times to the empty set. We see that $|\mathcal{P}_n\left(\varnothing\right)| = 2^{|\mathcal{P}_{n-1}(\varnothing)|}$.

$\mathcal{P}$4

$\mathcal{P}$4.a $(A\backslash B)\subseteq A$

Define $C := A\backslash B$. We wish to prove that $C\subseteq A$. By definition, C is made up of all the elements of A that don't appear in B. Therefore all the elements of C will be in A, proving that it is a subset.

$\mathcal{P}$4.b $(A \cup B)^C = A^C \cap B^C$

Define an arbitrary member x such that $x \in (A \cup B)^C$. By definition, $(x \notin A) \wedge (x \notin B)$. Therefore, by the definition of the complement of a set, $\left(x \in A^C\right) \wedge \left(x \in B^C\right)$. Since this works for an arbitrary x, $(A \cup B)^C \subseteq A^C \cap B^C$.

Similarly, define an arbitrary member y such that $y \in A^C \cap B^C$. Therefore, by the definition of the intersection, $\left(y \in A^C\right) \wedge \left(y \in B^C\right)$. By the definition of the complement $(y \notin A) \wedge (y \notin B)$. We can replace the conjunction operator with a set union using the definition of a union to write $y \notin (A \cup B)$. Using the definition of a complement again, $y \in (A \cup B)^C$. Since this works for an arbitrary y, $A^C \cap B^C \subseteq (A \cup B)^C$.

Putting these two arguments together shows that $(A \cup B)^C = A^C \cap B^C$.

$\mathcal{P}$5 $A \triangle B = (A \cup B) \backslash (A \cap B)$

Since we are already given a definition of the symmetric difference, it is sufficient to prove

$$(A \backslash B) \cup (B \backslash A) = (A \cup B) \setminus (A \cap B)$$

The symmetric difference only retains elements in A and B that are unique to them, relative to each other.

Consider an element x that is an arbitrary member of $A \triangle B$. That element, by definition is a member of exactly one set. If it is only a member of one set, then it is not in the intersection of the two sets. And is therefore also a member of $(A \cup B) \setminus (A \cap B)$. Therefore, $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$. We can repeat the same exact argument for a member of $(A \cup B) \setminus (A \cap B)$ to see that $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$. Therefore, $(A \backslash B) \cup (B \backslash A) = (A \cup B) \setminus (A \cap B)$.