$\mathcal{P}$1

$\mathcal{P}$1.a $\forall x \in \mathbb{R}, \left(x^2 < 73 \Rightarrow 0 < 1\right)$

Since 0 is always less than 1, we are done by trivial proof.

$\mathcal{P}$1.b $\forall x \in \mathbb{Z}, \left(-x^2 > 0 \Rightarrow x = 5\right)$

Multiplying both sides of the inequality and flipping the direction of the sign yields $x^2 < 0$. This is never true for any real x., so we are done by vacuous proof.

$\mathcal{P}$2

$\mathcal{P}$2.a $x \equiv 1 \pmod 2 \Rightarrow 7x - 5 \equiv 0 \pmod 2$

Since x is odd, it can be written as $x = 2k + 1$ for some $k \in \mathbb{Z}$. Substituting,

$$\begin{aligned} 7x - 5 &= 7\left(2k + 1\right) - 5 \\ &= 14k + 2 \\ &= 2\left(7k + 1\right) \end{aligned}$$

A number is even if it can be written in the form $2k'$ for some $k' \in \mathbb{Z}$. This is true for $7x - 5$ if x is odd and $k' = 7k + 1$.

$\mathcal{P}$2.b $a, c \equiv 1 \pmod 2 \Rightarrow ab + bc \equiv 0 \pmod 2$

We can factor the expression into $\left(a + c\right) b$. Since a and c are odd, we can write them as $a = 2k+1, b = 2k'+1$ for $k, k' \in \mathbb{Z}$. Substituting and simplifying, we get $b\left(2k + 2k' + 2\right) = 2b\left(k + k' + 1\right)$. Defining $k'' := b\left(k + k' + 1\right)$, we see that the entire expression can be written as $2k''$, proving the quantity to be even.

$\mathcal{P}$2.c $\exists x, y \in \mathbb{Z} : \forall k \in \mathbb{Z}, 2k + 1 = x^2 - y^2$

We prove this for an arbitrary odd number k. Since k is odd, we can write $k = 2x + 1$ for some $x \in \mathbb{Z}$. Fix y such that $y = x + 1$. We have

$$\begin{aligned} y^2 - x^2 &= \left(y - x\right)\left(y + x\right) \\ &= \left(2x + 1\right) \\ &= k \end{aligned}$$

$\mathcal{P}$3

$\mathcal{P}$3.a $x \equiv 1 \pmod 2 \iff x^3 \equiv 1 \pmod 2$

We first prove that if x is odd then x^3 is odd. If x is odd then we can write it as $2k+1$ for some $k \in \mathbb{Z}$. We have

$$\begin{aligned} x^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2\left(4k^3 + 6k^2 + 3k\right) + 1 \end{aligned}$$

We see that $x^3 = 2k' + 1$ for $k' = 4k^3 + 6k^2 + 3k$, making it even.

We then prove that if x^3 is odd then x is odd. Seeing this to be cumbersome, we prove the contrapositive: if x is even then x^3 is even. If x is even, then $x = 2n$ for some $n \in \mathbb{Z}$. Then $x^3 = 8n^3 = 2\left(4n^3\right)$. We see that x^3 can be written in the form of $2n'$, where $n = 4n^3$, making it even. Using the contrapositive statement, which we have proven to be true, we can then see that if x^3 is odd then x is also odd.

Putting these two arguments together, we see that x is odd if and only if x^3 is odd.

$\mathcal{P}$3.b $4 \mid x^2 \Rightarrow x \equiv 1 \pmod 2$

We prove the contrapositive: if x is even then 4 divides x^2. If x is even it can be written as $x = 2k$ for some $k' \in \mathbb{Z}$. Then $x^2 = 4k'^2$. Since x^2 can be written as $x^2 = 4k$ for $k = k'^2$, 4 divides x^2.

$\mathcal{P}$4

$\mathcal{P}$4.a No Largest Integer

We proceed by contradiction. Suppose that there is an $x \in \mathbb{Z}$ that is the largest integer. However, the integers are closed under addition, so there exists a $y = x + 1$ for all x. Moreover, $y > x$, so x isn't the largest integer, forming a contradiction.

$\mathcal{P}$4.b No Smallest Positive Rational

We proceed by contradiction. Suppose $x \in \mathbb{Q}$ is the smallest rational number. Since x is rational, it can be written as $\frac{p}{q}$ for $p, q \in \mathbb{Z}$. Conside the rational number $y := \frac{x}{2} = \frac{p}{2q}$. Clearly $y < x$, forming a contradiction.

$\mathcal{P}$4.c Product of Two Irrationals is Irrational

We proceed by contradiction. Assume that the product of two irrational numbers is irrational. Consider the irrational numbers $x := \pi$ and $y := \frac{1}{\pi}$. Their product xy is 1, which is a rational number, forming a contradiction.

$\mathcal{P}$4.d Sum of Rational and Irrational is Irrational

We proceed by contradiction. Assume that the sum of a rational number and an irrational number is rational. Choose $x \in \mathbb{R} \backslash \mathbb{Q}, y \in \mathbb{Q}$. Define $z := x + y \in \mathbb{Q}$. Since y and z are rational, we can write them as $x = \frac{p_x}{q_x}, z = \frac{p_z}{q_z}$

for some $p_x, q_x, p_z, q_z \in \mathbb{Z}$. We have

$$\begin{aligned} z &= x + y \\ x &= z - y \\ &= \frac{p_z}{q_z} - \frac{p_x}{q_x} \\ &= \frac{p_z q_x - p_x q_z}{q_z q_x} \\ &= \frac{p'}{q'} \end{aligned}$$

This implies that x is a rational number, since it can be written in the form of $\frac{p'}{q'}$, where $p', q' \in \mathbb{Z}$. This is a contradiction, so the sum must then be irrational.

$\mathcal{P}$5

Note: we define the absolute value $|x|$ as follows:

$$|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$

$\mathcal{P}$5.a Triangle Inequality

Using the definition of absolute value (defined above), we have

$$\begin{aligned} -|x| &\leqslant x \leqslant |x| \\ -|y| &\leqslant y \leqslant |y| \end{aligned}$$

Adding these inequalities up, we have

$$\begin{aligned} -(|x| + |y|) \leqslant x + y \leqslant (|x| + |y|) \\ |x + y| \leqslant |x| + |y| \end{aligned}$$

$\mathcal{P}$5.b Reverse Triangle Inequality

We use the substitution and then use the triangle inequality.

$$\begin{aligned} ||x| - |y|| &= ||x - y + y| - |y|| \\ &\leqslant ||x - y| + |y| - |y|| \\ &= ||x - y|| \\ &= |x - y| \end{aligned}$$