Relatively Innocuous Title

Definition 6.1. *A **relation** R between two sets A and B is a subset of $A \times B$. We say that if $(a, b) \in R$, a is **related** to b. We may also write aRb to signify the same thing.*

Note 6.2. *Sometimes, we may also use $\sim$ (`\sim`) as a symbol for a relation, because syntactically it is nice to see $a \sim b$. Also, it is common to let $A = B$, in which case we will say $\sim$ is a relation on A.*

Example: Suppose $\sim$ is a relation on $\mathbb{Z}$. We will say that a is related to b if $|a - b| = 1$. Thus, $\sim = \{(x, x+1), (x, x-1) \mid x \in \mathbb{Z}\}$.

Definition 6.3. *A relation $\sim$ is **reflexive** if for any a, $a \sim a$.*

Example: $A = \mathbb{Z}, \sim = \{(a, b) \mid a \text{ divides } b \text{ and } a, b \in \mathbb{Z}\}$

Nonexample: $A = \mathbb{Z}, \sim = \{(a, b) \mid a > b \in \mathbb{Z}\}$

Definition 6.4. *A relation $\sim$ is **symmetric** if for any a, b, $a \sim b \Rightarrow b \sim a$.*

Example: $A = [5], \sim = \{(1, 2), (2, 1)\}$

Nonexample: $A = \mathbb{Z}, \sim = \{(a, b) \mid a \text{ divides } b \text{ and } a, b \in \mathbb{Z}\}$

Definition 6.5. *A relation $\sim$ is **transitive** if for any a, b, c, $(a \sim b \wedge b \sim c) \Rightarrow a \sim c$.*

Example: $A = \mathbb{Z}, \sim = \{(a, b) \mid a \text{ divides } b \text{ and } a, b \in \mathbb{Z}\}$

Nonexample: $A = [3], \sim = \{(1, 2), (2, 3)\}$

We will now observe that if we constructed a bipartite, undirected graph with $|A| + |B|$ vertices, we could replicate a relation by basically letting $E = R$. If $A = B$, then we can talk about a directed graph with just $|A|$ vertices instead. Reflexivity implies that every vertex has a self-loop. Symmetry implies that the graph is basically undirected. Transitivity is hard to interpret with a graph, but essentially every "chain" of edges in your graph induces another edge.

Definition 6.6. *An **equivalence** relation is a relation that is reflexive, symmetric, and transitive.*

Note 6.7. *I always remember this because R(eflexive), S(ymmetric), and T(ransitive) are consecutive letters in the alphabet. Conveniently, the definitions also use 1, 2 and 3 variables respectively.*

We will discuss one particularly important equivalence relation: $\equiv_n = \{(a, b) \mid a, b \in \mathbb{Z}, n \mid a - b\}$. (Note that some authors use the alternate notation $a \equiv b \pmod{n}$.) Let's observe what happens with $n = 3$. We find that $0 \equiv_3 3 \equiv_3 6 \equiv_3 9 \ldots$. Also, because the relation is transitive, all pairs of elements just listed are related. The same happens for $1, 4, 7, 10, \ldots$ and $2, 5, 8, 11, \ldots$. Something interesting is happening here.

Definition 6.8. *A **partition** of a nonempty set A is a collection $\mathcal{A}$ of subsets of A satisfying the following three conditions:*

(i) $\varnothing \notin \mathcal{A}$.

(ii) *If $P, Q \in \mathcal{A}$ such that $P \neq Q$, then $P \cap Q = \varnothing$.*

(iii) $\bigcup_{P \in \mathcal{A}} P = A$.

In other words, the elements of $\mathcal{A}$ are disjoint, and their union is A itself.

Proposition 6.9. *An equivalence relation on A induces a partition on A. Two elements are in the same part of the partition exactly when they are related.*

Proof. Let $\sim$ be an equivalence relation on A, and consider the sets $A_a \coloneqq \{x \in A \mid x \sim a\}$. Then for any a, b in A, where $a \sim b$, we will prove $A_a = A_b$. Suppose $x \in A_a$, which means $x \sim a$. Since $a \sim b$, transitivity shows that $x \sim b$, and therefore $x \sim b$. Symmetrically the other direction is true, so $A_a = A_b$ iff $a \sim b$.

Let $\mathcal{A} = \{A_a \mid a \in A\}$. Note that $\varnothing \notin \mathcal{A}$ since if $A_a \in \mathcal{A}$, then $a \in A_a$. Pick some A_a, A_b, and suppose that there is some x in their intersection. Then $x \sim a$, and $x \sim b$. But this means that $a \sim b$, so $A_a = A_b$. Therefore there are no two distinct sets in $\mathcal{A}$ with nonempty intersection. Also, the union of all A_a must be A itself, as each a appears in at least A_a. $\square$

Definition 6.10. *We refer to each A_a above as an **equivalence class**.*

From our example before, we see that $\equiv_3$ generates three equivalence classes, and hence partitions $\mathbb{Z}$. Notice that everything in an equivalence class is related, and every pair of things in two different equivalence classes is not related. In general $\equiv_n$ generates n equivalence classes, and we call the set of the equivalence classes under $\equiv_n$ to be $\mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$. Also, for ease of notation, we give equivalence classes a "representative element," and in this case we would just use $[m]$ for the equivalence class of m.

Why do we care about equivalence relations and equivalence classes? When we have a set and define an equivalence relation on them, this gives us a way to define which elements are "equal" and, furthermore, define functions and operations on them. For example, in $\mathbb{Z}_3$, we can define addition as in $\mathbb{Z}$: $[n] + [m] = [n + m]$. When we do define this operation, however, we need to check that it is well-defined: that is, that the operation does not depend on the choice of representative. How can we be sure that if $n_1, n_2 \in [n]$, we have that $n_1 + m$ and $n_2 + m$ are both in the same equivalence class? This is something that we need to check! If $n' \in [n]$ and $m' \in [m]$ in $\mathbb{Z}_3$, then $n' = 3k + n$ and $m' = 3l + m$ for some $k, l \in \mathbb{Z}$. Then $n' + m' = 3(k + l) + (n + m)$, which shows that $n' + m'$ is in $[n + m]$. Hence this operation is well-defined. You should check that you can construct an addition table using this operation.

Also, this implies that the graph representing an equivalence relation is a bunch of K_i's glued together (recall that K_n is the fully connected graph with n vertices).

Definition 6.11. *A relation is **anti-symmetric** if for all $a \neq b$, either a is not related to b or b is not related to a.*

Definition 6.12. *A relation is a **partial ordering** if it is reflexive, anti-symmetric, and transitive.*

Note 6.13. *A partial ordering should be thought about like its name implies. It orders elements in some sense, but it is not always true that for any a, b, one of them is always larger than the other. In that sense, it is only partial.*

One example of a partial ordering is the usual order $\leq$ on $\mathbb{N}$, $\mathbb{Z}$, etc. It is easy enough to check this, so we will leave it as an exercise for you to verify. Next, let A be a nonempty set, and consider the relation given by $\subseteq$ on $\mathcal{P}(A)$. We will show that this is a partial order. First, we have that for any $X \subseteq A$, $X \subseteq X$, so our relation is reflexive. Next, if we have $X, Y \in \mathcal{P}(A)$ such that $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$, so our relation is antisymmetric. Finally, if $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$. Thus our relation is a partial order.

Definition 6.14. *A relation is a **strict partial ordering** if it is irreflexive, antisymmetric, and transitive.*

It is easy to verify that the usual order $<$ on $\mathbb{N}$, $\mathbb{Z}$, etc. and the strict subset relation $\subset$ on $\mathcal{P}(A)$ are strict partial orders. You should verify these yourself.

Definition 6.15. *Let A be a nonempty set and let $\prec$ be a partial order on A. If $B \subseteq A$, we say that $\alpha \in A$ is an **upper bound** for B if $b \prec \alpha$ for all $b \in B$. Furthermore, if α has the property that $\alpha \prec \beta$ for all upper bounds β for B, then we say that α is a **least upper bound** for B, or **supremum** of B, and write $\alpha = \sup B$ (`\sup B`). If $\alpha = \sup B \in B$, then we say that α is the **maximal element** of B.*

Lower bounds and greatest lower bounds (infimum, $\inf B$) are defined analogously. As an example, let $A = [5] = \{1, 2, \ldots, n\} \subseteq \mathbb{Z}$ under $\leq$ the usual order. Then we can see that 6 is an upper bound for A, and 5 is the least upper bound for A, and furthermore it is a maximum. As another example, consider $\mathcal{P}([5])$ under the $\subseteq$ relation. Define $\mathcal{Q} = \mathcal{P}([5]) \setminus \{[5]\}$. Then $[5]$ is itself the least upper bound of $\mathcal{Q}$. There are sets where the least upper bound doesn't exist. We will return to this idea in a later lecture.

Homework

1. Consider $[5] = \{1, 2, \ldots, 5\}$. Give an example of, or show that such a request impossible, a nonempty relation on $[5]$ that is:

   (a) not symmetric, but reflexive and transitive.

   (b) not transitive, but reflexive and symmetric.

   (c) not reflexive, but symmetric and transitive.

   (d) both symmetric and antisymmetric.

2. Determine if each of the following relation on a set A is an equivalence relation or not. If so, exhibit the equivalence classes. Justify each answer.

   (a) $A = \mathbb{R}^2$, $(a, b)R(c, d)$ if $a^2 + b^2 = c^2 + d^2$

   (b) $A = \mathbb{Q}$, $R = \varnothing$ the empty relation.

3. Prove or disprove the converse of Proposition 6.9: that is, a partition $\mathcal{A}$ on A induces an equivalence relation on A by $x \sim y$ if and only if there exists some $B \in \mathcal{A}$ such that B contains both x and y.

4. Consider the relation $D \coloneqq \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ divides } b\}$. Show that D is a partial ordering.

5. This problem will deal with what we call a total ordering.

   Definition 6.16. *A* ***total ordering****, or a* ***linear ordering*** *is a partial ordering such that any two elements are comparable; that is, for any a and b, either $a \prec b$ or $b \prec a$.*

   Suppose A and B are two sets, with total orderings $\prec_A$ and $\prec_B$ respectively. Define

   $$\prec_L := \{((a, b), (c, d)) \mid (a \neq c \wedge a \prec_A c) \vee (a = c \wedge b \prec_B d)\},$$

   and

   $$\prec_P := \{((a, b), (c, d)) \mid a \prec_A c \wedge b \prec_B d\}.$$

   Note 6.17. *$\prec_L$ is known as the lexicographic ordering, and $\prec_P$ is known as the product ordering.*

   (a) Describe in your own words how $\prec_L$ and $\prec_P$ work.

   (b) Show that $\prec_P$ is a partial ordering.

   (c) Show that $\prec_P \subseteq \prec_L$.

6. Let A be a nonempty set, and consider $\mathcal{P}(A)$ with $\subseteq$ being the partial ordering. Let $\mathcal{A}$ is any family of subsets of A. Find $\sup \mathcal{A}$ and $\inf \mathcal{A}$ and prove your answers.

7. Let A be a nonempty set, and consider $\mathcal{S} = \{\mathcal{A} \subseteq \mathcal{P}(A) \mid \mathcal{A} \text{ is a partition of } A\}$. Define a relation $\preceq$ on $\mathcal{S}$ by $\mathcal{B} \preceq \mathcal{C}$ if for each $C \in \mathcal{C}$, there exists some $B \in \mathcal{B}$ such that $C \subseteq B$. In this case, we say that $\mathcal{C}$ is a **refinement** of $\mathcal{B}$. Show that $\preceq$ defines a partial order on $\mathcal{S}$. What is the maximal element?