Foreword

Proofs are hard. In order of importance, keep the following things in mind:

1. Do not make assumptions. There are a lot of silly things we take for granted, but as long as we are in 300 or haven't previously proven it, do not make unfounded assumptions.
2. Use English effectively. I cannot parse your proof if it is a string of logical symbols, or a stream of mathematical facts. Proofs should look like an English paragraph (or more).
3. Do not write *War and Peace*. It is very difficult to accidentally write *War and Peace*, but some of you manage this. Cut your proofs down to the very bone! I want to see dense proofs where every single word is necessary. But make sure to not infringe on the previous two points when doing so.

Graphic Illustrations and Where to Find Them

Note 5.1. *Sorry, there are no illustrations in here.*

Definition 5.2. *A **graph** is a pair (V, E), where V is an arbitrary (finite) set, and $E \subseteq V \times V$. Recall that $\times$ is the Cartesian product of two sets. An element of V is known as a **vertex**, and an element of E is known as an **edge**.*

Definition 5.3. *An **undirected graph** is a graph $G = (V, E)$ where $\forall(a, b) \in E, (b, a) \in E$.*

Note 5.4. *By convention, we will just consider (a, b) and (b, a) to be equal when considering undirected graphs.*

Definition 5.5. *A **self-loop** is an edge (a, a).*

Definition 5.6. *A **simple graph** is an undirected graph with no self-loops.*

Example: Let $G = (V, E)$ be a graph where $V = [5]$ and $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$. This graph is known as C_5, the cycle graph with 5 vertices.

Example: Let $G = (V, E)$, where $V = [5]$ and $E = \{(i, j) \mid i, j \in V, i \neq j\}$. This graph is known as K_5, the connected graph with 5 vertices.

Example: Let $G = (V, E)$, where $V = [5]$ and $E = \{(i, i + 1) \mid 1 \leq i \leq 4\}$. This graph is known as P_5, the path graph with 5 vertices.

Example: Let $G = (V, E)$, where $V = [4]$ and $E = \{(1, i) \mid 2 \leq i \leq 4\}$. This graph is known as S_3, the star graph with three leaves. In particular, this graph is known as a "claw."

Definition 5.7. *A **path** in a graph is a sequence of unique vertices $v_1, \ldots, v_n$ such that each pair of consecutive vertices are connected.*

Example: The path graph P_n has a path from vertex 1 to vertex n.

Definition 5.8. *A **cycle** in a graph is path from u to v such that u and v are also connected. A graph which does not have a cycle is called **acyclic**.*

Example: C_5 has exactly one cycle of length 5. P_n and S_n are acyclic.

Definition 5.9. *A graph is **connected** if for any two vertices, there is a path between them.*

Definition 5.10. *A **tree** is a connected, acyclic graph.*

Example: P_n and S_n are trees. C_n is not a tree because it contains a cycle, for $n > 2$. K_n is not a tree because it contains many cycles, also for $n > 2$. $G = ([4], \{(1, 2), (3, 4)\})$ is not a tree because 1 and 4 are not connected.

Definition 5.11. *The **degree** of a vertex is the number of vertices it is connected to. If a vertex has degree 1, it is called a **leaf**.*

Definition 5.12. *The **well-ordering principle (WOP)** states that every non-empty set of positive integers contains a least element.*

Note 5.13. *Do not confuse the well-ordering principle with the well-ordering theorem, which states that every set can be well ordered (even $\mathbb{R}$)! We'll come back to this later when we later discuss order relations, but note that this is a highly nontrivial fact; in fact, this is equivalent to the Axiom of Choice.*

Proposition 5.14. *Every tree has at least two leaves.*

Proof. Let $G = (V, E)$ be a tree. Let S be the set of lengths of paths in G. S is a set of integers, bounded above by $|V|$. By WOP, it must have a maximum, m. So there exists some path $v_1, \ldots, v_m$, and there are no paths in G with length more than m.

Now, let s be an arbitrary vertex that v_1 is connected to.

The sequence $s, v_1, \ldots, v_m$ has length $m + 1$, and therefore cannot be path. Therefore s must be a duplicate, so $s = v_i$ for some i. Since there is a path from $v_1, \ldots, v_i$, and G is acyclic, v_i is necessarily v_2.

Thus, the only neighbor of v_1 is v_2, implying that v_1 has degree one, and is a leaf. The same applies to v_m, so v_1 and v_m are two leaves. □

Proposition 5.15. *Every tree with n vertices has $n - 1$ edges.*

Proof. We will proceed by induction. The graph with 1 vertex is connected and acyclic, so it is a tree. It has 0 edges, so the claim holds for $n = 1$.

Suppose that any tree with n vertices has $n-1$ edges. Pick an arbitrary tree $G = (V, E)$ with $|V| = n+1$. By Proposition 5.14, it has at least one leaf, v. Consider the graph G', obtained by deleting v and the single edge connected to it.

For any $s, t \in V \setminus \{v\}$, the fact that G is connected implies there is a path $v_1, \ldots, v_k$ where $v_1 = s$ and $v_k = t$. v cannot appear anywhere in the middle of this path, as it has degree one. Therefore G' is still connected, as the same path between any two vertices still exists.

The edges of G' are a subset of the edges of G, which means that there cannot be a cycle in G', as G would then also have a cycle. Therefore G' is a tree with n vertices.

By the inductive hypothesis, G' has $n - 1$ edges. Since we deleted exactly one edge to get G', G has n edges. □

Definition 5.16. *A **bipartite graph** is a graph $G = (V, E)$ where there exist two sets A and B, such that $A \cup B = V$, $A \cap B = \varnothing$, and $E \subseteq A \times B$. That is to say, there is a disjoint partition of the vertices such that the only edges are strictly between the two partitions of the vertices.*

Example: P_n is bipartite. Pick $A = \{2k + 1 \mid 1 \leq 2k + 1 \leq n\}, B = V \setminus A$. The only edges are between consecutive vertices, and the vertices in A and B respectively all have the same parity.

Example: S_n is bipartite. Pick $A = \{1\}, B = V \setminus A$. The only edges are between the leaves and the center, so the definition is satisfied.

Homework 5

1. Find the error in the proof of Proposition 5.14 (it is not true!). Respond in the homework channel on Discord rather than here, and discuss potential solutions.

2. Prove that the sum of degrees of vertices in a simple graph is even.

3. (a) Prove that a connected, simple graph with $|V| - 1$ edges is acyclic.
   (b) Prove that an acyclic, simple graph with $|V| - 1$ edges is connected.

4. Prove that a simple graph is bipartite iff it contains no odd-length cycles.

5. A simple graph is said to be k-colorable if there is a function $f : V \to [k]$ such that for any edge (a, b), $f(a) \neq f(b)$.

   (a) Show that a simple graph is 2-colorable iff it is bipartite.
   (b) † Show that a simple graph whose vertices all have degree at most k is $(k + 1)$-colorable.

Additional Exercises for Practice

6. *Warning 1: This problem is quite hard, do not attempt it unless you are confident with your answers to all other problems.*

   Warning 2: It is not recommended to read this.

   Life has been going well for you at work, with your $n+1$ other coworkers. It is just another Sunday afternoon. As usual, you are relaxing in the cafeteria, admiring the large red button in the middle. You walk around it, entranced by its roundness. You wonder if someday, you too could be as round as such a button.

   And that's when it happened.

   Suddenly, the lights dim, and you find yourself surrounded by darkness. It's hard to even make out your hands in front of your face. You figure you should head over to the electrical room and see if there's anything you can do.

   Joe, who had also been admiring the button with you, follows along. As you enter the electrical room, you see a couple other people trying to figure out what's wrong. Joe heads over to the back to see if there's anything he can do. After a brief moment, the lights return, and everyone breathes a sigh of relief. You yell over to Joe, "Hey, looks like we're all good!" But as you peek your head around the corner...

   You are just in time to see the lower half of Joe's body fall over, and the air vent next to him slam shut. You shriek wildly, sobbing. How could this happen to Joe? You think back to the time when you had tried to impress Joe Mother with card tricks. It will never be the same.

   Dashing out of the room, you know what you have to do. You head directly to the cafeteria, and slam the button with all your might. As the alarm pierces the air, the crew rushes to join you.

   Once everyone has gathered, you know there are no two ways about this. You say it bluntly for everyone to realize - "*There is an impostor sumongus.*"

   Accusations suddenly fly through the air. An accusation is a triple (u, v, w), where u accuses v of w sussiness, u is not equal to v, and w is a natural number. The result of the accusation (u, v, w) is that u's sussiness decreases by w and v's increases by w. Of course, no one sends any accusations towards you, due to your closeness with Joe. You don't accuse anyone else, either. Instead, you realize that you need to quell the situation, otherwise you have no chance of catching the sumongus.

   To do this, you whip out your famous deck of cards. Your best **card trick** to date is the one where you pick two participants, u and v, and you feed a card w to u. You then pretend to retrieve the same card from behind v's ear. Since this is very suspicious of v, v's sussiness increases by w, and u's sussiness decreases by w, where w is the number on the card. However, since you only have n cards, you can only perform the trick $n-1$ times. On the bright side at least, you can change the number on a card to be whatever you want. Formally, a card trick the same type as as an accusation.

   At the end of all your tricks, you would like it to be the case that each person has accrued a net sussiness of zero. How can you make sure of this?

   Formally, let $G = (V, E)$ be a weighted graph with no self-loops. A **weighted graph** is a graph where the edge set consists of triples (u, v, w), which indicate that there is an edge from u to v with weight w.

   Define $I_E(u)$ to be the sum of weights of edges in E going into u, and likewise, define $O_E(u)$ to be the sum of weights of edges in E going out of u. We wish to construct a set C of card tricks, such that for all $u \in V$, we have

   $$I_E(u) + I_C(u) = O_E(u) + O_C(u).$$

   Can you help avenge Joe?

   (a) †† In other words, C represents a set of card tricks taken place to resolve all sussiness, either directly or indirectly[1]. Show how to construct such a C, where $|C| \le |V| - 1$.

[1] This problem comes from the real life situation, where many people may be paying for one another, and a complex set of debts arises. It's nice to be able to resolve all debts with few payments, and it turns out you can always do it in $n-1$ payments!

(b) †† Now suppose that, we can only make card tricks between two people if one of them directly accused the other[2]. You can also assume this time, that the graph

$$H = (V, \{(u, v) \mid \exists w \in \mathbb{N} : (u, v, w) \in E \vee (v, u, w) \in E\})$$

is connected. Find a set C that satisfies all above conditions, and also has the property that for any $(u, v, w) \in C$, either $(u, v, x) \in E$ or $(v, u, x) \in E$ for some x.

[2]This can arise if the people paying have to friend each other on the same app in order to pay each other. So sometimes, it's not possible to make arbitrary payments between any two people.