Problem 11

Problem 11.a

We first find the probability of A_s. Since the ace of spades is only 1 card, $\mathbb{P}(A_s) = \frac{1}{52}$.

On the other hand, $A_s \cap B$ is the event that both cards drawn are spades and one of them is the ace of spades. This means the other card has 3 other options available for its suite. The ace of spades can either be the first or second, and adding together the respectively probabilities yields

$$\mathbb{P}(A_s \cap B) = \frac{1}{52} \cdot \frac{3}{51} + \frac{3}{52} \cdot \frac{1}{51} = \frac{6}{51 \cdot 52}$$

Therefore, $\boxed{\mathbb{P}(B|A_s) = \frac{2}{17}}$.

Problem 11.b

The probability of at least one ace being chosen is the complement of no aces being chosen. That probability is $\mathbb{P}(A^c) = \frac{48}{52} \cdot \frac{47}{51} = \frac{188}{221}$. Therefore, $\mathbb{P}(A) = 1 - \frac{188}{221} = \frac{33}{221}$.

The event $A \cap B$ implies that both cards are aces. Therefore,

$$\mathbb{P}(A \cap B) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{13 \cdot 17}$$

Finally, $\boxed{\mathbb{P}(B|A) = \frac{1}{33}}$.

Problem 24

Problem 24.a

We can use the law of total probability to solve this problem. However, we must first define some notation for convenience. Let W be the event that the ball selected from urn II is white. Let X_W, X_R be the events that the ball transferred was white or red, respectively. Using the law of total probability, we have

$$\begin{aligned}
\mathbb{P}(W) &= \mathbb{P}(W|X_W)\,\mathbb{P}(X_W) + \mathbb{P}(W|X_R)\,\mathbb{P}(X_R) \\
&= \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3}\frac{2}{3} \\
&= \frac{4}{9}
\end{aligned}$$

Therefore, $\boxed{\mathbb{P}(W) = \frac{4}{9}}$.

Problem 24.b

Urn II will then have 2 white balls and 1 red ball. Using the notation from the last subproblem, $\boxed{\mathbb{P}(W|X_W) = \frac{2}{3}}$.

Problem 31

We simply count the probability that each ball hasn't been touched, making sure to reduce the denominator by 1 each time. This implies the probability to be $\frac{12}{15} \cdot \frac{11}{14} \cdot \frac{10}{12} = \boxed{\frac{11}{21}}$.

Problem 32

Let X_1 be the event that the first box is selected. Let X_2 be the event that the second box is selected. Let B be the event that the selected marble is black. Let W be the event that the selected marble is white. We use to law of total probability to have

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B|X_1)\,\mathbb{P}(X_1) + \mathbb{P}(B|X_2)\,\mathbb{P}(X_2) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} \\ &= \boxed{\frac{7}{12}}\end{aligned}$$

For the next part of the problem, we use Bayes' theorem.

$$\begin{aligned}\mathbb{P}(X_1|W) &= \frac{\mathbb{P}(W|X_1)}{\mathbb{P}(X_1)}\mathbb{P}(W) \\ &= \frac{\frac{1}{2}}{\frac{1}{2}}\left(\mathbb{P}(W|X_1)\,\mathbb{P}(X_1) + \mathbb{P}(W|X_2)\,\mathbb{P}(X_2)\right) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} \\ &= \boxed{\frac{5}{12}}\end{aligned}$$