HW 7

Problem 66

Problem 66.a

Since 1 person enters every 2 minutes, 2.5 people enter every 5 minutes. Therefore, $\lambda = \frac{5}{2}$. Then, the probability that x people enter the casino is

$$\begin{aligned}\mathbb{P}\left(x\right) &= \frac{\left(\frac{5}{2}\right)^{x} e^{-\frac{5}{2}}}{x!} \\ \mathbb{P}\left(0\right) &= \boxed{e^{-\frac{5}{2}}} \\ &\approx 0.0821\end{aligned}$$

Problem 66.b

Let X be a random variable representing the number of people entering the gambling casino between 12:00 and 12:05. We use the method of complements.

$$\begin{aligned}\mathbb{P}\left(X \geqslant 4\right) &= 1 - \mathbb{P}\left(X < 4\right) \\ &= 1 - \mathbb{P}\left(0\right) - \mathbb{P}\left(1\right) - \mathbb{P}\left(2\right) - \mathbb{P}\left(3\right) \\ &= 1 - e^{-\frac{5}{2}} - \frac{5}{2}e^{-\frac{5}{2}} - \frac{25}{8}e^{-\frac{5}{2}} - \frac{125}{48}e^{-\frac{5}{2}} \\ &= \boxed{1 - \frac{443}{48}e^{-\frac{5}{2}}} \\ &\approx 0.242\end{aligned}$$

Problem 2

We first find the value of C. Since the probability distribution must be normalized,

$$\begin{aligned}\int\limits_{-\infty}^{\infty} f\left(x\right)\ dx &= \int\limits_{0}^{\infty} Cxe^{-\frac{x}{2}}\ dx \\ &= 4C \\ &= 1\end{aligned}$$

Therefore, $C = \frac{1}{4}$.

$$\begin{aligned}\mathbb{P}\left(X \geqslant 5\right) &= \int\limits_{5}^{\infty} \frac{1}{4}xe^{-\frac{x}{2}}\ dx \\ &= \boxed{\frac{7}{2}e^{-\frac{5}{2}}} \\ &\approx 0.287\end{aligned}$$

Problem 8

Let X be a random variable representing the lifetime of the tube.

$$\begin{aligned}\mathbb{E}\left(X\right) &= \int_{0}^{\infty} x^2 e^{-x}\ dx \\ &= \boxed{2}\end{aligned}$$

Problem 10

Let ℓ be a random variable representing the position of the random point chosen. The ratio of the shorter segment to the longer segment is then, $\min\left\{\frac{\ell}{L-\ell}, \frac{L-\ell}{L}\right\}$. We then have,

$$\begin{aligned}\frac{\ell}{L-\ell} &< \frac{1}{4} \\ 4\ell &< L - \ell \\ 5\ell &< L \\ \ell &< \frac{L}{5}\end{aligned}$$

$$\begin{aligned}\frac{L-\ell}{\ell} &< \frac{1}{4} \\ 4L - 4\ell &< \ell \\ 4L &< 5\ell \\ \ell &> \frac{4L}{5}\end{aligned}$$

We see that these two events are disjoint. The ratio between the two segments being less than $\frac{1}{4}$ is then the same as the random point being in the far left or far right fifth of the line segment. Therefore, the probability is

$$\begin{aligned}\mathbb{P}\left(\ell < \frac{L}{5}\right) + \mathbb{P}\left(\ell > \frac{4L}{5}\right) &= \frac{1}{5} + \frac{1}{5} \\ &= \boxed{\frac{2}{5}}\end{aligned}$$