HW 10

Problem 26

Problem 26.a

Since the random variables are independent, $f(A < a, B < b, C < c) = f_A(a) f_B(b) f_C(c)$. Therefore, is any of the marginal distributions are 0, then the whole joint distribution will be 0. Therefore,

$$f(A < a, B < b, C < c) \begin{cases} 1 & \forall k \in \{a, b, c\} : (0 < k < 1) \\ 0 & \exists k \in \{a, b, c\} : (k > 1 \vee k < 0) \end{cases}$$

Problem 26.b

We start with the law of total probability, and condition over possible values of A. Then,

$$\mathbb{P}\left(B^2 \geqslant 4AC\right) = \int_0^1 \mathbb{P}\left(B^2 \geqslant\right)$$

Problem 34

Define $Z := X + Y$. Then, Z is a normally distributed random variable with mean 20 and variance 8. Therefore, $\mathbb{P}(X + Y > x) = \mathbb{P}(Z > x)$. Then, defining Z' as a standard normal variable,

$$\begin{aligned} \mathbb{P}(Z > x) &= \mathbb{P}(X > 15) \\ &= 1 - \mathbb{P}(X < 15) \\ &= 1 - \mathbb{P}\left(Z' < \frac{5}{2}\right) \\ 1 - \mathbb{P}(Z < x) &= \\ 1 - \mathbb{P}\left(Z' < \frac{x - 20}{2\sqrt{2}}\right) &= \\ \mathbb{P}\left(Z' < \frac{x - 20}{2\sqrt{2}}\right) &= \mathbb{P}\left(Z' < \frac{5}{2}\right) \\ \frac{x - 20}{2\sqrt{2}} &= \frac{5}{2} \\ x &= \boxed{20 + 5\sqrt{2}} \end{aligned}$$

Problem 45

Problem 45.a

We first find $f_X(x)$ and $f_Y(y)$.

$$\begin{aligned} f_X(x) &= \int_0^\infty x e^{-x(y+1)}\, dy \\ &= e^{-x} \\ f_Y(y) &= \int_0^\infty x e^{-x(y+1)}\, dx \\ &= \frac{1}{(y+1)^2} \end{aligned}$$

Then, we can find the conditional densities with some ease.

$$
\begin{aligned}
f\left(X = x | Y = y\right) &= \frac{f\left(X = x, Y = y\right)}{f_Y\left(y\right)} \\
&= \boxed{\frac{x}{\left(y+1\right)^2} e^{-x(y+1)}} \\
f\left(Y = y | X = x\right) &= \frac{f\left(X = x, Y = y\right)}{f_X\left(x\right)} \\
&= \boxed{x e^{-xy}}
\end{aligned}
$$

Problem 45.b

Since $Z = XY$, the bounds of Z are 0 and ∞.

$$
\begin{aligned}
f\left(Z = z\right) &= \int\limits_0^\infty f\left(x, \frac{z}{x}\right) \frac{1}{|x|} \, dx \\
&= \int\limits_0^\infty e^{-z-x} \, dx \\
&= \boxed{e^{-z}}
\end{aligned}
$$

Problem 46

Usually, we first find the value of c by demanding that the sum of the density function over the region be equal to 1. However, we only want to find the conditional distribution, so the c will cancel out regardless of its value.

We first find the marginal distribution of X.

$$
\begin{aligned}
f_X\left(x\right) &= \int_{-x}^{x} c\left(x^2 - y^2\right) \, dy \\
&= \frac{4}{3} c x^3 e^{-x}
\end{aligned}
$$

Then, the conditional distribution is

$$
\begin{aligned}
f\left(Y = y | X = x\right) &= \frac{f\left(Y = y, X = x\right)}{f_X\left(x\right)} \\
&= \boxed{\frac{3}{4}\left(\frac{1}{x} - \frac{y^2}{x^3}\right)}
\end{aligned}
$$