Problem 5

There are $(9-2)+1=8$ possible options for the first number, 2 possible options for the second number, and $(9-1)+1=9$ possible options for the third number. Multiplying these, we have $8 \cdot 2 \cdot 9 = \boxed{144}$ total options.

If the first number is a 4, then there is only 1 option for the first number. The number of options for the second and third numbers do not change. Therefore, there are $1 \cdot 2 \cdot 9 = \boxed{18}$ total options.

Problem 8

Problem 8.a

There are no repeats, so there are 5 choices for the first letter, 4 choices for the second letter, 3 choices for the third letter, 2 choices for the fourth letter, and 1 choice for the fifth letter. In total, that means there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = \boxed{120}$ different letter arrangements.

Problem 8.b

We note that the P and O each appear twice, and the R, S, and E each appear once. We assign the doubly-appearing letters first, starting with the P's. Of the seven letters in each arrangement, 2 must be occupied by P's, so there are $\binom{7}{2}$ possible options for this. There are $7-2=5$ letters left, so there must be $\binom{5}{2}$ options to choose where to put the O's. There are then 3 letters left in the arrangement, so there are 3 options to choose where to put the R, 2 options to choose where to put the S, and 1 option to choose where to put the E. Multiplying everything out,

$$\begin{aligned}\binom{7}{2} \cdot \binom{5}{2} \cdot 3 \cdot 2 \cdot 1 &= 21 \cdot 10 \cdot 3 \cdot 2 \cdot 1 \\ &= \boxed{1260}\end{aligned}$$

Problem 11

Problem 11.a

If the relative order doesn't matter, then each book can be treated arbitrarily, regardless of its type. There are 6 books in total, so there are $6! = \boxed{720}$ different ways to order the books.

Problem 11.b

Since there is only 1 chemistry book, and the 3 novels and 2 math books must be together, we can instead organize objects we define to be collections. Let the first collection be the 1 chemistry book, let the second collection be the two math books, and let the third collection be the 3 novels. We can then organize the collections, and there are no restrictions on the relative ordering of the collections. There are 3 collections, so there must be $3! = \boxed{6}$ ways to arrange the collections.

Problem 11.c

Similar to the previous subproblem, the 3 novels can be treated as one collection. Therefore, the total number of things we need to order is the 1 collection of novels, 2 math books, and 1 chemistry book, totalling 4. There must be $4! = \boxed{24}$ ways to arrange the collections.

Problem 26

Problem 26.a

We use the binomial theorem.

$$\begin{aligned}3^n &= (2+1)^n \\ &= \sum_0^k \binom{n}{k} (2)^n (1)^{n-k} \\ &= \sum_0^k \binom{n}{k} (2)^n\end{aligned}$$

Here, we have used the fact that $1^x = 1$ for all natural x.

Problem 26.b

Using the fact that $1^y = 1$ for all natural y, we can write

$$\begin{aligned}
\sum_{n=0}^{k} \binom{n}{k} x^k &= \sum_{n=0}^{k} \binom{n}{k} x^k 1^{n-k} \\
&= \boxed{(x+1)^k}
\end{aligned}$$

Here, we have used the binomial theorem.