Problem 5

There are $(9-2)+1=8$ possible options for the first number, 2 possible options for the second number, and $(9-1)+1=9$ possible options for the third number. Multiplying these, we have $8\cdot 2\cdot 9=\boxed{144}$ total options.

If the first number is a 4, then there is only 1 option for the first number. The number of options for the second and third numbers do not change. Therefore, there are $1\cdot 2\cdot 9=\boxed{18}$ total options.

Problem 8

Problem 8.a

There are no repeats, so there are 5 choices for the first letter, 4 choices for the second letter, 3 choices for the third letter, 2 choices for the fourth letter, and 1 choice for the fifth letter. In total, that means there are $5\cdot 4\cdot 3\cdot 2\cdot 1=5!=\boxed{120}$ different letter arrangements.

Problem 8.b

We note that the P and O each appear twice, and the R, S, and E each appear once. We assign the doubly-appearing letters first, starting with the P's. Of the 7 letters in each arrangement, 2 must be occupied by P's, so there are $\binom{7}{2}$ possible options for this. There are $7-2=5$ letters left, so there must be $\binom{5}{2}$ options to choose where to put the O's. There are then 3 letters left in the arrangement, so there are 3 options to choose where to put the R, 2 options to choose where to put the S, and 1 option to choose where to put the E. Multiplying everything out,

$$\begin{aligned}\binom{7}{2}\cdot\binom{5}{2}\cdot 3\cdot 2\cdot 1 &= 21\cdot 10\cdot 3\cdot 2\cdot 1\\ &= \boxed{1260}\end{aligned}$$

Problem 11

Problem 11.a

If the relative order doesn't matter, then each book can be treated arbitrarily, regardless of its type. There are 6 books in total, so there are $6!=\boxed{720}$ different ways to order the books.

Problem 11.b

Since there is only 1 chemistry book, and the 3 novels and 2 math books must be together, we can instead organize objects we define to be collections. Let the first collection be the 1 chemistry book, let the second collection be the two math books, and let the third collection be the 3 novels. Within, each collection, there is only 1 way to organize the chemistry books, $3!=6$ ways to organize the novels, and $2!=2$ ways to organize the math books. We can then organize the collections. There are 3 of them, so there must be $3!\cdot=6$ ways to arrange the collections. Therefore, the total number of ways to organize the books is $3!\cdot 2!\cdot 3!=\boxed{72}$.

Problem 11.c

Similar to the previous subproblem, the 3 novels can be treated as one collection. The number of ways to organize the novels within the collection is $3!=6$. On a slightly broader level, there are the collection of novels, the 2 math books, and 1 chemistry book. There are 4 objects in total, so the number of ways to organize them is $4!=24$. The total number of ways to organize the books in this way is therefore $3!\cdot 4!=\boxed{144}$.

Problem 26

Problem 26.a

We use the binomial theorem.

$$\begin{aligned} 3^n &= (2+1)^n \\ &= \sum_{0}^{k} \binom{n}{k} (2)^n (1)^{n-k} \\ &= \sum_{0}^{k} \binom{n}{k} (2)^n \end{aligned}$$

Here, we have used the fact that $1^x = 1$ for all natural x.

Problem 26.b

Using the fact that $1^y = 1$ for all natural y, we can write

$$\begin{aligned} \sum_{n=0}^{k} \binom{n}{k} x^k &= \sum_{n=0}^{k} \binom{n}{k} x^k 1^{n-k} \\ &= \boxed{(x+1)^k} \end{aligned}$$

Here, we have used the binomial theorem.