HW 7

Problem 39

We must first solve for $\mathbb{E}\left(X^2\right)$.

$$\begin{aligned}\mathbb{E}\left(X^2\right) - \mathbb{E}\left(X\right)^2 &= \mathrm{Var}\left(X\right) \\ \mathbb{E}\left(X^2\right) - 1 &= 5 \\ \mathbb{E}\left(X^2\right) &= 6\end{aligned}$$

Problem 39.a

We use linearity of expectation, finding

$$\begin{aligned}\mathbb{E}\left(\left(2+X\right)^2\right) &= \mathbb{E}\left(4 + 4X + X^2\right) \\ &= 4 + 4\mathbb{E}\left(X\right) + \mathbb{E}\left(X^2\right) \\ &= \boxed{14}\end{aligned}$$

Problem 39.b

We use the formula for variance and then linearity of expectation, finding

$$\begin{aligned}\mathrm{Var}\left(4 + 3X\right) &= \mathbb{E}\left(\left(4+3X\right)^2\right) - \mathbb{E}\left(4+3X\right)^2 \\ &= \mathbb{E}\left(16 + 24X + 9X^2\right) - \left(4+3\right)^2 \\ &= 16 + 24 + 54 - 49 \\ &= \boxed{45}\end{aligned}$$

Problem 40

Let X be a binomially-distributed random variable, that tells us the number of white balls drawn from the urn. Then,

$$\begin{aligned}\mathbb{P}\left(X = 2\right) &= \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \\ &= \frac{6}{16} \\ &= \boxed{\frac{3}{8}}\end{aligned}$$

Problem 60

Let X be a random variable that represents the number of accidents that happen on a highway today.

Problem 60.a

We could directly compute this using an infinite series, but will instead use the principle of complements, finding

$$\begin{aligned}
\mathbb{P}(X \geqslant 3) &= 1 - \mathbb{P}(X < 3) \\
&= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) \\
&= 1 - e^{-3} - 3e^{-3} - \frac{9}{2}e^{-3} \\
&= \boxed{1 - \frac{17}{2}e^{-3}} \\
&\approx 0.577
\end{aligned}$$

Problem 60.b

Here, X is strictly greater than 0. Therefore,

$$\begin{aligned}
\mathbb{P}(X \geqslant 3 | X \geqslant 1) &= \frac{\mathbb{P}(X \geqslant 3)}{\mathbb{P}(X \geqslant 1)} \\
&= \frac{1 - \frac{17}{2}e^{-3}}{1 - \mathbb{P}(X = 0)} \\
&= \frac{1 - \frac{17}{2}e^{-3}}{1 - 3e^{-3}} \\
&= \boxed{\frac{2e^3 - 17}{2e^3 - 6}} \\
&\approx 0.678
\end{aligned}$$

Problem 63

Let C be a Poisson random variable representing the number of colds the individual has in a year. Let R be the event that the indvidual is in the population for which $\lambda = 3$. R^c is then the event that the individual is in the population for which $\lambda = 5$. We can then compute, using Bayes' theorem and then the law of total probability,

$$\begin{aligned}
\mathbb{P}(R | C = 2) &= \mathbb{P}(C = 2 | R) \frac{\mathbb{P}(R)}{\mathbb{P}(C = 2)} \\
&= \mathbb{P}(C = 2 | R) \frac{\mathbb{P}(R)}{\mathbb{P}(C = 2 | R)\mathbb{P}(R) + \mathbb{P}(C = 2 | R^c)\mathbb{P}(R^c)} \\
&= \left(\frac{9e^{-3}}{2}\right) \frac{\frac{3}{4}}{\frac{3}{4} \cdot \frac{9e^{-3}}{2} + \frac{1}{4} \cdot \frac{25e^{-5}}{2}} \\
&= \left(\frac{9e^{-3}}{2}\right) \frac{\frac{3}{4}}{\frac{27e^{-3}}{8} + \frac{25e^{-5}}{8}} \\
&= \frac{\frac{27e^{-3}}{8}}{\frac{27e^{-3}}{8} + \frac{25e^{-5}}{8}} \\
&= \boxed{\frac{27e^2}{27e^2 + 25}} \\
&\approx 0.889
\end{aligned}$$