Problem 1

Let X be a random variable denoting the winnings. Since doing otherwise would be cumbersome, we make a table illustrating what the possible values of X are, as well as what ball combinations they correspond to.

Value of X	**Ball 1**	**Ball 2**
4	Black	Black
2	Black	Orange
1	Black	White
0	Orange	Orange
-1	White	Orange
-2	White	White

Note that **Ball 1** and **Ball 2** may be freely switched.

This means,

$$\begin{aligned}
\mathbb{P}(X = 4) &= \frac{4}{14} \cdot \frac{3}{13} \\
&= \boxed{\frac{6}{91}} \\
\mathbb{P}(X = 2) &= 2 \cdot \frac{4}{14} \cdot \frac{2}{13} \\
&= \boxed{\frac{8}{91}} \\
\mathbb{P}(X = 1) &= 2 \cdot \frac{4}{14} \cdot \frac{8}{13} \\
&= \boxed{\frac{32}{91}} \\
\mathbb{P}(X = 0) &= \frac{2}{14} \cdot \frac{1}{13} \\
&= \boxed{\frac{1}{91}} \\
\mathbb{P}(X = -1) &= 2 \cdot \frac{8}{14} \cdot \frac{2}{13} \\
&= \boxed{\frac{16}{91}} \\
\mathbb{P}(X = -2) &= \frac{8}{14} \frac{7}{13} \\
&= \boxed{\frac{4}{13}}
\end{aligned}$$

Problem 6

There are 4 possible values of X: $\{3, 1, -1, -3\}$. The first and last values correspond to all heads and all tails, respectively. The second and third values correspond to 2 heads and 2 tails, respectively.

$$
\begin{aligned}
\mathbb{P}(X = 3) &= \left(\frac{1}{2}\right)^3 \\
&= \boxed{\frac{1}{8}} \\
\mathbb{P}(X = 1) &= \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) \\
&= \boxed{\frac{3}{8}} \\
\mathbb{P}(X = -1) &= \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) \\
&= \boxed{\frac{3}{8}} \\
\mathbb{P}(X = -3) &= \left(\frac{1}{2}\right)^3 \\
&= \boxed{\frac{1}{8}}
\end{aligned}
$$

Problem 21

Problem 21.a

The only possible values of X that are possible are contained in $\{1, -1, 3\}$. When $X = 1$, this means that you won the first game, or that you lost the first and won the second and third. When $X = -1$, this means that you only won one of the last two games, and lost the first one. When $X = -3$, this means that you lost all three games.

Since the only positive value attainable by X is 1, we only need to compute $\mathbb{P}(X = 1)$ to compute $\mathbb{P}(X > 0)$. However, we will still compute $\mathbb{P}(X = -1)$ and $\mathbb{P}(X = -3)$, since it is necessary information for part (c) of this question.

For the sake of notation, we let G_i be the event that the ith game was won for $i \in [1, 3] \cap \mathbb{Z}$, so G_i^c is the event that the ith game was lost. We know that the games are independent from each other, and that $\mathbb{P}(G_i) = \frac{9}{19}$ and $\mathbb{P}(G_i^c) = \frac{10}{19}$.

We then have

$$\begin{aligned}
\mathbb{P}(X=1) &= \mathbb{P}(G_1) + \mathbb{P}(G_1^c \cup G_2 \cup G_3) \\
&= \frac{9}{19} + \frac{10}{19} \cdot \left(\frac{9}{19}\right)^2 \\
&= \frac{9}{19} + \frac{10}{19} \cdot \frac{81}{361} \\
&= \frac{9}{19} + \frac{810}{6859} \\
&= \frac{4059}{6859} \\
\mathbb{P}(X=-1) &= \mathbb{P}(G_1^c \cup G_2 \cup G_3^c) + \mathbb{P}(G_1^c \cup G_2^c \cup G_3) \\
&= 2 \cdot \left(\frac{10}{19}\right)^2 \cdot \frac{9}{19} \\
&= \frac{1800}{6859} \\
\mathbb{P}(X=-3) &= \mathbb{P}(G_1^c \cup G_2^c \cup G_3^c) \\
&= \left(\frac{10}{19}\right)^3 \\
&= \frac{1000}{6859}
\end{aligned}$$

Since $X=1$ is the only possible value of X such that $X>0$,

$$\mathbb{P}(X>0) = \mathbb{P}(X=1) = \boxed{\frac{4059}{6859}}$$

We note that $\frac{4059}{6859} \approx 0.592 > \frac{1}{2}$.

Problem 21.b

It highly depends on the answer to part (c). A strategy is only a winning one if its expected value is positive, which may or may not be true. Although each individual probability is a component of the expected value, we can't say without first calculating the expected value.

In part (c), we will show that $\mathbb{E}(x) < 0$, so this isn't a winning strategy.

Problem 21.c

$$\begin{aligned}
\mathbb{E}(X) &= 1 \cdot \mathbb{P}(X=1) - 1\mathbb{P}(X=-1) - 3\mathbb{P}(X=-3) \\
&= \frac{1}{6859}(4059 - 1800 - 3 \cdot 1000) \\
&= \boxed{-\frac{39}{361}} \\
&\approx -0.108
\end{aligned}$$

In context, this means that we will lose about 11 cents each time this strategy is used. Since $\mathbb{E}(X) < 0$, this strategy is a losing strategy, on average.

Problem 25

Problem 25.a

When $X = 1$, only one of the coins may be heads. Therefore,

$$\begin{aligned}
\mathbb{P}(X = 1) &= \frac{3}{5} \cdot \frac{3}{10} + \frac{2}{5} \cdot \frac{7}{10} \\
&= \boxed{\frac{23}{50}}
\end{aligned}$$

Problem 25.b

We must first calculate $\mathbb{P}(X = 0)$ and $\mathbb{P}(X = 2)$.

$$\begin{aligned}
\mathbb{P}(X = 0) &= \frac{2}{5} \cdot \frac{3}{10} \\
&= \frac{3}{25} \\
\mathbb{P}(X = 2) &= \frac{3}{5} \cdot \frac{7}{10} \\
&= \frac{21}{50}
\end{aligned}$$

Now, we can calculate $\mathbb{E}(X)$.

$$\begin{aligned}
\mathbb{E}(X) &= 0 \cdot \frac{3}{25} + 1 \cdot \frac{23}{50} + 2 \cdot \frac{21}{50} \\
&= \frac{23}{50} + \frac{42}{50} \\
&= \boxed{\frac{13}{10}} \\
&= 1.3
\end{aligned}$$