HW 7

Problem 2

Problem 2.a

X_1-value	X_2-value	
$\mathbb{P}(X_1, X_2)$	0	1
0	$\frac{8}{13} \cdot \frac{7}{12} = \frac{14}{39}$	$\frac{8}{13} \cdot \frac{5}{12} = \frac{20}{39}$
1	$\frac{5}{13} \cdot \frac{8}{12} = \frac{10}{39}$	$\frac{5}{13} \cdot \frac{4}{12} = \frac{5}{39}$

Problem 2.b

X_1	X_2	X_3	$\mathbb{P}(X_1, X_2, X_3)$
0	0	0	$\frac{8}{13} \cdot \frac{7}{12} \cdot \frac{6}{11} = \frac{28}{143}$
1	0	0	$\frac{5}{13} \cdot \frac{8}{12} \cdot \frac{7}{11} = \frac{70}{286}$
1	1	0	$\frac{5}{13} \cdot \frac{4}{12} \cdot \frac{8}{11} = \frac{40}{429}$
1	1	1	$\frac{5}{13} \cdot \frac{4}{12} \cdot \frac{3}{11} = \frac{5}{143}$
0	1	1	$\frac{8}{13} \cdot \frac{5}{12} \cdot \frac{4}{11} = \frac{40}{429}$
1	0	1	$\frac{5}{13} \cdot \frac{8}{12} \cdot \frac{4}{11} = \frac{40}{429}$
0	1	0	$\frac{8}{13} \cdot \frac{5}{12} \cdot \frac{7}{11} = \frac{70}{143}$
0	0	1	$\frac{8}{13} \cdot \frac{7}{12} \cdot \frac{5}{11} = \frac{70}{286}$

Problem 8

Problem 8.a

Since the distribution must be normalized,

$$
\begin{aligned}
1 &= c \int_0^\infty \int_{-y}^{y} \left(y^2 - x^2\right) e^{-y} \, dx \, dy \\
&= \frac{4c}{3} \int_0^\infty y^3 e^{-y} \, dy \\
&= 8c
\end{aligned}
$$

Therefore, $\boxed{c = \frac{1}{8}}$

Problem 8.b

By definition,

$$\begin{aligned} f_X(x) &= \int\limits_{-\infty}^{\infty} f(x,y)\ dy \\ &= \frac{1}{8}\int\limits_{|x|}^{\infty} \left(y^2e^{-y} - x^2e^{-y}\right)\ dy \\ &= \frac{1}{4}\int\limits_{|x|}^{\infty} ye^{-y}\ dy \\ &= \frac{1}{4}|x|\,e^{-|x|} + \frac{1}{4}e^{-|x|} \\ &= \boxed{\frac{1}{4}(|x|+1)\,e^{-|x|}} \end{aligned}$$

Similarly,

$$\begin{aligned} f_Y(y) &= \frac{1}{8}\int\limits_{-y}^{y} f(x,y)\ dx \\ &= \boxed{\frac{1}{6}y^3e^{-y}} \end{aligned}$$

Note that this is only defined when $y \geqslant 0$. When $y < 0$, $f_Y(y) = 0$.

Problem 8.c

Using the definition of expected value,

$$\begin{aligned} \mathbb{E}(X) &= \int\limits_{0}^{\infty}\int\limits_{-y}^{y} xf(x,y)\ dx\ dy \\ &= \int\limits_{0}^{\infty}\int\limits_{-y}^{y} x\left(y^2e^{-y} - x^2e^{-y}\right)\ dx\ dy \end{aligned}$$

We note that $x\left(y^2e^{-y} - x^2e^{-y}\right)$ is an odd function in x, so when integrated over symmetric bounds, the integral is 0. Therefore, $\boxed{\mathbb{E}(X) = 0}$.

Problem 10

Problem 10.a

By definition, we simply adjust the bounds of one of the definite integrals so that we only evalute over the region where $X < Y$. This is equivalent to saying $Y > X$, so we can start the integral with respect to y at

x instead of at 0.

$$\begin{aligned}
\mathbb{P}(X < Y) &= \int_0^\infty \int_x^\infty e^{-(x+y)}\ dy\ dx \\
&= \int_0^\infty e^{-2x}\ dx \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

This makes sense, because by symmetry, the same number of values of x are less than y and greater than y.

Problem 10.b

We can find $\mathbb{P}(X < a)$ by integrating.

$$\begin{aligned}
\mathbb{P}(X < a) &= \int_0^\infty \int_0^a e^{-(x+y)}\ dx\ dy \\
&= \left(1 - e^{-a}\right) \int_0^\infty e^{-y}\ dy \\
&= \boxed{\left(1 - e^{-a}\right)}
\end{aligned}$$

Problem 20

We split this problem up into two subparts, since each subpart is only tangentially related to the other.

Problem 20.a

We find the marginal distrubtions for each variable, $f_X(x)$ and $f_Y(y)$, and compare their product to the original density function.

$$\begin{aligned}
f_X(x) &= \int_0^\infty xe^{-(x+y)}\ dy \\
&= xe^{-x}
\end{aligned}$$

Similarly,

$$\begin{aligned}
f_Y(y) &= \int_0^\infty xe^{-(x+y)}\ dx \\
&= e^{-y}
\end{aligned}$$

We see that $f_X(x) f_Y(y) = xe^{-(x+y)} = f(x, y)$. Therefore, X and Y are independent random variables.

Problem 20.b

We find each of the marginal distributions.

$$\begin{aligned} f_X\left(x\right) &= \int\limits_0^1 2\ dy \\ &= 2 \\ f_y\left(y\right) &= \int\limits_0^y 2\ dx \\ &= 2y \end{aligned}$$

We see that $f_X\left(x\right) f_Y\left(y\right) = 4y \neq f\left(x, y\right)$. Therefore, X and Y are not independent.