

Brachistochrone Problem

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Physics Club

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Getting Started

Lagrangians

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- More advanced

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- More advanced
- Goals

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- More advanced
- Goals
 - Get everyone to pass $F=ma$ exam

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 - Get everyone to pass $F=ma$ exam
 - USAPhO Qualifiers!!

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- Goals
 - Get everyone to pass $F=ma$ exam
 - USAPhO Qualifiers!!
- Prerequisites (recommended)

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 - Get everyone to pass $F=ma$ exam
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- Prerequisites (recommended)
 - Taken/currently taking a physics class

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- Prerequisites (recommended)
 - Taken/currently taking a physics class
 - Or...

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 - Get everyone to pass $F=ma$ exam
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- Prerequisites (recommended)
 - Taken/currently taking a physics class
 - Or...
 - Willingness to learn

PSA: Problems

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- On classroom

PSA: Problems

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- On classroom
- Due date: next meeting

PSA: Problems

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- On classroom
- Due date: next meeting
- We hope to continue this pattern for the rest of this year

Brachistochrone

What Do I mean?

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- Etymology

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 - Brachistos ($\beta\rho\alpha\chi\iota\sigma\tau\sigma$) means "shortest"

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 - Brachistos ($\beta\rho\alpha\chi\iota\sigma\tau\sigma$) means "shortest"
 - Chronos ($\chi\rho\omicron\nu\omicron\sigma$) means "time"

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 - Brachistos ($\beta\rho\alpha\chi\iota\sigma\tau\sigma$) means "shortest"
 - Chronos ($\chi\rho\omicron\nu\omicron\sigma$) means "time"
- A brachistochrone curve is the path such that a ball traveling along this path takes the least amount of time

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- This is our problem

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- This is our problem
- Formal problem statement

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- Formal problem statement
 - Constraints: given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

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- This is our problem
- Formal problem statement
 - Constraints: given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$
 - Find function $y = f(x)$ such that the time it takes for a ball to travel under the influence of gravity from P_1 to P_2

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- Let s be a position vector

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- Let s be a position vector
- Let v be the associated velocity vector

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- Let s be a position vector
- Let v be the associated velocity vector
- From last lecture, recall that

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} \Rightarrow t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$$

Energy Conservation

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- Kinetic energy $K = \frac{1}{2}mv^2$

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- Kinetic energy $K = \frac{1}{2}mv^2$
- Gravitational potential energy $U = mgy$

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- Kinetic energy $K = \frac{1}{2}mv^2$
- Gravitational potential energy $U = mgy$
- Conservation of energy means that these two are equal

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$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

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- Conservation of energy means that these two are equal

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

- We can substitute this into the last equation

Pythagorean Theorem

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$$ds^2 = dx^2 + dy^2$$

$$ds^2 = dx^2 \left(1 + \left(\frac{dy}{dx} \right)^2 \right)$$

$$ds^2 = dx^2 \left(1 + \left(\frac{dy}{dx} \right)^2 \right)$$

$$ds^2 = dx^2 (1 + y'^2)$$

$$ds = dx \sqrt{1 + y'^2}$$

Putting It All Together

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- Original equation:

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$$

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- Original equation:

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$$

- Conservation of energy:

$$v = \sqrt{2gy}$$

- Pythagorean theorem:

$$ds = dx \sqrt{1 + y'^2}$$

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$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

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$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...

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$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...
- picking a function $y = f(x)$ to minimize integral

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$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

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- picking a function $y = f(x)$ to minimize integral
- How do we do it???

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$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

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- How do we do it???
- Lagrangians