Brachistochrone Problem

Rajeev Atla

Problems

Definitions

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

Thank You

Brachistochrone Problem

Rajeev Atla

Physics Club

November 20, 2020

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Definitions

Catting Starte

Conservation of

Pythagorean

Lagrangians

Partial Derivatives

Beltrami's Identity

Brachistochrone Problem

Rajeev Atla

Administrivia

Definitions

Getting Starte

Conservation of

Pythagorean

Lagrangians

Partial Derivatives

Beltrami's Identity

Thank You

More advanced

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Definitions

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

Thank Var

- More advanced
- Goals

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

- More advanced
- Goals
 - Get everyone to pass F=ma exam

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Cattina Stanta

Conservation of

Pythagorean

Lagrangians

Partial Derivative

Beltrami's Identit

Chank Van

- More advanced
- Goals
 - Get everyone to pass F=ma exam
 - USAPhO Qualifiers!!

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Cattina Stanta

Conservation o

Pythagorean

Lagrangians

Partial Derivative

Beltrami's Identit

- More advanced
- Goals
 - Get everyone to pass F=ma exam
 - USAPhO Qualifiers!!
- Prerequisites (recommended)

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Getting Starter

Conservation o

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

- More advanced
- Goals
 - Get everyone to pass F=ma exam
 - USAPhO Qualifiers!!
- Prerequisites (recommended)
 - Taken/currently taking a physics class

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Getting Starter

Conservation o

Pythagorear Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

- More advanced
- Goals
 - Get everyone to pass F=ma exam
 - USAPhO Qualifiers!!
- Prerequisites (recommended)
 - Taken/currently taking a physics class
 - Or...

Brachistochrone Problem

Rajeev Atla

Administrivia

- C - - -

Getting Started

Conservation of Energy

Pythagorear Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

Chank You

- More advanced
- Goals
 - Get everyone to pass F=ma exam
 - USAPhO Qualifiers!!
- Prerequisites (recommended)
 - Taken/currently taking a physics class
 - Or...
 - Willingness to learn

PSA: Problems

Brachistochrone Problem

Rajeev Atla

alaa ta ta ka da da da

Problems

Onfinition

Carrier Crantal

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

• On classroom

PSA: Problems

Brachistochrone Problem

Rajeev Atla

dministrivi

Problems

Definitions

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

- On classroom
- Due date: next meeting

PSA: Problems

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation o Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

- On classroom
- Due date: next meeting
- We hope to continue this pattern for the rest of this year

What Do I mean?

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Definitions

Getting Starte

Conservation of

Pythagorean

Lagrangians

Partial Derivative

Beltrami's Identit

Thank Var

What Do I mean?

Brachistochrone Problem

Rajeev Atla

Problems

Definitions

Conservation o

Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank Var

Etymology

• Brachistos $(\beta \rho \alpha \chi \iota \sigma \tau \sigma)$ means "shortest"

What Do I mean?

Brachistochrone Problem

Rajeev Atla

dministrivi

Problems

Definitions

Getting Starte

Conservation of Energy

Pythagorear

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

- Brachistos ($\beta \rho \alpha \chi \iota \sigma \tau \sigma$) means "shortest"
- Chronos $(\chi \rho o \nu o \sigma)$ means "time"

What Do I mean?

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Definitions

Cetting Startes

Conservation of

Pythagorea

Lagrangians

Partial Derivative

Beltrami's Identit

Thank Var

- Etymology
 - Brachistos ($\beta \rho \alpha \chi \iota \sigma \tau \sigma$) means "shortest"
 - Chronos $(\chi \rho o \nu o \sigma)$ means "time"
- A brachistochrone curve is the path such that a ball traveling along this path takes the least amount of time

What Do I mean?

Brachistochrone Problem

Najeev Atio

Problems

Definitions

Getting Started

Conservation of Energy

Pythagoreau Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Chank Voi

- Brachistos ($\beta \rho \alpha \chi \iota \sigma \tau \sigma$) means "shortest"
- Chronos $(\chi \rho o \nu o \sigma)$ means "time"
- A brachistochrone curve is the path such that a ball traveling along this path takes the least amount of time
- This is our problem

What Do I mean?

Brachistochrone Problem

Rajeev Atla

Problems

Definitions

C-44:-- C4--4-

Conservation of

Pythagorean

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

- Brachistos ($\beta \rho \alpha \chi \iota \sigma \tau \sigma$) means "shortest"
- Chronos $(\chi \rho o \nu o \sigma)$ means "time"
- A brachistochrone curve is the path such that a ball traveling along this path takes the least amount of time
- This is our problem
- Formal problem statement

What Do I mean?

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Definitions

Getting Starter

Conservation of

Pythagorea Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

Thank You

- Brachistos ($\beta \rho \alpha \chi \iota \sigma \tau \sigma$) means "shortest"
- Chronos $(\chi \rho o \nu o \sigma)$ means "time"
- A brachistochrone curve is the path such that a ball traveling along this path takes the least amount of time
- This is our problem
- Formal problem statement
 - Constraints: given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

What Do I mean?

Brachistochrone Problem

Rajeev Atla

dministrivis

Problems

Definitions

Cattina Stanta

Conservation of

Pythagoreai Theorem

Lagrangians

raitiai Delivative

Beltrami's Identity

- Etymology
 - Brachistos $(\beta \rho \alpha \chi \iota \sigma \tau \sigma)$ means "shortest"
 - Chronos $(\chi\rho\sigma\nu\sigma\sigma)$ means "time"
- A brachistochrone curve is the path such that a ball traveling along this path takes the least amount of time
- This is our problem
- Formal problem statement
 - Constraints: given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$
 - Find function y = f(x) such that the time it takes for a ball to travel under the influence of gravity from P_1 to P_2

Getting Started

Brachistochrone Problem

Rajeev Atla

dministrivia

.

 ${\sf Getting}\ {\sf Started}$

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

• Let s be a position vector

Getting Started

Brachistochrone Problem

Rajeev Atla

Administrivi

Definitions

Getting Started

Conservation of Energy

Pythagorean

Lagrangians

Partial Derivative

Beltrami's Identit

- Let s be a position vector
- Let v be the associated velocity vector

Getting Started

Brachistochrone Problem

Rajeev Atla

Problems

Definitions

Getting Started

Conservation of

Pythagorean

Lagrangians

Partial Derivatives

Beltrami's Identit

- Let s be a position vector
- Let v be the associated velocity vector
- From last lecture, recall that

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} \Rightarrow t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$$

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

Brachistochrone Problem

Rajeev Atla

Administrivi

5 0 ...

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

• Kinetic energy $K = \frac{1}{2}mv^2$

Brachistochrone Problem

Rajeev Atla

Problems

Definitions

Getting Started

Conservation of Energy

Pythagorean

Lagrangians

Partial Derivative

Beltrami's Identit

- Kinetic energy $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy

Brachistochrone Problem

Rajeev Atla

Administrivi Problems

Definitions

Getting Started

Conservation of Energy

Pythagorear Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

- Kinetic energy $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy
- Conservation of energy means that these two are equal

Brachistochrone Problem

Rajeev Atla

Problems

Definitions

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

- Kinetic energy $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy
- Conservation of energy means that these two are equal

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

Brachistochrone Problem

Rajeev Atla

Problems

_ . _ .

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

Thank You

- Kinetic energy $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy
- Conservation of energy means that these two are equal

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

• We can substitute this into the last equation

Pythagorean Theorem

Brachistochrone Problem

Rajeev Atla

dministrivia

Carrier Charles

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

$$ds^{2} = dx^{2} + dy^{2}$$

$$ds^{2} = dx^{2} \left(1 + \left(\frac{dy^{2}}{dx^{2}} \right) \right)$$

$$ds^{2} = dx^{2} \left(1 + \left(\frac{dy}{dx} \right)^{2} \right)$$

$$ds^{2} = dx^{2} \left(1 + y'^{2} \right)$$

$$ds = dx \sqrt{1 + y'^{2}}$$

Putting It All Together

Brachistochrone Problem

Rajeev Atla

dministrivia

Droblome

Delinitions

Getting Started

Conservation of

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

• Original equation:

$$t_{12} = \int\limits_{P_1}^{P_2} \frac{ds}{v}$$

Putting It All Together

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

Thank You

Original equation:

$$t_{12} = \int\limits_{P_1}^{P_2} \frac{ds}{v}$$

• Conservation of energy:

$$v=\sqrt{2gy}$$

Putting It All Together

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank Yo

Original equation:

$$t_{12} = \int\limits_{P_1}^{P_2} \frac{ds}{v}$$

Conservation of energy:

$$v = \sqrt{2gy}$$

• Pythagorean theorem:

$$ds = dx \sqrt{1 + y'^2}$$

Lagrangians

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation of Energy

Pythagorean

Lagrangians

Partial Derivatives

Beltrami's Identit

Thank Vau

$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

Lagrangians

Brachistochrone Problem

Rajeev Atla

dministrivi:

Problems

D 0.....

Getting Started

Conservation of Energy

Pythagorea Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

• We want to minimize this by...

Brachistochrone Problem

Rajeev Atla

dministrivia

1 Toblettis

Getting Started

Conservation of Energy

Pythagorea Theorem

Lagrangians

Partial Derivatives

Beltrami's Identit

$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...
- picking a function y = f(x) to minimize integral

Brachistochrone Problem

Rajeev Atla

Administrivia

r robicino

. . .

Getting Started

Conservation of Energy

Pythagoreau Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...
- picking a function y = f(x) to minimize integral
- How do we do it???

Brachistochrone Problem

Rajeev Atla

Administrivi

_

Getting Started

Conservation o Energy

Pythagoreau Theorem

Lagrangians

Partial Derivatives

Beltrami's Identit

$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...
- picking a function y = f(x) to minimize integral
- How do we do it???
- Lagrangians

More About Lagrangians

Brachistochrone Problem

Raieev Atla

Administrivia

Problems

Getting Started

Conservation of Energy

Pythagorea

Lagrangians

Partial Derivatives

Beltrami's Identit

Thank You

• Let the Lagrangian be

$$\mathcal{L} = \sqrt{\frac{1 + y'^2}{2gy}}$$

More About Lagrangians

Brachistochrone Problem

Rajeev Atla

. Administrivia

Problems

Getting Started

Conservation of Energy

Pythagorea Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

• Let the Lagrangian be

$$\mathcal{L} = \sqrt{\frac{1 + y'^2}{2gy}}$$

• Remeber that y = f(x)

$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

• $f'(x) = \frac{df(x)}{dx}$ (Lagrangian notation)

Least Action Principle

Brachistochrone Problem

Rajeev Atla

Administrivia

Getting Started

Conservation of Energy

Pythagoreau

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

• We need to choose f(x) minimize the time

$$t_{12} = \int\limits_{P_1}^{P_2} \mathcal{L}(x) dx$$

Least Action Principle

Brachistochrone Problem

Rajeev Atla

Administrivia

Getting Started

Conservation of Energy

Pythagorear Theorem

Lagrangians

Partial Derivative

Beltrami's Identit

Thank You

• We need to choose f(x) minimize the time

$$t_{12} = \int\limits_{P_1}^{P_2} \mathcal{L}(x) dx$$

• Any ideas?

Least Action Principle

Brachistochrone Problem

Rajeev Atla

dministrivia

1 TODICITIS

Conservation of Energy

Pythagorea Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

Thank You

• We need to choose f(x) minimize the time

$$t_{12} = \int_{P_1}^{P_2} \mathcal{L}(x) dx$$

- Any ideas?
- Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'(x)}\right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$

Brachistochrone Problem

Rajeev Atla

dministrivi

Problems

Definitions

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identit

Thank Var

ullet Symbol is ∂

Brachistochrone Problem

Rajeev Atla

Administrivi

Problems

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identit

- Symbol is ∂
- Hold all other variables constant while taking a derivative

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Getting Starter

Conservation o

Pythagorean

Lagrangian

Partial Derivatives

Beltrami's Identit

- Symbol is ∂
- Hold all other variables constant while taking a derivative
- Let f(x,y) = 2x + 3y, what are $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$?

Brachistochrone Problem

Rajeev Atla

Administrivia

Definitions

Getting Started

Conservation o Energy

Pythagorear Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

- Symbol is ∂
- Hold all other variables constant while taking a derivative
- Let f(x,y) = 2x + 3y, what are $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$?

$$\frac{\partial f(x,y)}{\partial x} = 2$$

$$\frac{\partial f(x,y)}{\partial y} = 3$$

Brachistochrone Problem

Rajeev Atla

Administrivi

Definitions

Getting Started

Conservation o Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

- Symbol is ∂
- Hold all other variables constant while taking a derivative
- Let f(x,y) = 2x + 3y, what are $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$?

$$\frac{\partial f(x,y)}{\partial x} = 2$$

$$\frac{\partial f(x,y)}{\partial v} = 3$$

Beltrami's Identity

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation of Energy

Pythagorear

Lagrangians

Partial Derivatives

Beltrami's Identity

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial f'(x)} \right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$

$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

Beltrami's Identity

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

Thank You

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial f'(x)} \right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$
$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

• Anyone want to do this???

Beltrami's Identity

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation o Energy

Pythagorear Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'(x)}\right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$
$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

- Anyone want to do this???
- Time for a trick: Beltrami's Identity
 - Notice that $\mathcal{L}(x)$ doesn't explicitly depend on x

$$\mathcal{L} - f'(x) \frac{\partial \mathcal{L}}{\partial f'(x)} = C$$

Using Beltrami's Identity

Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivative

Beltrami's Identity

$$C = \mathcal{L} - f'(x) \frac{\partial \mathcal{L}}{\partial f'(x)}$$

$$\frac{\partial \mathcal{L}}{\partial f'(x)} = \frac{f'(x)}{\sqrt{(2gf(x))(1 + f'(x)^2)}}$$

$$C = \frac{1}{\sqrt{2gf(x)(1 + f'(x)^2)}}$$

$$\frac{1}{2gC^2} = f(x)(1 + f'(x)^2)$$

Lagrangians Solution

Brachistochrone Problem

Rajeev Atla

dministrivis

- ...

Definition

Catting Started

Conservation of

Energy

Lagrangian

Partial Derivative

Beltrami's Identity

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

Lagrangians Solution

Brachistochrone Problem

Rajeev Atla

dministrivia

D 11

Definitions

Getting Started

Conservation of

Pythagorear

Lagrangians

Partial Derivative

Beltrami's Identity

Thank You

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

So what's the solution???

Solution

Brachistochrone Problem

Rajeev Atla

dministrivia

Droblome

Delinitions

Getting Started

Conservation of Energy

Pythagorean

Lagrangian:

Partial Derivative

Beltrami's Identity

$$\frac{1}{2gC^2} = f(x) \left(1 + f'(x)^2 \right)$$

- So what's the solution???
- It can be shown that this is a cycloid curve

Solution

Brachistochrone Problem

Rajeev Atla

dministrivia

Definitions

Getting Started

Conservation of

Pythagorean

Lagrangians

Partial Derivatives

Beltrami's Identity

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

- So what's the solution???
- It can be shown that this is a cycloid curve
- Has to be parametrized

$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$

$$y(\theta) = \frac{1}{4gC^2} (1 - \cos \theta)$$

Solution

Brachistochrone Problem

Rajeev Atla

Administrivia

Problems

Delilitions

Getting Started

Conservation of Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

Thank You

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

- So what's the solution???
- It can be shown that this is a cycloid curve
- Has to be parametrized

$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$

$$y(\theta) = \frac{1}{4gC^2} (1 - \cos \theta)$$

• We can use $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ to find C

Visualizing the Solution

Brachistochrone Problem

Beltrami's Identity

$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$

$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$
$$y(\theta) = \frac{1}{4gC^2} (1 - \cos \theta)$$

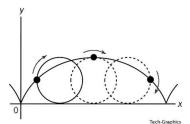
Visualizing the Solution

Brachistochrone Problem

Beltrami's Identity

$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$
$$y(\theta) = \frac{1}{4gC^2} (1 - \cos \theta)$$

$$y(\theta) = \frac{1}{4gC^2} (1 - \cos \theta)$$



Brachistochrone Problem

Rajeev Atla

dministrivia

Problems

Catting Storted

Conservation of

Energy

Pythagorean Theorem

Lagrangians

Partial Derivatives

Beltrami's Identity

Thank You