Brachistochrone Problem

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Physics Club

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This Section

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- More advanced

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- Goals

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  - Get everyone to pass F=ma exam

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  - Willingness to learn

PSA: Problems

- On classroom

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- We hope to continue this pattern for the rest of this year

Brachistochrone

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  - Constraints: given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$
  - Find function $y = f(x)$ such that the time it takes for a ball to travel under the influence of gravity from P_1 to P_2

Getting Started

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- From last lecture, recall that

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} \Rightarrow t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$$

Energy Conservation

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$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$
- We can substitute this into the last equation

Pythagorean Theorem

$$ds^2 = dx^2 + dy^2$$

$$ds^2 = dx^2 \left(1 + \left(\frac{dy^2}{dx^2}\right)\right)$$

$$ds^2 = dx^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$ds^2 = dx^2 \left(1 + y'^2\right)$$

$$ds = dx\sqrt{1 + y'^2}$$

Putting It All Together

- Original equation:

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More About Lagrangians

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- Let the Lagrangian be

$$\mathcal{L} = \sqrt{\frac{1 + y'^2}{2gy}}$$

- Remeber that $y = f(x)$

$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

- $f'(x) = \frac{df(x)}{dx}$ (Lagrangian notation)

Least Action Principle

- We need to choose $f(x)$ minimize the time

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- Any ideas?
- Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'(x)}\right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$

Partial Derivatives

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Beltrami's Idendtity

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'(x)}\right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$

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- Anyone want to do this???
- Time for a trick: Beltrami's Idendtity
  - Notice that $\mathcal{L}(x)$ doesn't *explicitly* depend on x

$$\mathcal{L} - f'(x)\frac{\partial\mathcal{L}}{\partial f'(x)} = C$$

Lagrangians

Using Beltrami's Idendtity

$$C = \mathcal{L} - f'(x)\frac{\partial \mathcal{L}}{\partial f'(x)}$$

$$\frac{\partial \mathcal{L}}{\partial f'(x)} = \frac{f'(x)}{\sqrt{(2gf(x))\left(1 + f'(x)^2\right)}}$$

$$C = \frac{1}{\sqrt{2gf(x)\left(1 + f'(x)^2\right)}}$$

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

Lagrangians

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$$x(\theta) = \frac{1}{4gC^2}(\theta - \sin\theta)$$
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- We can use $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ to find C

Lagrangian

Visualizing the Solution

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