#### Brachistochrone Problem

Rajeev Atla

### Administrivia

Definitions

Getting Started

Lagrangians

## Brachistochrone Problem

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Physics Club

November 20, 2020

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More advanced

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- More advanced
- Goals

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- More advanced
- Goals
  - ullet Get everyone to pass F=ma exam

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- More advanced
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- More advanced
- Goals
  - Get everyone to pass F=ma exam
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- Prerequisites (recommended)

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- More advanced
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  - Get everyone to pass F=ma exam
  - USAPhO Qualifiers!!
- Prerequisites (recommended)
  - Taken/currently taking a physics class

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- More advanced
- Goals
  - Get everyone to pass F=ma exam
  - USAPhO Qualifiers!!
- Prerequisites (recommended)
  - Taken/currently taking a physics class
  - Or...

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Getting Started

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- More advanced
- Goals
  - Get everyone to pass F=ma exam
  - USAPhO Qualifiers!!
- Prerequisites (recommended)
  - Taken/currently taking a physics class
  - Or...
  - Willingness to learn

## **PSA: Problems**

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### ${\sf Administrivia}$

Definition:

Getting Started

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On classroom

## **PSA: Problems**

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On classroom

• Due date: next meeting

## **PSA**: Problems

#### Brachistochrone Problem

### Administrivia

- On classroom
- Due date: next meeting
- We hope to continue this pattern for the rest of this year

What Do I mean?

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Definitions

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Getting Started

Lagrangians

Etymology

What Do I mean?

#### Brachistochrone Problem

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Definitions

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Etymology

• Brachistos  $(\beta \rho \alpha \chi \iota \sigma \tau \sigma)$  means "shortest"

What Do I mean?

#### Brachistochrone Problem

**Definitions** 

Etymology

- Brachistos  $(\beta \rho \alpha \chi \iota \sigma \tau \sigma)$  means "shortest"
- Chronos  $(\chi \rho o \nu o \sigma)$  means "time"

What Do I mean?

#### Brachistochrone Problem

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Definitions

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Getting Startet

- Etymology
  - Brachistos  $(\beta \rho \alpha \chi \iota \sigma \tau \sigma)$  means "shortest"
  - Chronos  $(\chi \rho o \nu o \sigma)$  means "time"
- A brachistochrone curve is the path such that a ball traveling along this path takes the least amount of time

What Do I mean?

#### Brachistochrone Problem

Definitions

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- This is our problem

What Do I mean?

#### Brachistochrone Problem

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Definitions

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Getting Started

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- This is our problem
- Formal problem statement

What Do I mean?

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  - Constraints: given two points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$

What Do I mean?

#### Brachistochrone Problem

Definitions

- Etymology
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- This is our problem
- Formal problem statement
  - Constraints: given two points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$
  - Find function y = f(x) such that the time it takes for a ball to travel under the influence of gravity from  $P_1$  to  $P_2$



# Getting Started

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Getting Started

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• Let s be a position vector

# Getting Started

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Getting Started

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- Let s be a position vector
- Let v be the associated velocity vector

# **Getting Started**

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Getting Started

Lagrangians

- Let s be a position vector
- Let v be the associated velocity vector
- From last lecture, recall that

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} \Rightarrow t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$$

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Lagrangians

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• Kinetic energy  $K = \frac{1}{2}mv^2$ 

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Lagrangian

- Kinetic energy  $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy

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Getting Started

\_agrangian:

- Kinetic energy  $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy
- Conservation of energy means that these two are equal

#### Brachistochrone Problem

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- Kinetic energy  $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy
- Conservation of energy means that these two are equal

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

#### Brachistochrone Problem

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\_agrangian

- Kinetic energy  $K = \frac{1}{2}mv^2$
- Gravitational potential energy U = mgy
- Conservation of energy means that these two are equal

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

• We can substitute this into the last equation

## Pythagorean Theorem

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Definitions

**Getting Started** 

$$ds^{2} = dx^{2} + dy^{2}$$

$$ds^{2} = dx^{2} \left( 1 + \left( \frac{dy^{2}}{dx^{2}} \right) \right)$$

$$ds^{2} = dx^{2} \left( 1 + \left( \frac{dy}{dx} \right)^{2} \right)$$

$$ds^{2} = dx^{2} \left( 1 + y'^{2} \right)$$

$$ds = dx \sqrt{1 + y'^{2}}$$

# Putting It All Together

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**Definitions** 

**Getting Started** 

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Original equation:

$$t_{12} = \int\limits_{P_1}^{P_2} \frac{ds}{v}$$

# Putting It All Together

#### Brachistochrone Problem

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Definitions

**Getting Started** 

Lagrangian

Original equation:

$$t_{12} = \int\limits_{P_1}^{P_2} \frac{ds}{v}$$

Conservation of energy:

$$v=\sqrt{2gy}$$

# Putting It All Together

#### Brachistochrone Problem

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Definitions

**Getting Started** 

Lagrangian

Original equation:

$$t_{12} = \int\limits_{P_1}^{P_2} \frac{ds}{v}$$

• Conservation of energy:

$$v = \sqrt{2gy}$$

• Pythagorean theorem:

$$ds = dx \sqrt{1 + y'^2}$$

## Lagrangians

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Lagrangians

$$t_{12} = \int\limits_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

# Lagrangians

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Definitions

Getting Started

Lagrangians

$$t_{12} = \int\limits_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

• We want to minimize this by...

## Brachistochrone Problem

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Getting Started

$$t_{12} = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...
- picking a function y = f(x) to minimize integral

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Getting Started

$$t_{12} = \int\limits_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...
- picking a function y = f(x) to minimize integral
- How do we do it???

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Definitions

Getting Started

$$t_{12} = \int\limits_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

- We want to minimize this by...
- picking a function y = f(x) to minimize integral
- How do we do it???
- Lagrangians

# More About Lagrangians

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Getting Started

Lagrangians

Let the Lagrangian be

$$\mathcal{L} = \sqrt{\frac{1 + y'^2}{2gy}}$$

# More About Lagrangians

#### Brachistochrone Problem

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Definitions

Getting Started

Lagrangians

• Let the Lagrangian be

$$\mathcal{L} = \sqrt{\frac{1 + y'^2}{2gy}}$$

• Remeber that y = f(x)

$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

•  $f'(x) = \frac{df(x)}{dx}$  (Lagrangian notation)

# Least Action Principle

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Definitions

Getting Started

Lagrangians

• We need to choose f(x) minimize the time

$$t_{12} = \int\limits_{P_1}^{P_2} \mathcal{L}(x) dx$$

# Least Action Principle

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Getting Started

Lagrangians

• We need to choose f(x) minimize the time

$$t_{12} = \int\limits_{P_1}^{P_2} \mathcal{L}(x) dx$$

• Any ideas?

# Least Action Principle

#### Brachistochrone Problem

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Getting Starter

Lagrangians

• We need to choose f(x) minimize the time

$$t_{12} = \int\limits_{P_1}^{P_2} \mathcal{L}(x) dx$$

- Any ideas?
- Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'(x)}\right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$

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Lagrangians

ullet Symbol is  $\partial$ 

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Getting Started

- ullet Symbol is  $\partial$
- Hold all other variables constant while taking a derivative

#### Brachistochrone Problem

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Getting Started

- Symbol is  $\partial$
- Hold all other variables constant while taking a derivative
- Let f(x,y) = 2x + 3y, what are  $\frac{\partial f(x,y)}{\partial x}$  and  $\frac{\partial f(x,y)}{\partial y}$ ?

#### Brachistochrone Problem

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Getting Started

- ullet Symbol is  $\partial$
- Hold all other variables constant while taking a derivative
- Let f(x,y) = 2x + 3y, what are  $\frac{\partial f(x,y)}{\partial x}$  and  $\frac{\partial f(x,y)}{\partial y}$ ?

$$\frac{\partial f(x,y)}{\partial x} = 2$$
$$\frac{\partial f(x,y)}{\partial y} = 3$$

#### Brachistochrone Problem

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Getting Started

- ullet Symbol is  $\partial$
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Beltrami's Idendtity

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Definitions

Getting Started

$$\frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial f'(x)} \right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$
$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

## Beltrami's Idendtity

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Lagrangians

$$\frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial f'(x)} \right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$

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• Anyone want to do this???

Beltrami's Idendtity

Brachistochrone Problem

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Definition

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$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'(x)}\right) = \frac{\partial \mathcal{L}}{\partial f(x)}$$
$$\mathcal{L}(x) = \sqrt{\frac{1 + f'(x)^2}{2gf(x)}}$$

- Anyone want to do this???
- Time for a trick: Beltrami's Idenditty
  - Notice that  $\mathcal{L}(x)$  doesn't explicitly depend on x

$$\mathcal{L} - f'(x) \frac{\partial \mathcal{L}}{\partial f'(x)} = C$$

Using Beltrami's Idendtity

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Definitions

Getting Started

$$C = \mathcal{L} - f'(x) \frac{\partial \mathcal{L}}{\partial f'(x)}$$
$$\frac{\partial \mathcal{L}}{\partial f'(x)} = \frac{f'(x)}{\sqrt{(2gf(x))(1 + f'(x)^2)}}$$
$$C = \frac{1}{\sqrt{2gf(x)(1 + f'(x)^2)}}$$
$$\frac{1}{2gC^2} = f(x)(1 + f'(x)^2)$$

# Lagrangians Solution

## Brachistochrone Problem

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Definitions

Getting Started

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

# Lagrangians Solution

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Definitions

Getting Started

Lagrangians

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

So what's the solution???

Solution

## Brachistochrone Problem

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Getting Started

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

- So what's the solution???
- It can be shown that this is a cycloid curve

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Definitions

Getting Started

$$\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$$

- So what's the solution???
- It can be shown that this is a cycloid curve
- Has to be parametrized

$$x(\theta) = \frac{1}{4gC^2} \left( \theta - \sin \theta \right)$$

$$y(\theta) = \frac{1}{4gC^2} (1 - \cos \theta)$$

Solution

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Definitions

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Lagrangi<u>ans</u>

- So what's the solution???
- It can be shown that this is a cycloid curve
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$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$

 $\frac{1}{2gC^2} = f(x)\left(1 + f'(x)^2\right)$ 

$$y(\theta) = \frac{1}{4\sigma C^2} (1 - \cos \theta)$$

• We can use  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  to find C

Visualizing the Solution

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Getting Starte

$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$
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Visualizing the Solution

## Brachistochrone Problem

$$x(\theta) = \frac{1}{4gC^2} (\theta - \sin \theta)$$
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