Linear Programming Problem (LPP)
(Solution procedure)

Before going to the solution procedure we need to have idea about some definitions/concepts that are related to the methods of solving LPPs.

- **Solution:** A set of values of decision variables satisfying all the constraints of an LPP is called a solution to that problem.
- **Feasible solution**: A solution which also satisfies the non-negativity restrictions of the problem is called a feasible solution.
- **Solution space**: An LPP may have more than one feasible solution. The set of all solution of an LPP is called solution space.
- **Optimal feasible solution:** A feasible solution which gives the optimum (maximum or minimum) value of the objective function is called an optimal feasible solution.

Consider the following LPP:

$$\begin{aligned} \text{Maximize } \; & Z = 3x_1 + 2x_2 \\ \text{Subject to } \; & x_1 + 3x_2 \leq 6, \\ & 2x_1 + 3x_2 \leq 9, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Here, $x_1 = 2$, $x_2 = 1$ is a feasible solution as it satisfy both the constraints of the problem, and for this solution $Z = 8$. Similarly, $x_1 = 1$, $x_2 = 1.5$ is an another feasible solution for which $Z = 6$. Also, $x_1 = 2$, $x_2 = 1.5$ is an another feasible solution for which $Z = 9$.

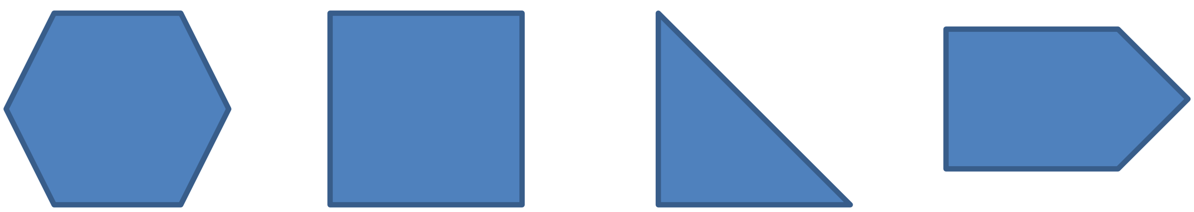
Now the question is how to find the feasible solution, i.e. the values of x_1 and x_2 which gives the maximum value of the objective function Z.

Some required definitions:

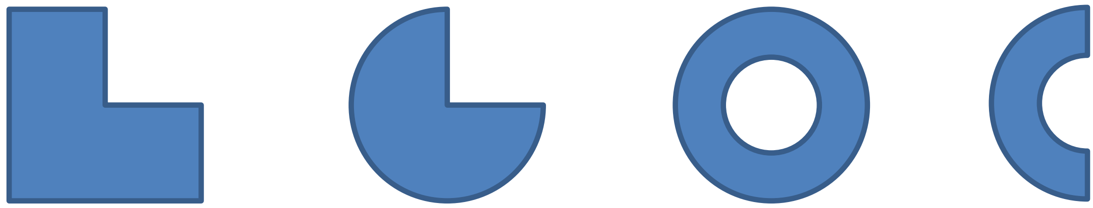
Convex set: A convex set S is a collection of points such that are any two points x_1, x_2 in S, the line segment joining them is also a member in the set S. That is the set S will be convex if for any $x_1, x_2 \in S$, $\lambda\, x_1 + (1 - \lambda)\, x_2 \in S$ for $0 \leq \lambda \leq 1$.

Line segment joining two points:

A region which is convex is called convex region. The following regions are convex.



The following regions are not convex:



Feasible region of an LPP: The set of all feasible solution of an LPP constitute a region which is called feasible region. Feasible region of an LPP is always convex.

❖ Graphical Method of solution of LPP:

In Class X, you have learnt how to graph an equation involving two variables x and y by constituting two perpendicular axis - an X axis and a Y axis. Graphical method is applicable to solve an LPP with only two decision variables. In the first step, all the constraints are written as equations to plot them in a graph. Next we have to determine the feasible region. Then the optimal solution can be obtained using the following theorem:

Theorem: If optimal solution of an LPP exists, then it must occur at a corner point (extreme point/vertex) of the feasible region.

Plot each of the given constraints, whether equations or inequalities, as equations.

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Identify the feasible region and find the corner points

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Determine the optimal solution (it will occur at a corner point of the feasibale region

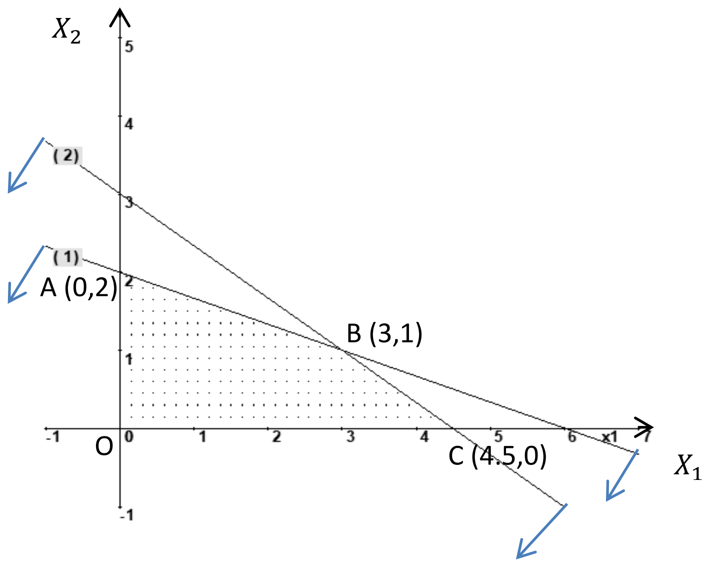
Let's consider the above given LPP. First we plot each of the given constraints as equation, i.e.

$x_1 + 3x_2 = 6 \qquad \cdots (1)$

$2x_1 + 3x_2 = 9 \qquad \cdots (2)$

x_1	6	0
x_2	0	2

x_1	4.5	0
x_2	0	3



In the above figure, every points on and

So here OABC is the feasible region.

However, one of the corner points or vertices will give us the optimal solution.

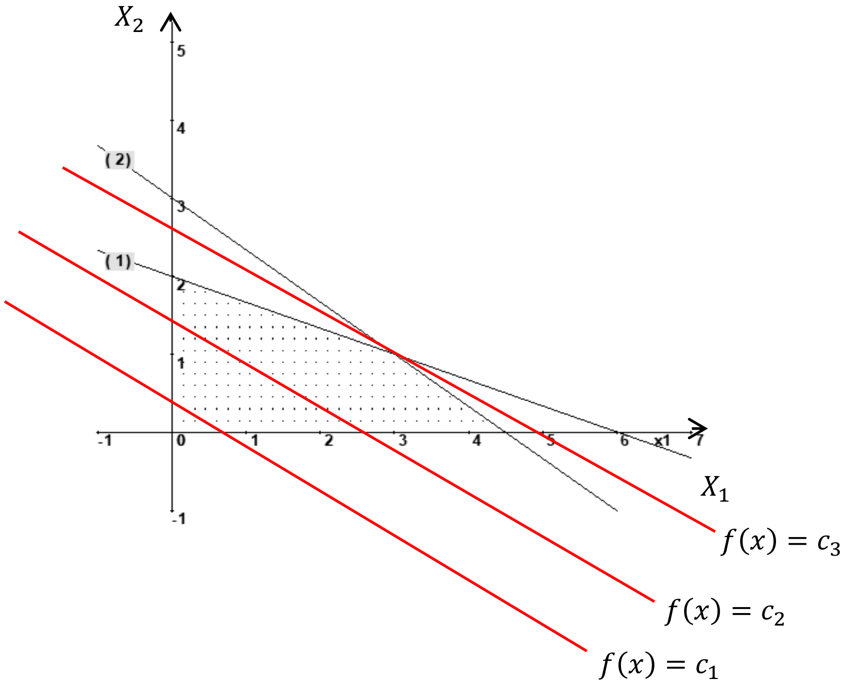
At the point A, i.e. for $x_1 = 0$, $x_2 = 2$, value of Z =4

At the point B, i.e. for $x_1 = 3$, $x_2 = 1$, value of Z =11

At the point C, i.e. for $x_1 = 4.5$, $x_2 = 0$, value of Z =13.5

At the point O, i.e. for $x_1 = 0$, $x_2 = 0$, value of Z =0

So the optimal solution is $x_1 = 4.5$, $x_2 = 0$ for which Max $Z = 13.5$.



Exercises: Solve the following problems by graphical method.

1. Maximize $Z = 4x_1 + 6x_2$
Subject to $x_1 + 3x_2 \leq 6$, $2x_1 + 3x_2 \leq 6$, $x_1, x_2 \geq 0$.

2. Reddy Mikks produces both exterior and interior points from two raw materials, M1 and M2. The following table provides the basic data of the problem

	Tons of raw material require for per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
Raw material M1	6	4	24
Raw material M2	1	2	6
Profit per ton ($1000)	5	4	

The maximum daily demand for interior product is 2 tons. Determine the daily amounts of interior and exterior paints to be produced to maximize the total daily profit.

3. Maximize $Z = 2x + 3y$

Subject to $x + y \le 30, y \ge 3, 0 \le y \le 12, x - y \ge 0, 0 \le x \le 20.$

4. Minimize $Z = x_1 + 2x_2$

Subject to $-x_1 + 2x_2 \le 6, 3x_1 + 2x_2 \ge 6, x_1 - 2x_2 \le 2,\ x_1, x_2 \ge 0.$

5. A farmer can buy two types of plant food, mix A and mix B. Each cubic metre of mix A contains 20 kg of phosphoric acid, 30 kg of nitrogen, and 5 kg of potash. Each cubic metre of mix B contains 10 kg of phosphoric acid, 30 kg of nitrogen and 10 kg of potash. The minimum monthly requirements are 460 kg of phosphoric acid, 960 kg of nitrogen, and 220 kg of potash. If mix A costs \$30 per cubic metre and mix B costs \$35 per cubic metre, how many cubic metres of each mix should the farmer blend to meet the minimum monthly requirements at a minimal cost? What is the cost?

❖ The simplex method

Consider the following LPP which is in standard form:

$$\begin{aligned}
&\text{Maximize } Z = 2x_1 + x_2 + 3x_3 \\
&\text{Subject to} \quad 2x_1 + 3x_2 - 2x_3 = 7, \qquad \cdots R_1 \\
&\qquad\qquad\quad x_1 - x_2 + x_3 = 1, \qquad \cdots R_2 \\
&\qquad\qquad\quad x_1, x_2, x_3 \ge 0.
\end{aligned}$$

Let's try to solve the constraints (which are a system of 2 linear equations with 3 decision variables) by the method of elimination of variables (pivot procedure)

$$\begin{aligned}
&x_1 + \tfrac{3}{2}x_2 - x_3 = \tfrac{7}{2} && R'_1 = \tfrac{1}{2}R_1 \\
&x_1 - x_2 + x_3 = 1 && R_2
\end{aligned}$$

$$\begin{aligned}
&x_1 + \tfrac{3}{2}x_2 - x_3 = \tfrac{7}{2} && R'_1 \\
&0 - \tfrac{5}{2}x_2 + 2x_3 = -\tfrac{5}{2} && R'_2 = R_2 - R'_1
\end{aligned}$$

$$\begin{aligned}
&x_1 + 0 + \tfrac{1}{5}x_3 = 2 && R''_1 = R'_1 + \tfrac{3}{5}R'_2 \\
&0 - x_2 + \tfrac{4}{5}x_3 = -1 && \tfrac{2}{5}R'_2
\end{aligned}$$

Thus the solution is $x_1 = 2 - \frac{1}{5}x_3$ and $x_2 = 1 + \frac{4}{5}x_3$, which means that we can take x_3 as any real value and this gives us the corresponding values of x_1 and x_2. Particularly, if we take $x_3 = 0$, then $x_1 = 2$ and $x_2 = 1$.

Here the solution $x_1 = 2, x_2 = 1$ and $x_3 = 0$ is called *basic feasible solution*. x_1 and x_2 are known as basic variable and x_3 is known as non-basic variable.

Basic feasible solution: For a system of m linear equations and n variables such that $n \ge m$, the solution obtained by setting $(n - m)$ variables equal to 0 (zero) is called basic solution. If this basic solution is also feasible, then it is called basic feasible solution (b.f.s).

The $(n-m)$ variables having value zero are called non-basic variables and the remaining m variables are called basic variables provided the column vectors corresponding to these variables are linearly independent.

Actually the column vectors corresponding to the basic variables must have to be linearly independent to have a solution of the problem.

Note: A basic feasible solution to an LPP correspond to an extreme point (vertex) of the feasible region and conversely.

The simplex procedure starts with finding an initial basic feasible solution, and then follows a step-by-step procedure of improving this initial b.f.s towards an optimal solution.

Next we discuss the step-by-step procedure of simplex method by solving a problem.

- **Simplex procedure (Maximization problem: for LPP with less than or equal to constraints)**

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3$$
$$\text{Subject to } x_1 + x_2 + x_3 \leq 9,\ \ 2x_1 + 3x_2 + 5x_3 \leq 30, 2x_1 - x_2 - x_3 \leq 8,$$
$$x_1, x_2, x_3 \geq 0.$$

Step 1. Covert the given problem into standard form.

Introducing slack variables s_1, s_2 and s_3, we have the standard form as follows:

Maximize $Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$
Subject to $x_1 + x_2 + x_3 + s_1 = 9,$
$2x_1 + 3x_2 + 5x_3 + s_2 = 30,$
$2x_1 - x_2 - x_3 + s_3 = 8,$
$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$

So the problem is now consists of 6 variables and 3 constraints.

Step 2. Start with an initial basic feasible solution and form the simplex table.

Find the initial b.f.s by setting $(6-3) = 3$ variables to zero, i.e. taking $x_1 = 0, x_2 = 0, x_3 = 0$, which yields the initial b.f.s as $s_1 = 9, s_2 = 30, s_3 = 8$ and Max $Z = 0$. We now form the following simplex table.

Table -1

	c_j	3	2	5	0	0	0		
c_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	b	Ratios
0	s_1	1	1	1	1	0	0	9	9/1
0	s_2	2	3	$\boxed{5}$	0	1	0	30	30/5 → Leaving variable
0	s_3	2	-1	-1	0	0	1	8	--
$c_j - z_j$		3	2	5 ↑ Entering variable	0	0	0		

Here z_j = sum of the components obtained by the multiplication of entries under x_j-column and coefficients of basic variables. E.g., here
$z_1 = 1 \cdot 0 + 2 \cdot 0 + 2 \cdot 0 = 0,$
$z_2 = 3\ 1 \cdot 0 + 3 \cdot 0 + (-1) \cdot 0 = 0$

[*$c_j - z_j$ is called net evaluation. It represents the net contribution (marginal improvement) in the value of the objective function for each unit of the corresponding variable x_j if it is introduced into the basis. For maximization problem, a positive number in $\boldsymbol{c_j - z_j}$ row indicates the amount by which the profit will be increased if a unit of the corresponding variable introduced into the solution. Similarly for minimization problem, a negative number in $\boldsymbol{c_j - z_j}$ row indicates the amount by which the cost will be decreased if a unit of the corresponding variable introduced into the solution.*]

Step 3. Iterate towards an optimal solution.

[*For maximization problem, if all the elements in $\boldsymbol{c_j - z_j}$ row are negative or zero, then the current solution is optimal (i.e. there is no scope of improvement in the value of the objective function). But in case any element is positive, the solution is not optimal, and so we have to improve the current b.f.s.*]

Since in the initial simplex table 1, some of the elements in $c_j - z_j$ row are positive, we can improve the current solution. In this case one of the current basic variables is replaced by a non-basic variable.

Selection of entering variable into the basis: Pick the column having the *most positive number in the $c_j - z_j$ row*. This column is called key (/ pivot) column. The corresponding variable heading that column will enter into the basis (this variable is called entering or incoming variable). **From the Table 1, we observe that $\boldsymbol{x_3}$ will enter into the basis**.

Selection of leaving variable from the basis: Divide the elements under the *b* column by the corresponding *positive elements* in the key column (i.e. consider only the *positive ratios*). The element in the key column for which this *ratio is minimum* is selected as the *key element*, and the basic variable correspond to this row will leave the basis. **From the Table 1, we observe that $\boldsymbol{s_2}$ will leave the basis**. Here **5** is the key element. (If all the elements in the key column are negative or zero, then the solution of the problem is unbounded.)

Updating the simplex table: Replace the leaving variable $\boldsymbol{s_2}$ in the basis by the entering variable $\boldsymbol{x_3}$. Note that the column vectors in the basis should remain linearly independent. So we make the column corresponding to the entering variable (here $\boldsymbol{x_3}$) into an identity column as like the column corresponding to the leaving variable (here $\boldsymbol{s_2}$). To do this first divide the elements in the x_3-row by 5.

Table -2

	$\boldsymbol{c_j}$	3	2	5	0	0	0		
c_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	b	Ratios
0	s_1	$\boxed{\frac{3}{5}}$	$\frac{2}{5}$	0	1	$-\frac{1}{5}$	0	3	5 $\rightarrow$
5	x_3	$\frac{2}{5}$	$\frac{3}{5}$	1	0	$\frac{1}{5}$	0	6	15
0	s_3	$\frac{12}{5}$	$-\frac{2}{5}$	0	0	$\frac{1}{5}$	1	14	35/6

Updated s_1-row =
old s_1-row −new x_3-row × **1**

$$1 - \frac{2}{5} \times \mathbf{1} = \frac{3}{5}$$

$$1 - \frac{3}{5} \times \mathbf{1} = \frac{2}{5}$$

$$\mathbf{1 - 1 \times 1 = 0}$$

$$1 - 0 \times \mathbf{1} = 1$$

$$0 - \frac{1}{5} \times \mathbf{1} = -\frac{1}{5}$$

$$0 - 0 \times \mathbf{1} = 0$$

$$9 - 6 \times \mathbf{1} = 3$$

$c_j - z_j$	1	-1	0	0	-1	0	

↑

By similarly manner as shown in the right box, update the elements in the s_3-row. (Updated s_3-row =old s_3-row +new x_3-row × **1**).
Next replace the leaving variable $\boldsymbol{s_1}$ in the basis by the entering variable $\boldsymbol{x_1}$ and update the table in similar manner. Then divide the elements in the x_1-row by 3/5.

Updated x_3-row =
old x_3-row – new x_1-row × $\frac{2}{5}$

$$\frac{2}{5} - 1 \times \frac{2}{5} = 0$$

$$\frac{3}{5} - \frac{2}{3} \times \frac{2}{5} = \frac{1}{3}$$

$$1 - 0 \times \frac{2}{5} = 1$$

$$0 - \frac{5}{3} \times \frac{2}{5} = -\frac{2}{3}$$

$$\frac{1}{5} - (-\frac{1}{3}) \times \frac{2}{5} = \frac{1}{3}$$

$$0 - 0 \times \frac{2}{5} = 0$$

$$6 - 5 \times \frac{2}{5} = 4$$

Table 3

	c_j	3	2	5	0	0	0	
c_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	b
3	x_1	1	$\frac{2}{3}$	0	$\frac{5}{3}$	$-\frac{1}{3}$	0	5
5	x_3	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	4
0	s_3	0	-2	0	-4	1	1	3
$c_j - z_j$		0	-5/3	0	-5/3	-2/3	0	

By similarly manner as shown in the right box, update the elements in the s_3-row. (Updated s_3-row =old s_3-row −new x_1-row × **12/5**).

Since all the $c_j - z_j$ values are negative or zero, so the current feasible solution is optimal. The optimal solution is

$x_1 = 5, x_2 = 0$, $x_3 = 4$ and Maximum $Z = 35$.

Exercises: Solve the following problems by simplex method.

Ex.1. Maximize $Z = 3x_1 + 2x_2$
Subject to $-x_1 + 2x_2 \leq 4, 3x_1 + 2x_2 \leq 14, x_1 - x_2 \leq 3, x_1, x_2 \geq 0$.

Ex.2 Maximize $Z = x_1 + 2x_2 + 3x_3$
Subject to $x_1 + 2x_2 + 3x_3 \leq 10, x_1 + x_2 \leq 5, \ x_1, x_2, x_3 \geq 0$.

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