Optimization Problems: Introduction

An optimization problem deals with optimization (maximization or minimization) of some objective function that may be subject to a set of constraints/restrictions.

Basic features:

A constrained optimization problem has three basic components:

- **i.** Decision or design variables
- **ii.** Objective (goal)
- **iii.** Constraints (restrictions)

Let's consider the following problem to discuss the above components:

Example. (Product-mix problem) Reddy Mikks produces both exterior and interior points from two raw materials, M1 and M2. The following table provides the basic data of the problem

	Tons of raw material require for per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
Raw material M1	6	4	24
Raw material M2	1	2	6
Profit per ton ($1000)	5	4	

The maximum daily demand for interior product is 2 tons. The company wants to determine the daily amounts of interior and exterior paints to be produced to maximize the total daily profit. Formulate this problem as an optimization problem (it will gives us a linear programming problem (LPP)).

i. Decision variable: The decision variables refer to those quantities we seek to determine in order to solve the problem. These variables represent the amount of a resource to use or the level of some activity for best outcome.

In the above example, one has to determine the '*daily amounts of exterior and interior paints*' to be produced so that the total daily profit is maximized. So there are two decision variables, each of them represents the '*daily amounts'* of paints to be produced.

Notation: x, y, z or $x_1, x_2, x_3, \ldots$ Like, let the daily amounts of exterior and interior paints to be produced are x ton and y ton respectively to maximize the total profit.

ii. Objective (goal): An optimization problem always has clearly identifiable and quantifiable goal or objective that could be maximization of profit/sales, minimization of cost or time, etc.

In the above example, the objective is to *maximize the total daily profit*. Profit per ton of exterior and interior paints is given. So the objective function will be $5x + 4y$ which we have to be maximized.

iii. Constraints: Constraints are a set of linear/nonlinear equations and/or inequalities in terms of the decision variables describing the limits on the available resources (e.g., money, amount of raw materials, number of working days, volume, space, etc.).

In the above example, maximum daily availabilities for the two types of raw materials create a set of constraints. It is given that maximum daily availability of raw material M1 is 24 tons. It is given that 6 tons and 4 tons of raw material M1 are required for production of per ton of exterior and interior paints respectively. So the mathematical form of this constraint will be $6x + 4y \leq 24$. Similarly, the limitation on availability of raw material M2 will create another constraint (write yourself).

The formulation of the problem in Ex.1 as an optimization is given as follows. Let the daily amounts of exterior and interior paints to be produced are x ton and y ton respectively to maximize the total profit. That is we have to find the values of x and y so as to:

$$\text{Maximize } Z = 5x + 4y \quad \text{(Objective function)}$$

Subject to the constraints:

$$\begin{aligned} & 6x + 4y \leq 24, \\ & x + 2y \leq 6, \\ & y \leq 2, \\ & x, y \geq 0. \qquad \text{(Feasibility, i.e. non-negativity conditions)} \end{aligned}$$

General form:

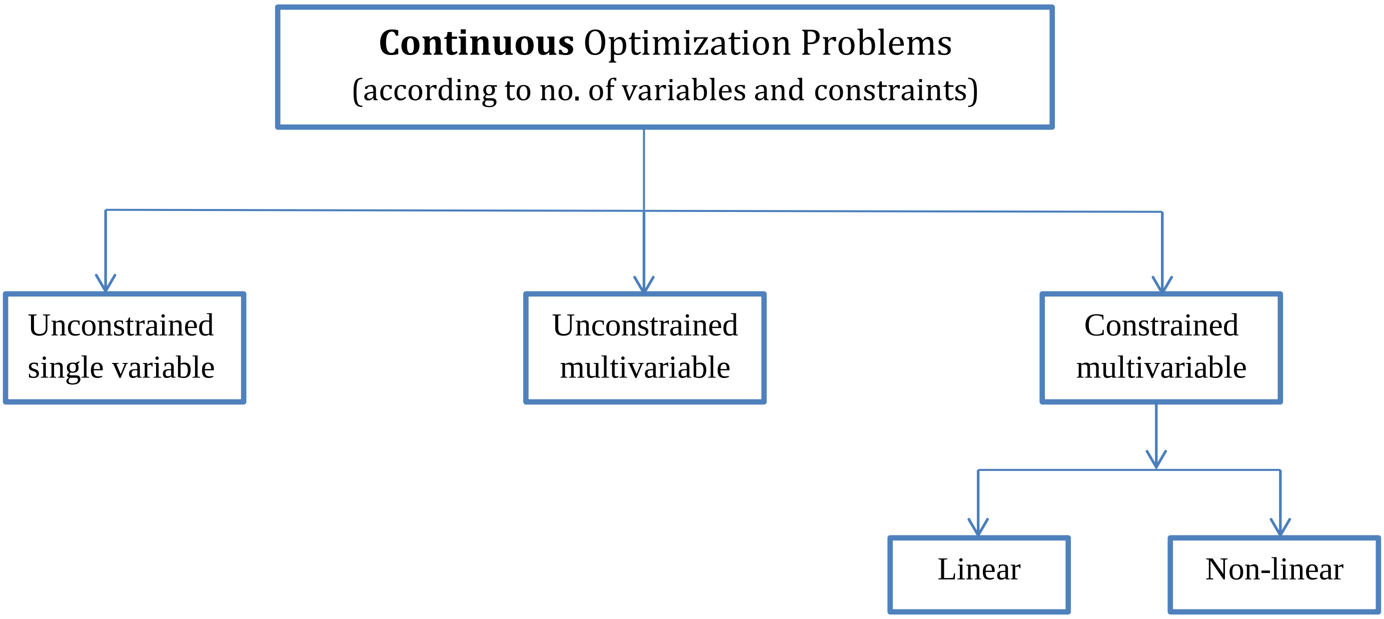
$$\text{Optimize (maximize/minimize) } Z = f(\boldsymbol{x})$$

$$\text{subject to} \qquad g_j(\boldsymbol{x})(=,\ \leq,\ \geq)\ 0,\ \ j = 1,2,\dots,m \qquad (1)$$

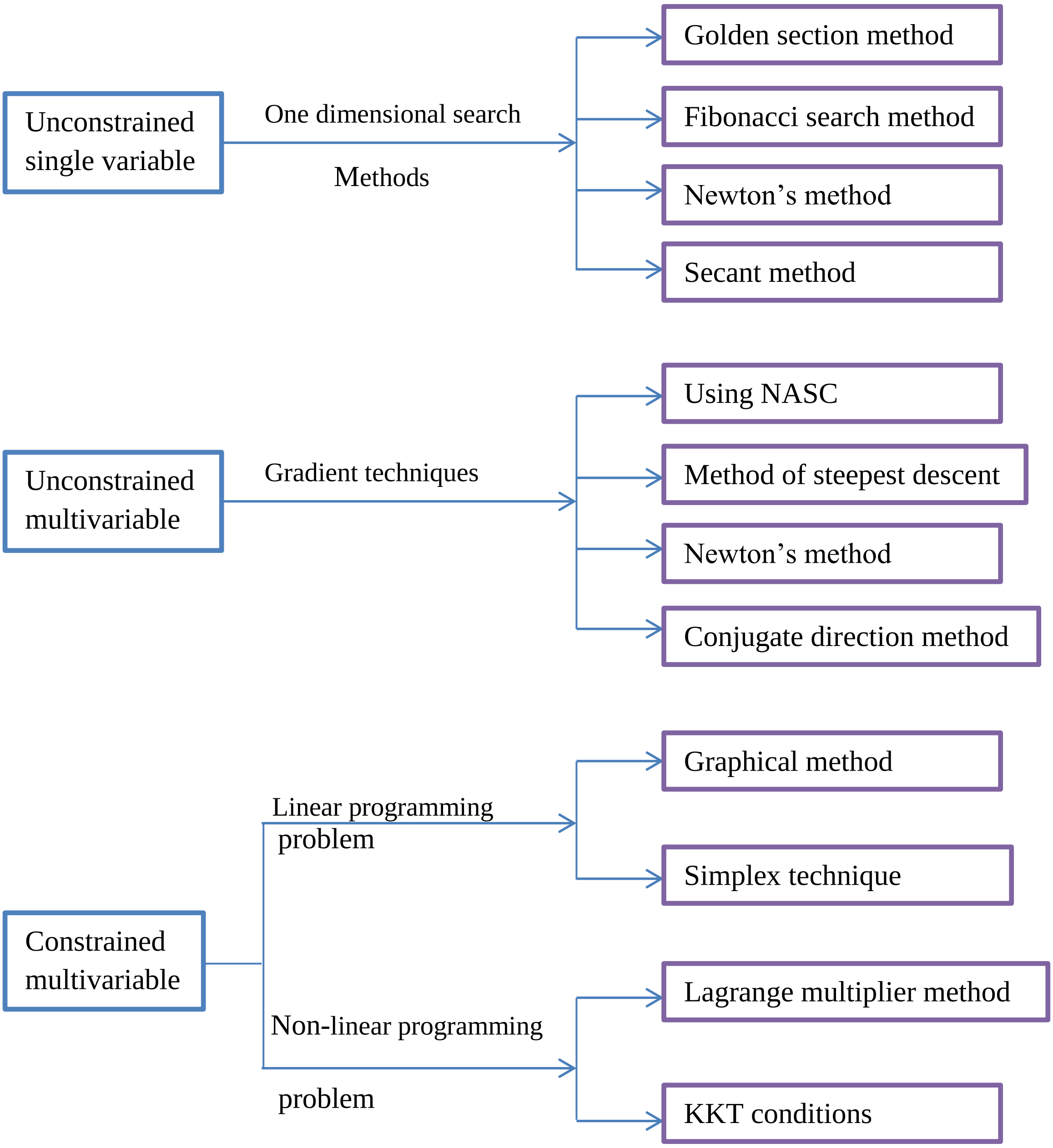
$$\text{where} \qquad \boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,\ \ f: \mathbb{R}^n \to \mathbb{R},$$

Optimal solution: Values of the decision or design variables $x_i,\ i = 1,2,\dots,n$ satisfying all the constraints for which the objective function $f(\boldsymbol{x})$ will give optimal value (i.e. maximum or minimum according to the problem) is called optimal solution. So to solve the problem (1), we have to find value of the design vector $\boldsymbol{x}$**,** say $\boldsymbol{x}^*$ satisfying all the given constraints such that $f(\boldsymbol{x})$ will attain its maximum or minimum value according to maximization and minimization problem respectively. Then $f(\boldsymbol{x}^*)$ will be the optimal objective value.

Note: If problem (1) consists of continuous decision variables, i.e. variables that take on a continuous range of values, then it refers to continuous optimization. If in problem (1), there is no constraint, then it refers to unconstrained optimization problem. If problem (1) consists of only one variable, then it refers to single variable problem, otherwise we call it multivariable optimization. If the objective function and the constraints are linear, then the problem is called linear programming problem (LPP), otherwise it will be non-linear programming problem. Here the word 'linear' indicates that 2nd or higher power of the variables or their products do not appear in the expressions of the objective function as well as of the constraints.



Some Methods:



An example of non-linear optimization problem (NLP): The problem formulated above is an LPP. Let us now consider the following engineering problem:

Problem: Formulate an optimization problem to find the dimensions of a box of largest volume that can be inscribed in a sphere of unit radius.

Solution: Let the center of the sphere be the origin of the Cartesian coordinate system, and (x_1, x_2, x_3) be an arbitrary point on the sphere. Then the sides of the box be $2x_1, 2x_2$ and $2x_3$. Then the volume of the box is given by $f(x) = 8x_1x_2x_3$. According to the given condition (unit sphere), $x_1^2 + x_2^2 + x_3^2 = 1$. So the problem becomes:

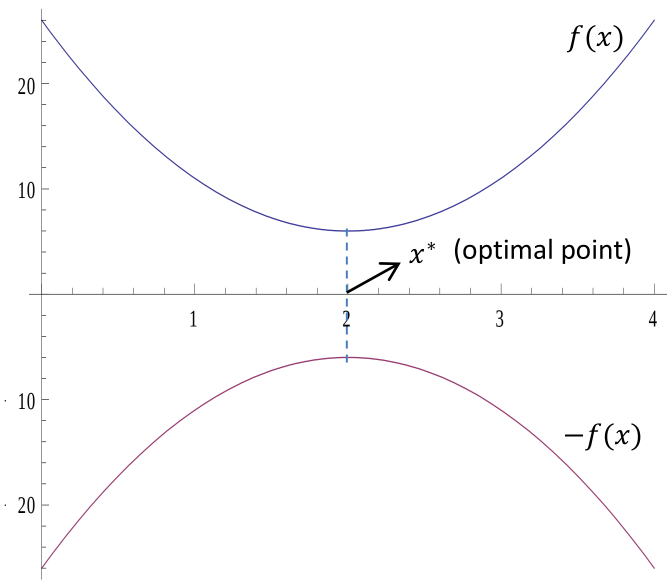
Maximize $f(x) = 8x_1x_2x_3$
subject to $x_1^2 + x_2^2 + x_3^2 = 1$
$x_1, x_2, x_3 > 0$ (feasibility condition).

Conversion of maximization (minimization) problem to minimization (maximization) problem:

- Minimization (maximization) of $f\ (x)$ is same as maximization (minimization) of $-f\ (x)$.

Note: Constant multiple of the objective function or addition to the objective function will not change the optimal solution, i.e.

$cf\ (x)$ and $c + f(x)$ will have the same optimal solution to that of $f\ (x)$, where c is constant.



The concept of feasible region or solution space, bounded and unbounded region, and active constraint will be discussed later.