

Assignment Ncert Exemplar

Rajeev Kumar
EE22BTECH11042

I. QUESTION:- GATE 2023.54

Suppose that X is a discrete random variable with the following probability mass

$$P(X = 0) = \frac{1}{2} (1 + e^{-1}) \quad (1)$$

$$P(X = k) = \frac{e^{-1}}{2k!} \text{ for } k = 1, 2, 3, \dots \quad (2)$$

Which of the following is/are true?

- 1) $E(X) = 1$
- 2) $E(X) < 1$
- 3) $E(X|X > 0) < \frac{1}{2}$
- 4) $E(X|X > 0) > \frac{1}{2}$

Solution:

1) **Theory:**

a) As we know,

$$E(X) = \sum k p_X(k) \quad (3)$$

Therefore,

$$E(X) = 0 \cdot \frac{1}{2} (1 + e^{-1}) + \sum_{k=1}^{\infty} \frac{k e^{-1}}{2k!} \quad (4)$$

$$= \sum_{k=1}^{\infty} \frac{e^{-1}}{2(k-1)!} \quad (5)$$

$$= \frac{1}{2e} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \quad (6)$$

$$= \frac{1}{2e} \cdot e \quad (\text{Using standard result of exponential series})$$

$$= \frac{1}{2} \quad (7)$$

b) To find $E(X|X > 0)$, first we need to find $\Pr(X|X > 0)$ which can be given as:

$$\Pr(X|X > 0) = \frac{\Pr(X = k)}{1 - \Pr(X = 0)} \quad (8)$$

$$= \frac{e^{-1}}{2k!} \cdot \frac{1}{(1 - \frac{1}{2}(1 + e^{-1}))} \quad (9)$$

$$= \frac{e^{-1}}{2k!} \cdot \frac{2}{(1 - e^{-1})} \quad (10)$$

$$= \frac{e^{-1}}{k!(1 - e^{-1})} \quad (11)$$

$$= \frac{1}{k!(e - 1)} \quad (12)$$

Therefore,

$$E(X|X > 0) = \sum_{k=1}^{\infty} k \frac{1}{k!(e - 1)} \quad (13)$$

$$= \frac{1}{e - 1} \sum_{k=1}^{\infty} \frac{1}{(k - 1)!} \quad (14)$$

$$= \frac{1}{e - 1} \cdot e \quad (\text{Using standard result of exponential series})$$

$$= 1.582 \quad (15)$$

Referring to equations (7) and (15), we get that option (2) and (4) are correct.

2) **Simulation:**

To make the simulation of the given question, generate a large set of random variables, say X , with the probability:

$$P(X = 0) = \frac{1}{2} (1 + e^{-1}) \quad (16)$$

$$P(X = k) = \frac{e^{-1}}{2k!} \text{ for } k = 1, 2, 3, \dots \quad (17)$$

In the simulation process, we first start by defining the cumulative distribution function (CDF) for the given random variable X , which is expressed as:

$$CDF = F_X(x) = \sum_{-\infty}^x f_x(t) dt \quad (\text{Here, } f_x \text{ is probability density function})$$

Once we have the CDF, we use the concept of Inverse Transform Sampling. This involves generating a uniform random variable U from the range $[0, 1]$ and then inverting the CDF to obtain X . The inversion is done using the formula:

$$X = F_X^{-1}(U) \quad (18)$$

In simpler terms, you're using U to navigate through the distribution's cumulative probabilities to find the corresponding value of X . It link these random numbers to specific distributions and it relies on a key formula:

$$X = F_X^{-1}(U) = -\frac{\ln(1 - U)}{\lambda} \quad (\lambda \text{ depends on probability given})$$

Using this method, random number is compared to cumulative probabilities (CDF) for various values of the random variable k until the CDF exceeds it. This process continues until a match is found and k is returned as the generated random variable.

- a) To get $E(X)$, take the weighted sum of all possible values of X including 0, each multiplied by its respective probability. Expression for this can be given as:

$$E(X) = \sum k p_X(k) \quad (19)$$

- b) As given in theory, to get $E(X|X > 0)$, take the weighted sum of all possible values of X excluding 0, each multiplied by its respective probability and dividing it by the probability of getting random variables greater than 0. Expression for this can be given as:

$$\Pr(X|X > 0) = \frac{\Pr(X = k)}{1 - \Pr(X = 0)} \quad (20)$$

$$(21)$$