Assignment
Ncert Exampler

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I. Question:- GATE 2023.54

Suppose that X is a discrete random variable with the following probability mass

$$P(X = 0) = \frac{1}{2}\left(1 + e^{-1}\right) \tag{1}$$

$$P(X = k) = \frac{e^{-1}}{2k!} \; for \; k = 1, 2, 3, \dots. \tag{2}$$

Which of the following is/are true?

1) $E(X) = 1$
2) $E(X) < 1$
3) $E(X|X > 0) < \frac{1}{2}$
4) $E(X|X > 0) > \frac{1}{2}$

Solution:

1) **Theory:**
   a) As we know,

$$E(X) = \sum k p_X(k) \tag{3}$$

Therefore,

$$E(X) = 0 \cdot \frac{1}{2}\left(1 + e^{-1}\right) + \sum_{k=1}^{\infty} \frac{ke^{-1}}{2k!} \tag{4}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-1}}{2(k-1)!} \tag{5}$$

$$= \frac{1}{2e} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \tag{6}$$

$$= \frac{1}{2e} \cdot e \qquad \text{(Using standard result of exponential series)}$$

$$= \frac{1}{2} \tag{7}$$

b) To find $E(X|X>0)$, first we need to find $\Pr(X|X>0)$ which can be given as:

$$\begin{aligned}
\Pr(X|X>0) &= \frac{\Pr(X=k)}{1-\Pr(X=0)} && (8)\\
&= \frac{e^{-1}}{2k!}\cdot\frac{1}{(1-\frac{1}{2}(1+e^{-1}))} && (9)\\
&= \frac{e^{-1}}{2k!}\cdot\frac{2}{(1-e^{-1})} && (10)\\
&= \frac{e^{-1}}{k!(1-e^{-1})} && (11)\\
&= \frac{1}{k!(e-1)} && (12)
\end{aligned}$$

Therefore,

$$\begin{aligned}
E(X|X>0) &= \sum_{k=1}^{\infty} k\frac{1}{k!(e-1)} && (13)\\
&= \frac{1}{e-1}\sum_{k=1}^{\infty}\frac{1}{(k-1)!} && (14)\\
&= \frac{1}{e-1}\cdot e && \text{(Using standard result of exponential series)}\\
&= 1.582 && (15)
\end{aligned}$$

Refering to equations (7) and (15), we get that option (2) and (4) are correct.

2) **Simulation:**
To make the simulation of the given question, generate a large set of random variables, say X, with the probability:

$$P(X=0) = \frac{1}{2}\left(1+e^{-1}\right) \tag{16}$$

$$P(X=k) = \frac{e^{-1}}{2k!}\ for\ k = 1, 2, 3, \dots. \tag{17}$$

To simulate random variables, you kickstart the process by setting the initial conditions for a random number generator, often using the current time to ensure different random sequences each time the program runs. In a loop, you generate a random number between 0 and 1. This whole technique is based on a fascinating concept known as Inverse Transform Sampling. According to this theorem, you have a uniform random variable $U \sim Unif(0,1)$, and you connect it to the distribution you want using

$$X = {F_X}^{-1}(U). \tag{18}$$

In simpler terms, you're using U to navigate through the distribution's cumulative probabilities to find the corresponding value of X. It link these random numbers to specific distributions and it relies on a key formula:

$$k = \frac{-ln(1-randomnumber)}{\lambda} \tag{19}$$

,where λ is a crucial parameter for the distribution you're simulating.
Using this method, random number is compared to cumulative probabilities (CDF) for various values of therandom variable k until the CDF exceeds it. This process continues until a match is found and k is returned as the generated random variable.

a) To get $E(X)$, take the weighted sum of all possible values of X including 0, each multiplied by its respective probability. Expression for this can be given as:

$$E(X) = \sum k p_X(k) \tag{20}$$

b) As given in theory, to get $E(X|X > 0)$, take the weighted sum of all possible values of X excluding 0, each multiplied by its respective probability and dividing it by the probability of getting random variables greater than 0. Expression for this can be given as:

$$\Pr(X|X > 0) = \frac{\Pr(X = k)}{1 - \Pr(X = 0)} \tag{21}$$

$$\tag{22}$$