Assignment
Ncert Exampler

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I. Question:- GATE 2023.54

Suppose that X is a discrete random variable with the following probability mass

$$P(X = 0) = \frac{1}{2}\left(1 + e^{-1}\right) \tag{1}$$

$$P(X = k) = \frac{e^{-1}}{2k!} \; for \; k = 1, 2, 3, \dots . \tag{2}$$

Which of the following is/are true?

1) $E(X) = 1$
2) $E(X) < 1$
3) $E(X|X > 0) < \frac{1}{2}$
4) $E(X|X > 0) > \frac{1}{2}$

Solution:

1) **Theory:**
   a) As we know,

$$E(X) = \sum k p_X(k) \tag{3}$$

Therefore,

$$E(X) = 0 \cdot \frac{1}{2}\left(1 + e^{-1}\right) + \sum_{k=1}^{\infty} \frac{k e^{-1}}{2k!} \tag{4}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-1}}{2(k-1)!} \tag{5}$$

$$= \frac{1}{2e} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \tag{6}$$

$$= \frac{1}{2e} \cdot e \qquad \text{(Using standard result of exponential series)}$$

$$= \frac{1}{2} \tag{7}$$

b) To find $E(X|X > 0)$, first we need to find $\Pr(X|X > 0)$ which can be given as:

$$\Pr(X|X > 0) = \frac{\Pr(X = k)}{1 - \Pr(X = 0)} \tag{8}$$
$$= \frac{e^{-1}}{2k!} \cdot \frac{1}{(1 - \frac{1}{2}(1 + e^{-1}))} \tag{9}$$
$$= \frac{e^{-1}}{2k!} \cdot \frac{2}{(1 - e^{-1})} \tag{10}$$
$$= \frac{e^{-1}}{k!(1 - e^{-1})} \tag{11}$$
$$= \frac{1}{k!(e - 1)} \tag{12}$$

Therefore,

$$E(X|X > 0) = \sum_{k=1}^{\infty} k \frac{1}{k!(e - 1)} \tag{13}$$
$$= \frac{1}{e - 1} \sum_{k=1}^{\infty} \frac{1}{(k - 1)!} \tag{14}$$
$$= \frac{1}{e - 1} \cdot e \qquad \text{(Using standard result of exponential series)}$$
$$= 1.582 \tag{15}$$

Refering to equations (7) and (15), we get that option (2) and (4) are correct.

2) **Simulation:**
To make the simulation of the given question, generate a large set of random variables, say X, with the probability:

$$P(X = 0) = \frac{1}{2}\left(1 + e^{-1}\right) \tag{16}$$
$$P(X = k) = \frac{e^{-1}}{2k!} \; for \; k = 1, 2, 3, \dots . \tag{17}$$

In the simulation process, we first start by defining the cumulative distribution function (CDF) for the given random variable X, which is expressed as:

$$CDF = F_X(x) = \int_{-\infty}^{x} f_x(t)dt \qquad \text{(Here, } f_x \text{ is probability density function)}$$

Once we have the CDF, we use the concept of Inverse Transform Sampling. This involves generating a uniform random variable U from the range $[0, 1]$ and then inverting the CDF to obtain X. The inversion is done using the formula:

$$X = {F_X}^{-1}(U) \tag{18}$$

In simpler terms, you're using U to navigate through the distribution's cumulative probabilities to find the corresponding value of X. It link these random numbers to specific distributions and it relies on a key formula:

$$X = {F_X}^{-1}(U) = -\frac{ln(1 - U)}{\lambda} \qquad (\lambda \text{ depends on probability given})$$

Using this method, random number is compared to cumulative probabilities (CDF) for various values of therandom variable k until the CDF exceeds it. This process continues until a match is found and k is returned as the generated random variable.

a) To get $E(X)$, take the weighted sum of all possible values of X including 0, each multiplied by its respective probability. Expression for this can be given as:

$$E(X) = \sum kp_X(k) \tag{19}$$

b) As given in theory, to get $E(X|X > 0)$, take the weighted sum of all possible values of X excluding 0, each multiplied by its respective probability and dividing it by the probability of getting random variables greater than 0. Expression for this can be given as:

$$\Pr(X|X > 0) = \frac{\Pr(X = k)}{1 - \Pr(X = 0)} \tag{20}$$

$$\tag{21}$$