

# 10.3.5.3

EE24BTECH11028 - Jadhav Rajesh

## Question:

Solve the system of linear equations:

$$8x + 5y - 9 = 0 \quad (0.1)$$

$$3x + 2y - 4 = 0 \quad (0.2)$$

## Step 1: Represent the system in matrix form

The system of equations can be written as:

$$A\mathbf{x} = \mathbf{b}, \quad (0.3)$$

where

$$A = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}, x = \begin{pmatrix} x \\ y \end{pmatrix}, b = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \quad (0.4)$$

## Step 2: Perform LU Decomposition

We decompose the matrix  $A$  into the product of a lower triangular matrix  $L$  and an upper triangular matrix  $U$ , i.e.,

$$A = LU \quad (0.5)$$

where

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} \quad (0.6)$$

Now, let's compute the LU decomposition step by step.

First, we find the elements of  $U$  and  $L$ :

$$u_{11} = a_{11} = 8, u_{12} = a_{12} = 5. \quad (0.7)$$

Next, we compute  $l_{21}$  and  $u_{22}$ :

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{3}{8}, \quad (0.8)$$

$$u_{22} = a_{22} - l_{21}u_{12} = \frac{1}{8}. \quad (0.9)$$

So the LU decomposition is:

$$L = \begin{pmatrix} 1 & 0 \\ \frac{3}{8} & 1 \end{pmatrix}, U = \begin{pmatrix} 8 & 5 \\ 0 & \frac{1}{8} \end{pmatrix}, \quad (0.10)$$

### Step 3: Solve for $\mathbf{x}$ using $LU$ decomposition

Now we solve the system in two steps using forward substitution and backward substitution.

First, solve  $L\mathbf{y} = \mathbf{b}$  for  $\mathbf{y}$ :

$$L = \begin{pmatrix} 1 & 0 \\ \frac{3}{8} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \quad (0.11)$$

This gives:

$$y_1 = 9, \frac{3}{8}y_1 + y_2 = 4 \quad (0.12)$$

$$y_2 = \frac{5}{8}. \quad (0.13)$$

Thus,  $\mathbf{y} = \begin{pmatrix} 9 \\ \frac{5}{8} \end{pmatrix}$ .

Next, solve  $U\mathbf{x} = \mathbf{y}$  for  $\mathbf{x}$ :

$$U = \begin{pmatrix} 8 & 5 \\ 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ \frac{5}{8} \end{pmatrix} \quad (0.14)$$

This gives:

$$\frac{y}{8} = \frac{5}{8} \Rightarrow y = 5, \quad (0.15)$$

$$8x + 5y = 9 \Rightarrow x = -2. \quad (0.16)$$

Thus, the solution is  $x = -2$  and  $y = 5$ .

### LU Decomposition using Doolittle's algorithm:

The LU decomposition can be efficiently computed using Doolittle's algorithm. This method generates the matrices  $L$  (lower triangular) and  $U$  (upper triangular) such that  $A = LU$ . The elements of these matrices are calculated as follows:

Elements of the  $U$  Matrix:

For each column  $j$ :

$$U_{ij} = A_{ij} \text{ if } i = 0, \quad (0.17)$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik}U_{kj} \text{ if } i > 0. \quad (0.18)$$

Elements of the  $L$  Matrix:

For each row  $i$ :

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \text{ if } j = 0, \quad (0.19)$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik}U_{kj}}{U_{jj}} \text{ if } j > 0. \quad (0.20)$$

This systematic approach ensures that the matrix  $A$  is decomposed into  $L$  and  $U$  without requiring row swaps, provided  $A$  is nonsingular.

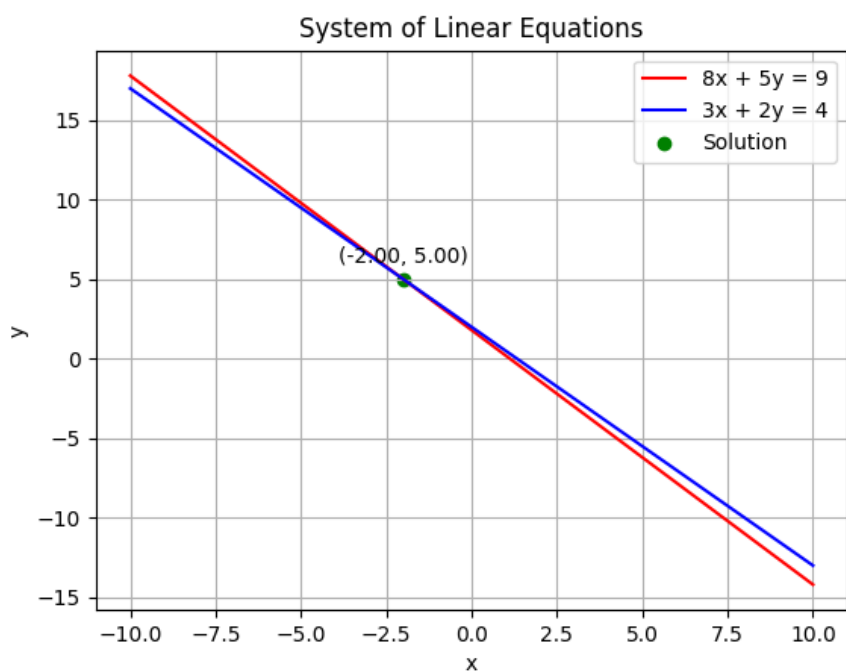


Fig. 0.1: Solving the system of equations