9.3.12

EE24BTECH11028 - Jadhav Rajesh

Question: Which of the following differential equations has $y = x$ as one of its particular solution?

$$(C)\,\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0 \tag{0.1}$$

Solution: NUMERICAL METHOD

Consider,

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0 \tag{0.2}$$

Assuming the initial conditions $y(0) = 0$ and $y'(0) = 1$.
HOMOGENEOUS PART:

The associated homogeneous equation is:

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0. \tag{0.3}$$

Assume a power series solution:

$$y_h = \sum_{n=0}^{\infty} a_n x^n. \tag{0.4}$$

The derivatives are:

$$\frac{dy_h}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y_h}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \tag{0.5}$$

Substitute into the homogeneous equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0. \tag{0.6}$$

Rewriting terms, we derive the recurrence relation:

$$a_{n+2} = \frac{n-2}{(n+2)(n+1)} a_{n-1}. \tag{0.7}$$

Apply the initial conditions

The initial conditions are: $y(0) = 0 \quad \Rightarrow \quad a_0 = 0$,
$y'(0) = 1 \quad \Rightarrow \quad a_1 = 1$.

Using the recurrence relation:

$$a_{n+2} = \frac{n-2}{(n+2)(n+1)} a_{n-1}, \tag{0.8}$$

we compute the coefficients:

1) For $n = 0$: $a_0 = 0$
2) For $n = 1$: $a_1 = 1$
3) For $n = 2$: $a_2 = \frac{0-2}{(2+2)(2+1)} a_1 = \frac{-2}{12} = -\frac{1}{6}$
4) For $n = 3$: $a_3 = \frac{1-2}{(3+2)(3+1)} a_2 = \frac{-1}{20} . \left(-\frac{1}{6}\right) = \frac{1}{120}$
5) For $n = 4$: $a_4 = \frac{2-2}{(4+2)(4+1)} a_3 = 0$

The pattern is:

$$a_{2k} = 0, \quad a_{2k+1} = \frac{(-1)^k}{(2k+1)!}. \tag{5.1}$$

Therefore, the homogeneous solution is:

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \tag{5.2}$$

figure/fig.png

Fig. 5.1: Numerical Solution