12.6.6.18

EE24BTECH11028 - Jadhav Rajesh

QUESTION: Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and greatest volume of a cylinder is $\frac{4}{27}\pi h^3 tan^2\alpha$.

SOLUTION:

THEORETICAL SOLUTION:

Geometry of the cone and cylinder

Cone Dimensions	Cylinder Dimensions
Height of the cone: h	Height of the cylinder: x (to be determined)
Semi-vertical angle: α	Radius of the cylinder: r
Radius of the base of the cone: $R = h\tan\alpha$	$r = (h - x)\tan\alpha$

Volume of the cylinder

The volume V of the cylinder is given by:

$$V = \pi r^2 x \tag{0.1}$$

Substitute $r = (h - r)\, tan\alpha$

$$V = \pi\left[(h - x)\, tan\alpha\right]^2 x \tag{0.2}$$

Simplify

$$V = \pi (h - x)^2 \left(tan^2\alpha\right) x \tag{0.3}$$

Expand $(h - x)^2$

$$V = \pi tan^2\alpha \left(h^2 - 2hx + x^2\right) x \tag{0.4}$$

Distribute x

$$V = \pi tan^2\alpha \left(h^2 x - 2hx^2 + x^3\right) \tag{0.5}$$

Maximize the volume

To maximize V, take its derivative with respect to x and set it equal to zero

$$\frac{dv}{dx} = \pi tan^2\alpha \left(h^2 - 4hx + 3x^2\right) \tag{0.6}$$

Set $\frac{dv}{dx} = 0$

$$h^2 - 4hx + 3x^2 = 0 \tag{0.7}$$

Factorize

$$(h - x)(3x - h) = 0 \tag{0.8}$$

This gives two solutions

$$x = h \; or \; x = \frac{h}{3} \tag{0.9}$$

Since $x = h$ corresponds to the cylinder degenerating into a point, the height of the cylinder of greatest volume is

$$x = \frac{h}{3} \tag{0.10}$$

Greatest volume of the cylinder
Substitude $x = \frac{h}{3}$ into the volume formula

$$V = \pi tan^2\alpha\left(h^2x - 2hx^2 + x^3\right) \tag{0.11}$$

Substitude $x = \frac{h}{3}$

$$V = \pi tan^2\alpha\left(h^2 \cdot \frac{h}{3} - 3h\left(\frac{h}{3}\right)^2 + \left(\frac{h}{3}\right)^3\right) \tag{0.12}$$

$$V = \frac{4}{27}\pi h^3 tan^2 \tag{0.13}$$

Final Result:
The height of the cylinder of greatest volume is

$$x = \frac{h}{3} = h_c \tag{0.14}$$

The greatest volume of the cylinder is

$$V = \frac{4}{27}\pi h^3 tan^2\alpha \tag{0.15}$$

Computational Solution:
Gradient Ascent method:

The iterative update rule for maximizing V with respect to h_c is:

$$h_{c,n+1} = h_c + \alpha \frac{dV}{dh_c}\Big|_{h_c=h_{c,n}} \tag{0.16}$$

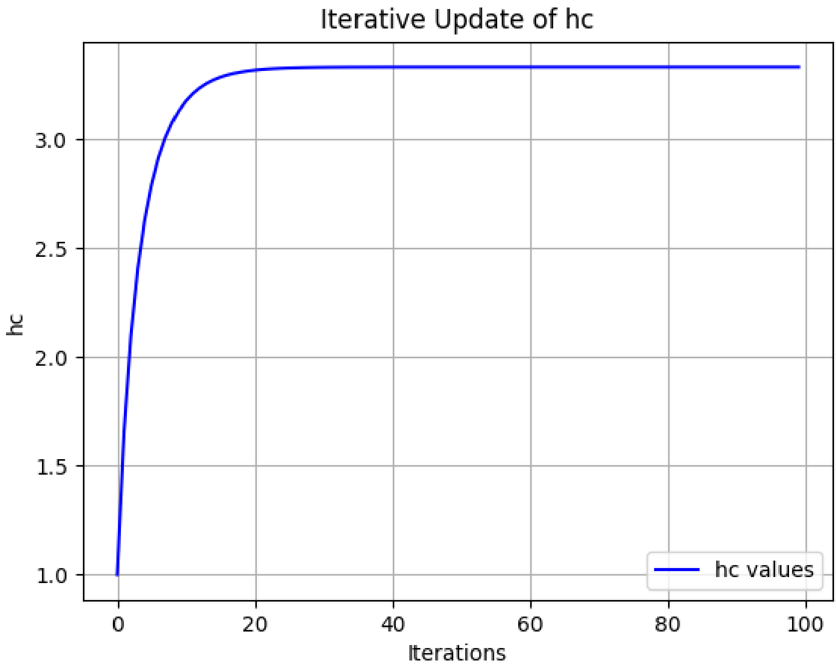
where α is the learning rate.
Compute $\frac{dV}{dh_c}$:

$$V = \pi h^2 tan^2\alpha \left(1 - \frac{2h_c^2}{h} + \frac{3h_c^3}{h^2}\right) \tag{0.17}$$

$$\frac{dV}{dh_c} = \pi h^2 tan^2\alpha \left(1 - \frac{4h_c}{h} + \frac{3h_c^2}{h^2}\right) \tag{0.18}$$

Thus, the update rule becomes:

$$h_{c,n+1} = h_{c,n} + \alpha\pi h^2 tan^2\alpha \left(1 - \frac{4h_{c,n}}{h} + \frac{3h_{c,n}^2}{h^2}\right) \tag{0.19}$$



Iterative Update of hc
3.0
2.5
2.0
1.5
1.0
hc
0
20
40
60
80
100
Iterations
hc values