Theory of Black-Scholes Model

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The model for stock price evolution is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \ldots.(1)$$

and a riskless bond, B, grows at a continuously compounding rate r. The Black-Scholes pricing theory then tells us that the price of a vanilla option, with expiry T and pay-off f , is equal to

$$e^{-rT}E(f(S_T))$$

where the expectation is taken under the associated risk-neutral process,

$$dS_t = rS_t dt + \sigma S_t dW_t \quad \ldots.(2)$$

We solve equation (2) by passing to the log and using Ito's lemma; we compute

$$d\ log(S_t) = (r - \frac{1}{2}\sigma^2)dt \ + \ \sigma dW_t$$

As this process is constant-coefficient, it has the solution

$$log(S_t) = log(S_0) + (r - \frac{1}{2}\sigma^2)t \ + \ \sigma W_t$$

Note that, here W_t is a Brownian motion, $W_t \sim \sqrt{T}N(0,1)$.

and hence

$$log(S_T) = log(S_0) + (r - \frac{1}{2}\sigma^2)T \ + \ \sigma\sqrt{T}N(0,1)$$

or equivalently,

$$S_T = S_0(e^{\{(r-\frac{1}{2}\sigma^2)T \ + \ \sigma\sqrt{T}N(0,1)\}})$$

The price of a vanilla option is therefore equal to

$$e^{-rT}E(f(S_0 e^{\{(r-\frac{1}{2}\sigma^2)T \ + \ \sigma\sqrt{T}N(0,1)\}}))$$

where $f(S)$, $\{f(S) = (S-K)_+ = max\{0, S(T) - K\}\}$ is the pay-off function where, K =**Strike Price** which is fixed at a previous point of time, lets say time 0.