Hence the general solution is

$$y=\left(C_1\cos\frac{a}{\sqrt{2}}x+C_2\sin\frac{a}{\sqrt{2}}x\right)e^{-\frac{a}{\sqrt{2}}x}+\left(C_3\cos\frac{a}{\sqrt{2}}x+C_4\sin\frac{a}{\sqrt{2}}x\right)e^{\frac{a}{\sqrt{2}}x},$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Ex. 4. *Solve the equation* $(D^2+1)^3(D^2+D+1)^2y=0$, *where* $D\equiv\dfrac{d}{dx}$.

Here the auxiliary equation is $(m^2+1)^3(m^2+m+1)^2=0$.

Therefore $m=\pm i, \pm i, \pm i$ and $m=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}, -\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$.

Hence the general solution is

$$y=(C_1+C_2x+C_3x^2)\cos x+(C_4+C_5x+C_6x^2)\sin x$$
$$+\left\{(C_7+C_8x)\cos\frac{\sqrt{3}}{2}x+(C_9+C_{10}x)\sin\frac{\sqrt{3}}{2}x\right\}e^{-\frac{1}{2}x},$$

in which $C_1, C_2, \ldots\ldots, C_{10}$ are arbitrary constants.

Note that the general solution contains ten arbitrary constants, which is the same as the order of the given equation.

Note. The values of the arbitrary constants and hence the particular solution of the equation can be determined from given conditions.

Examples V(A)

Solve the following differential equations (1 – 9) :

1. $4\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-3y=0$.
2. $\{D^2+(a+b)D+ab\}y=0$.
3. $\dfrac{d^2y}{dx^2}-24\dfrac{dy}{dx}+144y=0$.
4. $\dfrac{d^2y}{dx^2}+8\dfrac{dy}{dx}+25y=0$.
5. $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}-6y=0$.
6. $(D^2+4)(D^2+1)y=0$.
7. $\dfrac{d^4y}{dx^4}-\dfrac{d^3y}{dx^3}-9\dfrac{d^2y}{dx^2}-11\dfrac{dy}{dx}-4y=0$.
8. $\dfrac{d^4y}{dx^4}-4\dfrac{d^3y}{dx^3}+8\dfrac{d^2y}{dx^2}-8\dfrac{dy}{dx}+4y=0$.
9. $(D-1)^3(D^2-4)(D+2)y=0$.

10. Solve the equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ and find the partic[ular] solution if $y = 3$ when $x = 0$ and $y = 8$ when $x = \log 2$.

11. Solve : $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$.

If $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$, find the particular solut[ion].

12. Solve $\frac{d^2y}{dx^2} - 4y = 0$, satisfying $y = 5$ and $\frac{dy}{dx} = 2$ when $x = 0$.

13. Solve $\frac{d^2x}{dt^2} + 4t = 0$, satisfying $x = 4$, $\frac{dx}{dt} = 3$ when $t = 0$.

[B. H. 1980]

14. Solve $(D + 1)^2 y = 0$, satisfying $y = 2\log 2$ when $x = \log 2$ an[d] $y = \frac{4}{3}\log 3$ when $x = \log 3$.

15. Show that the solution of the equation $\frac{d^2u}{dx^2} + 6\frac{du}{dx} + 9u = 0$ with $u(0) = 2$ and $\frac{du}{dx}(0) = 0$, is $u(x) = 2(1 + 3x)e^{-3x}$

16. Find the particular solution of the equation

$$l\frac{d^2\theta}{dt^2} + g\theta = 0,$$

satisfying $\theta = \alpha$ and $\frac{d\theta}{dt} = 0$ when $t = 0$. [C. H. 1991]

17. Find y in terms of x from the equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, satisfying $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.

18. If, in the differential equation $\frac{d^4y}{dx^4} - n^4y = 0$, $\frac{dy}{dx} = y = 0$ when $x = 0$ and $x = l$, then prove that

$$y = A(\cos nx - \cosh nx) + B(\sin nx - \sinh nx)$$

and $\cos nl \,.\, \cosh nl = 1$, where A and B are constants. [C. H. 1994]

19. If $\dfrac{d^2x}{dt^2} = -n^2x$ and $x = a$, $\dfrac{dx}{dt} = u$ when $t = 0$, then show that the maximum value of x is $\sqrt{a^2 + \dfrac{u^2}{n^2}}$. [B. H. 1985]

20. For the equation

$$\frac{d^2s}{dt^2} + \frac{g}{e}(s - l) = 0, \ (l, g, e \text{ being constants}),$$

find s and $\dfrac{ds}{dt}$, if $s = s_0$ and $\dfrac{ds}{dt} = 0$ when $t = 0$. [C. H. 1980]

Answers

1. $y = Ae^{\frac{1}{2}x} + Be^{-\frac{3}{2}x}$.
2. $y = Ae^{-ax} + Be^{-bx}$.
3. $y = (C_1 + C_2x)e^{12x}$.
4. $y = e^{-4x}(A\cos 3x + B\sin 3x)$.
5. $y = Ae^{-x} + Be^{2x} + Ce^{-3x}$.
6. $y = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x$.
7. $y = (C_1 + C_2x + C_3x^2)e^{-x} + C_4e^{4x}$.
8. $y = e^x\{(C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x\}$.
9. $y = (C_1 + C_2x + C_3x^2)e^x + (C_4 + C_5x)e^{-2x} + C_6e^{2x}$.
10. $y = Ae^{2x} + Be^{3x}$; $y = 4e^{2x} - e^{3x}$
11. $y = (A + Bx)e^{-2x}$; $y = (1 + 2x)e^{-2x}$.
12. $y = 3e^{2x} + 2e^{-2x}$.
13. $x = 4\cos 2t + \frac{3}{2}\sin 2t$.
14. $y = 4xe^{-x}$.
16. $\theta = \alpha\cos\sqrt{\frac{g}{l}}\,t$.
17. $y = e^{-x} - e^{-2x}$.
20. $s = (s_0 - l)\cos\sqrt{\frac{g}{e}}\,t + l$, $\dfrac{ds}{dt} = (l - s_0)\sqrt{\frac{g}{e}}\sin\sqrt{\frac{g}{e}}\,t$.

5.7. Symbolic operator $\dfrac{1}{f(D)}$.

We use the expression $\dfrac{1}{f(D)}X$ to denote a function of x, which does not contain any arbitrary constant and which gives X when operated with $f(D)$. Thus, since

$$(D^2 - D)(x^2 - x) = 3 - 2x,$$

we have
$$\frac{1}{D^2 - D}(3 - 2x) = x^2 - x.$$