Proof: Let $f(x, y)$ be an integrating factor of the given differential equation, then $f(x, y)(Mdx + Ndy)$ will be an exact differential, i.e., there must be a function $u(x, y)$ such that

$$f(x, y)(Mdx + Ndy) = du.$$

Multiplying the above equation by a function $g(u)$, we get

$$g(u)\, f(Mdx + Ndy) = g(u)\, du = d[G(u)]\ (\text{say}).$$

Since the expression in the right-hand side is an exact differential, the expression in the left-hand side will also be an exact differential. This shows that $g(u)\, f(x, y)$ is also an integrating factor of the given equation. Since the function $g(u)$ is arbitrary, the given equation has infinite number of integrating factors.

3.2 Rules for Finding the Integrating Factors

An equation of the form $Mdx + Ndy = 0$ has always integrating factors. Here we shall discuss the rules for finding them.

3.2.1 By inspection

Integrating factors of the equation $Mdx + Ndy = 0$ can be found by rearranging its terms, or by dividing it by suitable function of x, y. Here we give a list of expressions of exact differentials which would help in solving an equation of the above form.

(i) $d\left(\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$, (ii) $d\left(\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}$,

(iii) $d(xy) = xdy + ydx$, (iv) $\dfrac{1}{2} d(x^2 + y^2) = xdx + ydy$

(v) $d[\ln xy] = \dfrac{xdy + ydx}{xy}$, (vi) $d\left[\ln\left(\dfrac{x}{y}\right)\right] = \dfrac{ydx - xdy}{xy}$,

(vii) $d\left[\ln\left(\dfrac{y}{x}\right)\right] = \dfrac{xdy - ydx}{xy}$, (viii) $d\left[\dfrac{1}{2}\ln(x^2 + y^2)\right] = \dfrac{xdx + ydy}{x^2 + y^2}$,

(ix) $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2 + y^2}$, (x) $d\left(\tan^{-1}\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{x^2 + y^2}$

(xi) $d[\sin^{-1} xy] = \dfrac{xdy + ydx}{\sqrt{1 - x^2y^2}}$ (xii) $d\left[\sin^{-1}\dfrac{y}{x}\right] = \dfrac{xdy - ydx}{x\sqrt{x^2 - y^2}}$.

ILLUSTRATIVE EXAMPLES

(i) Solve: $xdy + ydx = k(xdy - ydx)$.

Solution

Dividing by xy, we get

$$\frac{d(xy)}{xy} = k\left(\frac{xdy - ydx}{xy}\right) \text{ or, } d[\ln xy] = k\, d\left(\ln \frac{y}{x}\right)$$

Integrating, we get

$$\ln xy = k\ln\frac{y}{x} + \ln C \quad \text{or,} \quad xy = C\left(\frac{y}{x}\right)^k,$$ where C is an arbitrary constant.

(ii) Solve: $(x^2+y^2)(xdx+ydy)+xdy-ydx=0$.

Solution

We rewrite the equation as $\frac{1}{2}d\left(x^2+y^2\right)+\frac{xdy-ydx}{x^2+y^2}=0$.

Integrating, we get $x^2+y^2+2\tan^{-1}\frac{y}{x}=C$, where C is an arbitrary constant.

(iii) Solve: $x^2\frac{dy}{dx}+xy+2\sqrt{1-x^2y^2}=0$.

Solution

We rewrite the equation as $x(xdy+ydx)+2\sqrt{1-x^2y^2}\,dx=0$

$$\text{or,} \quad \frac{xdy+ydx}{\sqrt{1-x^2y^2}}+2\frac{dx}{x}=0 \qquad \text{or,} \quad d\left[\sin^{-1}xy\right]+2d\left[\ln x\right]=0.$$

Integrating we get $\sin^{-1}xy+2\ln x=C$, where C is an arbitrary constant.

(iv) Solve: $xdx+ydy+\left(1+\frac{y^2}{x^2}\right)(ydx-xdy)=0$.

Solution

We rewrite the equation as $\frac{xdx+ydy}{x^2+y^2}+\frac{ydx-xdy}{x^2}=0$

$$\text{or,} \quad d\left[\frac{1}{2}\ln\left(x^2+y^2\right)\right]-d\left(\frac{y}{x}\right)=0.$$

Integrating we get $\ln(x^2+y^2)=2\frac{y}{x}+C$, where C is an arbitrary constant.

(v) Solve: $xdy-ydx=2\sqrt{x^2-y^2}\,dx$.

Solution

We rewrite the equation as $\frac{xdy-ydx}{x\sqrt{x^2-y^2}}=\frac{2}{x}dx$ or, $d\left[\sin^{-1}\frac{y}{x}\right]=2d(\ln x)$

Integrating we get $\sin^{-1}\frac{y}{x}=2\ln x+C$, where C is an arbitrary constant.

3.2.2 If the equation $Mdx+Ndy=0$ is a homogeneous equation of degree n and $Mx+Ny\neq 0$, then $\frac{1}{Mx+Ny}$ will be an integrating factor.

Proof: The condition of integrability of the equation $\frac{Mdx+Ndy}{Mx+Ny}=0$

is $\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right)=\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)$.

Now, $\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right)$

$$=\frac{(Mx+Ny)\frac{\partial M}{\partial y}-M\left(x\frac{\partial M}{\partial y}+y\frac{\partial N}{\partial y}+N\right)}{(Mx+Ny)^2}=\frac{Ny\frac{\partial M}{\partial y}-M\left(y\frac{\partial N}{\partial y}+N\right)}{(Mx+Ny)^2}$$

$$=\frac{N\left(x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}\right)-M\left(y\frac{\partial N}{\partial y}+N\right)-Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2}$$

$$=\frac{(n-1)MN-My\frac{\partial N}{\partial y}-Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2}$$

and $\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)$

$$=\frac{(Mx+Ny)\frac{\partial N}{\partial x}-N\left(x\frac{\partial M}{\partial x}+M+y\frac{\partial N}{\partial x}\right)}{(Mx+Ny)^2}$$

$$=\frac{Mx\frac{\partial N}{\partial x}-N\left(x\frac{\partial M}{\partial x}+M\right)}{(Mx+Ny)^2}$$

$$=\frac{M\left(x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}\right)-N\left(x\frac{\partial M}{\partial x}+M\right)-My\frac{\partial N}{\partial y}}{(Mx+Ny)^2}$$

$$=\frac{(n-1)MN-My\frac{\partial N}{\partial y}-Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2}.$$

Thus from (2) and (3) we see that the condition of integrability is satisfied, hence $\frac{1}{Mx+Ny}$ is an integrating factor of the given equation.

Alternative Method: We rewrite $Mdx + Ndy$ as

$$Mdx+Ndy=\frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right].$$

Dividing by $Mx + Ny$ as $Mx + Ny \neq 0$, we get

$$\frac{Mdx+Ndy}{Mx+Ny}=\frac{1}{2}\left[d(\ln xy)+\frac{Mx-Ny}{Mx+Ny}d\left(\ln\frac{x}{y}\right)\right].$$

If $Mx + Ny$ be a homogeneous function, then $Mx - Ny$ is also so and hence we assume

$$\frac{Mx-Ny}{Mx+Ny}=f\left(\frac{x}{y}\right)=f\left(e^{\ln\frac{x}{y}}\right)=F\left(\ln\frac{x}{y}\right).$$

So $$\frac{Mdx+Ndy}{Mx+Ny}=\frac{1}{2}\left[d(\ln xy)+F\left(\ln\frac{x}{y}\right)d\left(\ln\frac{x}{y}\right)\right].$$

This shows that $\frac{Mdx+Ndy}{Mx+Ny}$ is integrable, so $\frac{1}{Mx+Ny}$ is an integrating factor of the equation

$$Mdx+Ndy=0.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $x^2dy-(xy+2y^2)dx=0.$

Solution

The given equation is a homogeneous equation and $Mx+Ny=-x^2y-2y^2x+x^2y$ $=-2y^2x\neq 0$, so $-\frac{1}{2y^2x}$ is an integrating factor of the equation. Multiplying the equation by $-\frac{1}{2y^2x}$, we get

$$\frac{x^2dy-(xy+2y^2)dx}{y^2x}=0 \quad \text{or,} \quad \frac{x(xdy-ydx)}{y^2x}-2\frac{dx}{x}=0$$

$$\text{or,} \quad \frac{xdy-ydx}{y^2}-2\frac{dx}{x}=0 \quad \text{or,} \quad d\left(\frac{x}{y}\right)+2d(\ln x)=0.$$

Integrating, we get $\frac{x}{y}+2\ln x=C$, where C is an arbitrary constant.

(ii) Solve: $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0.$

Solution

The given equation is a homogeneous equation and

$$Mx+Ny=x^3y-2x^2y^2-x^3y+3x^2y^2=x^2y^2\neq 0,$$

so $\frac{1}{x^2y^2}$ is an integrating factor of the equation. Multiplying the equation by $\frac{1}{x^2y^2}$, we get

$$\frac{(x^2y-2xy^2)dx-(x^3-3x^2y)dy}{x^2y^2}=0. \quad \text{or,} \quad \left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$$

$$\text{or,} \quad \frac{ydx-xdy}{y^2}-\frac{2}{x}dx+\frac{3}{y}dy=0 \quad \text{or,} \quad d\left(\frac{x}{y}\right)-2d(\ln x)+3d(\ln y)=0.$$

Integrating, we get

$$\frac{x}{y}-2\ln x+3\ln y=C, \text{ where } C \text{ is an arbitrary constant.}$$

(iii) Solve: $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0.$

Solution

The given equation is a homogeneous equation and

$$Mx + Ny = y^3x - 2x^3y - x^3y + 2xy^3 = 3xy(y^2 - x^2) \neq 0,$$

so $\dfrac{1}{xy(y^2 - x^2)}$ is an integrating factor of the equation. Multiplying the equation by $\dfrac{1}{xy(y^2 - x^2)}$, we get

$$\frac{(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy}{xy(y^2 - x^2)} = 0$$

or,
$$\frac{(y^3 - yx^2)dx + (xy^2 - x^3)dx + xy^2dy - yx^2dx}{xy(y^2 - x^2)} = 0$$

or,
$$\frac{dx}{x} + \frac{dy}{y} + \frac{1}{2}\frac{d(y^2 - x^2)}{(y^2 - x^2)} = 0.$$

Integrating; we get $\ln x^2 + \ln y^2 + \ln(y^2 - x^2) = \ln C$

or, $x^2y^2(y^2 - x^2) = C$, where C is an arbitrary constant.

3.2.3: If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of the equation $Mdx + Ndy = 0$.

Proof: If $\mu(x)$ be an integrating factor of the equation $Mdx + Ndy = 0$, then

$$\mu Mdx + \mu Ndy = 0$$

is an exact equation, i.e., we must have

$$\mu\frac{\partial}{\partial y}M = \frac{\partial}{\partial x}(\mu N).$$

This gives
$$N\frac{d\mu}{dx} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

or,
$$\frac{d\mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx = f(x)\,dx.$$

Integrating, we get

$$\ln\mu = \int f(x)\,dx \quad \text{i.e.,} \quad \mu(x) = e^{\int f(x)dx}$$

Alternative Method: Given $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(x)$

i.e., $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} + Nf(x)dx.$

Multiplying both sides by $e^{\int f(x)\,dx}$, we get

$$\frac{\partial M}{\partial y} e^{\int f(x)\,dx} = \frac{\partial N}{\partial x} e^{\int f(x)\,dx} + N f(x) e^{\int f(x)\,dx}\, dx$$

or, $$\frac{\partial}{\partial y}\left(M e^{\int f(x)\,dx} \right) = \frac{\partial}{\partial x}\left(N e^{\int f(x)\,dx} \right).$$

This proves exactness of the equation

$$(Mdx + Ndy)\, e^{\int f(x)\,dx} = 0.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $\left(xy^2 - e^{\frac{1}{x^3}} \right) dx - x^2 y dy = 0.$

Solution

Here $M = xy^2 - e^{\frac{1}{x^3}}$, $N = -x^2 y$ and $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{2xy + 2xy}{-x^2 y} = -\frac{4}{x}$. So, $e^{-\int \frac{4}{x} dx} = e^{-4 \ln x} = \frac{1}{x^4}$ is an integrating factor of the equation. Multiplying the equation by $\frac{1}{x^4}$, we get

$$\frac{\left(xy^2 - e^{1/x^3} \right) dx - x^2 y dy}{x^4} = 0$$

or, $$\frac{xy(ydx - xdy)}{x^4} - \frac{1}{x^4} e^{1/x^3} dx = 0$$

or, $$-\frac{y}{x} d\left(\frac{y}{x} \right) + \frac{1}{3} e^{\frac{1}{x^3}} d\left(\frac{1}{x^3} \right) = 0.$$

Integrating, we get $-\frac{y^2}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = C$, where C is an integrating factor.

(ii) Solve: $xdy - \left(4y + x^6 e^x \right) dx = 0.$

Solution

Here $M = -x^6 e^x - 4y$, $N = x$ and $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{-4-1}{x} = -\frac{5}{x}$. So, $e^{-\int \frac{5}{x} dx} = e^{-5 \ln x} = \frac{1}{x^5}$ is an integrating factor of the equation. Multiplying the equation by $\frac{1}{x^5}$, we get

$$\frac{xdy - (4y + x^6 e^x) dx}{x^5} = 0 \quad \text{or,} \quad \frac{xdy - 4ydx}{x^5} - xe^x dx = 0$$

or, $$\frac{x^4 dy - 4x^3 y dx}{x^8} - xe^x dx = 0 \quad \text{or,} \quad d\left(\frac{y}{x^4} \right) - xe^x dx = 0.$$

Integrating, we get $yx^{-4} - (x-1)e^x = C$, where C is an arbitrary constant.

(iii) Solve: $(x^3 + xy^4)dx + 2y^3dy = 0.$

Solution

Here $M = x^3 + xy^4$, $N = 2y^3$ and $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4xy^3 - 0}{2y^3} = 2x.$

So, $e^{\int 2x dx} = e^{x^2}$ is an integrating factor of the equation. Multiplying the equation by e^{x^2} we get

$$(x^3 + xy^4)e^{x^2}dx + 2y^3e^{x^2}dy = 0 \quad \text{or,} \quad x^2e^{x^2}d(x^2) + d\left(y^4e^{x^2}\right) = 0.$$

Integrating, we get

$$(x^2 - 1)e^{x^2} + y^4e^{x^2} = C \text{ or, } y^4 + x^2 = 1 + Ce^{-x^2},$$ where C is an arbitrary constant.

3.2.4: If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of y alone, say $f(y)$, then $e^{-\int f(y)dy}$ is an integrating factor of the equation $Mdx + Ndy = 0$.

Proof: If $\mu(y)$ be an integrating factor of the equation $Mdx + Ndy = 0$, then

$$\mu Mdx + \mu Ndy = 0$$

is an exact equation, i.e., we must have

$$\frac{\partial}{\partial y}(\mu M) = \mu\frac{\partial N}{\partial x}.$$

This gives
$$M\frac{d\mu}{dy} = -\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

or,
$$\frac{d\mu}{\mu} = -\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dy = -f(y)dy$$

Integrating, we get

$$\ln\mu = -\int f(y)\,dy \text{ i.e., } \mu(y) = e^{-\int f(y)\,dy}.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $xydx + (2x^2 + 3y^2 - 20)\,dy = 0.$

Solution

Here $M = xy$, $N = 2x^2 + 3y^2 - 20$, so $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(x - 4x) = -\frac{3}{y}.$

Hence, $e^{\int\frac{3}{y}dy} = e^{3\ln y} = y^3$ is an integrating factor of the equation. Multiplying the equation by y^3, we get

$$xy^4dx + (2x^2 + 3y^2 - 20)y^3dy = 0 \text{ or, } \frac{1}{2}d(x^2y^4) + 3y^5dy - 20y^3dy = 0.$$

Integrating, we get

$$x^2y^4 + y^6 - 10y^4 = C,$$ where C is an arbitrary constant.

(ii) Solve: $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0.$

Solution

Here, $M=xy^3+y$, $N=2x^2y^2+2x+2y^4$, so

$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{xy^3+y}(3xy^2+1-4xy^2-2)=-\frac{1}{y}$. Hence, $e^{\int\frac{dy}{y}}=e^{\ln y}=y$ is an integrating factor of the equation. Multiplying the equation by y, we get

$$(xy^3+y)ydx+2(x^2y^2+x+y^4)ydy=0$$

or, $$xy^4dx+2x^2y^3dy+y^2dx+2xydy+y^5dy=0$$

or, $$\frac{1}{2}d(x^2y^4)+d(xy^2)+y^5dy=0.$$

Integrating, we get

$$3x^2y^4+6xy^2+y^6=C,\text{ where } C \text{ is an arbitrary constant.}$$

(iii) Solve: $(2xy^4e^y+2xy^3+y)dx+(x^2y^4e^y-x^2y^2-3x)dy=0.$

Solution

Here, $M=2xy^4e^y+2xy^3+y$, $N=x^2y^4e^y-x^2y^2-3x.$

So, $$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

$$=\frac{1}{2xy^4e^y+2xy^3+y}(8xy^3e^y+2xy^4e^y+6xy^2+1-2xy^4e^y+2xy^2+3)$$

$$=\frac{8xy^3e^y+8xy^2+4}{2xy^4e^y+2xy^3+y}=\frac{4}{y}.$$

Hence, $e^{-\int\frac{4}{y}dy}=e^{-4\ln y}=\frac{1}{y^4}$ is an integrating factor of the equation. Multiplying the equation by $\frac{1}{y^4}$, we get

$$\frac{1}{y^4}(2xy^4e^y+2xy^3+y)dx+\frac{1}{y^4}(x^2y^4e^y-x^2y^2-3x)dy=0$$

or, $$d(x^2e^y)+\frac{2xydx-x^2dy}{y^2}+\frac{y^3dx-3xy^2dy}{y^6}=0$$

or, $$d\left(x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}\right)=0.$$

Integrating, we get

$$x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}=C,\text{ where } C \text{ is an integrating constant.}$$

3.2.5: If $Mx - Ny \neq 0$, then $\dfrac{1}{Mx - Ny}$ is an integrating factor of the equation $Mdx +$ $Ndy = 0$, where $M = yf(xy)$, $N = xg(xy)$.

Proof: If $\dfrac{1}{Mx - Ny}$ is an integrating factor of the equation $Mdx + Ndy = 0$, then

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{yf(xy)\,dx + xg(xy)\,dy}{xy(f(xy) - g(xy))} = \frac{yfdx + xgdy}{xy(f-g)} = \frac{f}{x(f-g)}dx + \frac{g}{y(f-g)}dy$$

will be exact.

Now $$\frac{\partial}{\partial y}\left(\frac{f}{x(f-g)}\right) = \frac{1}{x}\,\frac{(f-g)\dfrac{\partial f}{\partial y} - f\dfrac{\partial}{\partial y}(f-g)}{(f-g)^2} = \frac{1}{x}\,\frac{f\dfrac{\partial g}{\partial y} - g\dfrac{\partial f}{\partial y}}{(f-g)^2}$$ and

$$\frac{\partial}{\partial x}\left(\frac{g}{y(f-g)}\right) = \frac{1}{y}\,\frac{(f-g)\dfrac{\partial g}{\partial x} - g\dfrac{\partial}{\partial x}(f-g)}{(f-g)^2} = \frac{1}{y}\,\frac{f\dfrac{\partial g}{\partial x} - g\dfrac{\partial f}{\partial x}}{(f-g)^2}.$$

So, $$\frac{\partial}{\partial y}\left(\frac{f}{x(f-g)}\right) - \frac{\partial}{\partial x}\left(\frac{g}{y(f-g)}\right)$$

$$= \frac{1}{xy}\,\frac{y\left(f\dfrac{\partial g}{\partial y} - g\dfrac{\partial f}{\partial y}\right) - x\left(f\dfrac{\partial g}{\partial x} - g\dfrac{\partial f}{\partial x}\right)}{(f-g)^2}$$

$$= \frac{1}{xy}\,\frac{g\left(x\dfrac{\partial f}{\partial x} - y\dfrac{\partial f}{\partial y}\right) + f\left(y\dfrac{\partial g}{\partial y} - x\dfrac{\partial g}{\partial x}\right)}{(f-g)^2}$$

$= 0$, since $x\dfrac{\partial f}{\partial x} = xy\,f' = yx\,f' = y\dfrac{\partial f}{\partial y}$ and $y\dfrac{\partial g}{\partial y} = x\dfrac{\partial g}{\partial x}$.

Hence, the equation $\dfrac{f}{x(f-g)}dx + \dfrac{g}{y(f-g)}dy = 0$, i.e., $\dfrac{1}{Mx - Ny}(Mdx + Ndy) = 0$ is exact. In orther words, $\dfrac{1}{Mx - Ny}$ is an integrating factor of the equation $Mdx + Ndy = 0$.

Alternative Method: We rewrite $Mdx + Ndy$ as

$$Mdx + Ndy = \frac{1}{2}\left[(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right].$$

Dividing by $Mx - Ny$, as $Mx - Ny \neq 0$, we get

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left[\frac{Mx + Ny}{Mx - Ny}\,d(\ln xy) + d\left(\ln\frac{x}{y}\right)\right].$$

$$= \frac{1}{2}\left[\frac{f + g}{f - g}\,d(\ln xy) + d\left(\ln\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[F(xy)d(\ln xy) + d\left(\ln\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[F(e^{\ln xy})\,d(\ln xy) + d\left(\ln\frac{x}{y}\right)\right].$$

This shows that $\dfrac{Mdx + Ndy}{Mx - Ny}$ is integrable, so $\dfrac{1}{Mx - Ny}$ is an integrating factor of the equation

$$Mdx + Ndy = 0.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0.$

Solution

The given equation is of the form $Mdx + Ndy = 0$, where $M = yf(xy)$ and $N = xg(xy)$. Now $Mx - Ny = 3x^3y^3 \neq 0$, so $\dfrac{1}{3x^3y^3}$ is an integrating factor of it. Multiplying the equation by $\dfrac{1}{3x^3y^3}$, we get

$$\frac{y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy}{3x^3y^3} = 0$$

or,

$$\frac{ydx + xdy + xy(2ydx - xdy)}{x^2y^2} = 0$$

or,

$$\frac{d(xy)}{(xy)^2} + 2\frac{dx}{x} - \frac{dy}{y} = 0.$$

Integrating, we get

$$-\frac{1}{xy} + 2\ln x - \ln y = C, \text{ where } C \text{ is an arbitrary constant.}$$

(ii) Solve: $(x^2y^2 + xy + 1)\,ydx + (x^2y^2 - xy + 1)\,xdy = 0.$

Solution

The given equation is of the form $Mdx + Ndy = 0$, where $M = yf(xy)$ and $N = xg(xy)$. Now $Mx - Ny = 2x^2y^2 \neq 0$, so $\dfrac{1}{2x^2y^2}$ is an integrating factor of it. Multiplying the equation by $\dfrac{1}{2x^2y^2}$, we get

$$\frac{(x^2y^2 + xy + 1)\,ydx + (x^2y^2 - xy + 1)\,xdy}{x^2y^2} = 0$$

or, $$\left(1+\frac{1}{x^2y^2}\right)(ydx+xdy)+\frac{dx}{x}-\frac{dy}{y}=0$$

or, $$\left(1+\frac{1}{(xy)^2}\right)d(xy)+\frac{dx}{x}-\frac{dy}{y}=0.$$

Integrating, we get
$$xy-\frac{1}{xy}+\ln\left(\frac{x}{y}\right)=C, \text{ where } C \text{ is an arbitrary constant.}$$

(iii) Solve: $y(1+2xy)\,dx+x(1+2xy-x^3y^3)\,dy=0.$

Solution

The given equation is of the form $Mdx + Ndy = 0$, where $M = yf(xy)$ and $N = xg(xy)$. Now $Mx - Ny = x^4y^4 \neq 0$, so $\frac{1}{x^4y^4}$ is an integrating factor of it. Multiplying the equation by $\frac{1}{x^4y^4}$, we get
$$\frac{y(1+2xy)\,dx+x(1+2xy-x^3y^3)\,dy}{x^4y^4}=0$$

or, $$\left(\frac{1}{(xy)^4}+\frac{2}{(xy)^3}\right)d(xy)-\frac{dy}{y}=0.$$

Integrating, we get
$$\frac{1}{3x^3y^3}+\frac{1}{x^2y^2}+\ln y=C, \text{ where } C \text{ is an arbitrary constant.}$$

3.2.6: If $\frac{1}{M-N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of the form $f(x+y)$, then the equation $Mdx + Ndy = 0$ has an integrating factor of the form $e^{-\int f(x+y)\,d(x+y)}$.

Proof: If $\mu(x+y)$ be an integrating factor of the differential equation $Mdx + Ndy = 0$, then
$$\mu Mdx + \mu Ndy = 0$$
is an exact equation, i.e., we must have
$$\frac{\partial}{\partial x}(\mu N)=\frac{\partial}{\partial y}(\mu M)$$

or, $$N\frac{\partial \mu}{\partial x}+\mu\frac{\partial N}{\partial x}=M\frac{\partial \mu}{\partial y}+\mu\frac{\partial M}{\partial y}$$

or, $$\mu\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=(M-N)\,\mu'$$

or, $$\frac{d\mu}{\mu}=\frac{1}{M-N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)d(x+y)=-f(x+y)\,d(x+y)$$

or, $$\mu(x+y)=e^{-\int f(x+y)\,d(x+y)}.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $(y-1)dx-(x+1)dy=0.$

Solution

Here, $M=y-1$, $N=-x-1$, so $\frac{1}{M-N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-2}{x+y}.$

Hence, $e^{-\int\frac{2}{x+y}d(x+y)}=\frac{1}{(x+y)^2}$ is an integrating factor. Multiplying the equation by $\frac{1}{(x+y)^2}$, we get

$$\frac{(y-1)dx-(x+1)dy}{(x+y)^2}=0 \text{ or, } \frac{ydx-xdy-dx-dy}{(x+y)^2}=0$$

$$\text{or, } -\frac{xdy-ydx}{x^2}\frac{1}{\left(1+\frac{y}{x}\right)^2}-\frac{d(x+y)}{(x+y)^2}=0 \text{ or, } \frac{d\left(\frac{y}{x}\right)}{\left(1+\frac{y}{x}\right)^2}+\frac{d(x+y)}{(x+y)^2}=0.$$

Integrating, we get

$$-\frac{1}{1+\frac{y}{x}}-\frac{1}{x+y}=-C \text{ or, } x+1=C(x+y), \text{ where } C \text{ is an arbitrary constant.}$$

(ii) Solve: $(x^2+2)dy+(y^2+2)dx=0.$

Solution

Here $M=y^2+2$, $N=x^2+2$, so $\frac{1}{M-N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-2(y-x)}{(y-x)(y+x)}=\frac{-2}{x+y}.$

Hence, $e^{-\int\frac{2}{x+y}d(x+y)}=\frac{1}{(x+y)^2}$ is an integrating factor. Multiplying the equation by $\frac{1}{(x+y)^2}$, we get

$$\frac{(x^2+2)dy+(y^2+2)dx}{(x+y)^2}=0$$

$$\text{or, } \frac{x^2dy+y^2dx}{(x+y)^2}+2\frac{d(x+y)}{(x+y)^2}=0$$

$$\text{or, } \frac{x(x+y)dy+y(x+y)dx-xyd(x+y)}{(x+y)^2}+2\frac{d(x+y)}{(x+y)^2}=0$$

$$\text{or, } \frac{(x+y)(xdy+ydx)-xyd(x+y)}{(x+y)^2}+2\frac{d(x+y)}{(x+y)^2}=0$$

or, $$\frac{(x+y)d(xy)-xyd(x+y)}{(x+y)^2}+2\frac{d(x+y)}{(x+y)^2}=0$$

or, $$d\left(\frac{xy}{x+y}\right)-2d\left(\frac{1}{x+y}\right)=0.$$

Integrating, we get

$$\frac{xy-2}{x+y}=C,\text{ where } C \text{ is an integrating constant.}$$

(iii) Solve: $(2xy-x^2-x)dy+(2xy-y^2-y)dx=0.$

Solution

Here, $M=2xy-y^2-y$, $N=2xy-x^2-x$, so $\frac{1}{M-N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-4}{x+y+1}$.

Hence $e^{-\int\frac{4}{x+y+1}d(x+y)}=\frac{1}{(x+y+1)^4}$ is an integrating factor. Multiplying the equation by $\frac{1}{(x+y+1)^4}$, we get

$$\frac{(2xy-x^2-x)dy+(2xy-y^2-y)dx}{(x+y+1)^4}=0$$

or, $$\frac{3xy(dx+dy)-(x+y+1)xdy-(x+y+1)ydx}{(x+y+1)^4}=0$$

or, $$\frac{3xyd(x+y+1)-(x+y+1)(xdy+ydx)}{(x+y+1)^4}=0$$

or, $$\frac{3xy(x+y+1)^2d(x+y+1)-(x+y+1)^3d(xy)}{(x+y+1)^6}=0$$

or, $$d\left(\frac{xy}{(x+y+1)^3}\right)=0.$$

Integrating, we get

$$\frac{xy}{(x+y+1)^3}=C,\text{ where } C \text{ is an integrating constant.}$$

3.2.7: If $\frac{1}{xM-yN}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of the form $f(xy)$, then the equation $Mdx+Ndy=0$ has an integrating factor of the form $e^{-\int f(xy)d(xy)}$.

Proof: If $\mu(xy)$ be an integrating factor of the differential eqnation $Mdx+Ndy=0$, then

$$\mu Mdx+\mu Ndy=0$$

is an exact equation, i.e., we must have

$$\frac{\partial}{\partial x}(\mu N)=\frac{\partial}{\partial y}(\mu M) \text{ or, } N\frac{\partial \mu}{\partial x}+\mu\frac{\partial N}{\partial x}=M\frac{\partial \mu}{\partial y}+\mu\frac{\partial M}{\partial y}$$

$$\text{or, } Ny\mu'+\mu\frac{\partial N}{\partial x}=Mx\mu'+\mu\frac{\partial M}{\partial y} \text{ or, } \mu\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=(xM-yN)\mu'$$

$$\text{or, } \frac{d\mu}{\mu}=\frac{1}{xM-yN}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)d(x+y)=-f(xy)d(xy) \text{ or, } \mu(xy)=e^{-\int f(xy)d(xy)}.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $y(1+xy)dx+x(1-xy)dy=0$.

Solution

Here $M=y(1+xy)$, $N=x(1-xy)$, so $\dfrac{1}{yN-xM}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=-\dfrac{2}{xy}$.

Hence, $e^{-\int\frac{2}{xy}d(xy)}=\dfrac{1}{x^2y^2}$ is an integrating factor of the equation. Multiplying the equation by $\dfrac{1}{x^2y^2}$, we get

$$\frac{y(1+xy)dx+x(1-xy)dy}{x^2y^2}=0$$

$$\text{or, } \frac{ydx+xdy}{x^2y^2}+\frac{dx}{x}-\frac{dy}{y}=0$$

$$\text{or, } \frac{d(xy)}{(xy)^2}+\frac{dx}{x}-\frac{dy}{y}=0.$$

Integrating we get

$$\ln\frac{x}{y}-\frac{1}{xy}=C, \text{ where } C \text{ is an arbitrary constant.}$$

(ii) Solve: $(x^4y^2-y)dx+(x^2y^4-x)dy=0$.

Solution

Here $M=x^4y^2-y$, $N=x^2y^4-x$, so $\dfrac{1}{yN-xM}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=-\dfrac{2}{xy}$.

Hence, $e^{-\int\frac{2}{xy}d(xy)}=\dfrac{1}{x^2y^2}$ is an integrating factor of the equation. Multiplying the equation by $\dfrac{1}{x^2y^2}$, we get

$$\frac{(x^4y^2-y)dx+(x^2y^4-x)dy}{x^2y^2}=0$$

$$\text{or,}\quad x^2dx + y^2dy - \frac{xdy + ydx}{x^2y^2} = 0$$

$$\text{or,}\quad d(x^3) + d(y^3) - 3\frac{d(xy)}{(xy)^2} = 0.$$

Integrating, we get

$$x^3 + y^3 + \frac{3}{xy} = C, \text{ where } C \text{ is an arbitrary constant.}$$

(iii) Solve: $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0.$

Solution

Here $M = xy^2 + 2x^2y^3$, $N = x^2y - x^3y^2$, so $\frac{1}{yN - xM}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{3}{xy}$.

Hence, $e^{-\int \frac{3}{xy} d(xy)} = \frac{1}{x^3y^3}$ is an integrating factor of the equation. Multiplying the equation by $\frac{1}{x^3y^3}$, we get

$$\frac{(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy}{x^3y^3} = 0$$

$$\text{or,}\quad \frac{ydx + xdy}{x^2y^2} + 2\frac{dx}{x} - \frac{dy}{y} = 0$$

$$\text{or,}\quad \frac{d(xy)}{(xy)^2} + 2d(\ln x) - d(\ln y) = 0.$$

Integrating, we get

$$\ln\frac{x^2}{y} - \frac{1}{xy} = C, \text{ where } C \text{ is an arbitrary constant.}$$

3.2.8: If $\frac{y^2}{xM + yN}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of the form $f\left(\frac{x}{y}\right)$, then the equation $Mdx + Ndy = 0$ has an integrating factor of the form $e^{\int f\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right)}$.

Proof: If $\mu\left(\frac{x}{y}\right)$ be an integrating factor of the differential equation $Mdx + Ndy = 0$, then

$$\mu Mdx + \mu Ndy = 0$$

is an exact equation, i.e., we must have

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$

$$\text{or,}\quad N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x} = M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y}$$

$$\text{or,}\quad \frac{N\mu'}{y} + \mu\frac{\partial N}{\partial x} = -\frac{M\mu' x}{y^2} + \mu\frac{\partial M}{\partial y}$$

or, $\mu\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=-\frac{(xM+yN)\mu'}{y^2}$

or, $\frac{d\mu}{\mu}=-\frac{y^2}{xM+yN}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)d\left(\frac{x}{y}\right)=f\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)$

or, $\mu\left(\frac{x}{y}\right)=e^{\int f\left(\frac{x}{y}\right)d\left(\frac{x}{y}\right)}$.

ILLUSTRATIVE EXAMPLES

(i) Solve: $\left(\frac{2x^2}{y}+\frac{x}{y}\right)dx+2xdy=0.$

Solution

Here $M=\frac{2x^2}{y}+\frac{x}{y}$, $N=2x$.

So $\frac{y^2}{xM+yN}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{y^3}{x^2(2x+1)+2xy^2}\left(-\frac{x(2x+1)}{y^2}-2\right)=-\frac{y}{x}$.

So, $e^{-\int\frac{y}{x}d\left(\frac{x}{y}\right)}=e^{-\ln\frac{x}{y}}=\frac{y}{x}$ is an integrating factor of the equation. Multiplying the equation by $\frac{y}{x}$, we get

$$\left(\frac{2x^2}{y}+\frac{x}{y}\right)\frac{y}{x}dx+2x\frac{y}{x}dy=0$$

or, $(2x+1)\,dx+2ydy=0.$

Integrating we get

$$x^2+y^2+x=C, \text{ where } C \text{ is an arbitrary constant.}$$

(ii) Solve: $\left(2y+\frac{y}{x}\right)dx+\frac{2y^2}{x}dy=0.$

Solution

Here $M=2y+\frac{y}{x}$, $N=\frac{2y^2}{x}$.

So $\frac{y^2}{xM+yN}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{xy^2}{(2xy+y)x+2y^3}\left(2+\frac{1}{x}+\frac{2y^2}{x^2}\right)=\frac{y}{x}$.

So, $e^{\int\frac{y}{x}d\left(\frac{x}{y}\right)}=e^{\ln\frac{x}{y}}=\frac{x}{y}$ is an integrating factor of the equation. Multiplying the equation by $\frac{x}{y}$, we get

$$\left(2y+\frac{y}{x}\right)\frac{x}{y}dx+\frac{2y^2}{x}\frac{x}{y}dy=0 \text{ or, } (2x+1)\,dx+2ydy=0.$$

Integrating we get

$$x^2 + y^2 + x = C, \text{ where } C \text{ is an arbitrary constant.}$$

3.2.9: If $\dfrac{x^2}{xM + yN}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of the form $f\left(\dfrac{y}{x}\right)$, then the equation $Mdx + Ndy = 0$ has an integrating factor of the form $e^{-\int f\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)}$.

Proof: If $\mu\left(\dfrac{y}{x}\right)$ be an integrating factor of the differential equation $Mdx + Ndy = 0$, then

$$\mu M dx + \mu N dy = 0$$

is an exact equation, i.e., we must have

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$

$$\text{or,} \quad N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x} = M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y}$$

$$\text{or,} \quad \frac{-N\mu' y}{x^2} + \mu\frac{\partial N}{\partial x} = \frac{M\mu'}{x} + \mu\frac{\partial M}{\partial y}$$

$$\text{or,} \quad \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{(xM + yN)\,\mu'}{x^2}$$

$$\text{or,} \quad \frac{d\mu}{\mu} = \frac{x^2}{xM + yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d\left(\frac{y}{x}\right) = -f\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)$$

$$\text{or,} \quad \mu\left(\frac{y}{x}\right) = e^{-\int f\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)}.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $\left(\dfrac{2x^2}{y} + \dfrac{x}{y}\right)dx + 2xdy = 0.$

Solution

Here $M = \dfrac{2x^2}{y} + \dfrac{x}{y}$, $N = 2x$.

So $\dfrac{x^2}{xM + yN}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{yx^2}{x^2(2x+1) + 2xy^2}\left(-\dfrac{x(2x+1)}{y^2} - 2\right) = -\dfrac{x}{y}.$

So, $e^{\int \frac{x}{y} d\left(\frac{y}{x}\right)} = e^{\ln\frac{y}{x}} = \dfrac{y}{x}$ is an integrating factor of the equation. Multiplying the equation by $\dfrac{y}{x}$, we get

$$\left(\frac{2x^2}{y} + \frac{x}{y}\right)\frac{y}{x}dx + 2x\frac{y}{x}dy = 0 \text{ or, } (2x+1)\,dx + 2ydy = 0.$$

Integrating, we get

$$x^2 + y^2 + x = C, \text{ where } C \text{ is an arbitrary constant.}$$

(ii) Solve: $\left(2y+\frac{y}{x}\right)dx+\frac{2y^2}{x}dy=0.$

Solution

Here $M=2y+\frac{y}{x},\ N=\frac{2y^2}{x}.$

So $\frac{x^2}{xM+yN}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{x^3}{(2xy+y)x+2y^3}\left(2+\frac{1}{x}+\frac{2y^2}{x^2}\right)=\frac{x}{y}.$

So, $e^{-\int\frac{x}{y}d\left(\frac{y}{x}\right)}=e^{-\ln\frac{y}{x}}=\frac{x}{y}$ is an integrating factor of the equation. Multiplying the equation by $\frac{y}{x}$, we get

$$\left(2y+\frac{y}{x}\right)\frac{x}{y}dx+\frac{2y^2}{x}\frac{x}{y}dy=0 \text{ or, } (2x+1)dx+2ydy=0.$$

Integrating we get

$$x^2+y^2+x=C, \text{ where } C \text{ is an arbitrary constant.}$$

3.2.10: For the equation $x^a y^b(mydx+nxdy)=0$, $x^{m-1-a}y^{n-1-b}\varphi(x^m y^n)$, is an integrating factor where $\varphi(x^m y^n)$ is an arbitrary function.

Proof: Multiplying the equation $x^a y^b(mydx+nxdy)=0$ by, $x^{m-1-a}y^{n-1-b}\varphi(x^m y^n)$, we get

$$x^{m-1}y^{n-1}\varphi(x^m y^n)(mydx+nxdy)=0$$
$$\text{or, } \varphi(x^m y^n)\left[y^n d(x^m)+x^m d(y^n)\right]=0$$
$$\text{or, } \varphi(x^m y^n)d(x^m y^n)=0$$
$$\text{or, } d\left[f(x^m y^n)\right]=0, \text{ where } f'(x^m y^n)=\varphi(x^m y^n).$$

This proves the exactness of the equation

$$x^{m-1}y^{n-1}\varphi(x^m y^n)(mydx+nxdy)=0.$$

Putting $a=0=b$ in the above results we see that $x^{m-1}y^{n-1}\varphi(x^m y^n)$ is an integrating factor of the equation

$$mydx+nxdy=0.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $x^2y^3(5ydx+3xdy)=0.$

Solution

Here $a=2, b=3, m=5, n=3$, so $x^{m-1-a}y^{n-1-b}\varphi(x^m y^n)$, i.e., $x^2y^{-1}\varphi(x^5y^3)$ is an integrating factor of the equation. Multiplying the equation by $x^2y^{-1}\varphi(x^5y^3)$, we get

$$x^4y^2(5ydx+3xdy)\varphi(x^5y^3)=0 \text{ or, } \varphi(x^5y^3)d(x^5y^3)=0.$$

Integrating, we get

$$\int \varphi(x^5y^3)\,d(x^5y^3) = C, \text{ where } \varphi(x^5y^3) \text{ is an arbitrary function}$$

of x^5y^3 and C is an arbitrary constant.

(ii) Solve: $x^2y^3(5ydx + 3xdy) = 0$.

Solution

Here $a = 2, b = 3, m = 5, n = 3$, so $x^{km-1-a}y^{kn-1-b}$, i.e., $x^{5k-3}y^{3k-4}$, where k has any value, is an integrating factor of the equation. Multiplying the equation by $x^{5k-3}y^{3k-4}$, we get

$$x^{5k-1}y^{3k-1}(5ydx + 3xdy) = 0 \text{ or, } d(x^{5k}y^{3k}) = 0.$$

Integrating, we get

$$x^{5k}y^{3k} = C, \text{ where } C \text{ is an arbitrary constant.}$$

3.2.11: For the equation $x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0$, where $mq \neq np$, x^hy^k will be an integrating factor, if

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \text{ and } \frac{c+h+1}{p} = \frac{d+k+1}{q}.$$

Proof: If x^hy^k is an integrating factor of the equation

$$x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0,$$

then

$$x^{h+a}y^{k+b}(mydx + nxdy) + x^{h+c}y^{k+d}(pydx + qxdy)$$

would be an exact differential. In other words, we must have

$$\frac{\partial}{\partial y}\left(mx^{h+a}y^{k+b+1} + px^{h+c}y^{k+d+1}\right) = \frac{\partial}{\partial x}\left(nx^{h+a+1}y^{k+b} + qx^{h+c+1}y^{k+d}\right)$$

or,

$$m(k+b+1)\,x^{h+a}y^{k+b} + p(k+d+1)\,x^{h+c}y^{k+d}$$
$$= n(h+a+1)\,x^{h+a}y^{k+b} + q(h+c+1)\,x^{h+c}y^{k+d}$$

Comparing both sides we get

$$m(k+b+1) = n(h+a+1) \text{ and } p(k+d+1) = q(h+c+1)$$

or,

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \text{ and } \frac{c+h+1}{p} = \frac{d+k+1}{q}.$$

These equations will give the values of h, k, i.e., the integrating factor x^hy^k. Now, the above equations in h, k are solvable, if the coefficient determinant does not vanish, i.e., if

$$\begin{vmatrix} n & -m \\ q & -p \end{vmatrix} \neq 0, \text{ i.e., if } mq \neq np.$$

ILLUSTRATIVE EXAMPLES

(i) Solve: $2ydx + 3xdy + 2xy(3ydx + 4xdy) = 0.$

Solution

Here $a = 0, b = 0, c = 1, d = 1, m = 2, n = 3, p = 6, q = 8$ and $mq - np = 16 - 18 \neq 0.$
Hence, $x^h y^k$ will be an integrating factor where h, k satisfy

$$\frac{h+1}{2} = \frac{k+1}{3} \text{ and } \frac{h+2}{6} = \frac{k+2}{8},$$

$$\text{or, } \quad 3h - 2k + 1 = 0 \text{ and } 4h - 3k + 2 = 0.$$

Solving, we get

$$\frac{h}{-4+3} = \frac{k}{4-6} = \frac{1}{-9+8}, \text{ i.e., } h = 1, k = 2.$$

Multiplying the equation by xy^2, we get

$$xy^2(2ydx + 3xdy) + 2x^2y^3(3ydx + 4xdy) = 0$$

$$\text{or, } \quad d(x^2y^3) + 2d(x^3y^4) = 0.$$

Integrating, we get

$$x^2y^3 + 2x^3y^4 = C, \text{ where } C \text{ is an arbitrary constant.}$$

(ii) Solve: $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0.$

Solution

Rewriting the equation as $y(ydx - xdy) + 2x^2(ydx + xdy) = 0$ we see that, here

$$a = 0,\ b = 1,\ c = 2,\ d = 0,\ m = 1,\ n = -1,\ p = 1,\ q = 1$$

and $mq - np = 1 + 1 \neq 0$. Hence $x^h y^k$ will be an integrating factor where h, k satisfy

$$\frac{h+1}{1} = \frac{k+2}{-1} \text{ and } \frac{h+3}{1} = \frac{k+1}{1},$$

$$\text{or, } \quad h + k + 3 = 0 \text{ and } h - k + 2 = 0.$$

Solving, we get

$$h = -\frac{5}{2}, k = -\frac{1}{2}.$$

Multiplying the equation by $x^{-\frac{5}{2}}y^{-\frac{1}{2}}$, we get

$$x^{-\frac{5}{2}}y^{\frac{1}{2}}(ydx - xdy) + 2x^{-\frac{1}{2}}y^{-\frac{1}{2}}(ydx + xdy) = 0$$

$$\text{or, } \quad x^{-\frac{5}{2}}y^{\frac{3}{2}}dx - x^{-\frac{3}{2}}y^{\frac{1}{2}}dy + 2x^{-\frac{1}{2}}y^{\frac{1}{2}}dx + 2x^{\frac{1}{2}}y^{-\frac{1}{2}}dy = 0$$

$$\text{or, } \quad 4d\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) - \frac{2}{3}d\left(x^{-\frac{3}{2}}y^{\frac{3}{2}}\right) = 0.$$

Integrating, we get

$$6x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{-\frac{3}{2}}y^{\frac{3}{2}} = C, \text{ where } C \text{ is an arbitrary constant.}$$

EXERCISE

1. Solve the following equations:

(i) $xdy - ydx = (x^2 + y^2)dx$,　　(ii) $xdy - ydx = \sqrt{x^2 + y^2}\,dx$,

(iii) $xdy - ydx = x\sqrt{x^2 + y^2}\,dx$,　　(iv) $xdx + ydy = \sqrt{x^2 + y^2}\,dx$,

(v) $xdy - ydx = \cos\frac{1}{x}dx$,

[*Hints*: Multiply both sides by $\frac{1}{x^2}$ and put $z = \frac{1}{x}$.]

(vi) $xdy + ydx = y^2 \ln x\, dx$,

(vii) $(x + y)(dx - dy) = dx + dy$,

[*Hints*: Rewrite the equation as $dx - dy = \frac{d(x+y)}{x+y}$]

(viii) $(xdx + ydy)(x^2 + y^2) + xdy - ydx = 0$,

(ix) $xdy - ydx = 2\sqrt{x^2 - y^2}\,dx$,

(x) $\frac{x + yy'}{xy' - y} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$, where $y' \frac{dy}{dx}$,

[*Hints*: Rewrite the equation as $\frac{d\left(\sqrt{x^2 + y^2}\right)}{\sqrt{1 - x^2 - y^2}} = \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}$.]

(xi) $xdx + ydy + a^2 \frac{ydx - xdy}{x^2 + y^2} = 0$,

(xii) $xdx + ydy = m(xdy - ydx)$,

[*Hints*: Rewrite the equation as $\frac{d(x^2 + y^2)}{x^2 + y^2} = 2m\frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}$.]

(xiii) $x\cos\frac{y}{x}(ydx + xdy) = y\sin\frac{y}{x}(xdy - ydx)$,

(xiv) $x^2dy - (xy + 2y^2)dx = 0$,　　(xv) $2xydy + (y^2 - 3x^2)dx = 0$,

(xvi) $(x^2 + y^2)dy - 2xydx = 0$,　　(xvii) $(x^3 + y^3)dx - xy^2dy = 0$,

(xviii) $x^2ydx - (x^3 + y^3)dy = 0$,

(xix) $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$,

(xx) $y(xy + 1)dx + x(1 - xy)dy = 0$,

(xxi) $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0$,

(xxii) $3x^2ydx + (x^3 + y^3)dy = 0$,

(xxiii) $(1 + 3x^2 + 6xy^2)dx + (1 + 3y^2 + 6yx^2)dy = 0$,

(xxiv) $(x^3y^3 + x^2y^2 + xy + 1)\,ydx + (x^3y^3 - x^2y^2 - xy + 1)\,xdy = 0,$

(xxv) $(x^2 + y^2 + 1)dy - 2xydx = 0,$

(xxvi) $(x^2 + y^2 + 1)dx + x(x - 2y)dy = 0,$

(xxvii) $(x^2 + y^2 + x)dx + xydy = 0,$

(xxviii) $(x^3 - 2y^3)dx + 2xy^2dy = 0,$

(xxix) $(x^3 + xy^4)dx + 2y^3dy = 0,$

(xxx) $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0,$

(xxxi) $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0,$

(xxxii) $(3x + 2y^2)\,ydx + 2x(2x + 3y^2)dy = 0,$

(xxxiii) $(3x^2y^3 + 2x)\,ydx + (2xy^3 - 1)\,x^2dy = 0,$

(xxxiv) $(2x^2 - 3y^3)\,ydx + (3x^2 + 2y^3)\,xdy = 0,$

(xxxv) $(y + 2x^2)\,ydx + (2x^2 - y)\,xdy = 0,$

(xxxvi) $x(3ydx + 2xdy) + 8y^4(ydx + 3xdy) = 0,$

(xxxvii) $x^2y(2ydx - xdy) + (ydx + 3xdy) = 0,$

(xxxviii) $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0,$

(xxxix) $x(4ydx + 2xdy) + y^3(3ydx + 5xdy) = 0,$

(xl) $3ydx + 5xdy + xy^3(ydx + 2xdy) = 0,$

(xli) $\left(\dfrac{3}{x} + 2y^4\right)dx + \left(\dfrac{3}{y} - xy^3\right)dy = 0,$

(xlii) $x^2(xdx + ydy) + 2y(xdy - ydx) = 0,$

[*Hints*: Rewrite the equation as $\dfrac{d(x^2 + y^2)}{\sqrt{x^2 + y^2}} + 4\dfrac{\dfrac{y}{x}d\left(\dfrac{y}{x}\right)}{\sqrt{1 + \left(\dfrac{y}{x}\right)^2}} = 0$.]

2. Show that e^{x^2} is an integrating factor of the equation $(x + y^4)\,xdx + 2y^3dy = 0,$ and hence solve it.

3. Show that $\dfrac{1}{x(x^2 - y^2)}$ is an integrating factor of the equation $(x^2 + y^2)dx - 2xydy = 0$ and hence solve it.

4. Show that $\dfrac{1}{(x + y + 1)^4}$ is an integrating factor of the equation $(2xy - y^2 - y)dx + (2xy - x^2 - x)dy = 0$. Hence solve it.

5. Show that $\frac{x}{x-1}$ is an integrating factor of the equation $x(x-1)\frac{dy}{dx}-y=[x(x-1)]^2$ and hence solve it.

[*Hints*: In the problem no 2-5, multiply the equation by the I.F. and then prove the exactness of the resulting equations.]

6. If $(x^2+y^2)^k$ be an integrating factor of the equation $(x+y)\,dx+(y-x)\,dy=0$, find k, and hence solve it.

7. If $x^a y^b$ be an integrating factor of the equation $\left(\frac{3}{x}+2y^4\right)dx+\left(\frac{3}{y}-xy^3\right)dy=0$, then find a, b.

8. If $x^a y^b$ be an integrating factor of the equation $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0$, then find a, b.

9. If $x^a y^b$ be an integrating factor of the equation $(x-y^2)\,dx+2xy\,dy=0$, find a, b.

ANSWERS

1. (i) $\tan^{-1}\frac{y}{x}=x+C$, (ii) $y+\sqrt{x^2+y^2}=Cx^2$,

(iii) $y=x\sinh(x+C)$, (iv) $\sqrt{x^2+y^2}=x+C$,

(v) $\frac{y}{x}+\sin\frac{1}{x}=C$, (vi) $1+Cxy=y(1+\ln x)$,

(vii) $x-y=\ln(x+y)+C$, (viii) $x^2+y^2+2\tan^{-1}\frac{y}{x}=C$,

(ix) $y=x\sin(\ln Cx^2)$, (x) $x^2+y^2=\sin^2\left(\tan^{-1}\frac{y}{x}+C\right)$,

(xi) $x^2+y^2+2a^2\tan^{-1}\left(\frac{x}{y}\right)=C$, (xii) $\ln(x^2+y^2)=C-2m\tan^{-1}\left(\frac{x}{y}\right)$,

(xiii) $xy\cos\frac{y}{x}=C$, (xiv) $x+y\ln x^2=Cy$,

(xv) $x(x^2-y^2)=C$, (xvi) $x^2-y^2=Cy$,

(xvii) $y^3=3x^3\ln x+Cx^3$, (xviii) $y=ce^{\frac{x^3}{3y^3}}$,

(xix) $\ln\frac{x^2}{y}-\frac{1}{xy}=C$, (xx) $\ln\frac{x}{y}-\frac{1}{xy}=C$,

(xxi) $(x+y)(1+2xy)=C$, (xxii) $y(4x^3+y^3)=C$,

(xxiii) $x+y+x^3+y^3+3x^2y^2=C$, (xxiv) $x^2y^2-1=2xy(\ln y+C)$,

(xxv) $x^2-y^2=1+Cx$ (xxvi) $x(x+y)=1+Cx+y^2$,

(xxvii) $3x^4 + 6x^2y^2 + 4x^3 = C$, (xxviii) $x = ce^{\frac{-2y^3}{3x^3}}$,

(xxix) $e^{x^2}(x^2 - 1 + y^4) = C$, (xxx) $x^3y^3 + x^2 = Cy$,

(xxxi) $3x^2y^4 + 6xy^2 + y^6 = C$, (xxxii) $x^3y^4 + x^2y^6 = C$,

(xxxiii) $6\sqrt{xy} - \left(\frac{x}{y}\right)^{\frac{3}{2}} = C$, (xxxiv) $5x^{-\frac{36}{13}}y^{\frac{24}{13}} + 12x^{-\frac{10}{13}}y^{-\frac{15}{13}} = C$,

(xxxv) $6\sqrt{xy} - \left(\frac{y}{x}\right)^{\frac{3}{2}} = C$, (xxxvi) $x^3y^3 + 4x^2y^7 = C$,

(xxxvii) $4x^{\frac{10}{7}}y^{-\frac{5}{7}} - 5x^{-\frac{4}{7}}y^{-\frac{12}{7}} = C$, (xxxviii) $x^2y^3 + 2x^3y^4 = C$,

(xxxix) $x^3y^2(x + y^3) = C$, (xl) $x^3y^5(4 + xy^3) = C$,

(xli) $\frac{1}{x^3y^3} + \frac{y}{x^2} = C$, (xlii) $(x^2 + y^2)(x + 2)^2 = Cx^2$,

6. $k = -1$, $\ln(x^2 + y^2) - 2\tan^{-1}\frac{y}{x} = C$,

7. $a = -4, b = -4$,

8. $h = -\frac{5}{2}, k = -\frac{1}{2}$,

9. $a = -2, b = 0$.