Integrating both sides, we get

$$xe^{-\frac{1}{y}} = \int \frac{1}{y^3} e^{-\frac{1}{y}} dy + C = -\int ze^z dz + C, \text{ where } z = -\frac{1}{y}$$

$$= -(z-1)e^z + C = \left(\frac{1}{y}+1\right)e^{-\frac{1}{y}} + C$$

or, $$x = \left(\frac{1}{y}+1\right) + Ce^{\frac{1}{y}}.$$

11. Solve the equation: $\left(1+y^2\right)dx - \left(\tan^{-1} y - x\right)dy = 0.$

Solution

We rewrite the equation as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}.$$

This is a linear differential equation, whose integrating factor is

$$e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}.$$

Multiplying the equation by $e^{\tan^{-1} y}$, we get

$$\left(\frac{dx}{dy} + \frac{x}{1+y^2}\right)e^{\tan^{-1} y} = \frac{\tan^{-1} y}{1+y^2}e^{\tan^{-1} y} \quad \text{or,} \quad \frac{d}{dy}\left(xe^{\tan^{-1} y}\right) = \frac{\tan^{-1} y}{1+y^2}e^{\tan^{-1} y}.$$

Integrating both sides, we get

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy = \int ze^z dz, \text{ where } z = \tan^{-1} y$$

$$= (z-1)e^z + C = \left(\tan^{-1} y - 1\right)e^{\tan^{-1} y} + C$$

or, $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}.$

12. Solve the equation: $\dfrac{dy}{dx} + \dfrac{y}{\sqrt{\left(1-x^2\right)^3}} = \dfrac{x+\sqrt{1-x^2}}{\left(1-x^2\right)^2}.$

Solution

The given equation is a linear differential equation whose integrating factor is

$$e^{\int \frac{dx}{\sqrt{\left(1-x^2\right)^3}}} = e^{\int \sec^2 \theta d\theta}, \text{ where } x = \sin\theta$$

$$= e^{\tan\theta} = e^{\frac{x}{\sqrt{1-x^2}}}.$$

Multiplying the given equation by $e^{\frac{x}{\sqrt{1-x^2}}}$, we get

$$\left(\frac{dy}{dx}+\frac{y}{\sqrt{(1-x^2)^3}}\right)e^{\frac{x}{\sqrt{1-x^2}}}=\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}e^{\frac{x}{\sqrt{1-x^2}}}$$

or, $$\frac{d}{dx}\left(ye^{\frac{x}{\sqrt{1-x^2}}}\right)=\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}e^{\frac{x}{\sqrt{1-x^2}}}.$$

Integrating, we get

$$ye^{\frac{x}{\sqrt{1-x^2}}}=\int\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}e^{\frac{x}{\sqrt{1-x^2}}}dx$$

$$=\int\frac{\sin\theta+\cos\theta}{\cos^3\theta}e^{\tan\theta}d\theta,\ \text{where } x=\tan\theta$$

$$=\int(1+z)e^z dz,\ \text{where } z=\tan\theta$$

$$=ze^z+C=\tan\theta\, e^{\tan\theta}+C$$

$$=\frac{x}{\sqrt{1-x^2}}e^{\frac{x}{\sqrt{1-x^2}}}+C$$

or, $$y=\frac{x}{\sqrt{1-x^2}}+Ce^{-\frac{x}{\sqrt{1-x^2}}}.$$

13. Solve the equation: $\frac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0.$

Solution

We rewrite the equation as

$$\frac{dy}{dx}+2x\tan^{-1}y(1+y^2)=x^3(1+y^2).$$

Dividing $1+y^2$, we get

$$\frac{1}{1+y^2}\frac{dy}{dx}+2x\tan^{-1}y=x^3.$$

Putting $z=\tan^{-1}y$, we get

$$\frac{dz}{dx}+2xz=x^3.$$

This is a linear differential equation whose integrating factor is

$$e^{\int 2x dx}=e^{x^2}.$$

42. $\left[1+x\left(1+y^2\right)\right]dy+y\left(1+y^2\right)dx=0,$

43. $y\ln y\,dx+(x-\ln y)\,dy=0,$

44. $\sin 2x\dfrac{dy}{dx}=y+\tan x,$

45. $dx+xdy=e^{-y}\sec^2 y\,dy,$

46. $\left(x^2y^3+2xy\right)dy=dx,$

47. $\left(x+2y^3\right)\dfrac{dy}{dx}=y,$

48. $\dfrac{dy}{dx}+\dfrac{1}{x}=\dfrac{e^y}{x^2},$

49. $\dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)\,e^x\sec y,$

50. $\dfrac{dy}{dx}+\dfrac{y\ln y}{x}=\dfrac{y(\ln y)^2}{x^2},$

51. $\dfrac{dy}{dx}+\dfrac{xy}{1-x^2}=x\sqrt{y},$

52. $\sin x\dfrac{dy}{dx}-y\cos x+y^2=0,$

53. $xy-\dfrac{dy}{dx}=y^3e^{-x^2},$

54. $\dfrac{dz}{dx}+\dfrac{z}{x}\ln z=\dfrac{z(\ln z)^2}{x^2},$

55. $x\dfrac{dy}{dx}+3y=x^3y^2,$

56. $\left(xy+x^3y^3\right)\dfrac{dy}{dx}=1,$

[*Hints*: Rewrite the equation as $\dfrac{1}{x^3}\dfrac{dx}{dy}-\dfrac{y}{x^2}=y^3$ and put $z=\dfrac{1}{x^2}$]

57. $\left(x^2+y^2+2y\right)dy+2xdx=0,$

58. $y\left(2xy+e^x\right)dx-e^x dy=0,$

59. Show that the general solution of the equation

$$\frac{dy}{dx}+P(x)\,y=Q(x)$$

can be written in the form $y=k\,(u-v)+v,$ where k is a constant and u, v are two particular solutions of the equation.

[*Hints*: If u v are two particular solutions of the equation, then any linear combination of these, say $y=au+bv$, where a, b are arbitrary constants, will be a general solution of the given equation. Choosing $a=k$, $b=1-k$, we see that

$$y=ku+(1-k)\,v=k\,(u-v)+v$$

is a general solution of the given equation.]

60. Show that the solution of the equation

$$\frac{dy}{dx}+P(x)\,y=Q(x)$$

can be put in the form

$$y=\frac{Q}{P}-e^{-\int P\,dx}\left[\int e^{\int P\,dx}\,d\left(\frac{Q}{P}\right)+C\right].$$

[*Hints*: Let $z=y-\dfrac{Q}{P}$, then $\dfrac{dz}{dx}=\dfrac{dy}{dx}-\dfrac{d}{dx}\left(\dfrac{Q}{P}\right)$.

Multiplying the above equation by e^{x^2}, we get

$$\left(\frac{dz}{dx}+2xz\right)e^{x^2}=x^3e^{x^2} \quad \text{or,} \quad \frac{d}{dx}\left(ze^{x^2}\right)=x^3e^{x^2}.$$

Integrating both sides, we get

$$ze^{x^2}=\int x^3e^{x^2}dx+C$$

$$=\frac{1}{2}\int ue^{u}du+C, \text{ where } u=x^2$$

$$=\frac{1}{2}(u-1)e^{u}+C=\frac{1}{2}\left(x^2-1\right)e^{x^2}+C$$

or, $\tan^{-1}y=\frac{1}{2}\left(x^2-1\right)+Ce^{-x^2}.$

14. Solve the equation: $\frac{dy}{dx}=y^2+\frac{1}{2x^2}$.

Solution

$$\frac{dy}{dx}=y^2+\frac{1}{2x^2}$$

$$\frac{dy}{dx}=\frac{2x^2y^2+1}{2x^2} \qquad x\frac{dy}{dx}+y=\frac{2x^2y^2+1}{2x}+y=\frac{2x^2y^2+2xy+1}{2x}$$

$$\Rightarrow d(xy)=\frac{1+(1+2xy)^2}{4x}dx \Rightarrow d(2xy+1)=\frac{1+(1+2xy)^2}{2x}dx$$

$$\frac{d(2xy+1)}{1+(1+2xy)^2}=\frac{dx}{2x}.$$

Integrating, we get

$$\tan^{-1}(1+2xy)=\frac{1}{2}\ln x+C.$$

15. Show that in an equation of the form $yf(xy)\,dx+xg(xy)\,dy=0$ the variables can be separated by the substitution $xy=v$. Hence find the solution of the equation

$$\left(x^3y^3+x^2y^2+xy+1\right)ydx+\left(x^3y^3-x^2y^2-xy+1\right)xdy=0.$$

Solution

Putting $xy=v$, i.e., $x\frac{dy}{dx}+y=\frac{dv}{dx}$ in the given equation, we get

$$yf(v)+g(v)\left[\frac{dv}{dx}-y\right]=0$$

$$\text{or,}\quad y[f(v)-g(v)]+g(v)\frac{dv}{dx}=0 \quad \text{or,}\quad \frac{g(v)\,dv}{v[f(v)-g(v)]}+\frac{dx}{x}=0.$$

(b) Find a solution of the differential equation

$$\frac{dy}{dx} - y\tan x = 0$$

in the form $y = y_1(x)$, say. Hence solve

$$\frac{dy}{dx} - y\tan x = \cos x$$

by the substitution $y = y_1(x)v(x)$. [C.H. 199

(c) When P and Q are functions of x alone or constants, solve:
(i) $\frac{dy}{dx} + Py = Q$ and (ii) $\frac{dy}{dx} + Py = Qy^n$. [C.H. 199

2. Solve by the method of variation of parameters (i.e., accepting a solution of reduced equation):

(a) $\frac{dy}{dx} - 5y = \sin x$.

(b) $\frac{dy}{dx} + 6y = 18e^{3x}$.

(c) $\frac{dx}{dt} + \frac{1}{t}x = t^2$.

(d) $\frac{dy}{dt} + e^t y = e^t$.

3. (a) $\frac{dy}{dx} + y\cot x = 2\cos x$.

(b) $\frac{dy}{dx} + \frac{1-2x}{x^2}y = 1$.

4. (a) $(1-x^2)\frac{dy}{dx} - xy = 1$.

(b) $\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$.

5. (a) $\frac{dy}{dx} + \frac{n}{x}y = \frac{a}{x^n}$.

(b) $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^3}$.

6. (a) $x\cos x\frac{dy}{dx} + (x\sin x + \cos x)y = 1$. [C.H. 19

(b) $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$.

7. $x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = ax^3$.

8. $\sqrt{(a^2+x^2)}\frac{dy}{dx} + y = \sqrt{(a^2+x^2)} - x$.

9. $\frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^{\frac{5}{2}}$.

10. $a\frac{du}{d\theta} + a(\tan\theta)u = \tan\theta$.

11. $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$.

12. $x\log x\frac{dy}{dx} + y = 2\log x$.

13. $(x+y+1)\,dy = dx$.

14. $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$.

15. $2(1-xy)\frac{dy}{dx} = y^2$.

16. $(x+2y^3)\frac{dy}{dx} = y$.

17. $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$.

18. $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$.

19. (a) $3\frac{dy}{dx} + \frac{2}{x+1}y = \frac{x^3}{y^2}$.

(b) $x\frac{dy}{dx} + y = xy^2$. [C.H. 1980]

20. (a) $3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^3 = ax^3$.

(b) $2x^2\frac{dy}{dx} = xy + y^2$. [C.H. 1983]

21. $x\frac{dy}{dx} + y = y^2\log x$. [C.H. 1993]

22. $\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$. [C.H. 1984]

23. $x\frac{dy}{dx} + y = y^2x^2\cos x$.

24. $dx + x\,dy = e^{-y}\sec^2 y\,dy$.

25. $(x^2y^3 + xy)\,dy = dx$.

26. $(x^3y^2 + 2xy)\,dy = dx$. [C.H. 1985, '95]