Examples II (D)

Solve the following equations (1 – 25) :

1. (a) $\cos x \dfrac{dy}{dx} + y \sin x = 1$.

   (b) $x \cos x \dfrac{dy}{dx} + y(x \sin x + \cos x) = 1$.

2. $\dfrac{dy}{dx} + 2xy = e^{-x^2}$.

3. $(x^2 + 1)\dfrac{dy}{dx} + 2xy = 4x^2$. [B. H. 1987]

4. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

5. $y + 2\dfrac{dy}{dx} = y^3(x - 1)$. [C. H. 1984]

6. $\sec^2 y \dfrac{dy}{dx} + 2x \tan y = x^3$.

7. $2x^2 \dfrac{dy}{dx} = xy + y^2$.

8. $x \dfrac{dy}{dx} + y = xy^2$.

9. $(x + y + 1)\, dy = dx$.

10. $(x + 2y^3)\dfrac{dy}{dx} = y$.

11. $dx + x\, dy = e^{-y} \sec^2 y\, dy$. [C. H. 1985, 1995]

12. $(x^2 y^3 + 2xy)\, dy = dx$.

13. $(x^2 y^3 + xy)\, dy = dx$.

14. $x^2 y - x^3 \dfrac{dy}{dx} = y^4 \cos x$.

15. $\dfrac{dy}{dx} - \dfrac{1}{1+x} \tan y = (1 + x)\, e^x \sec y$.

16. $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2$. [B. H. 1991 ; V. H. 1997]

17. $y(2xy + e^x)\, dx - e^x\, dy = 0$. [C. H. 1994]

18. $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$.

19. $\dfrac{dy}{dx} + \dfrac{1}{x} \tan y = \dfrac{1}{x^2} \tan y \sin y$.

20. $(x^2+1)\dfrac{dy}{dx} - 2xy = (x^4 + 2x^2 + 1)\cos x$.

21. $\dfrac{dy}{dx} = (\sin x - \sin y)\dfrac{\cos x}{\cos y}$.

22. $\dfrac{dy}{dx} + y\dfrac{df}{dx} = f(x)\dfrac{df}{dx}$. 23. $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$.

24. $\dfrac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0$.

25. $t\dfrac{dx}{dt} + 2x = 3t^3x^{\frac{4}{3}}$

Solve the following equations by the method of variation of parameters (26 – 29) :

26. $(1+x)\dfrac{dy}{dx} - xy = 1 - x$.

27. $x(x^2+1)\dfrac{dy}{dx} = y(1-x^2) + x^3\log x$.

28. $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$. 29. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x\sec y$.

30. (*a*) Find the solution of the differential equation $\dfrac{dy}{dx} = y\tan x$ in the form $y = y_1(x)$, say.

Hence solve $\dfrac{dy}{dx} - y\tan x = \cos x$ by the substitution

$$y = y_1(x)\,v(x).$$ [C. H. 1990]

(*b*) Reduce the differential equation $\dfrac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^3$ to a linear form and hence solve it. [C. H. 2005]

31. (*a*) Solve $(1 + y + x^2y)\,dx + (x + x^3)\,dy = 0$ and find the particular solution, if $y = 1$ when $x = 1$.

(*b*) Find the particular solution of the equation

$$y^2\,dx + \left(x - \frac{1}{y}\right)dy = 0, \text{ if } y = 1 \text{ when } x = 2.$$

32. If $2\dfrac{dr}{d\theta} + r\tan\theta = \dfrac{1}{r\cos\theta}$ and $r = 1$ when $\theta = 0$, then show that $r = \sqrt[4]{2}$ when $\theta = \frac{1}{4}\pi$.

33. Show that the equation of the curve, whose slope at any point (x, y) is equal to $(y + 2x)$ and which passes through the origin, is

$$y = 2(e^x - x - 1).$$ [B. H. 1990]

34. Show that the equation of the curve, whose slope at any point (x, y) is equal to $xy(x^2y^2 - 1)$ and which passes through the point $(0, 1)$, is $x^2y^2 = 1 - y^2$.

35. Find the curve, for which the radius of curvature varies as the square of the perpendicular upon the normal.

[Here $r\dfrac{dr}{dp} = k\,|\,r\sin(90^\circ - \phi)\,|^2$, k being a constant.]

36. Show that the curve, for which the sum of the reciprocals of the radius vector and the polar sub-tangent is constant, is of the form $r^{-1} = a^{-1} + ce^{\theta}$, c being arbitrary.

Answers

1. (a) $y\sec x = \tan x + c$. (b) $xy = \sin x + c\cos x$.

2. $ye^{x^2} = x + c$.

3. $3(x^2+1)y = 4x^3 + c$.

4. $y = \tan x - 1 + ce^{-\tan x}$.

5. $y^2(x + ce^x) = 1$.

6. $2\tan y = x^2 - 1 + ce^{-x^2}$.

7. $x - y = cy\sqrt{x}$.

8. $xy(c - \log x) = 1$.

9. $x + y + 2 = ce^y$.

10. $x = y(y^2 + c)$.

11. $x = e^{-y}(c + \tan y)$.

12. $\dfrac{1}{x} = \dfrac{1}{2}(1 - y^2) + ce^{-y^2}$.

13. $\dfrac{1}{x} = 2 - y^2 + ce^{-\frac{1}{2}y^2}$

14. $x^3 = y^3(3\sin x + c)$.

15. $\sin y = (1+x)(e^x + c)$.

16. $2x = (1 + 2cx^2)\log y$.

17. $y^{-1}e^x = c - x^2$.

18. $y^2(2x + c) = e^{x^2}$.

19. $\dfrac{1}{x\sin y} = \dfrac{1}{2x^2} + c$.

20. $y = (x^2+1)(\sin x + c)$.

21. $\sin y = \sin x - 1 + ce^{-\sin x}$.

22. $y = ce^{-f(x)} + f(x) - 1$.

23. $e^y = (e^x - 1) + ce^{-e^x}$.

24. $2\tan^{-1}y = x^2 - 1 + 2ce^{-x^2}$.

25. $7x^{-\frac{1}{3}} = ct^{\frac{2}{3}} - 3t^3$.

26. $y(1+x) = x + Ce^x$.

27. $y(x^2+1) = Cx + \dfrac{x^3}{2}\log x - \dfrac{x^3}{4}$.

28. $2x = e^y(1 + Cx^2)$.

29. $\sin y = (1+x)(C + e^x)$.

30. (a) $2y = \sec x\left\{\left(x + \frac{1}{2}\sin 2x\right) + c\right\}$. (b) $(y-x)^2(ce^{x^2} - x^2 - 1) = 1$.

31. (a) $xy = c - \tan^{-1}x$; $xy = 1 + \dfrac{\pi}{4} - \tan^{-1}x$. (b) $x - 1 = \dfrac{1}{y}$.

35. $r^2 - p^2 = \dfrac{p}{k} + \dfrac{1}{2k^2} + ce^{2kp}$.

A factor, that will make the first term an exact differential, is $x^{2k_1-2-1}\,y^{3k_1-1}$ and a factor, that will make the second term an exact differential, is $x^{-3k_2-1}\,y^{2k_2-3-1}$, where k_1 and k_2 have values such that these two factors are same. This gives $2k_1-3=-3k_2-1$ and $3k_1-1=2k_2-4$, that is, $2k_1+3k_2-2=0$ and $3k_1-2k_2+3=0$.

These give $k_1=-\frac{5}{13}$, $k_2=\frac{12}{13}$. Thus the common factor is $x^{-\frac{49}{13}}\,y^{-\frac{28}{13}}$.

Multiplying by this integrating factor, the equation becomes

$$\left(2x^{-\frac{23}{13}}y^{-\frac{15}{13}}-3x^{-\frac{49}{13}}y^{\frac{24}{13}}\right)dx+\left(3x^{-\frac{10}{13}}y^{-\frac{28}{13}}+2x^{-\frac{36}{13}}y^{\frac{11}{13}}\right)dy=0.$$

Here we see that $\dfrac{\partial M}{\partial y}=-\dfrac{30}{13}x^{-\frac{23}{13}}y^{-\frac{28}{13}}-\dfrac{72}{13}x^{-\frac{49}{13}}y^{\frac{11}{13}}$

and $\dfrac{\partial N}{\partial x}=-\dfrac{30}{13}x^{-\frac{23}{13}}y^{-\frac{28}{13}}-\dfrac{72}{13}x^{-\frac{49}{13}}y^{\frac{11}{13}}$.

Therefore the equation is now exact.

Its primitive, as done in Ex. 6, is

$$2\left(-\frac{13}{10}\right)x^{-\frac{10}{13}}y^{-\frac{15}{13}}-3\left(-\frac{13}{36}\right)x^{-\frac{36}{13}}y^{\frac{24}{13}}=c_1,\quad c_1 \text{ being a constant}$$

or, $-\dfrac{13}{5}x^{-\frac{10}{13}}y^{-\frac{15}{13}}+\dfrac{13}{12}x^{-\frac{36}{13}}y^{\frac{24}{13}}=c_1$

or, $5x^{-\frac{36}{13}}y^{\frac{24}{13}}-12x^{-\frac{10}{13}}y^{-\frac{15}{13}}=c$, where $c=60\,c_1$ is another constant

Examples II (C)

Solve the following differential equations (1 – 36) :

1. $(x+2y^3)\dfrac{dy}{dx}=y$.
2. $\cot y\,dx-\tan x\,dy=0$.
3. $(x+y)\,dy+(y-x)\,dx=0$.
4. $y\,dx+x\,dy=xy\,(dy-dx)$.
5. $x\,dx+y\,dy+k\,(x\,dy-y\,dx)=0$.
6. $x\,dy-y\,dx-\cos\frac{1}{x}\,dx=0$.
7. $\sin x\dfrac{dy}{dx}+y^2=y\cos x$
8. $x\dfrac{dy}{dx}+y=y^2\log x$. [B. H. 1989, 200…]
9. $x^2\dfrac{dy}{dx}+xy+2\sqrt{1-x^2y^2}=0$.

10. $(\,xy\cos xy+$ …
11. $x\,(y\,dx+x\,dy$ …
12. $(x^4y^2-y)\,d$ …
13. $y\,(2x^2y+e^x$ …
14. $\cos x\,(\cos x-$ …
15. $(x^2-4xy-2$ …
16. $(\sin x\cos y+$ …
17. $(1+4xy+2y$ …
18. $(1+xy)\,y\,dx$ …
19. $(1+3x^2+6x$ …
20. $(\log y+\frac{1}{x})\,dx$ …
21. $(2xy+e^x)\,y$ …
22. $(e^x\sin y+e$ …
23. (a) $\dfrac{dy}{dx}\sin x-$ …
(b) $(y^2e^{xy^2}+4$ …
24. (a) $x^2y\,dx-($ …
(b) (x^4+y^4) …
25. $y\,(xy+2x^2y^2$ …
26. $(x^2y^2+xy+$ …
27. $3(x^2+y^2)\,dx$ …
28. $(y+\frac{1}{3}y^3+\frac{1}{2}x$ …
29. $(xy^3+y)\,dx+$ …
30. $(xy^2-x^2)\,dx$ …

10. $(xy \cos xy + \sin xy)\,dx + x^2 \cos xy\,dy = 0$.

11. $x(y\,dx + x\,dy) \cos \frac{y}{x} = y(x\,dy - y\,dx) \sin \frac{y}{x}$.

12. $(x^4 y^2 - y)\,dx + (x^2 y^4 - x)\,dy = 0$. [C. H. 1991]

13. $y(2x^2 y + e^x)\,dx - (e^x + y^3)\,dy = 0$.

14. $\cos x(\cos x - \sin \alpha \sin y)\,dx$
$+ \cos y(\cos y - \sin \alpha \sin x)\,dy = 0$.

15. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$.

16. $(\sin x \cos y + e^{2x})\,dx + (\cos x \sin y + \tan y)\,dy = 0$.

17. $(1 + 4xy + 2y^2)\,dx + (1 + 4xy + 2x^2)\,dy = 0$.

18. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$.

19. $(1 + 3x^2 + 6xy^2)\,dx + (1 + 3y^2 + 6x^2 y)\,dy = 0$.

20. $(\log y + \frac{1}{x})\,dx + (\frac{x}{y} + 2y)\,dy = 0$.

21. $(2xy + e^x)\,y\,dx - e^x\,dy = 0$.

22. $(e^x \sin y + e^{-y})\,dx + (e^x \cos y - xe^{-y})\,dy = 0$.
[N. B. H. 1987]

23. (a) $\frac{dy}{dx} \sin x - y \cos x + y^2 = 0$.

(b) $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$.
[C. H. 1998]

24. (a) $x^2 y\,dx - (x^3 + y^3)\,dy = 0$.

(b) $(x^4 + y^4)\,dx - xy^3\,dy = 0$. [B. H. 1997]

25. $y(xy + 2x^2 y^2)\,dx + x(xy - x^2 y^2)\,dy = 0$.
[V. H. 1992 ; C. H. 1994]

26. $(x^2 y^2 + xy + 1)\,y\,dx + (x^2 y^2 - xy + 1)\,x\,dy = 0$.

27. $3(x^2 + y^2)\,dx + x(x^2 + 3y^2 + 6y)\,dy = 0$.

28. $(y + \frac{1}{3} y^3 + \frac{1}{2} x^2)\,dx + \frac{1}{4}(x + xy^2)\,dy = 0$.

29. $(xy^3 + y)\,dx + 2(x^2 y^2 + x + y^4)\,dy = 0$.

30. $(xy^2 - x^2)\,dx + (3x^2 y^2 + x^2 y - 2x^3 + y^2)\,dy = 0$.

31. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$.

32. (*a*) $(y^3 - 2x^2 y)\,dx + (2xy^2 - x^3)\,dy = 0$.

(*b*) $(y^2 + 2x^2 y)\,dx + (2x^3 - xy)\,dy = 0$.

33. $(3x^2 y^4 + 2xy)\,dx + (2x^3 y^3 - x^2)\,dy = 0$.

34. $(xy^2 - e^{1/x^3})\,dx - x^2 y\,dy = 0$. [C. H. 19

35. $3x^2 y\,dx + (x^3 + y^3)\,dy = 0$.

36. (*a*) $(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2 y^4 e^y - x^2 y^2 - 3x)\,dy = 0$.

(*b*) $(2x^2 y^2 + y)\,dx - (x^3 y - 3x)\,dy = 0$.

37. Show that the equation of the curve passing through the p
(2, 1) and having its gradient $\left(-\dfrac{x+y}{x}\right)$ at any point (x, y) is

$$x^2 + 2xy = 8.$$

38. (*a*) Show that the particular solution of the equation
$(ye^{xy} - 2y^3)\,dx + (xe^{xy} - 6xy^2 - 2y)\,dy = 0$, when $x = 0$, $y = 1$
is
$$e^{xy} = 2xy^3 + y^2 - 3.$$

(*b*) Show that the equation $3y(x^2 - 1)\,dx = (3x - 8y - x^3)$ i
exact and its particular solution when $x = 0$, $y = 1$ is

$$xy(x^2 - 3) = 4(1 - y^2).$$

39. (*a*) Show that $[x(x^2 - y^2)]^{-1}$ is an integrating factor of
equation $(x^2 + y^2)\,dx - 2xy\,dy = 0$ and hence solve the equation

(*b*) Show that $x(x - 1)^{-1}$ is an integrating factor of the equa

$$x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2.$$

(*c*) Prove that $(x + y + 1)^{-4}$ is an integrating factor of
equation $(2xy - y^2 - y)\,dx + (2xy - x^2 - x)\,dy = 0$ and hence s
it.

[C. H. 1993, 2001; K. H.]

(*d*) Prove that e^{x^2} is an integrating factor of the equation

$$(x^2 + xy^4)\,dx + 2y^3\,dy = 0.$$

40. (*a*) If $x^\alpha y^\beta$ be an integrating factor of the equation
$(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$, then find α and

(*b*) If $x^\alpha y^\beta$ be an integrating factor of the equa
$(-3x^{-1} - 2y^4)\,dx + (-3y^{-1} + xy^3)\,dy = 0$, then find α and β

Answers

1. $y^3 = x + cy$.
2. $\sin x \cos y = c$.
3. $x^2 - y^2 - 2xy = c$.
4. $\log(xy) = y - x + c$.
5. $\frac{1}{2}\log(x^2 + y^2) + k \tan^{-1}\frac{y}{x} = c$.
6. $\frac{y}{x} + \sin\frac{1}{x} = c$.
7. $\sin x = y(x + c)$.
8. $y(1 + \log x) = 1 + cxy$.
9. $\sin^{-1} xy + 2\log x = c$.
10. $x \sin xy = c$.
11. $xy = c \sec\left(\frac{y}{x}\right)$.
12. $x^3 + y^3 + \frac{3}{xy} = c$.
13. $\frac{e^x}{y} + \frac{2x^3}{3} - \frac{y^2}{2} = c$.
14. $2(x + y) + \sin 2x + \sin 2y - 4 \sin\alpha \sin x \sin y = c$.
15. $x^3 + y^3 = 6xy(x + y) + c$.
16. $\frac{1}{2}e^{2x} - \cos x \cos y + \log \sec y = c$.
17. $x + 2x^2y + 2xy^2 + y = c$.
18. $x = cye^{\frac{1}{xy}}$.
19. $x + y + x^3 + y^3 + 3x^2y^2 = c$.
20. $x \log y + \log x + y^2 = c$.
21. $x^2 + \frac{e^x}{y} = c$.
22. $e^x \sin y + xe^{-y} = c$.
23. (a) $\sin x = y(x + c)$. (b) $e^{xy^2} + x^4 - y^3 = c$.
24. (a) $y = ce^{\frac{x^3}{3y^3}}$. (b) $y^4 = x^4(4\log x + c)$.
25. $\log\frac{x^2}{y} = \frac{1}{xy} + c$.
26. $xy + \log\frac{x}{y} = \frac{1}{xy} + c$.
27. $x(x^2 + 3y^2) = ce^{-y}$.
28. $x^4y(3 + y^2) + x^6 = c$.
29. $3x^2y^4 + 6xy^2 + 2y^6 = c$.
30. $\left(\frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + \frac{1}{6}y^2 - \frac{1}{18}y + \frac{1}{108}\right)e^{6y} = c$.
31. $xy + y^2 + \frac{2x}{y^2} = c$.
32. (a) $x^2y^2(x^2 - y^2) = c$. (b) $6\sqrt{xy} - x^{-\frac{3}{2}}y^{\frac{3}{2}} = c$.
33. $x^3y^3 + x^2 = cy$.
34. $3y^2 - 2x^2e^{\frac{1}{x^3}} = cx^2$.
35. $4x^3y + y^4 = c$.