2023

MATHEMATICS — HONOURS

Paper : CC-7

(ODE and Multivariate Calculus - I)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ denotes the set of real numbers and $\mathbb{N}$ denotes the set of natural numbers.

Group - A

(Marks : 20)

1. Answer the following multiple-choice questions with only one correct option. Choose the correct option and justify. $(1+1)\times10$

(a) The values of parameters a and b for which the differential equation

$$\left(ax^2y+y^3\right)dx+\left(\frac{1}{3}x^3+bxy^2\right)dy=0$$

is exact are

(i) $a=3,\ b=3$ (ii) $a=1,\ b=1$

(iii) $a=1,\ b=3$ (iv) $a=3,\ b=1$.

(b) The solution of $\frac{dy}{dx}=\frac{(1-x)}{y}$ represents

(i) a family of circle centre at (1, 0)

(ii) a family of circle centre at (0, 0)

(iii) a family of circle centre at (–1, 0)

(iv) a family of straight line with slope –1.

(c) Which of the following differential equations is linear?

(i) $\frac{dy}{dx}+2xy^2=5e^x$ (ii) $x^2\frac{dy}{dx}+3x\sin y=x^{\frac{2}{3}}$

(iii) $\frac{dy}{dx}+3y\cos x=e^{-x^2}$ (iv) $\left(\frac{dy}{dx}\right)^2+5xy=\log_e x$.

(d) The Wronskian of the functions $y_1 = \sin x$ and $y_2 = \sin x - \cos x$ is

(i) 0 (ii) 1

(iii) $\sin^2 x$ (iv) $\cos^2 x$.

(e) Determine the nature of the critical point (0, 0) of the following plane autonomous system :

$$\dot{x} = 2x - 3y,\ \dot{y} = x + 4y.$$

(i) stable spiral (ii) unstable spiral

(iii) saddle point (iv) stable node.

(f) Which one of the following is correct for the linear differential equation :

$$\left(x^2 - 3x\right)\frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + y = 0\,?$$

(i) $x = 0$ is an ordinary point (ii) $x = 3$ is an ordinary point

(iii) $x = 0$ is a regular singular point (iv) $x = 0$ is an irregular singular point.

(g) Domain of definition of the function $f(x, y) = \dfrac{1}{\sqrt{36 - x^2 - y^2}} + \log_e\left(x^2 + y^2\right)$ is

(i) $\left\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \le 36\right\}$ (ii) $\left\{(x, y) \in \mathbb{R}^2 : 0 \le x^2 + y^2 < 36\right\}$

(iii) $\left\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 36\right\}$ (iv) $\left\{(x, y) \in \mathbb{R}^2 : 0 \le x^2 + y^2 \le 36\right\}$.

(h) Evaluate : $\displaystyle\lim_{\substack{x\to\infty \\ y\to 2}} \frac{xy+4}{x^2+2y^2}$

(i) ∞ (ii) 0

(iii) 1 (iv) does not exist.

(i) The directional derivative of $f(x, y) = e^x - x^2 y$ at (1, 2) in the direction of $2\hat{i} + \hat{j}$ is

(i) $\frac{1}{\sqrt{5}}(2e+9)$ (ii) $\frac{1}{\sqrt{5}}(2e-9)$

(iii) $\frac{1}{\sqrt{5}}(e+9)$ (iv) $\frac{1}{\sqrt{5}}(2e-13)$.

(j) For the function $f(x, y) = 2x^4 - 3x^2 y + y^2$ has

(i) a maximum at (0, 0) (ii) a minimum at (0, 0)

(iii) neither maxima nor minima at (0, 0) (iv) none of these.

Group - B

(Marks : 30)

Answer ***any six*** questions.

2. Show that the following equation is not exact.

$$\left(xy^2 - y^2 - y^5\right)\frac{dy}{dx} + \left(1 + y^3\right) = 0$$

Find an integrating factor and hence solve the equation. 1+2+2

3. (a) Solve : $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$.

(b) Verify the existence and uniqueness of the solution of the differential equation :

$$\frac{dy}{dx} = \frac{y}{x},\ y(0) = 0.$$ 3+2

4. Reduce the equation $xp^2 - 2yp + x + 2y = 0\left(p = \frac{dy}{dx}\right)$ to Clairaut's form by the substitution $x^2 = u,\ y - x = v$ and hence solve it. Also find the singular solution (if it exists). 2+2+1

5. Solve by using the method of variation of parameters, the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$$ 5

6. Solve the following equation by the method of undetermined coefficients :

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x$$ 5

7. Solve the equation $\frac{dy}{dx} + y\log y = \frac{y}{x^2}(\log y)^2$ 5

8. Find the general solution of the ordinary differential equation :

$$(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$$ 5

9. Solve the following system by operator method :

$$\frac{dx}{dt} + 4x + 3y = t$$

$$\frac{dy}{dt} + 2x + 5y = e^t$$ 5

10. Determine the nature and stability of the critical point (0, 0) of the following system :

$$\frac{dx}{dt} = 2x + 4y$$

$$\frac{dy}{dt} = -2x + 6y$$

Also draw rough sketch of the corresponding phase portraits. 3+2

11. Solve the equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 + 2\right)y = 0$ in series about the ordinary point $x = 0$. 5

Group - C

(Marks : 15)

Answer ***any three*** questions.

12. (a) Using definition calculate $f_x(0, 0)$ and $f_y(0, 0)$ from the function defined by

$$f(x, y) = \frac{xy^2}{x^2 + y^4} \text{ for } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0.$$

(b) Show that the set $S = \left\{\left(\frac{1}{m}, \frac{1}{n}\right) \in \mathbb{R}^2 : m, n \in \mathbb{N}\right\}$ is closed. 3+2

13. Let $f(x, y)$ be continuous at an interior point (a, b) of domain of definition of f and $f(a, b) \neq 0$. Show that $f(x, y)$ maintains same sign in a neighbourhood of (a, b). What can be said about the sign of f in a neighbourhood of (a, b) if $f(a, b) = 0$? 3+2

14. If $z = z(u, v)$, where $u = x^2y$ and $v = 3x + 2y$ show that

(i) $\frac{\partial^2 z}{\partial y^2} = x^4\frac{\partial^2 z}{\partial u^2} + 4x^2\frac{\partial^2 z}{\partial u \partial v} + 4\frac{\partial^2 z}{\partial v^2}$

(ii) $\frac{\partial^2 z}{\partial x \partial y} = 2x^3y\frac{\partial^2 z}{\partial u^2} + \left(3x^2 + 4xy\right)\frac{\partial^2 z}{\partial u \partial v} + 2x\frac{\partial z}{\partial u} + 6\frac{\partial^2 z}{\partial v^2}$. 2+3

15. Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1). 5

16. Use Lagrange's method of multipliers to find the maximum and minimum values of the function $f(x, y) = 7x^2 + 8xy + y^2$ subject to the condition $x^2 + y^2 = 1$. 5