ILLUSTRATIVE EXAMPLES

1. Solve: $y = p^2 x + p$.

Solution

Differentiating the given equation with respect to x, we get

$$p = \frac{dy}{dx} = p^2 + (2px + 1)\frac{dp}{dx}$$

or, $$p(1-p)\frac{dx}{dp} = (2px+1)$$

or, $$\frac{dx}{dp} + \frac{2x}{p-1} = \frac{1}{p(1-p)}.$$

This is a linear differential equation whose integrating factor is

$$e^{\int \frac{2dp}{p-1}} = e^{2\ln(p-1)} = (p-1)^2.$$

Multiplying the above equation by $(p-1)^2$, we get

$$\frac{d}{dp}\left[x(p-1)^2\right] = \frac{1}{p} - 1.$$

Integrating, we get $$x(p-1)^2 = \ln p - p + C$$

or, $$x = \frac{\ln p - p + C}{(p-1)^2}.$$

Hence p-eliminant of $x = \dfrac{\ln p - p + C}{(p-1)^2}$ and $y = p^2 x + p$ will give the solution.

2. Solve: $y = 2px + p^2$

Solution

Differentiating both sides with respect to x, we get

$$p = \frac{dy}{dx} = 2p + 2(x+p)\frac{dp}{dx} \quad \text{or,} \quad \frac{dx}{dp} + \frac{2x}{p} = -2.$$

This is a linear differential equation whose integrating factor is $e^{\int \frac{2}{p} dp} = p^2$. Multiplying the above equation by p^2, we get

$$\frac{d}{dp}\left(xp^2\right) = -2p^2.$$

Integrating, we get

$$xp^2 = -\frac{2}{3}p^3 + C \text{ or, } x = -\frac{2}{3}p + Cp^{-2}.$$

Using this in the given equation, we get

$$y = -\frac{1}{3}p^2 + \frac{2C}{p}.$$

Hence, the p-eliminant of $x = -\frac{2}{3}p + Cp^{-2}$ and $y = -\frac{1}{3}p^2 + \frac{2C}{p}$ will give th general solution.

3. Solve: $y = \ln\left(p^3 + p\right)$.

Solution

Differentiating with respect to x, we get

$$p = \frac{dy}{dx} = \frac{3p^2 + 1}{p^3 + p}\frac{dp}{dx} \quad \text{or,} \quad dx = \frac{3p^2 + 1}{\left(p^2 + 1\right)p^2}dp$$

Integrating, we get

$$x = 2\tan^{-1} p - \frac{1}{p} + C.$$

Hence, the p-eliminant of $x = 2\tan^{-1} p - \frac{1}{p} + C$ and $y = \ln\left(p^3 + p\right)$ will give general solution.

4. Solve: $y = px + \sqrt{a^2p^2 + b^2}$.

Solution

The given equation is in Clairaut's form. Differentiating both sides with respec x, we get

$$p = \frac{dy}{dx} = p + \left(x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}}\right)\frac{dp}{dx}$$

$$\text{or,} \quad \left(x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}}\right)\frac{dp}{dx} = 0.$$

From $\frac{dp}{dx} = 0$, we get $p = C$. Using this in the given equation we get gen solution as

$$y = Cx + \sqrt{a^2C^2 + b^2}.$$

From $x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}} = 0$, we get $x = -\frac{a^2p}{\sqrt{a^2p^2 + b^2}}$. Eliminating p from the g equation with the aid of it we get the singular solution as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2p^2}{a^2p^2 + b^2} + \left(-\frac{a^2p^2}{\sqrt{a^2p^2 + b^2}} + \sqrt{a^2p^2 + b^2}\right)^2 \frac{1}{b^2} = 1.$$

$= 0$

$p^{n-1} = -x$

$p = 1$

5. Reduce the differential equation $xy(y - px) = x + py$ to Clairaut's form by the substitution $x^2 = u,\ y^2 = v$ and then solve it.

Solution

From $x^2 = u,\ y^2 = v$, we get $\frac{ydy}{xdx} = \frac{dv}{du}$. Writing $\frac{dy}{dx} = p$ and $\frac{dv}{du} = P$, we get

$$xv - uxP = x + xP \text{ or, } v = uP + P + 1.$$

This is Clairaut's form. So the general solution is

$$v = uC + C + 1 \text{ or, } y^2 - 1 = C(x^2 + 1).$$

6. Reduce the differential equation $(y + xp)^2 = x^2 p$ to Clairaut's form by the substitution $xy = v$ and then solve it.

Solution

From $xy = v$, we get $xp + y = \frac{dv}{dx} = P(\text{say})$. Using this in the given equation, we get

$$P^2 = x(P - y) = xP - v \text{ or, } v = xP - P^2.$$

This is in Clairaut's form, So the general solution is

$$v = xC - C^2 \text{ or, } xy = xC - C^2.$$

7. Reduce the different equation $y^2 \ln y = xyp + p^2$ to Clairaut's form by the substitution $\ln y = v$ and then solve it.

Solution

From $\ln y = v$, we get $\frac{p}{y} = \frac{dv}{dx} = P(\text{say})$. Using this in the given equation we get

$$y^2 v = xy^2 P + y^2 P^2 \text{ or, } v = xP + P^2.$$

This is in Clairaut's form. So the general solution is

$$v = xC + C^2 \text{ or, } \ln y = xC + C^2.$$

8. Reduce the differential equation

$(x^2 + y^2)(1 + p)^2 - 2(x + y)(1 + p)(x + yp) + (x + yp)^2 = 0$ to Clairaut's form by the substitution $x + y = u,\ x^2 + y^2 = v$ and then solve it.

Solution

Form $x + y = u,\ x^2 + y^2 = v$ we get

$$2xdx + 2ydy = dv \text{ and } dx + dy = du.$$

Dividing we get

$$P = \frac{dv}{du} = \frac{2(x + yp)}{1 + p}.$$

Using this in the given equation we get

$$v - uP + P^2 = 0 \text{ or, } v = uP - P^2.$$

This is in Clairaut's form. So the general solution is

$$v = uC - C^2 \text{ or, } x^2 + y^2 = (x + y)C - C^2.$$

9. Reduce the different equation $x^2p^2 + y(2x + y)p + y^2 = 0$ to Clairaut's form by the substitution $y = u, xy = v$ and then solve it. Obtain also the singular solution, if any.

Solution

From $y = u, xy = v$, we get $P = \frac{dv}{du} = x + y\frac{dx}{dy} = x\frac{y}{p}$ or $p = \frac{y}{P - x}$. Using this in the given equation, we get

$$\frac{x^2y^2}{(P-x)^2} + y(2x + y)\frac{y}{P - x} + y^2 = 0$$

$$\text{or, } \quad x^2 + (2x + y)(P - x) + (P - x)^2 = 0$$

$$\text{or, } \quad xy = yP + P^2$$

$$\text{or, } \quad v = uP + P^2.$$

Differentiating with respect to u, we get

$$(u + 2P)\frac{dP}{du} = 0.$$

If $\frac{dP}{du} = 0$, then $P = c$. So the general solution is

$$v = uc + c^2 \text{ or, } xy = yc + c^2.$$

If $u + 2P = 0$, then using this in $v = uP + P^2$ we get the singular solution as

$$v = -\frac{u^2}{2} + \frac{u^2}{4} = -\frac{u^2}{4}$$

or, $u^2 + 4v = 0$ or, $y^2 + 4xy = 0$

or, $y + 4x = 0$, since $y \neq 0$.

10. Solve: $y = xp^2 + p^4$.

Solution

The given equation is in Lagrange's form, where $\phi(p) = p^2, g(p) = p^4$. Differentiating both sides with respect to x, we get

$$\frac{dx}{dp} - \frac{2}{1-p}x = \frac{4p^2}{1-p}.$$

This is a linear differential equation whose integrating factor is

$$e^{\int \frac{2}{p-1}dp} = e^{2\ln(p-1)} = (p-1)^2.$$

Integrating, we get

$$x(p-1)^2 = \frac{4}{3}p^3 - p^4 + \frac{C}{3} \quad \text{or,} \quad x = \frac{4p^3 - 3p^4 + C}{3(p-1)^2}.$$

So,

$$y = \frac{4p^3 - 3p^4 + C}{3(p-1)^2}p^2 + p^4.$$

Hence, p, eliminant of $x = \dfrac{4p^3 - 3p^4 + C}{3(p-1)^2}$ and $y = \dfrac{4p^3 - 3p^4 + C}{3(p-1)^2}p^2 + p^4$ will give the general solution.

4.5 Equations Homogeneous in x, y

If the equation $f(x, y, p) = 0$ is homogeneous in x, y, then it can be expressed as

$$F\left(\frac{y}{x}, p\right) = 0,$$

which in turn can be expressed either as $p = \phi\left(\dfrac{y}{x}\right)$ or, as $\dfrac{y}{x} = \psi(p)$.

We have already discussed the method of solving the equation $p = \phi\left(\dfrac{y}{x}\right)$. Now, differentiating

$$y = x\psi(p)$$

with respect to x, we get

$$p = \frac{dy}{dx} = \psi(p) + x\psi'(p)\frac{dp}{dx} \quad \text{or,} \quad \frac{dx}{x} = \frac{\psi'(p)}{p - \psi(p)}dp,$$

which can be integrated easily. The p-eliminant of the solution of this equation and the given equation give the general solution.

ILLUSTRATIVE EXAMPLES

1. Solve: $y = (p^2 + p)x$.

Solution

The given equation is of the form $y = x\psi(p)$, where $\psi(p) = p^2 + p$. Differentiating with respect to x, we get

$$\frac{dx}{x} = \frac{\psi'(p)}{p - \psi(p)}dp = -\frac{2p+1}{p^2}dp.$$

Integrating, we get

$$\ln x = -2\ln p + \frac{1}{p} + \ln C \quad \text{or,} \quad x = \frac{C}{p^2}e^{\frac{1}{p}}.$$

Using this in the given equation we get $y = \dfrac{C(p+1)}{p}e^{\frac{1}{p}}$. The p-eliminant of the two will give the general solution.

EXERCISE

- Solve the following equations:

1. $p^2 + 2xp - 3x^2 = 0,$
2. $p^2 - 2p\cosh x + 1 = 0,$
3. $p(p + x) = y(x + y),$
4. $p(p + y) = x(x + y),$
5. $p(p - y) = x(x + y),$
6. $px - y = py\sqrt{x^2 - y^2},$
7. $x^2p^2 + xyp - 6y^2 = 0,$
8. $x^2p^2 - 2xyp + 2y^2 - x^2 = 0,$
9. $p^3 - (x^2 + xy + y^2)p^2 + xy(x^2 + xy + y^2)p - x^3y^3 = 0,$
10. $y = 2px + yp^2,$
11. $y = 2px + y^2p^3,$
12. $y^2 \ln y = xpy + p^2,$
13. $y = x - p^2,$
14. $x(1 + p^2) = 1,$
15. $x = \tan^{-1} p + \dfrac{p}{1 + p^2},$
16. $x = py + ap^2,$
17. $p^2 - 2xp + 1 = 0,$
18. $p^3y^2 - 2px + y = 0,$
19. $y = 3px + 6p^2y^2,$
20. $x = ap^2 + bp^3,$
21. $y = x + p^3,$
22. $xp^2 - 2yp + ax = 0,$
23. $y = 2px + \tan^{-1}(xp^2),$
24. $y = 2px + f(xp^2),$
25. $y + px = x^4p^2,$
26. $y = 3px + 4p^2,$
27. $y = \dfrac{x}{p} + p,$
28. $y = p\tan p + \ln(\cos p),$
29. $y = \sin p - p\cos p,$
30. $y = p\sin p + \cos p,$
31. $y = p - \ln(p^2 - 1),$
32. $y = a\sqrt{p^2 + 1},$
33. $y = 2p + 3p^2,$
34. $y = ap^2 + bp^3,$
35. $y = a + bp + dp^2,$
36. $p^2y^2\cos^2\alpha - 2px\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0,$
37. $y = px + \dfrac{a}{p},$
38. $y = px + a\sqrt{p^2 + 1},$
39. $y = px + p(1 - p),$
40. $y = px + \sin^{-1} p,$
41. $y = px - p^2,$
42. $y = px + p^n,$
43. $py = p^2(x - b) + a,$
44. $(x - a)p^2 + (x - y)p - y = 0.$

- Reduce the following equations to Clairaut's form using suitable substitutions and solve: [Use $x^2 = u$, $y^2 = v$ in 45-49]

45. $x^2(y - px) = p^2 y$

46. $(px - y)(x - py) = 2p$,

47. $xyp^2 - (x^2 + y^2 - 1)p + xy = 0$,

48. $xy(y - px) = x + yp$,

49. $y^2 = \left(\frac{px}{x}\right)x^2 + f\left(\frac{py}{x}\right)$. [Use $x = u$, $y^2 = v$ in 50 and 51]

50. $y = 2px - p^2 y$,

[Use $x = u$, $xy = v$ in 52]

51. $y = 2px + y^2 p^3$,

52. $(px + y)^2 = py^2$,

53. $(px + y)^2 = px^2$, [Use $x = u$, $xy = v$]

- Reduce the following equations to Clairaut's form under the given substitutions and solve:

54. $e^{by}(a - bp) = f\left(pe^{by-ax}\right), \left[e^{ax} = u, e^{by} = v\right]$,

55. $e^{3x}(p-1) + p^3 e^{2y} = 0 \left[e^x = u, e^y = v\right]$,

56. $e^{4x}(p-1) + p^2 e^{2y} = 0 \left[e^{2x} = u, e^{2y} = v\right]$,

57. $y^2(y - xp) = x^4 p^2 \left[x = \frac{1}{u}, y = \frac{1}{v}\right]$,

58. $(x^2 - a^2)p^2 - 2xyp - x^2 = 0 \left[x^2 = u\right]$,

59. $y = 2px + ayp^2 \left[y^2 = v\right]$,

60. $y = 2px + y^2 p^3 \left[y^2 = v\right]$,

61. $y^2 \ln y = pxy + p^2 \left[\ln y = v\right]$,

62. $(px^2 + y^2)(px + y) = (p+1)^2 \left[u = xy, v = x + y\right]$,

63. $x^2 p^2 + yp(2x + y) + y^2 = 0$, $\left[y = u, xy = v\right]$,

64. $xp^2 - 2yp + x + 2y = 0$ $\left[x^2 = u, y - x = v\right]$.

ANSWERS

1. $(2y + 3x^2 - C)(2y - x^2 - C) = 0$,

2. $(y - e^x - C)(y + e^{-x} - C) = 0$,

3. $(y - Ce^x)(y + x - Ce^{-x} - 1) = 0$,

4. $(x^2 + y^2 - C)(2xy + x^2 - C) = 0$,

5. $(x^2+2y-C)(y+x-Ce^x+1)=0$, 6. $x = y\cosh(y+C)$,

7. $(yx^3-C)\left(\frac{y}{x^2}-C\right)=0$, 8. $\sin^{-1}\frac{y}{x}=\pm\ln Cx$,

9. $(x^3-3y+C)\left(e^{\frac{x^2}{2}}+Cy\right)(xy+Cy+1)=0$,

10. $y^2=2xC+C^2$, 11. $y^2=2xC+C^3$,

12. $\ln y = Cx+C^2$,

13. $y=C-\frac{p^2}{2}-p-\ln(p-1), x=C-2\{p+\ln(p-1)\}$,

14. $x(1+p^2)=1, y=C-\tan^{-1}p+\frac{p}{1+p^2}$,

15. $x=\tan^{-1}p+\frac{p}{1+p^2}, y=C-\frac{1}{1+p^2}$,

16. $x=\frac{p(C+a\sin^{-1}p)}{\sqrt{1-p^2}}, y=\frac{C+a\sin^{-1}p}{\sqrt{1-p^2}}-ap$,

17. $x=\frac{p}{2}+\frac{1}{2p}, y=\frac{p^2}{4}-\frac{1}{2}\ln p+C$,

18. $y^2=2Cx-C^3$, 19. $y^3=3Cx+6C^2$,

20. $x=ap^2+bp^3, y=\frac{2a}{3}p^3+\frac{3b}{4}p^4+C$,

21. $x=\frac{3}{2}p^2+3p+3\ln(p-1)+C, y=p^3+\frac{3}{2}p^2+3p+3\ln(p-1)+C$,

22. $C^2x^2-2yC+a=0$, 23. $y=2\sqrt{Cx}+\tan^{-1}C$,

24. $y=2C\sqrt{x}+f(C^2)$, 25. $xy+C=C^2x$,

26. $x=-\frac{8}{5}p+Cp^{-3/2}, y=\frac{3C}{\sqrt{p}}-\frac{4}{5}p^2$,

27. $x=\frac{p(C+\cosh^{-1}p)}{\sqrt{p^2-1}}, y=p+\frac{(C+\cosh^{-1}p)}{\sqrt{p^2-1}}$,

28. $x=\tan p+C, y=p\tan p+\ln(\cos p)$,

29. $x=C-\cos p, y=\sin p-p\cos p$,

30. $x=C+\sin p, y=p\sin p+\cos p$,

31. $x = \ln\frac{p(p+1)}{p-1} + C,\ y = p - \ln(p^2 - 1),$

32. $x = C + a\sinh^{-1} p,\ y = a\sqrt{p^2+1},$

33. $x = C + 6p + 2\ln p,\ y = 2p + 3p^2,$

34. $y = ap^2 + bp^3,\ x = 2ap + \frac{3}{2}bp^2 + C,$

35. $y = a + bp + dp^2,\ x = b\ln p + 2dp + C$

36. $x^2 + y^2 - 2Cx\sec\alpha + C^2 = 0,$

37. $y = Cx + \frac{a}{C}$, singular solution $y^2 = 4ax,$

38. $y = Cx + a\sqrt{C^2+1}$, singular solution $x^2 + y^2 = a^2,$

39. $y = Cx + C(1-C)$, singular solution $(x+1)^2 = 4y,$

40. $y = Cx + \sin^{-1} C$, singular solution $y = \sqrt{x^2-1} + \sin^{-1}\frac{\sqrt{x^2-1}}{x},$

41. $y = Cx - C^2$, singular solution $x^2 = 4y,$

42. $y = Cx + C^n$, singular solution $n^n y^{n-1} + x^n(n-1)^{n-1} = 0,$

43. $Cy = C^2(x-b) + a$, singular solution $y^2 = 4a(x-b),$

44. $(x-a)C^2 + (x-y)C - y = 0$, singular solution $(x+y)^2 = 4ay,$

45. $y^2 = Cx^2 + C^2,$

46. $C^2x^2 + C(2 - x^2 - y^2) + y^2 = 0,$

47. $y^2 = Cx^2 + \frac{C}{C-1},$

48. $y^2 = Cx^2 + C + 1,$

49. $y^2 = Cx^2 + f(C),$

50. $y^2 = Cx - \frac{1}{4}C^2,$

51. $y^2 = Cx + \frac{1}{8}C^3,$

52. $xy = Cy - C^2,$

53. $xy = Cx - C^2,$

54. $e^{by} = e^{ax}C + \frac{1}{a}f\left(\frac{a}{b}C\right),$

55. $e^y = Ce^x + C^3,$

56. $e^{2y} = Ce^{2x} + C^2,$

57. $x = Cy + xyC^2,$

58. $2Cy = x^2 - a^2 - C^2,$

59. $y^2 = xC - \frac{a}{4}C^2,$

60. $y^2 = xC + \frac{1}{8}C^3,$

61. $\ln y = Cx + C^2,$

62. $x + y = Cxy + C^2,$

63. $xy = yC + C^2,$

64. $2C^2x^2 - 2C(y-x) + 1 = 0.$