Lecture Notes on

Gravitational Lensing

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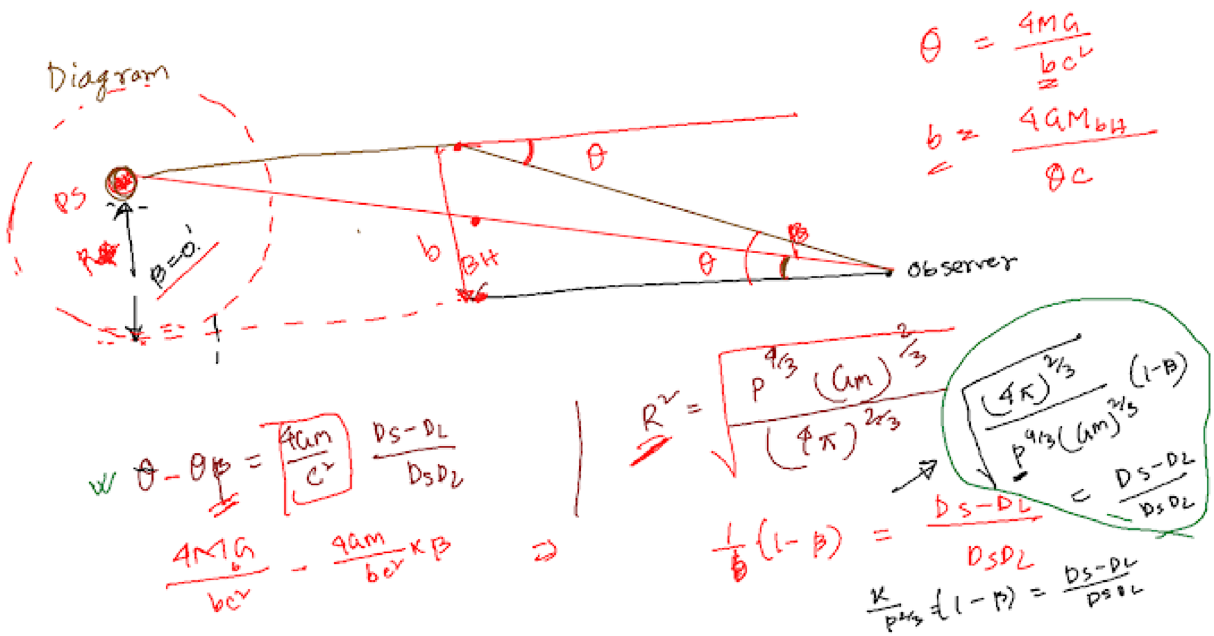
These notes cover basic Gravitational Lensing, with a focus on Astronomy Olympiads. I have included various derivations,planning to eventually add my work and what I learn from the research group '**BDLensing**'.



Some doodles from BDOAA Camp 2022

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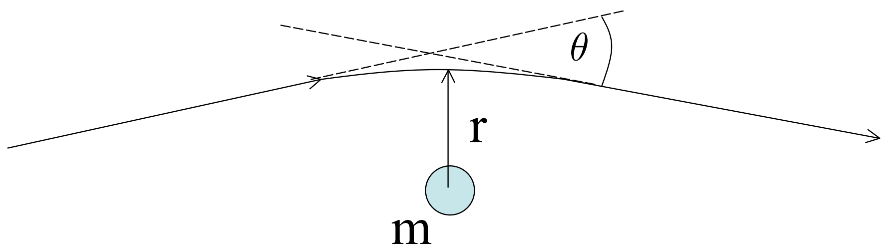
1 Introduction

The deflection of light by a gravitational field was first predicted by Einstein in 1912, a few years before the publication of General Relativity in 1916. A massive object that causes a light deflection behaves like a classical lens. This prediction was confirmed by Sir Arthur Stanley Eddington in 1919.

We'll now use dimensional analysis to determine the form of the equation describing the deflection angle due to gravity for a light ray passing by a star (or other objects) of mass m. First, let's define the angle θ as the angle between the directions of the ray of light when it is asymptotically far from the star (coming towards the star and going away from the star), as shown in



Note that when angles appear in an equation, they should always be expressed in radians. An angle expressed in radians is dimensionless. Therefore, the deflection angle θ is dimensionless.
On which physical variables might the deflection angle depend? Our physical intuition tells us that the angle should depend on the mass of the star m and on the distance of the ray of light from the star. Let's define r to be the distance of the closest approach of the ray to the star as shown in the sketch above. If we proceed with our dimensional analysis at this point, we will find that there is no dimensionally consistent form for the equation expressing θ in terms of m and r, just as we found that the period of oscillation of a pendulum could not be expressed in terms of m and l alone. So, again there must be a dimensional constant that we need to include. Since the deflection of light is due to gravity, we might suspect that the angle depends on the gravitational constant G. What are the dimensions of G? Recall that the equation for the gravitational force between two massive objects of mass m_1 and m_2 a distance r apart is given by $F = \frac{Gm_1m_2}{r^2}$. Therefore,

$$[G] = \left[\frac{Fr^2}{m_1m_2}\right] = \mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}$$

where we used $[F] = \mathrm{MLT}^{-2}$. Now let's try to find the equation for θ:

$$\theta = km^\alpha r^\beta G^\gamma$$

The equation relating dimensions is

$$\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0 = \mathrm{M}^\alpha\mathrm{L}^\beta(\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2})^\gamma$$

Equating the exponents of the basic dimensions M, L, and T, we get

$$\begin{gathered}\text{Exponents of } M \to 0 = \alpha - \gamma,\\ \text{Exponents of } L \to 0 = \beta + 3\gamma,\\ \text{Exponents of } T \to 0 = -2\gamma.\end{gathered}$$

But the last equation gives us $\gamma = 0$, the second one gives us $\beta = 0$ and the first one gives $\alpha = 0$! So, we must still be missing a physical variable or a dimensional constant. Which dimensional constant is most likely to be relevant for the case of the bending of light by gravity? How about the speed of light, c? Let's try it:

$$\theta = km^\alpha r^\beta G^\gamma c^\delta$$

The equation relating dimensions is now

$$\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0 = \mathrm{M}^\alpha \mathrm{L}^\beta (\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2})^\gamma (\mathrm{L}\mathrm{T}^{-1})^\delta$$

Equating the exponents of the basic dimensions M, L and T, we get

$$\begin{gathered}
\text{Exponents of } M \to 0 = \alpha - \gamma, \\
\text{Exponents of } L \to 0 = \beta + 3\gamma + \delta, \\
\text{Exponents of } T \to 0 = -2\gamma - \delta.
\end{gathered}$$

So now we have three equations in four unknowns. The four exponents α, β, γ, and δ are constrained but are not uniquely determined. Each of the three equations involves γ, so let's express the other three exponents in terms of γ. From the first equation, $\alpha = \gamma$. From the last equation, $\delta = -2\gamma$. And from the second equation, $\beta = -\delta - 3\gamma = 2\gamma - 3\gamma = -\gamma$. Therefore, the equation for the bending angle is of the form

$$\theta = km^\gamma r^{-\gamma} G^\gamma c^{-2\gamma} = k\left(\frac{mG}{rc^2}\right)^\gamma$$

Actually, there could be more than one term in the equation for θ, each with a different value of the exponent γ and the constant k, but each term must have the above form. In fact, there could be an infinite number of terms (an infinite series), in which case the right-hand side might be a function of $\frac{mG}{rC^2}$ that can be represented as a series expansion. So, we have not uniquely determined the form of the equation for θ but we can already draw some conclusions from the above equation. For example, we can see that the bending angle depends on the ratio m/r; if m and r are both changed by the same factor, the bending angle will be the same.

We can go further and restrict γ by using physical intuition. First, we expect that θ approaches zero as m becomes very small or as r becomes very large. If γ were negative, then θ would approach infinity as m became small or r became large. Therefore, γ must be a positive exponent: $\gamma > 0$.

To further restrict γ, we can try to apply physical intuition to the derivative of θ with respect to m or r. Since the ratio m/r appears in the equation for θ, let's consider the derivative with respect to $x = \frac{mG}{rc^2}$:

$$\frac{d\theta}{dx} = \gamma k x^{\gamma-1}, \quad \gamma > 0$$

Our physical intuition might tell us that in the limit of $x = \frac{mG}{rc^2}$ becoming very small, the change in θ with respect to a change in m/r should become small, but should not vanish. Therefore, we want the exponent of $\frac{mG}{rc^2}$ to be zero in the equation for $\frac{d\theta}{dx}$. So γ must equal 1 and the equation for θ, at least for small values of the dimensionless combination of variables $\frac{mG}{rc^2}$, is

$$\theta = k\frac{mG}{rc^2}$$

I admit that this last argument is a bit of a stretch...
What about the dimensionless constant k? A survey of all the equations that you will learn in the Introductory Physics sequence will convince you that the dimensionless constants in physical equations are always of order 1. And the equation we just derived is no exception. It turns out that $\theta = k\frac{mG}{rc^2}$ with $k = 4$.

2 Gravitational Focusing

In most derivation, we accounted only for physical collisions, ignoring the fact that particles can also "collide" gravitationally. That is, their trajectories can be significantly deflected by gravitational attraction, approximating a physical collision. This is called gravitational focusing. Here we will derive a cross-section that accounts for this effect.

a. Consider two identical particles (labeled 1 and 2) of mass M and radius R. 1 passes by 2 with impact parameter b and velocity v_0. Estimate the change in velocity, Δv, of 1 as it is deflected from its original trajectory due to the gravitational pull of 2.

Assuming that the effect on the trajectory of particle 1 occurs at a distance of $2b$, which should be familiar, we have a time of action given by

$$\Delta t = \frac{2b}{v_0}$$

where we have simply used dimensional analysis or the fact that velocity is the distance over time. The acceleration during this time interval is approximately constant with a value of

$$a = \frac{GM}{b^2}$$

so that the change in velocity is given by

$$\Delta v = a\Delta t$$

$$\boxed{\Delta v = \frac{2GM}{bv_0}}$$

We can consider the time of interaction to be $\Delta t = 2b/c$ where we've made an OOM assumption about how close the particle must be to really feel the defection due to gravity. Therefore, the change in its velocity is given by

$$\Delta v = \frac{2GM}{bc}$$

b. Gravitational focusing becomes significant when the change in velocity is on order-of-magnitude equal to the initial velocity (i.e. effect on order unity, $\Delta v \approx v_0$). How large does the impact parameter b need to be for this to occur?

$$\Delta v = v_0$$

$$\frac{2GM}{bv_0} = v_0 \implies \boxed{b = \frac{2GM}{v_0^2}}$$

Note. It is sometimes stated that the deflection of light by gravity was first predicted with general relativity. This is something of an oversimplification. In Newtonian theory, you could imagine light as made of particles that traveled at the speed c, and such particles would feel gravity just like anything else. If they passed at an impact parameter b from a point mass M, they would experience a total deflection angle in radians of

The defection angle, with trig and a small angle approximation, is $\alpha = v/c$, we have

$$\alpha = \frac{2GM}{bc^2}$$

Using full general relativity, we'd arrive at a similar answer off by a factor of 2,

$$\alpha = \frac{4GM}{bc^2} \tag{1}$$

Bending of Light by the Sun

Before we calculate the angle by which light is bent when passing close to the sun, let's review the historical significance of this phenomenon. One of the first tests of Einstein's theory of general relativity was a measurement of the bending of starlight as it passed by the edge of the sun. The difficulty with this measurement is that it is normally impossible to see a star when it is in line with the edge of the sun. It was necessary to wait for a solar eclipse so that the measurement could be made since we can see the stars during a solar eclipse! After Einstein first presented his ideas on light deflection, an expedition to the location of the next solar eclipse was prevented by war. A few years later, in 1919, another total solar eclipse occurred and his theory was tested and verified. Now let's calculate the bending angle. For a light ray passing near the edge of the sun, $r_\odot$ is the radius of the sun ($r_\odot = 6.96 \times 10^8$ m) and $m_\odot$ is the mass of the sun ($m_\odot = 1.99 \times 10^{30}$ kg). The bending angle is

$$\begin{aligned}
\alpha &= \frac{4mG}{rc^2} \\
&= \frac{4 \times 1.99 \times 10^{30} \times 6.67 \times 10^{-11}\ \mathrm{m}^3/(\mathrm{kg} \cdot \mathrm{s}^2)}{6.96 \times 10^8\ m \times (3.0 \times 10^8\ \mathrm{m/s})^2} \\
&= 8.5 \times 10^{-6} \text{ radians} \\
&= 8.5 \text{ microradians}
\end{aligned}$$

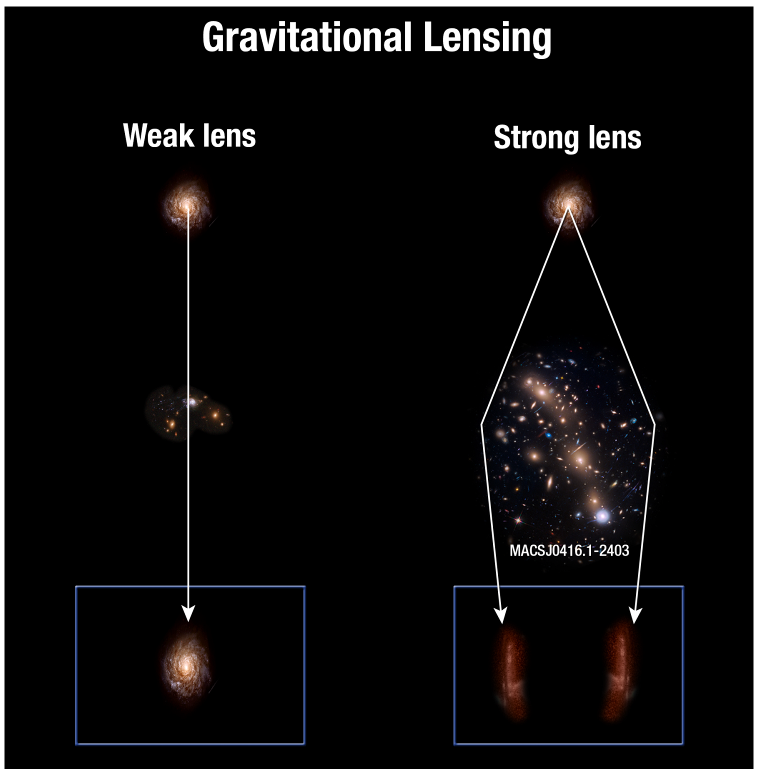
We can convert this into degrees by multiplying by $180°/\pi$ to get $\beta = 0.0005°$ or 5/10,000 of a degree. To get a sense of the size of this angle, let's compare it to the angular diameter of the sun $\Delta\theta_\odot$:

$$\begin{aligned}
\Delta\theta_\odot &= \frac{\text{diameter of the sun}}{\text{distance from earth to sun}} \\
&= \frac{2 \times 6.96 \times 10^8\ \mathrm{m}}{1.49 \times 10^{11}\ \mathrm{m}} \\
&= 0.0093 \text{ radians} \times \frac{180°}{\pi} \\
&= 0.54°
\end{aligned}$$

Therefore, the angular diameter of the sun is about half a degree. So the angular shift of the starlight passing near the sun's edge is about 1/1000 of the angular diameter of the sun itself. This shift would certainly not be apparent to the naked eye. Precise measurements of images in a telescope are required to measure this shift relative to the angular position of other stars.

3 Gravitational Lensing

Gravitational lensing is a consequence of one of the most famous predictions of Einstein's General Relativity—the idea that light is bent in a gravitational field. Indeed, the first calculation showing that gravitational bending of starlight could act as a lens was produced by Einstein himself, although he did somewhat pessimistically conclude that "there is no great chance of observing this phenomenon". The first gravitationally lensed quasar, Q0957+561, was discovered by Walsh et al. in 1979.



There are three main forms of gravitational lensing:

1. In strong lensing, the lens is a large mass, the geometry is favorable, and the deflection is comparatively large. The observer sees two or more separate images of the source.
2. In weak lensing, the lens is a large mass, but the geometry is less favorable. The image of the source is mildly distorted, with a tendency to smear into an arc centered on the lens center: an effect known as shear. This means that the alignment of the background objects appears non-random, so shear can be measured statistically even if the distortions of individual objects are too small to be identified directly
3. In microlensing, the lens is a small mass (usually a star), so that although the geometry is extremely favorable—source, lens, and observer in a straight line—the deflection, distortion,

and multiple images caused by lensing cannot be resolved. Instead, the image of the source appears to brighten for the duration of the lensing event (since the source, lens, and observer all have relatively proper motions, the alignment that creates the microlens is temporary).

Microlensing has been used to search for astrophysical dark objects—MACHOs—in the Galactic halo. Strong and weak lensing can be used to map the mass distributions of clusters of galaxies; over larger areas weak lensing can also map large-scale structures.

It may help to draw a diagram (in fact, it's probably necessary). Let's start with the simplest form of the lensing effect:

The Einstein ring is the simplest and most symmetrical effect of gravitational lens, generated when the source, the lens, and the observer are perfectly aligned. As its name suggests, the Einstein ring corresponds to the situation where the observer detects the light deviated by the lens as if it were a circumference or ring. Figure 1 illustrates this situation, where the Einstein ring is represented by a dashed yellow line circumference, centered on the source and being looked at from the side. It can be observed that the rays of light that travel directly to the observer (continuous red lines) subtend an angle θ_E, known as Einstein angle. Which is the characteristic magnitude of the Einstein ring. It also can be appreciated the deviation angle α calculated in the previous section. Again, this angle is very accentuated in the figure, because, in practice, α and θ_E are on the order of microseconds of arc.

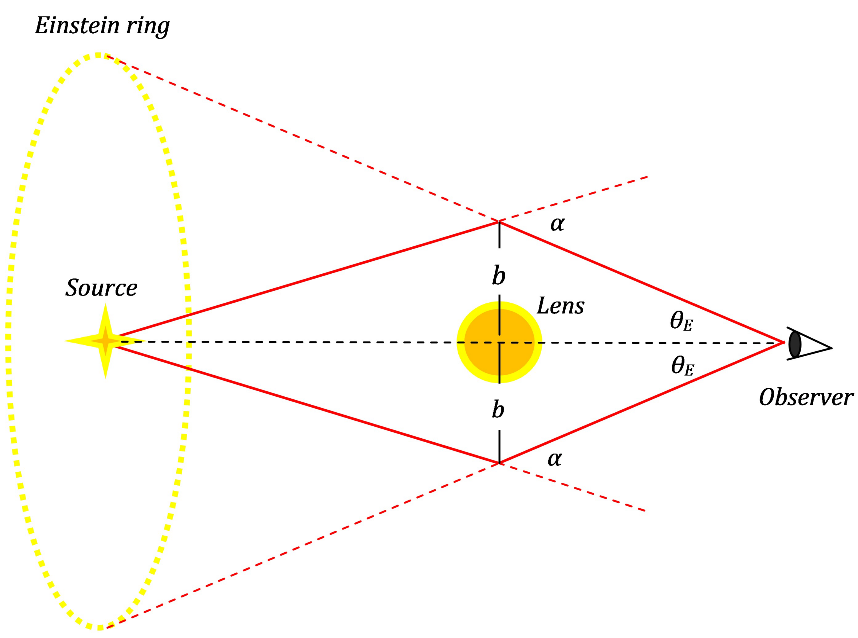


Figure 1: A source, a lens and an observer are perfectly aligned, which creates an Einstein ring. The drawing simplifies the phenomenon by showing only two of the light rays that form the ring (continuous red lines), that subtend an Einstein's angle θ_E.

Figure 5 shows a version of Figure 1 which is more useful to calculate θ_E. Here are shown the distance between the source and the lens, d_{LS}, the distance between the lens and the observer, d_L, the distance between the source and the observer, d_S, and the angle β between the undisturbed ray of light and the straight line between the source and the lens. To calculate θ_E, let us note that by the exterior angle theorem we must have that:

$$\alpha = \theta_E + \beta \tag{2}$$

If M is the mass of the lens, combining equations (1) and (2) we get:

$$\alpha = \frac{4GM}{c^2 b} = \theta_E + \beta \tag{3}$$

Because α and θ_E are so small, when expressing them in radians we can use the approximations:

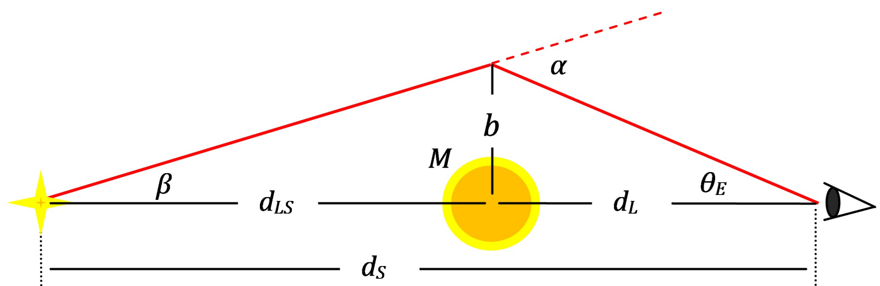


Figure 2: Reduced version of the Figure 1, where we show Einstein angle, θ_E, the distance between the source and the lens, d_{LS}, the distance between the lens and the observer, d_L, the distance between the source and the observer, d_S, and the angle β between the undisturbed ray of light and the straight line between the source and the lens.

$$\theta_E \approx \tan\theta_E = \frac{b}{d_L}, \qquad \beta \approx \tan\beta = \frac{b}{d_{LS}}$$

Introducing these expressions into equation 3

$$\frac{4GM}{cb^2} = \frac{b}{d_L} + \frac{b}{d_{LS}}$$

Solving for b:

$$b = \left(\frac{4GM}{c^2}\frac{d_L d_{LS}}{d_L + d_{LS}}\right)^{1/2} = \left(\frac{4GM}{c^2}\frac{d_L d_{LS}}{d_S}\right)^{1/2}$$

To express this result in *rad*, we divide member for member by the distance between the observer and the lens, obtaining Einstein angle:

$$\theta_E = \frac{b}{d_L} = \left(\frac{4GM}{c^2}\frac{d_{LS}}{d_L d_S}\right)^{1/2}$$

Now MORE complicated geometry! FROM IOAA 2007

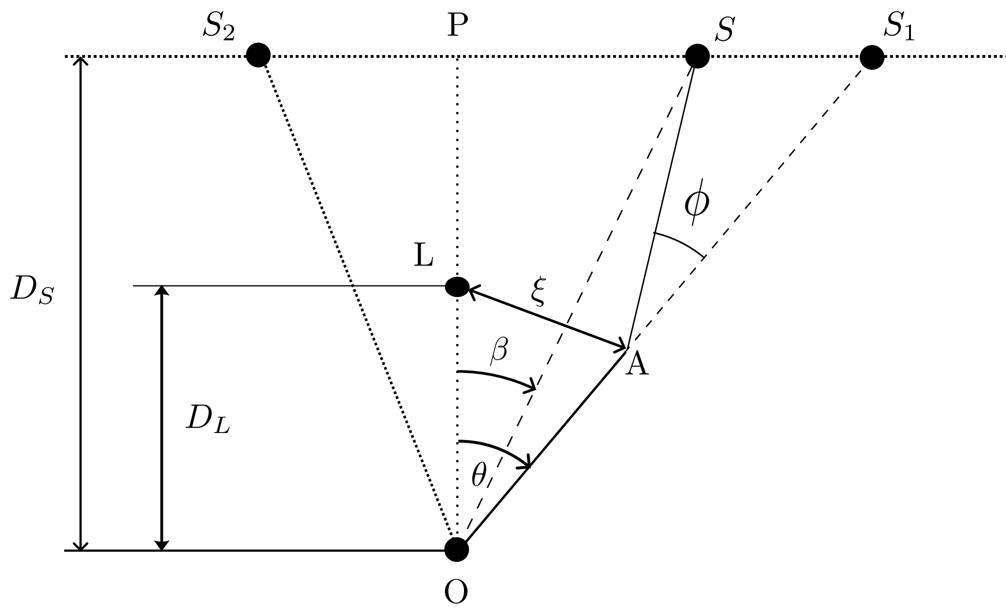


Figure 3: Geometric Model of Gravitational Lensing

$$
\begin{aligned}
SA &\approx PL = D_S - D_L \\
PS_1 &= PS + SS_1 \\
\implies D_S\theta &= D_S\beta + (SA)\cdot\phi \\
&= D_S\beta + (D_S - D_L)\cdot\phi \\
\theta &= \beta + \frac{D_S - D_L}{D_S}\cdot\phi \\
\phi &= \frac{4GM}{\xi c^2}
\end{aligned}
$$

From Newtonian gravity, a photon moving past a mass, M, with an impact parameter, ξ will undergo an acceleration perpendicular to the direction of its motion. Under the Born approximation,

$$\frac{dv_\perp}{dt} = \frac{GM}{r^2}\sin\theta$$

Integrated over the path, the total deflection is

$$
\begin{aligned}
v_\perp &= \frac{GM}{c}\int_{-\infty}^{\infty}\frac{1}{x^2+\xi^2}\cdot\frac{\xi}{\sqrt{\xi^2+x^2}}dx \\
&= \frac{GM\xi}{c}\int_{-\infty}^{\infty}(x^2+\xi^2)^{-\frac{3}{2}}dx \\
&= \frac{GM\xi}{c}\cdot\frac{2}{\xi^2} = \frac{2GM}{\xi c}
\end{aligned}
$$

So the Newtonian deflection is

$$\phi = \frac{v_\perp}{c} = \frac{2GM}{\xi^2 c}$$

But Einstein showed that when you consider space-time together, the Einstein angle is twice that found in Newtonian physics

$$\phi = \frac{v_\perp}{c} = \frac{4GM}{\xi^2 c}$$

Note that x is much smaller than the distance to the source or the lens. Hence one generally makes a "thin lens" approximation and assumes that the entire deflection occurs instantaneously. One never models 3-D mass distributions; 2-D models will do.

Now,

$$\begin{aligned}
\implies \theta - \beta &= \frac{D_S - D_L}{D_S} \cdot \frac{4GM}{\xi c^2} \\
\theta - \beta &= \frac{D_S - D_L}{D_S} \cdot \frac{4GM}{D_L \theta \cdot c^2} \\
\theta^2 - \beta\theta &= \frac{D_S - D_L}{D_S D_L} \cdot \frac{4GM}{c^2} \qquad (4)
\end{aligned}$$

In figure 3, for an isolated point source S, there will be two images (S_1 and S_2) formed by the gravitational lens. For a perfect alignment in which $\beta = 0$, we have $\theta = \pm\theta_E$, where

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{(D_L - D_S)}{D_S D_L}} \qquad (5)$$

Here, we need to define a few things for simplification. We define a distance

$$D \equiv \frac{D_{L-S}}{D_L D_S}$$

and an Einstein ring radius

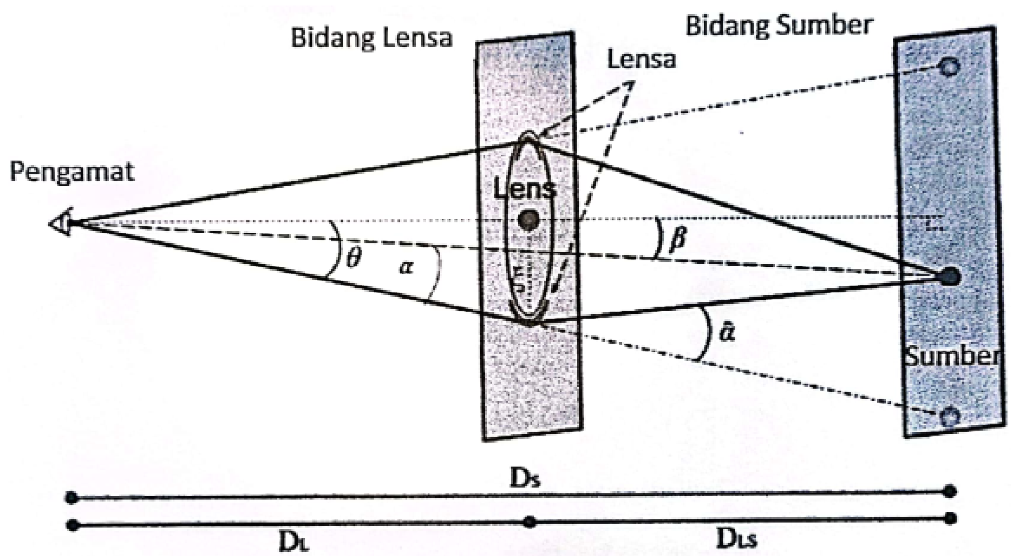
$$\boxed{\theta_E = \left(\frac{4GM}{Dc^2}\right)^{1/2}}$$

3.1 Problem 1:

The gravitational lens is one of the confirmations of Einstein's General Theory of Relativity (TRU) which explains the relationship between the curvature of space-time and mass. The existence of mass between the observer and the light source will cause the curvature of space-time, then the curvature makes the photon trajectory from the background light source can be focused to the observer. As an illustration, see the image below which is a geometric scheme of the formation of the gravitational lens phenomenon. D_L, D_S, and D_{LS}, respectively, are the distances from the observer to the lens, the observer to the source, and the lens to the source. In addition, the angles measured from the line of sight are defined, namely the line connecting the observer to the center of the lens forward to the source plane, β the angular position of the unlensed source, θ the angular position of the formed image, α the deflection angle, and $\widehat{\alpha}$ is the reduced deflection angle.

a. Given the reduced deflection angle of the TRU is

$$\hat{\alpha} = \frac{4GM_L}{c^2 \xi}$$

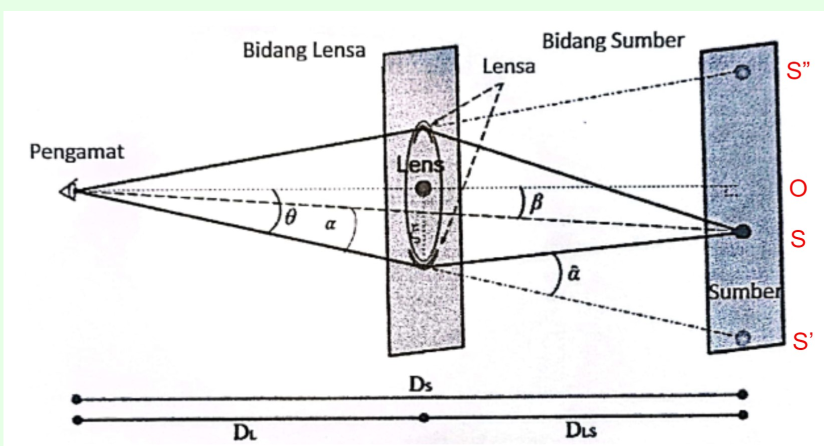


with G the gravitational constant, c the speed of light, M_L the mass of the lens object, and ξ the *impact* parameter, derive the gravitational lens equation in the form of a relationship between β as a function of θ (the observed angle) using the geometric relationship illustrated by the figure above!

b. Assuming that a dwarf star of mass $0.5M_\odot$ located at a distance of 5.8 kpc from an observer on Earth has lensed a background star around the center of the Milky Way galaxy. Calculate the angular radius formed when the background star is exactly in the line of sight (called the Einstein radius)! Express it in milliarcseconds!

c. Can the angular size obtained from question 8b be observed using the *Hubble Space Telescope* (HST) with an angular resolution of 0.03 arcseconds?

Please note that the image provided is not to scale. The first key to answering this question is to realize that the angles given in the sketch $(\theta, \alpha, \beta, \widehat{\alpha})$ are very small angles, because the distances (D_L, D_S, D_{LS}) depicted are actually very large distances (cosmological scale; distances between galaxies).



Since the angle is very small, we can use the small angle approximation, $\sin\theta \approx \tan\theta \approx \theta_{\text{rad}}$, so if we use radians we can write,

$$\xi = \theta \cdot D_L$$
$$OS = \beta \cdot D_S$$
$$SS' = \widehat{\alpha} \cdot D_{LS}$$
$$SS' = \alpha \cdot D_S$$

and we also have connection,

$$D_S = D_L + D_{LS}$$
$$OS' = OS + SS'$$
$$\theta = \beta + \alpha$$

a. We can derive β as a function of θ

$$\begin{aligned}
\beta &= \theta - \alpha \\
\beta &= \theta - \frac{SS'}{D_S} \\
\beta &= \theta - \frac{\widehat{\alpha} \cdot D_{LS}}{D_S} \\
\beta &= \theta - \frac{D_{LS}}{D_S}\frac{4GM_L}{c^2\xi} \\
\beta &= \theta - \frac{D_{LS}}{D_S}\frac{4GM_L}{c^2 \cdot \theta \cdot D_L} \\
\beta &= \theta - \frac{D_{LS}}{D_S \cdot D_L}\frac{4GM_L}{c^2\theta}
\end{aligned}$$

b. We have derived $\beta(\theta)$ in the previous problem, so for the case where the source is directly behind the lens, $\beta = 0$, we can derive a formula called the *Einstein radius/Einstein ring*,

$$\begin{aligned}
\theta &= \frac{D_{LS}}{D_S \cdot D_L}\frac{4GM_L}{c^2\theta} \\
\theta^2 &= \frac{D_{LS}}{D_S \cdot D_L}\frac{4GM_L}{c^2} \\
\theta_E &= \sqrt{\frac{D_{LS}}{D_S \cdot D_L}\frac{4GM_L}{c^2}}
\end{aligned}$$

For a lens with mass $M_L = 0.5M_\odot$, lens distance $D_L = 5.8$ kpc, source distance (distance from the center of the galaxy) $D_S = 8$ kpc, and lens-to-source distance $D_{LS} = 8 - 5.8 = 2.2$ kpc, we can calculate the Einstein radius,

$$\theta_E = 2.13 \times 10^{-9} \text{ rad} = 0.44 \text{ milliarcseconds.}$$

c. The Einstein radius of the object (0.44 *mas*) is much smaller than the angular resolution of the HST (30 *mas*), so it cannot be separated.

3.2 Problem 2: Microlensing

Note. Originally conceived as a way to discover Massive Compact Halo Objects (MACHOS) if stellar-mass black holes or other massive dark objects comprise the dark matter in galaxies, microlensing is so named because of the small size of the resulting Einstein rings. Such microlensing of background stars is detected in wide-area, high-cadence surveys toward the bulge of the Galaxy and the Magellanic Clouds, but it is inferred that the lensing object is also a normal star (or a brown dwarf). Because of the relative motion between the lens and the source, a microlensing event has a characteristic and symmetric curve of magnification that is easy to recognize and discriminate from those of other (intrinsic) variable stars.

The MACHO project imaged stars in the Large Magellanic Cloud looking for evidence of microlensing from MACHOs in the Milky Way. The MACHO project lasted for $t_{tot} = 6$ years and they re-imaged the same areas of sky about once every $\Delta t = 2$ days on average. The MACHO project did not find any MACHOs. Over what mass range did they exclude the possibility of MACHOs? Assume that each MACHO is $D_{\rm MACHO} = 20$ kpc from us. Assume that every star in the LMC is $D_* = 50$ kpc away and has a relative velocity of $v_{\rm rel} = 50$ km/s.

[Hint: compare the microlensing timescale to the length of the MACHO project and the length over which MACHO returned to the same area of sky.]

The MACHO project could have only found MACHOs with microlensing timescales between Δt and $t_{\rm tot}$: any shorter and they would have happened too quickly to be seen; any longer and they wouldn't have noticed an increase in flux. So now we need to calculate the masses corresponding to these microlensing timescales.

The microlensing timescale Δt is the time that it takes the lens to move by an Einstein radius:

$$\Delta t = \frac{\theta_E D_L}{v_{\rm rel}}$$

where D_L is the distance to the lens and $v_{\rm rel}$ is the relative velocity of the background star to the lens. Using the definition of the Einstein radius we can rewrite the microlensing timescale in terms of the mass:

$$\Delta t = \sqrt{\frac{4GM}{c^2 D}}\frac{D_{\rm macho}}{v_{\rm rel}}$$

$$D = \frac{D_{\rm macho} D_{ast}}{D_{\rm macho} - D_{ast}}$$

where I have plugged in the distance to the MACHO as the lens distance and the distance to the background star as the source distance, respectively. Solving for the mass gives

$$M = \frac{\Delta t^2 v_{\rm rel}^2}{D_{\rm macho}^2}\frac{c^2 D}{4G}$$

Plugging in Δt, I find the minimum mass to be $3\times10^{-5} M_\odot$; plugging in t_{tot}, I find a maximum mass of $40 M_\odot$. This means that the MACHO survey excluded dark matter particle masses between $3\times10^{-5} M_\odot$ and $40 M_\odot$. This is interesting because there aren't any other strong constraints on the masses of dark matter particles until they exceed $10^3 M_\odot$.

This has led people to propose that dark matter may be composed of primordial black holes in the mass range $40 < M < 10^3 M_\odot$ (although other ideas for the origin of dark matter are much more popular).

From the equations (4) and (5) above we arrive at LENS equation,

$$
\begin{aligned}
\theta^2 - \beta - \theta_E^2 &= 0 \\
\implies \theta_{1,2} &= \frac{\beta \pm \sqrt{\beta^2 + 4\theta_E^2}}{2} \\
\theta_1 &= \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \theta_E^2} \\
\theta_2 &= \frac{\beta}{2} - \sqrt{\left(\frac{\beta}{2}\right)^2 + \theta_E^2}
\end{aligned} \tag{6}
$$

This corresponds to one image above the source (positive root) and one below (negative root). In general, no matter the complexity, there are always an even number of images for a non-transparent lens, and an odd-number for a transparent lens. It may help to be familiar with some of the limiting cases:

- $\beta = 0$: We have $\theta = \theta_E$. Therefore, we see a full Einstein ring centered on the lensing object.
- $\beta \ll \theta_E$: In this case, we see two images. They lie on a line that runs vertically perpendicular to our line of sight. One image is just above (outside of) the (unlit) Einstein ring, the other is just inside of the ring but below the source. Mathematically, $\theta_1 \approx \theta_E + \beta/2$ and $\theta_2 \approx -\theta_E + \beta/2$.
- $\beta = \theta_E$: In this case, $\theta_1 \approx 3\theta_E/2$ and $\theta_2 \approx -\theta_E/2$. This means there are again two images, one above and one below the source, with the one above outside of the ring and the one below inside of it.
- $\beta \gg \theta_E$: Lensing is insignificant. For strong lensing, β must be less than or comparable to θ_E.

Let us introduce, "impact parameter", the angular separation between lens and source in units of the Einstein radius: $\eta = \dfrac{\beta}{\theta_E}$. In Fig 3, $PS \equiv \eta$. Dividing the above equation (6) by β

$$
\begin{aligned}
\theta_{1,2} &= \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \theta_E^2} \\
\frac{\theta_{1,2}}{\beta} &= \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\theta_E}{\beta}\right)^2} \\
\frac{\theta_{1,2}}{\beta} &= \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{\eta^2}} \\
&= \frac{1}{2}\left(1 \pm \frac{\sqrt{\eta^2 + 4}}{\eta}\right)
\end{aligned}
$$

Again,

$$\theta^2 - \beta\theta - \theta_E^2 = 0$$

$$
\begin{aligned}
(\theta+\Delta\theta)^2-(\beta+\Delta\beta)(\theta+\Delta\theta)-\theta_E^2&=0\\
(\theta^2-\beta\theta-\theta_E^2)+2\theta\Delta\theta-\beta\Delta\theta-\theta\Delta\beta&=0\\
(2\theta-\beta)\Delta\theta-\theta\Delta\beta&=0
\end{aligned}
$$

Thus, we get,

$$
\begin{aligned}
\frac{\Delta\theta}{\Delta\beta}&=\frac{\theta}{2\theta-\beta}\\
\left[\frac{\Delta\theta}{\Delta\beta}\right]_{\theta=\theta_{1,2}}&=\frac{\theta_{1,2}}{2\theta_{1,2}-\beta}\\
&=\frac{\theta_{1,2}/\beta}{2\theta_{1,2}/\beta-1}\\
&=\frac{\frac{1}{2}\left(1\pm\frac{\eta^2+4}{\eta}\right)}{1-\frac{1}{2}\left(1\pm\frac{\eta^2+4}{\eta}\right)}\\
&=2\left(\frac{1}{2}\left(1\pm\frac{\eta^2+4}{\eta}\right)-1\right)\\
&=\frac{\eta\pm\sqrt{\eta^2+4}}{2(\eta\pm\sqrt{\eta^2+4}-\eta)}\\
&=\frac{1}{2}\left(1\pm\frac{\eta}{\sqrt{\eta^2+4}}\right)
\end{aligned}
$$

Simply differentiating,

$$
\frac{d\theta_{1,2}}{d\beta}=\frac{1}{2}\pm\frac{1}{2\sqrt{1+\frac{4}{\eta^2}}}
$$

The magnification of an image is defined by the ratio between the solid angles of the image and the source, since the surface brightness is conserved.

$$
\textbf{lensing magnification, } \mathcal{M}=\frac{\text{Angular size of the image}}{\text{Angular size of the source}}
$$

The relationship between θ and β implies that one image of the lens images must be magnified. Consider how the angles map:
Without the lens, photons would fall within an area $dA=\beta\Delta\vartheta\Delta\beta$. With the lens, these same photons fall within an area $dA'=\theta\Delta\vartheta\Delta\theta$. So the light is "focused" from area dA to area dA', and the ratio of these two areas is simply

$$
\mathcal{M}=\frac{dA'}{dA}=\left|\frac{\theta\Delta\vartheta\Delta\theta}{\beta\Delta\vartheta\Delta\beta}\right|=\left|\frac{\theta\Delta\theta}{\beta\Delta\beta}\right|=\left|\frac{\theta}{\beta}\frac{d\theta}{d\beta}\right|
$$

For a circularly symmetric lens,

$$
\begin{aligned}
\mathcal{M}_{1,2}&=\frac{\theta_{1,2}}{\beta}\cdot\frac{d\theta_{1,2}}{d\beta}\\
&=\frac{1}{2}\left(1\pm\frac{\sqrt{\eta^2+4}}{\eta}\right)\times\left(\frac{1}{2}\pm\frac{1}{2\sqrt{1+\frac{4}{\eta^2}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2} \pm \frac{\sqrt{1+\frac{4}{\eta^2}}}{2}\right) \times \left(\frac{1}{2} \pm \frac{1}{2\sqrt{1+\frac{4}{\eta^2}}}\right) \\
&= \frac{\eta^2+2}{2\eta\sqrt{\eta^2+4}} \pm \frac{1}{2}
\end{aligned}
$$

The magnification of one image (the one inside the Einstein radius) is negative. This means it has negative parity: it is mirror-inverted. The sum of the absolute values of the two image magnifications is the measurable total magnification:

$$
\boxed{\mathcal{M} = |\mathcal{M}_1| + |\mathcal{M}_2| = \frac{\eta^2+2}{\eta\sqrt{\eta^2+4}}}
$$

Note that this value is (always) larger than one! The difference between the two image magnifications are unity:

$$
\mathcal{M}_+ - \mathcal{M}_- = 1
$$

> **Note.** Because each problem and diagram addresses different notations and I am too lazy to unify all the notations, you may encounter different notations for the same thing in this lecture.

3.3 Problem 3: Magnification Factors in Gravitational Lensing

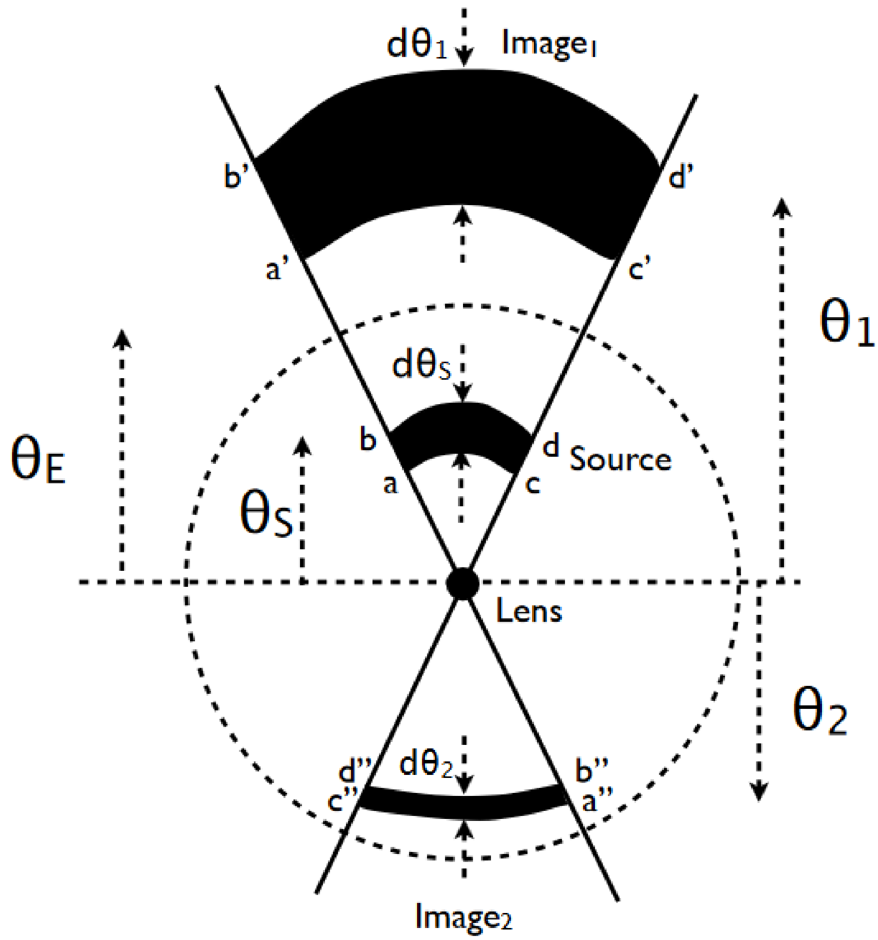


Figure 4: Gravitational lensing. Deviations from perfectly circular arcs are due solely to the instructor's ineptitude with Keynote.

Figure 4 was sketched in class. The figure shows a source lensed by a mass M. The source is in the shape of a thin circular arc of (angular) radius θ_s and radial width $d\theta_s \ll \theta_s$.

The observer does not see the source, but sees instead its two lensed images, located above and below the source. Each lensed image, like the parent source, is also a thin circular arc. Image$_1$ has radius θ_1 and radial width $d\theta_1$, and *image*$_2$ has radius θ_2 and radial width $d\theta_2$.

Note that each point $\{a, b, c, d\}$ on the source maps to a corresponding point on *image*$_1$ $\{a', b', c', d'\}$ and *image*$_2$ $\{a'', b'', c'', d''\}$.

This problem works out the area of *image*$_1$, A_1, and its relation to the source area, A_s —and likewise for *image*$_2$. The total area of the two images, $A_1 + A_2$, divided by the area of the source, A_s, gives the *total magnification* $\mathcal{M}$ as a function of θ_s (how displaced the source is relative to the lens).

$$\mathcal{M} = \frac{A_1 + A_2}{A_s}$$

Please recall that the Einstein ring is NOT actually illuminated unless $\theta_s = 0$ (the source is directly behind the lens). Nevertheless, we can always draw the location of the Einstein ring for reference, and this is done in Figure 4 — see the dashed circle. **For this entire problem, treat $\theta_E > 0$, the angular radius of the Einstein ring, as a given known quantity.**

(a) Consider $image_1$. It is magnified azimuthally compared to the source by a factor $\mathcal{M}_{az,1}$. In other words, the arclength b'–d' is larger than the arclength $b - d$ by a factor $\mathcal{M}_{az,1}$ (the same magnification factor relates arclengths a'–c' and $a - c$).

Write down $\boldsymbol{\mathcal{M}_{az,1}}$ in terms of $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_s}$.

$$b - d = \theta_s \cdot \phi_c$$
$$b' - d' = \theta_1 \cdot \phi_c$$

Azimuthal magnification,

$$\mathcal{M}_{az,1} = \frac{b' - d'}{b - d} = \frac{\theta_1 \cdot \phi_c}{\theta_s \cdot \phi_c}$$

$$\boxed{\mathcal{M}_{az,1} = \frac{\theta_1}{\theta_s}}$$

(b) Rewrite the lens equation (derived previously)

$$\theta^2 - \theta_s\theta - \theta_E^2 = 0 \tag{7}$$

as

$$\theta_s = \theta - \frac{\theta_E^2}{\theta} \tag{8}$$

Here θ can refer either to θ_1 or θ_2.
Combine 7 with 8 to **write down $\boldsymbol{\mathcal{M}_{az,1}}$ in terms of $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_E}$.**

$$\begin{aligned}\mathcal{M}_{az,1} &= \frac{\theta_1}{\theta_s} \\ &= \frac{\theta_1}{\theta_1 - \frac{\theta_E^2}{\theta_1}} \qquad [\because \theta \text{ can be } \theta_1, \theta_2] \\ &= \frac{\theta_1^2}{\theta_1^2 - \theta_E^2}\end{aligned}$$

(c) Now consider the radial magnification of $image_1$. The radial segment a'–b' is larger (or possibly smaller — you will find out) than the radial segment a–b by a factor $\mathcal{M}_{rad,1}$.

Write down $\mathcal{M}_{rad,1}$ in terms of $d\theta_1$ and $d\theta_s$. Then take the derivative of the rewritten lens equation (8) to **solve for $\boldsymbol{\mathcal{M}_{rad,1}}$ in terms of $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_E}$.**

$a' - b' = d\theta_1$ and $a - b = d\theta_s$

$$\therefore \mathcal{M}_{rad,1} = \frac{a' - b'}{a - b} = \frac{d\theta_1}{d\theta_s}$$

From equation (8): $\theta_s = \theta_1 - \frac{\theta_E^2}{\theta_1}$. Now differentiating,

$$\begin{aligned}
\implies \frac{d\theta_s}{d\theta_1} &= \frac{d}{d\theta_1}\left(\theta_1 - \frac{\theta_E^2}{\theta_1}\right) \\
&= \frac{d}{d\theta_1}\theta_1 - \frac{d}{d\theta_1}\left(\frac{\theta_E^2}{\theta_1}\right) \\
&= 1 + \theta_E^2 \cdot \frac{d}{d\theta_1}\left(\theta_1^{-1}\right) \\
&= 1 - \theta_E^2 \cdot (-\theta_1^{-2}) \\
&= 1 - \left(-\frac{\theta_E^2}{\theta_1^2}\right) \\
&= 1 + \frac{\theta_E^2}{\theta_1^2}
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{d\theta_1}{d\theta_s} = \frac{1}{d\theta_s/d\theta_1} &= \frac{1}{1 + \frac{\theta_E^2}{\theta_1^2}} \\
&= \frac{\theta_1^2}{\theta_1^2 + \theta_E^2}
\end{aligned}$$

$$\boxed{\mathcal{M}_{rad,1} = \frac{\theta_1^2}{\theta_1^2 + \theta_E^2}}$$

(d) The total magnification $\mathcal{M} = \mathcal{M}_{az,1} \cdot \mathcal{M}_{rad,1}$. Show that

$$\mathcal{M}_1 = \left(1 - \frac{\theta_E^4}{\theta_1^4}\right)^{-1} \tag{9}$$

Having shown this is true for $\mathcal{M}_1$, simply replace subscript 1 with subscript 2 to write down the analogous expression for $\mathcal{M}_2$.

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_{az,1} \cdot \mathcal{M}_{rad,1} \\
&= \left(\frac{\theta_1^2}{\theta_1^2 - \theta_E^2}\right) \cdot \left(\frac{\theta_1^2}{\theta_1^2 + \theta_E^2}\right) \\
&= \frac{\theta_1^4}{\theta_1^4 - \theta_E^4} \\
&= \left(\frac{\theta_1^4 - \theta_E^4}{\theta_1^4}\right)^{-1}
\end{aligned}$$

$$\boxed{\mathcal{M}_{1,2} = \left(1 - \frac{\theta_E^4}{\theta_{1,2}^4}\right)^{-1}}$$

(e) Solve the original lens equation (7) for θ_1 and θ_2 in terms of θ_E and θ_S. Prove that $\theta_1 \geq \theta_E$ ($image_1$ is always located outside the Einstein ring) and $\theta > \theta_2 \geq -\theta_E$ ($image_2$ is always located inside the Einstein ring). Thereby also show that $\mathcal{M}_1 \geq 1$ **(*image*$_1$ is always magnified)**.

(Note that $\theta_2 < 0$ and $\mathcal{M}_2 < 0$ which simply means that $image_2$ is flipped relative to the source; see Figure 4. If you get the right answers for $\mathcal{M}_2$ and θ_2, you should find that $|\mathcal{M}_2|$ can be > 1 or < 1 — i.e., $image_2$ can either be magnified or de-magnified depending on θ_s. But you don't have to show this.)

From 7 we can solve quadratic equation, $ax^2 + bx + c = 0$ to find the roots,

$$\begin{aligned}
a &= 1 \\
b &= -\theta_s \\
c &= -\theta_E^2 \\
\implies x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x \equiv \theta &= \frac{\theta_s \pm \sqrt{\theta_s^2 + 4\theta_E^2}}{2} \\
\therefore \theta_1 &= \frac{\theta_s + \sqrt{\theta_s^2 + 4\theta_E^2}}{2}
\end{aligned}$$

θ_1 will be minimum when $\theta_s = 0$; the source is directly behind the lens.

$$\therefore \theta_{1,\text{min}} = \frac{\sqrt{4\theta_E^2}}{2} = \theta_E$$

$$\boxed{\theta_1 \geq \theta_E}$$

Similarly, the minimum value of θ_2 is at $\theta_s = 0$

$$\therefore \theta_{2,\text{min}} = -\frac{\sqrt{4\theta_E^2}}{2} = -\theta_E$$

As the minimum values is negative, the maximum value will be 0,

$$\boxed{0 > \theta_2 \geq -\theta_E}$$

Now,

$$\mathcal{M}_1 = \left(1 - \frac{\theta_E^4}{\theta_1^4}\right)^{-1}, \quad \text{as } \theta_1 \geq \theta_E$$

$$0 \leq 1 - \frac{\theta_E^4}{\theta_1^4} \leq 1$$

$$\left(1 - \frac{\theta_E^4}{\theta_1^4}\right)^{-1} \geq 1$$
$$\therefore \boxed{\mathcal{M}_1 \geq 1}$$

(f) By factoring (9) (it is the difference of two perfect squares), and combining with the original lens equation (7) divided by θ^2, **rewrite $\mathcal{M}_1$ as**

$$\mathcal{M}_1 = \left(\frac{\theta_1}{\theta_S}\right)^2 \left(2\frac{\theta_1}{\theta_S} - 1\right)^{-1}$$

Also, write down the analogous expression for $\mathcal{M}_2$ by simply swapping out sub-script 1 for subscript 2.

Again from magnification,

$$\begin{aligned}\mathcal{M}_1 &= \left[1 - \left(\frac{\theta_E^2}{\theta_1^2}\right)^2\right]^{-1} \\ &= \left[\left(1 + \frac{\theta_E^2}{\theta_1^2}\right)\left(1 - \frac{\theta_E^2}{\theta_1^2}\right)\right]^{-1}\end{aligned}$$

The original lens equation,

$$\begin{aligned}\theta^2 - \theta_s\theta - \theta_E^2 &= 0 \\ 1 - \frac{\theta_s}{\theta_1} - \frac{\theta_E^2}{\theta_1^2} &= 0 \\ \frac{\theta_E^2}{\theta_1^2} &= 1 - \frac{\theta_s}{\theta_1} \qquad (10)\end{aligned}$$

Substituting the value of Eq. 10 into magnification equation,

$$\begin{aligned}\mathcal{M}_1 &= \left[\left(1 + 1 - \frac{\theta_s}{\theta_1}\right)\left(1 - 1 + \frac{\theta_s}{\theta_1}\right)\right]^{-1} \\ &= \left[\left(2 - \frac{\theta_s}{\theta_1}\right)\left(\frac{\theta_s}{\theta_1}\right)\right]^{-1} \\ &= \left[\left(\frac{2\theta_1 - \theta_s}{\theta_1}\right)\left(\frac{\theta_s}{\theta_1}\right)\right]^{-1} \\ &= \left[\frac{2\theta_1\theta_s - \theta_s^2}{\theta_1^2}\right]^{-1} \\ &= \left[\frac{\theta_s^2\left(\frac{2\theta_1}{\theta_s} - 1\right)}{\theta_1^2}\right]^{-1}\end{aligned}$$

So we arrive at,

$$\boxed{\mathcal{M}_{1,2} = \left(\frac{\theta_{1,2}}{\theta_S}\right)^2 \left(2\frac{\theta_{1,2}}{\theta_S} - 1\right)^{-1}}$$

(g) Define $u \equiv \theta_s/\theta_E$. Rewrite your answer for θ_1 and θ_2 in (e) to find that

$$\frac{\theta_{1,2}}{\theta_s} = \frac{1}{2}\left(1 \pm \frac{\sqrt{u^2+4}}{u}\right)$$

(h) **Combine (f) and (g) to find**

$$\mathcal{M}_{1,2} = \frac{1}{2} \pm \frac{u^2+2}{2u\sqrt{u^2+4}}$$

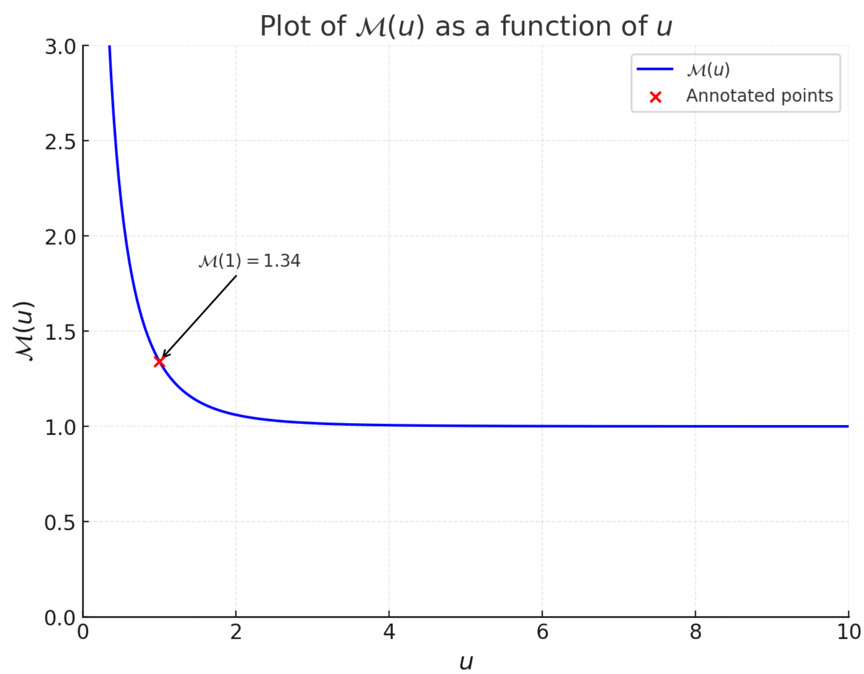
(i) **Use (h) to solve for the TOTAL magnification:**

$$\mathcal{M} = \mathcal{M}_1 + |\mathcal{M}_2| = \frac{u^2+2}{u\sqrt{u^2+4}}$$

> **Note.** Previously derived in this note!

Note the absolute value of $\mathcal{M}_2$ — see the parenthetical remark under part (e) above.

Plot $\mathcal{M}$ as function of u between $u = 0$ and $u = \infty$, and annotate on your plot the precise values of $\mathcal{M}$ for $u = 0.1$ and $u = 1$.



https://cosmo.nyu.edu/blanton/astrophysics/ps11.pdf
https://cosmo.nyu.edu/blanton/astrophysics/ans11.pdf

4 Gravitational Lensing Brazil Problem

(50 points) Gravitational lensing is a phenomenon in which light from a distant source can be deflected by the curvature of spacetime caused by a very massive object, such as a black hole or another compact object along the line of sight between an observer and a distant object. This was first observed during the 1919 solar eclipse in Sobral, Ceará, and on the island of Príncipe, São Tomé and Príncipe, where the observed positions of stars behind the Sun differed from their astrometric positions, following Einstein's earlier predictions.
At the time of the eclipse, the deflection of light was also predicted using Newtonian mechanics.

(a) **(12 points)** Using only Newtonian mechanics, show that the deflection of light can be, with good approximation, given by

$$\theta_N = \frac{2GM}{rc^2},$$

where r is the closest distance between the lensing object, with mass M, and the apparent path of the light, with θ in radians. In this case, the observer and source are not near the massive object.
In the case where the observer, massive object, and source are aligned, the light from the source is deflected, according to general relativity, by an angle (in radians) given by

$$\alpha = \frac{4GM}{r_E c^2}$$

where r is now called the Einstein radius (r_E) and has the same previous meaning.

(b) **(6 points)** Draw a diagram to describe the physical scenario of an ideal lens system (observer, lens, and source in a straight line). Draw the light path and mark the quantities α and r_E. Also, mark the Einstein angular radius θ_E (the angular deflection of the source image as seen from Earth), the distance to the lens D_O, and the distance to the source D_F, measured from Earth.

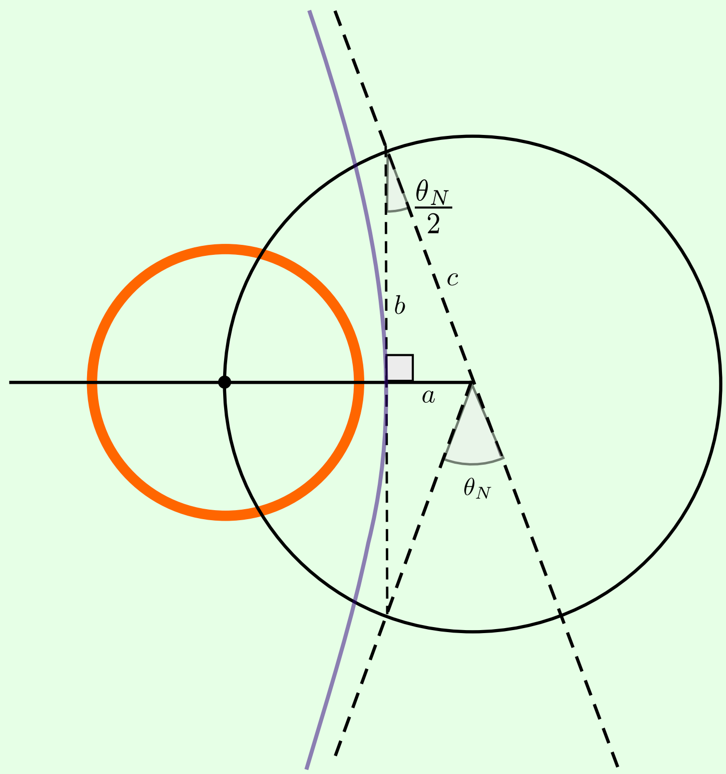
(c) **(6 points)** Sketch the image of the source (for example, a star) as seen by an observer on Earth when the source, lensing object, and observer are aligned.

(d) **(6 points)** Sketch the image of the source (for example, a star) as seen by an observer on Earth in the non-ideal case where the source, lensing object, and observer are not perfectly aligned. Draw a diagram of the system as done in part (b).

Gravitational lensing has been proposed as a method for detecting massive compact halo objects (MACHOs) in our galaxy, which may be a candidate for dark matter. These objects are often dark stellar remnants, such as neutron stars and black holes. As MACHOs and stars orbit the galaxy, there is a chance that a lensing event will occur when a black hole or neutron star passes in front of a background star.

(e) **(12 points)** In the case where the source, lens, and observer are aligned, given a measurement of α and r_E, calculate the Schwarzschild radius of the lensing object in terms of θ_E, D_O, and D_F, assuming the lensing object has a mass of the order of a few solar masses, D_O and D_F are of the order of 10^{18} meters, and D_F is greater than D_O.

(f) **(8 points)** Suppose we have an event in which a lensing object of $3.0M_\odot$ and 2.6×10^{18} meters away from Earth passes in front of a star at a distance of 9.2×10^{18} meters. This occurs in such a way that the ideal configuration takes place. What is the Einstein angular radius θ_E (as seen from Earth)?



$$\sin\left(\frac{\theta_N}{2}\right) = \frac{\alpha}{c} = \frac{1}{e} \implies \theta_N \approx \frac{2}{e}$$

$$e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^2}}$$

$$L = mv_t r = mcr$$

$$\frac{E}{m} = \frac{c^2}{2} - \frac{GM}{r} = \frac{c^2 r - 2GM}{2r}$$

$$a = -\frac{GM}{2} \cdot \frac{m}{E} = -\frac{GM}{2} \cdot \frac{2r}{c^2 r - 2GM} = \frac{GMr}{2GM - c^2 r}$$

$$e = \sqrt{1 - \frac{L^2}{GMm^2 a}} = \sqrt{1 - \frac{1}{GMm^2} \cdot m^2 c^2 r^2 \cdot \frac{2GM - c^2 r}{GMr}}$$

$$= \sqrt{1 + \frac{c^4 r^3 - 2c^2 r^2 GM}{G^2 M^2 r}}$$

$$\approx \sqrt{\frac{c^4 r^2}{G^2 M^2}} \approx \frac{c^2 r}{GM} \Rightarrow e \approx \frac{c^2 r}{GM}$$

$$\theta_N = \frac{2}{e} \implies \theta_N \approx \frac{2GM}{rc^2}$$

Gravitational Lensing by Iran

1. **Geometric Features of the Images**
   - **(a)** By considering light as a particle (photon), derive the amount of deflection due to a gravitational field. The value predicted by general relativity is twice that of the Newtonian prediction.
   - **(b)** In the case where the lens and the source lie on the same straight line, show that the image of the source becomes a ring. Derive the angular radius of this ring (Einstein radius).
   - **(c)** Now assume the lens and source are not aligned. Let θ be the angular position of the image and β the angular position of the source. Derive the Einstein radius in terms of these parameters. Show that two images form.
   - **(d)** Define the magnification factor η as:

$$\eta = \frac{\theta}{\beta}$$

     Derive this in the limit where $\delta\theta \ll \theta$ and $\delta\beta \ll \beta$.

2. **Amplification and Brightness of the Source**
   - **(a)** Derive a formula for the increase in source brightness as a function of β (angular separation).
   - **(b)** Express the apparent brightness of the source as a function of time.
   - **(c)** The apparent magnitude data of a star from photometry is given. The star is 50 kiloparsecs away and affected by lensing. The minimum angular distance from the lens is 0.5 milliarcseconds. If the lens (a black hole) is 35 kiloparsecs from Earth, estimate the mass of the black hole.
   - **(d)** Estimate the velocity of the star using the data.

3. **More General Cases**
   - **(a)** Consider a galaxy undergoing lensing. Place a polar coordinate system such that the angle measurement direction lies along the line connecting the source and the lens. Show:

$$\theta_E^4 = \theta^4 + \theta^2\beta^2 - 2\beta\theta^3 \cos\gamma$$

     where θ_E is the Einstein radius.
   - **(b)** Assume the lens is at redshift z_1 and the source is at redshift z_2. Assuming a flat universe and neglecting radiation, derive the lens equation in terms of z_1 and z_2.
   - **(c)** For extended source and lens objects, show that the lens equation becomes vectorial:

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}$$

     and that:

$$\vec{\alpha} = \frac{D_{LS}}{D_{OS}}\hat{\alpha}$$

   - **(d)** For a galaxy with circular symmetry and surface mass density σ, show that:

$$\hat{\alpha} = \frac{4\pi\sigma^2}{c^2}$$

and derive:

$$\beta = \theta \left(1 - \frac{\theta_E}{\theta}\right)$$

Also determine θ_E.

4. **Applied Astrophysical Problem**

   (a) Consider a neutron star of mass $M = 3.5\, M_\odot$ and density $\rho = 9018\, \mathrm{kg/m^3}$. Observations suggest the presence of a planet around it. Using gravitational lensing, design a method to detect this planet.

   (b) Assume the neutron star emits in two cones with vertex angle 30°. Can the Hubble Space Telescope resolve this planet? Given:

   - Distance to the star: 50 kpc
   - Wavelength: 9 nm

5. **Bonus Question**

   (a) What is the probability of microlensing occurring for a given number of background stars? Assume the stars are at fixed distances and microlensing happens when their angular separation from the lens is less than θ_E. Estimate the order of magnitude of this probability.

5 IOAA 2023

Data Analysis 2: 'Isolated black hole'

In 2022, two independent groups reported the discovery of an isolated black hole based on observations of the gravitational microlensing event OGLE-2011-BLG-0462. In this problem, we will analyze data from the Hubble Space Telescope to reproduce their findings.
Gravitational microlensing occurs when the light of a distant star (the 'source') is bent and magnified by the gravitational field of an intervening object (the 'lens'). The characteristic angular scale of gravitational microlensing events, called the angular Einstein radius θ_E, depends on the mass M and distance D_L from the Earth to the lens:

$$\theta_E = \sqrt{\frac{4GM\left(D_S - D_L\right)}{c^2 D_S D_L}},$$

where D_S is the distance to the source star. For typical microlensing events observed in the Milky Way, the source stars are in the Galactic bulge, near the Galactic center, so $D_S \approx 8\,\mathrm{kpc}$.

(a) Calculate the angular Einstein radius in milliarcseconds (mas) for an example lens of $1M_\odot$ located at a distance of $1\,\mathrm{kpc}$ (2 points)

Suppose that at time t the lens and the source are separated by an angle $\theta \equiv u(t)\theta_E$ on the sky. Two images of the source are created on a line through the positions of the source and the lens, at angular distances θ_+ and θ_- from the lens given by:

$$\theta_\pm = \frac{1}{2}\left(u \pm \sqrt{u^2+4}\right)\theta_E.$$

Note. Remember earliar in this lecture we defined $u = \dfrac{\theta}{\theta_E}$ and derived,

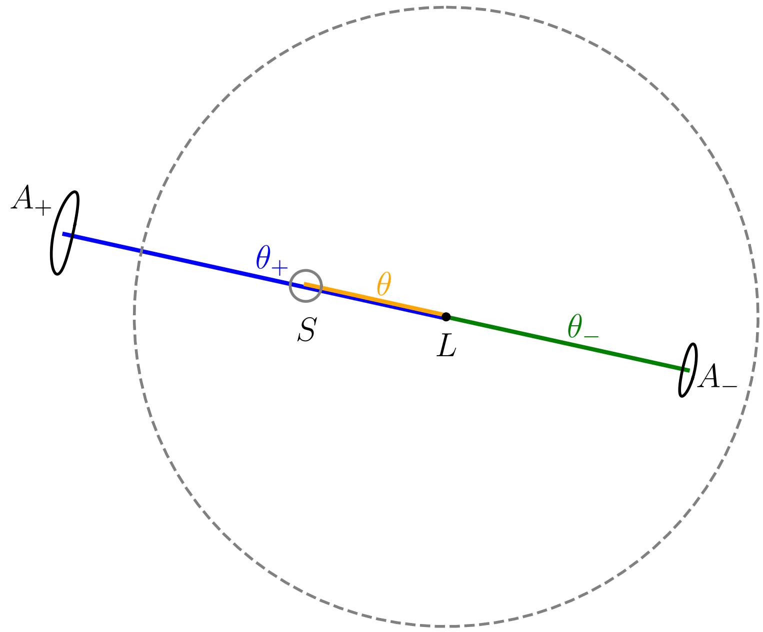
$$\frac{\theta_\pm}{\theta} = \frac{1}{2}\left(1 \pm \frac{\sqrt{u^2+4}}{u}\right)$$

So the equation provided in this question is nothing new!

These two images are magnified, relative to the unlensed brightness of the source. The absolute magnification of the images is:

$$A_\pm = \frac{1}{2}\left(\frac{u^2+2}{u\sqrt{u^2+4}} \pm 1\right).$$

The image below shows the geometry of the event. The position of the lens is marked as L, the unlensed position of the source is marked as S, while A_+ and A_- mark the positions of the two images of the source. The dashed circle has a radius of one Einstein radius.



(b) Current telescopes cannot normally resolve this pair of images, but only measure the position of the image centroid, i.e. the brightness-weighted mean of the positions of the two images. Derive an expression for the angular separation θ_c of the image centroid relative to the lens as a function of u and θ_E. **(8 points)**

$$\begin{aligned}
\theta_c &= \frac{\theta_+ A_+ + \theta_- A_-}{A_+ + A_-} \\
&= \frac{\frac{1}{4}\left(u+\sqrt{u^2+4}\right)\left(\frac{u^2+2}{u\sqrt{u^2+4}}+1\right)+\frac{1}{4}\left(u-\sqrt{u^2+4}\right)\left(\frac{u^2+2}{u\sqrt{u^2+4}}-1\right)}{\frac{u^2+2}{u\sqrt{u^2+4}}}\theta_E \\
&= \frac{\left(u+\sqrt{u^2+4}\right)\left(u^2+2+u\sqrt{u^2+4}\right)+\left(u-\sqrt{u^2+4}\right)\left(u^2+2-u\sqrt{u^2+4}\right)}{4(u^2+2)}\theta_E \\
&= \frac{2u(u^2+2)+2u(u^2+4)}{4(u^2+2)}\theta_E \\
&= \frac{2u(2u^2+6)}{4(u^2+2)}\theta_E \\
&= \frac{u(u^2+3)}{u^2+2}\theta_E
\end{aligned}$$

(c) Derive an expression for the source deflection $\Delta\theta$, i.e. the difference between the location of the centroid and the unlensed position of the source, as a function of u and θ_E. What is the source deflection when the lens and the source are nearly perfectly aligned ($u \approx 0$)? **(4 points)**

$$\Delta\theta = \theta_c - \theta = \frac{u(u^2+3)}{u^2+2}\theta_E - u\theta_E = \frac{u(u^2+3)-u(u^2+2)}{u^2+2}\theta_E = \frac{u}{u^2+2}\theta_E$$

$$\Delta\theta(u=0) = 0$$

> so there is no deflection when the lens and the source are nearly perfectly aligned.

The source and lens are moving relative to each other in the sky. Thus, both the total magnification of the images and the position of the centroid changes with time, resulting in observable photometric and astrometric microlensing effects. For now, we assume that the source-lens relative motion is rectilinear.

The plot below shows the light curve of the gravitational microlensing event OGLE-2011-BLG-0462, discovered by the OGLE sky survey led by astronomers from the University of Warsaw. The solid line shows the best-fitting light curve model. The Einstein timescale of the event, i.e. the time needed for the source to move by one angular Einstein radius relative to the lens, was $t_E = 247$ days. The event peaked on 21 July 2011 (HJD = 2455763). The minimal separation between the lens and the source was $u_0 \approx 0$.

6 Lensing Research Notes [Advanced]

6.1 Basics of Light Deflection

Einstein's general theory of relativity predicts that a mass M can bend light. The deflection angle, α, of a light ray passing near a point mass M is given by:

$$\alpha = \frac{4GM}{c^2}\frac{1}{b},$$

where b is the impact parameter (the closest approach of the light ray to the lens mass).

6.2 Lens Equation

Gravitational lensing is described by the **lens equation**, which relates the angular position of the source ($\vec{\beta}$) to the observed angular position of the image ($\vec{\theta}$):

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}),$$

where:

- $\vec{\beta}$: True angular position of the source,
- $\vec{\theta}$: Observed angular position of the lensed image,
- $\vec{\alpha}(\vec{\theta})$: Deflection angle at $\vec{\theta}$.

For an extended mass distribution, the deflection angle is proportional to the **surface mass density** (Σ) of the lens, integrated over the lens plane.

6.3 Physical Quantities and Key Equations

1. **Deflection Angle ($\vec{\alpha}$)**:

$$\vec{\alpha} = \nabla\psi(\vec{\theta}),$$

   where the gradient is taken with respect to the angular coordinates in the lens plane.
2. **Convergence** (κ)
   The convergence quantifies how much the mass distribution focuses light. It is directly related to the second derivative of the gravitational lensing potential:

$$\kappa(\vec{\theta}) = \frac{1}{2}\nabla^2\psi(\vec{\theta}),$$

   where:
   - $\kappa(\vec{\theta})$: Convergence at angular position $\vec{\theta}$,
   - $\psi(\vec{\theta})$: Gravitational lensing potential,
   - ∇^2: Laplacian operator, which is taken with respect to the angular coordinates in the lens plane.
3. **Surface Mass Density (Σ) and Convergence (κ)**:

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{\text{crit}}},$$

where Σ_{crit} is the critical surface density given by:

$$\Sigma_{\text{crit}} = \frac{c^2}{4\pi G}\frac{D_s}{D_l D_{ls}}.$$

where:

- c: Speed of light,
- G: Gravitational constant,
- D_s: Angular diameter distance to the source,
- D_l: Angular diameter distance to the lens,
- D_{ls}: Angular diameter distance between the lens and the source.

4. **Shear** (γ)
   The gravitational shear describes distortions (stretching or compressing) of lensed images. It is derived from the second derivatives of the gravitational lensing potential:

$$\gamma_1 = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right), \quad \gamma_2 = \frac{\partial^2\psi}{\partial x \partial y}.$$

Here:

- γ_1: Shear component along the x- and y-axes,
- γ_2: Shear component related to cross terms between x and y,
- ψ: Gravitational lensing potential.

The total shear is given by:

$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}.$$

6.4 Derivation of Critical Surface Density

The **critical surface density** (Σ_{crit}) is the surface mass density required to produce strong lensing effects. It is defined as the mass per unit area in the lens plane that causes the formation of multiple images. The relation between Σ_{crit} and the Einstein radius arises from the fact that the total mass inside the Einstein radius corresponds to the lens' effective mass:

$$M = \int_0^{\theta_E} \Sigma \, dA,$$

where dA is the area element in the lens plane. For distant objects, a lens is likely to be complex, as in the mass distribution of a galaxy. Since we are using a thin lens approximation, the lens' 3-D mass distribution is irrelevant: what's important is the 2-D mass surface density (mass/area), Σ

$$M(\vec{\xi}) = 2\pi \int_0^{\xi} \Sigma(\xi')\xi' d\xi'$$

Now consider that the relation between β and θ is

$$\beta = \theta - \frac{D_{LS}}{D_S \cdot D_L}\frac{4GM_L}{c^2\theta}$$

where $\xi = D_S\theta$. For simplicity, suppose we are dealing with a constant surface density lens, i.e., $M = \pi\xi^2\Sigma$. The observed deflection angle will be

$$\begin{aligned}\theta - \beta &= \left(\frac{D_{SL}}{D_L \cdot D_S} \cdot \frac{4\pi G\xi^2\Sigma}{c^2}\right)\frac{1}{\theta} \\ &= \left(\frac{D_{SL}}{D_L \cdot D_S} \cdot \frac{4\pi G D_S^2\theta^2\Sigma}{c^2}\right)\frac{1}{\theta} \\ &= \left(\frac{D_L \cdot D_{LS}}{D_S} \cdot \frac{4\pi G\,\Sigma}{c^2}\right)\theta\end{aligned}$$

Now suppose $\beta = 0$. For the above equation to be true, From geometric considerations and substituting the Einstein radius formula, the critical surface density is derived as:

$$\Sigma_{\text{crit}} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}},$$

where:

- c^2/G: Combines the relativistic and gravitational effects,
- $D_S/(D_L D_{LS})$: Accounts for the lens-source geometry.

Note than when one observes an Einstein ring-like object, the mass inside the ring

$$M(\theta) \approx \pi\,(D_S\theta)^2\,\Sigma_{\text{crit}}$$

Key Intuitions

1. **Physical Meaning**: Σ_{crit} is the threshold surface density for lensing to produce strong effects, such as multiple images or arcs. If $\Sigma > \Sigma_{\text{crit}}$, strong lensing occurs. Otherwise, only weak distortions are observed.
2. **Dependency on Geometry**: The critical density depends on the angular diameter distances, meaning the configuration of the lens, source, and observer plays a crucial role in determining the strength of the lensing effect.
3. **Units**: Σ_{crit} has units of surface density, such as $M_\odot/\text{pc}^2$.

This derivation is essential for calculating lensing effects and understanding the lensing strength of a mass distribution.

Gravitational Lensing Potential and Critical Surface Density

The **gravitational lensing potential** is a scalar potential used to describe the bending of light in the presence of a gravitational field caused by a massive object (e.g., a galaxy, cluster, or black hole). This potential is fundamental in modeling gravitational lensing phenomena and allows us to calculate image distortions, magnifications, and deflections.

6.5 Definition of the Gravitational Lensing Potential

The lensing potential, ψ, is a two-dimensional scalar field defined in the lens plane. It encapsulates the effect of the gravitational field of the lensing mass on the trajectory of light. The potential is given by:

$$\psi(\vec{\theta}) = \frac{1}{\pi} \int \kappa(\vec{\theta}') \ln |\vec{\theta} - \vec{\theta}'| \, d^2\theta',$$

where:

- $\vec{\theta}$: Angular position of the light ray in the lens plane,
- $\kappa(\vec{\theta}')$: Dimensionless surface mass density (convergence) at position $\vec{\theta}'$,
- $\ln |\vec{\theta} - \vec{\theta}'|$: Logarithmic kernel representing the influence of mass at $\vec{\theta}'$ on $\vec{\theta}$,
- $d^2\theta'$: Element of area in the lens plane.

6.6 Significance of the Lensing Potential

The gravitational lensing potential is crucial in determining several key lensing quantities:

- **Deflection Angle ($\vec{\alpha}$)**:
$$\vec{\alpha}(\vec{\theta}) = \nabla \psi(\vec{\theta}).$$
This describes the change in direction of light due to the gravitational field.
- **Convergence (κ) and Shear (γ)**:
$$\begin{aligned} \kappa &= \frac{1}{2} \nabla^2 \psi, \\ \gamma_1 &= \frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right), \\ \gamma_2 &= \frac{\partial^2 \psi}{\partial x \partial y}. \end{aligned}$$
Convergence describes isotropic magnification, while shear describes image distortion.
- **Magnification (μ)**:
$$\mu = \frac{1}{(1 - \kappa)^2 - \gamma^2}.$$
This measures the amplification or de-amplification of light due to the lens.

6.7 Applications in Astronomy

The gravitational lensing potential allows astronomers to:

1. Map the distribution of dark matter in galaxy clusters using strong and weak lensing,
2. Measure cosmological parameters such as the Hubble constant (H_0) through time delays in multiple images,
3. Study the properties of distant galaxies magnified by the gravitational lensing effect.