

SNU AO Seminar Notes

Astronomical Observation and Lab at Seoul National University

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Introduction

In math, people prefer to use \log_e or just \log rather than \ln . But due to its simplicity (especially when typing), I will stick to the usage of \ln which is identical to \log_e . But to avoid any confusion, I will always specify the base 10 (\log_{10}).

In astronomy, people use \equiv as “is defined as” or any kind of “trivial by the definition” cases. But I will use $:=$ for the definition and use \equiv to mean “equality holds trivially by the definition”. For example, under 1-D linear homogeneous isotropic matter, linear electric polarization is written as $P \equiv \epsilon_0 \chi E$ or $\chi := P/\epsilon_0 E$. The latter is the *definition* of electric susceptibility, while the former is *trivially true by the definition*.

In this note, I used some math-like notations like theorem (Thm). However, the theorem here is not necessarily the same as that of mathematics. I tried to put some important assumptions and summary notes to theorem

The reference “Walpole” is Walpole et al. 2013, Essentials of Probabilities and Statistics.

Chapter 1

Statistics - Basic

In any chapter in this book, especially in this chapter, I will assume (1) you have really basic knowledge on statistics, e.g., what “Gaussian”, “probability”, “standard normal distribution” means and/or (2) you know how to use Google so that you can find concepts which you don’t know.

Let’s start with a question.

Ex 1

For the **same star**, consider the following three scenarios:

- Case A: Researcher 1 says $m_1 = 14.0^m \pm 0.1^m$; Researcher 2 says $m_2 = 15.0^m \pm 1.0^m$.
- Case B: Researcher 1 says $m_1 = 14.0^m \pm 0.1^m$; Researcher 2 says $m_2 = 15.0^m \pm 0.3^m$.
- Case C: Researcher 1 says $m_1 = 14.0^m \pm 0.1^m$; Researcher 2 says $m_2 = 15.0^m \pm 0.1^m$.

For each of the three cases: Are the two studies coincide? If so, with how much confidence would you say so? Wait, but what does that “ \pm ” sign means in rigorous mathematical sense?

Some of the previous course takers answered that they coincide, because the “3-sigma rule” says, e.g., for case A, $m_1 = 14.0^m \pm 0.3^m$ & $m_2 = 15.0^m \pm 3.0^m$ overlaps with each other. Crudely speaking, this makes sense, but it is of course not the “publication level” reasoning.

To give you the answer: It is an ill-defined question. The answer can change based on the number of observations each researcher used for the determination of the magnitude. If we assume that number is 5 for both researchers, for example, we can give some meaningful answers: We *cannot reject* $m_1 = m_2$ for case A & B with 90 % confidence, and we *can reject* $m_1 = m_2$ for case C with 90 % confidence. Depending on the confidence level, the answer may change. Also the expression that *can/cannot reject* is very important! You shouldn’t say you *accept* $m_1 = m_2$. Let me explain why throughout this chapter.

1.1 The n - σ

Maybe you are familiar of the 1- σ , 2- σ , and 3- σ words (cf. fig. 1.1). For example, mean ± 1 - σ contains 68.27...% of the total area. Similarly 3- σ contains 99.73...% of it, so it is very unlikely to get a sample outside of the mean ± 3 - σ .

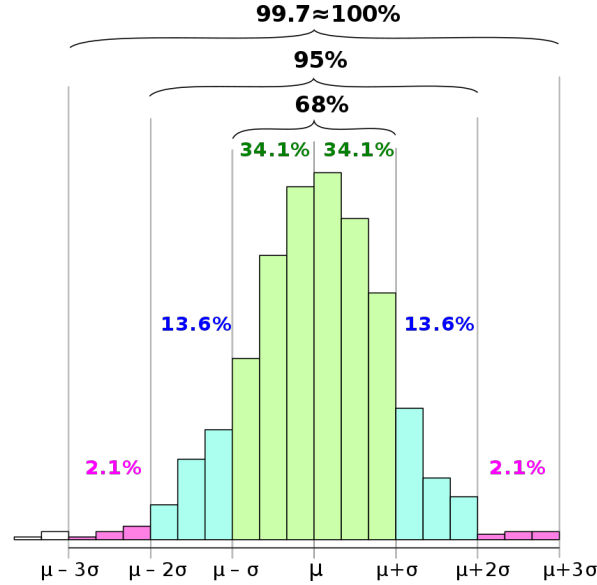


Figure 1.1: A test sampling from Gaussian (normal) distribution (from Wikimedia).

In this sense, the “ $n\text{-}\sigma$ ” is defined as the x -axis value which gives the integrated area of certain value. More precisely, we have to call this confidence interval (CI):

Def 1 ($n\text{-}\sigma$ Confidence Interval; CI)

1- σ CI is the integrated area is 0.6827... of the total area. Similarly, 2- σ CI is that with 0.9545..., 3- σ CI is that with 0.9973..., etc.

Mathematically it is more correct to call, e.g., 1- σ CI as the 68.27...% CI. We also define the significance level α as 100% minus this percentage, i.e., the 1- σ CI is of significance level $\alpha = 1 - 0.6827... = 0.3173...$:

$$1\text{-}\sigma \text{ CI} \equiv 68.27...\% \text{ CI} \equiv \text{CI of significance level } 0.3173... \quad (1.1)$$

In natural sciences we often use $n\text{-}\sigma$ CI, but in mathematics and other branches of sciences and engineering, the 90 %, 95 %, and 99 % CIs, i.e., CIs with significance level $\alpha = 0.10, 0.05$, and 0.01 are used more often.

Because when we say “ $n\text{-}\sigma$ ”, we are just omitting the term “CI” at the end of it, this is not n times the standard deviation (σ): What we mean by “ $n\text{-}\sigma$ ” is actually “ $n\text{-}\sigma$ CI” and is *not necessarily* $n \times \sigma$.

They are the same for, e.g., Gaussian distribution. For Gaussian (normal) distribution, which is a distribution that appears very often in natural sciences, such a 2- σ is nothing but 2 times the standard deviation (σ). For almost all of the distributions other than Gaussian, this is not the case.

1.2 The Meaning of \pm Sign

Simple answer: The \pm sign means the $1\text{-}\sigma$ confidence interval (CI). It is also simply called as the “error-bar”.

Honestly speaking, the \pm sign, i.e., the $1\text{-}\sigma$ CI, does *not* fully describe the uncertainty. The best way to show or describe the uncertainty is to show a graph of probability distribution function (pdf or p.d.f.) as in fig. 1.2. In many but not all publications or books, people do not show this, because (1) the distribution is very similar to Gaussian or certain widely known distribution and it is clearly written in the text or assumed as all the readers know that, (2) detailed calculation is not of interest and/or does not affect the final result, (3) the writer is lazy.

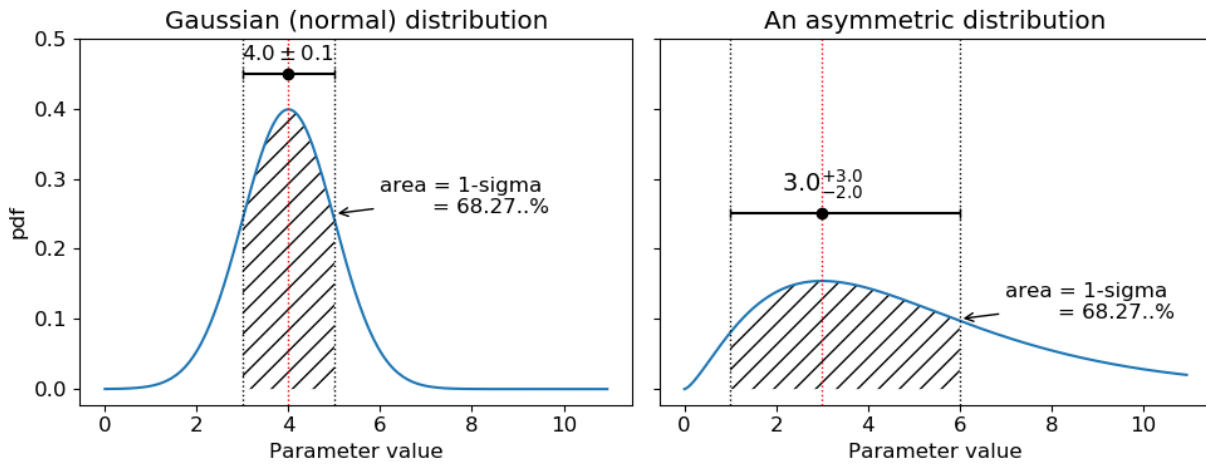


Figure 1.2: The probability distribution function (pdf) of two examples of distributions, namely, Gaussian and an asymmetric distributions. I used a chi-square distribution for the latter for plotting purpose.

The uncertainty ranges shown in the figure (4.0 ± 0.1 and $3.0^{+3.0}_{-2.0}$) are determined such that the lower/upper bounds include $68.27\cdots\%$ of the total area. It is trivial for a Gaussian distribution: mean $\pm 1\text{-}\sigma$. But if the distribution is asymmetric (right panel of the figure), there are few choices to set such bounds. First and the most widely used one is to set the bounds such that the distance between upper/lower bounds are minimized (but include $68.27\cdots\%$ of the total area). Second is to make it symmetric while include $68.27\cdots\%$ of the total area, so that you can use \pm sign for simplicity ($3.0 \pm 2.XX$ for example). Third choice would be something like FWHM: find x -axis values such that the pdf value is $0.5 \times \text{pdf}_{\text{max}}$. The last two bounds are simple to calculate but not as accurate as the first choice.

Very assymmetric probability distributions appear in cosmological sciences* and exhaustive statistical analyses are conducted on such models to reject cosmological models. The reason is that, in cosmology, we have only one single sample (our universe), and the statistical analyses given the observational data is of utmost importance. Of course similar exhaustive statistical analyses should be conducted on any research field if the data/object/target is of such importance.

*Google “posterior distribution cosmological constants”.

1.3 Caveat on the “ n - σ CI” Notation

Let me emphasize again: What we mean by “ n - σ ” is actually “ n - σ CI” and is *not necessarily* $n \times \sigma$.

It maybe is tempting to convert the 1- σ CI to the 3- σ CI. For instance, from fig. 1.2, 4.0 ± 0.1 and $3.0_{-2.0}^{+3.0}$ to 4.0 ± 0.3 and $3.0_{-6.0}^{+9.0}$. This is true for the former (Gaussian) but wrong for the latter (non-Gaussian). The lower bound of the second parameter now become negative ($3.0 - 6.0 = -3.0$), which is not even physically correct if this parameter is defined to be positive. What you have to do is, get the 3- σ CI which should contain $0.9973 \cdots$ of the total area from the distribution given in fig. 1.2. There can be at least three choices to do it, which I mentioned in the previous section.

1.4 Central Limit Theorem (CLT)

The Central Limit Theorem (CLT) is at the heart of all the observational or experimental sciences. It is stated as*

Thm 1 (*Central Limit Theorem; CLT*)

Consider a random sample (observation, measurement, etc) with size n and the mean value of this sample is \bar{X} . Then $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ approaches the standard normal distribution (\mathcal{Z}) as $n \rightarrow \infty$, where μ and σ^2 are the (finite) mean and variance of the population:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{Z} \quad (\text{as } n \rightarrow \infty). \quad (1.2)$$

There is one more important theorem, which we usually skip to mention:

Thm 2 (*Sample Variance Distribution*)

Consider a random sample (observation, measurement, etc) with size n and the sample variance is S^2 . Then $(n-1)S^2/\sigma^2$ follows a chi-squared distribution with degrees of freedom $(n-1)$, where σ^2 is the variance of the population:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{(n-1)}^2 \quad (1.3)$$

Also the definition of the Student t -distribution[†]:

*For strict definitions of random sample, population, independence, the statement of CLT in mathematical senses, etc, I recommend mathematical statistics textbooks.

[†]This definition is directly copied from Walpole et al. p.177

Def 2 (Student t -distribution)

Let Z be a standard normal random variable and V a chi-squared random variable with v degrees of freedom. If Z and V are independent, then the distribution of the random variable T , where

$$T = \frac{Z}{\sqrt{V/v}} \quad (1.4)$$

is given by the density function

$$h(t) = \frac{\Gamma[\frac{v+1}{2}]}{\Gamma[\frac{v}{2}]\sqrt{\pi v}} \left(1 + \frac{t^2}{v}\right)^{-\frac{(v+1)}{2}}, \quad -\infty < t < \infty. \quad (1.5)$$

This is known as the Student t -distribution with v degrees of freedom.

We can see that the t -distribution approaches the standard normal distribution as the degrees of freedom $v \rightarrow \infty$, because the power term takes the $e^{-t^2/2}$ form.

Thm 3 (Practical Usage of the CLT)

Consider a random sample (observation, measurement, etc) with size n large enough and the mean value of this sample is \bar{X} . If μ , σ^2 , and S^2 are the (finite) mean and variance of the population, and the sample variance, respectively:

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T_{(n-1)}. \quad (1.6)$$

Sometimes people empirically say $n \geq 30$ is enough. This really depends on the situation and I will skip this issue here. The sketch of the proof of the theorem is simple. By the definition of the t -distribution and Thm 2,

$$T = \frac{Z}{\sqrt{V/v}} = \frac{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} = \frac{\bar{X} - \mu}{S/\sqrt{n}} \quad (1.7)$$

so the theorem is plausible. For your information, the sample variance is defined as

$$S^2 := \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad (1.8)$$

and the sample standard deviation S is the square root of this.

Note here the difference between Thm 1 and Thm 3: σ is changed to S , and the standard normal distribution is changed to the t -distribution of $(n-1)$ degrees of freedom. Since the true variance σ^2 is unknown, it is difficult to use the CLT (Thm 1) directly. But thanks to Thm 2, we can use the variance of the sample, S^2 , which is measurable, and utilize the t -distribution, which is slightly bothersome than the standard normal distribution but still useful.

Also be aware that CLT says the *expectation value of the mean* is normally distributed, not the *sample* is so.

Ex 2 (Coin Tossing and the CLT)

Consider a coin-tossing experiment and assign ± 1 to heads and tails. Each experiment will give you

± 1 , but never 0. Meanwhile, we know a fair coin should have $\mu = 0$ (same probability of heads and tails). After $n = 100$ experiments, say you obtained $\bar{X} = 0.01$ and $S = 0.1$, so

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{0.01 - \mu}{0.1/\sqrt{100}} = 1 - 100\mu .$$

Since $T \sim t_{99}$, the significance level $\alpha = 0.05$ confidence interval, i.e., the confidence interval containing $100(1 - \alpha)\% = 95\%$, can be

$$t_{99,0.025} < T < t_{99,0.975} \quad \rightarrow \quad -1.9842 < 1 - 100\mu < 1.9842 \quad \rightarrow \quad \mu \in [-0.0098, 0.0298] .$$

The notation $t_{\nu,x}$ means the input argument of the t -distribution with the degrees of freedom ν such that $\int_{-\infty}^{t_{99,x}} h(t)dt = x$. Thus, to get the two-tail CI of significance level α , we need to calculate $t_{\nu,\alpha/2}$ and $t_{\nu,1-\alpha/2}$. For a symmetric distribution like t here, $t_{\nu,1-\alpha/2} = -t_{\nu,\alpha/2}$. This is very widely used standard notation.

If you just calculated without pondering about the meaning of the calculation, you may be surprised that your next experiment gives either $+1$ or -1 , while the expectation is $\mu \in [-0.0098, 0.0298]$. This is because the result from CLT is, as described, about the *mean* of the samples, not the *single sample*. Thus, the error-bar from the CLT is not necessarily predicting the possible range of *future experiments*, but it just confines the *position of the mean*.

The range which predicts the future experiments is called the prediction interval. You may learn the prediction interval (PI) to clarify this difference.

The two-tail significance level α interval of the t -distribution calculable in python by

```
from scipy.stats import t
alpha = 0.95
nu = 99

# Note that the alpha below is NOT the significance level,
# but the area under the curve!
lo, hi = t.interval(alpha=alpha, df=nu)
print(f"Confidence Interval: [{lo:.4f}, {hi:.4f}]")
```

Note: Uncertainty of Median

The $1\text{-}\sigma$ error-bar of the median is more difficult to handle than that of the mean (CLT: Thm 1). But we can get a result from simplifying assumptions, which are not necessarily true for real observations:

Thm 4 (*Uncertainty of Median*)

Consider $n(\gg 1)$ samples are independently drawn from $X \sim \mathcal{G}(\mu, \sigma^2)$, a general continuous distribution with finite mean and variance μ and σ^2 with pdf $p(x)$. The uncertainty of the median estimator is

$$\Delta_{\text{med}} \approx \frac{1}{2p(\nu_0)\sqrt{n-1}} \quad (1.9)$$

and if $\mathcal{G} = \mathcal{N}$, i.e., a normal distribution,

$$\Delta_{\text{med}} \approx \sqrt{\frac{\pi/2}{n-1}} s = \sqrt{\frac{\pi}{2}} \frac{n}{n-1} \Delta_{\text{mean}} \approx 1.25 \Delta_{\text{mean}} , \quad (1.10)$$

where s is the sample standard deviation and $\Delta_{\text{mean}} = s/\sqrt{n}$ is the error-bar of the mean from Thm 3.

Proof of general distribution : To prove it, say the true median is ν_0 and samples $\{x_1, \dots, x_n\}$ are sorted as increasing order. For an integer m , set $n = 2m + 1$, because $n \gg 1$. The sample median is then $\nu = x_{m+1}$. The probability of having sample median ν is calculable by noting that we have to sample m of sample with $x_i \leq \nu$ and the other m with $x_i > \nu$:

$$f(\nu) = \frac{n!}{m!m!} q^m (1-q)^m ,$$

where $q = q(\nu) = \mathbb{P}\{X_i \leq \nu\} = \int_{-\infty}^{\nu} p(x) dx$. $p(x)$ is the pdf of the normal distribution.

The Taylor expansion of q at $\nu = \nu_0$:

$$\begin{aligned} q(\nu) &= q(\nu_0) + \frac{1}{1!} q'(\nu_0)(\nu - \nu_0) + O((\nu - \nu_0)^2) \\ &\approx \frac{1}{2} + q'(\nu_0)(\nu - \nu_0) \\ &\equiv \frac{1}{2} + p(\nu_0)(\nu - \nu_0) . \end{aligned}$$

Then

$$\begin{aligned} f(\nu) &\approx \frac{n!}{m!m!} \left[\frac{1}{2} + p(\nu_0)(\nu - \nu_0) \right]^m \left[\frac{1}{2} - p(\nu_0)(\nu - \nu_0) \right]^m \\ &= \frac{n!}{m!m!} \left[\frac{1}{4} (1 - 4p(\nu_0)^2(\nu - \nu_0)^2) \right]^m \\ &= \frac{n!}{m!m!4^m} \left[1 - \frac{4mp(\nu_0)^2(\nu - \nu_0)^2}{m} \right]^m \\ &\approx \frac{n!}{m!m!4^m} e^{-4mp(\nu_0)^2(\nu - \nu_0)^2} . \end{aligned}$$

This has the form of Gaussian distribution ($\propto e^{-(x-\mu)^2/2\sigma^2}$) with mean ν_0 and variance $\frac{1}{4(n-1)p(\nu_0)^2}$ since $m = (n-1)/2$. Thus,

$$\nu \sim \mathcal{N}\left(\nu_0, \frac{1}{4(n-1)p(\nu_0)^2}\right) \rightarrow \Delta_{\text{med}} \approx \frac{1}{2p(\nu_0)\sqrt{n-1}} . \quad (1.11)$$

Q.E.D.

Proof of Gaussian distribution : Now if $p(x)$ is a Gaussian, median is equal to mean, so the pdf becomes $p(\nu_0) = p(\mu) = \frac{1}{\sqrt{2\pi}\sigma}$, and

$$\nu \sim \mathcal{N}\left(\nu_0, \frac{\pi/2}{n-1}\sigma^2\right) .$$

If we denote $A = \sqrt{(n-1)/(\pi/2)}$, the statistic $Z = A\frac{\nu - \nu_0}{\sigma}$ will follow the standard normal distribution. On the other hand, $V = \frac{(n-1)s^2}{\sigma^2}$ follows the chi-squared distribution of degrees of freedom $(n-1)$ by Thm 2. Then

$$T = \frac{Z}{\sqrt{V/(n-1)}} = A \frac{\nu - \nu_0}{s}$$

will follow a t -distribution with degrees of freedom $(n-1)$ by Def 2. T is nearly a Gaussian when n is large (rule-of-thumb: when $n \gtrsim 30$), so

$$\nu \dot{\sim} \mathcal{N}\left(\nu_0, \frac{\pi/2}{n-1}s^2\right) \rightarrow \Delta_{\text{med}} \approx \sqrt{\frac{\pi/2}{n-1}}s . \quad (1.12)$$

Q.E.D.

1.5 Meaning of the Confidence Interval

I circunvented the definition of the CI so far. It is defined as

Def 3 (*Confidence Interval; CI*)

A confidence interval of significance level α of a parameter X is the interval of X value such that if we conduct the identical parameter estimation process in many *parallel universes* (i.e., ensemble) which have the identical true value X_{true} , fraction of $(1 - \alpha)$ of such universes could have calculated a CI such that the X_{true} is included in that CI.

This is not a practically meaningful definition, but philosophically important. Let me give an example to elaborate the meaning of this.

Ex 3 (*Meaning of the Confidence Interval*)

Imagine there are many *parallel universes* and the same observer observes the same star in each universe. Because of random errors (which we will learn later but that includes Poisson photon noise, readout noise, etc), the observer at each universe will obtain slightly different observational results. The observers will observe the star for n times, and calculate their own mean and error-bar as \bar{X} and S/\sqrt{n} from their observations. Note again that S/\sqrt{n} is the uncertainty of the *mean* value.

For visualization, the true magnitude of the star is $\mu = 14.0^{\text{m}}$ and true standard deviation of the observation $\sigma = 0.2^{\text{m}}$ with $n = 9$ are used for the generation of fig. 1.3. The universe ID 0 of both the upper and lower panels show roughly $m = 14.0^{\text{m}} \pm 0.1^{\text{m}}$. That is, when the true magnitude and true standard deviation are given as 14.0^{m} and 0.2^{m} , an observer at one of the universes (in our case universe ID 0) got $m = 14.0^{\text{m}} \pm 0.1^{\text{m}}$, and that is our universe.

I will close this section with an excerpt from Walpole p.234:

The interpretation of a CI is often misunderstood. It is tempting to conclude that the parameter falls inside the CI with probability of(, e.g.,) 0.95. (But this is not true.) A CI merely suggests that if the experiment is conducted and data are observed again and again, about 95% of such intervals will contain the true parameter.

1.6 Answer to the Question

In statistics, the *null hypothesis*, denoted H_0 , is the one that is the simplest, and that has value if it is *rejected* (not *accepted*). The *alternative hypothesis*, H_1 , is another possibility if H_0 is not true. In the simplest case, H_1 is the complementary of H_0 . For example, $H_0 : \mu = 0$ and $H_1 : \mu \neq 0$.

Let's consider the opening question: $m = 14.0^{\text{m}} \pm 0.1^{\text{m}}$. Now you understand that 14.0^{m} means that the mean value* from the observation, and $\pm 0.1^{\text{m}}$ means the uncertainty or the error-bar of the Gaussian probability distribution of that mean value, S/\sqrt{n} . If it were not Gaussian, the author should have given more information. Note that the sample standard deviation is \sqrt{n} times the error-bar.

*It can actually be the median value, most probable value, or whatever the representative value.

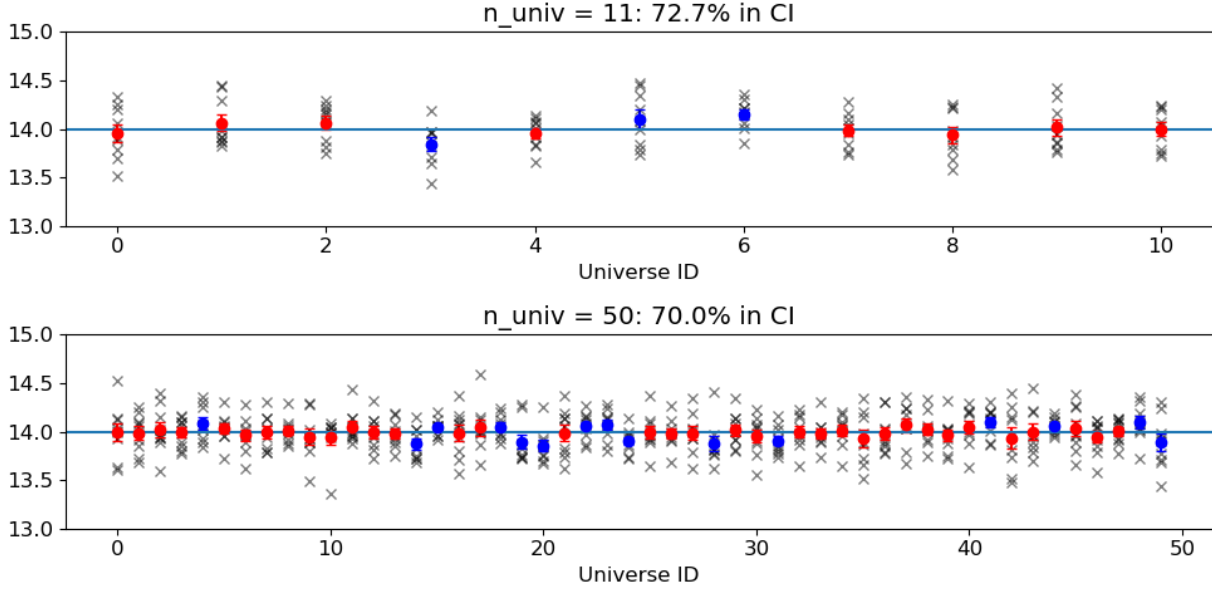


Figure 1.3: Simulation to show the concept of CI and CLT. Black crosses are the observation in each universe, and the circles are the mean and its uncertainty from the CLT S/\sqrt{n} . Red circles are the universes where the confidence interval contains the true mean $\mu = 14.0^m$ and blue circles are the others. In the title, the fraction of the universes which contains the true mean within their own confidence intervals out of `n_univ` simulated universes is shown. This fraction approaches $0.6827 \dots$ as the number of universe gets infinity.

To answer the question, we set the hypotheses*:

$$H_0 : m_1 = m_2 \quad ; \quad H_1 : m_1 \neq m_2 , \quad (1.13)$$

where $m_1 = 14.0^m \pm 0.1^m$ and m_2 is the other (from researcher 2). m_1 and m_2 here means the “true” magnitude of the star, not the measured value! If it were the measured value, they are just different, since one is 14.0 and the other is 15.0. The two true means may be different because (1) researchers may have mistakenly observed different star, (2) the star is maybe a variable (including binary), (3) exoplanet occulted some part of the star, or any other scenario is possible.

Now we have to choose which formula to use. When two measurements with $\bar{x}_1 \pm s_1$ and $\bar{x}_2 \pm s_2$ from the number of observations n_1 and n_2 are given, and μ_1 and μ_2 are the true means of two sampling distributions that will be tested under the hypothesis testing, there are two different formulae:

- The true variances are unknown and different (Satterthwhite approximation[†]):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} \quad (1.14)$$

*There are some “standard” ways to set the hypotheses, and it may seem very sudden if you are not familiar with these. Please refer to basic applied statistics textbooks for more examples, e.g., p.246 and p.264 of Walpole et al.

[†]SatterthwhiteFE (1946, Biometrics Bulletin, 2, 110), “An Approximate Distribution of Estimates of Variance Components.”

follows the t -distribution with degrees of freedom

$$\nu \approx \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}} . \quad (1.15)$$

- The true variances are unknown but the same:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{1/n_1 + 1/n_2}} \quad (1.16)$$

follows the t -distribution with degrees of freedom $\nu = n_1 + n_2 - 2$ and

$$s_p = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{\nu} . \quad (1.17)$$

Which one will you choose? To answer our original question, it is more reasonable to assume the true variances are not necessarily identical, because they may have used different instruments at different sky conditions. So I will use the first choice.

To apply the formula, you need the number of observations from each researcher, i.e., the n_1 and n_2 values. As I mentioned, it is an ill-defined question since this numbers are not given. So let me assume $n_1 = n_2 = n = 5$. The question can be reformulated as $\bar{x}_1 = 14.0$, $\bar{x}_2 = 15.0$, $s_1 = 0.1\sqrt{n}$, and $s_2 = 1.0\sqrt{n}$, $0.3\sqrt{n}$, $0.1\sqrt{n}$, where I dropped the magnitude sign for brevity. Under the null hypothesis ($H_0 : \mu_1 = \mu_2$), $\mu_1 - \mu_2 = 0$. For the three s_2 values,

$$\begin{aligned} t &= \frac{(14.0 - 15.0) - 0}{\sqrt{(s_1^2 + s_2^2)/n}} & \rightarrow & \quad t = & -0.995, & -3.162, & -7.07 \\ \nu &= (s_1^2 + s_2^2)(n - 1) & \rightarrow & \quad \nu = & 4.08, & 4.32, & 8.0 \\ & & & \approx & 4, & 4, & 8 \end{aligned} \quad (1.18)$$

The values are in the order for $s_2 = (1.0, 0.3, 0.1)\sqrt{n}$ cases. The (two-tail) significance level α test is listed in table 1.1.

Table 1.1: The confidence interval calculation for the example question.

CI %	significance level α and $\frac{\alpha}{2}$	$\nu = 4$ case	$\nu = 8$ case
90 %	0.10 & 0.05	$[-2.1318, 2.1318]$	$[-1.8595, 1.8595]$
95 %	0.05 & 0.025	$[-2.7764, 2.7764]$	$[-2.3060, 2.3060]$
99 %	0.01 & 0.005	$[-4.6041, 4.6041]$	$[-3.3554, 3.3554]$

The answers to the questions are, assuming $n_1 = n_2 = 5$,

- Case A ($t = -0.995$ with $\nu = 4$): The t value is inside of all three CIs. Thus, we cannot reject H_0 under 90 %, 95 %, and 99 % CI criterion (cannot reject under the significance level $\alpha = 0.10, 0.05, 0.01$).
- Case B ($t = -3.162$ with $\nu = 4$): The t value is inside of 90 % and 95 % CI but outside of the 99 % CI. Thus, we reject H_0 with $\alpha = 0.10$ and 0.05 , but cannot reject it with $\alpha = 0.01$.
- Case C ($t = -7.07$ with $\nu = 8$): We reject H_0 with all the three $\alpha = 0.10, 0.05, 0.01$.

By *rejecting the null hypothesis*, $H_0 : m_1 = m_2$, we mean that it is unlikely that the true magnitude value of the two studies are identical. By *failing in rejecting the null hypothesis*, we mean that it is impossible to conclude whether m_1 and m_2 are the same under the given significance level.

The results are as expected: For Case A, the error-bar of m_2 is too large, which means it is almost impossible to reject the claim that $m_1 = m_2$, so the we always fail to reject the null hypothesis. For Case C, the error-bars are too small and thus m_1 is of course vastly different from m_2 , so the null hypothesis is always rejected.

1.7 Poisson Noise

The Poisson process is a fancy naming for some special “counting”. In astronomy, we count photons (well, actually the CCD counts the electrons) over time, in social sciences we can count the death toll over time, in experiments we could count the lattice of a randomly shaped crystal over the distance. A formal definition is*

Def 4 (*Poisson Process 1*)

- The Poisson process $N(t)$ for $t \geq 0$ (t can be time, distance, or similar things), with rate λ is defined by
1. $N(0) = 0$
 2. $N(t)$ has independent increment
 3. $N(t_2) - N(t_1)$ follows Poisson distribution of rate $\lambda(t_2 - t_1)$ for $t_1 < t_2$.

It can be proven that it is identical to

Def 5 (*Poisson Process 2*)

- The Poisson process $N(t)$ for $t \geq 0$ (t can be time, distance, or similar things), with rate λ is defined by
1. $N(0) = 0$
 2. $N(t)$ has independent increment
 3. $\mathbb{P}(N(h) = 1) = \lambda h + o(h)$ for small h .
 4. $\mathbb{P}(N(h) \geq 2) = o(h)$ for small h .

Here $o(h)$ is any function such that $\lim_{h \rightarrow 0} o(h)/h = 0$. The last two items in this second definition can be re-phrased like this:

3. For a very small interval h , the probability of one single “counting” (Poisson process) occurs in that interval is proportional to the length of h .
4. For a very small interval h , the probability of two or more “counting” (Poisson process) occurs in that interval is nearly 0.

These must be independent of whether the counting happened outside of the interval, because the second item of the definition states the increment is independent.

*See, e.g., http://dept.stat.lsa.umich.edu/~ionides/620/notes/poisson_processes.pdf and https://www.probabilitycourse.com/chapter11/11_1_2_basic_concepts_of_the_poisson_process.php

Ex 4 (Photon Counting and Poisson Process)

The photon counting process of astronomy resembles the Poisson Process. If $N(t)$ is the number of photons or electrons under the exposure time t , it is trivial that the first two items of the definition is satisfied. Next, use the rephrased version of the next items. The number of photons hitting CCD with infinitesimal time interval $h = dt$ must be 0 or 1, but not larger than 1 (e.g., if a pixel count is 10,000 after 10 sec of exposure, we can take $h \ll 10\text{ s}/10,000 = 1\text{ ms}$ to meet this condition). Thus, photon counting is a Poisson process.

The Poisson process follow the Poisson distribution:

Def 6 (Poisson Distribution)

The Poisson process of rate λ has the following pdf

$$p(x; \lambda t) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \quad (1.19)$$

for non-negative integer x .

Thm 5 (Poisson Distribution Mean and Variance)

The mean, variance, and the standard deviation of the Poisson distribution $p(x; \lambda t)$ are

$$\text{mean} = \lambda t \quad ; \quad \text{var} = \lambda t \quad ; \quad \text{std} = \sqrt{\lambda t} . \quad (1.20)$$

The reason we use λt not just λ is clear from Thm 5: the mean changes as t changes, and we want to express $p(x; \text{mean})$ rather than $p(x; \text{mean}/t)$. For instance, if the photon influx is $\lambda = 10$ photons/s, the mean of photon count is a function of time as λt , so for $t = 10\text{ s}$, the pdf is $p(x; 100\text{ photons})$. If the particles are distributed with mean density along the line of sight $\lambda = 0.1$ particle/m, the mean of particles along the line of sight is a function of the reaching distance λd , so the pdf for $d = 5\text{ m}$ is $p(x; 0.5\text{ particles})$.

Thm 6 (Poisson to Gaussian)

As $x \rightarrow \infty$, the Poisson distribution approaches normal distribution $\mathcal{N}(\lambda t, \lambda t)$.

Although I will omit here, the proof uses Stirling's formula $x! \approx \sqrt{2\pi x} x^x e^{-x}$ and $\ln[(1+\varepsilon)^{\lambda t(1+\varepsilon)+1/2}] \approx \lambda t\varepsilon + \lambda t\varepsilon^2/2$, where $x = \lambda t(1+\varepsilon)$ with $\lambda t \gg 1$ and $\varepsilon \ll 1$. Stirling's formula has relative error of less than 1 % when $x = 1$ and is almost negligible for any x of interest in many cases. Considering all the approximations in the proof, we can safely assume that Poisson distribution is quite Gaussian (normal) in many astronomical contexts, unless x is very small ($\lesssim 10..?$).

Ex 5 (Photon Counting and Poisson Noise)

We saw the photon or electron counting is a Poisson process. Thus, we can use eqs. (1.19) and (1.20). For example, if we collected 10,000 electrons, the $1\text{-}\sigma$ uncertainty ($1\text{-}\sigma$ CI) or the standard deviation of this counting is $\sqrt{10,000} = 100$, so we denote $N = 10,000 \pm 100$. Since the N is large enough, Thm 6 says this is just a Gaussian distribution of mean 10,000 and standard deviation 100 (1 % of the mean). Since magnitude is $(\text{const}) - 2.5 \log_{10}(\text{count})$, differentiation gives a first-order estimation of the uncertainty in the magnitude $\Delta m = -\frac{2.5}{\ln 10} \frac{\Delta(\text{count})}{\text{count}} = 0.01^{\text{m}}$ (error-bar), which is small enough for some scientific purposes.

The next example is not directly related to Poisson process, but related to the binomial distribution (Bernoulli process) which has the pdf of $b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$ for getting x outcomes from n trials with p probability to have outcome. The pdf of binomial process approaches Poisson pdf with $p(x; np)$ when $n \rightarrow \infty$, $p \rightarrow 0$ but np is finite. Thus, the mean, variance, and standard deviation of such binomial distribution is calculable using Thm 5.

Ex 6 (*Babies in Hospital: Provided by Prof. Jae-Kwang Kim at KAIST*)

There are two hospitals A and B. New babies are born as either sex chromosome XX/XY with probability 50:50. Everyday A gives birth to 45 babies and B to 15 babies. Each hospital recorded the number of days when more than 60 % of babies are girl.

1. Which hospital will have more such days?
2. With how much confidence can you say so?

1.8 Error Propagation

When we have to calculate the uncertainty of a parameter derived from other parameters that have their own uncertainties, what we usually use is the *error propagation*. For example, $g' = 14.0 \pm 0.1$ and $r' = 15.0 \pm 0.1$, what is the uncertainty of the $g' - r'$ color? Many astronomers will say it is -1.0 ± 0.14 , where the error here is $\sqrt{\sigma_{g'}^2 + \sigma_{r'}^2}$ (σ is the error-bar). If the temperature is $T = 100 \pm 1\text{K}$, what is the fractional uncertainty of bolometric luminosity ($L = \sigma_{\text{SB}} T^4$)? Now people will say $\Delta L/L \approx \frac{4\sigma_{\text{SB}} T^3 \Delta T}{\sigma_{\text{SB}} T^4} = 4\Delta T/T = 4\%$.

This is, however, only an approximation, and in reality, we need more complicated calculation. Note that it is impossible to know how accurate this approximation is, before you conduct detailed calculation. The reason astronomers use it in spite of the pitfall is because most cases we are only interested in the rough estimation of the error-bar.

1.9 The Chi-Square Minimization

You may have heard of it, or simply the *least square* something. Least square or the chi-square minimization is a process to find the set(s) of model parameters which has the biggest (or bigger than threshold) *likelihood* in Bayesian statistics. The set of “best” model parameters is the one with the largest likelihood, so we call it the maximum likelihood estimator, **MLE**.

Consider independent and identically distributed (*i.i.d.*) random variables*, e.g., flux as a function of λ . It is independent because the measurement at each wavelength itself is independent from that of any other wavelength. It is identically distributed because and we will assume each error-bar is Gaussian. The second assumption is not necessarily correct, and actually that happens many times (e.g., flux is Gaussian distributed but magnitude is not, because it is log of Gaussian distribution).

*Random variables X_i for $i = 0, \dots, N$ are called i.i.d. if they (1) follow identical probability distribution, say $f(x)$, and (2) $f(X = x) = f(X_1 = x_1) \times \dots \times f(X_N = x_N)$, where $x = (x_1, \dots, x_N)$ (same for X).

$$P(\theta|D, I) = \frac{P(D|\theta, I)P(\theta|I)}{P(D, I)} \quad (1.21)$$

$$P(\theta|D, I) \propto P(D|\theta, I)P(\theta|I) \quad (1.22)$$

(x_i, y_i) , independent, $y \sim N(y_i, \sigma_i^2)$

$$\sum_i (y_i - f(x_i|\theta))^2 \quad (1.23)$$

$$P(D = \{y_i\}) = \Pi_i A e^{-(f(x_i|\theta) - y_i)^2 / 2\sigma_i^2} \quad (1.24)$$

$$\ln P(D = \{y_i\}) = \ln A \sum_i -(f(x_i|\theta) - y_i)^2 / 2\sigma_i^2 \quad (1.25)$$

Maximizing P == maximizing $\ln P$ == minimizing summation.

$$\chi^2 = \sum_i \frac{(y_i - f(x_i|\theta))^2}{\sigma_i^2} \quad (1.26)$$

Chapter 2

Idea of Photometry

2.1 The Point-Spread Function (PSF)

Stars must be point sources. But from our experience, we know that they are never a point, but a usually nearly circular extended sources on the CCD. The image on the CCD when a single perfect point source is imaged, is called the **point spread function (PSF)**. There are basically two mechanisms responsible for this: diffraction and seeing. Although diffraction can also be a part of seeing, it is somewhat special than others: It is just impossible to eliminate the diffraction, while other sources of seeing can be removed (see below).

2.1.1 Diffraction

When a light ray passes through a finite-sized aperture, diffraction must occur. Due to this, a point source is blurred to a finite size after it passed through an aperture (e.g., telescope or DSLR camera aperture).

Ex 7 (*Diffraction*)

For $D = 1$ m in visible (550 nm wavelength), the diffraction limit is $1.22 \frac{\lambda}{D} = 0.14$ arcsec. You may be familiar with this formula. Using the proportionality, you can memorize it as $\theta_{\min}^{(550 \text{ nm})} = \frac{0.14}{D[\text{m}]} \text{ arcsec}$.

It is more complicated in reality: the mirrors and other optical instruments located in the optical path affect the final shape of a point source by diffractions. See fig. 2.1 for the examples of simulated PSFs from optics, i.e., when the seeing effects are neglected.

As you can see in fig. 2.1, although the detailed PSF are different (mid row), the long-range features are weaker compared to the central nearly circular features (bottom row) in cases except for the last column, which is not of our interest.

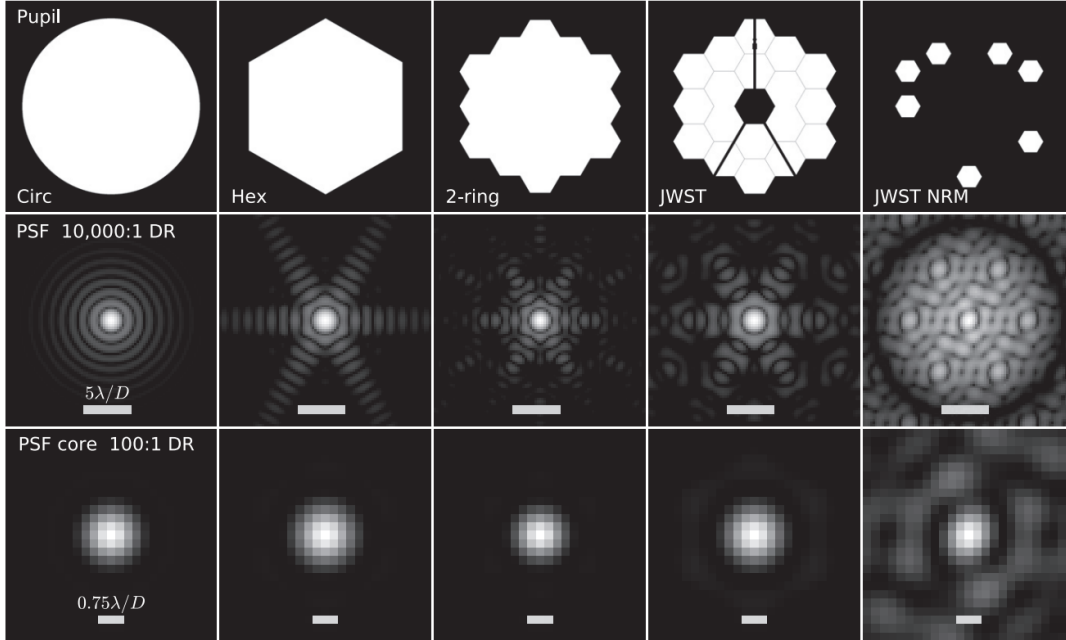


Figure 2. Comparison of the point spread functions (PSFs) of different shaped apertures. The top row shows apertures as labeled. The middle row shows the corresponding PSFs with 10,000:1 dynamic range at consistent stretch and pixel scale. The bottom row shows the cores of the same PSFs with 100:1 dynamic range. Note the relatively small core and deep null surrounding the *JWST* NRM PSF.

Figure 2.1: Example of PSFs. For circular aperture, the PSF shows concentric diffraction pattern as we learned in general physics course. For JWST case, you can see the secondary mirror and the mirror holding pipe (black things) makes the PSF different compared to the 2-ring case. The bottom row is not just a zoom-in of the PSF, but also the stretch (contrast) is different from the mid row (DR means the dynamic range). Direct excerpt from Figure 2 of FordKES+, 2014, ApJ, 783, 73.

2.1.2 Seeing

Even if we ignore the diffraction, a point source is blurred due to two main reasons: (1) air turbulence and (2) discontinuous movement of the telescope since the telescope gear is not a perfectly smooth continuous ideal machine. The blurring caused by these are called *natural (atmospheric)* and *telescopic seeing*, respectively. The atmospheric seeing makes a point source an extended circular source (for details, search for the theory of, e.g., Andrei Kolmogorov), while the telescope seeing *may* make it very elongated or irregular depending on the quality of the tracking system. If the tracking system is well established on a telescope, the final telescope seeing effect is quite circular.

All these effects can be removed if we move our telescope in space. The atmospheric turbulence is now gone, and the telescope can be very stable such that the telescopic seeing is ignorable. This is what is called the diffraction-limited case.

2.1.3 Summary of PSF

To summarize:

$$\begin{aligned} \text{PSF} &= \text{diffraction} + \text{seeing} \\ \text{seeing} &= \text{atmospheric seeing} + \text{telescopic seeing} \end{aligned}$$

and practically speaking,

$$\text{PSF shape} \approx \text{circular} .$$

This is why we can talk about the circular aperture photometry. Also

$$\begin{aligned} &\text{diffraction} \ll \text{other sources of seeing} \\ &(\text{unless adaptive optics or space telescope used}) . \end{aligned}$$

Actually, for many ground-based observations, it is not necessarily important to distinguish diffraction from other seeing effects. I feel like people tend to use the word “PSF” when the PSF is measured rather accurately (e.g., PSF photometry), and use the word “seeing” when we assumed circular PSF (e.g., aperture photometry).

2.1.4 Seeing Disc Size

As mentioned above, when we assume circular PSF (as we will do frequently in aperture photometry), we use the word “seeing” for the PSF. *Seeing disc* is the word for the circular shape of the PSF, and the *seeing disc size* is a measure of the size of that circle. The definition of the “size” is not trivial, but many people use the FWHM of the stellar radial profile*. This seeing disc size measurement is used to find the focal position and give a sense to the data quality.

The seeing disc FWHM is dependent on wavelength, weather, airmass, etc. At Seoul National University observatories in Seoul, it is roughly 1 to 3 arcsec at Bldg. 46 and 3 to 6 arcsec at Bldg. 45. For Subaru telescope at Hawai’i, for example, it is less than 1 arcsec†. Compared to the diffraction size θ_{\min} , these are much larger. It can go even below if we use adaptive optics (then diffraction \approx total seeing and non-circular PSF may be important).

The fact that the seeing disc size (i.e., size of the circular PSF) on the CCD is normally much larger than the diffraction scale hint that atmospheric and telescope seeing are very important. Since the telescope seeing is roughly constant over time for continuous tracking of a good telescope, the change in seeing can also strongly dependent on the weather condition. So astronomers use seeing disk size (usually rough estimate of FWHM) as a proxy of the weather condition.

It can also be used as a data-quality-indicator.

Ex 8 (*Contamination by Other Object*)

If you have your faint target near a bright stellar source with 1 arcsec separation and if the seeing disc FWHM is roughly 1 arcsec, your faint target is contaminated by the nearby bright star. A possibility is to do photometry of “your target + bright star” and subtract the flux of the “bright star” from catalog or future/previous observations. In this case, although you can get the flux of the target, its uncertainty may increase significantly.

*Radial profile means the pixel values of the star image as a function of the distance from the center of the star: $I(r)$. From many free/commercial products, it is rather easily drawn, such as **ginga**, **SAO ds9**, or **Maxim DL 6**. They also automatically calculates FWHM for you. The details on how to find the “center” will be studied later. Here, just think about something like Gaussian curve.

†<https://www.naoj.org/Observing/Telescope/ImageQuality/Seeing/>

Ex 9 (*Slit Width Determination*)

For spectroscopy, seeing disc size may determine the slit width you have to use to collect the target's light into the slit. It will be dealt in the spectroscopy chapter.

Ex 10 (*Finding Focal Position from Seeing FWHM*)

First find a random bright star that do not vary over short period of time (e.g., about 1 hour). You then expose and check the FWHM of the PSF (assuming it is circular). Now tune the focal position, e.g., by moving the position of the secondary mirror, and expose again, and check the FWHM again. Doing this repeatedly, you can get the FWHM as a function of the focal position. When the FWHM becomes the minimum, that is *the* focal position (fig. 2.2). Normally this curve is roughly second order polynomial.

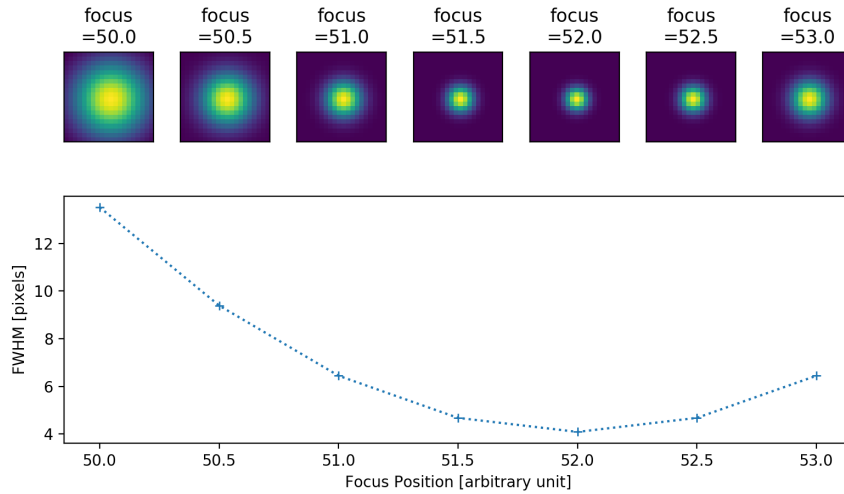


Figure 2.2: Finding focal position. While tuning the focal position value (x-axis), check the FWHM of the stellar profile. In this fake simulation, it is found that the focal position of 52.0 is *the* focal position with FWHM ~ 4 pixels. Multiplying with the pixel scale will give the FWHM in angular units (e.g., if the pixel scale is 0.8 arcsec/pixel, FWHM after the focusing is $4 \text{ pixel} \times 0.8 \text{ arcsec/pixel} = 3.2 \text{ arcsec}$)

Our goal in observation is to “gather all the photons from the target”. For a faint source, you have to tune the focal position very accurately, so that the signal from the source gathers to as small number of pixels as possible. Otherwise, the signal will be buried in sky values or other noise sources (dark current or readnoise) and our goal may not be achieved. The non-circular PSFs for high-quality data shares this same philosophy: the “sprays” of PSF should be analyzed. But for a very bright source, you may even intentionally defocus before the exposure. This is because we can avoid saturation and blooming from the central pixels while the signal is so strong that the noise is overwhelmed by the source signal. Then our goal is achieved even if we forget about the accurate focal position or accurate PSF.

2.2 Centroid

Even though we know the accurate PSF, we cannot use it for photometry if we cannot determine the center or the origin. There are various ways to determine this “origin”, but the most widely used one is centroiding (center of mass).

Let I_i for $i = 1, 2, \dots, N$ means the i -th pixel with $(x, y) = (x_i, y_i)$ of the original $N_x \times N_y$ 2-D pixel array so that $N = N_x N_y$. Say we set a threshold I_0 (e.g., background value, signal-to-noise ratio of 3, etc). Then change $I_i = 0$ if $I_i < I_0$. For $\xi = x$ or y , the centroid is by definition

$$\xi_c = \frac{\sum_i I_i \xi_i}{\sum_i I_i} . \quad (2.1)$$

The uncertainty of this is, considering ξ_i 's are fixed with no uncertainty,

$$(\Delta \xi_c)^2 = \left(\frac{\partial \xi_c}{\partial I_1} \right)^2 (\Delta I_1)^2 + \dots + \left(\frac{\partial \xi_c}{\partial I_N} \right)^2 (\Delta I_N)^2 . \quad (2.2)$$

But

$$\frac{\partial \xi_c}{\partial I_k} = \frac{\partial}{\partial I_k} \left(\frac{I_1 \xi_1 + \dots + I_N \xi_N}{I_1 + \dots + I_N} \right) = \frac{\xi_k}{\sum_i I_i} - \xi_c , \quad (2.3)$$

so

$$(\Delta \xi_c)^2 = \sum_k \left(\frac{\xi_k}{\sum_i I_i} - \xi_c \right)^2 (\Delta I_k)^2 = \frac{\sum_k (\xi_k - \xi_c)^2 (\Delta I_k)^2}{\sum_i I_i} . \quad (2.4)$$

Pixels near the object are likely to have high pixel values, so we approximate the uncertainty ΔI_i is just the Poisson noise of the pixel value (which is source + sky + read noise), i.e., in eq. (1.20), $\Delta I_i = \sqrt{I_i}$. Then expanding the square and using the definition of ξ_c , we get

$$\Delta \xi_c = \sqrt{\frac{\sum_k (\xi_k - \xi_c)^2 I_k}{\sum_i I_i}} = \sqrt{\frac{s_c^2}{\sum_i I_i}} \quad (2.5)$$

where

$$s_c^2 = \frac{\sum_k \xi_k^2 I_k}{\sum_i I_i} - \xi_c^2 . \quad (2.6)$$

There are some other techniques which involves Gaussian fitting: marginalize the image onto the x - and y -axes, and do the 1-D Gaussian fitting to each of them. Also the sigma-clipping inside the centroiding box (cbox) can be used (e.g., Ma+ 2009, Opt.Exp., 17, 8525).

2.3 Aperture Sum

We need first to find the total flux inside the aperture. The sky subtraction will be done later. Consider a very simple case in fig. 2.3: The centroid is calculated following eq. (2.1), and it is trivial that the centroid is at the center of the shown image. If we put circular aperture of radius 1 pixel (middle panel), the pixel contribution from the pixel with value 10 (right panel) is

$$\text{pixel value} \times \frac{\text{area in aperture}}{\text{total area}} = 10 \times \frac{\pi/4}{1} = 7.854 . \quad (2.7)$$

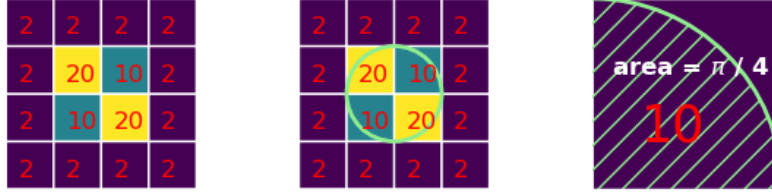


Figure 2.3: Left: An example pixel values near the source. Middle: An aperture of radius 1 pixel shown as green circle. Right: A zoom in of one pixel with pixel value 10.

Doing the same calculation for all the 4 pixels inside the aperture, you will get $N_{\text{apsum}} = 47.12$, where apsum means the “aperture sum”.

One thing to note is the uncertainty in position. In this 4 by 4 pixel example, $\sum_i I_i = 84$ and $\xi_c = (x_c, y_c) = (1.5, 1.5)$ under the 0-indexing. For Δx_c , we consider only the x -coordinate for eq. (2.5) calculation. Note that in that case, $\sum_k (x_k^2 I_k)$ for all the $4 \times 4 = 16$ pixels is nothing but $\sum_{k=1}^4 (x_k^2 I'_k)$ where $I' = [8, 34, 34, 8]$ is the marginalized intensity along the x direction from fig. 2.3. Then $\Delta x_c = 0^2 \times 8 + 1^2 \times 34 + 2^2 \times 34 + 3^2 \times 8 = 242$. This must be the same for the y -axis, and thus $s_c^2 = 242/84 - 1.5^2 = 0.63$, so $\Delta x_c = \Delta y_c = \sqrt{0.63/84} = 0.087$ pixel.

Here, I did not touch the topics such as (1) how to select the aperture radius, r_{ap} and (2) in reality, we need to fit some function to the radial profile, but which function should we use? These will be dealt in the next chapter.

Since this aperture sum includes not only the source but also the sky values (sky value is always non-zero for ground-based observations), we need to subtract the sky.

2.4 Sky Estimation

Estimation of the sky is one of the most trickiest part in observational astronomy. There are two most widely used methods: annulus and 2D interpolation. I will first explain few formulae, and then explain these two methods.

2.4.1 Simple Parametric Sky Estimation

There have been some parametric methods to estimate correct sky value. The most widely used ones include the “MMM” relation, or the “mean-median-mode” relation, and regard the mode (most frequent value) as the constant sky value. This has been used by virtually all astronomical softwares, including IRAF, IDL, SExtractor, and python packages. It is the simplest and does not require much computational speed, while gives reasonable results in many cases.

Why mode? The mission in determining the *constant* sky value is to find a *robust*, i.e., trustful constant value. Average (mean) is very vulnerable to few outliers even though we did sigma-clipping, and the median is less robust than mode, because if faint object with S/N ratio ~ 1 is in the sky

region, the median can also be overestimate the sky. Thus, mode is the most widely used simple statistic for sky estimation.

The determination of the modal value is very difficult, because the it is not simple to define it for the *sampled data*. If the distribution is not a mathematically continuous one, i.e., if it is discrete as we always encounter after sampling process, the mode is the largest value when we draw a *histogram*. The histograms are basically sensitive to the subjective selection of bins, and you can change the modal value to an unexpected value depending on the bin you choose (see fig. 2.4).

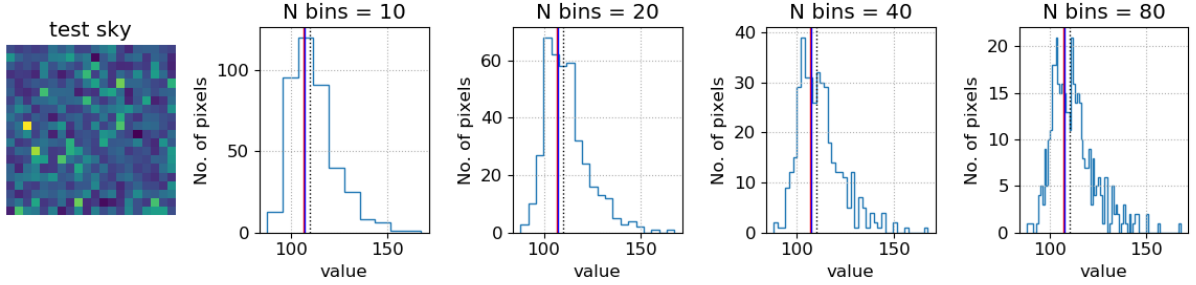


Figure 2.4: Left: A fake sky of 20 by 20 pixels generated by $100 + \mathcal{N}(0, 5^2) + 5\chi_2^2 + 3\mathcal{U}(0, 1)$ (normal distribution, chi-squared distribution, and uniform distribution, respectively). Others: The histogram of the pixel values with different bin numbers with fixed bin width. The black dotted lines is the median, and the red and blue solid lines are the mode estimation from eqs. (2.8) and (2.9), respectively (almost indistinguishable). As can be seen, the mode value does not lie at the same position, and it even become bimodal when the number of bins is 80.

Since 1800s, people empirically knew that the mode has the following relationship which holds for a moderately asymmetric distribution:

$$\text{mode} \approx 3 \times \text{median} - 2 \times \text{mean} , \quad (2.8)$$

or equivalently, *the distance between the mode and median is two-thirds of that between mode and mean*. DoodsonAT (1917, Biometrika Trust, 11, 425) first gave mathematical proof of it using some mathematical tricks including Laplace's and De Morgan's (1836), and it is recognized by many statisticians including KendallMG (1943; or the second edition of it in 1945, The Advanced Theory of Statistics 2e, Vol 1, p.35). People now refer to more recent references like KendallMG+StuartK (1977, The Advanced Theory of Statistics, Vol. 1. Charles Griffin & Co., London).

SEExtractor (BertinE+ArnoutsS, 1996, A&AS, 117, 393) uses a variant of this:

$$\text{mode} \approx 2.5 \times \text{median} - 1.5 \times \text{mean} . \quad (2.9)$$

There is no clear reason provided in the original paper nor the SEx user manual, but it is just descibed that it was found to be more accurate when they tested. The estimation process works like this: Get the mean, median, and sample standard deviation (mean, med, std, respectively) from the sky pixel values after sigma-clipping. If $\frac{\text{mean}-\text{med}}{\text{std}} > 0.3$, use $\text{mode} \approx \text{med}^*$. Otherwise, use eq. (2.9).

*Although the publication and SEx manual says they use mean if $\frac{\text{mean}-\text{med}}{\text{std}} > 0.3$, but the SEx software actually returns med, not mean (described in photutils v0.6, Background Estimation (photutils.background))

IDL MMM uses eq. (2.8). IRAF also uses eq. (2.8) internally, but it gives the mean value if $\text{mean} < \text{med}$. In astropy-related packages (e.g., `photutils`), it is the user's choice. In all cases, you can set the sigma-clipping factors, e.g., a such that data outside $\text{med} \pm a \times \text{std}$ can be rejected, and n , the maximum iterations this process should be done until no more data are rejected. Whether to use $\text{med} \pm a \times \text{std}$ or $\text{mean} \pm a \times \text{std}$, whether to give asymmetric a values (upper and lower sigma-clipping), etc are up to the user.

I recommend to follow what SEx does, with sigma-clipping of $a \sim 3$ and $n \geq 5$, because that became a virtual standard in parametric sky estimation.

2.4.2 Other Ways for Sky Estimation

As can be seen from fig. 2.4, the sky histogram is mostly skewed towards right. Because of this, BijaouiA (1980, A&A, 84, 81) first introduced to fit the sky histogram with Gaussian multiplied by Laplace:

$$p(I) = \frac{e^{\sigma^2/2a^2}}{a} e^{-(I-s)/a} \text{erfc}\left(\frac{\sigma}{a} - \frac{I-s}{\sigma}\right) \quad (2.10)$$

where

$$\text{erfc}(x) := \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-t^2/2} dt. \quad (2.11)$$

IrwinMJ (1985, MNRAS, 214, 575) argued that this method for modal estimation (Bayesian maximum likelihood is used) is better compared to mean, median, and Gaussian fitting. But because of the complexity of the calculation, Irwin used simpler method, i.e., get the smoothed histogram near the estimated mode and fit a Gaussian. BeardSM+ (1990, MNRAS, 247, 311) used slightly different method: smooth the (sky pixel) frequency histogram with moving box filter, find the mode estimate with it, sample the pixels nearby it which has pixel values larger than half of it. Then it does a cubic polynomial fit to the unsmoothed original histogram and finds the peak from this polynomial as the mode estimate.

For more, you may refer to Appendix A of AkhlaghiM and IchikawaT (2015, ApJS, 220, 1) and section 3.1 of MasiasM+ (2012, MNRAS, 422, 1674).

2.4.3 Annulus Sky

The simplest way is to use circular annulus. This method assumes the sky is nearly constant near the target of interest, PSF is circular, and there is no nearby celestial object. Under these simplifying assumptions, the sky must be azimuthally symmetric and a function of radius centered on the centroid of the target. If we set the inner and outer radii r_{in} and r_{out} much larger than the seeing disc size to define the circular annulus, the pixel values within that annulus are now the “sample” of sky values, and should not have clear tendency along the radial and azimuthal direction. fig. 2.5 shows the result of centroiding and sky estimation.

Consider we obtained the sky value m_{sky} somehow, e.g., median of sky, eq. (2.9) after sigma-clipping, etc. There can be many ways to estimate the uncertainties of this m_{sky} . Let me call this Δm_{sky} . The fluctuation of sky value can critically affect the final result of the photometry. But

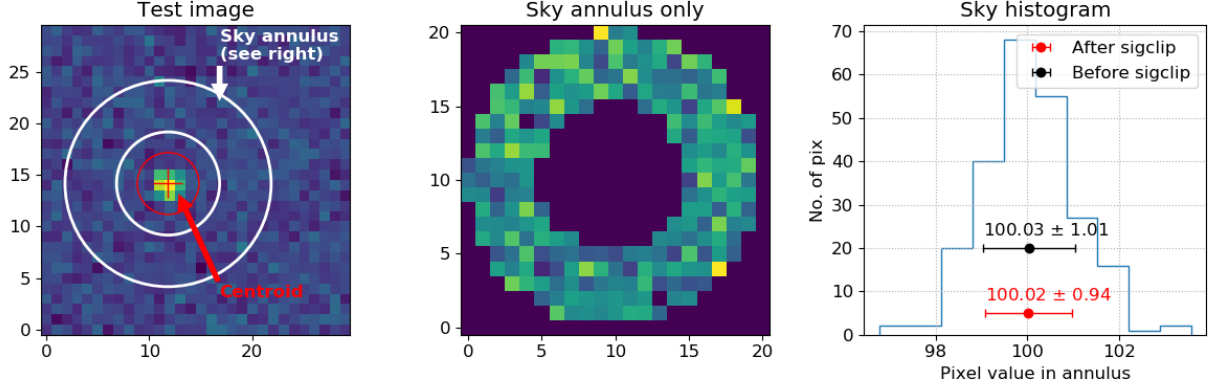


Figure 2.5: Left: The test image. Centroid is determined by eq. (2.1) and annulus is defined as $r_{\text{in}} = 5$ and $r_{\text{out}} = 10$ pixels, respectively. Middle: Plotting the sky annulus region only, to just visualize the random fluctuations in the sky region. Clearly, there is no radial trend and it seems quite symmetric along the azimuth. Right: The histogram of the sky pixel values in the annulus. The red/black markers are the median and sample standard deviation of the sky values after and before sigma-clipping, respectively. The sigma clip was done with 3-sigma 50-iterations.

once a robust way to estimate the sky is set and if the target is bright enough, the uncertainty due to the sky value is not so large. Thus, people do not care too much about the accurate pdf of the m_{sky} value. The most widely used way is to use the practical CLT (Thm 3) to m_{sky} , treating as if it is the mean of the sky samples (although we almost never use the mean for m_{sky}). Then

$$\Delta m_{\text{sky}} = \frac{s_{\text{sky}}}{\sqrt{n_{\text{sky}}}}. \quad (2.12)$$

Here, s_{sky} and n_{sky} are the sample standard deviation and the number of the sky pixels survived after the sigma-clipping, respectively. So we are assuming here that the mode has similar uncertainty as the mean.

2.4.4 Source Extractor Sky

We will not cover Source Extractor (SE or SEx or SExtractor) deeply. But simply put, it works like the description below. Say we have the image of 1000 by 1000 pixel.

1. **Meshing:** Chop the image by given box size. For example, if you set the box size as (50, 25), 20 by 40 (in total 800) “pads” or “meshes” will be made. Each pad will of course have 50 by 25 (in total 1250) pixels.
2. **Filtering:** Determine the size of the median filter. If the size is (3, 3), for instance, a 3 by 3 median filter will be used to reduce sharp noised pixels in each pad (similar to Gaussian convolution). The filter size should be determined as a function of seeing disc size.
3. **Rejecting Meshes:** In this median-filtered pad, the sigma-clipping is done. If too many pixels inside a pad is rejected, that pad is rejected for the next background estimation step. Such a threshold, say exclusion percentile, must be given by the user. For example, if that percentile is 10 %, any of the 800 mesh which rejected more than 10 % of its pixels from sigma-clipping (10 % out of 1250 pixels, i.e., 125 pixels) will be regarded as “bad region” for the sky estimation.

4. **Sky at each Mesh** (bottom middle panel of fig. 2.6): Now come back to the original un-filtered meshes. From the pixel values, estimate the sky value and its uncertainty following eq. (2.9) and the description below the equation. For the meshes which are flagged as “bad” from previous step are not used for this process. As a result, you will have 20 by 40 array of sky values, while some elements may be empty or NaN, if some meshes are flagged as “bad”.
5. **Interpolation** (top middle panel of fig. 2.6): From the estimated single sky value at each mesh, i.e., the 20 by 40 array, we do interpolation to estimate the sky at all the 1000 by 1000 pixels. The estimation uncertainties should also be calculated for each pixel.

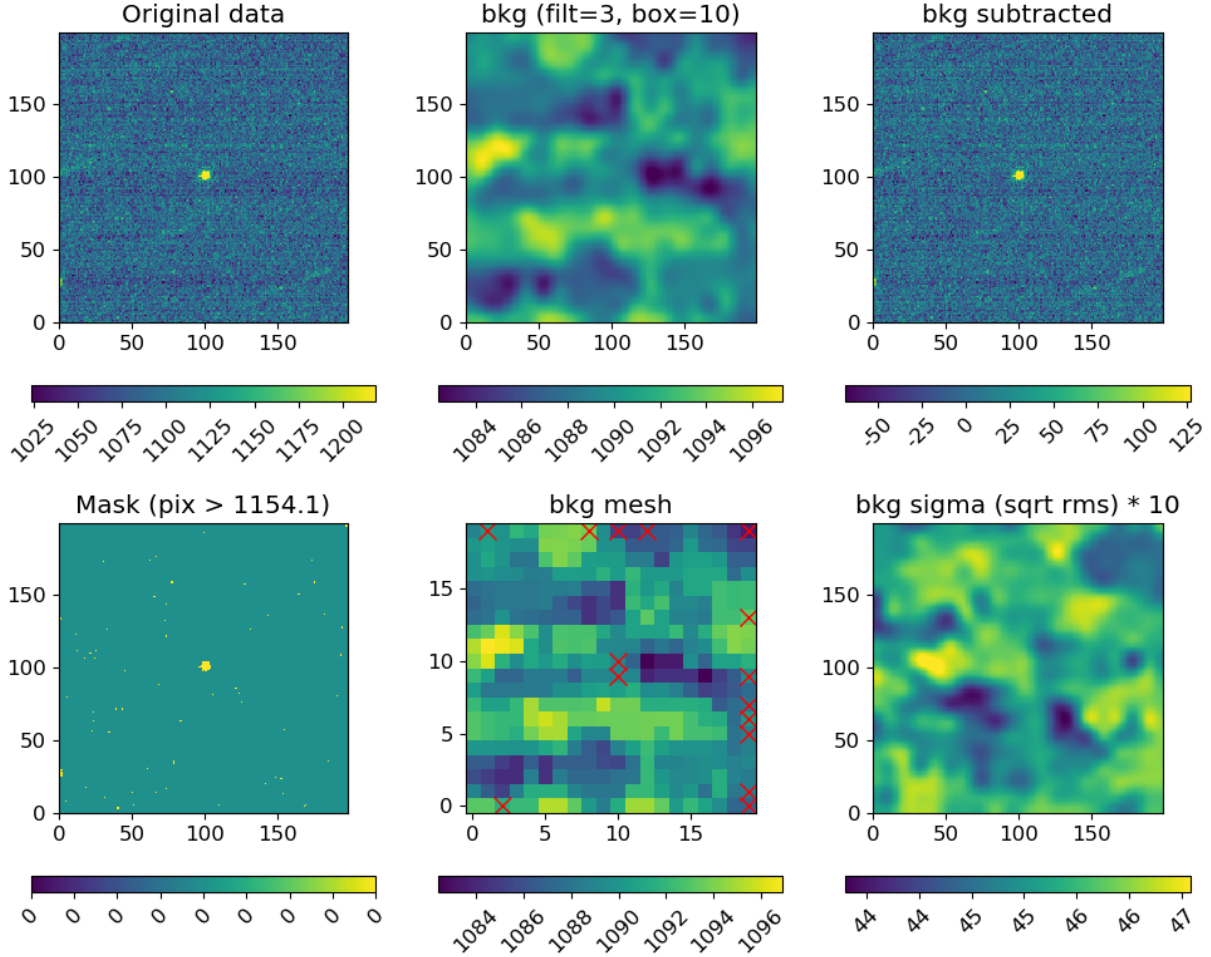


Figure 2.6: Example of the SExtractor background estimation usage. The original data is the targeto-centric combined image from 13 KMTNet images from SAAO observatory. The pixels with value 1154.1, which is the med + 3std after the sigma-clipping (3-sigma, 5-iter clipping) are masked because those can be stars (bottom left). The background meshes with the estimated sky values (eq. (2.9)) are shown in bottom center, and the interpolated one is given in top middle. Note that the sky level changes at maximum around 12 ADU out of the pixel value 1000+. Normally this should be smaller like 5 ADU or 2 ADU within this small region (200 by 200 pixel). The background sigma shown in bottom right is around 4 ADU (note the figure is 10 times the sigma), but many times the sky sigma is smaller than this.

After the sky subtraction, I plotted the Box plot of the image with x-axis 70 to 90 and y-axis 50

to 150 (python indexing) in fig. 2.7.

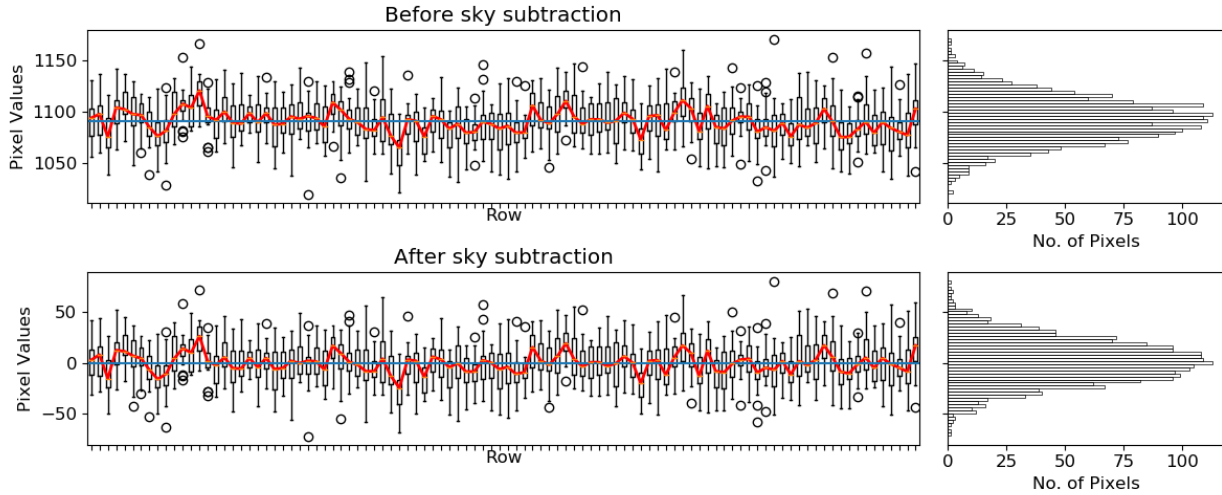


Figure 2.7: The Box plot and pixel value histogram of before/after the sky subtraction, of a region defined by `data[50:150, 70:90]` (python indexing). The horizontal axes of Box plots correspond to the “row” or the y index. Blue horizontal lines are the median of all the pixels in that region, and red solid lines are the medians

As can be noticed, SEx background estimation*, unlike the annulus sky estimation technique, is estimating the sky values at all the pixels, not for each source. Thus, it takes much longer time. Also you can see that lots of parameters (rejection percentile, box size, filter size, interpolation method and related parameters, etc) should be set for SEx to work! In some cases, astronomers run SExtractor again and again to do get expected results, and find best combination of such parameters. But these parameters of course depend on the instruments, sky conditions, etc, so if the results are suspicious, some fine tuning must be made or you have to find different ways to do photometry than SExtractor.

2.5 Aperture in Real Reduction

We most frequently use circular aperture photometry. To determine how large the aperture should be, we need information about the radial profile of the PSF. Historically, there are three important radial profiles: Gaussian, Moffat, and Penny:

1. Gaussian. The Gaussian function.
2. Moffat. A purely empirical function.
3. Penny. A purely empirical function.

(links to Wikipedia)

Among parameters for PSFs, one that is very widely used in astronomy is the FWHM (full width at half-maximum), as described before. FWHM is defined as the distance between the two points which have the value $\frac{1}{2}f_{\max}$, i.e., $f(r = \text{HWHM}) = \frac{1}{2}f_{\max}$ for $\text{HWHM} = \text{FWHM}/2$. FWHM is

*We tend to call “background” rather than “sky” when we are dealing with SExtractor.

also used for Lorentzian profile, which has infinite standard deviation (the second centered moment diverges, so the standard deviation is not defined, but FWHM can). The profiles of PSFs and their FWHMs are described below.

In the descriptions below, I will assume the center \mathbf{r}_0 is clearly fixed and the PSF is azimuthally symmetric (circular) and thus $r = |\mathbf{r} - \mathbf{r}_0|$ is the only free parameter except for other parameters described in the model.

2.5.1 Gaussian

The circular, i.e., azimuthally symmetric Gaussian PSF is like this:

$$f_{\text{Gauss}}(r|A, \sigma) = Ae^{-r^2/2\sigma^2} \quad (2.13)$$

Here, r is the distance from the center (r_0), and σ is the standard deviation of the profile. This is the most famous form when it comes to numerical calculation, because all the normalization constants are not so important in PSF fitting.

The normalization constant $A = \left(\frac{1}{\sqrt{2\pi}\sigma^2}\right)^N$ for an N -D Gaussian, such that the integration is normalized, i.e., $\int_{-\infty}^{+\infty} f_{\text{Gauss}}(r)2\pi r dr = 1$. The 2-D Gaussian of integrated flux I is, therefore,

$$f_{\text{Gauss}}(r|I, \sigma) = \frac{I}{2\pi\sigma^2}e^{-r^2/2\sigma^2} \quad (2.14)$$

This is useful because the total integrated flux is an input parameter. Conversion between this and eq. (2.13) is very easy.

The FWHM is calculated by setting $f_{\text{Gauss}}(r = \text{FWHM}) = \frac{A}{2}$. First we obtain $(\text{FWHM})^2 = (2\ln 2)\sigma^2$ and thus,

$$\text{FWHM}(\text{circGauss}) := 2\sqrt{2\ln 2}\sigma. \quad (2.15)$$

Re-formulating eq. (2.14) with FWHM (F for brevity):

$$f_{\text{Gauss}}(r|I, F) = I \frac{4\ln 2}{\pi F^2} e^{-(4\ln 2)r^2/F^2}. \quad (2.16)$$

This is not widely used, although both I and F are what we're most interested in. Rather, people tend to convert F to σ and use eq. (2.13) or eq. (2.14).

2.5.2 Moffat

The first suggestion of using Moffat profile was made in 1969 by A. F. J. Moffat in Moffat AFJ 1969, A&A, 3, 455 (). It is purely an empirical function, and is one of the most widely accepted form, because of its simple form yet powerful in fitting many circular profiles. Comparison of notations is given in section 2.5.2.

The circular, i.e., azimuthally symmetric Moffat PSF is:

$$f_{\text{Moffat}}(r|A, R, \beta) = A \left[1 + \left(\frac{r}{R} \right)^2 \right]^{-\beta}. \quad (2.17)$$

Table 2.1: Moffat profile comparison. **astropy**'s notation is strikingly confusing, because of its **alpha** notation, which completely goes against the original Moffat paper and other software implementations (e.g., IRAF). Note: **AsPyLib** is developed by J. Caron in Python 2 and the development is halted since 2013.

Symbol	astropy	IRAF, AsPyLib	Description
r			radial distance from the center ($r_0 = (x_0, y_0)$)
R	gamma	alpha	core width
β	alpha	beta	power

The parameter R is called the core width and β is called the power. As in Gaussian case, this is the most widely used form in numerical calculation.

Normalizing as we did for Gaussian, $\int_{-\infty}^{+\infty} f_{\text{Moffat}}(r) 2\pi r dr = 1$, we have the constant $A = \frac{\beta-1}{\pi R^2}$. Therefore, Moffat profile has a restriction that $\beta > 1$. The Moffat profile for the integrated flux I is, similar to Gaussian,

$$f_{\text{Moffat}}(r|I, R, \beta) = I \frac{\beta-1}{\pi R^2} \left[1 + \left(\frac{r}{R} \right)^2 \right]^{-\beta} \quad (2.18)$$

Similar to Gaussian, this is also widely used.

The FWHM is calculated by setting $f_{\text{Moffat}}(\text{HWHM}) = \frac{A}{2}$. As we did in Gauss example, we obtain $\text{HWHM}^2 = R^2(2^{1/\beta} - 1)$,

$$\text{FWHM}(\text{circMoffat}) = 2R\sqrt{2^{1/\beta} - 1} . \quad (2.19)$$

Using FWHM (F for brevity), we re-formulate Moffat profile eq. (2.18)*:

$$f_{\text{Moffat}}(r|I, F, \beta) = I \frac{4(2^{1/\beta} - 1)(\beta - 1)}{\pi F^2} \left[1 + 4(2^{1/\beta} - 1) \left(\frac{r}{F} \right)^2 \right]^{-\beta} . \quad (2.20)$$

Thm 7 (High power Moffat is Gaussian)

For $\beta \rightarrow \infty$,

$$f_{\text{Moffat}}(r|I, F, \infty) = f_{\text{Gauss}}(r|I, F) . \quad (2.21)$$

Practically $\beta \gtrsim 100$ is enough for this approximation.

Proof: For $\beta \rightarrow \infty$, we approximate $2^{1/\beta} - 1 \approx \frac{\ln 2}{\beta}$, so

$$f_{\text{Moffat}}(r|I, F, \infty) \approx I \frac{\tilde{A}}{\pi} \lim_{\beta \rightarrow \infty} \frac{\beta-1}{\beta} \left[1 + \frac{\tilde{A} r^2}{\beta} \right]^{-\beta} ,$$

for $\tilde{A} = \frac{4 \ln 2}{F^2}$. Note that $\lim_{\beta \rightarrow \infty} \left[1 + \frac{\tilde{A}}{\beta} \right]^{-\beta} \equiv e^{-\tilde{A}}$ and $\frac{\beta-1}{\beta}$ becomes unity. Substituting these, we get the approximated formula identical to eq. (2.16), or to eq. (2.13) with eq. (2.15). Q.E.D.

*Trujillo+2001, MNRAS, 328, 977

2.5.3 Penny

It was Franz*, who first used Lorentzian-like profile for stars. Then Penny† used it with linear interpolation in sky level to do the UBV photometry for visual binaries close to each other. After that, Penny and Dickens‡, using their own package (**STARMAN**), introduced Gaussian and Lorentzian mixed model§.

The original profile suggested by Penny is a sum of a circular Gaussian and elliptical modified-Lorentzian:

$$f(r|A, Q, \sigma_1, \sigma_2, \sigma_3, \sigma_4, P, \theta, P_Q, R_Q) = A \left\{ \frac{1}{1 + \left[\left(\frac{x'}{\sigma_1} \right)^2 + \left(\frac{y'}{\sigma_2} \right)^2 \right]^{\frac{P}{2}} \left(1 + \sqrt{\left(\frac{x'}{\sigma_3} \right)^2 + \left(\frac{y'}{\sigma_4} \right)^2} \right)} + Q \exp \left[- \left(\frac{r}{R_Q} \right)^{P_Q} \right] \right\}, \quad (2.22)$$

for

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \cos \theta & \sin \theta \\ -\cos \theta & \sin \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\mathbf{r}_0 = (x_0, y_0)$ is the center, and θ is the tilting angle of the Lorentzian. The notation $\sigma_{1,2,3,4}$ are R_{maj} (**RX**), R_{min} (**RY**), RP_{maj} (**PRX**), and RP_{min} (**PRY**) in the **POORMAN**'s notations. It is no more widely used, because of the large number of free parameters.

In IRAF, it is implemented¶ in two ways: **penny1** and **penny2**.

||

Simple way of using it is, assuming $Q = 0$, $P = 2$, and $\sigma_3, \sigma_4 \rightarrow \infty$, so that

$$f(r|A, \theta, \sigma_1, \sigma_2), \quad (2.23)$$

2.5.4 IRAF

In IRAF **imexamine** task, the centroiding is done by fitting Gaussian or Moffat to the pixel values. The weighting for each pixel is

$$w = \begin{cases} r^{-2} & r \leq \text{HWHM} \\ e^{-(\frac{r}{\text{HWHM}} - 1)^2} & r > \text{HWHM} \end{cases}, \quad (2.24)$$

*FranzOG 1973, JRASC, 67, 81

†PennyAJ 1979, MNRAS, 187, 829

‡PennyAJ & DickensRJ, 1986, MNRAS, 220, 845, which is cited for more than 110 times as of Oct 2019

§Unfortunately, however, both the original Penny & Dickens paper and Penny (1995, StaUN 141) pp.44–45 have so many typos in the star profile formulae...

¶The source code is at `noao.digiphot.daophot.dailib.profile.x`.

||Although there is no reference, “Photometry using IRAF v. 2.5.1” (2012-03-25) by Keunhong Park, Jinhyuk Ryu, Insung Jang, and Ho Seong Hwang notes that (p. 8): “**penny1** or **penny2** of IRAF fits the analytical stellar profiles well, while **moffat15** or **moffat25** fits the stellar profiles well if observed on cloudy nights.” I cannot even understand the meaning of it...

which is different from usual least square statistic (which uses $w = 1/\sigma^2$).

Two more parameters are calculated, namely, ellipticity e and angle pa :

$$e = \sqrt{\frac{(M_{xx} - M_{yy})^2 + 4M_{xy}^2}{M_{xx} + M_{yy}}} \quad ; \quad \text{pa} = \frac{1}{2} \text{atan}\left(\frac{2M_{xy}}{M_{xx} - M_{yy}}\right), \quad (2.25)$$

where the moments

$$M_{ij} = \frac{\sum Iij}{\sum I} \quad (2.26)$$

for pixel value I and $i, j \in [x, y]$.

In realistic photometry, such as IRAF `psfmeasure` task, we must use $I - \text{sky}$ instead of I .

Some old packages including ROMAFOT, STARMAN, DAOPHOT, DoPHOT, WOLF, LUND, CAPELLA are summarized in this lecture note and StetsonPB 1992, ASPC, 25, 297

Chapter 3

Standardization

3.1 Problem Statement

The standardization process is a process to convert the instrumental values to physical values. In CCD, what is being read is the potential (\propto number of electrons \propto flux). But what does a CCD pixel value mean? 1 ADU can mean 1 Jy at one CCD but at different one it can mean 5 Jy because it is designed to be insensitive to photons for some reason. Thus, what astronomers do is

1. Make a list of objects which have known flux (e.g., star A has spectrum of blahblah, and it has V_0 magnitude or flux I_0 in the V-band). These stars are called **standard stars**.
2. Observe the target and the standard stars simultaneously.

Now the power of CCD comes in: It's highly linear, i.e., the pixel counts of N (of the target of interest) and N_0 are very much proportional to the original flux, I and I_0 . Thus, you can use Pogson's formula, because what it requires is only the ratio of flux:

$$V - V_0 = -2.5 \log_{10} \frac{I}{I_0} = -2.5 \log_{10} \frac{N}{N_0} . \quad (3.1)$$

Ex 11 (*Simplest Standardization*)

If the aperture photometry gave pixel count of 1000 for a standard star of $V_0 = 10.00^m$ and the object had pixel count of 500, the above formula will give $V = 10.75^m$.

In practice*, it is very difficult to use these formula. Because

1. The atmosphere exists. The magnitude we observe on ground is different from the one we would have observed outside of the atmosphere (space). This gives the k' and k'' terms in eq. (3.2) below.
2. The CCD is not perfect. For example, if it is more sensitive to redder wavelength, making red stars brighter than they should be. This gives the k term in eq. (3.2) below.

After these are considered, we can obtain the following second-order approximation of the standard magnitude of an object seen on CCD:

$$M_f = m_f - k'_f X - k''_f X C + z_f + k_f C \equiv m_{0f} + z_f + k_f C , \quad (3.2)$$

*From here, I extensively referred to Ch. 6 of "A Practical Guide to Lightcurve Photometry and Analysis" by Brian D. Warner, 2e.

where

$$m_f = m_{0f} + k'_f X + k''_f X C \quad (3.3)$$

and

- f : The filter (V, B, g', etc).
- X : airmass (secant of zenith angle).
- M_f : The standard apparent magnitude (or the true apparent magnitude) at filter f .
- m_f : The instrumental magnitude, i.e., $-2.5 \log_{10} N$.
- m_{0f} : The extra-atmospheric magnitude, i.e., the m_f value we would have obtained if we were in space ($X = 0$).
- C : The true color index, e.g., $B - V$ or $r' - i'$.
- k'_f : The first order extinction coefficient at filter f .
- k''_f : The second order extinction coefficient at filter f .
- z_f : The zero point at filter f .
- k_f : The system transform coefficient.

I will explain the terms related to m_{0f} and then the other terms. Note that lower- and upper-cased letters are used for the instrumental and true magnitudes, respectively (e.g., v and V , b and B , $m_{g'}$ and $M_{g'}$, etc).

From eq. (3.2)*:

$$V - V_0 = (v - v_0) + k'_V(X - X_0) + k''_V(XC - X_0C_0) + k_V(C - C_0) + \Delta z_V \neq v - v_0. \quad (3.4)$$

So the calculation given in the example is true only if the airmass of the object and standard star are identical AND the true color indices of them are identical. Otherwise, we cannot simply equate the right hand side of eq. (3.1) ($= v - v_0$) to the left hand side ($= V - V_0$). In space, $X = 0$ and $\Delta z \approx 0$, so only the k_f term remains. This is why space observation is powerful.

3.2 Understanding the Standardization Formula

3.2.1 Atmospheric Extinction

The atmospheric extinction is dependent on the wavelength as in fig. 3.1. The extinction is severe at smaller wavelength, and that is why the sun look redder when it rises or sets (i.e., when airmass is larger).

Consider an object with spectrum $S_0(\lambda)$ is at an airmass of X and underwent atmospheric extinction described by optical depth of $\tau(\lambda)$. The magnitude at filter with profile $f(\lambda)$ (say the filter has non-zero profile at $\lambda \in (\lambda_1, \lambda_2)$) will have the following Pogson's relation:

$$m_f - m_{0f} = -2.5 \log_{10} \left(\frac{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) e^{-\tau(\lambda)X} d\lambda}{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) d\lambda} \right). \quad (3.5)$$

* z_f must be a constant unless the device is affected by external disturbance in our simple model dealt in this chapter. In reality it is true that this zero point fluctuate at each exposure, and that is because of the imperfect readout process of CCD electronics. $\Delta z_V \approx 0$ is assumed in this chapter. Although you may be uncomfortable, but this is what is assumed even in professional space telescope data reduction processes, so please bear with me.

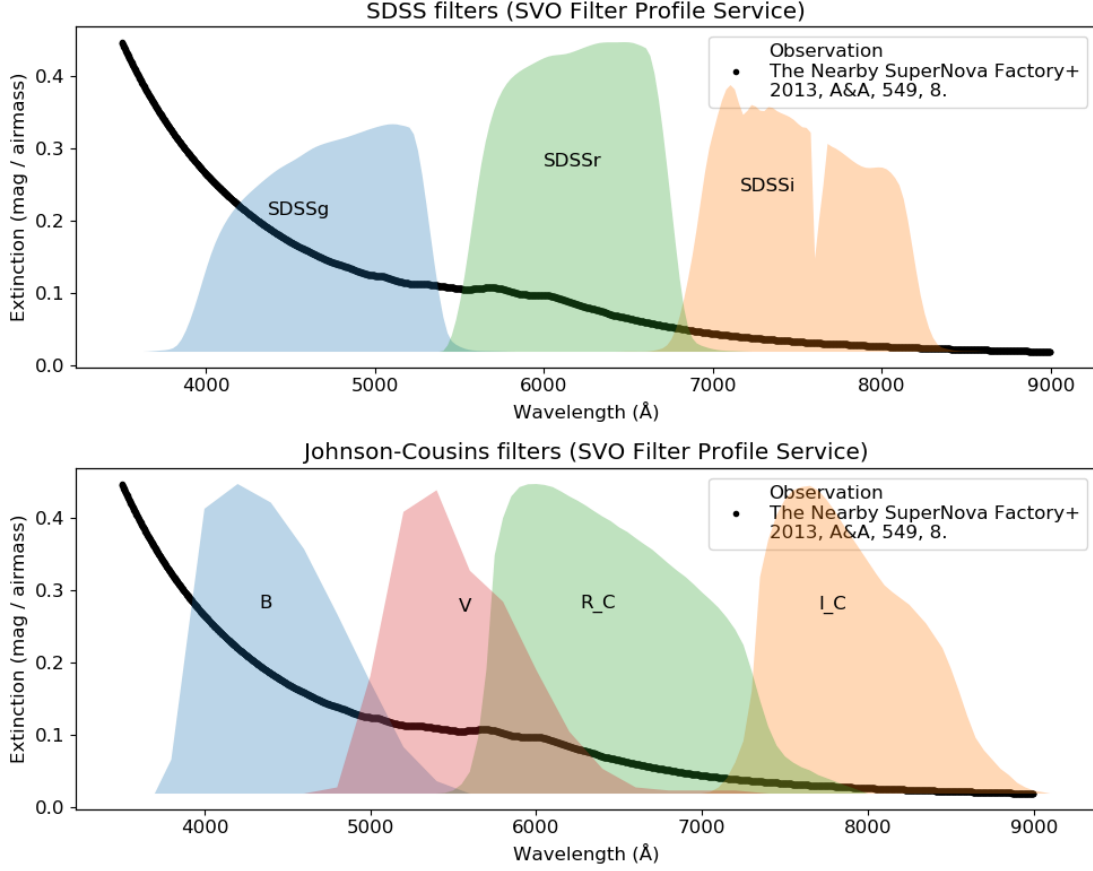


Figure 3.1: The atmospheric extinction as a function of wavelength at Mauna Kea, based on some 4285 standard star spectra obtained on 478 nights spread over a period of 7 years obtained by the Nearby SuperNova Factory using the SuperNova Integral Field Spectrograph. (excerpt from The Nearby SuperNova Factory+ 2013, A&A, 549, 8). The SDSS and Johnson-Cousins filters' filter profiles are overplotted.

Here $e^{-\tau(\lambda)X}$ is actually an approximation of $e^{\int -\tau(\lambda)dX}$. Using Taylor expansions, $e^{-\tau(\lambda)X} \approx 1 - \tau(\lambda)X$ and $\log_{10}(1 - A) \approx -A/\ln 10$,

$$m_f - m_{0f} \approx -2.5 \log_{10} \left(1 - \frac{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) \tau(\lambda) d\lambda}{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) d\lambda} X \right) \approx \frac{2.5}{\ln 10} \frac{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) \tau(\lambda) d\lambda}{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) d\lambda} X. \quad (3.6)$$

Remembering $I = I_0 e^{-\tau X}$, we have $\Delta m = -2.5 \log_{10}(I/I_0) = 1.086 \tau X$. Then the y -axis of fig. 3.1 is $\sim 1.1\tau$, so you can roughly understand that the y -axis of the figure is similar to τ . Therefore, $\tau X \ll 1$ is reasonable. In cases such as short wavelength (shorter than B/g) and high airmass observation, this assumption may break down. The error due to the approximation, however, may not be severe compared to other error sources (e.g., changing weather).

Now we want to further assume that, $\tau(\lambda) \approx c_1 + c_2 \lambda$ within the wavelength range of (λ_1, λ_2) . This is similar to approximating the black markers in fig. 3.1 within each filter as a line because its

y-axis is nothing but 1.1τ . Then

$$m_f - m_{0f} \approx 2.5 \left(c_1 + c_2 \frac{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) \lambda d\lambda}{\int_{\lambda_1}^{\lambda_2} S_0(\lambda) f(\lambda) d\lambda} \right) X. \quad (3.7)$$

If the filter is fixed (e.g., V-band or SDSS g' filter, etc), the only unknown thing in the second term in the parenthesis is $S_0(\lambda)$, i.e., the spectral shape. If it is a black body spectrum, it is true that the spectral shape has a one-to-one relationship with the color index*. Even if it is not a perfect black body, it is reasonable to assume the spectral shape, $S_0(\lambda)$, and color index, C , have *nearly* one-to-one relationship. Thus, C is an indicator of $S_0(\lambda)$, so the second term is roughly a function of C , say $c_2 \tilde{S}(C)$. The final assumption we make here is that the second term is $c_2 \tilde{S}(C) \approx c_3 + c_4 C$ as the first-order approximation. Here c_3 and c_4 depends on the *filter* profile, but not on the spectral shape, under our simplifying assumptions. We denote them with subscript f which means that filter:

$$m_f - m_{0f} \approx 2.5(c_1 + c_{3f} + c_{4f}C)X \equiv k'_f X + k''_f CX \quad (3.8)$$

These are the origins of k'_f and k''_f in eq. (3.2).

To illustrate the result, I used the SDSS filter system as shown in fig. 3.2 and calculated how much magnitude extinction happens depending on the black body temperature:

Temperature	$m_{g'} - m_{0g'}$	$m_{r'} - m_{0r'}$	$m_{i'} - m_{0i'}$
3000 K	0.142 ^m	0.081 ^m	0.034 ^m
6000 K	0.158 ^m	0.084 ^m	0.035 ^m
20 000 K	0.171 ^m	0.087 ^m	0.036 ^m

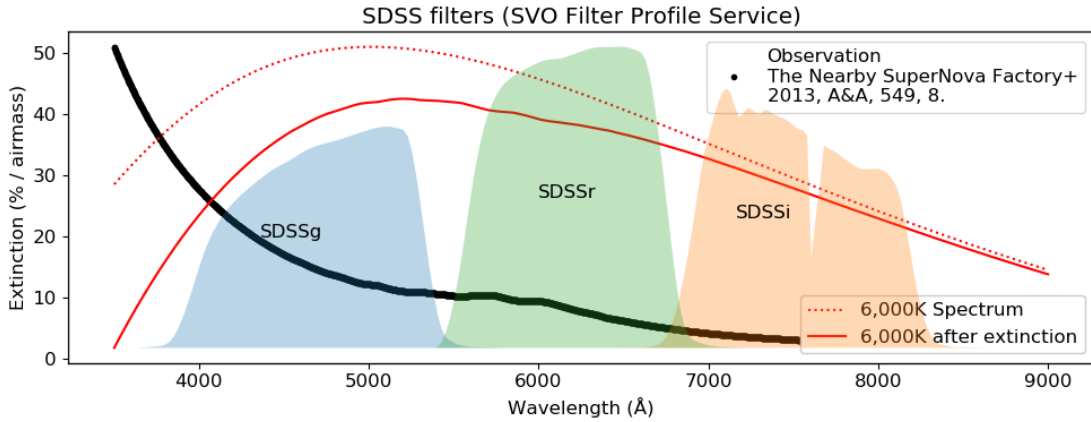


Figure 3.2: A black body radiation spectrum ($T = 6000$ K) before and after the extinction at SDSS bands. Note that the y-axis is changed to % per airmass (cf. fig. 3.1) by $10^{-0.4\Delta m}$.

*This is not strictly true. For example, at temperatures between $T = 15000$ K and 40000 K, the color indices are nearly constant as $B - V = -0.20$ and -0.30 . Furthermore, color index is not one-to-one with temperature between $30,000$ and $35,000$ K, i.e., different temperature (different $S(\lambda)$) can have the same color index. This happens because filter profile is not “flat” as a function of λ .

As can be seen, the extinction is stronger for high temperature objects (lower color index), so likely $k_f'' < 0$. The difference of extinction between the objects gets smaller as we look at longer wavelength. Thus, $k_f'' \sim 0$ except for B or u filters.

These facts can be understood intuitively. The higher the temperature the spectrum will have more fraction of energy at shorter wavelength. Considering that the atmospheric extinction is stronger at shorter wavelength (see black markers in fig. 3.2), high temperature object will get more “penalty” when it comes into the atmosphere. That is why high temperature object is more strongly extincted. But as can be seen, the extinction gets nearly constant at wavelengths of r and i bands. Then this effect will get weaker and that is the reason for smaller discrepancy in Δm values at longer wavelength.

SmithJA+ (2002, AJ, 123, 2121) determined the coefficients for SDSS filters at 1.0 m Ritchey–Chrétien telescope at the USNO Flagstaff Station, from the observations in 1998–2000:

Table 3.1: The extinction coefficients of SDSS from SmithJA+ (2002, AJ, 123, 2121).

Parameter	u'	g'	r'	i'	z'
k_f'	$> +0.5$	$+0.20 \pm 0.05$	$+0.10 \pm 0.05$	$+0.05 \pm 0.05$	$+0.05 \pm 0.05$
k_f'' method 1	-0.021 ± 0.003	-0.016 ± 0.003	-0.004 ± 0.003	$+0.006 \pm 0.003$	$+0.003 \pm 0.003$
k_f'' method 2	-0.032	-0.015	0.000	$+0.005$	$+0.006$

The k_f' value fluctuates much for each night (see Fig 6 of the original publication), so I just took representative values from visual inspection. The k_f'' values were obtained by two independent pipelines, `superExcal` (method 1) and `solve_network` (method 2). Both are quite consistent except for the u filter.

The atmospheric extinction coefficients (k_f' , k_f'') all change as a function of time. We just hope they are reasonably constant during our observation of our targets and standard stars. From experience, we know k_f'' is almost very small for filters longer than u or B-band, but is not necessarily ignorable because $k_f''XC$ might be larger than the accuracy you want to obtain.

Ex 12 (Ignoring the Second Order Extinction Term)

Consider an observation at $X = 2$ of $C = 0.2$ star. If you ignore the second order extinction term, you are making $k_f''XC = 0.4k_f''$ of uncertainty. According to table 3.1, this is most likely smaller than 0.01 magnitude.

The Sun has $C = g' - r' \sim 0.5$, and the error becomes $1.0k_f''$ and it is $< 0.01^m$ only for riz bands.

The red M0 stars have $C = g' - r' \lesssim 1.5$ and the error is now $3.0k_f''$. The accuracy of 0.01^m can be achieved in riz bands, but risky.

The calculation above is only for SDSS observatory at altitude of 2.3 km. But fortunately, BuchheimB (2005, SASS, 24, 111) found that the $k_f'' \lesssim 0.005$ for V band even at many low-altitude observatories (including Bochum observatory at altitude 200 m, Vainu Bappu Observatory at altitude 700 m), so likely we expect the correction from the second-order extinction term is small enough.

3.2.2 Transformation Coefficient

The sensitivity of the optics is also a function of λ . The argument is identical to atmospheric extinction, but there is no X (similar parameter will be something like the optical depth of materials blocking CCD pixel, but that should be a device-dependent constant). Then the same logic leads us to the conclusion that there should be a color term which tunes the final output of the CCD count, and that is the $k_f C$ (“transformation”) term.

The transformation coefficient, which I denoted k_f , is fortunately nearly constant for the given device. Warner argues that it is enough to update k_f (Warner uses notation of T_f) value only about 2–4 times a year, unless you physically changed the device elements (e.g., filter, CCD, lens, etc). Moreover, from experience, we know that this is nearly zero: $|k_f| \lesssim 0.1$. Many cases $|k_f| \lesssim 0.01$. Since the range of color indices are $\max(\Delta C) \lesssim 1^m$, we have $|k_f C| \lesssim 0.1$, and in many cases, $|k_f C| \lesssim 0.01$.

3.2.3 A Note on Linearity

As we noted at the beginning part of this chapter, CCD is highly linear. That means, $N = gN_e = \alpha N_\gamma$ where N is the pixel count (after *bias* and *dark* subtraction), N_e is the total number of photo-electrons, g is the electronic gain of the CCD (a constant; unit of counts per electrons), and N_γ is the photon incident to the CCD from the true photon number $N_{\gamma 0}$. No higher-order terms, no other constants. The α value may differ from pixel to pixel due to the inhomogeneity of optics or CCD pixels, but they are homogeneized by the so-called *flat fielding*, so here I can safely say it is strictly constant over all the pixels.

Any kind of extinction (atmosphere or optics in front of the CCD) is multiplicative only to the linear term, i.e., $N_\gamma = N_{\gamma 0} \times \text{something}_1$ (e.g., $e^{-\tau}$). There is neither higher-order terms like $N_{\gamma 0}^2$ nor addition of constant. Therefore, $N = \text{something}_2 \times N_{\gamma 0}$ and the instrumental magnitude

$$m_f := -2.5 \log_{10} N = \text{something}_3 - 2.5 \log_{10} N_{\gamma 0} \equiv \text{something}_3 + M_f . \quad (3.9)$$

Thus, thanks to the linearity of CCD, we have no additional coefficient *multiplied* in front of m_f or M_f . If, for example, N were $\alpha N_\gamma + \alpha'$, or there were other terms in the extinction (proportional to N^2 or a constant radiation from the optics), this simple relation wouldn't hold.

To emphasize, *you should not worry about whether to multiply something in front of M_f or m_f to satisfy eq. (3.2)*. Their coefficients *must be unity*. From our experiences, most observational experts and electrical engineers would say that you should only care about this if you are sure that some parts of the optics have serious problems (e.g., your CCD underwent serious problem and shows non-linearity).

3.3 Differential Photometry

Now that we justified the usage of eq. (3.2), let's find the cases of application. The simplest case is the *differential photometry*.

If there are many celestial objects (of course including your target) in the field of view with known standard magnitudes, we can use them as standard stars. Although there can be variable stars and galaxies*, if most of the objects with known magnitudes are non-variable stars, those *outliers* will be smoothed out. Thus, we just assume all the celestial objects in the field of view with known standard magnitudes as “standard stars”.

This technique is very widely used in variable star and asteroidal light curve observations. This is widely used than normal (absolute) photometry, because it is annoying and difficult to observe standard stars at different airmasses while observing your target, which requires telescope time and human labor. In asteroidal science, even single-filter differential photometry is frequently used.

3.3.1 Multi-Filter Differential Photometry

When we have more than one filter observation, we can get do the photometry in a rather complete way. As a simple example, consider a 2-filter observation. Rearranging eq. (3.2) gives

$$M_f - m_f = z_f - k'_f X + (k_f - k''_f X)C. \quad (3.10)$$

In the same FOV, they should have almost identical airmass and zero point. If we observe the same field with filter x (at airmass of X_x) and y (at airmass of X_y), and denote $C_{xy} = M_x - M_y$,

$$\begin{cases} M_x - m_x &= (z_x - k'_x X_x) + (k_x - k''_x X_x)C_{xy} = a_x + b_x C_{xy} \\ M_y - m_y &= (z_y - k'_y X_y) + (k_y - k''_y X_y)C_{xy} = a_y + b_y C_{xy} \end{cases} \quad (3.11)$$

Note here that, for field stars with known standard magnitudes, M_x , M_y , and thus $C_{xy} = M_x - M_y$ should all be known for N “standard stars” in the image. The m_x and m_y are direct results from the photometry, so these should also be known from the image. Also, the values involved in a and b factors, there is no color-term, i.e., they should be identical for all the objects in the image[†]. Then what we have to determine are the two sets of constants, (a_x, b_x) and (a_y, b_y) .

If we plot $M_x - m_x$ as ordinate and C_{xy} as abscissa for N field stars (the “standard stars”), the y intercept is a_x and the slope is b_x . Same goes for the y filter. So (a_x, b_x) and (a_y, b_y) are determined with the uncertainties.

Then for the target, what we have to do is to solve

$$\begin{cases} M_x^{\text{target}} - m_x^{\text{target}} &= a_x + b_x C_{xy}^{\text{target}} \\ M_y^{\text{target}} - m_y^{\text{target}} &= a_y + b_y C_{xy}^{\text{target}} \end{cases} \quad (3.12)$$

*Galaxies can have spectra significantly different from those of black bodies. Therefore, the coefficients k_f and k''_f , which are approximations of spectral shape (i.e., not k'_f), should be different from those derived from standard stars, which are black bodies. But mostly this effect is not serious because, as we discussed before, both $k_f C$ and $k''_f X C$ terms are anyway very small. For this reason, people use k_f and k''_f derived from standard stars for their target galaxies (or any non-black body like spectra). If you really worry about this, you must conduct spectroscopic observation, not broad-band photometry.

[†]Of course we are assuming the coefficients, i.e., weather and CCD properties are constant over the time of x and y image exposures.

or since $C_{xy}^{\text{target}} = M_x^{\text{target}} - M_y^{\text{target}}$,

$$C_{xy}^{\text{target}} = \frac{m_x^{\text{target}} - m_y^{\text{target}} + (a_x - a_y)}{1 - (b_x - b_y)}. \quad (3.13)$$

Putting this back to the original equation,

$$\begin{cases} M_x^{\text{target}} = m_x + a_x + b_x \frac{m_x^{\text{target}} - m_y^{\text{target}} + (a_x - a_y)}{1 - (b_x - b_y)} \\ M_y^{\text{target}} = m_y + a_y + b_y \frac{m_x^{\text{target}} - m_y^{\text{target}} + (a_x - a_y)}{1 - (b_x - b_y)} \end{cases} \quad (3.14)$$

Now we have the standard magnitude of the target in both x and y filters.

What if we have more than 2 filters, say x , y , and z ? Solve for x and y as above by using (a_x, b_x) . Then solve for y and z , but this time using color as C_{yz} , not C_{xy} , using (a'_x, b'_x) . Theoretically $a_x = a'_x$, because there is no color term. However, $b_x \neq b'_x$, because k''_y is defined for C_{xy} for the first case, but it is defined for C_{yz} for the second case. In reality, because what we get is only the best-fit values with uncertainties, a_x and a'_x can also be different (likely within certain amount of error-bar).

3.3.2 Single-Filter Differential Photometry

Sometimes

1. we don't have multi-filter (also called the “multi-color”) observation data of the target, or
2. the linear fit to $M - m$ VS C_{xy} is not possible because there are only few stars and/or narrow color range.

For the first case, although we can get (a_x, b_x) , we don't know the C_{xy} (mathematically identical to state that we don't have parameters with subscript y to plug into eq. (3.14)). For the second case, it is just impossible to get (a_x, b_x) . Then we have to assume a little more and do a *single-filter* differential photometry. This is used in the cases when $b_f C = (k_f - k''_f X)C$ is smaller than the photometric accuracy we seek for.

Ex 13 (*Can we really ignore that term?*)

Consider observatins made in wavelength ranges around V or gri bands. From table 3.1, $|k''_f| \sim 0.000-0.020$ and we expect that $k_f \lesssim 0.02$ for most optics. Since k''_f is mostly negative, $k_f - k''_f X$ is likely be positive, but not always (see table 3.1). Also we have $1 < X \lesssim 2$ in most cases. Then if you play with many combinations of numbers, $b_f C = (k_f - k''_f X)C \lesssim 0.05C$ and most likely much smaller than that. Note that the color index is mostly $-1 \lesssim C \lesssim +1$.

Now we simply drop that term because the uncertainty due to this term will be smaller than the final uncertainty. Thus,

$$M_f - m_f = (k_f - k''_f X)C + z_f - k'_f X \approx z_f - k'_f X \equiv Z_f. \quad (3.15)$$

Sometimes people call Z_f just “zero point”, although rigorously the zero point should be z_f .

In the same FOV, they should have almost identical airmass and zero point. Therefore, under the same filter f , we have $Z_f \simeq \text{const}$ for all the objects (of course including our target) in the same

CCD image. This is the different point from previous multi-filter observation, and this happens because we ignored color-dependent terms. What we have to do here is to find that constant.

There are three identical processes to do that, but each of these will show you slightly different impression:

Table 3.2: Differential Photometry Linearity Curves

x -axis	y -axis	Appearance	Comments
M_f	m_f	straight line slope of unity	$ \text{slope} - 1 \gtrsim 0.01$ means somethings are critically wrong (assumptions or your device)
M_f	$M_f - m_f$	constant value regardless of M_f	Scatter is larger for large M_f objects (\because faint object's mag is less accurate)
C	$M_f - m_f$	constant value regardless of M_f	If you can see a clear trend, the color-terms in eq. (3.2) was not negligible.

People many times call the first graph as *linearity curve*, and I will call any process to investigate these three graphs (or other related plots) as linearity curve analysis.

For the first case: I have seen some extreme cases when $|\text{slope} - 1| \sim 0.02$, but that was only when I intentionally did not remove non-stellar objects from the curve*. If I correctly remove them, the slope never deviated more than 0.005.

If graphs do not look as expected, that means (1) many field objects are variable or highly non-blackbody-like galaxies so that they are not suitable as “standard stars” and the basic assumptions of eq. (3.2) break down, (2) the color term is not actually negligible so eq. (3.15) is not a good approximation, (3) catalog M_f suffer from unknown uncertainties or systematic errors so the catalog value is not reliable, (4) m_f was measured incorrectly, (5) many other possibilities.

3.4 Photometry Using Standard Stars

The idea is similar to multi-filter differential photometry, but we use the real standard stars. Except for few fields of standard stars (where many standard stars are located within small FOV), *there is only one single standard star in a CCD image*. This makes it different from the differential photometry.

Consider a standard star with ID = i is observed for N_i times, and denote each observation as j ($j = 1, \dots, N_i$), and write the parameters of eq. (3.2) as $M_{f,i}$, C_i (M and C are fixed values for a standard star, so no j is needed), $m_{f,i}^{(j)}$, $X_{f,i}^{(j)}$, etc. We here assume the zero point, z_f , and the

*So if graphs don't appear as they should, please first check you have correctly removed non-stellar objects from your linearity curve analysis.

coefficients k_f , k'_f , and k''_f are almost constant over the night*. Then

$$\begin{cases} M_{f,i} - m_{f,i}^{(1)} &= z_f + k'_f X_{f,i}^{(1)} + k''_f X_{f,i}^{(1)} C_i + k_f C_i \\ M_{f,i} - m_{f,i}^{(2)} &= z_f + k'_f X_{f,i}^{(2)} + k''_f X_{f,i}^{(2)} C_i + k_f C_i \end{cases} \quad (3.16)$$

for the first two observations of the same standard star (ID = i) at filter f . Subtracting two,

$$\Delta m_{f,i}^{(1,2)} = (k'_f + k''_f C_i) \Delta X_{f,i}^{(1,2)} \quad \rightarrow \quad \left(\frac{\Delta m}{\Delta X} \right)_{f,i}^{(1,2)} = k'_f + k''_f C_i . \quad (3.17)$$

Of course $\Delta m_{f,i}^{(1,2)} = m_{f,i}^{(1)} - m_{f,i}^{(2)}$ and $\Delta X_{f,i}^{(1,2)} = X_{f,i}^{(1)} - X_{f,i}^{(2)}$.

Therefore, if we plot $\left(\frac{\Delta m}{\Delta X} \right)_{f,i}$ as a function of color C_i of the standard stars with different color indices (colors of them are all known), the linear fit will give intercept of k'_f and slope of k''_f . For star i , we have N_i observations, so we have $\binom{N_i}{2} = N_i(N_i - 1)/2$ points at the single C_i value. If we have N standard stars of wide color range, we can have $N \sum_i \binom{N_i}{2}$ points to fit the linear line.

*Even SDSS standard stars were also observed and analyzed under this assumption. See SmithJA (2002, AJ, 123, 2121)

Chapter 4

Statistics - Bayesian

Spectroscopy is a great starting point for Bayesian statistics to those who're interested in astronomy. This is because it has plenty of data points along the x -axis (e.g., wavelength) with error-bars, and we usually fit a simple analytic function to the dataset.

Consider you have the 1-D spectrum as in fig. 4.1. The big question is this: **How likely is that the peak is an actual line(s), not due to random noise?**

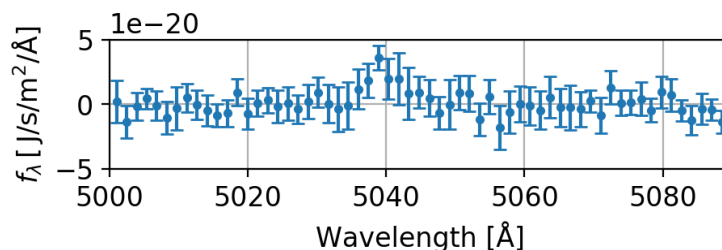


Figure 4.1: The data extracted from Greco+2018ApJ, without Gaussian fit. Note that $1 \text{ erg/s/cm}^2/\text{\AA} = 1 \text{ mW/m}^2/\text{\AA}$. See the notebook `Spectroscopy_Simulation` for the codes I used in this chapter.

To test this, let's set up hypotheses: The null hypothesis (H_0) is that there is no line, while the alternative hypothesis (H_1) is that there *is* a line*. Note that we haven't specified the properties of the line (amplitude, width, etc) yet. Then our strategy must be

1. We have to check which hypothesis is more likely (assuming there're only two possibilities, H_0 and H_1).
2. If H_0 is more likely, we accept the null hypothesis, i.e., “we argue the non-existence of line in this data with certainty of xxx”.
3. If H_1 is more likely, we have to find the line properties (amplitude, width, etc) in the form of $x \pm dx$.

The first is called the **Model Selection** and the third is called the **Parameter Estimation**. If the second is the case, things are simple: no line! That's all. Model selection is *discrete* (finite number of different hypotheses), while parameter estimation is usually *continuous*.

*Reminder: *the null hypothesis is what we want to reject*.

4.1 Bayes Theorem

Thm 8 (*Bayes' Theorem*)

If H , D , and I are the hypothesis, data, and prior information, respectively, Bayes' theorem states

$$P(H|D, I) = \frac{P(H|I) \times P(D|H, I)}{P(D|I)} . \quad (4.1)$$

Usually it is simply put as

$$P(H|D, I) \propto P(H|I) \times P(D|H, I) . \quad (4.2)$$

The **posterior** $P(H|D, I)$ is understood as “the probability that the hypothesis is true, given the data equals to D and we have prior information I .” The **prior** $P(H|I)$ can be quite subjective, since it means “the probability that the hypothesis is true.” In most cases, we take either uniform prior or Jeffrey's prior (see below). The denominator, $P(D|I) := \int P(D|H, I)P(H|I)dH$ is understood as the normalization constant for the posterior.

The term $P(D|H, I)$ is called the **likelihood**, and this is what we have calculated during our high school course works:

Ex 14 (*High school exam problem: calculating liklihood*)

Given the hypothesis that a person has possibility 0.5 to win a game, what is the probability for her to win 3 games in a row?

In this case, the hypothesis is $H = (p = 0.5)$, and the data $D = (\text{win 3 in a row})$. The prior knowledge is that “the game rule does not change”, “the person doesn't use foul measures to make the possibility to change over time”, etc, which are just trivial assumptions. Most importantly, we have to assume that the results of each game should be *independent*, so that we can use the multiplication law. Therefore,

$$P(D|H, I) = (1/2)^3 = 1/8 .$$

In real scientific problems, I can be something like “the fundamental physical laws do not change over time”, “this galaxy is spiral for sure (if you're working on rotational curves of S galaxies, you may assume your samples are definitely S, not E or Irr)”, etc.

When you are finding the mean of the data, it is a single parameter case, and therefore, H just means the mean value, so you can write something like this: $P(H|D, I) = P(m|D, I)$. When you have multiple parameters, usually people denote the set of paramers as $\vec{\theta}$, so $P(H|D, I) = P(\vec{\theta}|D, I)$. If we're fitting the data with, say, a 3rd order polynomial with 4 parameters, we can understand that I includes “the data is described by a 3rd order polynomial with 4 parameters”.

4.2 Towards the Model Selection

4.2.1 Model Selection Concept

The term “odd” is used for $\frac{\text{probability}}{1-\text{probability}}$. In this case the $P(H_1|I)/P(H_0|I)$ is called the prior odds because $P(H_1|I) + P(H_0|I) = 1$, as there is no other possibility other than H_0 and H_1 . Similarly, since $P(D|H_0, I) + P(D|H_1, I) = 1$, the LHS is also called the posterior odds.

Now come back to the original question: model selection. The model selection is done based on the **odds ratio**:

$$R := \frac{P(H_1|D, I)}{P(H_0|D, I)} = \frac{P(H_1|I)}{P(H_0|I)} \times \frac{P(D|H_1, I)}{P(D|H_0, I)} = \text{odds}(\text{prior}) \times B_{10} . \quad (4.3)$$

Here B_{10} is called the **Bayes' factor**.

If $R > 1$, we select H_1 over H_0 and vice versa. If $R = 1$, we can't give any conclusion. Equivalently, we can use $\ln R$ and compare it with 0. This is because we frequently get extremely large or small R values (e.g., 10^{-40}), which is inappropriate for some computer programming.

4.2.2 Prior Selection

There are two major priors: uniform and Jeffreys' prior.

- **Uniform prior** assumes uniform probability for the model within reasonable range.
- **Jeffreys' prior** assumes the probability proportional to the determinant of the Fisher information matrix. Simply put, for the parameters like standard deviation, if the value is small (has more information), we give more weight to it.

Another possibility is that iteratively updated prior based on the accumulated data, as sometimes used in artificial intelligence.

Ex 15 (*Uniform prior*)

Consider an emission line fitting to the 1-D spectrum. If you are sure that the line must be Gaussian with center $\lambda_c \in [650, 655]$ nm, the uniform prior of the line center will be $\lambda_c \sim \mathcal{U}(650, 655)$, which has the pdf of $p(\lambda_c) = 1/5$ for $\lambda_c \in [650, 655]$ nm and 0 otherwise. That is, uniform prior regards it is equally likely to have any value within the bound specified by the user.

4.2.3 Likelihood Calculation

Assume all data values, e.g., the f_λ values, are independent*. Denoting the i -th pixel's value as D_i ,

$$P(D|H, I) = P(D_1, \dots, D_N|H, I) = P(D_1|H, I) \times \dots \times P(D_N|H, I) .$$

Under H_0 (no emission line), $P(D_i|H_0, I)$ is the probability to measure D_i electrons when there is only the pixel noise σ_i (no actual line). Each pixel we have Poissonian noise which is nearly Gaussian, and sky estimation noise term, which is difficult to quantify but we usually assume it is Gaussian, and the Gaussian readout noise in standard CCD. Therefore, each pixel is assumed to have a Gaussian noise. Thus,

$$P(D_i|H_0, I) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left\{-\frac{(D_i - 0)^2}{2\sigma_i^2}\right\} \quad \rightarrow \quad P(D|H_0, I) = \prod_{i=1}^N P(D_i|H_0, I)$$

*A strange thing happens on CCD sometimes maybe? If this is true, then nothing is independent. See BooneK+18 PASP (2018PASP...130f4504B) "A Binary Offset Effect in CCD Readout and Its Impact on Astronomical Data".

Then the **log-likelihood** is

$$\ln P(D|H_0, I) = C_\sigma - \sum_{i=1}^N \frac{D_i^2}{2\sigma_i^2}, \quad (4.4)$$

where $C_\sigma := \sum \ln(\sqrt{2\pi}\sigma_i)^{-1}$ is a constant. Note that even the second term is a calculable constant once the data is given.

For H_1 , we can just change the zero mean to the Gaussian line profile. Consider the best-fit line profile is described as $g(x|\vec{\theta}_0)$ for $\vec{\theta}_0 = (A, w, \lambda_c)$ (amplitude, width sigma, central wavelength). Then

$$\ln P(D|H_1, I) = C_\sigma - \sum_{i=1}^N \frac{(D_i - g(x_i|\vec{\theta}_0))^2}{2\sigma_i^2} \equiv C_\sigma - \frac{1}{2}\chi^2. \quad (4.5)$$

Here, χ^2 is the usual chi-square statistic from the data and model, because in our case, the error-bars are independent and normally distributed. Finding the best-fit parameter set, $\vec{\theta}_0$, is done by the least-square fitting (also called the χ^2 -minimization).

Although it is mathematically too trivial, let me put another theorem to emphasize.

Thm 9

Maximizing (log-)likelihood is identical to minimizing χ^2 , when the error-bars are independent and follows Gaussian.

4.2.4 Model Selection Calculation

Now let's do the real calculation to compare H_0 and H_1 . Because I want to use amplitude rather than flux as a free paramter, the gaussian function will be $g(x|\vec{\theta}_0) = Ae^{-(\lambda-\lambda_c)^2/2w^2}$. The best fit Gaussian function to the data shown in fig. 4.1 is found to have the following parameters:

$$\text{amplitude} = 3.213 \times 10^{-20} \quad ; \quad \lambda_c = 5039.4 \text{ \AA} \quad ; \quad w = 2.33 \text{ \AA} \quad (4.6)$$

with the integrated flux $\log_{10}(F_{\text{OIII}}/\text{mW m}^{-2}) = -15.73$, because $F = A\sqrt{2\pi w^2}$. This matches well with the original publication -15.7 ± 0.1 .

From the data, $C_\sigma = 2739.234$. The log-likelihood of H_0 and H_1 are

$$\begin{aligned} \ln P(D|H_0, I) &= C_\sigma - \sum_{i=1}^N \frac{D_i^2}{2\sigma_i^2} &&= 2720.810 \\ \ln P(D|H_1, I) &= C_\sigma - \sum_{i=1}^N \frac{(D_i - g(x_i|\text{amplitude}, \lambda_c, w))^2}{2\sigma_i^2} &&= 2730.136. \end{aligned}$$

Assume the probability of H_0 being true is the same as that of H_1 being true. The Bayes' ratio in eq. (4.3) therefore becomes just a Bayes' factor:

$$R = B_{10} = \frac{e^{2730.136}}{e^{2720.810}} = 1.12 \times 10^4 \gg 1, \quad (4.7)$$

which means H_1 is extremely more likely. Thus, we now believe there must be an emission line, and have to find the CI of the parameters (e.g., flux value).

4.2.5 Model Selection with AIC, BIC

But wait, is this all? No.

The pitfall of this naïve approach using Bayes' ratio is that, you can minimize χ^2 to 0, by increasing the number of fitting parameters. When the number of free parameters is equal to the number of data points, you must be able to make a function f such that $D_i - f(\lambda_i)$ is always 0. Does the N -parameter model a better choice than 1-parameter case, for example, if just the ratio is large? It can't be.

The “number of paramter” problem is not an easy thing to solve, but we have simple rule-of-thumbs: The Akaike Information Criteria, AIC, and the Bayesian Information Criteria, BIC. The derivations are not shown here, because it can become a bit lengthy while that derivation itself is not at the heart of the understanding.

Thm 10 (*Bayesian Information Criterion; BIC*)

For the $N(\rightarrow \infty)$ data points, the BIC for the model with n free parameters denoted as $\vec{\theta}$ is given as

$$\text{BIC} := n \ln N - 2 \ln P(D|\vec{\theta}_0, I) = n \ln N + \chi_{\min}^2 - 2C_{\sigma} , \quad (4.8)$$

where $\vec{\theta}_0$ is the best-fit parameter which results in the minimum χ^2 , or maximum likelihood ($P(D|H, I)$). For the same data, comparing with two different models, the difference in the BICs is used:

$$\Delta \text{BIC} = (n_1 - n_2) \ln N - 2 \ln \frac{P(D|\vec{\theta}_{0,1}, I)}{P(D|\vec{\theta}_{0,2}, I)} = (n_1 - n_2) \ln N + (\chi_{\min,1}^2 - \chi_{\min,2}^2) . \quad (4.9)$$

The second equalities above including χ^2 hold only if the error-bars are independent and normally distributed.

Some people prefer to define in the opposite sign and/or half of this value to remove the factor 2 in front of the log-likelihood.

Note that BIC is usable only if $N \gg n$. Also the prior distribution only affects when you find $\vec{\theta}$, not when calculating the BIC.

Since RafteryAE's work*, the following criteria for choosing models are widely used:

$ \Delta \text{BIC} \in$	$[0, 2]$	$[2, 6]$	$[6, 10]$	$10+$
Evidence	Not worth more than a bare mention	Positive	Strong	Very Strong

*RafteryAE (1995) “Bayesian Model Selection in Social Research”, Sociological Methodology, 25, 111

Thm 11 (*Akaike Information Criterion; AIC*)

For the $N(\rightarrow \infty)$ data points, the BIC for the model with n free parameters denoted as $\vec{\theta}$ is given as

$$\text{AIC} := 2n - 2 \ln P(D|\vec{\theta}_0, I) = 2n + \chi_{\min}^2 - 2C_\sigma, \quad (4.10)$$

where $\vec{\theta}_0$ is the best-fit parameter which results in the minimum χ^2 , or maximum likelihood ($P(D|H, I)$). For the same data, comparing with two different models, the difference in AICs is used:

$$\Delta\text{AIC} = 2(n_1 - n_2) - 2 \ln \frac{P(D|\vec{\theta}_{0,1}, I)}{P(D|\vec{\theta}_{0,2}, I)} = 2(n_1 - n_2) + (\chi_{\min,1}^2 - \chi_{\min,2}^2). \quad (4.11)$$

The second equalities above including χ^2 hold only if the error-bars are independent and normally distributed.

Although I am not so familiar with hard-core statistics, it seems like there are debates about which should be preferred (BIC or AIC). BIC is known to prefer the “true model”, if it is in our set of alternative hypotheses, with probability 1 when $N \rightarrow \infty$, while AIC doesn’t. On the other hand, if the data is too few (note that the difference in BIC and AIC is the $\ln N$ term), BIC tend to seek for the model which explains that small dataset, so it can prefer worse model which tries to explain the bad data points than AIC.

Now let’s calculate ΔBIC and ΔAIC for our sample data to test H_0 VS H_1 . As before, $C_\sigma = 2739.234$, and log-likelihoods are 2720.810 and 2730.136. Then because H_0 has no parameter and H_1 has three paramters, and $N = 61$,

$$\Delta\text{BIC} = (0 - 3) \ln 61 - 2(2720.810 - 2730.136) = 6.32$$

$$\Delta\text{AIC} = 2(0 - 3) - 2(2720.810 - 2730.136) = 12.65$$

The fact that these are positive means we should prefer the H_1 , but not as strong as what we’ve seen from the odd’s ratio.

4.3 Towards the Parameter Estimation

4.3.1 Brute-Force

From the last section, we learned we have a clear evidence that there is a line. Then how can we estimate the line properties with uncertainties? This is the same as we did in the statistics chapter. First is the **brute-force** fixed-grid search scheme

In the chi-square sense, what we have to do are

1. Calculate the chi-square statistic at each parameter space position.
2. Keep only those with $\chi^2 < \chi_{\min}^2 + \Delta(n_\theta, \alpha)$
 - Δ : inverse cdf (cumulative distribution function) of χ^2 distribution.
 - α : significance level ($\alpha = 0.6827$ for 1-sigma)
 - n_θ : number of free parameters.
 - In python, you can do `delta = scipy.stats.chi2.ppf(0.6827, n_param)`
3. These are the models “within 1-sigma level confidence interval.”
4. Get the min/max of each of the parameters and set these as lower/upper limit of the parameters.

5. The *center* of the parameters can be obtained by simple maximum likelihood estimation, such as least-square fitting.

For the 1-sigma contour of 2 parameters, $\chi^2 < \chi_{\min}^2 + 2.30$. The marginalized pdfs is usually drawn together to grasp the distribution of the parameters. If you have used chi-square statistic, you can use the fact that $P \propto e^{-\chi^2/2}$. Define $\bar{P} = e^{-\chi^2/2}$. Then the normalization constant will be $A = \sum_{\text{parameters}} \bar{P}$, i.e., you can use $P = \bar{P}/A$ as the normalized probability values.

In this example, because the authors mentioned that the w parameter is obtained from the $H\alpha$ fitting, I just fixed the w value as the best fit value. From zooming in the original paper, I couldn't find any difference from theirs to ours.

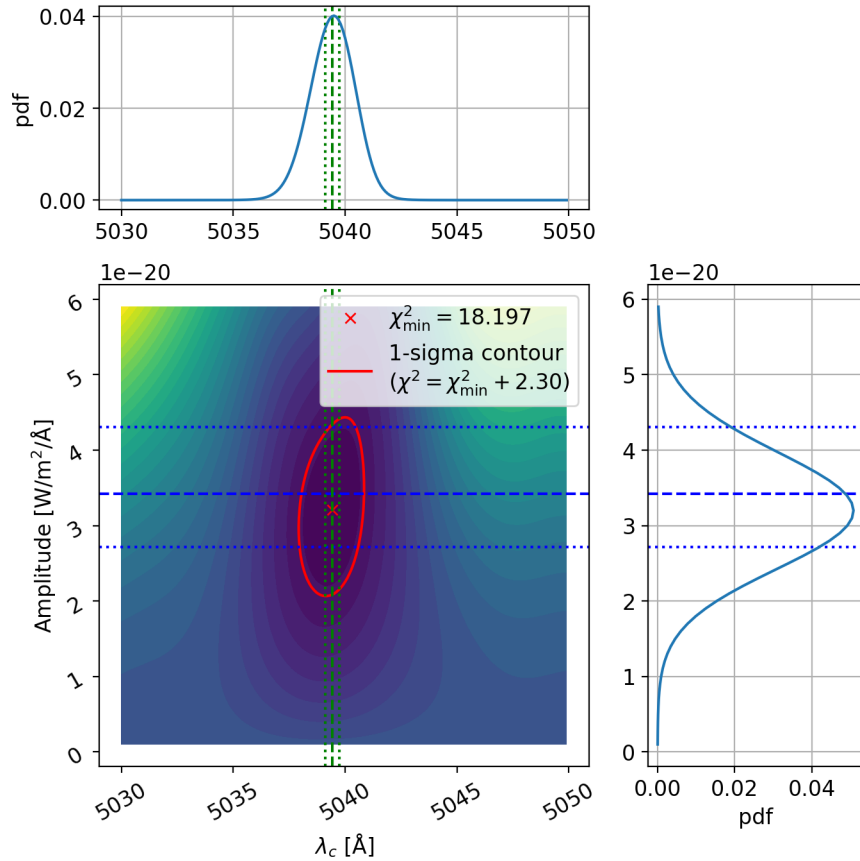


Figure 4.2: Our fitting of Gaussian curve to the emission line. The 2-D contour shows the χ^2 map in the 2-D parameter space. The 1-D plots show marginalized pdf from $P = e^{-\chi^2/2}/\bar{P}$.

In the figure, the green vertical lines:

- dashed = our best fit central wavelength
- dotted = the uncertainty range from the paper's redshift uncertainty measured from $H\alpha$ line (1742 ± 19 km/s)
- Because their uncertainty is from $H\alpha$, which has signal much better than O^{2+} , the error-bar is much smaller than ours.

and the blue horizontal lines:

- dashed = paper's flux ($10^{-15.7}$ mW/m²) converted to amplitude

- dotted = paper’s uncertainty around the paper’s value.
- The difference between our fit and paper’s value is only 0.03 dex (maybe the authors obtained the identical value but just dropped the significance numbers)
- Error bars seem slightly underestimated, but in reality if we use error bar of 0.14 dex in log scale, it’s similar to ours. Maybe the authors just did not care about such detailed numbers, which is understandable.

Notes

The original paper argue that they used gaussian with flat spectrum*. When I tried this, however, the best fit constant is $-1.36\text{e-}21$, which must be a marginally visible negative shift to the fitted Gaussian, but I cannot find this by zooming in the paper’s plot. Thus, I guess they may have just ignored this flat offset or set a bound that this constant must be positive, so that the best fit value is just 0.

4.3.2 Markov Chain Monte Carlo (MCMC)

Markov chain is a fancy name of “memoryless”:

$$p(x_5|x_4, x_3, x_2, x_1) = p(x_5|x_4) \quad (4.12)$$

The Monte Carlo (MC)[†] came from the casino “Monte Carlo” in Monaco. Stanislaw Ulam and John von Neumann, in 1940, wanted to find best-fit parameters to make a better nuclear weapon at the Los Alamos National Laboratory. They arrived at the idea of (now-called) Monte Carlo, but wanted it to be secret to enemies, so they chose the secret name MC, where Ulam’s uncle used to play gamble.

When you have, e.g., 7 parameters to fit 100 data points, and if the model is too complicated, the usual grid searching in the previous section is computationally impossible. Therefore, the “hopping” in the N-dimensional parameter space is suggested and that’s MC. Currently I don’t have complete plan to cover MCMC in AO class, but you may refer to many available packages from websites[‡]. I recommend you to try `emcee` (although I used `pymc3`, but I feel `emcee` is more standard and safe as it’s classical. `pymc3` is too much a black box to my eyes, and my friends kind of agreed).

*From the paper: “When fitting the [O III] $\lambda 5007$ line, we assume a flat continuum plus a single Gaussian line profile with standard deviation given by the $\text{H}\alpha$ fit.”

[†]Historical background adopted from KalinovaV’s lecture note 2017 Feb.

[‡]A great compilation is available here