ASTR 3890 - Selected Topics: Data Science for Large Astronomical Surveys (Spring 2022)

Time Series Analysis: II

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recap: Time Series

A time series is a sequence of random variables $\{\mathbf{X}_t\}_{t=1,2,\cdots}$.

Thus, a time series is a **series of data points ordered in time**. The time of observations provides a source of additional information to be analyzed.

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Since there may be an infinite number of random variables, we consider **multivariate distributions of random vectors**, that is, of finite subsets of the sequence $\{\mathbf{X}_t\}_{t=1,2,\cdots}$.

A **time series model** for the observed data $\{x_t\}$ is defined to be a specification of all of the joint distributions of the random vectors $\mathbf{X} = (X_1, \cdots, X_n)^{\mathrm{T}}$, $n = 1, 2, \cdots$ of which $\{x_t\}$ are possible realizations, that is, at all of these probabilities

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n), \ -\infty < x_1, \cdots, x_n < \infty, \\ n = 1, 2, \cdots .$$

Covariance Matrix

For a N-dimensional random vector $\mathbf{X}$ (such as a time series), one can calculate the **covariance matrix** which gives the covariance between each pair of elements of a given random vector:

If the entries in the column vector $\mathbf{X} = (X_1, X_2, ..., X_n)^{\mathrm{T}}$ are random variables, each with finite variance and expected value, then the covariance matrix $K_{\mathbf{XX}}$ is the matrix whose (i, j) entry is the covariance

$$K_{X_i X_j} = \mathrm{cov}[X_i, X_j] = \mathrm{E}[(X_i - \mathrm{E}[X_i])(X_j - \mathrm{E}[X_j])]$$

where the operator E denotes the **expectation value** of its argument.

Cross-Covariance Matrix

By comparison, the notation for the cross-covariance matrix between two vectors $\mathbf{X}, \mathbf{Y}$ is

$$cov(\mathbf{X}, \mathbf{Y}) = \mathrm{K}_{\mathbf{XY}} = E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^{\mathrm{T}}\right].$$

In detail, the cross-covariance matrix is calculated as:

variance:

$$var(\mathbf{X}) = \frac{\sum_i^N (X_i - \bar{X})^2}{N-1}$$

covariance:

$$cov(\mathbf{X}, \mathbf{Y}) = \frac{\sum_i^N (X_i - \bar{X}) \cdot (Y_i - \bar{Y})^2}{N-1}$$

cross-covariance matrix:

$$C(\mathbf{X}, \mathbf{Y}) = \begin{bmatrix} var(\mathbf{X}) & cov(\mathbf{X}, \mathbf{Y}) \\ cov(\mathbf{X}, \mathbf{Y}) & var(\mathbf{Y})) \end{bmatrix}$$

$$C(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \begin{bmatrix} var(\mathbf{X}) & cov(\mathbf{X}, \mathbf{Y}) & cov(\mathbf{X}, \mathbf{Z}) \\ cov(\mathbf{X}, \mathbf{Y}) & var(\mathbf{Y}) & cov(\mathbf{Y}, \mathbf{Z}) \\ cov(\mathbf{X}, \mathbf{Z}) & cov(\mathbf{Y}, \mathbf{Z}z) & var(\mathbf{Z}) \end{bmatrix}$$

Autocovariance Function

As a time series usually involves a large (infinite in theory) number of random variables, there is a very large number of pairs of variables. So the **autocovariance function** is defined as an extension of the covariance matrix. It is usually denoted by the Greek letter γ and we write

$$\begin{aligned}\gamma(x_{t+\tau}, x_t) &= \mathrm{Cov}(X_{t+\tau}, X_t) \\ &= \frac{1}{n}\sum_{t=1}^{n-|\tau|}(x_{t+|\tau|} - \bar{x}),\ -n < \tau < n\end{aligned}$$

for all indices t and lags τ and where

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t.$$

Stationary Time Series

Definition

A time series $\{X_t\}$ is called **weakly stationary** or just stationary if

- $\mathrm{E}(X_t) = \mu_{X_t} = \mu < \infty$, that is, the expectation of X_t is finite and is not depending on t and
- $\gamma(x_{t+\tau}, x_t) = \gamma_\tau$, that is, for each τ, the autocovariance of $X_{t+\tau, X_t}$ is not depending on t.

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A time series $\{X_t\}$ is called **strictly stationary** if the random vectors $(X_{t_1}, \cdots, X_{t_n})^{\mathrm{T}}$ and $(X_{t_1+\tau}, \cdots, X_{t_n+\tau})^{\mathrm{T}}$ have the same joint distribution for all sets of indices $\{t_1, \cdots t_n\}$ and for all integers τ and $n > 0$. It is written as

$$(X_{t_1}, \cdots, X_{t_n})^{\mathrm{T}} \stackrel{\mathrm{d}}{=} (X_{t_1+\tau}, \cdots, X_{t_n+\tau})^{\mathrm{T}},$$

where $\stackrel{\mathrm{d}}{=}$ means *equal in distribution*.

Stationary Time Series

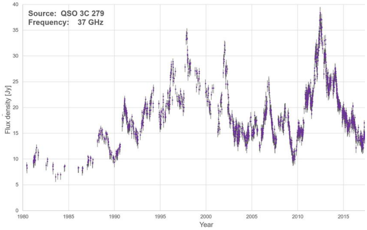
Properties of a Strictly Stationary Time Series:

- The random variables X_t are identically distributed for all t.
- Pairs of random variables $(X_t, X_{t+\tau})^{\mathrm{T}}$ are identically distributed for all t and τ, that is $(X_t, X_{t+\tau})^{\mathrm{T}} \stackrel{d}{=} (X_1, X_{1+\tau})^{\mathrm{T}}$
- The series X_t is a weakly stationary time series if $\mathrm{E}(X_t^2) < \infty$ for all t.
- Weak stationarity does not imply strict stationarity.

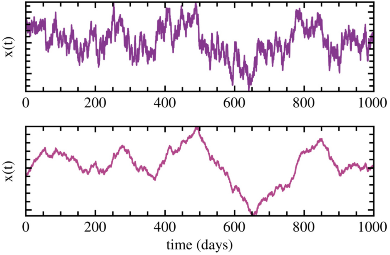
Describing Light Curves as Stochastic Processes

If a system is always variable, but the variability is not (infinitely) predictable, then we have a **stochastic process** (leading to a time series). These processes can also be characterized, but only statistically, not deterministically.

observed quasar light curve:



simulated light curves time series generated by a stochastic process (Moreno et al. 2019):



Stochastic Processes

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Stochastic variability includes behavior that is not predictable forever as in the periodic case, but unlike temporally localized events, variability is always present.

Typically, the **underlying physics** is so complex that we cannot deterministically predict future values. Despite their seemingly irregular behavior, there are a number of ways on how to **quantify and characterize** the data.

Autoregressive Models

Processes that are not periodic, but that nevertheless *retain memory* of previous states, can be described in terms of **autoregressive models**.

Autoregressive models (AR) with dependencies on k past values are called **autoregressive process of order** k and denoted as AR(k). A generalization is called the continuous autoregressive process, CAR(k).

Autoregressive Models

For **linear regression**, we are predicting the dependent variable from the independent variable

$$y = mx + b.$$

For **auto-regression**, the dependent and independent variable is the same and we are predicting a future value of y based on k past values of y:

$$y_i = a_i y_{i-1} + \ldots = \sum_{j=1}^{k} a_j y_{i-j} + \epsilon_i$$

where a_i is the **lag coefficient**.

Autoregressive Models

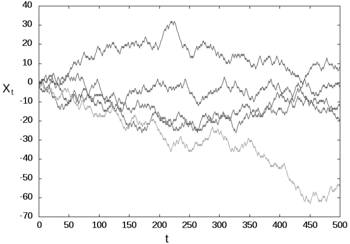
A **random walk** is an example of such a process (with $a_i = 1$, $k = 1$); every value is given by the preceeding value plus noise.

The model can then be written as

$$X_t = X_{t-1} + Z_t,$$

where Z_t is a white noise variable with zero mean and variance σ^2. This model is *not stationary*.

different realizations of a 1D Random Walk time series with 500 time steps



Random Walk

Repeatedly substituting for past variables results in

$$\begin{aligned} X_t &= X_{t-1} + Z_t \\ &= X_{t-2} + Z_{t-1} + Z_t \\ &= X_{t-3} + Z_{t-2} + Z_{t-1} + Z_t \\ &\vdots \\ &= X_0 + \sum_{j=0}^{t-1} Z_{t-j}. \end{aligned}$$

If the initial value X_0 is fixed, then the mean value of X_t is equal to X_0, that is,

$$\mathrm{E}(X_t) = \mathrm{E}\left(X_0 + \sum_{j=0}^{t-1} Z_{t-j}\right) = X_0.$$

Random Walk

The mean is constant, but the variance and covariance depend both on time, not just on the lag.
Since the white noise variables Z_t are uncorrelated, we obtain

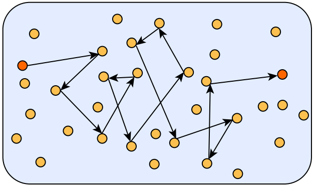
$$\begin{aligned}\mathrm{Var}(X_t) &= \mathrm{Var}\left(X_0 + \sum_{j=0}^{t-1} Z_{t-j}\right) = \mathrm{Var}\left(\sum_{j=0}^{t-1} Z_{t-j}\right) \\ &= \sum_{j=0}^{t-1} \mathrm{Var}\left(Z_{t-j}\right) = t\sigma^2\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(X_t, X_{t-\tau}) &= \mathrm{Cov}\left(\sum_{j=0}^{t-1} Z_{t-j}, \sum_{k=0}^{t-\tau-1} Z_{t-\tau-k}\right) \\ &= \mathrm{E}\left\{\left(\sum_{j=0}^{t-1} Z_{t-j}\right)\left(\sum_{k=0}^{t-\tau-1} Z_{t-\tau-k}\right)\right\} \\ &= \min(t, t-\tau)\sigma^2.\end{aligned}$$

Damped Random Walk

The **Ornstein-Uhlenbeck process** is a stochastic process that originally describes the velocity of a massive Brownian particle under the influence of friction.



Brownian motion is the random motion of particles suspended in a medium.

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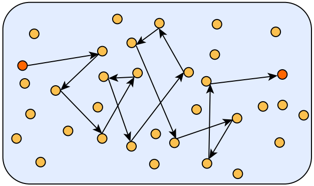


Brownian motion is the random motion of particles suspended in a medium.

The process can be considered as a modification of the random walk but with a **tendency to move back towards a central location**, with a greater **attraction** when the process is further away from the center. The process is stationary.

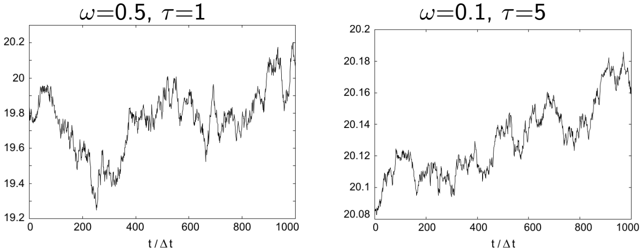
Damped Random Walk

An Ornstein-Uhlenbeck process x_t satisfies the following stochastic differential equation:

$$\mathrm{d}x_t = \tau(\mu - x_t)\mathrm{d}t + \omega\,\mathrm{d}W_t$$

where $\tau > 0$, μ and $\omega > 0$ are parameters and W_t denotes the Wiener process (random walk in continuous time).
The stationary (long-term) variance is given by $\mathrm{Var}(x_t) = \frac{\omega^2}{2\tau}$.

two realizations of a 1D Damped Random Walk, each with a mean of 20



Damped Random Walk

The damped random walk can also be described by its covariance matrix:

$$S_{ij} = \sigma^2 \exp(-|t_{ij}/\tau|)$$

where σ and τ are the model parameters.
σ^2 controls the short timescale covariance ($t_{ij} \ll \tau$), which decays exponentially on a timescale given by τ which is called the characteristic timescale (relaxation time, or damping timescale).

With this, the **autocorrelation function for a damped random walk** is

$$\mathrm{ACF}_{\mathrm{DRW}}(t) = \exp(-t/\tau).$$

Moving Average Process

A **moving average (MA) process** is similar to an AR process, but the value at each time step depends not on the value of previous time step, but rather the **perturbations from previous time steps**.

MA processes are defined by

$$y_i = \epsilon_i + \sum_{j=1}^{q} b_j \epsilon_{i-j}.$$

So, for example, an MA(q=1) process would look like

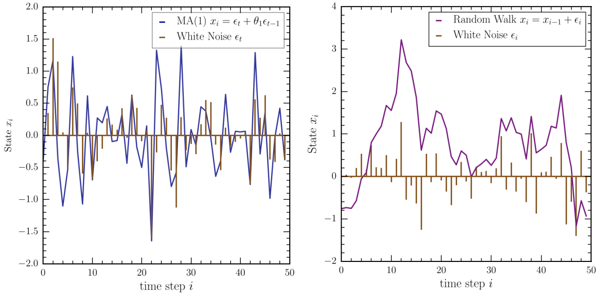
$$y_i = \epsilon_i + b_1 \epsilon_{i-1},$$

whereas an AR(p=2) process would look like

$$y_i = a_1 y_{i-1} + a_2 y_{i-2} + \epsilon_i$$

Moving Average Process

Thus in an MA process a 'shock/impulse' affects only the current value and q values into the future. In an *AR* process a 'shock/impulse' affects all future values. These two plots show the difference between an MA(1) process and an AR(1) (random walk) process:



credit: Moreno et al. (2019)

(Auto-)Correlation Function

problem:

We observe a (stochastically varying) quasar which has both **line and continuum emission** and where the line emission is stimulated by the continuum.

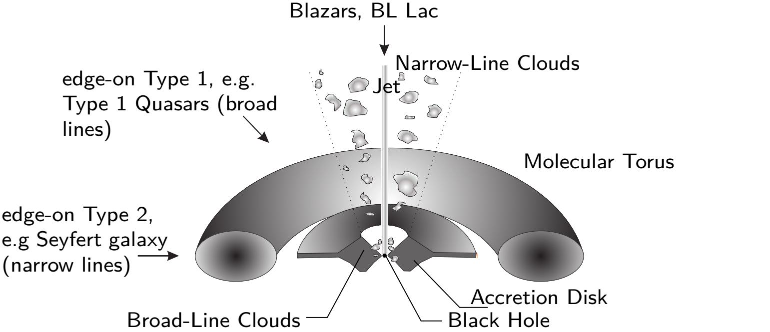
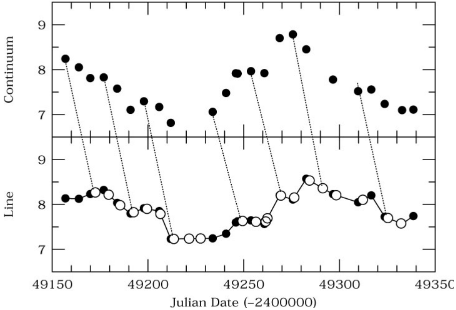


Illustration of the unified AGN model for the radio-quiet case. The classes of AGN are indicated by the viewing angle shown by arrows. Own diagram based on Elvis+2000.

(Auto-)Correlation Function

The physical separation between the regions that produce each type of emission causes a delay between the light curves:



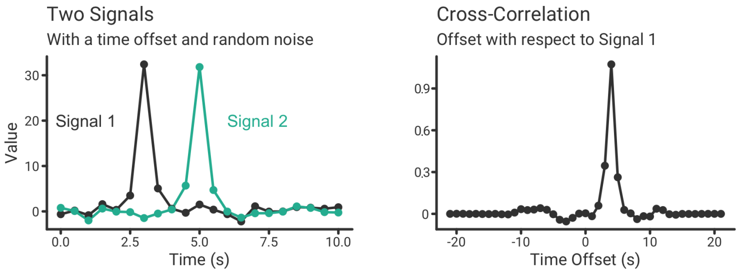
credit: Peterson+2001

We measure line and continuum emission and want to compute the time lag t_{lag}.

Correlation Function

To find a time lag between two time series, we can compute the **correlation function**. If one time series is derived from another simply by shifting the time axis by t_{lag}, then their (cross-)correlation function will have a peak at $\Delta t = t_{\text{lag}}$.

Computing the correlation function is basically the mathematical processes of convolution, i.e., sliding the two curves over each other and computing the degree of similarity for each step in time:



Correlation Function

The **(cross-)correlation function** between time series $f(t)$ and $g(t)$ is defined as

$$\mathrm{CCF}(\Delta t) = \frac{\lim_{T\to\infty} \frac{1}{T} \int_T f(t)\, g(t+\Delta t) dt}{\sigma_f \sigma_g}$$

σ_f and σ_g are the standard deviations of $f(t)$ and $g(t)$, respectively. With this normalization, the correlation function is unity for $\Delta t = 0$.

Correlation Function

For the **autocorrelation function (ACF)**, we take our correlation function from above and set $f(t) = g(t)$.

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Whereas the correlation function yields information about a possible time lag between $f(t)$ and $g(t)$, the autocorrelation function yields information about the variable timescales present in a process. When $f(t)$ values are uncorrelated (e.g., due to white noise without any signal), $\text{ACF}(\Delta t) = 0$ except for ACF(0)=1. For processes that *retain memory* of previous states only for some characteristic time τ, the autocorrelation function vanishes for $\Delta t \ll \tau$.

Correlation Function

Let $x(t_i)$ and $y(t_i)$ be two **discrete, evenly sampled time series**.
In this case, the cross-correlation function as a function of time lag τ is

$$\mathrm{CCF}(\tau) = \frac{1}{N} \sum_{i=1}^{N} \frac{\left[x(t_i) - \bar{x}\right]\left[y(t_i - \tau) - \bar{y}\right]}{\sigma_x \sigma_y},$$

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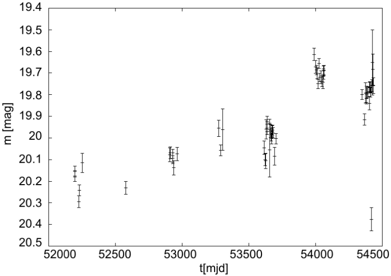
The **accuracy** with which the peak of a CCF can be determined when the data points are evenly spaced has been dealt with by Tonry+1979. The accuracy which is achieved is proportional to the ratio of the height of the peak to the noise level in the CCF divided by the half-width at half-maximum of the peak in the CF.

Correlation Function for Unevenly Sampled Time Series

In practice, for time series being **long-time astronomical data** or other measurements, such as those produced by quasar monitoring programs, $x(t_i)$ and $y(t_j)$ are usually **not known at regular intervals of time**. A considerable body of theory has been developed for this case, how one can carries out time series analysis based on CCF even in this cases.



A quasar light curve from SDSS Stripe 82.

Correlation Function for Unevenly Sampled Time Series

There are in general two approaches for dealing with the uneven sampling: the **Discrete Cross Correlation Function (DCCF)** and interpolation methods like the **Interpolated Cross-Correlation Function (ICCF)**.

These two methods are briefly described in the following.

The Discrete Cross Correlation Function (DCCF)

In the DCCF method (Edelson+1988), a contribution to the DCCF is **calculated only using the actual data points**. Each pair of points, one from each of the light curves, gives one correlation value at a lag corresponding to the time separation. For two light curves with N and M data points respectively this gives an Unbinned Cross Correlation Function (UCCF),

$$\mathrm{UCCF}_{ij} = \frac{(x_i - \bar{x})(y_j - \bar{y})}{\sigma_x \sigma_y}.$$

The DCCF is then obtained by averaging the UCCF in time lag bins. This results in

$$\mathrm{UCCF}_{ij} = \frac{1}{N} \frac{(x_i - \bar{x})(y_j - \bar{y})}{\sigma_x \sigma_y}.$$

where N is the number of pairs.

A point to notice is that a large variation in observational coverage over the light curve can have a strong effect on the DCCF amplitudes.

The Interpolated Cross Correlation Function (ICCF)

In the ICCF method (Gaskell+1986), the light curve is **linearly interpolated** and resampled onto a regular grid. It is common to calculate the ICCF twice, where interpolation is done in each of the light curves, one at a time, and average these.

For two time series x_i and y_i, in the case of interpolation in time series y_i, the cross-correlation function is approximated by

$$\mathrm{ICCF}_{xy}(\Delta t) = \frac{1}{N}\sum_{i=1}^{N} \frac{(x_{t_i} - \bar{x})\{L\left[y_{t_i+\Delta t}\right] - \bar{y}\}}{\sqrt{\sum_{i=1}^{N}(x_{t_i} - \bar{x})^2}\sqrt{\sum_{i=1}^{N}(y_{t_i} - \bar{y})^2}}$$

where L indicates a piecewise linear interpolation of series y at time $t_i + \Delta t$ and N is the number of data points in x.

The piecewise linear interpolation of y is done by

$$L\left[y_{t_i+\Delta t}\right] = \begin{cases} y_{t_i+\Delta t} & \text{if observed for } t_i + \Delta t \\ y_{t_{i-1}} \frac{t_{i+1}-(t_i+\Delta t)}{t_{i+1}-t_{i-1}} + y_{t_{i+1}} \frac{(t_i+\Delta t)-t_{i-1}}{t_{i+1}-t_{i-1}} & \text{else} \end{cases}$$

where $y_{t_{i-1}}$ and $y_{t_{i+1}}$ are indicating the nearest data points available.

Structure Function

The **structure function** is another quantity that is frequently used in astronomy and is related to the ACF:

$$\mathrm{SF}(\Delta t) = \mathrm{SF}_\infty[1 - \mathrm{ACF}(\Delta t)]^{1/2},$$

where SF_∞ is the standard deviation of the time series as evaluated on timescales much larger than any charateristic timescale, τ.

The ACF for a Damped Random Walk (DRW) is given by

$$\mathrm{ACF}(t) = \exp(-t/\tau),$$

where τ is the characteristic timescale (i.e., the damping timescale). Remember that a DRW modeled as an AR(1) has $a_1 = \exp(-1/\tau)$. The **structure function for a DRW** can then be written as

$$\mathrm{SF}(t) = \mathrm{SF}_\infty[1 - \exp(-t/\tau)]^{1/2}.$$

Break & Questions

afterwards we continue with `lecture_9.ipynb` from the `github` repository