Bivariate Random Variables:-

Define two dimensional random variable?

Let X & Y be two random variables defined on the same sample S, then the function (X, Y) that assigns a point in R^2 (R×R), is called two dimensional random variable or bivariate random variables.

We shall now consider Discrete & continuous random variables cases separately.

(a) **Discrete case:-** Let X, Y be two discrete r.v, we define the joint prob. f^n. of X & Y by:

$$P(X=x, Y=y) = f(x,y).$$

where (i) $f(x,y) \geq 0$

(ii) $\sum_x \sum_y f(x,y) = 1$

A joint prob. f^n. of X & Y can be rep. by a joint prob. table as below:

X \ Y	y_1	y_2	---	y_n	Totals $= f_X(x)$
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	--	$f(x_1,y_n)$	$f_1(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$	---	$f(x_2,y_n)$	$f_1(x_2)$
!	!	!	---	---	--
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$	--	$f(x_m,y_n)$	$f_1(x_m)$
Totals $= f_Y(y)$	$f_2(y_1)$	$f_2(y_2)$	--	$f_2(y_n)$	1

Marginal Prob. f^n. of X: The probability that $X = x_j$ is obtained by adding all the entries in the row corresponding to x_j & is given by :-

$$P(X = x_j) = f_1(x_j) = \sum_{k=1}^{n} f(x_j, y_k)$$

For $j = 1, 2, \ldots, m$, they are indicated by the entry totals in extreme right hand column or margins of the table. Thats why $f_1(x_j)$ or simply $f_X(x)$ is called marginal prob. f^n of X.

Illy **Marginal Prob. f^n. of Y:** - is obtained at $Y = y_k$ by adding all the entries in the column corresponding to y_k & is given by :-

$$P(Y = y_k) = f_2(y_k) = \sum_{j=1}^{m} f(x_j, y_k). \quad \text{for } k = 1, 2, \ldots, n.$$

The joint dist^n. f^n of X & Y is defined as:

$$F(x, y) = P(X \le x, Y \le y) = \sum_{u \le x} \sum_{v \le y} f(u, v).$$

(b) **Continuous Case:-** The joint prob. density f^n. for continuous r.v. X & Y is defined by:

(i) $f(x, y) \geq 0$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

The prob X lies b/w a & b while Y lies b/w c & d, is given by:

$$P(a<X<b, c<Y<d) = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dx\,dy$$

The joint distn. fn of X & Y in the conti. case is defined by:-

$$F(x,y) = P(X\le x, Y\le y) = \int_{u=-\infty}^{x}\int_{v=-\infty}^{y} f(u,v)\,du\,dv \quad \text{—(1)}$$

* The density fn is obtained by differentiating the distribution fn. w.r.t x & y. i.e $\frac{\partial^2 F(x,y)}{\partial x\,\partial y}$

* $$P(X\le x) = F_1(x) = \int_{u=-\infty}^{x}\int_{v=-\infty}^{\infty} f(u,v)\,du\,dv$$

& $$P(Y\le y) = F_2(y) = \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{y} f(u,v)\,du\,dv$$

$F_1(x)$ & $F_2(y)$ are the marginal distn. fn of X & Y resp.

* $$f_X(x) = f_1(x) = \frac{dF_1}{dx} = \int_{v=-\infty}^{\infty} f(x,v)\,dv$$

$$f_Y(y) = f_2(y) = \frac{dF_2}{dy} = \int_{u=-\infty}^{\infty} f(u,y)\,du$$

$f_1(x)$ & $f_2(y)$ are called marginal density fn. of X & Y resp.

Q.1) For the following bivariate probability distn of X & Y, find :-

(i) $P(X \le 1, Y = 2)$ (ii) $P(X \le 1)$ (iii) $P(Y \le 3)$

(iv) $P(X < 3, Y \le 4)$

sol

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Soln The marginal distn are given below :-

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$P(X=0) = \frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$P(X=1) = \frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$P(X=2) = \frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$ $P(Y=1)$	$\frac{3}{32}$ $P(Y=2)$	$\frac{11}{64}$ $P(Y=3)$	$\frac{13}{64}$ $P(Y=4)$	$\frac{6}{32}$ $P(Y=5)$	$\frac{16}{64}$ $P(Y=6)$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(i) $P(X \le 1, Y = 2) = P(X=0, Y=2) + P(X=1, Y=2)$
$= 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(X \le 1) = P(X=0) + P(X=1) = \frac{8}{32} + \frac{10}{16} = \frac{28}{32} = \frac{7}{8}$

(iii) $P(Y \le 3) = \cancel{P(Y=0)} + P(Y=1) + P(Y=2) + P(Y=3)$.
$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$.

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(iv) $P(X<3, Y\leq 4) = P(X=0,Y=1)+P(X=0,Y=2)+P(X=0,Y=3)+P(X=0,Y=4)$
$+P(X=1,Y=1)+P(X=1,Y=2)+P(X=1,Y=3)+P(X=1,Y=4)$
$+P(X=2,Y=1)+P(X=2,Y=2)+P(X=2,Y=3)+P(X=2,Y=4)$

$= 0+0+\frac{1}{32}+\frac{2}{32}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}+\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64} = \frac{36}{64}$

$\Rightarrow P(X<3, Y\leq 4) = \cancel{\frac{17}{11}}\ \cancel{\frac{9}{17}} = \frac{9}{16}$

Conditional Distribution:

(a) Discrete case:- Let (X, Y) be a discrete two dimensional random variable. Then the conditional discrete density f^n or conditional prob. mass f^n of X, given $Y=y$, denoted by $f_{X|Y}(x|y)$ or $f(x|y)$ is defined as

$$f_{X|Y}(x|y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{f(x,y)}{f_2(y)}$$

$$\Rightarrow \boxed{f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}}$$

Illy for Y & given $X=x$, we have

$$f_{Y|X}(y|x) = \frac{P(X=x, Y=y)}{P(X=x)} = \frac{f(x,y)}{f_1(x)}$$

$$\Rightarrow \boxed{f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}}$$

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(b) Continuous case :-

$$F_{X|Y}(x|y) = P(X \le x \mid Y=y) = \frac{\int_{v=-\infty}^{\infty}\int_{u=-\infty}^{x} f(u,v)\,du\,dv}{\int_{v=-\infty}^{\infty}\int_{u=x}^{x+dx} f(u,v)\,du\,dv}$$

& a similar expression for $F_{Y|X}(y|x)$.

Ques 1) The joint prob. distn. of two r.v. X & Y is given by :

$P(X=0, Y=1) = \frac{1}{3}$, $P(X=1, Y=-1) = \frac{1}{3}$ & $P(X=1, Y=1) = \frac{1}{3}$.

Find (i) Marginal distn of X & Y.

(ii) the condition prob. distn of X given $Y=1$.

Soln $P(X=0) = P(X=0, Y=1) + P(X=0, Y=-1)$

$= \frac{1}{3} + 0 = \frac{1}{3} \Rightarrow \boxed{P(X=0) = \frac{1}{3}}$

$P(X=1) = P(X=1, Y=1) + P(X=1, Y=-1) = \frac{1}{3} + \frac{1}{3}$

$\Rightarrow \boxed{P(X=0) = \frac{2}{3}}$

Therefore we have following table

Y \ X	0	1	Marginal(Y)
-1	0	$\frac{1}{3}$	$\frac{1}{3} = P(Y=-1)$
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3} = P(Y=1)$
Marginal(X)	$\frac{1}{3}$ $P(X=0)$	$\frac{2}{3}$ $P(X=1)$	1

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(i) Marginal dist^n of X is given by:-

x:	0	1
$P(X=x)$:	$\frac{1}{3}$	$\frac{2}{3}$

Marginal dist^n of Y is given by:-

y	-1	1
$P(Y=y)$	$\frac{1}{3}$	$\frac{2}{3}$

(ii) Conditional prob. dist^n of X given $Y=1$ is:

$$P(X=x \mid Y=1) = \frac{P(X=x, Y=1)}{P(Y=1)}$$

$$\therefore P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$\& \; P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Thus the conditional dist^n of X given, $Y=1$ is:

x	0	1
$P(X=x \mid Y=1)$	$\frac{1}{2}$	$\frac{1}{2}$

Q2) A two dimensional R.V. (X, Y) have a bivariate dist^n given by:

$$P(X=x, Y=y) = \frac{x^2+y^2}{32}, \text{ for } x=0,1,2,3 \;\&\; y=0,1.$$

Find the marginal dist^n of X & Y.

Sol^n We are given that $P(X=x, Y=y) = \frac{x^2+y^2}{32}$,

for $x=0,1,2,3$ & $y=0,1$.

$\therefore P(X=0, Y=0) = 0 \quad , \quad P(X=0, Y=1) = \frac{0+1}{32} = \frac{1}{32}$.

Similarly we can find all the values, express these values in tabular form:

Y \ X	0	1	2	3	Total $f_Y(y)$
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$	$\frac{14}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{18}{32}$
Total $f_X(x)$	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{19}{32}$	1

The marginal distn of X is given by:

x :	0	1	2	3
$P(X=x)$:	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{19}{32}$

The marginal distn of Y is given by:-

y :	0	1
$P(Y=y)$:	$\frac{14}{32}$	$\frac{18}{32}$

Q3] Let X & Y are two R.V. having joint density fn:

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0 \le x < 2,\ 2 \le y < 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $P(X<1 \cap Y<3)$

(b) $P(X+Y<3)$ (c) $P(X<1 / Y<3)$.

Solⁿ: (a) $P(X<1 \cap Y<3) = \int_{-\infty}^{1}\int_{-\infty}^{3} f(x,y)\,dx\,dy$

$$= \int_{x=0}^{1}\int_{y=2}^{3} \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \int_{x=0}^{1} \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

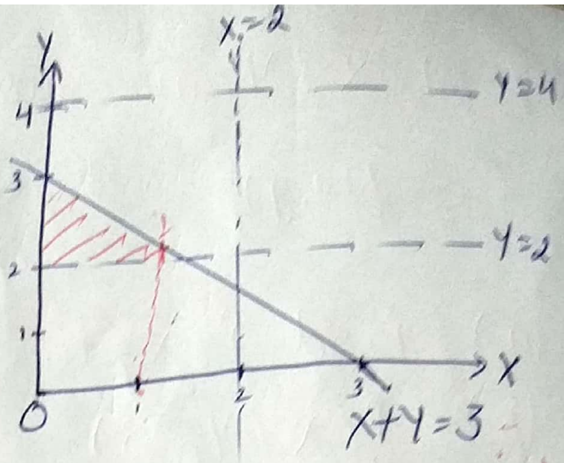
$$= \int_{x=0}^{1} \frac{1}{8}\left[18 - 3x - \frac{9}{2} - 12 + 2x + 2\right] dx$$

$$= \int_{x=0}^{1} \frac{1}{8}\left[\frac{7}{2} - x\right] dx = \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = \frac{3}{8}$$

$$\Rightarrow \boxed{P(X<1 \cap Y<3) = \frac{3}{8}}$$

(b) $P(X+Y<3) = \int_{x=0}^{1}\int_{y=2}^{3-x} \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \int_{x=0}^{1} \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$



$$= \int_{x=0}^{1} \frac{1}{8}\left[18-6x-3x+x^2-\frac{(3-x)^2}{2}-12+2x+\frac{4}{2}\right]dx$$

$$= \int_{x=0}^{1} \frac{1}{8}\left[\frac{7}{2}-4x+\frac{x^2}{2}\right]dx$$

$$= \frac{1}{8}\left[\frac{7}{2}x-\frac{4x^2}{2}+\frac{x^3}{6}\right]_0^1 = \frac{1}{8}\times\frac{10}{6} = \frac{5}{24}$$

$$\Rightarrow \boxed{P(X+Y<3)=\frac{5}{24}}$$

(c) $P(X<1 \mid Y<3) = \dfrac{P(X<1 \cap Y<3)}{P(Y<3)}$

$$= \frac{\int_{x=0}^{1}\int_{y=2}^{3}\frac{1}{8}(6-x-y)\,dx\,dy}{\int_{x=0}^{2}\int_{y=2}^{3}\frac{1}{8}(6-x-y)\,dx\,dy} = \frac{\frac{3}{8}}{\frac{5}{8}}$$

$$\Rightarrow \boxed{P(X<1 \mid Y<3) = \frac{3}{5}}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

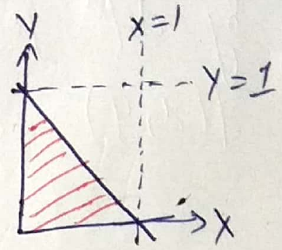
Q4) Let X & Y have a joint prob distn fn as

$$f(x,y)=\begin{cases} 6x^2y, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find $P(X+Y<1)$ (b) Find $P(X>Y)$

(c) Find $P(X<1 \mid Y<2)$.

Soln: (a) $P(X+Y<1) = \int_0^1 \int_0^{1-x} 6x^2y\,dy\,dx$

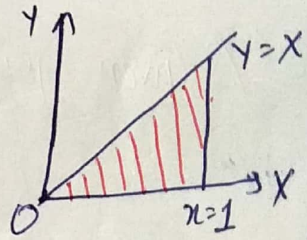


$$= \int_0^1 6x^2\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

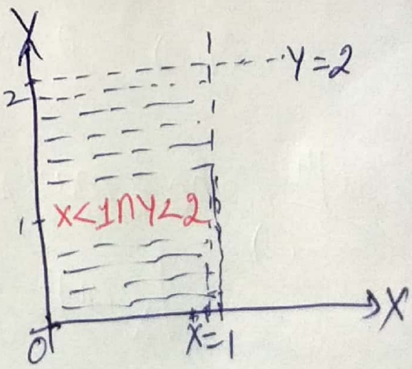
$$= \int_0^1 3x^2(1-x)^2 dx = \frac{1}{10}$$

(b) $P(X>Y) = \int_{x=0}^{1}\int_{y=0}^{x} 6x^2y\,dx\,dy = \int_{x=0}^{1} 6x^2\left[\frac{y^2}{2}\right]_0^x dx$

$$= \int_0^1 3x^4\,dx = \frac{3}{5}$$



(c) $P(X<1 \mid Y<2) = \dfrac{P(X<1 \cap Y<2)}{P(Y<2)}$



$$= \frac{\int_{x=0}^{1}\int_{y=0}^{1} 6x^2y\,dy\,dx + \int_{x=0}^{1}\int_{y=1}^{2} 0\cdot dx\,dy}{\int_{x=0}^{1}\int_{y=0}^{1} 6x^2y\,dy\,dx + \int_{x=0}^{1}\int_{y=1}^{2} 0\cdot dy\,dx}$$

$$= \frac{1}{1} \Rightarrow \boxed{P(X<1 \mid Y<2) = 1}$$

Q5/:- The joint distⁿ fⁿ of Bivariate R.V (X, Y) is given by

$$F_{XY}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find joint prob. density fⁿ. of (X, Y)
(ii) Find probability $(1<X<3,\ 1<Y<2)$

Solⁿ: (i) By defⁿ. joint prob. density fⁿ of (X, Y) is given by:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\, \partial y} = e^{-(x+y)}$$

(ii) Also $P(1<X<3,\ 1<Y<2) = \int_{x=1}^{3}\int_{y=1}^{2} f(x,y)\,dy\,dx$

$$= e^{-2} - e^{-3} - e^{-4} + e^{-5} = 0.074.$$

Q6/ Given joint prob. distⁿ fⁿ. of Bivariate R.V (X,Y) as

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) marginal prob. distⁿ fⁿ. of X & Y.
(ii) Conditional density of X given $Y=y$.
(iii) $P(X \le \frac{1}{2} \mid Y = \frac{1}{2})$

Solⁿ: (i) The marginal prob. distⁿ fⁿ. of R.V. X is given by:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{2}{3}\int_0^1 (x+2y)\,dy = \frac{2}{3}(x+1).$$

Illy Marginal prob. distⁿ fⁿ of R.V. Y is given by:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{2}{3}\int_0^1 (x+2y)\,dx = \frac{1}{3}(1+4y).$$

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(ii) The conditional prob. distn. fn X given Y=y is

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(1+4y)} = \frac{2x+4y}{1+4y}, \quad 0<x<1.$$

& $f_{X|Y}(x,y) = 0$, otherwise.

(iii) $$P\left(X \le \tfrac{1}{2} \,\middle|\, Y = \tfrac{1}{2}\right) = \frac{P\left(X \le \frac{1}{2} \cap Y = \frac{1}{2}\right)}{P\left(Y = \frac{1}{2}\right)}$$

$$= \frac{\int_0^{1/2}\int_0^{1/2} \frac{2}{3}(x+2y)\,dy\,dx}{\int_0^{1}\int_0^{1/2} \frac{2}{3}(x+2y)\,dy\,dx} = \frac{\frac{5}{8}}{\frac{3}{2}} = \frac{5}{12}.$$

Q7) Obtain the marginal & conditional prob. fn. if the joint density fn is:

$$f(x,y) = \begin{cases} 2(2-x-y) ; & 0 \le x \le y < 1 \\ 0 , & \text{elsewhere.} \end{cases}$$

Soln: Marginal Prob. fn of X:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 2(2-x-y)\,dy$$

$$= 2\left[2y - xy - \frac{y^2}{2}\right]_x^1 = 2\left[\frac{3}{2} - x - 2x + \frac{3}{2}x^2\right]$$

$$\Rightarrow \begin{cases} f_X(x) = x^2 - 6x + 3 , & \text{for } 0 \le x \le 1 \\ f_X(x) = 0 , & \text{elsewhere.} \end{cases}$$

Marginal Prob. f^n of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 2(2-x-y)\,dx.$$

$$= 2\left[2x - \frac{x^2}{2} - xy\right]_0^y = 4y - 3y^2.$$

$$\Rightarrow f_Y(y) = \begin{cases} 4y-3y^2, & \text{for } 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Conditional Prob. density function:-

The conditional density f^n of X given Y, $(0 \le y \le 1)$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2(2-x-y)}{4y-3y^2}, \quad 0 \le x \le 1.$$

The conditional density f^n of Y given X, $(0 \le x \le 1)$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2(2-x-y)}{x^2-6x+3}, \quad 0 \le y \le 1.$$

Conditional Expectations

Defn: If X is a discrete r.v. & $P_{X|Y}(x|y)$ is conditional p.m.f X given $Y=y$ at x then the conditional expectation of r.v. X is defined as:

$$E(X|Y=y) = \sum_x x\, P_{X|Y}(x|y) = \sum_x x\, f_{X|Y}(x|y)$$

In case of Continuous

$$E(X|Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$$

Conditional Variance:-

$$V(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2.$$

$$\text{IIly } V(X|Y=y) = E(X^2|Y=y) - [E(X|Y=y)]^2.$$

$\frac{\Sigma u^2}{n} - \left(\frac{\Sigma u}{n}\right)^2$

Q1: The joint p.d.f of Bivariate r.v X & Y is given by:

$$f_{XY}(x,y) = \begin{cases} 6xy(2-x-y); & 0<x<1, \ 0<y<1 \\ 0 & ; \text{ otherwise.} \end{cases}$$

Find conditional expectation of X given Y=y, where $0<y<1$

~~Soln: Marginal prob density fn. of Y~~ ~~is given~~ by:

$\cancel{f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 6xy(2-x-y)dx}$

Soln: Conditional prob. of X given Y=y is:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{6xy(2-x-y)}{\int_0^1 6xy(2-x-y)dx}$$

$$= \frac{6xy(2-x-y)}{6y\left[x^2 - 2\frac{x^3}{3} - \frac{x^2}{2}y\right]_0^1}$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{6x(2-x-y)}{4-3y}$$

$$\therefore E(X|Y=y) = \int_0^1 x \cdot f_{X|Y}(x|y)dx$$

$$= \int_0^1 x \cdot \frac{6x(2-x-y)}{4-3y}dx = \frac{5-4y}{8-6y}$$

$$\Rightarrow \boxed{E(X|Y=y) = \frac{5-4y}{8-6y}}$$

Q2] If $f(x,y)=2$; $0<x<1$, $0<y<1$.

Find (i) $E(Y|X)$ (ii) $E(X|Y)$

Sol^n (i) $E(Y|X) = \int_0^1 y\, f(x,y)\, dy = \int_0^1 2y\,dy = [y^2]_0^1 = 1$.

(ii) $E(X|Y) = \int_0^1 x\cdot f(x,y)\, dx = \int_0^1 2x\, dx = [x^2]_0^1 = 1$.

Q3] The Prob. density f^n of two r.v. X & Y is given by

$$f(x,y) = \begin{cases} 2 & 0<x<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find conditional mean & variance of $X=x$ & $Y=y$.

Sol^n $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{2}{\int_0^1 2\,dx} = 1$.

$\therefore E(X|Y=y) = \int_0^1 x\cdot f_{X|Y}(x|y) = \int_0^1 x\cdot 1\, dx = \frac{1}{2}$

Illy $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{2}{\int_0^1 2\cdot dy} = 1$

$\therefore E(Y|X=x) = \int_0^1 y\cdot f_{Y|X}(y|x)\, dy = \int_0^1 y\cdot 1\, dy = 1/2$.

Conditional variance:

$Var(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2$
$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$.

where $E(Y^2|X=x) = \int_0^1 y^2 f_{Y|X}(y|x)dy$

$= \int_0^1 y^2 \cdot 1\, dy = 1/3$

$Var.(X|Y=y) = E(X^2|Y=y) - [E(X|Y=y)]^2$

$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$

IIly $E(X^2|Y=y) = \int_0^1 x^2 f_{X|Y}(x|y)dx = \int_0^1 x^2 \cdot 1\, dx = \frac{1}{3}.$

Q4] Let $f(x,y) = \begin{cases} 8xy; & 0 \le x \le 1, \ 0 \le y \le x \\ 0, & \text{elsewhere} \end{cases}$

Find (i) Marginal density of X

(ii) " " of Y.

(iii) Conditional density of X

(iv) " " " Y.

(v) $E(Y|X)$

(vi) $E(Y|X=x)$.

Soln (i) Marginal density of X is given by:

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$= 8\int_0^x xy\, dy = 4x^3.$

(ii) Marginal density of Y is given by:

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = 8y\int_0^1 x\,dx = 4y.$

(iii) Conditional density of X: (16)

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{8xy}{4y} = 2x$$

(iv) Conditional density of Y:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}$$

(v) $E(Y|X) = \int_0^x y\, f(x,y)\,dy = \int_0^x y \cdot 8xy\,dy = \frac{8x^4}{3}$

(vi) $E(Y|X=x) = \int_0^x y \cdot f_{Y|X}(y|x)\,dy = \int_0^x y\left(\frac{2y}{x^2}\right)dy$

$$\Rightarrow \boxed{E(Y|X=x) = \frac{2}{3}x}$$

35] The joint density f^n of two r.v. of X & Y is given by:

$$f(x,y) = \begin{cases} xy/96 \,; & 0<x<4 \,;\ 1<y<5 \\ 0 \,, & \text{otherwise} \end{cases}$$

Find (a) $E(X)$

(b) $E(XY)$

$\underline{\underline{Sol^n}}$: (a) $E(X) = \int_{-\infty}^{\infty} x \, f_X(x)\, dx$

where $f_X(x) = \int_0 \frac{xy}{96} dy$
$= \frac{xy^2}{192}$.

$= \int_0^1 x \cdot \frac{xy^2}{192} dx$.

$= \frac{y^2}{192} \int_0^1 x^2 dx$.

$= \frac{x^3 y^2}{576}$.

(b) $E(XY) = \int_1^5 \int_0^1 xy \, f(x,y)\, dx\, dy$

$= \int_1^5 \int_0^1 xy \left(\frac{xy}{96}\right) dx\, dy$

$= \frac{1}{96} \left[\int_1^5 \int_0^1 x^2 y^2 \, dx\, dy \right] = \frac{1}{96} \int_1^5 y^2 \left[\frac{x^3}{3}\right]_0^1 dy$

$= \frac{1}{96} \int_1^5 \frac{y^2}{3} dy \Rightarrow \frac{1}{96 \times 3} \left[\frac{y^3}{3}\right]_1^5 = \frac{31}{24 \times 9}$.

$\Rightarrow \boxed{E(XY) = \frac{31}{216}}$

Q67 If X & Y are two r.v having the joint density f^n $f(x,y) = \frac{1}{27}(2x+y)$

where x & y can assume only integer value 0, 1, 2. Find the conditional $dist^n$ of Y for X = x.

Sol^n: $P(X=0, Y=0) = \frac{1}{27}(0+0) = 0$

$P(X=0, Y=1) = \frac{1}{27}(0+1) = \frac{1}{27}$

Illy we can calculate all the values & form a table:

Y \ X	0	1	2	Total $p_X(x)$
0	0	$\frac{2}{27}$	$\frac{4}{27}$	$6/27$
1	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{5}{27}$	$9/27$
2	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{6}{27}$	$12/27$
$p_Y(y)$	$3/27$	$9/27$	$15/27$	1

The conditional $dist^n$ of Y for X = x is given by:

$$P_{Y|X}(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$P(Y=0 \mid X=0) = \frac{P(Y=0, X=0)}{P(Y=0)} = 0$$

$$P(Y=0 \mid X=1) = \frac{P(Y=0, X=1)}{P(X=0)} = \frac{2/27}{9/27} = \frac{2}{9}$$

$$P(Y=0 \mid X=2) = \frac{P(Y=0, X=2)}{P(X=2)} = \frac{4/27}{15/27} = \frac{4}{15}$$

$$P(Y=1 \mid X=0) = \frac{P(Y=1, X=0)}{P(X=0)} = \frac{1/27}{3/27} = 1/3$$

$P(Y=1 \mid X=1) = 1/3$

$P(Y=1 \mid X=2) = 1/3$

$P(Y=2 \mid X=0) = 2/3$

$P(Y=2 \mid X=1) = 4/9$

$P(Y=2 \mid X=2) = 2/5$

Two random variables X & Y with joint probability density function $f_{XY}(x,y)$ & marginal prob. distn. fn. $f_X(x)$ & $f_Y(y)$ resp. are said to be independent iff: ⑫

$$\boxed{f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

<u>Ques</u> The random variables X & Y are jointly distributed as:

$$f(x,y) = e^{-(x+y)} \quad x > 0, y > 0.$$

(a) Are X & Y independent?
(b) Find $P(X>1)$
(c) Find $P(X<Y \mid X<2Y)$
(d) Find $P(1<X+Y<2)$.

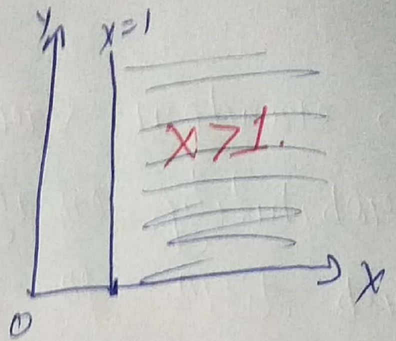
<u>Soln.</u> Here $f_X(x) = \int_{y=0}^{\infty} e^{-(x+y)} dy = e^{-x} \int_0^{\infty} e^{-y} dy = e^{-x}$

& $f_Y(y) = \int_{x=0}^{\infty} e^{-(x+y)} dx = e^{-y}$.

(a) Thus $f(x,y) = e^{-(x+y)} = e^{-x} \cdot e^{-y}$
$= f_X(x) f_Y(y)$.

$\Rightarrow$ X & Y are independent.

(b) $P(X>1) = \int_{x=1}^{\infty}\int_{y=0}^{\infty} e^{-(x+y)}\,dy\,dx$



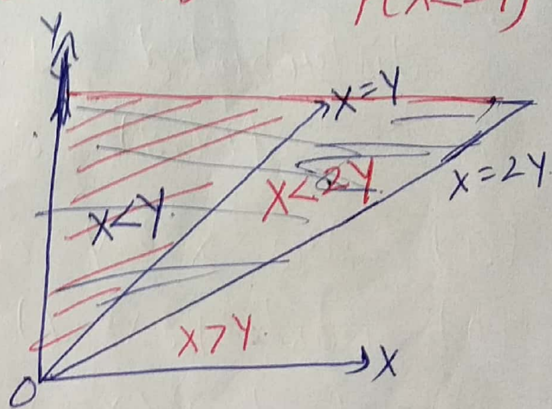
$= \int_{x=1}^{\infty} e^{-x}\,dx$

$= \left.\frac{e^{-x}}{-1}\right|_{1}^{\infty} = \frac{1}{e}$

(c) $P(X<Y \mid X<2Y) = \frac{P(X<Y \cap X<2Y)}{P(X<2Y)} = \frac{P(X<Y)}{P(X<2Y)}$

Now $P(X<Y)$

$= \int_{y=0}^{\infty}\int_{x=0}^{y} e^{-(x+y)}\,dx\,dy$



$= \int_{y=0}^{\infty} e^{-y}\left[\int_{x=0}^{y} e^{-x}\,dx\right]dy$

$\Rightarrow \boxed{P(X<Y) = \frac{1}{2}}$

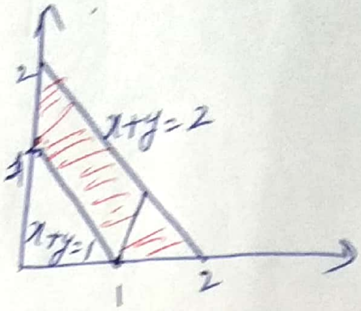
& $P(X<2Y) = \int_{y=0}^{\infty}\int_{x=0}^{2y} e^{-(x+y)}\,dx\,dy = 2/3$

$\Rightarrow P(X<2Y) = 2/3$

$\Rightarrow \boxed{P(X<Y \mid X<2Y) = \frac{1/2}{2/3} = 3/4}$

(d) $P(1 < X+Y < 2) = \int_{x=0}^{2}\int_{y=0}^{2-x} f(x,y)\,dy\,dx - \int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)\,dy\,dx.$

$= \int_{x=0}^{2}\left\{\int_{y=0}^{2-x} e^{-x-y}\,dy\right\}dx - \int_{x=0}^{1}\left\{\int_{y=0}^{1-x} e^{-x-y}\,dy\right\}dx$



$= \int_{x=0}^{2}\left\{e^{-x}(-e^{-y})\right\}_0^{2-x} dx - \int_{x=0}^{1}\left\{e^{-x}(-e^{-y})\right\}_0^{1-x} dx$

$= \{[-e^{-x} - xe^{-2}] - (-1-0)\} - \{(-e^{-1} - e^{-1}) - (-1-0)\}$

$= -3e^{-2} + 1 + 2e^{-1} - 1 = \frac{2}{e} - \frac{3}{e^2}.$

Ques: For the bivariate density f^n.

$$f(x,y) = \begin{cases} k(2x+3y)\,e^{-y/2}, & 0<x<2,\ y>0 \\ 0, & \text{elsewhere} \end{cases}$$

Show that $f(x,y) = f(x)\,f(y)$, k being a constant.

Sol^n. Marginal densities are given by:

$$f_X(x) = \int_0^{\infty} k(2x+3y)\,e^{-y/2}\,dy \quad \text{—— (1)}$$

$$f_Y(y) = \int_0^{2} k(2x+3y)\,e^{-y/2}\,dx \quad \text{—— (2)}$$

But firstly, we find the value of k then put these values in (1) & (2).

Now $\int_0^{2}\int_0^{\infty} k(2x+3y)\,e^{-y/2}\,dy\,dx = 1$

Do yourself

Ques: The joint density f^n of two R.V. X & Y is given by:-

$$f(x,y) = \begin{cases} \frac{xy}{96}, & 0<x<1,\ 1<y<5 \\ 0, & \text{elsewhere} \end{cases}$$

Find (a) $E(X)$ (b) $E(XY)$

Sol^n Already done one Page no. 10