Imperfect Bifurcation in a non-autonomous nonlinear system describing asymmetric water wheels

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Abstract

We derived a water wheel model from first principles under the assumption of an asymmetric water wheel for which the water inflow rate is in general unsteady (modeled by an arbitrary function of time). In this paper our results suggest that the given equation has an imperfect bifurcation.

$\dot{a_1} = wb_1 - ka_1 + p_1(t)$

$\dot{b_1} = -wa_1 - kb_1 + q_1(t)$

$I\dot{w} = -vw + \pi gRa_1$

Keywords

Asymmetric water wheel, unsteady water inflow rate, fixed points, linearization of the system, Imperfect bifurcation

Fixed points and linearize the system to figure out bifurcation

We know that fixed points occur where $\dot{a_1}, \dot{b_1}, I\dot{w}$ all are zero simultaneously for finding the fixed points.

We have $-vw + \pi gRa_1 = 0 \quad \Rightarrow w = \frac{\pi gRa_1}{v}$

We have $-wa_1 - kb_1 + q_1(t) = 0 \quad \Rightarrow b_1 = \frac{q_1(t)}{k} - \frac{\pi gRa_1^2}{vk}$

Now put the value of w and b_1 in $wb_1 - ka_1 + p_1(t) = 0$, we get

$$\pi^2 g^2 R^2 a_1^3 + a_1(v^2k^2 - \pi gRvq_1(t)) - v^2kp_1(t) = 0 \tag{1}$$

Equation (1) is in cubic polynomial form and we can find the roots by solving the depressed cubic equation of the form $y^3 + Ay = B$

We have $\pi^2 g^2 R^2 a_1^3 + a_1(v^2k^2 - \pi gRvq_1(t)) - v^2kp_1(t) = 0$

$\Rightarrow a_1^3 + a_1(\frac{v^2k^2 - \pi gRvq_1(t)}{\pi^2 g^2 R^2}) = \frac{v^2kp_1(t)}{\pi^2 g^2 R^2}$; It is in the form of $y^3 + Ay = B$

These are the following steps involved in solving the depressed cubic. We are left with the solving a depressed cubic equation of the form $y^3 + Ay = B$.

We will find s and t so that

$$3st = A \tag{2}$$

$$s^3 - t^3 = B \tag{3}$$

It turns out that $y = s - t$ will be a solution of the depressed cubic. Let's check that: Replacing A, B and y as indicated transforms our equation into

$(s-t)^3 + 3st(s-t) = s^3 - t^3$

This is true since we can satisfy the left side by using the binomial formula to

$(s^3 - 3s^2t + 3st^2 - t^3) + (3s^2t - 3st^2) = s^3 - t^3$

How can we find s and t satisfying (2) and (3)? Solving the second equation for s and substituting into (3) yields

$(\frac{A}{3t})^3 - t^3 = B$

Simplifying, this turns into the "tri-quadratic" equation $t^6 + Bt^3 - \frac{A^3}{27} = 0$;

Which using the substitution $u = t^3$ becomes the quadratic equation$u^2 + Bu - \frac{A^3}{27} = 0$

From this, we can find a value for u by the quadratic formula, then obtain t, afterwards s and we're done.

Now, we solve $a_1^3 + a_1(\frac{v^2k^2 - \pi gRvq_1(t)}{\pi^2 g^2 R^2}) = \frac{v^2kp_1(t)}{\pi^2g^2R^2}$ by depressed cubic method. We will find s and t so that

$$3st = \frac{v^2k^2 - \pi gRvq_1(t)}{\pi^2 g^2 R^2} \tag{4}$$

$$s^3 - t^3 = \frac{v^2kp_1(t)}{\pi^2g^2R^2} \tag{5}$$

We have $3st = \frac{v^2k^2 - \pi gRvq_1(t)}{\pi^2 g^2 R^2} \Rightarrow s = \frac{v^2k^2 - \pi gRvq_1(t)}{3t\pi^2 g^2 R^2}$

Now, put the value of s in equation (5), we get:-

$(\frac{v^2k^2 - \pi gRvq_1(t)}{3t\pi^2 g^2 R^2})^3 - t^3 = \frac{v^2kp_1(t)}{\pi^2g^2R^2}$

Let $\frac{v^2k^2 - \pi gRvq_1(t)}{3\pi^2 g^2 R^2} = a$, we get :-

$$\frac{a^3}{t^3} - t^3 = \frac{v^2kp_1(t)}{\pi^2g^2R^2} \tag{6}$$

Multiply by t^3 in equation (6), we get

$t^6 + \frac{v^2kp_1(t)t^3}{\pi^2g^2R^2} - a^3 = 0$; Let $t^3 = y$, we get

$$y^2 + \frac{v^2kp_1(t)y}{\pi^2g^2R^2} - a^3 = 0 \tag{7}$$

Find the roots of equation (7) by using $y = \frac{-b+\sqrt{b^2-4ac}}{2a}$ if quadratic equation of the form $ay^2 + by + c = 0$

Note:-We will discard the negative root.

The root of equation (7) is given by

$y = (-\frac{v^2kp_1(t)}{\pi^2g^2R^2} + \sqrt{\frac{v^2k^2p_1^2(t)}{\pi^4g^4R^4} + 4a^3}) * \frac{1}{2}$

Let $b = \sqrt{\frac{v^2k^2p_1^2(t)}{\pi^4g^4R^4} + 4a^3}$

We will discard the negative root, then take the cube root to obtain t

$\Rightarrow t \ = \ \sqrt[3]{\frac{1}{2} * (\frac{-v^2kp_1(t)}{\pi^2g^2R^2} + b)}$

Now, $s^3 = \frac{v^2kp_1(t)}{\pi^2g^2R^2} + (\frac{-v^2kp_1(t)}{\pi^2g^2R^2} + b) * \frac{1}{2}$

$\Rightarrow s^3 = \frac{v^2kp_1(t)}{2\pi^2g^2R^2} + \frac{b}{2} \Rightarrow s \ = \ \sqrt[3]{\frac{1}{2} * (\frac{v^2kp_1(t)}{\pi^2g^2R^2} + b)}$

Our solution a_1 for the depressed cubic equation is the difference of s and t i.e.

$\Rightarrow a_1 \ = \sqrt[3]{\frac{1}{2} * (\frac{v^2kp_1(t)}{\pi^2g^2R^2} + b)} - \sqrt[3]{\frac{1}{2} * (\frac{-v^2kp_1(t)}{\pi^2g^2R^2} + b)}$

After obtaining the value of a_1, we get w because $w = \frac{\pi gRa_1}{v}$ and also we obtain b_1 because $b_1 = \frac{q_1(t)}{k} - \frac{\pi gRa_1^2}{vk}$

In this way we obtained a_1, w and b_1 the fixed point of waterwheel equation.

Linearized system

Consider the system

$\dot{x} = f(x, y)$

$\dot{y} = g(x, y)$; and suppose that $(\bar{x}, \bar{y})$ is a fixed point i.e. $f(\bar{x}, \bar{y}) = 0, \quad g(\bar{x}, \bar{y}) = 0$

Let $u = x - \bar{x} \ \ and \ \ v = y - \bar{y}$ denote the components of a small disturbance from the fixed point.

The linearized system of the above system are as follows :

$$\dot{u} = \frac{\delta f}{\delta x}u + \frac{\delta f}{\delta y}v; \qquad and \quad \dot{v} = \frac{\delta g}{\delta x}u + \frac{\delta g}{\delta y}v$$

In similar way we linearize our following system i.e.

$\dot{a_1} = wb_1 - ka_1 + p_1(t)$

$\dot{b_1} = -wa_1 - kb_1 + q_1(t)$

$I\dot{w} = -vw + \pi gRa_1$

$$M = \begin{bmatrix} -k & w & b_1 \\ -w & -k & -a_1 \\ \pi gR & 0 & -v \end{bmatrix}$$

Find the determinant value of the above matrix, we get :-

$$\triangle \ = \pi gR(-wa_1 + kb_1) - v(k^2 + w^2) \tag{8}$$

Now putting the value of w and b_1 in the equation (8) we get :-

$$\Rightarrow \pi gR(\frac{-2\pi gRa_1^2}{v} + q_1) - v(k^2 + w^2) \tag{9}$$

By considering equation (9) we can claim the stability analysis of the above waterwheel equation. Here we have unknown value of q_1 and v and based on these value of q_1 and v, we can say about the stability analysis of the given system.

i.e. If we have $\triangle$ is negative then the fixed point is a saddle point.

Note:- Consider $\dot{X} = h + rx - x^3$, if $h = 0$ then we have supercritical pitchfork bifurcation.If $h \neq 0$ then imperfect bifurcation come in light.

After converting the waterwheel equation we obtain the form of $\dot{X} = h + rx - x^3$. Hence the waterwheel equation has an imperfect bifurcation.