Indian Institute of Technology Madras

Department of Data Science and Artificial Intelligence

DA5000: Mathematical Foundations of Data Science

Tutorial I

Solutions

1. Let $A \in R^{m\times n}$, and let $\mathbf{e}_i$ denote the standard basis column vectors in $R^{n\times 1}$, for each $1 \le i \le n$.

   Define $\mathbf{b}_i := A\mathbf{e}_i$, for each $1 \le i \le n$.

   By our hypothesis, $\mathbf{e}_i$ is a solution of $C\mathbf{x} = \mathbf{b}_i$, for each i.

   Hence, each $\mathbf{e}_i$ satisfies
   $$(A - C)\mathbf{x} = \mathbf{0},$$
   which is deduced by subtracting the matrix equations.

   Therefore, the dimension of $N(A - C)$ (null space) is n.

   By the rank-nullity theorem, we have
   $$\begin{aligned} \operatorname{rank}(A - C) &= 0 \\ \implies (A - C) &= 0 \\ \implies A &= C. \end{aligned}$$

2. **(a) False**: If the columns of a matrix are dependent, it means that at least one column can be expressed as a linear combination of the other columns. However, this does not necessarily mean that the rows are dependent as well.

   Let A be a $m \times n$ matrix, where the columns are dependent. This means there exist scalars $c_1, c_2, \ldots, c_n$ not all zero such that:
   $$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$
   where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of A.

   However, the rows of A might still be independent. A counterexample will illustrate this:

   Counterexample:

   Consider:
   $$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

The columns are still dependent as $2\mathbf{b}_1 + \mathbf{b}_3 = \mathbf{b}_2$, but the rows are independent.

Thus, column dependence does not imply row dependence.

(b) False: The column space of a matrix is defined as the span of its columns, and the row space is defined as the span of its rows. Since a 2×2 matrix has 2 columns and 2 rows, the column space and the row space will not necessarily be the same.

Counterexample:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The column space is spanned by:

$$\mathrm{Col}(A) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right)$$

The row space is spanned by:

$$\mathrm{Row}(A) = \mathrm{span}\left(\begin{pmatrix} 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 \end{pmatrix}\right)$$

However, the specific vectors spanning these spaces are different, and their respective spans lie in different spaces.

Thus, the column space and row space are generally different in nature, and while their dimensions might coincide, they aren't "the same" space.

(c) True: The dimension of a vector space is the number of vectors in any basis for that space. For a 2×2 matrix, the column space and the row space will each have at most 2 linearly independent vectors, and therefore the dimension of both spaces will be at most 2.

Elaboration:

Given any $m \times n$ matrix, the rank of the matrix, which is the dimension of the column space (and also the row space), is equal to the maximum number of linearly independent columns (or rows) in the matrix.

For a 2×2 matrix, the maximum rank can be 2. Hence, the dimension of both the column space and row space is at most 2.

(d) False: The columns of a matrix do not necessarily form a basis for the column space. A basis is a set of linearly independent vectors that span the entire space. The columns may be dependent, in which case they do not form a basis for the column space.

Elaboration:

A basis for a vector space must consist of linearly independent vectors. If the columns of a matrix are dependent, they cannot form a basis for the column space.

Counterexample:

Consider the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

The columns are dependent since $\mathbf{a}_2 = 2\mathbf{a}_1$ and $\mathbf{a}_3 = 3\mathbf{a}_1$.

The column space of A is the span of the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, but the columns of A are not independent and hence do not form a basis.

Therefore, while the column space is spanned by the columns, the actual columns themselves may not form a basis unless they are linearly independent.

3. **(a) If $m = n$ then the row space of A equals the column space.**

False. The row space and column space of a matrix are generally different subspaces. For example, consider the identity matrix I of size 2×2:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The row space of I is spanned by $\{(1,0),(0,1)\}$, and the column space is also spanned by $\{(1,0),(0,1)\}$. However, this is a special case. For a general 2×2 matrix like:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The row space is spanned by $\{(1,2),(3,4)\}$ and the column space is spanned by $\{(1,3),(2,4)\}$, which are different.

(b) The matrices A and $-A$ share the same four subspaces.

True. The four fundamental subspaces of a matrix A are the column space, the null space, the row space, and the left null space. These subspaces are determined by the linear transformations associated with A and are not affected by scalar multiplication. Therefore, A and $-A$ share the same four subspaces.

(c) If A and B share the same four subspaces then A is a multiple of B.

False. Sharing the same four subspaces does not necessarily imply that one matrix is a scalar multiple of the other. For example, consider the matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Both matrices have the same row space, column space, null space, and left null space, but A is not a scalar multiple of B.

4. **(a) All diagonal matrices:**

Every diagonal matrix is of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

and so can be written as a linear combination of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since this system of generators is clearly minimal (none of these matrices is a linear combination of the others), this is a basis. Therefore, the dimension of the space of diagonal matrices is 3.

(b) All symmetric matrices ($A^T = A$):

Every symmetric matrix is of the form

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

and so can be written as a linear combination of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, since no matrix here is a linear combination of the others, this is a basis. Therefore, the dimension of the space of symmetric matrices is 6.

(c) All skew-symmetric matrices ($A^T = -A$):

Every skew-symmetric matrix is of the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

and so can be written as a linear combination of

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since these matrices again form a minimal set of generators, they are a basis. Therefore, the dimension of the space of skew-symmetric matrices is 3.

5. **Column Space** $C(A)$: The subspace of R^m spanned by the columns of A.

   Left Null Space $N(A^T)$: The null space of A^T, i.e., $N(A^T) = \{\mathbf{y} \in R^m \mid A^T\mathbf{y} = \mathbf{0}\}$.

   We need to prove that any vector in the column space $C(A)$ is orthogonal to any vector in the left null space $N(A^T)$. Let $A \in R^{m\times n}$. Let $\mathbf{v} \in C(A)$ and $\mathbf{w} \in N(A^T)$. Then:

$$\mathbf{v} = A\mathbf{x} \quad \text{for some } \mathbf{x} \in R^n$$

$$A^T\mathbf{w} = \mathbf{0}$$

$$\mathbf{v} = A\mathbf{x} \quad \text{for some vector } \mathbf{x} \in R^n$$
$$\mathbf{w}^T A = \mathbf{0}^T \quad \text{since } \mathbf{w} \in N(A^T)$$

   We calculate the dot product of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{w}^T\mathbf{v} = \mathbf{w}^T(A\mathbf{x})$$

   Since $\mathbf{w}^T A = \mathbf{0}^T$, we have:

$$\mathbf{w}^T(A\mathbf{x}) = (\mathbf{w}^T A)\mathbf{x} = \mathbf{0}^T\mathbf{x} = 0$$

   Therefore, $\mathbf{w}^T\mathbf{v} = 0$, which implies that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

   Since $\mathbf{v}$ and $\mathbf{w}$ were arbitrary vectors in $C(A)$ and $N(A^T)$, respectively, we have shown that every vector in $C(A)$ is orthogonal to every vector in $N(A^T)$. This proves that $C(A)$ and $N(A^T)$ are orthogonal complements in R^m.

6. Usually, randomly chosen entries of a 3×3 matrix will give a matrix with linearly independent columns, or in other words, a matrix of full rank (regular matrix).

   Therefore we have:

$$\begin{aligned} \dim \mathcal{C}(A) &= 3 & \dim \mathcal{C}(A^T) &= 3 \to \text{rank } r \\ \dim \mathcal{N}(A) &= n - 3 & \dim \mathcal{N}(A^T) &= m - r \\ &= 3 - 3 = 0 & &= 3 - 3 = 0 \end{aligned}$$

   On the other hand, when entries of a 3×5 matrix are randomly chosen, we will have 3 linearly independent columns, but since there are 5 columns in R^3, chosen columns of that matrix will be linearly dependent.

   Therefore:

$$\begin{aligned} \dim \mathcal{C}(A) &= 3 & \dim \mathcal{C}(A^T) &= 3 \to \text{rank } r \\ \dim \mathcal{N}(A) &= n - 3 & \dim \mathcal{N}(A^T) &= m - r \\ &= 5 - 3 = 2 & &= 3 - 3 = 0 \end{aligned}$$

7. Let A be an $m \times n$ matrix of rank r.

   The column space $\mathcal{C}(A)$ is the subspace of R^m spanned by the columns of A.

   The row space $\mathcal{C}(A^T)$ is the subspace of R^n spanned by the rows of A.

   The rank of A, denoted as $\text{rank}(A)$, is the maximum number of linearly independent columns of A. Equivalently, the rank is also the maximum number of linearly independent rows of A.

   Since the rank of A is the number of linearly independent columns, the dimension of the column space $\mathcal{C}(A)$ is r. Similarly, since the rank of A is the number of linearly independent rows, the dimension of the row space $\mathcal{C}(A^T)$ is also r.

   Therefore, we have:

$$\dim(\mathcal{C}(A)) = \text{rank}(A) = r$$

$$\dim(\mathcal{C}(A^T)) = \text{rank}(A) = r$$

   Since both the column space $\mathcal{C}(A)$ and the row space $\mathcal{C}(A^T)$ have the same dimension r, we conclude that:

$$\dim(\mathcal{C}(A)) = \dim(\mathcal{C}(A^T))$$

   Thus, the dimensions of the column space $\mathcal{C}(A)$ and the row space $\mathcal{C}(A^T)$ are equal, completing the proof.