Johnsons Algorithm (Sparse Graphs)

* Negative edges ✓
* Negative cycles ×

If all edges are positive → Dijkstras

$O((V^2+E) \times V) = O(V^3)$ (array implementation of EXTRACT-MIN)

$O((V\log + E) \times V) = O(V^2 \log V + EV)$

$\underset{\text{un}}{}$ $\qquad E \downarrow \ll V^2$

Make all edges positive can be done in $O(EV)$ time

(Why not simply go through E, take the most negative edge and shift all the edges up?)

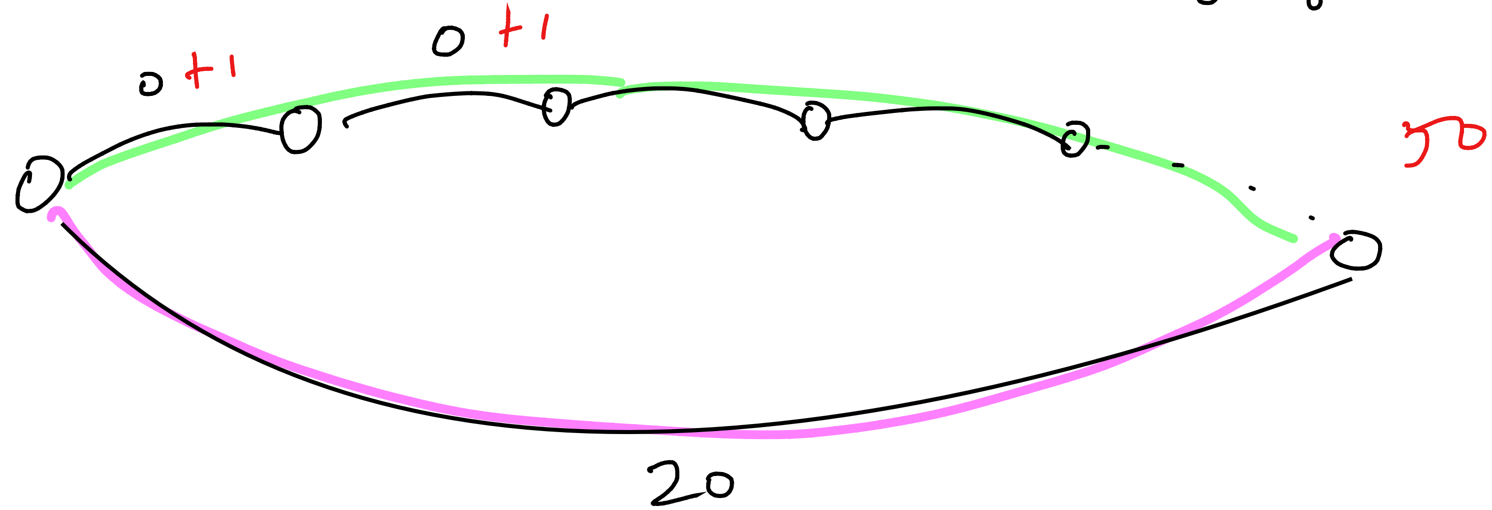
⇓

Will not work

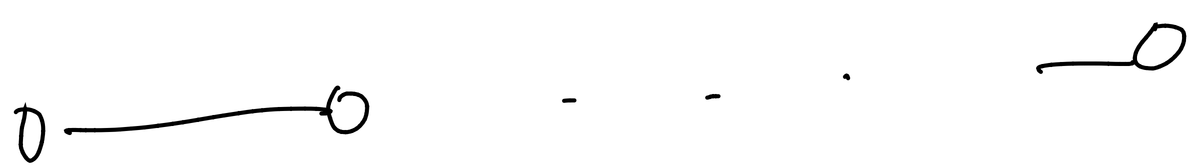
adjust = $\min_{\substack{(u,v):\\ w(u,v)<0}} w(u,v)$

$w(p) + q \underbrace{|\text{adjust}|}$ is different for diff paths

Moral: Eventhough every edge is shifted up by a constant adjust = 1



Exercise: Find the graph such that only one path is there between any two vertices.



every path is not shifted up by the same thing.

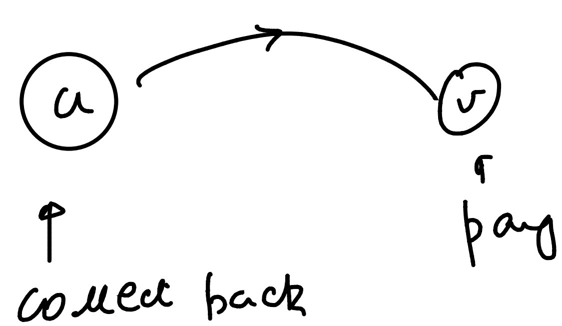
Reweighing: Take some $h : V \to \mathbb{R}$

new deposite   old

$$\hat{w}(u,v) = \underbrace{w(u,v) + h(u)} - h(v)$$

safety / security deposit

Rethink on this interpretation { paid when going out and collected back while coming in



Reweighting does not alter

$$p = \langle v_0, v_1, \ldots, v_k \rangle$$

$$
\begin{aligned}
\hat{\omega}(p) = \; & \hat{\omega}(v_0, v_1) && = w(v_0, v_1) + h(v_0) - \cancel{h(v_1)} \\
+ \; & \hat{\omega}(v_1, v_2) && + \; w(v_1, v_2) + \cancel{h(v_1)} - h(v_2) \\
+ \; & \vdots && + \; \vdots \\
+ \; & \hat{\omega}(v_{k-1}, v_k) && + \; w(v_{k-1}, v_k) + h(v_{k-1}) - h(v_k)
\end{aligned}
$$

$$= w(v_0, v_1) + \cdots + w(v_{k-1}, v_k) + h(v) - h(v)$$

$$\hat{\omega}(p) = w(p) + \underbrace{h(v_0) - h(v_k)}$$

penalty does not depend on hop lengths

Altering weights does not alter the ordering of shortest path

Pick any p, p' between v_0 and v_k

then if $\hat{w}(p') < \hat{w}(p)$

$$w(p') + h(v_0) - h(v_k) < w(p) + h(v_0) - h(v_k)$$

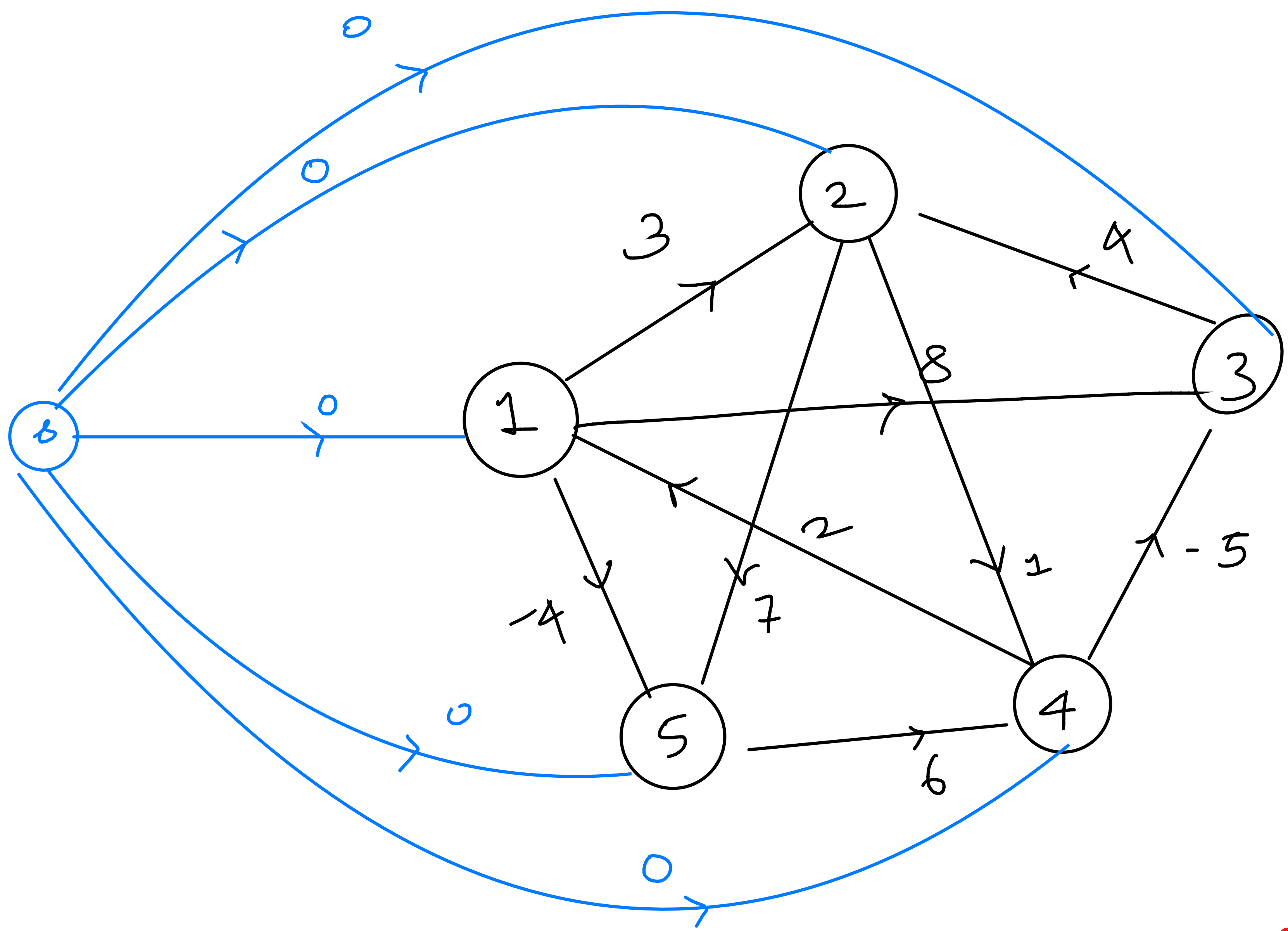
$$\Rightarrow \quad w(p') < w(p)$$

Reweighing Strategy:

Introduce a dummy edge s (dummy source)

$G' = (V', E')$, $V' = V \cup \{s\}$

$E' = E \cup \{(s, v), v \in V\}$



* NO incoming edges for s.

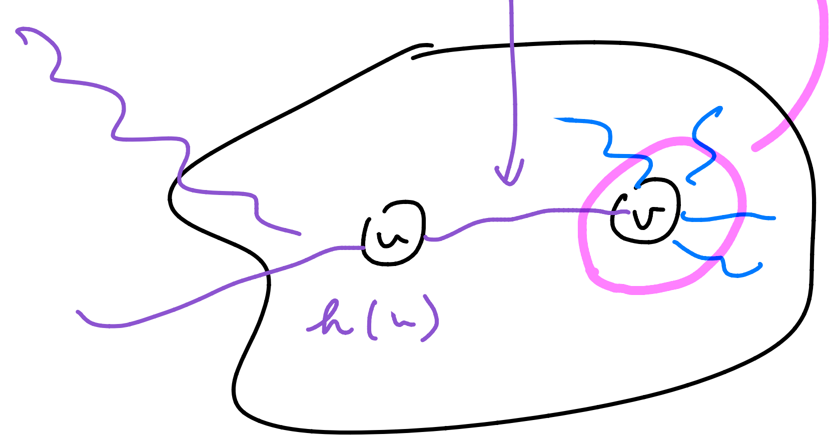
Idea: Solve single source shortest path $\delta(s, v)$

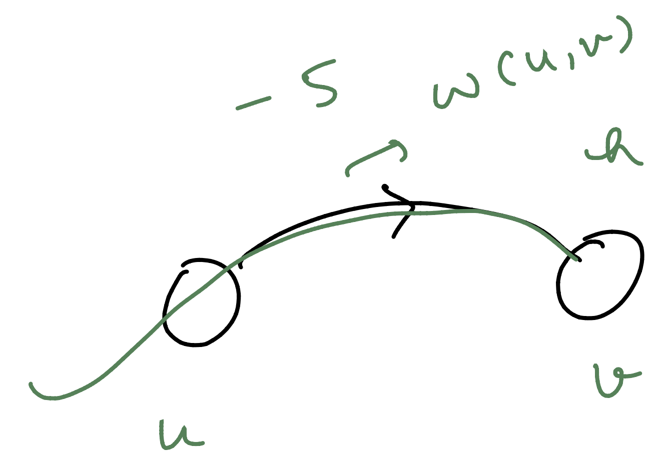
Set $h(v) = \delta(s, v)$

$h(v)$ = Shortest way to reach v from anywhere in the original graph G

Best way to reach v (via u) from anywhere in G

$$h(v) \leq h(u) + w(u,v)$$



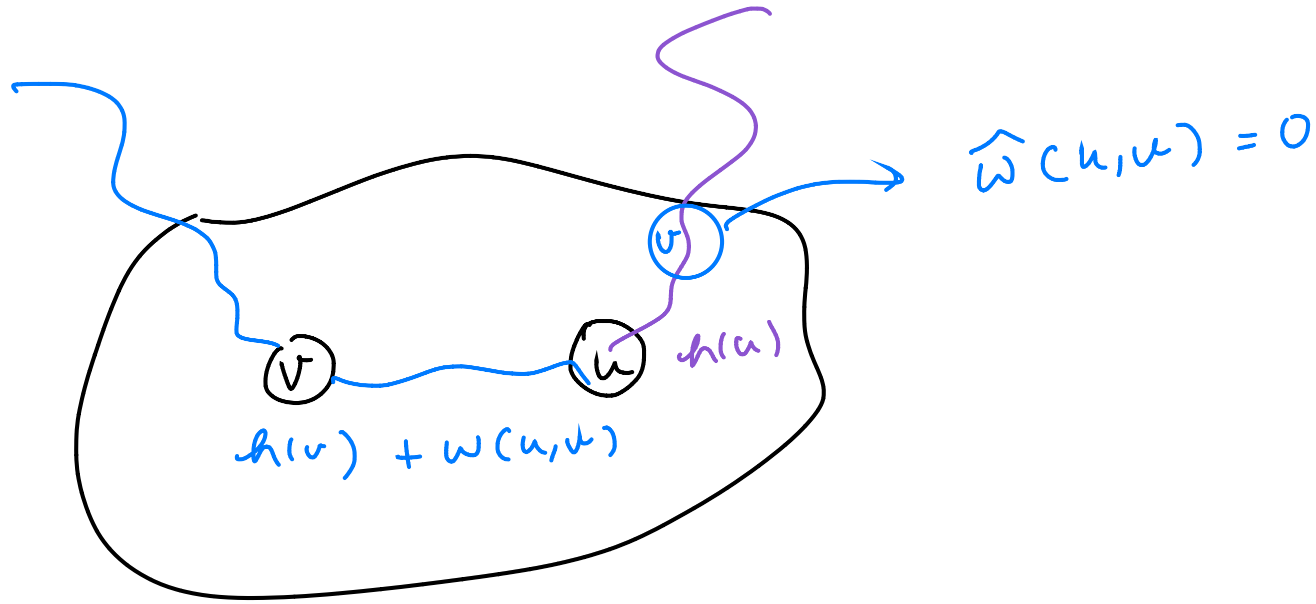


$$\underbrace{w(u,v) + h(u)}_{\text{Best possible via } u} - \underbrace{h(v)}_{\text{Best possible over all}} \geq 0$$

Similar to the Q-values/Advantage in Reinforcement learning

$$\boxed{\hat{w}(u,v) \geq 0}$$

→ Differential paid when reaching via u over and above optimal way to reach v from anywhere



v
u
v
h(u)
h(v) + w(u,v)
ŵ(u,v) = 0