Sample Question Paper

DA5000 - Mathematical Foundations of Data Science

Instructions:

- Answer all parts of the question in the same place, else they won't be evaluated.
- Begin answering each new question on a separate page, each violation will invite a penalty.

1. (5M) For minimizing $f(x) = \frac{1}{2}x^2$, consider the gradient descent algorithm $x_{k+1} = x_k - \alpha \nabla f(x_k)$. Which of the following are true? *Justify your answers*

   (a) It converges to a local minimum for $\alpha < 2$.

   (b) It onverges in finite steps to a local minimum for all $\alpha < 2$.

   (c) It converges monotonically without oscillations for $\alpha = 1.5$.

   (d) It converges geometrically to a local minimum if it converges.

2. (5M) Solve the following quadratic programming problem in $\mathbb{R}^2$.

$$\begin{aligned} \text{minimize } & x^2 + y^2 - 2x - 2y \\ \text{subject to } & x \geq 0, y \geq 0, \\ & x + y \leq 4, x + 2y \leq 6. \end{aligned}$$

   At the optimal, which are the active constraints.

3. (2M) Project $\mathbf{a}_1 = (1, 0)$ onto $\mathbf{a}_2 = (1, 2)$. Project it back onto $\mathbf{a}_1$. Draw these projections and multiply the projection matrices P_1, P_2. Is this multiplication a projection?

4. (2 x 5M) Short Questions

   (a) A 2×2 symmetric matrix has eigen vectors $v_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 8 \\ -a \end{bmatrix}$, corresponding to two distinct eigen values. Find the value of a.

   (b) A linear transformation $A : \mathbb{R}^{12} \to \mathbb{R}^{15}$ is defined by a 15×12 matrix. Let the dimension of kernel be 3, i.e. $N(A) = 3$. Compute the dimension $dim(N(A^T)^\perp)$.

   (c) Find a basis for the subspace defined by the plane $x + 2y - 3z - t = 0$ in $\mathbb{R}^4$. What is the dimension of this subspace?

   (d) A matrix $A \in \mathbb{R}^{n \times n}$ is said to be Idempotent if $A^2 = A$. Find the possible eigenvalues of an Idempotent matrix A.

   (e) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$. Compute the dimension of its kernel or null space.

5. (5M) Verify the KKT conditions and find the Lagrange multipliers for the following function at $x = (1, 0)$

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{1}{8}\right)^4$$

subject to

$$\begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geq 0$$

6. (2M) Find the minimum of the following function:

$$f(x) = (x_1 - 1)^2 + x_2^2$$

subject to

$$x_1 - x_2^2 \leq 0$$

7. (3M) Prove that a symmetric matrix $\mathbf{A}$ is positive definite if and only if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for every column $\mathbf{x} \neq 0$ in R^n.

8. (2M) Given two matrices

$$A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

   (a) Verify the Rank-Nullity theorem for A and B

   (b) Find the bases for the four fundamental subspaces of A and B and verify the relations between them.