INDIAN INSTITUTE OF TECHNOLOGY MADRAS

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Roll No. D A 2 4 M 0 1 5

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NA

Answer on both sides of the paper including the space below

Q1

a) $C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

✓

So rank of $C = 1$

2

Using Rank - Nullity theorem

$\dim(\text{Rank}(C)) + \dim(N(C)) = n = 4$

$1 + \dim(N(C)) = 4$

$\boxed{\dim(N(C)) = 3}$

Q b

c (iii) True

$|a_{ii}| \leqslant \sum_{i \neq j} |a_{ij}|$

~~d (i) & (ii)~~

~~(iii) is false because it does not passes from origin~~

~~for (ii) $x_1 + x_2 + \dots\dots x_n = 1$~~

~~It might be possible that all x_1 to x_{n-2} is zero & x_{n-1} is negative of x_n. But it~~

d) (i) & (ii) passes from origin. So it is a subspace.

(iii) does not passes from origin. So it is not a subspace.

e) (i) $\{\sin x ; \cos x, x \sin x\}$

$a \sin x + b \cos x + c x \sin x = 0$ for only $a = b = c = 0$

So $\{\sin x, \cos x, x \sin x\}$ is linearly independent

(ii) $\{\sin^2 x, \cos^2 x, \cos 2x\}$

$a \sin^2 x + b \cos^2 x + c \cos 2x = 0$ for $c = 1, b = -1, a = +1$

$+\sin^2 x - \cos^2 x + \cos 2x = 0$ $\quad [\cos 2x = \cos^2 x - \sin^2 x]$

$-(\cos^2 x - \sin^2 x) + \cos 2x = 0$

$-\cos 2x + \cos 2x = 0$

$\cos 2x$ is a linear combination of $\sin^2 x$ & $\cos^2 x$

So $\sin^2 x$, $\cos^2 x$ & $\cos 2x$ is linearly dependent.

f) $f(x_1, x_2) = x_1^2 + x_2^2$

subject $x_1 \geq 1$

$x_2 \geq 2$

$x_1 + x_2 \leq 5$

Sol^n:

Lagrangian

$$L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = x_1^2 + x_2^2 + \lambda_1(1 - x_1) + \lambda_2(2 - x_2) + \lambda_3(x_1 + x_2 - 5)$$

derivative w.r.t x_1

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 + \lambda_3 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_2 + \lambda_3 = 0$$

KKT Conditions

① $2x_1 - \lambda_1 + \lambda_3 = 0$

② $2x_2 - \lambda_2 + \lambda_3 = 0$

③ $x_1 \geq 1$

④ $x_2 \geq 2$

⑤ $x_1 + x_2 \leq 5$

⑥ $\lambda_1 (1 - x_1) = 0$

⑦ $\lambda_2 (2 - x_2) = 0$

⑧ $\lambda_3 (x_1 + x_2 - 5) = 0$

⑨ $\lambda_1, \lambda_2, \lambda_3 \geq 0$

case I $\quad \lambda_1 > 0, \ \lambda_2 > 0 \quad \lambda_3 = 0$

$1 - x_1 = 0 \qquad\qquad 2 - x_2 = 0$

$x_1 = 1 \ — (A) \qquad x_2 = 2 \ — (B)$

Put (A) & (B) in eqⁿ ① & ②

$2x_1 - \lambda_1 + \lambda_3 = 0 \quad \Rightarrow \quad 2 - \lambda_1 + 0 = 0$

$2x_2 - \lambda_2 + \lambda_3 = 0 \quad \Rightarrow \quad 4 - \lambda_2 + 0 = 0$

$\lambda_1 = 2$

$\lambda_2 = 4$

$\lambda_1 = 2, \ \lambda_2 = 4, \ \lambda_3 = 0$.

Critical Point $(x_1 = 1, x_2 = 2)$ satisfies all the above ~~9 condition~~ 9 KKT condition.

So $f(x_1, x_2) = x_1^2 + x_2^2$

$= (1)^2 + (2)^2$

$= 5$

case II $\quad \lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_3 > 0$

$1 - x_1 = 0 \qquad 2 - x_2 = 0 \qquad x_1 + x_2 - 5 = 0$

$x_1 = 1 \qquad x_2 = 2 \qquad x_1 + x_2 = 5$

$(x_1, x_2) = (1, 2)$ does not satisfy for the 3rd active constraints $x_1 + x_2 - 5 = 0$ as $\lambda_3 >$

So $(x_1, x_2) = (1, 2)$ is not the critical point when all constraints are active.

So, global minimum = 5 (from case I)

7) $f(x) = x^4 - 4x^2 + 2$

find derivative of f with respect to x

$$\frac{df}{dx} = 4x^3 - 8x = 0$$

$$\Rightarrow x(4x^2 - 8) = 0$$

$$\Rightarrow x = 0, \quad 4x^2 = 8$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

So $x = 0$, $x = \sqrt{2}$, $x = -\sqrt{2}$

~~Send~~ Second order derivative

$$f''(x) = \frac{d^2f}{dx^2} = 12x^2 - 8$$

f'' at $x = 0$ at $x = \sqrt{2}$ at $x = -\sqrt{2}$

$f''(0) = -8 < 0$ $f''(\sqrt{2}) = 12(\sqrt{2})^2 - 8 = 16 > 0$ $f''(-\sqrt{2}) = 12(-\sqrt{2})^2 - 8 = 16 > 0$

At $x = 0$, $f(x)$ is maximum

~~At $x = 0$, Saddle point~~

$x = +\sqrt{2}$, $x = -\sqrt{2}$ function is having minimum

So global minimum with occur at two point $x = \sqrt{2}, -\sqrt{2}$

$$f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 2$$

$$= 4 - 8 + 2$$

$$= -2$$

$$f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2 + 2$$

$$= 4 - 8 + 2$$

$$= -2$$

So global minimum is -2.

2

2

h) $f($

(i) (

(j) (

2

h) $f(x,y) = x^2 + 4y^2$

direction $= (-1, -0.5)$

Initial point $x_0 = (1,2)$

Step size, $\alpha = 0.1$

$x_{k+1} = x_k + \alpha p_k$

Let's take $k=0$

$x_1 = x_0 + \alpha p_0$

$x_1 = (1,2) + 0.1(-1, -0.5)$

2

$= (1 - 0.1, \cancel{2} - 0.05)$

$x_1 = (0.9, 1.95)$

Function value after the 1st step

$f(0.9, 1.95) = (0.9)^2 + 4(1.95)^2$

$= 0.81 + 3.8025 \times 4$

$= 0.81 + 15.21$

$= 16.02$ Ans

=8

0

Q1

(ii) (iii) Both (i) & (ii) must hold.

(j) $f(x,y) = x\sin(y) + x^2$

2 $(x_0, y_0) = (1,2)$

$\alpha = 0.01$

Sol

As we know

$x_{k+1} = x_k + \alpha \nabla f(x_k)$

$$\nabla f(x,y) = \begin{bmatrix} \sin(y) + 2x \\ x\cos y \end{bmatrix}$$

$$\nabla f(1,2) = \begin{bmatrix} \sin(2) + 2 \\ \cos(2) \end{bmatrix}$$

gradient in the steepest descent direction

$$\nabla f(1,2) = \begin{bmatrix} \sin(2)+2 \\ \cos(2) \end{bmatrix}$$

Q2

① $\min_{x,y} \ (x-1)^2 + (y-2)^2$

s.t $(x-1)^2 = 5y$

$L(x,y,\lambda) = (x-1)^2 + (y-2)^2 + \lambda(5y-(x-1)^2)$ ~~$+\lambda((x-1)^2 - 5y)$~~

$\frac{\partial L}{\partial x} = 2(x-1) - \lambda(2(x-1)) = 0$

$\Rightarrow 2(x-1)(1-\lambda) = 0$ —— (A)

$\frac{\partial L}{\partial y} = 2(y-2) + \lambda(+5) = 0$

KKT conditions

① $2(x-1) + \lambda(2(x-1)) = 0$

② $2(y-2) - 5\lambda = 0$

③ $(x-1)^2 - 5y = 0$

~~④ $\lambda \geq 0$~~ ④ $\lambda > 0$ not necessary

from eq (A)

either $x = 1$ or ~~$\lambda = -1$~~

~~from eq^n ③~~ ~~$y = -\frac{1}{2}$~~

~~$y = 0$~~ ~~x is not a real.~~

$\lambda = -4/5$

from eqⁿ (1)

$2(x-1)(1-\lambda)=0$

Either $x=1$, or $\lambda=1$

if $x=1$

then $\boxed{y=0}$

$\lambda=\frac{2(y-2)}{5}$

$=\frac{2(0-2)}{5}$

$\boxed{\lambda=\frac{-4}{5}}$

does not satisfy the 4th KKT condition

$2(y-2)-5\lambda=0$

$2(y-2)=5\lambda$

$2(y-2)=5$

$y=\frac{5}{2}+2$

$\boxed{y=\frac{9}{2}}$

$(x-1)^2=5y$

$(x-1)^2=5\times\frac{9}{2}=\frac{45}{2}$

$x-1=\pm\sqrt{\frac{45}{2}}$

$x=\sqrt{\frac{45}{2}}+1,\ -\sqrt{\frac{45}{2}}+1$

Points

$\left(\sqrt{\frac{45}{2}}+1,\frac{9}{2}\right),\left(-\sqrt{\frac{45}{2}}+1,\frac{9}{2}\right)$

for solution

$f(x,y)=(x-1)^2+(y-2)^2$

at $(x,y)=\left(\sqrt{\frac{45}{2}}+1,\frac{9}{2}\right)$

$f\left(\sqrt{\frac{45}{2}}+1,\frac{9}{2}\right)=\left(\sqrt{\frac{45}{2}}+1-1\right)^2+\left(\frac{9}{2}-2\right)^2$

$=\frac{45}{2}+\frac{25}{4}$

$=\frac{115}{4}$

At $(x,y) = \left(-\sqrt{\frac{45}{2}}+1, \frac{9}{2}\right)$

$$f\left(-\sqrt{\frac{45}{2}}+1, \frac{9}{2}\right) = \left(-\sqrt{\frac{45}{2}}+1-1\right)^2 + \left(\frac{9}{2}-2\right)^2$$

$$= \left(\frac{45}{2}\right) + \frac{25}{4}$$

$$= \frac{115}{4}$$

Both points $\left(-\sqrt{\frac{45}{2}}+1, \frac{9}{2}\right)$ & $\left(\sqrt{\frac{45}{2}}+1, \frac{9}{2}\right)$

are the solution.

Q.3
$$\min_{x_1, x_2} f(x) = -2x_1 + x_2 + 2$$
s.t $(1-x_1)^3 \geq x_2$

$x_2 + 0.25x_1^2 \geq 1$

Sol

$$L(x_1, x_2, \lambda_1, \lambda_2) = -2x_1 + x_2 + 2 + \lambda_1(x_2 - (1-x_1)^3) + \lambda_2(1 - x_2 - 0.25x_1^2)$$

$$\frac{\partial L}{\partial x_1} = -2 + 3\lambda_1(1-x_1) - \frac{1}{2}\lambda_2 x_1 = 0$$

$$\frac{\partial L}{x_2} = 1 + \lambda_1 - \lambda_2 = 0$$

KKT conditions:

① $-2 + 3\lambda_1(1-x_1) - \frac{\lambda_2}{2}x_1 = 0$

② $1 + \lambda_1 - \lambda_2 = 0$

③ $(1-x_1)^3 \geq x_2$

④ $x_2 + 0.25 x_1^2 \geq 1$

⑤ $\lambda_1 (x_2 - (1-x_1)^3) = 0$

⑥ $\lambda_2 (1 - x_2 - \frac{1}{4} x_1^2) = 0$

⑦ $\lambda_1 \geq 0, \ \lambda_2 \geq 0$

Case I ~~$\lambda_1 > 0, \lambda_2 \neq 0$~~ $\lambda_1 = 0, \lambda_2 > 0$

~~$x_2 = 1 - x_1$~~ $1 - x_2$

Case I $\quad \lambda_1 \neq 0, \ \lambda_2 = 0$

Condition 2 violates

Case II $\quad \lambda_2 > 0, \ \lambda_1 = 0$

$-2 + \frac{\lambda_2}{2} x_1 = 0 \qquad 1 - \lambda_2 = 0$

$\boxed{\lambda_2 = 1}$

(1) $\therefore -2 + \frac{x_1}{2} = 0$

$\frac{x_1}{2} = 2 \qquad x_1 = -4.$

$\boxed{x_1 = 4}$

$1 - x_2 - \frac{x_1^2}{4} = 0$

$x_2 = 1 - \frac{x_1^2}{4}$

$= 1 - \frac{4^2}{4}$

$= 1 - 4$

$\boxed{x_2 = -3}$

not all cases considered

All conditions valid at $x_1 = 4, \ x_2 = -3$

$\lambda_1 = 0, \ \lambda_2 = 1$

Optimal solution

$f(x_1, x_2) = -2x_1 + x_2 + 2$

$f(4, -3) = -2(4) - 3 + 2$

$= -8 - 3 + 2 = -9$ Ans.

0.5 Yes, Lineal Independent constraint qualification condition hold at $(x_1, x_2) = (4, -3)$. Justification incorrect

Q5

A) $rank(A+B) \leq rank(A) + rank(B)$

Let the columns of A & B be $a_1, a_2, \ldots a_n$ & $b_1, b_2, \ldots b_n$ respectively. The rank of A&B are the dimensions of span$\{a_1, a_2 \ldots a_n\}$ & span$\{b_1, b_2, \ldots, b_n\}$

Now, the rank of A+B is the dimensions of span of $\{a_1+b_1, a_2+b_2, \ldots, a_n+b_n\}$

0.5 Since, span$\{a_1+b_1, a_2+b_2, \ldots, a_n+b_n\} \subseteq$ span$\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$

how?

$rank(A+B) \leq rank(A) + rank(B)$

Hence Proved.

$$\boxed{rank(A+B) \leq rank(A) + rank(B)}$$

B) $|rank(A) - rank(B)| \leq rank(A-B)$

Let the columns of A & B be $a_1, a_2, \ldots a_n$ & $b_1, b_2, \ldots, b_n$ respectively. The rank of A & B are the dimensions of span $\{a_1, a_2 \ldots a_n\}$ & span$\{b_1, b_2, \ldots, b_n\}$

0.4

Now the rank of A−B is the dimensions of span of $\{a_1-b_1, a_2-b_2, \ldots a_n-b_n\}$.

how Since, span$\{a_1-b_1, a_2-b_2, \ldots a_n-b_n\} \supseteq$ span$\{a_1, a_2, a_3 \ldots a_n, b_1, b_2 \ldots b_n\}$

$rank(A-B) \geq rank(A) - rank(B)$

$rank(A) - rank(B) \leq rank(A-B)$

Hence Proved.

1

6) $\min_{x} \frac{1}{2} x^T P x + c^T x + r$

s.t $x^T x \leq 1$

Sol $L(x, \lambda) = \frac{1}{2} x^T P x + c^T x + r + \lambda(1 - x^T x)$

$\frac{\partial L}{\partial x} = xP + c - 2\lambda x = 0$

$\Rightarrow x(P - 2\lambda) + c = 0$

$\Rightarrow x(P - 2\lambda) = -c$

$$\boxed{x^* = -(P - 2\lambda)^{-1} c}$$

Hence Proved.

Type of mathematical program = Quadratic Programming.

Q.4

Vector space $P = \{\beta_1, \beta_2, \ldots, \beta_r\}$ — (1)

$Q = \{\beta_1, \beta_2 \ldots \beta_r, \omega\}$ — (2)

Proof:

~~Let's say span (P) = R~~

Span of $P \{\beta_1, \beta_2, \ldots \beta_r\} \subseteq$ Span of $Q(\beta_1, \beta_2 \ldots \beta_r, \omega)$

So, span of P & Q will be equal if ω is a linear combination of vectors of P.

So if ω is a linear combination of vectors of $P \{\beta_1, \beta_2 \ldots \beta_r\}$ then we can remove ω from Q.

Then $P = \{\beta_1, \beta_2 \ldots, \beta_r\}$ — (3)

$Q = \{\beta_1, \beta_2 \ldots, \beta_r\}$ — (4)

from eqn ③ & ④

We can say that ~~p~~ P & Q have same vector space. i.e. $P = Q = \{p_1, p_2, \ldots p_r\}$

So, we can say ~~both~~ that

$\text{Span}(P)$ is equal to $\text{Span}(Q)$

Hence proved..