Indian Institute of Technology Madras

Department of Data Science and Artificial Intelligence

Sample Questions on Optimization

Problems

1. Find the local extrema of the following functions and classify the points as minimum or maximum.
   (a) $f(x) = 4x^3 - 3x^2 + 2x - 1$
   (b) $f(x) = \sin x + \cos x$
   (c) $f(x) = \frac{x^2-1}{x}$
   (d) $f(x, y) = x^2 + y^2$
   (e) $f(x, y) = 2x^2 + 2xy + 2y^2 - 6x$
2. For the function denoted by $f(x) = x^3 - 5x^2 + cx + 1$, where c is a constant, find and classify the local extrema in terms of the constant c.
3. Show that the function $f(x_1, x_2) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$ has only one stationary point and that it is neither a maximum nor minimum, but a saddle point. Sketch the contour line of f.
4. Suppose that $f(x) = x^T Qx$, where Q is an $n \times n$ symmetric positive semidefinite matrix. Show the function $f(x)$ is convex on the domain I^n using the definition of a convex function.
5. Determine the stationary points and classify their nature for the function, $f(x) = x^4 + y^4 - 36xy$
6. Solve the following optimization problems by hand(s) and also draw the feasible regions:
   - Find the maximum of the following function:

$$f(x) = 1 - 8x + 2x^2 - \frac{10}{3}x^3 + \frac{1}{4}x^4 + \frac{4}{5}x^5 - \frac{1}{6}x^6$$

$$f(x_1, x_2) = x_1 + x_2 \text{ subject to } x_1^2 + x_2^2 - 1 = 0$$

   - Verify the KKT conditions and find the Lagrange multipliers for the following function at $x = (1, 0)$

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{1}{8}\right)^4$$

   subject to

$$\begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geq 0$$

- Find the minimum of the following function:

$$f(x) = (x_1 - 1)^2 + x_2^2$$

subject to

$$x_1 - x_2^2 \leq 0$$

7. Consider the following optimization problem

$$\begin{aligned} \min \quad & x_1^2 + 2x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ \text{s.t.} \quad & x_2 - x_1^2 \geq 1 \end{aligned}$$

Find out the optimal point that satisfies the first order KKT condition. Verify the solution using python code.

8. Find the solution for the following problem.

(a) Minimum risk problem

$$\begin{aligned} \min \quad & \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \\ \text{s.t.} \quad & \sum_{1}^{n} x_i = 1 \end{aligned}$$

where $\mathbf{Q}$ is the co-variance matrix of return.

(b) Minimum risk for specified return ($R_p = 5$)

$$\begin{aligned} \min \quad & \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \\ \text{s.t.} \quad & \sum_{1}^{n} x_i = 1 \\ & \sum_{1}^{n} \bar{r}_i x_i \geq R_p \end{aligned}$$

(c) For $\mathbf{Q} = \begin{bmatrix} 1.5 & 0.1 \\ 0.1 & 1.2 \end{bmatrix}$ and $\bar{r}_i = \begin{bmatrix} 5 & 4.9 \end{bmatrix}^{\mathrm{T}}$.

9. Consider the function $f(x_1, x_2) = (x_1 + x_2^2)^2$. At the point $x = (1, 0)^T$. We consider the search direction $p = (-1, 1)^T$. Show that p is a descent direction and find all minimizers of $f(x_k + \alpha p_k)$.