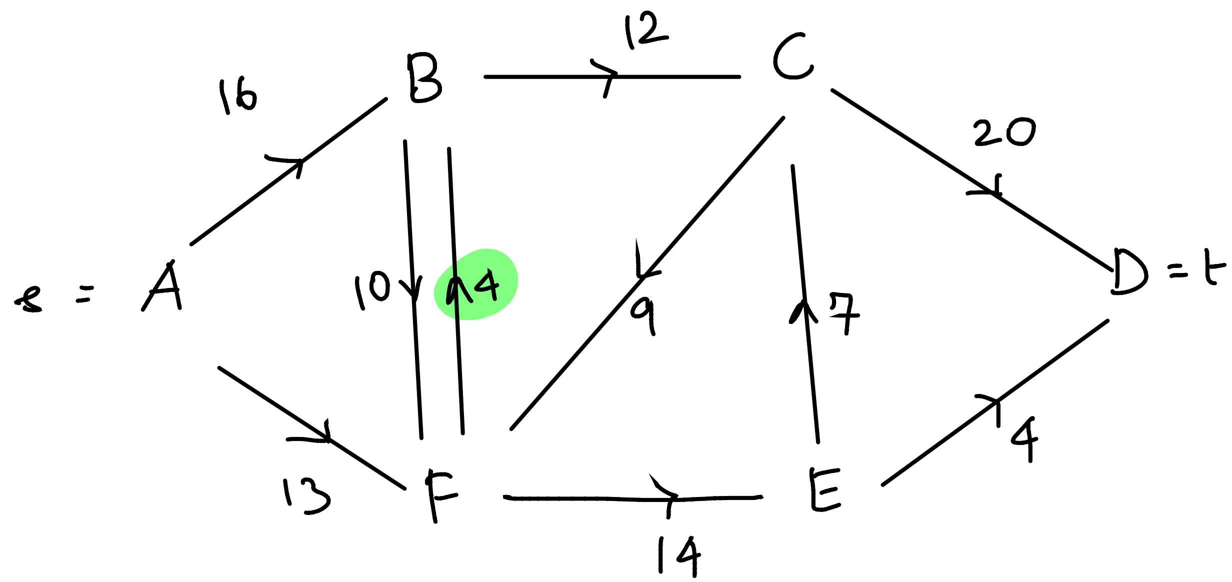
Residual Network

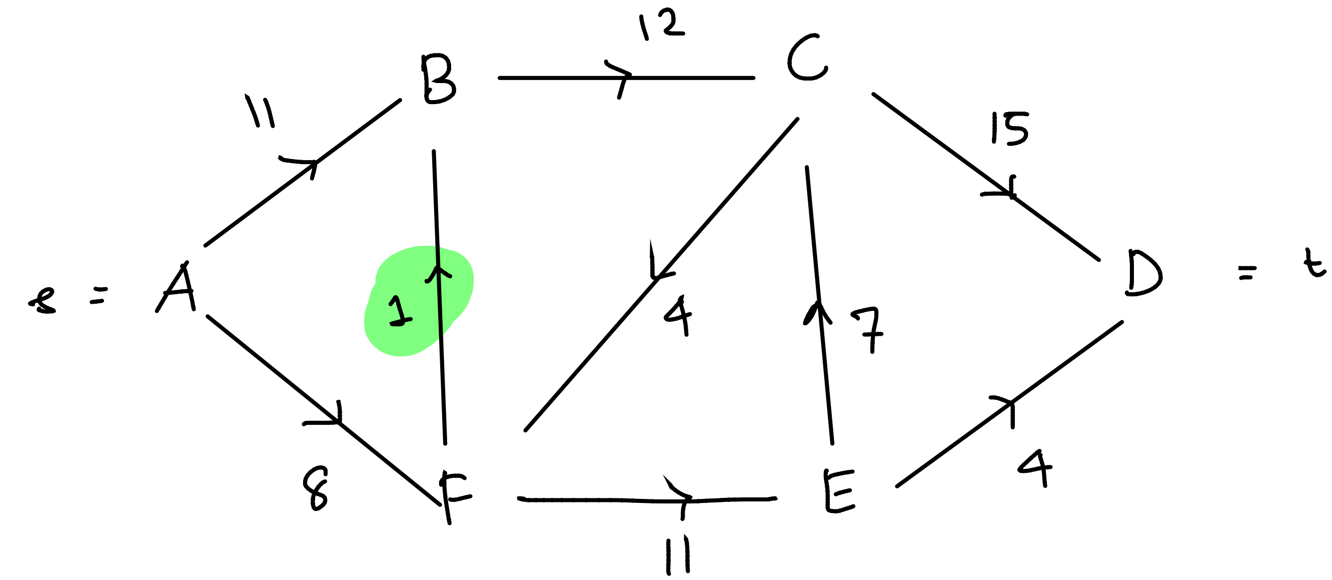
Residual capacity :

$$c_f(u,v) = c(u,v) - f(u,v)$$

Edge capacity

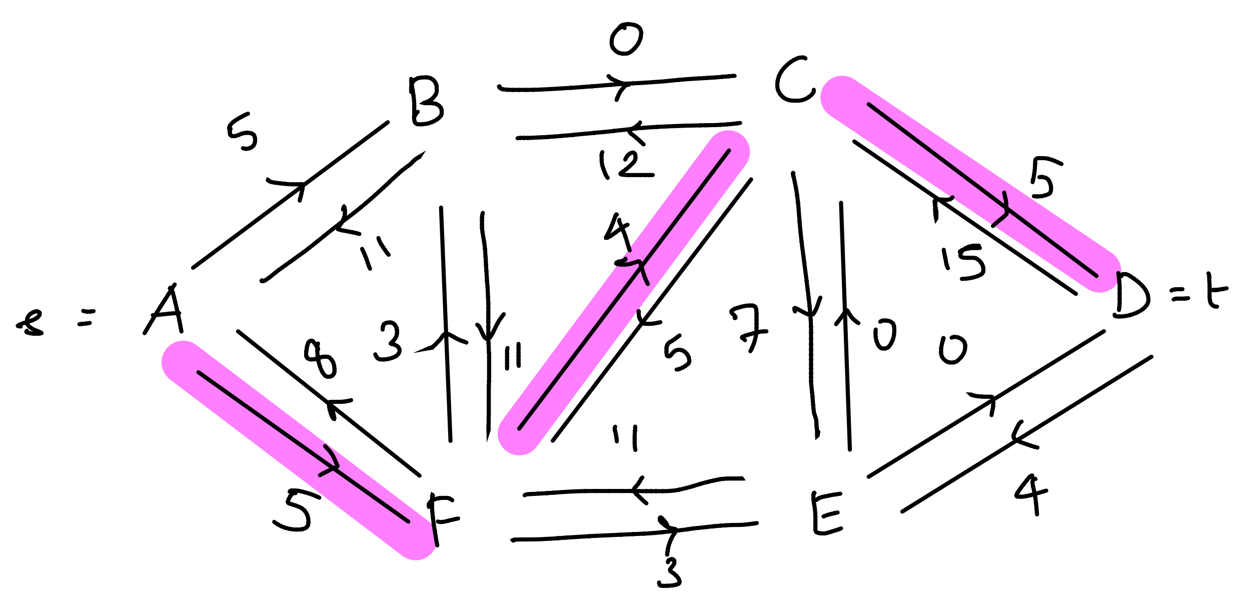


Flow f



Residual Graph

$$G_f = (V, c_f)$$

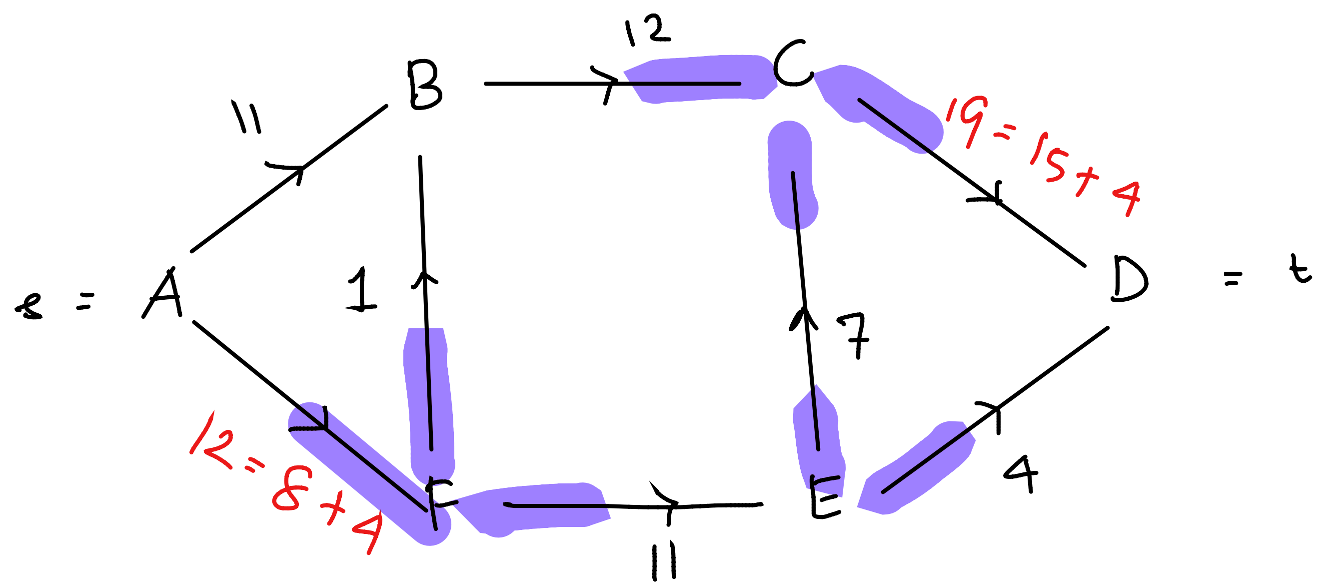


$$c_f(A,F) = 13 - 8 = 5, \quad c_f(F,A) = 0 - f(F,A) = 0 - (-f(A,F))$$

$$= f(A,F)$$

An Augmenting path is A F C D, path capacity = 4

New flow:



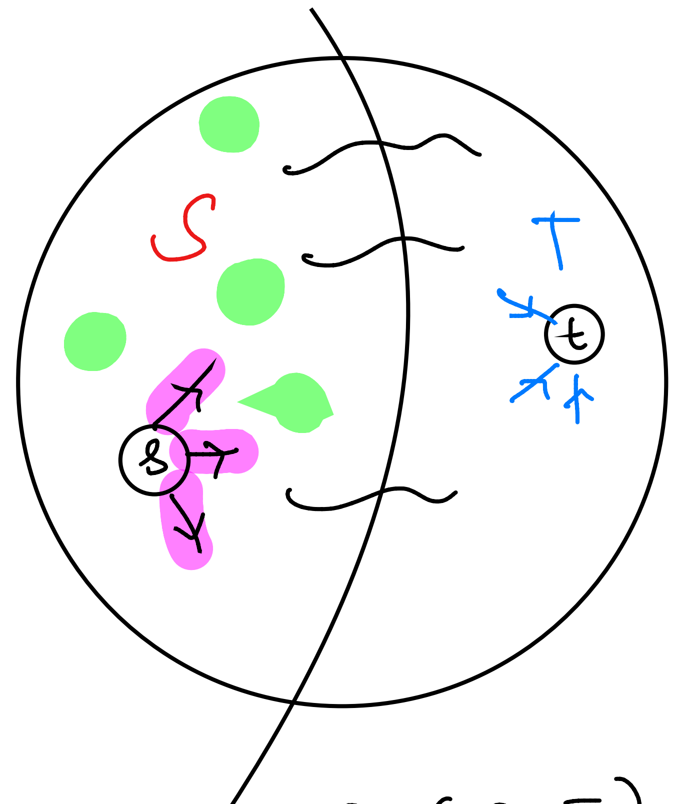
can calculate a new residual capacity

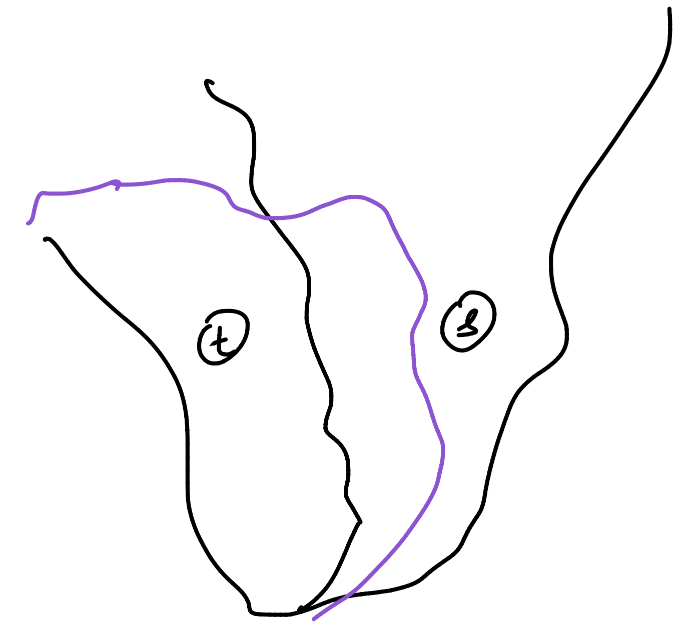
Fact: $|f|$ was for G, $|f'|$ was for G_f

$$|f + f'| = |f| + |f'|$$

cut : (S, T)

$s \in S$ $\quad$ $t \in T$





$$f(S, V) = f(S, V - S + S) = f(S, T) + \underbrace{f(S, S)}_{=0}$$

$$= f(S, V - S)$$

$$f(S, T) = f(S, V - S)$$

$$= f(S, V) - \underbrace{f(S, S)}_{=0}$$

$$= f(s, V) + \underbrace{f(S - s, V)}_{=0}$$

$$= |f|$$

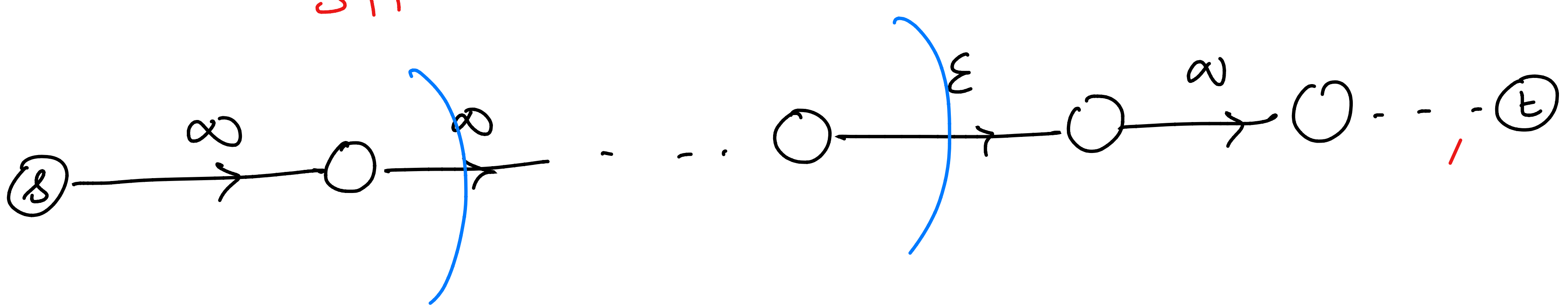
Fact :
$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v)$$

$$\leq \sum_{u \in S} \sum_{v \in T} c(u,v)$$

$$= c(S,T)$$

$$\Rightarrow \quad |f| \leq \underbrace{c(S,T)}_{\text{varies}}$$

$$\Rightarrow \quad |f| \leq \min_{S,T} c(S,T)$$



Max-Flow-Min-Cut

The following are equivalent

- f is a maximum flow in G

$\Downarrow$

- residual G_f has no augmenting path

$\Downarrow$

- $|f| = c(S,T)$ for some cut (S,T) of G

$(1) \Rightarrow (2)$

$\overline{(2)} \Rightarrow \overline{(1)}$ say there exists an augmenting path in G_f

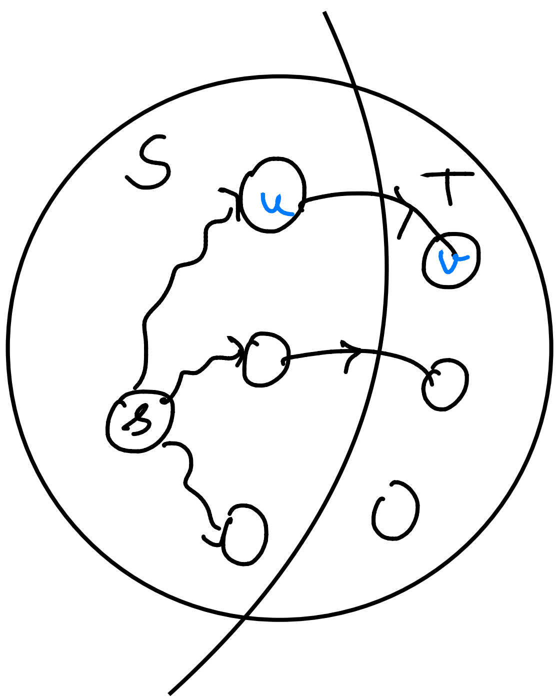
then $|f+f'| = |f| + |f'| > |f| \Rightarrow f$ is not maximum

(2) ⇒ (3)

Push as much tie boundary

$S = \{ v \in V :$ there is a path from s to v in $G_f \}$

$S \neq V$ (because no augmenting path, t cannot be reached by any path. So definitely t is not in S)



$u \in S, \; v \in T$

$c_f(u,v) = 0$

$c_f(u,v) = c(u,v) - f(u,v)$

$\Rightarrow \; f(u,v) = c(u,v)$

$|f| = f(S,T) = c(S,T)$

(3) ⇒ (1) equality is achieved ⇒ flow is maximum

FORD-FULKERSON (G, s, t)

for each edge $(u,v) \in E$

do $f(u,v) \leftarrow 0$

$f(v,u) \leftarrow 0$

while there exists a path from s to t in G_f

do $c_f(p) \leftarrow \min \{ c_f(u,v) : (u,v)$ is in $p \}$

for each edge (u,v) in p:

do $f(u,v) \leftarrow f(u,v) + c_f(p)$

$f(v,u) \leftarrow -f(u,v)$