

ASSIGNMENT 4
MTH102A

- (1) Let $\{w_1, w_2, \dots, w_n\}$ be a basis of a finite dimensional vector space V . Let v be a non zero vector in V . Show that there exists w_i such that if we replace w_i by v in the basis it still remains a basis of V .

Solution. Let $v = \sum_{i=1}^n a_i w_i$ for some $a_1, \dots, a_n \in \mathbb{F}$. Since v is non-zero, $a_i \neq 0$ for some $1 \leq i \leq n$. Assume $a_1 \neq 0$. Write $w_1 = \frac{1}{a_1}v - \sum_{i=2}^n \frac{a_i}{a_1}w_i$. Replace w_1 by v . Clearly $\{v, w_2, \dots, w_n\}$ spans V . Now we show that this set is L.I.. Let b_1, \dots, b_n be such that $b_1 v + \sum_{j=2}^n b_j w_j = 0$. Then $b_1 \sum_{i=1}^n a_i w_i + \sum_{j=2}^n b_j w_j = 0$ which implies $b_1 a_1 w_1 + \sum_{j=2}^n (b_1 a_j + b_j) w_j = 0$. Since w_1, w_2, \dots, w_n are LI we have $b_1 a_1 = 0, b_1 a_j + b_j = 0$ for $2 \leq j \leq n$. Since $a_1 \neq 0$ we have $b_j = 0$ for all $1 \leq j \leq n$.

- (2) Find the dimension of the following vector spaces :

- (i) X is the set of all real upper triangular matrices,
- (ii) Y is the set of all real symmetric matrices,
- (iii) Z is the set of all real skew symmetric matrices,
- (iv) W is the set of all real matrices with $Tr(A) = 0$

Solution. Let E_{ij} be the matrix with ij^{th} entry one and all other entries are zero, F_{ij} be the matrix with ij^{th} and ji^{th} entries are 1 and all other entries are zero and for $i \neq j$ define D_{ij} to be the matrix with ij^{th} entry 1, ji^{th} entry -1 and all other entries are zero.

- (i) The set $\{E_{ij}; i \leq j\}$ forms a basis for X . Hence $\dim(X) = n + \frac{n^2-n}{2} = \frac{n(n+1)}{2}$.
- (ii) The set $\{F_{ij}; i \leq j\}$ is a basis for Y . Hence $\dim(Y) = \frac{n(n+1)}{2}$.
- (iii) For a real skew-symmetric matrix all diagonal entries are zero. Then the set $\{D_{ij}; i < j\}$ is a basis. Hence $\dim(Z) = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$.
- (iv) Let A be a matrix with trace zero. Then $\sum_{i=1}^n a_{ii} = 0$. So $a_{11} = -(a_{22} + \dots + a_{nn})$. The set $\{E_{ij}, E_{ji} : i \neq j\} \cup \{E_{ii} - E_{i+1, i+1} : 1 \leq i \leq n-1\}$ is a basis of W . Hence $\dim(W) = n^2 - 1$.

- (3) Let $\mathcal{P}_2(X, \mathbb{R})$ be the vector space of all polynomials in X of degree less or equal to 2. Show that $B = \{X + 1, X^2 - X + 1, X^2 + X - 1\}$ is a basis of $\mathcal{P}_2(X, \mathbb{R})$. Determine the coordinates of the vectors $2X - 1, 1 + X^2, X^2 + 5X - 1$ with respect to B .

Solution. First we show that the set B is L.I.. Let $a_0, a_1, a_2 \in \mathbb{R}$ such that $a_0(X+1) + a_1(X^2 - X + 1) + a_2(X^2 + X - 1) = 0$. Then $a_0 + a_1 - a_2 + (a_0 - a_1 + a_2)X + (a_1 + a_2)X^2 = 0$. From here we will get $a_0 = a_1 = a_2 = 0$.

Let $p(X) = a_0 + a_1X + a_2X^2$ be any element in $\mathcal{P}_2(X, \mathbb{R})$. Let $b_0, b_1, b_2 \in \mathbb{R}$ such that $p(X) = a_0 + a_1X + a_2X^2 = b_0(X+1) + b_1(X^2 - X + 1) + b_2(X^2 + X - 1)$ then we get $b_0 = \frac{a_0+a_1}{2}, b_1 = \frac{a_0-a_1+2a_2}{4}, b_2 = \frac{a_1-a_0+2a_2}{4}$. Hence B spans $\mathcal{P}_2(X, \mathbb{R})$.

We have $2X - 1 = \frac{1}{2}(X+1) - \frac{3}{4}(X^2 - X + 1) + \frac{3}{4}(X^2 + X - 1)$. So $[2X - 1]_B = (\frac{1}{2}, -\frac{3}{4}, \frac{3}{4})$.

$1 + X^2 = \frac{1}{2}(X+1) + \frac{3}{4}(X^2 - X + 1) + \frac{1}{4}(X^2 + X - 1)$. So $[1 + X^2]_B = (\frac{1}{2}, \frac{3}{4}, \frac{1}{4})$.

$X^2 + 5X - 1 = 2(X+1) - 1(X^2 - X + 1) + (X^2 + X - 1)$. So $[X^2 + 5X - 1]_B = (2, -1, 1)$.

(4) Let W be a subspace of a finite dimensional vector space V

(i) Show that there is a subspace U of V such that $V = W + U$ and $W \cap U = \{0\}$,

(ii) Show that there is no subspace U of V such that $W \cap U = \{0\}$ and $\dim(W) + \dim(U) > \dim(V)$.

Solution.

(i) Let $\dim(V) = n$, since V is finite dimensional, W is also finite dimensional. Let $\dim(W) = k$ and let $B = \{w_1, \dots, w_k\}$ be a basis for W . In case $k = n$ we take $U = \{0\}$. If $k < n$ we extend B to a basis $B_1 = \{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ of V . Let U be the subspace of V generated by $\{v_{k+1}, \dots, v_n\}$.

Let $v \in V$. Then there exist scalars $a_1, \dots, a_n \in \mathbb{F}$ such that $v = a_1w_1 + \dots + a_kw_k + a_{k+1}v_{k+1} + \dots + a_nv_n = (a_1w_1 + \dots + a_kw_k) + (a_{k+1}v_{k+1} + \dots + a_nv_n) \in W + U$.

Now we show that $W \cap U = \{0\}$. Let $v \in W \cap U$. Then there exist scalars a_1, \dots, a_k and b_{k+1}, \dots, b_n such that $a_1w_1 + \dots + a_kw_k = v = b_{k+1}v_{k+1} + \dots + b_nv_n$. We have $a_1w_1 + \dots + a_kw_k - b_{k+1}v_{k+1} - \dots - b_nv_n = 0$. Since B_1 is a basis, $a_1 = \dots = a_k = b_{k+1} = \dots = b_n = 0$. Hence $W \cap U = \{0\}$.

(ii) Let $W \cap U = \{0\}$. Then $\dim(W + U) = \dim(W) + \dim(U) - \dim(W \cap U) = \dim(W) + \dim(U)$. So if $\dim(W) + \dim(U) > \dim(V)$ we have $\dim(W + U) > \dim(V)$, which is a contradiction as $W + U$ is a subspace of V .

(5) Decide which of the following maps are linear transformations:

(i) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + 2y, z, |x|)$,

(ii) Let $M_n(\mathbb{R})$ be the set of all $n \times n$ real matrices and $T : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ defined by

(a) $T(A) = A^T$,

(b) $T(A) = I + A$, where I is the identity matrix of order n ,

(c) $T(A) = BAB^{-1}$, where $B \in M_n(\mathbb{R})$ is an invertible matrix.

Solution.

(i) Not a linear transformation.

Take $a = -1$. Then $T(a(1, 0, 1)) = T(-1, 0, -1) = (-1, -1, 1) \neq aT((1, 0, 1)) = -1(1, 1, 1) = (-1, -1, -1)$.

(ii) (a) Linear Transformation.

Let $A, B \in M_n(\mathbb{R})$ and $a \in \mathbb{R}$. Then $T(A + aB) = A + aB^T = A^T + aB^T$.

(b) Not a linear transformation.

Let O be the zero matrix. Then $T(O) = I \neq O$.

(c) Linear transformation.

Let $P, Q \in M_n(\mathbb{R})$ and $a \in \mathbb{R}$. Then $T(P + aQ) = B(P + aQ)B^{-1} = BPB^{-1} + aBQB^{-1} = T(P) + aT(Q)$.

(6) Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be defined by $T(z) = \bar{z}$. Show that T is \mathbb{R} -linear but not \mathbb{C} -linear.

Solution. Let $a + ib, c + id \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. Then $T((a + ib) + \alpha(c + id)) = T(a + \alpha c + i(b + \alpha d)) = a + \alpha c - i(b + \alpha d) = (a - ib) + \alpha(c - id) = T(a + ib) + \alpha T(c + id)$. Hence T is \mathbb{R} -linear whereas $T(i \cdot i) = T(-1) = -1 \neq iT(i) = i(-i) = 1$, so it is not \mathbb{C} -linear.

(7) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that $T(1, 0, 0) = (1, 0, 0)$, $T(1, 1, 0) = (1, 1, 1)$, $T(1, 1, 1) = (1, 1, 0)$. Find $T(x, y, z)$, $\text{Ker}(T)$, $R(T)$ (Range of T). Prove that $T^3 = T$.

Solution. Let $(x, y, z) \in \mathbb{R}^3$. Then $(x, y, z) = (x - y)(1, 0, 0) + (y - z)(1, 1, 0) + z(1, 1, 1)$. We have $T((x, y, z)) = (x - y)T(1, 0, 0) + (y - z)T(1, 1, 0) + zT(1, 1, 1) = (x - y)(1, 0, 0) + (y - z)(1, 1, 1) + z(1, 1, 0) = (x, y, y - z)$.

Let $x = (a, b, c) \in \text{Ker}(T)$, then $T(x) = (a, b, b - c) = (0, 0, 0)$ implies $(a, b, c) = (0, 0, 0)$. Hence $\text{Ker}(T) = \{(0, 0, 0)\}$.

$R(T) = \{(x, y, y - z); x, y, z \in \mathbb{R}\} = \text{Span}\{(1, 0, 0), (0, 1, 1), (0, 0, -1)\}$.

$T^3((x, y, z)) = T^2((x, y, y - z)) = T((x, y, y - (y - z))) = T((x, y, z))$.

(8) Find all linear transformations from \mathbb{R}^n to \mathbb{R} .

Solution. For $a_1, \dots, a_n \in \mathbb{R}$, the map defined by $T((x_1, \dots, x_n)) = a_1x_1 + \dots + a_nx_n$ is a linear transformation. We claim that all the linear transformations are of this form for some $a_1, \dots, a_n \in \mathbb{R}$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ be the n -tuple with all components equal to 0, except the i -th, which is 1. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}$ be a linear map. Suppose $T(e_i) = a_i$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq n$. Then $T(x_1, x_2, \dots, x_n) = T(x_1e_1 + x_2e_2 + \dots + x_n e_n) = a_1x_1 + \dots + a_nx_n$.

- (9) Let $\mathcal{P}_n(X, \mathbb{R})$ be the vector space of all polynomials in X of degree less or equal to n . Let T be the differentiation transformation from $\mathcal{P}_n(X, \mathbb{R})$ to $\mathcal{P}_n(X, \mathbb{R})$. Find $Range(T)$ and $Ker(T)$.

Solution. If $f(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n \in \mathcal{P}_n(X, \mathbb{R})$ then $T(f) = a_1 + 2a_2X + \cdots + na_nX^{n-1}$. So $Range(T) = Span\{1, X, \cdots X^{n-1}\}$.

$Ker(T) = \{f : T(f) = f' = 0\}$. So $Ker(T) = \{f(X) : f(X) = c \text{ for } c \in \mathbb{R}\} = Span\{1\} = \mathbb{R}$.