

ASSIGNMENT 3
MTH102A

- (1) In \mathbb{R} , consider the addition $x \oplus y = x + y - 1$ and the scalar multiplication $\lambda.x = \lambda(x-1)+1$. Prove that \mathbb{R} is a vector space over \mathbb{R} with respect to these operations. What is the additive identity (the $\mathbf{0}$ vector in the definition) in this case ?

Solution: Easy verification. Here the $\mathbf{0}$ vector is $1 \in \mathbb{R}$.

- (2) Show that $W = \{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\}$ is a subspace of \mathbb{R}^4 spanned by vectors $(1, 0, 0, -1), (0, 1, 0, 1), (0, 0, 1, 1)$.

Solution:

$$\begin{aligned}(x_1, x_2, x_3, x_4) &\in \{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} \\ \Leftrightarrow (x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3, -x_1 + x_2 + x_3) \text{ as } x_4 = -x_1 + x_2 + x_3 \\ &= x_1(1, 0, 0, -1) + x_2(0, 1, 0, 1) + x_3(0, 0, 1, 1)\end{aligned}$$

Moreover, $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\}$ is a subspace of \mathbb{R}^4 because it is a linear span of vectors in \mathbb{R}^4 .

- (3) Describe all the subspaces of \mathbb{R}^3 .

Solution: $\{0\}$ and \mathbb{R}^3 are the trivial subspaces of \mathbb{R}^3 . Any line passing through origin is an one-dimensional subspace of \mathbb{R}^3 and any plane passing through origin is a 2-dimensional subspace in \mathbb{R}^3 . We claim that these are all the subspaces of \mathbb{R}^3 .

Let W be a non-trivial subspace of \mathbb{R}^3 . If $\dim(W) = 1$ choose a basis $\{v\}$ of W . Then $W = \{a.v : a \in \mathbb{R}\}$. So W represents a line passing through origin in the direction of v . If $\dim(W) = 2$ then choose a basis $\{v_1, v_2\}$ of W . Then $W = \text{Span}\{v_1, v_2\} = \{av_1 + bv_2 : a, b \in \mathbb{R}\}$. So W represents a plane passing through origin with normal vector $v_1 \times v_2$.

- (4) Find the condition on real numbers a, b, c, d so that the set $\{(x, y, z) | ax + by + cz = d\}$ is a subspace of \mathbb{R}^3 .

Solution: Let $W = \{(x, y, z) | ax + by + cz = d\}$. If W is a subspace then $(0, 0, 0) \in W$ and so $d = 0$.

- (5) Discuss the linear dependence/independence of following set of vectors:

(i) $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ in \mathbb{R}^3 as a vector space over \mathbb{R} ,

Ans: Linearly independent since the determinant of the matrix formed by taking $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ as row vectors is non zero. So they are linearly independent.

(ii) $\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (3, 2, 1, 0)\}$ in \mathbb{R}^4 as a vector space over \mathbb{R} ,

Ans: Linearly dependent since $(3, 2, 1, 0) = (1, 0, 0, 0) + (1, 1, 0, 0) + (1, 1, 1, 0)$.

(iii) $\{(1, i, 0), (1, 0, 1), (i + 2, -1, 2)\}$, in \mathbb{C}^3 as a vector space over \mathbb{C} ,

Ans: $(i + 2, -1, 2) = i(1, i, 0) + (1, 0, 1)$. So they are linearly dependent.

(iv) $\{(1, i, 0), (1, 0, 1), (i + 2, -1, 2)\}$, in \mathbb{C}^3 as a vector space over \mathbb{R} ,

Ans: If $a(1, i, 0) + b(1, 0, 1) + c(i + 2, -1, 2) = 0$ for $a, b, c \in \mathbb{R}$. Then we have $a = b = c = 0$. So they are linearly independent.

(v) The sets $\{1, \sin x, \cos x\}$ and $\{2, \sin^2 x, \cos^2 x\}$ in the vector space of real valued functions $F = \{f : f : \mathbb{R} \rightarrow \mathbb{R}\}$.

Solution: Suppose $a.1 + b.\sin x + c.\cos x = 0$. Then the identity is true for all $x \in \mathbb{R}$.

For $x = 0$ we have $a + c = 0$. For $x = \pi/2$ we have $a + b = 0$ and for $x = -\pi/2$ we have $a - b = 0$. From these linear equations we have $a = b = c = 0$. So the set $\{1, \sin x, \cos x\}$ is linearly independent. On the other hand we have $2\sin^2 x + 2\cos^2 x - 2 = 0$. So the set $\{2, \sin^2 x, \cos^2 x\}$ is linearly dependent.

(v) $\{u + v, v + w, w + u\}$ in a vector space V given that $\{u, v, w\}$ is linearly independent.

Ans: If $a(u + v) + b(v + w) + c(w + u) = 0$ for some scalars a, b, c . Then we have $a + b = b + c = a + c = 0$ and hence $a = b = c = 0$. So $\{u + v, v + w, w + u\}$ is linearly independent.

(6) Let $W_1 = \text{Span}\{(1, 1, 0), (-1, 1, 0)\}$ and $W_2 = \text{Span}\{(1, 0, 2), (-1, 0, 4)\}$. Prove that $W_1 + W_2 = \mathbb{R}^3$.

Solution: The three vectors $(1, 1, 0), (-1, 1, 0), (1, 0, 2)$ are in $W_1 + W_2$ and are linearly independent. So $\text{Span}\{(1, 1, 0), (-1, 1, 0), (1, 0, 2)\} = W_1 + W_2 = \mathbb{R}^3$.

(7) Find 3 vectors u, v and w in \mathbb{R}^4 such that $\{u, v, w\}$ is linearly dependent whereas $\{u, v\}$, $\{u, w\}$, and $\{v, w\}$ are linearly independent. Extend each of the linearly independent sets to a basis of \mathbb{R}^4 .

Solution: Let $u = (1, 0, 0, 0), v = (0, 1, 0, 0)$ and $w = (1, 1, 0, 0)$. Then since $w = u + v$, the set $\{u, v, w\}$ is linearly dependent whereas the sets $\{u, v\}$, $\{u, w\}$, and $\{v, w\}$ are linearly independent.

Extending $\{u, v\}$ we have the set $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ a basis of \mathbb{R}^4 .

Extending $\{v, w\}$ we have the set $\{((0, 1, 0, 0), (1, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1))\}$ a basis of \mathbb{R}^4 .

Extending $\{u, w\}$ we have the set $\{((1, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1))\}$ a basis of \mathbb{R}^4 .

- (8) Let A be a $n \times n$ matrix over \mathbb{R} . Then A is invertible iff the row vectors are linearly independent over \mathbb{R} iff the column vectors are linearly independent over \mathbb{R} .

Solutions: We know that A is invertible iff the row reduced echelon form in the identity matrix iff the system $Ax = 0$ has only the trivial solution $x = 0$. Let C_1, C_2, \dots, C_n be the column vectors of A . So A is invertible iff $b_1C_1 + b_2C_2 + \dots + b_nC_n = 0$ for some $b_i \in \mathbb{R}$ implies $b_i = 0$ for all i . So A is invertible iff the column vectors of A are linearly independent over \mathbb{R} .

A is invertible iff A^T is invertible. So the row vectors of A are linearly independent over \mathbb{R} iff the column vectors of A are linearly independent over \mathbb{R} .

- (9) Determine if the set $T = \{1, x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2\}$ is a basis for the vector space of polynomials in x of degree ≤ 4 . Is this set a basis for the vector space of polynomials in x of degree ≤ 3 ?

Solution If $a.1 + b(x^2 - x + 5) + c(4x^3 - x^2 + 5x) + d(3x + 2) = 0$ then equating the coefficients we get $a = b = c = d = 0$. So the set is linearly independent. But $x^4 \notin \text{Span}(T)$. So it is not a basis for the vector space of polynomials in x of degree ≤ 4 . However since dimension of the vector space of polynomials in x of degree ≤ 3 is 4 the linearly independent set $T = \{1, x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2\}$ is a basis.