Maximum Flow

Flow Network : $G = (V, E)$

* directed Graph
* edge capacity $c(u,v) \geq 0$

Capacity is diametrically opposite to cost

Capacity : No Edge $\Rightarrow$ capacity $= 0$

cost : No Edge $\Rightarrow$ cost $= \infty$

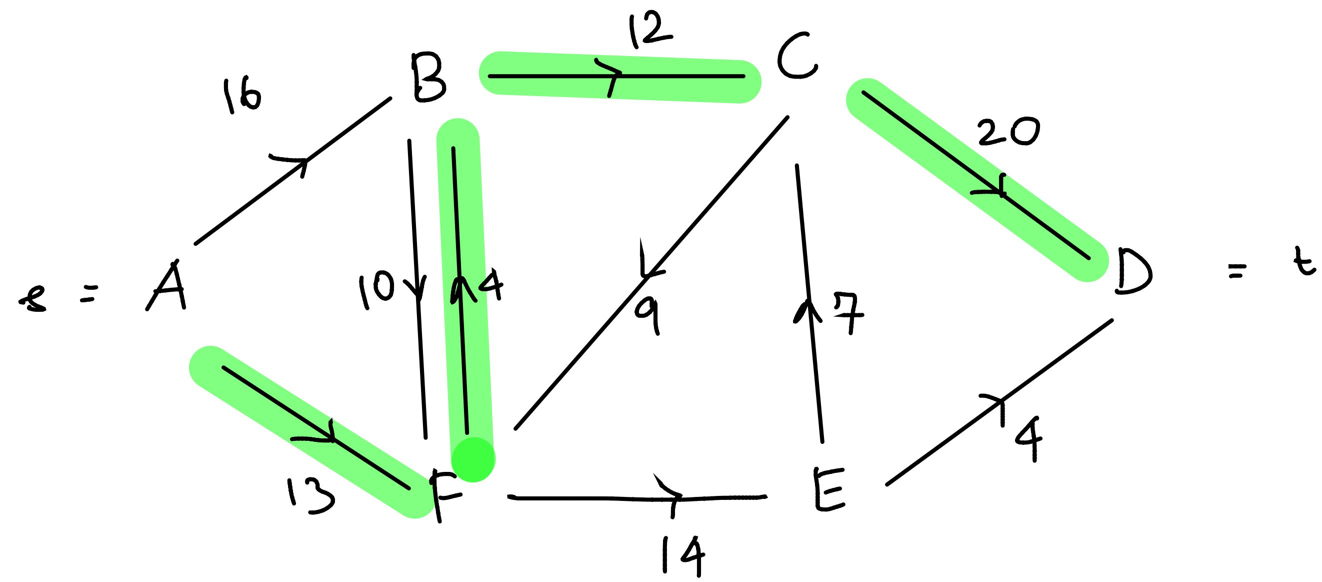
Capacity $= 0$ $\Rightarrow$ Bad for flow

cost $= 0$ $\Rightarrow$ Good for shortest paths

source : s , terminal / destination / sink : t
vertex vertex

Goal : Need to send across as many "things" from s to t

Example :



the flow is limited by $c(F, B) = 4$ (bottleneck)

Assume : Every vertex lies on some path between s and t

Flow: $f : V \times V \to \mathbb{R}$

Properties

* capacity constraint : $f(u, v) \leq c(u, v)$ (cannot push more than an capacity)

* skew symmetry : $f(u, v) = -f(v, u)$ (reverse flow)

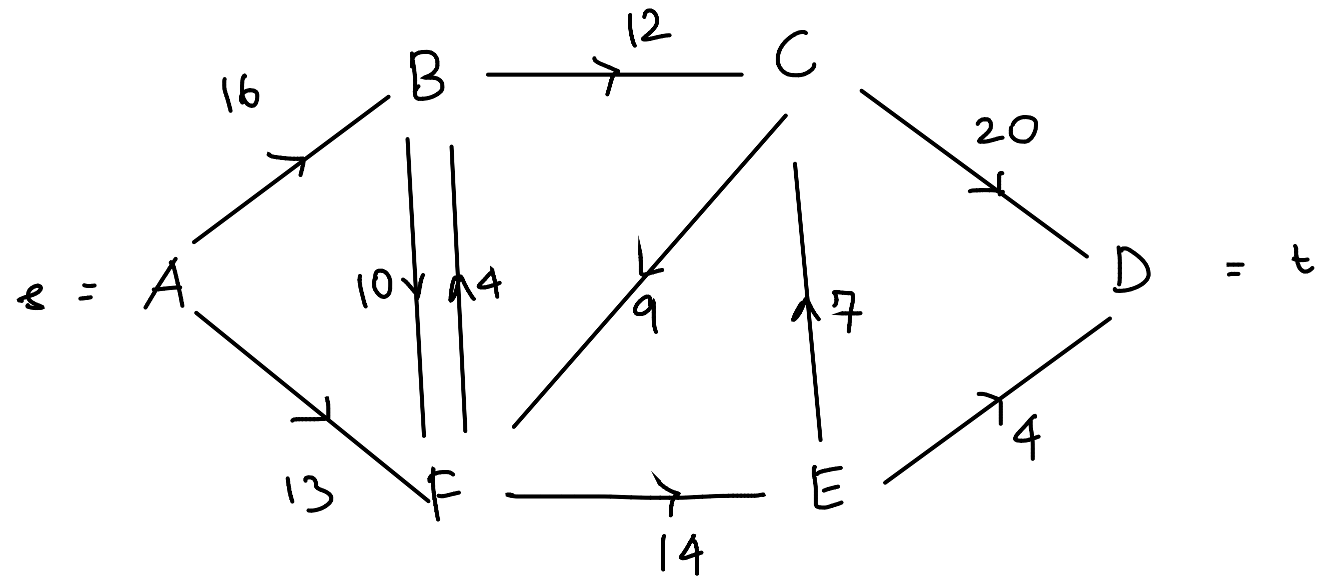
If (u, v) are not adjacent $f(u, v) = 0 = -f(u, v)$

* Conservation : $\forall u \in V - \{s, t\}$ (no accumulation in the vertex) other than s, t

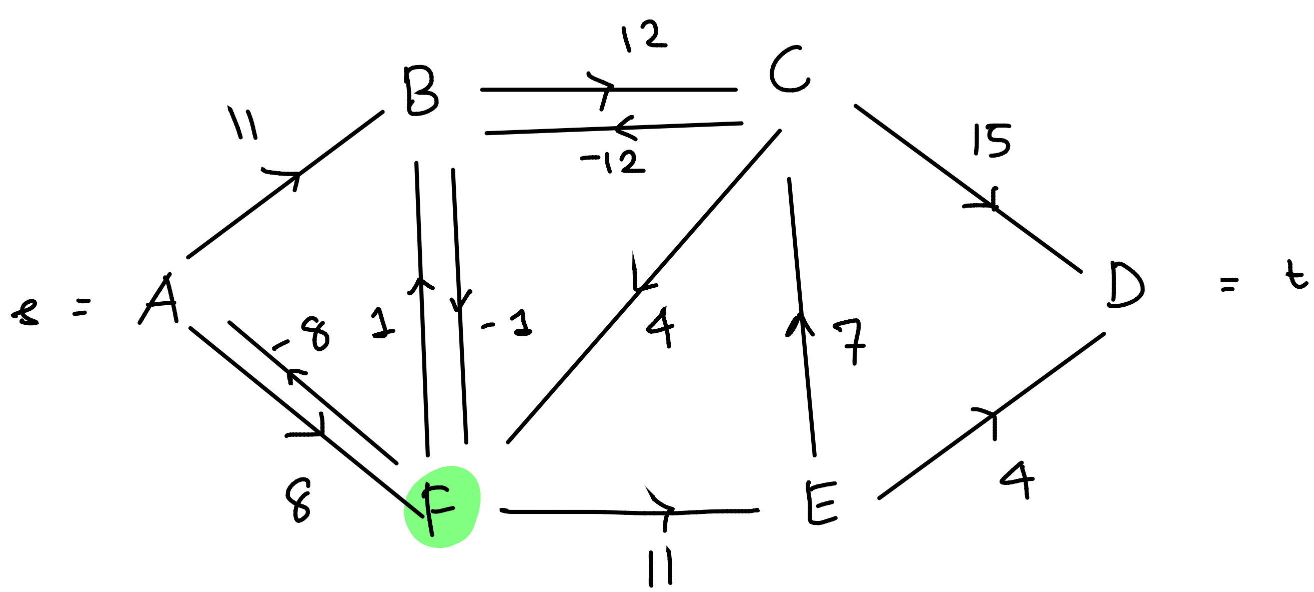
$$\sum_{v \in V} f(u, v) = 0$$

what comes in goes out

Example :



Flow



* Other negative edges have not been marked

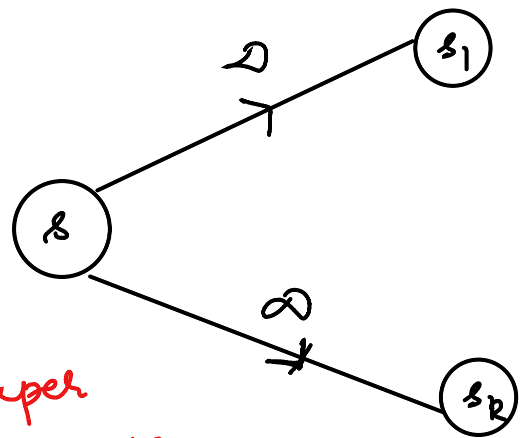
in : 8 + 4 = 1 + 11 : out

Flow conservation at F

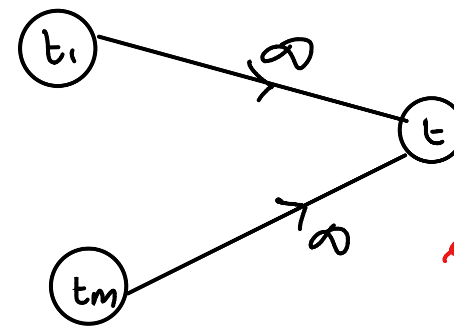
$$f(F, V) = \sum_{v \in V} f(F, V)$$

$$= f(F, A) + f(F, B) + f(F, E) + f(F, C)$$

Multi-Source Multi-Sink



super source



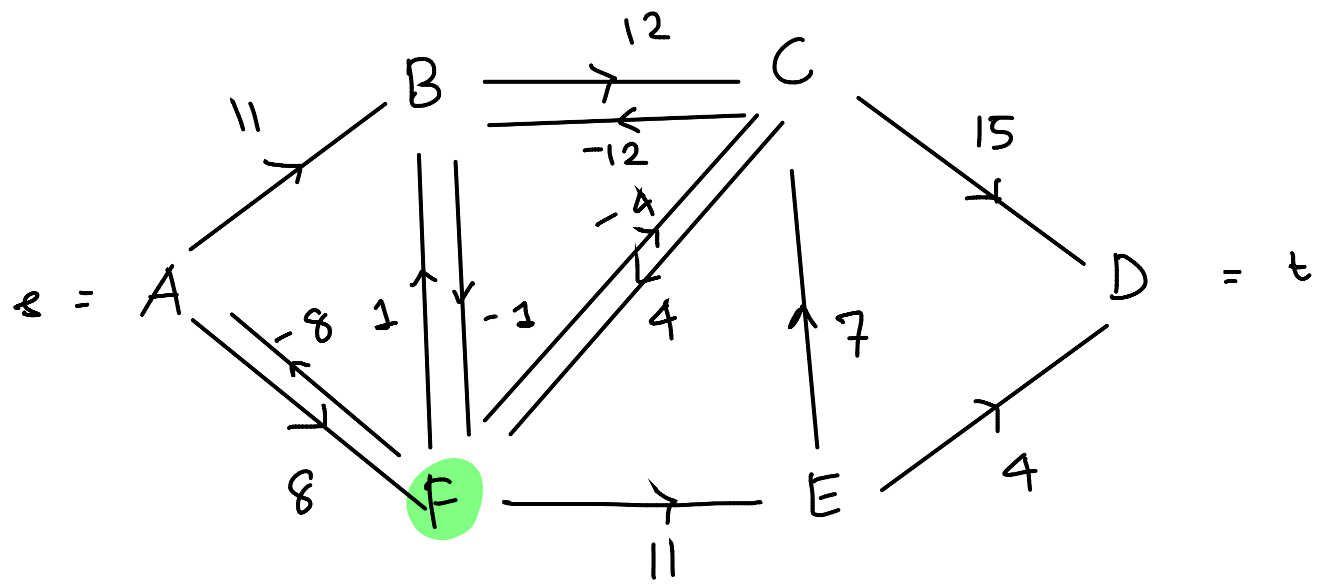
super terminal

Value of flow: $|f| = \sum_{v \in V} f(s, v)$ (the quantity that leaves the source)

Notation I: $f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$

Flow Conservation: $\forall u \in V - \{s, t\}$, $f(u, V) = 0$

$$f(u, V) = \sum_{v \in V} f(u, V) = 0$$



Flow conservation at F

$$f(F, V) = \sum_{v \in V} f(F, v)$$

$$= f(F, A) + f(F, B) + f(F, E) + f(F, C)$$

$$= -8 + 1 + 11 - 4 = 0$$

Notation II:

$$f(s, V - s) = f(s, V - \{s\})$$

Lemma (Implicit Notation)

- $f(X,X) = 0$ $\qquad f(X,X) = \sum_{x \in X} \sum_{x' \in X} f(x,x')$

terms occur in pairs of $f(u,v) + f(v,u) = 0$

- $f(X,Y) = -f(Y,X)$ $\qquad f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$ (each term is flipped)

$$= \sum_{x \in X} \sum_{y \in Y} -f(y,x)$$

$$= -f(Y,X)$$

- X, Y, Z s.t $X \cap Y = \phi$

$$f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$$

$$\underbrace{\sum_{x' \in (X \cup Y)}}_{\substack{\text{since } X \cap Y = \phi \\ x' \in X \text{ or } x' \in Y}} \sum_{z \in Z} f(x', z) = \sum_{x \in X} \underbrace{\sum_{z \in Z}}_{x' \text{ comes from } X} f(x,z) + \underbrace{\sum_{y \in Y} \sum_{z \in Z}}_{x' \text{ comes from } Y} f(y,z)$$

term by term it matches.

Value of flow in implicit notation

$$|f| = f(s, V)$$ what leaves the source

$$f(V,V) = f(\underbrace{s}_{X} \cup \{\underbrace{V-\{s\}}_{Y}\}, V) = f(s,V) + f(V-s, V)$$

$$f(s,V) = \overset{=0}{f(V,V)} - f(V-s, V)$$

$$V-\{s\} = \{V-\{s\}-\{t\}\} \cup \{t\}$$

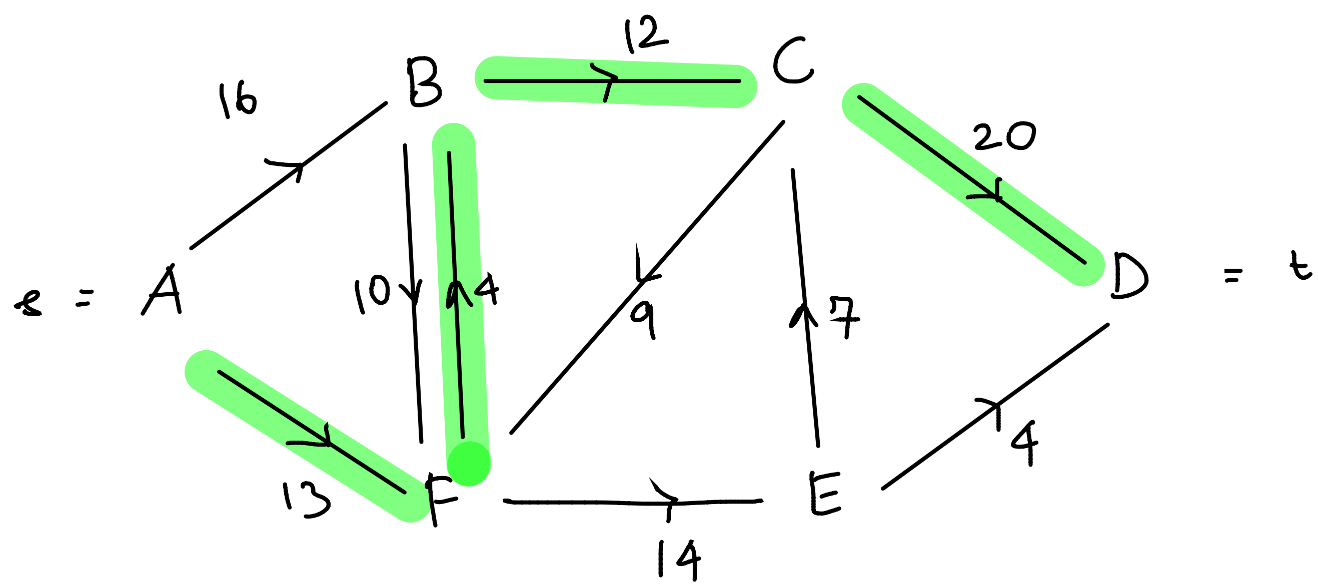
$$= f(V, V-s)$$

$$= \overset{=0\ (\text{Flow conservation})}{f(V, V-s-t)} + f(V, t)$$

$$= f(V, t)$$ what comes inside the terminal.

Augmenting path from s to t

Say initially we did not assign any flow, then A F B C D is an augmenting path with capacity $c_p = 4$



FORD-FULKERSON-METHOD

initialise flow f to be 0

while there exists an augmenting path p

do augment flow f along p

return f