Project Title:
Damping Modelling in Dynamic System by Using Fractional Derivatives

by

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Introduction - Dynamic System with Fractional Order Damper.

- For fractional derivative damper, damping force is proportional to fractional derivative of displacement.

$$\therefore F_{DM} = c\frac{d^{\alpha}x}{dt^{\alpha}}........\left(\alpha \in (0,1)\right)$$

where c is damping coefficient and ‘α’ is called as fractional order.

- The differential equation is given by,

$$m\ddot{x} + c\frac{d^{\alpha}x}{dt^{\alpha}} + kx = F(t)$$

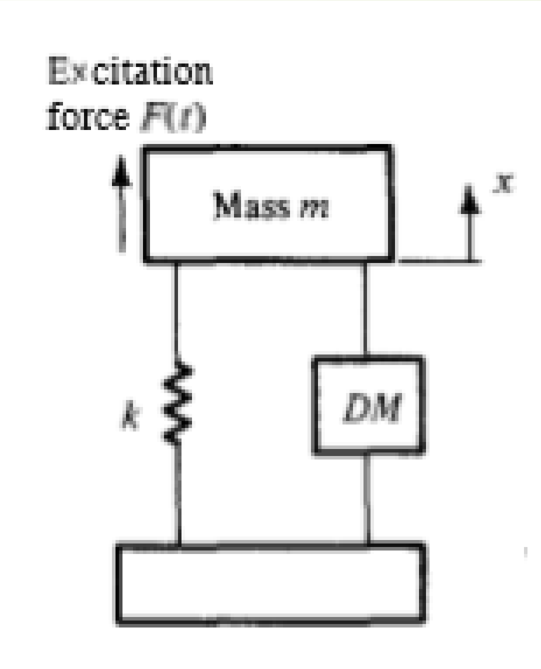

Fig. 1: Dynamic System

Objectives

1. To solve the fractional differential equation.

$$mD^2x(t) + cD^\alpha x(t) + kx(t) = F(t)$$

2. Response analysis and damping model identification.
3. Study of transmissibility for fractional derivative damping.
4. Study of dynamic vibration absorber for fractional derivative damping.

Literature Survey

1. Sakakibara, S. (2004). The Institute for Mathematical Sciences.

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- The fractional differential equation for damped oscillation is solved using Laplace Transform for any arbitrary order of fractional derivative.
- It is shown that there is power-law decay as compared with exponential decay of the usual viscous damping.

3. Rekhviashvili, S., Pskhu, A., Agarwal, P., & Jain, S. (2019). Turkish Journal of Physics.

- The relation between the order of fractional differentiation in the equation of motion and Q-factor of an oscillator is suggested.

4. Podlubny, I. (1998). Mathematics in Science and Engineering.

- Backward Difference method (BDM) is given to solve an FDE.

5. Liu, L., & Duan, J. (2015). Open Mathematics.

- The solution of the fractional vibration equation is investigated, where the damping term is characterized by the Caputo fractional derivative.

Literature Survey

6. Garrappa, R., Kaslik, E., & Popolizio, M. (2019). Mathematics Journal.

- A guide is provided to the evaluation of fractional integrals and derivatives of some elementary functions and studied the action of different derivatives on the same function.

7. Gómez-Aguilar, J., Yépez-Martínez, H., Calderón-Ramón, C., Cruz-Orduña, I., Escobar-Jiménez, R., & Olivares-Peregrino, V. (2015). Entropy.

- The fractional equations of the mass-spring-damper system with Caputo and Caputo–Fabrizio derivatives are presented. An analytical solution is obtained, and the resulting equations are given in terms of the Mittag–Leffler function.

8. Torvik, P., & Bagley, D. (1987). ASME, Design Engineering Division.

- They used a constitutive equation that involves generalized or fractional derivatives for the description and prediction of the time-dependent behavior of materials of interest for damping applications is described.

9. Das, S. (2011). Springer-Verlag Berlin Heidelberg.

- The basic introduction, dealing with the development of the fractional calculus is given. Several definitions of fractional differintegrations and the most popular ones are introduced here; it gives the feel of fractional differentiation of some functions, i.e., how they look. It also gives an overview of the application of fractional calculus.

Backward difference method (BDM) to solve the fractional differential equation (FDE).

- BDM is based on Grunwald-Letnikov’s definition of a fractional derivative. So, Grunwald-Letnikov's definition of a fractional derivative is given as:

$$D^{\alpha}(x(t)) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)} x(t-jh) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{m} w_j^{(\alpha)} x_{m-j}$$

- Where, $w_j^{(\alpha)} = (-1)^j \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)}$
- So, in the Backward Difference Method, we approximate a fractional derivative by removing the limit in Grunwald-Letnikov's definition.
- $D_t^{\alpha}\big(x(t)\big) \approx h^{-\alpha} \Delta_t^{\alpha}\big(x(t)\big) = h^{-\alpha} \sum_{j=0}^{m} w_j^{(\alpha)} x_{m-j}$
- Now, the fractional differential equation to be solved is given as:
- $m\ddot{x} + cD_t^{\alpha} x(t) + kx = f(t), \qquad (t > 0)$
- With initial conditions: - $x(0) = a, \quad \dot{x}(0) = b$

Backward difference method (BDM) to solve the fractional differential equation (FDE).

- Here, $f(t)$ is zero for free vibration case.
- Now by using BDM definition for the fractional derivative in our fractional differential equation and using by using backward finite difference formula for the integer-order derivative we get,
- $mh^{-2}(x_m - 2x_{m-1} + x_{m-2}) + ch^{-\alpha}\sum_{j=0}^{m} w_j^{(\alpha)} x_{m-j} + kx_m = f_m$
- After simplifying and solving for x_m, following algorithm is

  obtained by using BDM approximation.
- $x_0 = a, \quad x_1 = bh + x_0,$
- $x_m = \dfrac{h^2(f_m - kx_{m-1}) + m(2x_{m-1} - x_{m-2}) - ch^{2-\alpha}\sum_{j=1}^{m} w_j^{(\alpha)} x_{m-j}}{m + ch^{2-\alpha}}$

Free vibration response computed through Backward Difference Method.

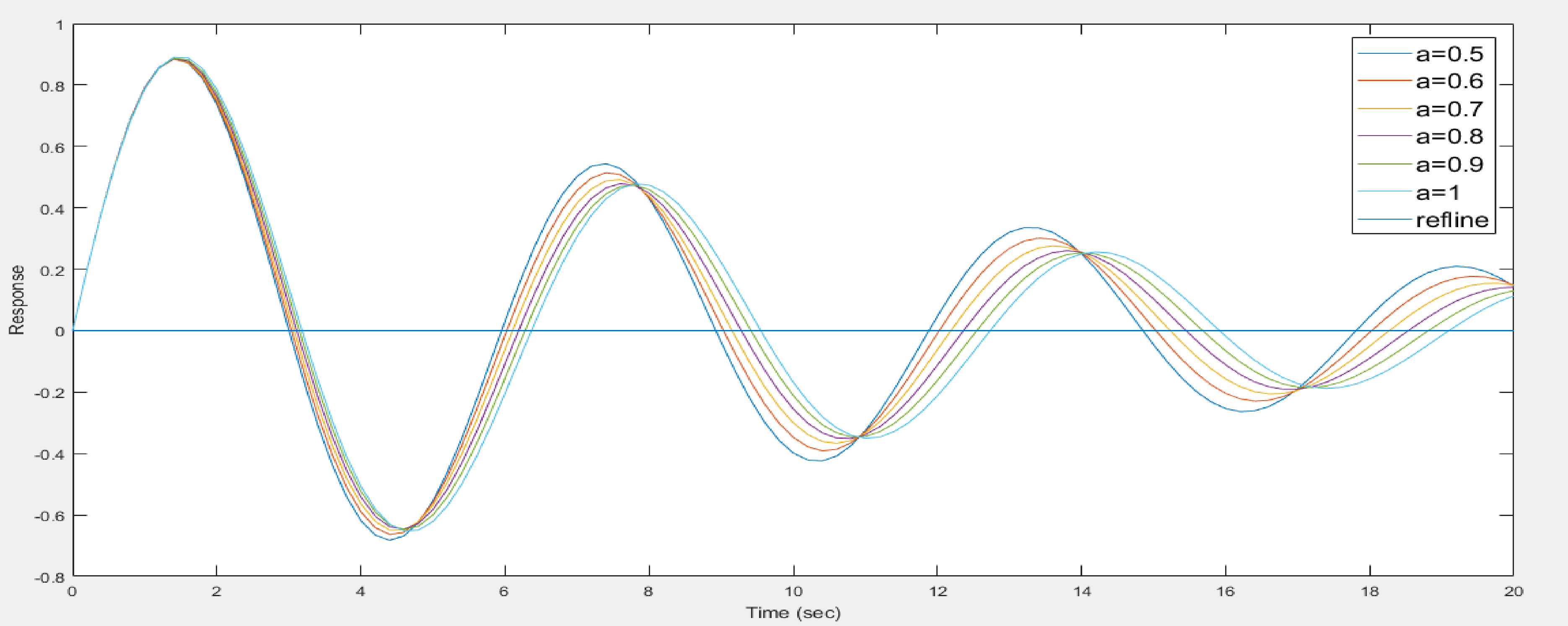


Fig. 2: Free vibration response vs. time

- Following observation are made from the graphs:
  - Amplitude decreases as alpha increases.
  - Decay is faster as alpha increases.

Response analysis (Free Vibration)

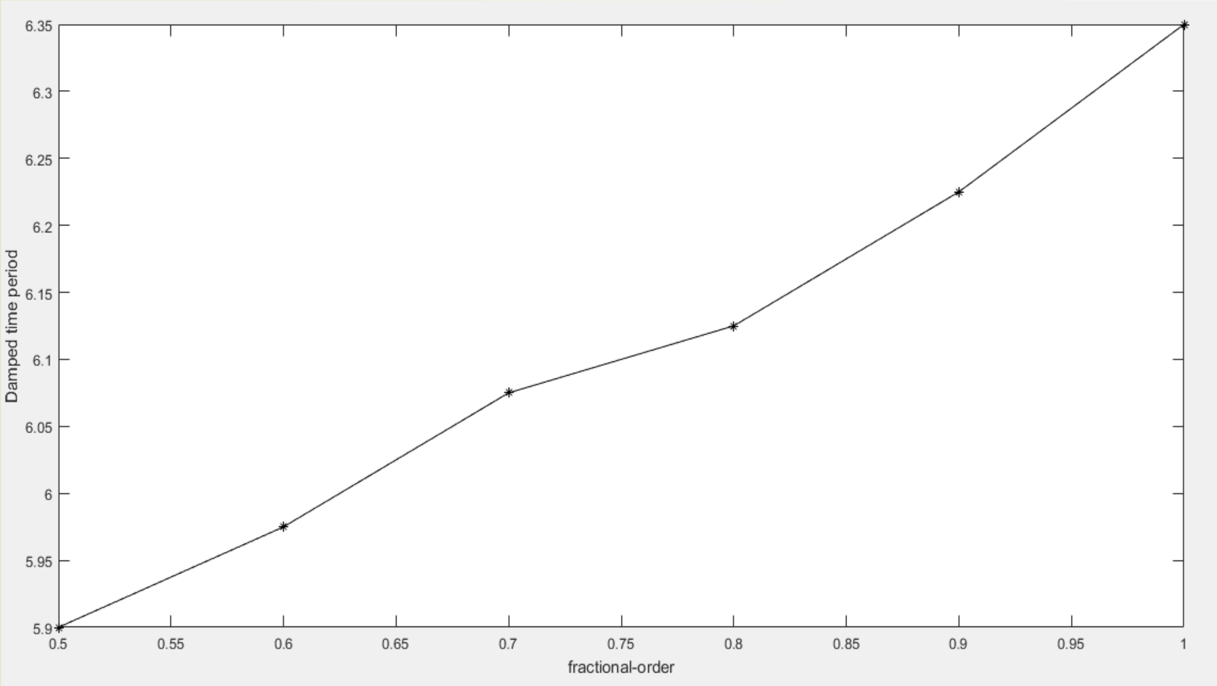


Fig. 3: Damped time period vs. fractional-order

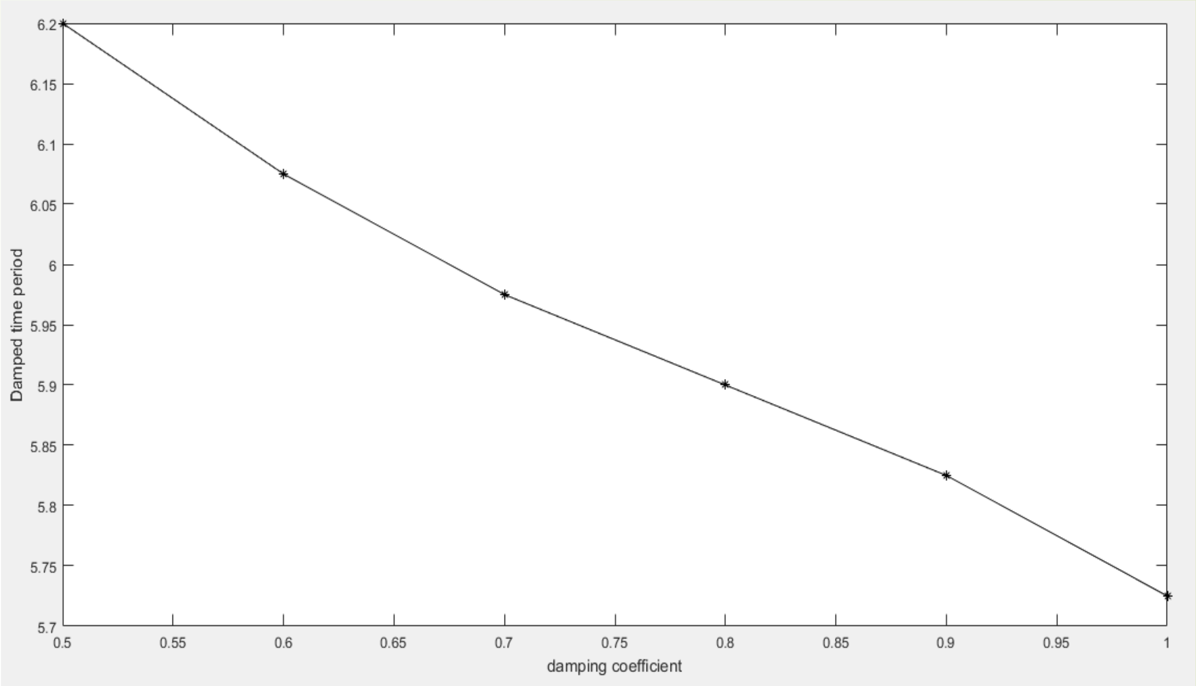


Fig. 4: Damped time period vs. damping coefficient

- From Fig. 3, we can say that the damped time period remains more or less constant for constant alpha and increases as alpha increases. $\left(F_{DM} = c\frac{d^{\alpha}x}{dt^{\alpha}}\right)$.
- From Fig. 4, we can say that the damped time period decreases as the damping coefficient increases which is just opposite to the behavior in case of viscous damping. $\left(F_{DM} = c\frac{d^{\alpha}x}{dt^{\alpha}}\right)$.

Response analysis (forced vibration)

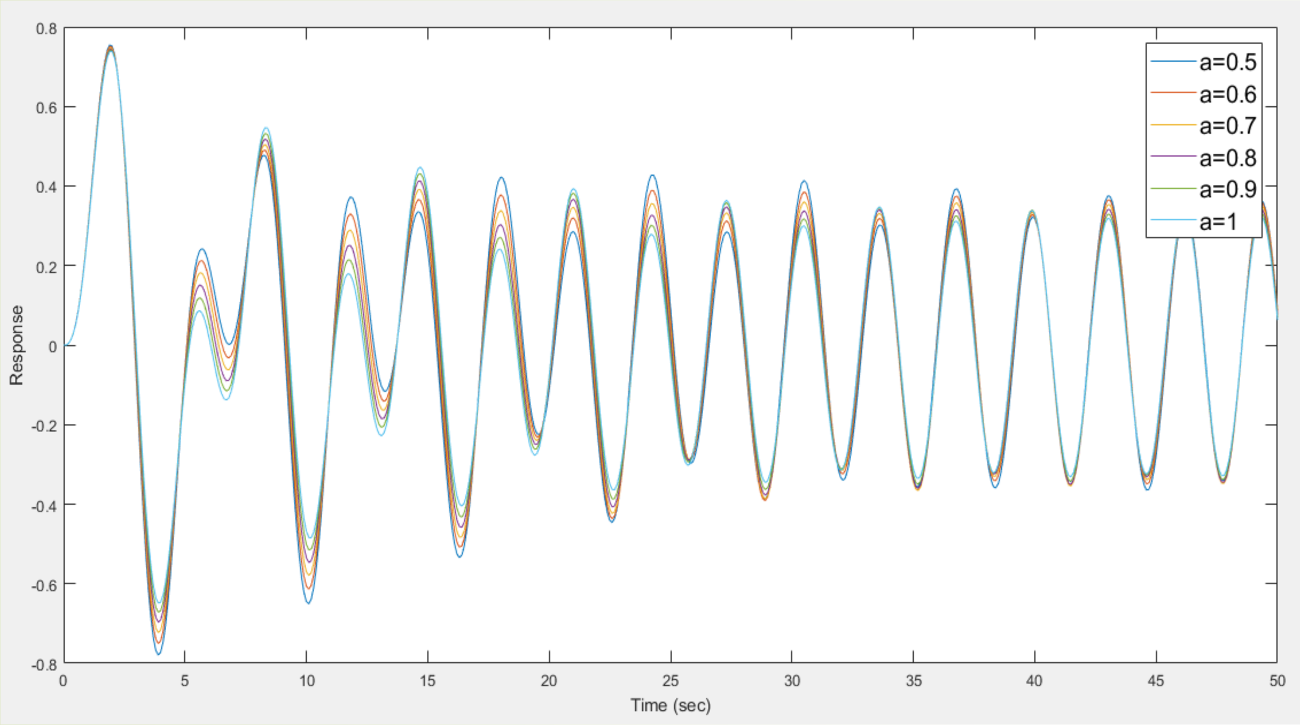


Fig. 5: Forced vibration response vs. time

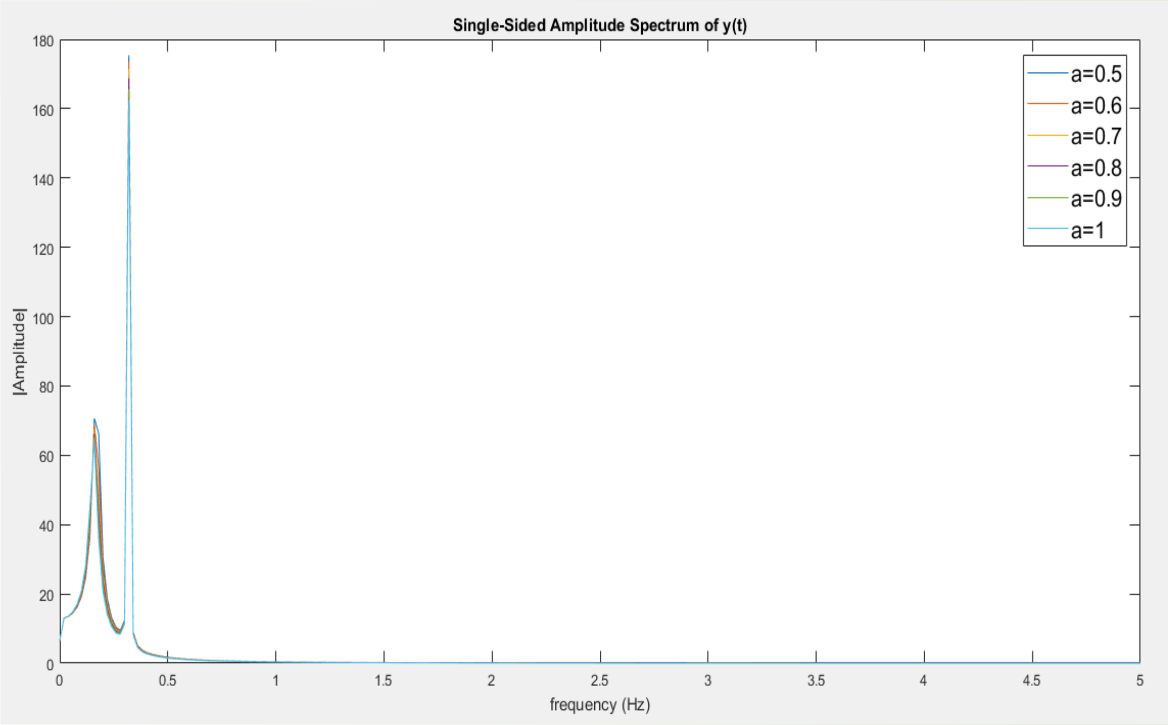


Fig. 6: FFT of forced vibration response

- From Fig. 5, we can say that the response follows forcing function after the initial transition period.
- We can also see that the amplitude of vibration decreases as fractional order value increases.
- From the FFT of the forced vibration response, we see that there are two frequency components. In which one frequency component is forcing function frequency and another frequency component is natural frequency.

Estimation of fractional order coefficient (α) and damping coefficient (c) from forced vibration response.

- $C_{eq} = c\omega^{\alpha-1}\sin(\alpha\pi/2)$
- $\sin(^{\alpha\pi}/_{2}) = \frac{DAF_{r=1}}{DAF_{max}} = \frac{X_{r=1}}{X_{max}}$, $\xi = \frac{\omega_n^{1-\alpha}}{DAF_{r=1}}$
- Verification of the relationship using the numerically obtained solution.

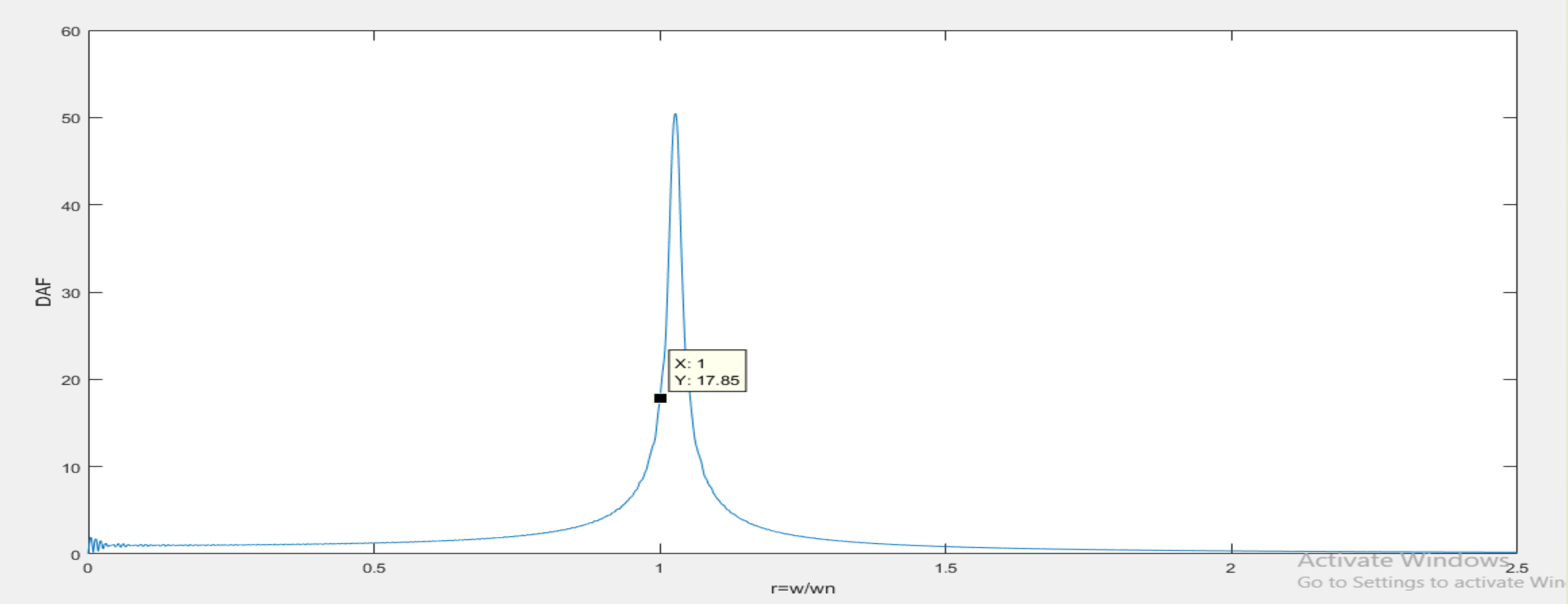


Fig. 7: Dynamic Amplification Factor (DAF) vs. Frequency Ratio (r)

- DAF at r=1 and maximum DAF is noted for each fractional order ‘α’ from the above graph and is given as below.

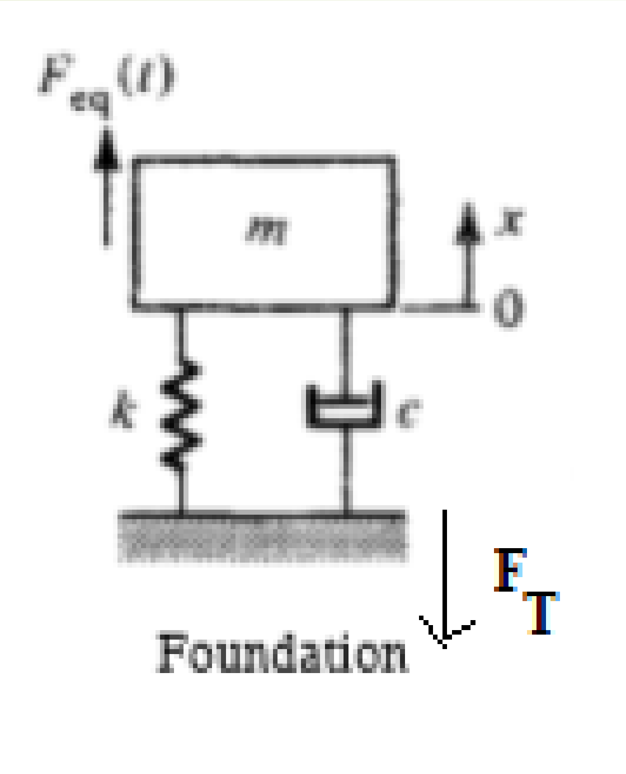
Results - Response analysis (forced vibration)

‘α’	DAF at r=1	Max DAF	‘α’ calculated
0.2	17.8500	50.4700	0.23
0.3	16.3317	34.0549	0.318
0.4	15.2202	24.9213	0.418
0.5	14.2244	19.5376	0.519
0.6	13.2928	16.0059	0.623
0.7	12.4173	13.6566	0.726
0.8	11.5962	11.9099	0.853
0.9	10.8274	10.8737	0.941
1.0	10.1078	10.1414	0.982

- Then ‘α’ is calculated from derived formula and verified with ‘α’ assumed.
- As the table shows, ‘α’ calculated and ‘α’ assumed values are matching, which verifies the derived formula.

Transmissibility for fractional damping.

- $Transmissibility = \frac{|F_T|}{F_0} = \frac{\sqrt{1+4\alpha\xi\omega_n^{\alpha-1}+4\xi^2\omega_n^{2(\alpha-1)}r^{2\alpha}}}{\sqrt{(1-r^2)^2+4(1-r^2)\alpha\xi\omega_n^{\alpha-1}r^\alpha+4\xi^2\omega_n^{2(\alpha-1)}r^{2\alpha}}}$



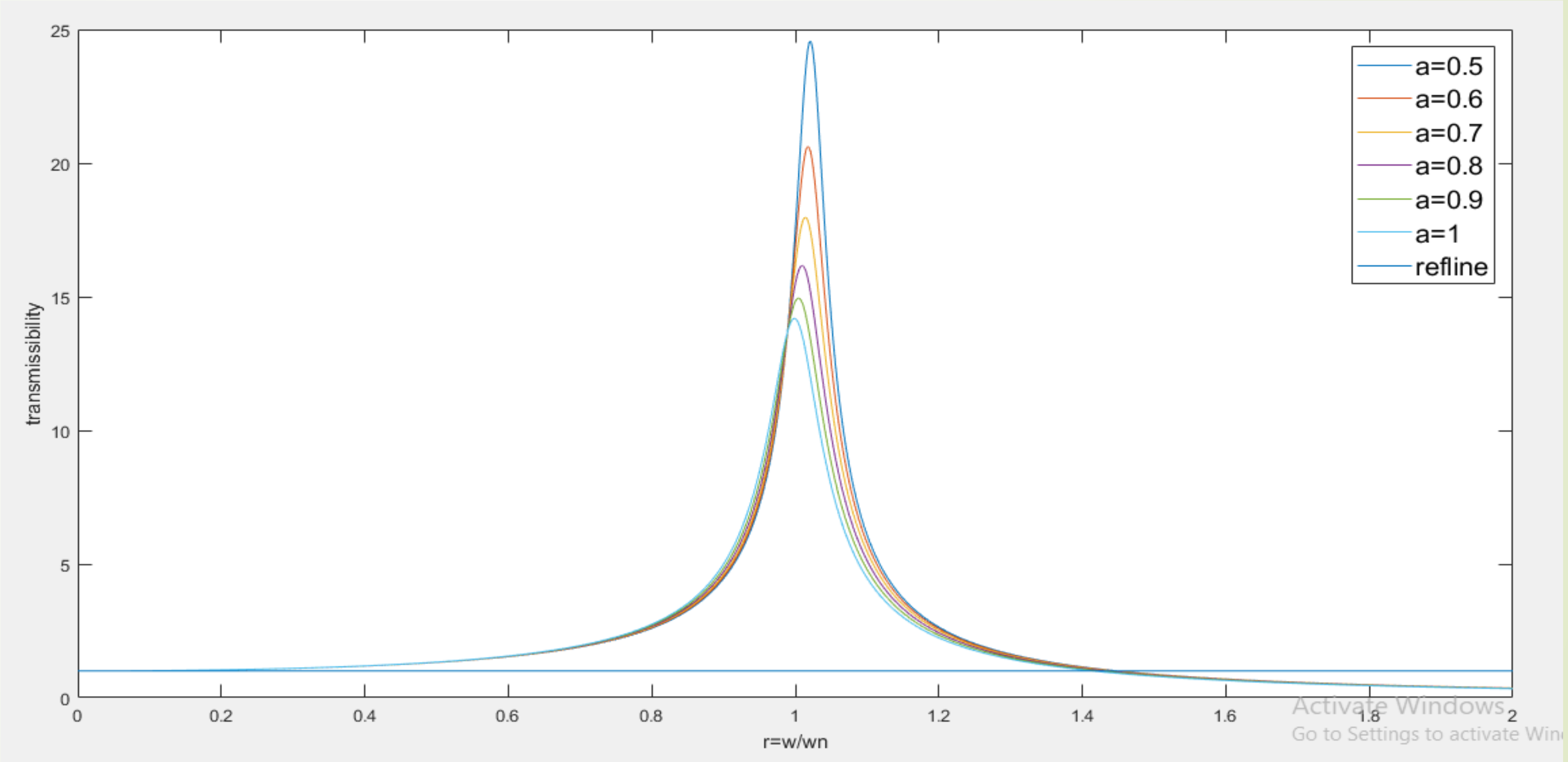


Fig. 8: Dynamic System

Fig. 9: Transmissibility Plot

- Fig. 9 shows that as fractional order is increasing force transmitted to the foundation is decreasing. Also, as damping is increasing the force transmitted to the foundation is decreasing.
- Unlike the viscous damping, the curves do not intersect at $r = \sqrt{2}$.

Fractional dynamic vibration absorber

- Fractional damping is considered for the dynamic vibration absorber and damping is considered zero for the main dynamic system as shown in the below figure.

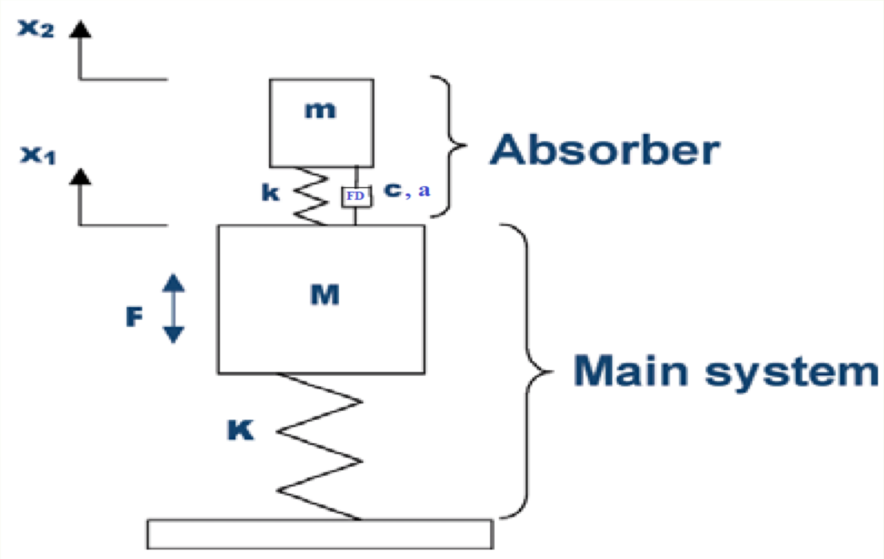


Fig. 10: Dynamic System with Fractional Dynamic Vibration Absorber.

$$\frac{X_1}{X_{st}}=\sqrt{\frac{\left(f^2-g^2\right)^2+4A\xi g^{\alpha}\Omega_n^{\alpha-1}\left(f^2-g^2\right)+\left(2\xi g^{\alpha}\Omega_n^{\alpha-1}\right)^2}{\left\{\left(1-g^2\right)\left(f^2-g^2\right)-\mu f^2 g^2\right\}^2+\left\{2\xi g^{\alpha}\Omega_n^{\alpha-1}\left(1-g^2-\mu g^2\right)\right\}^2+4A\xi g^{\alpha}\Omega_n^{\alpha-1}\left(1-g^2-\mu g^2\right)\left(\left(1-g^2\right)\left(f^2-g^2\right)-\mu f^2 g^2\right)}}$$

- This is the amplitude ratio X_1/X_{st} of the main mass as a function of the six essential variables μ, $c/c_c=\xi$, f, g, Ω_n, and α.

Fractional dynamic vibration absorber

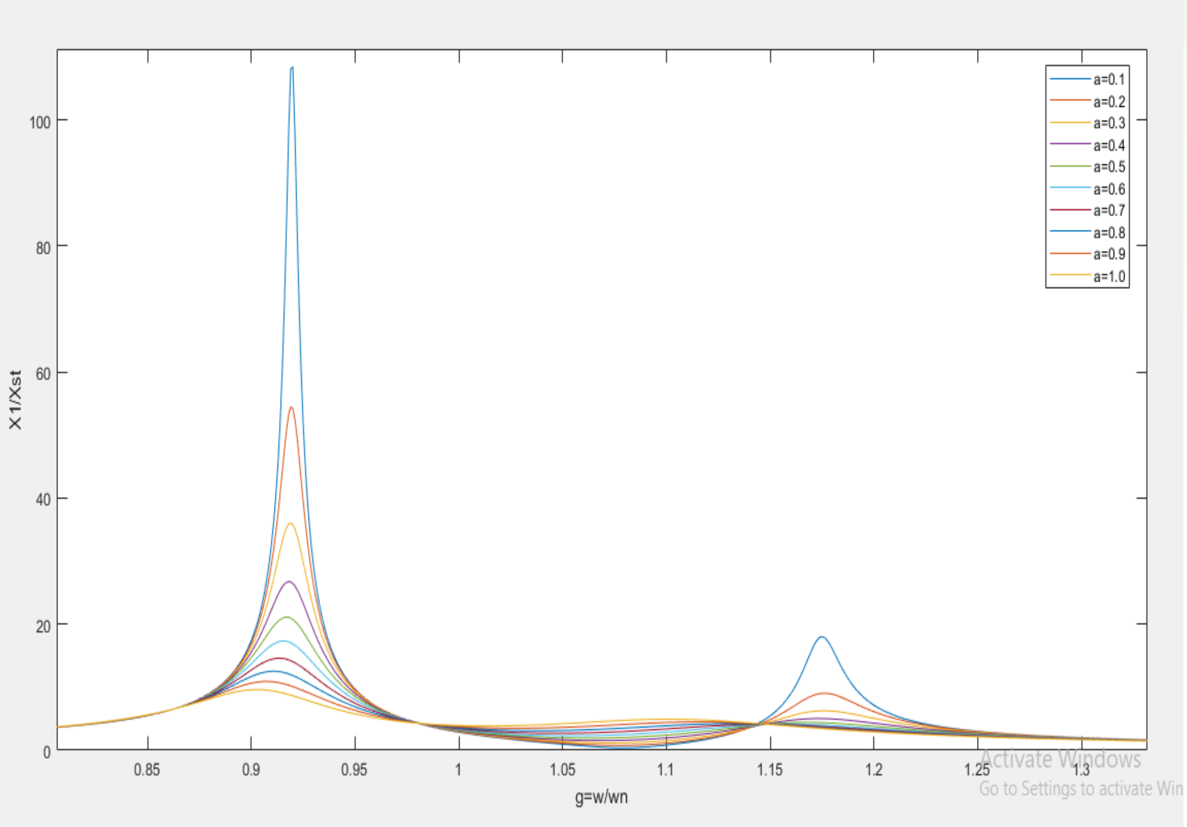


Fig. 6.2. Amplitudes of the main mass for various values of fractional-order from 0.1 to 1.

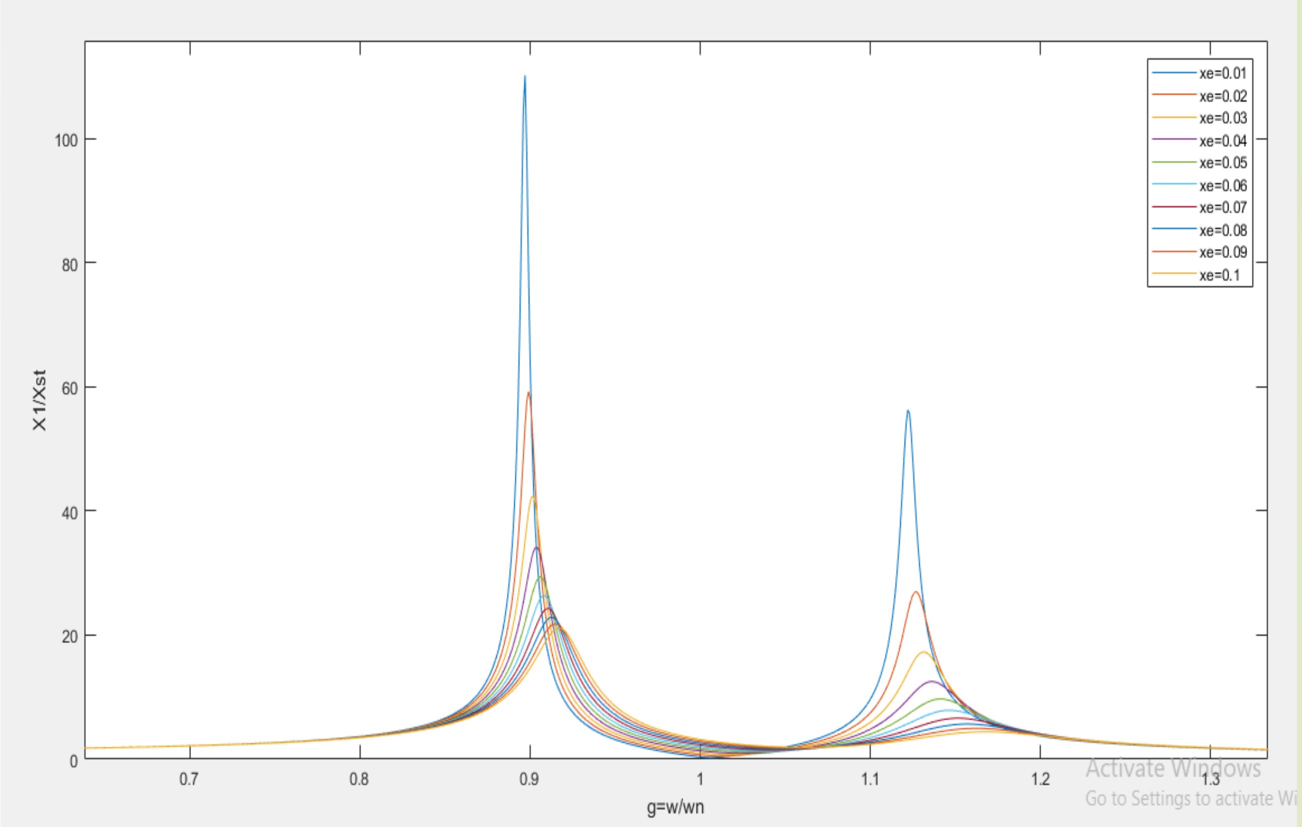


Fig. 12: Amplitudes of the main mass for various values of absorber damping from 0.01 to 0.1.

- Fig. 11 shows that as fractional order is increasing amplitude of vibration of the main mass is decreasing.
- Fig. 12 shows that as damping is increasing the amplitude of vibration of the main mass is decreasing.
- Unlike the viscous damping, the curves do not intersect at two fixed points.

Conclusion

- Fractional derivative damping follows power-law decay.
- The damped time period decreases as damping increases. There is no critical damping.
- By knowing the forced vibration response of the fractional damped system we can find the natural frequency of the system by taking FFT of the response.
- By using the dynamic amplification factor of a given dynamic system with fractional derivative damping we can find both fractional-order and damping coefficient.
- Also, it is seen that for both transmissibility and dynamic vibration absorber, viscous damping is better than fractional derivative damping for dynamic vibration absorber.

References

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THANK YOU