Damping Modeling in Dynamic Systems by using Fractional Derivative

Project report submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirements for the award of the degree

Bachelor of Technology
In
Mechanical Engineering

by

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2019-2020

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Declaration

I, Rajshekhar Lagshetti, hereby declare that this project work titled "Damping Modeling in Dynamic Systems by using Fractional Derivative" is carried out by me in the Department of Mechanical Engineering of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institution / University.

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Certificate

This is to certify that the project titled "Damping Modeling in Dynamic Systems by using Fractional Derivative", submitted by **Rajshekhar Lagshetti** in partial fulfillment of the requirements for the award of the degree of **Bachelor of Technology in Mechanical Engineering**, VNIT Nagpur. The work is comprehensive, complete, and fit for final evaluation.

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Acknowledgment

I wish to express my deep sense of gratitude and indebtedness to Prof. A. Chatterjee, for introducing the present topic and for his inspiring guidance, constructive criticism and valuable suggestion throughout this project work.

I also extend my sincere thanks to all my friends who have patiently helped us in accomplishing this undertaking. I also thank our HOD Prof. V. R. Kalamkar for his constant support throughout the project work.

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ABSTRACT

Damping is the most unknown thing in any dynamic system but is the most important thing to know. This project is a step forward to know damping in a much better way. Already many researchers have worked extensively in the field of vibration damping. Different damping models are in place to describe the damping but all have some limitations.

The fractional derivative damping model is first applied to study viscoelastic damping. In this project, the fractional derivative damping model is applied to damping in any dynamic system. Fractional-order derivative is just a generalization to the integer-order derivative. The integer order derivative does not consider the history of the system. Fractional derivative considers the history of the system.

In the fundamental equation of motion of the dynamic system, the viscous damping term (assumed proportional to the first-order derivative of displacement) is replaced by fractional-order derivative term. The resulting equation is solved in MATLAB and the solution is studied.

Results show that there is power-law decay of the free vibration response of the system. The damped time period decreases as damping increases. There is no critical damping. Two frequency components are seen in forced vibration response in which one frequency component is the natural frequency of the system. These results are just opposite to the behavior in the case of ordinary viscous damping.

Analytically power-law decay characteristic is identified for free vibration response. The equivalent damping coefficient is found out for the fractional-order system. Mathematical relation is derived for finding fractional order of the system if forced vibration response is known. Transmissibility and fractional dynamic vibration absorber are also studied for the fractional damped system.

LIST OF FIGURES

Figure	Title	Page No.
1.1	Dynamic System	4
3.1	Free vibration response for viscous damping.	14
3.2	Graph of free vibration response $y(t)$ vs. time t for different values of α from 0.5 to 1.	17
3.3	Graph of free vibration response $y(t)$ vs. time t for different values of c from 0.5 to 1.	19
3.4	Graph of free vibration response $y(t)$ vs. time t for large values of c=1, 10, 100.	20
3.5	Graph of free vibration response $y(t)$ vs. time t showing agreement between BDM and Laplace Method.	21
4.1	Graph of forced vibration response $y(t)$ vs. time t for different values of α from 0.5 to 1.	27
4.2	Fast Fourier Transform of forced vibration response $y(t)$.	28
4.3	The graph between Dynamic Amplification Factor (DAF) vs. Frequency Ratio (r)	33
5.1	Transmissibility plot for different values of fractional order from 0.5 to 1	38
5.2	Transmissibility plot for different values of the damping coefficient from 0.1 to 0.5	38
6.1	Dynamic System with Fractional Dynamic Vibration Absorber.	41
6.2	Amplitudes of the main mass for various values of fractional-order from 0.1 to 1.	45
6.3	Amplitudes of the main mass for various values of absorber damping from 0.01 to 0.1.	45

LIST OF TABLES

Table	Title	Page No.
3.1	The damped time period for different α from 0.5 to 1.	18
3.2	The damped time period for different c from 0.5 to 1.	19
4.1	Values of α assumed and α calculated from the derived relationship.	33

NOMENCLATURE

1. t……………………………………………Time.
2. $x(t), y(t)$………………………………...Displacement.
3. α……………………………………………Fractional-order of the derivative.
4. D^{α}…………………………………………..The fractional derivative of fractional-order α.
5. $D^{-\alpha}$………………………………………The fractional integral of fractional-order of α.
6. Γ……………………………………………Gamma Function.
7. m.………………......................................Mass.
8. c……………………………………………Damping Coefficient.
9. k……………………………………………Stiffness.
10. ω ………………………………………..Forcing function frequency.
11. ω_n………………………………………Natural frequency.
12. F(t)……………………………………...External forcing function.
13. ${}_{o}^{RL}D_{t}^{\alpha}$ ……………………………………Reimann-Liouville fractional derivative.
14. ${}^{C/GL}D_0{}^{\alpha}$ ………………………………...Caputo/Grunwald-Letinkov fractional derivative.
15. X(s)…………......................................Laplace Transform of x(t).
16. F_T………………………………………Force Transmitted.
17. F_0………………………………………The amplitude of the forcing function.
18. M………………………………………..Main mass.
19. K……………….....................................Main-spring stiffness.
20. μ = m/M…………………………………..Mass ratio = absorber mass/main mass.
21. ω_a^2 = k/m.………....................................The natural frequency of absorber.
22. Ω_n^2 = K/M………………………………The natural frequency of the main system.
23. f = ω_a/Ω_n………………………………Frequency ratio (natural frequencies).
24. g = ω/Ω_n………………………………Forced frequency ratio.
25. $X_{st} = F_0/K$……………………………Static deflection of the system.
26. c_c = 2mΩ_n………………………………Critical damping.
27. L………………………………………..Laplace Transform.
28. E_d………………………………………Energy dissipated

29. c_{eq}……………………………………….Equivalent damping coefficient

30. FOC…………………………………….Fractional order calculus.

31. IOC……………………………………..Integer order calculus.

32. BDM…………………………………….Backward difference method.

33. FDE…………………………………….Fractional differential equation.

34. FFT……………………………………..Fast Fourier transform.

35. DAF_{max}………………………………….Maximum dynamic amplification factor.

36. $DAF_{r=1}$…………………………………..Dynamic amplification factor at r=1.

37. GL………………………………………Grunwald-Letinkov.

38. RL………………………………………Reimann-Liouville.

INDEX

CHAPTER	TITLE	PAGE NO.
	DECLARATION	
	CERTIFICATE	
	ACKNOWLEDGMENT	
	ABSTRACT	I
	LIST OF FIGURES	II
	LIST OF TABLES	III
	NOMENCLATURE	IV-V
1	**INTRODUCTION AND OUTLINE OF THE PROJECT** 1.1 Motivation (why fractional derivative damper model?) 1.2 Problem definition And Objectives 1.2.1 Problem definition 1.3 Outline of the project	1-5
2	**LITERATURE SURVEY**	6-11
3	**RESPONSE ANALYSIS (FREE VIBRATION)**	12-23
	3.1 Backward difference method (BDM) to solve the fractional differential equation (FDE).	
	3.2. MATLAB code to solve the fractional differential equation (FDE) using the Backward difference method (BDM) for free vibration.	
	3.3. MATLAB code to solve the FDE using Laplace transform method for free vibration.	
	3.4. Power-law decay derivation for free vibration.	
4	**RESPONSE ANALYSIS (FORCED VIBRATION)**	24-34
	4.1. MATLAB code for Forced Vibration Response using Backward Difference Method (BDM).	

	4.2. Relationship to estimate the fractional-order and damping coefficient using a forced vibration response.	
	4.3. Verification of the relationship using the numerically obtained solution.	
5	**TRANSMISSIBILITY FOR FRACTIONAL DAMPING (FRACTIONALLY DAMPED VIBRATION ISOLATOR)**	35-39
	5.1. Force Transmissibility derivation for Fractionally Damped Vibration Isolator.	
	5.2. MATLAB Code to plot transmissibility graph.	
6	**FRACTIONAL TUNED MASS DAMPER (FRACTIONAL DYNAMIC VIBRATION ABSORBER)**	40-46
	6.1. Derivation for finding the amplitude of the main mass (X_1)	
	6.2. MATLAB Code to plot the amplitude of the main mass (X_1).	
7	**CONCLUSION AND FUTURE SCOPE**	47-48
	REFERENCES	49-50

CHAPTER 1

INTRODUCTION AND OUTLINE OF THE PROJECT

Damping is always present in the motion of structures. Therefore, to fully model the dynamic behavior of a structure, its damping properties have to be taken into account as well as its stiffness and mass. There are several ways to model damping, each having its advantages and disadvantages. The fractional derivative model of damping is considered to study dynamic systems.

Fractional order calculus (FOC) is three centuries old as the conventional calculus, but not very popular among science and/or engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity) (Das, 2011).

The applications of FOC in engineering were delayed because FOC has multiple definitions, there is not a simple geometrical interpretation and the Integer order calculus (IOC) seems, at first sight, to be enough to solve engineering problems. However, many natural phenomena may be better described by a FOC formulation, because it takes into account the past behavior and it is compact when expressing high-order dynamics. Some common definitions of FOC are listed as follows (Gutierrez et al., 2010):

1. Riemann-Liouville:

Integral:

$$D_a^{-\alpha}\left(f(t)\right)=\frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau \quad \text{.........}(1.1)$$

Derivative:

$$D^{\alpha}(f(t))=\frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau\right], m\in \mathrm{Z}^+, m-1<\alpha\le m \quad \text{.........}(1.2)$$

2. Caputo:

$$D^{\alpha}(f(t))=\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau \text{.........}(1.3)$$

3. Grunwald-Letnikov:

Integral:

$$D^{-\alpha}(f(t)) = \lim_{h\to 0} h^{\alpha} \sum_{m=0}^{\left\lceil \frac{t-a}{h} \right\rceil} \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)} f(t-mh) \quad \ldots\ldots(1.4)$$

Derivative:

$$D^{\alpha}(f(t)) = \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{\left\lceil \frac{t-a}{h} \right\rceil} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha+1-m)} f(t-mh) \quad \ldots\ldots(1.5)$$

The simplest model of a single degree of freedom with fractional derivative damping corresponds to the following equation for the displacement $x(t)$,

$$mD^2x(t) + cD^{\alpha}x(t) + kx(t) = F(t) \quad \ldots\ldots(1.6)$$

where m is mass, k is spring constant, c is the damping coefficient, and $F(t)$ is the external force applied to the system. D denotes the ordinary time derivative d/dt, whereas $D^{\alpha}$is the fractional time derivative of order α, whose definition is given above.

1.1 Motivation (why fractional derivative damper model?)

There are several ways to model damping, some of them are as follows (Tse et al., 1978):

1. Viscous Damping:

$$\mathrm{F_{DM}} = \mathrm{c\dot{x}} \quad \ldots\ldots(1.7)$$

where $F_{DM}$is the damping force.

This means that the damping is proportional to frequency, which results in very high damping at high frequencies.

2. Coulomb Damping:

$$F_{DM} = \mu N \,\mathrm{sgn}(\dot{x}) \quad \ldots\ldots(1.8)$$

3. Quadratic Damping:

$$F_{DM} = c\dot{x}^2 \,\mathrm{sgn}(\dot{x}) \quad \ldots\ldots(1.9)$$

4. Hysteretic Damping:

$$\mathrm{F_{DM}} = \mathrm{jhx} \quad \ldots\ldots(1.10)$$

In this model the ordinary modulus of elasticity is replaced by a complex-valued modulus, implying that the damping becomes independent of frequency. This model works well for harmonic vibration, but it leads to noncausal responses to transient loads.

Reasons to use fractional derivative model:

- All the above-mentioned damping models have limited applications.
- By using fractional derivative in stress-strain relation, it has been observed that only five parameters often suffice in the stress-strain relationship to get a good fit to experimental data (Bagley & Torvik, 1983).
- The fractional derivative model results in moderate damping at high frequencies and leads to causal responses to transient loads (Bagley & Torvik, 1987).
- In 1936, Gemant observed that the harmonic motion of viscoelastic bodies behaves like frequency raised to fractional powers and suggested the use of fractional derivatives.

1.2 Problem definition and Objectives

The dynamic system with fractional damping is modeled as given below:

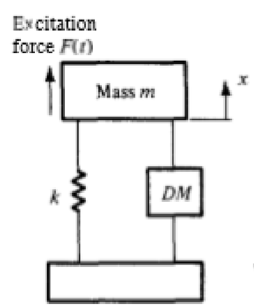

Fig. 1.1: Dynamic System

Here DM represents Damping Mechanism representing Fractional Damping. Therefore, Damping force is given by

$$F_{DM} = c\frac{d^{\alpha}(x(t))}{dt^{\alpha}} = cD^{\alpha}(x(t)) \quad \text{...(1.11)}$$

Here, F_{DM} is Damping Force, 'c' is damping coefficient and 'α' is fractional order of the derivative and $\alpha \in [0,1]$.

Therefore, the equation of motion for the above system is given as:

$$m\ddot{x} + cD^{\alpha}(x(t)) + kx = F(t) \quad \text{...(1.12)}$$

Where,

‘m’ is mass,

‘k’ is stiffness,

‘x(t)’ is displacement and

‘F(t)’ is the external forcing function.

1.2.1 Problem definition:

- To solve the fractional differential equation.

$$mD^2x(t) + cD^\alpha x(t) + kx(t) = F(t) \ldots\ldots\ldots\ldots(1.13)$$

- Write a MATLAB code for solving the equation.
- Comparison with the experimental data.
- Analysis and detection of the results.

1.3 Outline of the project:

The book is organized into 7 chapters. Chapter 1 is the basic introduction, dealing with the development of the fractional calculus. Several definitions of fractional differintegrations and the most popular ones are introduced here. Chapter 2 is a literature survey. Chapter 3 is a response analysis for free vibration, in which first, free vibration response for viscous damping is studied. Then free vibration response for fractional derivative damping is studied. The backward difference method is used to solve the fractional differential equation and with the help of MATLAB code, free vibration response is studied. Also, the Laplace transform method is used as given the literature to verify the results of the backward difference method. Then power-law decay characteristics are identified and derived. Chapter 4 is a response analysis for forced vibration, in which again the backward difference method is used to find the forced vibration response. Then by using a forced vibration response, the relationship to estimate the fractional-order and damping coefficient is derived and verified using the backward difference method. Chapter 5 is the study of transmissibility for fractional derivative damping. Chapter 6 is the study of fractional dynamic vibration absorber. Chapter 7 gives the conclusion and future scope of the project.

CHAPTER 2

LITERATURE SURVEY

1. Sakakibara, S. (2004). Fractional Derivative Models of Damped Oscillations. The Institute for Mathematical Sciences, Research Volume 1385 2004 65-76.

- The fractional differential equation for damped oscillation is solved using Laplace Transform for any arbitrary order of fractional derivative.
- They showed that there is power-law decay as compared with exponential decay of the usual viscous damping.

2. Sakakibara, S. (1997). Properties of Vibration with Fractional Derivative Damping of Order 1/2. JSME International Journal.

- Solution found out using Laplace Transform is studied for various damping ratio values.
- It is observed that there is no critical value of the damping coefficient that distinguishes the pattern of damping.
- It is observed that time period and amplitude decrease as damping increases.
- It is observed that the peak of resonance shifts towards higher frequencies as damping increases.

3. Rekhviashvili, S., Pskhu, A., Agarwal, P., & Jain, S. (2019). Application of the fractional oscillator model to describe damped vibrations. Turkish Journal of Physics.

- The relation between the order of fractional differentiation in the equation of motion and Q-factor of an oscillator is suggested.

4. Podlubny, I. (1998). Fractional Differential Equations. Mathematics in Science and Engineering, Volume 198. Pg. No. 223-235.

- Backward Difference method (BDM) is given to solve an FDE.
- BDM is based on Grunwald-Letnikov's definition of a fractional derivative.

5. Liu, L., & Duan, J. (2015). A detailed analysis for the fundamental solution of fractional vibration equation. Open Mathematics.

- The solution of the fractional vibration equation is investigated, where the damping term is characterized by the Caputo fractional derivative.
- It is shown that the solution is ultimately positive, and ultimately decreases monotonically and approaches zero.

6. Garrappa, R., Kaslik, E., & Popolizio, M. (2019). Evaluation of Fractional Integrals and Derivatives of Elementary Functions: Overview and Tutorial. Mathematics Journal.

- A guide is provided to the evaluation of fractional integrals and derivatives of some elementary functions and studied the action of different derivatives on the same function.
- They observed how Riemann–Liouville and Caputo's derivatives converge, on long times, to the Grünwald–Letnikov derivative which appears as an ideal generalization of standard integer-order derivatives although not always useful for practical applications.

7. Gómez-Aguilar, J., Yépez-Martínez, H., Calderón-Ramón, C., Cruz-Orduña, I., Escobar-Jiménez, R., & Olivares-Peregrino, V. (2015). Modeling of a Mass-Spring-Damper System by Fractional Derivatives with and without a Singular Kernel. Entropy.

- The fractional equations of the mass-spring-damper system with Caputo and Caputo–Fabrizio derivatives are presented.
- An analytical solution is obtained, and the resulting equations are given in terms of the Mittag–Leffler function.

8. Torvik, P., & Bagley, D. (1987). Fractional derivatives in the description of damping: materials and phenomena. ASME, Design Engineering Division.

- They used a constitutive equation that involves generalized or fractional derivatives for the description and prediction of the time-dependent behavior of materials of interest for damping applications is described.

- Such relationships that have origins in observed tendencies towards power-law behavior rather than exponential response are shown to be effective descriptors of the dynamic behavior of real materials.

9. Fenander, A. (1996). Modal Synthesis When Modeling Damping by Use of Fractional Derivatives. AIAA journal.

- The fractional derivative damping model and some of its properties are discussed.
- Here modal synthesis is used to solve the equations of motion in the time domain.

10. Das, S. (2011). Functional Fractional Calculus for System Identification and Controls. Springer-Verlag Berlin Heidelberg.

- The basic introduction, dealing with the development of the fractional calculus is given. Several definitions of fractional differintegrations and the most popular ones are introduced here; the chapter gives the feel of fractional differentiation of some functions, i.e., how they look.
- It gives an overview of the application of fractional calculus.

11. Gutiérrez, R., Rosário, J., & Machado, J. (2010). Fractional Order Calculus: Basic Concepts and Engineering Applications. Mathematical Problems in Engineering.

- This paper introduces the fundamentals of the FOC and some applications in systems identification, control, mechatronics, and robotics, where it is a promissory research field.

12. Torvik, P., & Bagley, D. (1983). Fractional Calculus – A Different Approach to the Analysis of Viscoelastically Damped Structures. AIAA Journal.

- Fractional calculus is used to construct stress-strain relationships for viscoelastic materials.
- These relationships are used in the finite element analysis of viscoelastically damped structures and closed-form solutions to the equations of motion are found.

- It is shown that very few empirical parameters are required to model the viscoelastic material and calculate the response of the structure for general loading conditions.

13. You, H., Shen, Y., & Yang, S. (2015). Optimal design for fractional-order active isolation system. Advances of Mechanical Engineering.

- The optimal control of a fractional-order active isolation system is researched based on the optimal control theory, and the effect of fractional-order derivative on a passive isolation system is also analyzed. The mechanical model is established where viscoelastic features of isolation materials are described by fractional-order derivative.
- It is shown that both the fractional coefficient and the fractional-order could affect not only the resonance amplitude through the equivalent linear damping coefficient but also the resonance frequency by the equivalent linear stiffness.
- The statistical responses of the displacement and velocity for passive and active vibration isolation systems subjected to random excitation are also presented, which further verifies the excellent performance of fractional-order derivative in vibration control engineering.

14. Rudinger, F. (2006). Tuned mass damper with fractional derivative damping. Engineering Structures.

- Optimal parameters for the tuned mass damper are obtained numerically by optimizing the effective damping ratio of the system.
- It is shown that the structural damping has very little influence on the optimal parameters.
- Furthermore, it is demonstrated that the effect of the damper is the same for different values of the fraction in the fractional derivative. This implies that this tuned mass damper with a fractional derivative damping element introduces the same reduction in the structural vibration as a conventional tuned mass damper if properly tuned.
- Simple approximate analytical expressions for optimal parameters are obtained by a frequency domain approach, in which the force acting between the structure and the

secondary mass is assumed to be equal to the force of a conventional tuned mass damper at resonance.

15. Barone, G., Lo Iacono, F., Navarra, G. (2014). Passive control of fractional viscoelastic structures by Fractional Tuned Mass Dampers.

- Numerical simulations on a fractional single degree of freedom structures excited by stochastic loads are presented by solving the fractional differential equations of motion of the system equipped with the damper device in the time domain and evaluating the statistics of the response by stochastic analysis in the frequency domain.
- Sensitivity analysis with respect to the mass ratio between the primary mass and the Fractional Tuned Mass Damper and with respect to the damper parameters is reported in terms of the performance index.

16. Den Hartog, J. Mechanical Vibrations. Pg. No. 93-104

- Analysis of viscous tuned mass damper is done.

CHAPTER 3

RESPONSE ANALYSIS (FREE VIBRATION)

Before solving the fractional differential equation based on fractional damping let us solve the ordinary differential equation with viscous damping.

The differential equation for free vibration based on viscous damping is given by:

$$m\ddot{x}+c\dot{x}+kx=0 \quad \text{.......................................}(3.1)$$

It is a differential equation of the second order. Its solution will be of the form

$$x=Ae^{s_1 t}+Be^{s_2 t} \quad \text{.......................................}(3.2)$$

where A and B are some constants, s_1 and s_2 are the roots of the auxiliary equation

$$s^2+\frac{c}{m}s+\frac{k}{m}=0 \quad \text{.......................................}(3.3)$$

i.e., $s_{1,2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}$ ....(3.4)

The ratio of $\left(\frac{c}{2m}\right)^2$ to $\left(\frac{k}{m}\right)$ represents the degree of dampness provided in the system and its square root is known as damping factor of damping ratio ξ, i.e.,

$$\xi=\sqrt{\frac{(c/2m)^2}{k/m}}=\frac{c}{2\sqrt{km}} \quad \text{.......................................}(3.5)$$

Thus when

$\xi=1,$ the damping is critical

$\xi>1,$ the system is overdamped

$\xi<1,$ the system is under-damped

Equation (3.1) can also be written as

$$\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2x=0 \quad \text{.......................................}(3.6)$$

and $s_{1,2}=\left(-\xi\pm\sqrt{\xi^2-1}\right)\omega_n$ ....(3.7)

The exact solution of Eq. (3.6) will depend upon whether the roots $s_{1,2}$ are real or imaginary.

1). $\xi>1,$ i.e., the system is overdamped.

The roots of the auxiliary equation are real.

Therefore, the solution is

$$x = Ae^{s_1 t} + Be^{s_2 t}$$
$$= Ae^{(-\xi+\sqrt{\xi^2-1})\omega_n t} + Be^{(-\xi-\sqrt{\xi^2-1})\omega_n t} \quad \text{.....(3.8)}$$

Constants A and B can be determined from the initial conditions. This is the equation of an aperiodic motion, i.e., the system cannot vibrate due to over-damping. The magnitude of the resultant displacement approaches zero with time.

2). $\xi < 1$, i.e., the system is under-damped.

The roots of the auxiliary equation are imaginary.

$$s_{1,2} = (-\xi \pm i\sqrt{1-\xi^2})\omega_n \quad \text{.....(3.9)}$$

$$\therefore x = Ae^{(-\xi+i\sqrt{1-\xi^2})\omega_n t} + Be^{(-\xi-i\sqrt{1-\xi^2})\omega_n t} = e^{-\xi\omega_n t}\left\{Ae^{(i\sqrt{1-\xi^2})\omega_n t} + Be^{(-i\sqrt{1-\xi^2})\omega_n t}\right\} \quad \text{.....(3.10)}$$

By putting $\sqrt{1-\xi^2}\omega_n = \omega_d$ and by using Euler's formula we get,

$$x = e^{-\xi\omega_n t}\left\{A\cos(\omega_d t) + B\sin(\omega_d t)\right\} \quad \text{.....(3.11)}$$

The resultant motion is oscillatory with decreasing amplitudes having a frequency of ω_d.

Ultimately, the motion dies down the time.

3). $\xi = 1$, i.e., damping is critical.

The roots of the auxiliary equation are equal, each being equal to $-\omega_n$ and the solution is

$$x = (A+Bt)e^{-\omega_n t} \quad \text{.....(3.12)}$$

The motion is aperiodic. The displacement will be approaching to zero with time.

Figure (3.1) shows the characteristics of motion for three different cases discussed.

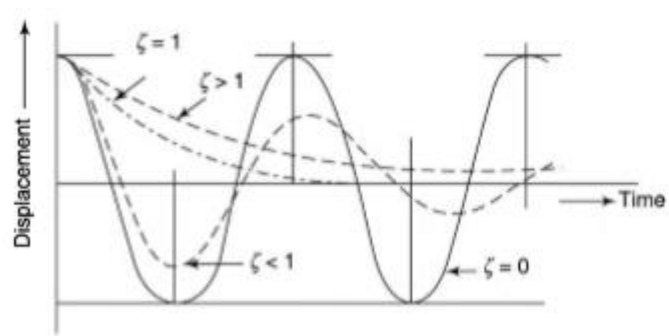

Fig. 3.1 Free vibration response for viscous damping.

3.1 Backward difference method (BDM) to solve the fractional differential equation (FDE).

BDM is based on Grunwald-Letnikov's definition of a fractional derivative.

So, Grunwald-Letnikov's definition of a fractional derivative is given as:

$$D^{\alpha}(x(t)) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left\lceil \frac{t-a}{h} \right\rceil} (-1)^{j} \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)} x(t-jh) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{m} w_j^{(\alpha)} x_{m-j} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.13)$$

Where, $w_j^{(\alpha)} = (-1)^j \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha+1-j)}$……………………………………………………..(3.14)

So, in the Backward Difference Method, we approximate a fractional derivative by removing the limit in Grunwald-Letnikov's definition. Therefore,

$$D_t^{\alpha}\big(x(t)\big) \approx h^{-\alpha}\Delta_t^{\alpha}\big(x(t)\big) = h^{-\alpha}\sum_{j=0}^{m} w_j^{(\alpha)} x_{m-j} \ldots\ldots\ldots\ldots\ldots\ldots(3.15)$$

Now, the fractional differential equation to be solved is given as:

$$m\ddot{x} + cD_t^{\alpha}x(t) + kx = f(t), \qquad (t > 0) \ldots\ldots\ldots\ldots\ldots\ldots(3.16)$$

With initial conditions: - $x(0) = a, \quad \dot{x}(0) = b$

Here, $f(t)$ is zero for free vibration case.

Now by using BDM definition for the fractional derivative in our fractional differential equation and using by using backward finite difference formula for the integer-order derivative we get,

$$mh^{-2}(x_m - 2x_{m-1} + x_{m-2}) + ch^{-\alpha}\sum_{j=0}^{m} w_j^{(\alpha)} x_{m-j} + kx_m = f_m \ldots\ldots\ldots\ldots\ldots\ldots(3.17)$$

After simplifying and solving for x_m, following algorithm is obtained by using BDM approximation.

$$x_0 = a, \quad x_1 = bh + x_0,$$
$$x_m = \frac{h^2(f_m - kx_{m-1}) + m(2x_{m-1} - x_{m-2}) - ch^{2-\alpha}\sum_{j=1}^{m} w_j^{(\alpha)} x_{m-j}}{m + ch^{2-\alpha}} \tag{3.18}$$

3.2. MATLAB code to solve the fractional differential equation (FDE) using the Backward difference method (BDM) for free vibration.

```
%% mD^2x(t)+cD^ax(t)+kx(t)=F(t)
clc
clear all
%% Values of m, c, k, alpha, t0, T
m=1;
c=0.2;
k=1;
t0=0;
T=20;
n=200;
h=(T-t0)/n;
%% Forcing function
F=zeros(n+1,1);
%% Initial Conditions
y(1)=0;
y(2)=y(1)+(1*h);
% a=1;
for a=0.5:0.1:1
    for j=1:n
        w(j)=((-1)^j)*((gamma(a+1))/

             ((gamma(j+1))*(gamma(a-j+1))));
    end
    for i=3:n+1
        B=0;
        for j=2:i
```

```
            B=B+(w(j-1)*y(i+1-j));
        end
            y(i)=(((h^2*(F(i)))+(m*(2*y(i-1)-y(i-
  2)))- (c*h^(2-a)*B))/(m+(k*(h^(2-
  a)))+k*(h^2)));
    end
    plot(t0:h:T,y);
    hold on;
    xlabel('Time (t)');
    ylabel('y(t)');
    %% Curve fitting X=At^C
    [X,t]=findpeaks(y,t0:h:T);
    % Td=diff(t);
    a1=log(X');
    a2=log(t');
    a3=length(a2);
    a4=[ones(a3,1),a2];
    phi=inv(a4'*a4)*a4'*a1;
    A(((10*a)-4))=exp(phi(1));
    C(((10*a)-4))=phi(2);
end
```

Results of MATLAB code 1: -

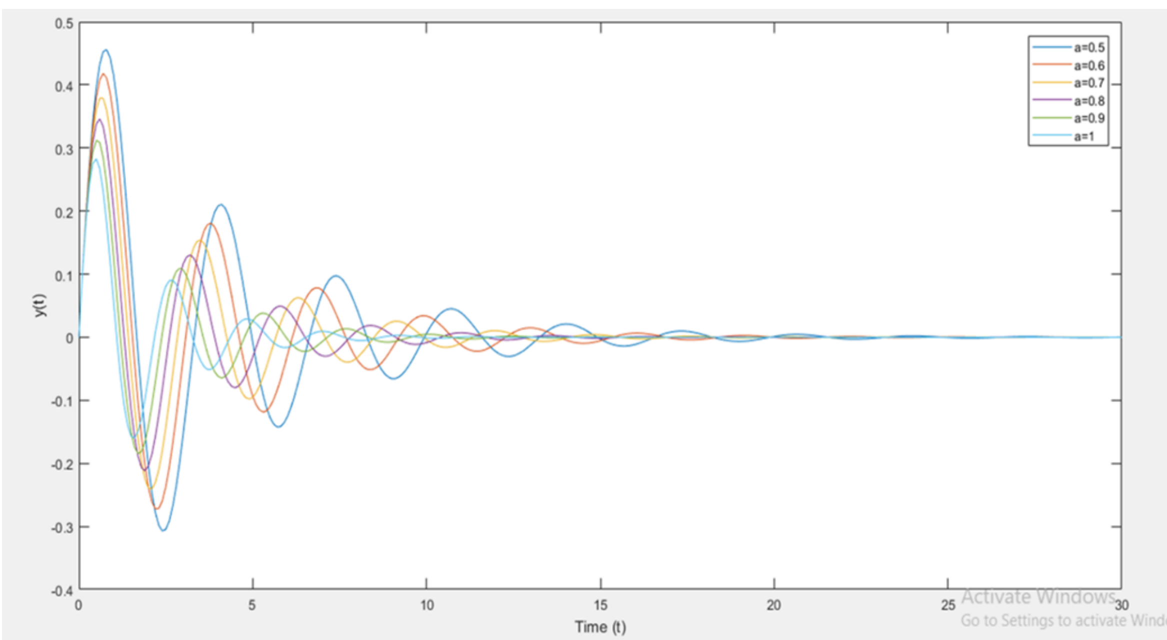

Fig. 3.2: Graph of free vibration response $y(t)$ vs. time t for different values of α from 0.5 to 1.

Following observation are made from the graph:

- Amplitude decreases as alpha increases.
- Decay is faster as alpha increases.

By measuring the peak to peak distance in the above graph we get the following damped time period values for each value of fractional order α.

Table 3.1: The damped time period for different α from 0.5 to 1.

α	Damped Time Period (Td)			
0.5	5.9	5.9	5.9	5.9
0.6	6.0	6.0	6.0	5.9
0.7	6.1	6.1	6.0	6.1
0.8	6.1	6.1	6.2	6.1
0.9	6.2	6.3	6.2	6.2
1.0	6.3	6.4	6.3	6.4

- From the above table, we can say that the damped time period is constant for constant alpha and increases as alpha increases.
- By doing curve fitting through the peak points we can say that decay is power-law (At^{-C}).

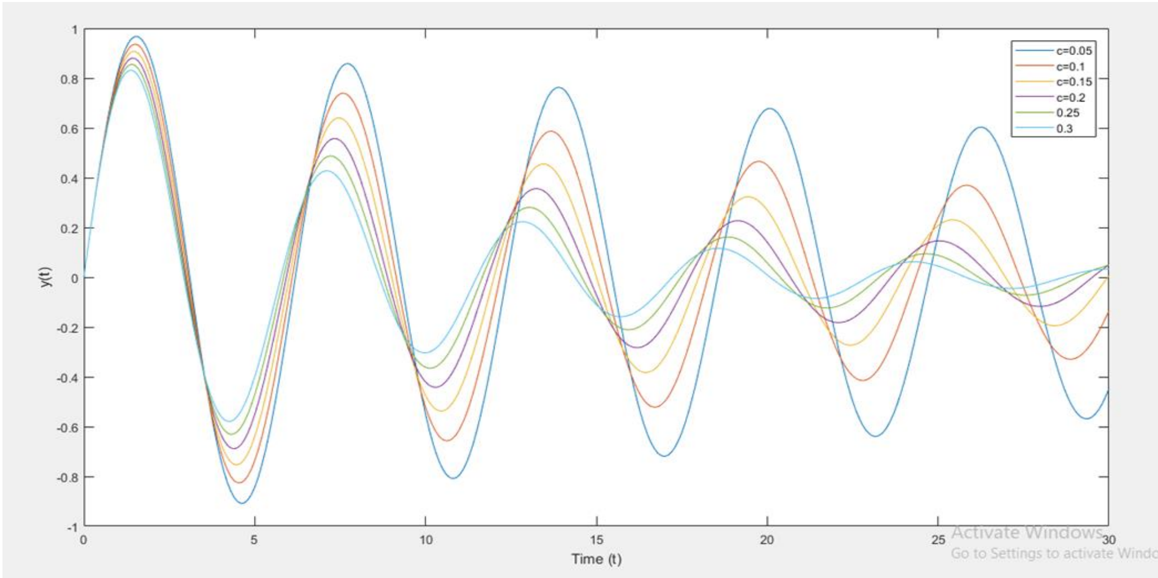

Fig. 3.3: Graph of free vibration response $y(t)$ vs. time t for different values of c from 0.5 to 1.

By measuring the peak to peak distance in the above graph we get the following damped time period values for each value of the damping coefficient c.

Table 3.2: The damped time period for different c from 0.5 to 1.

c	Damped Time Period (Td)			
0.05	6.2	6.2	6.2	6.2
0.10	6.1	6.1	6.0	6.1
0.15	6.0	5.9	6.0	6.0
0.20	5.9	5.9	5.9	5.9
0.25	5.8	5.8	5.8	5.9
0.30	5.7	5.7	5.8	5.7

- From the above table, we can say that the damped time period decreases as the damping coefficient increases which is just opposite to the behavior in case of viscous damping.

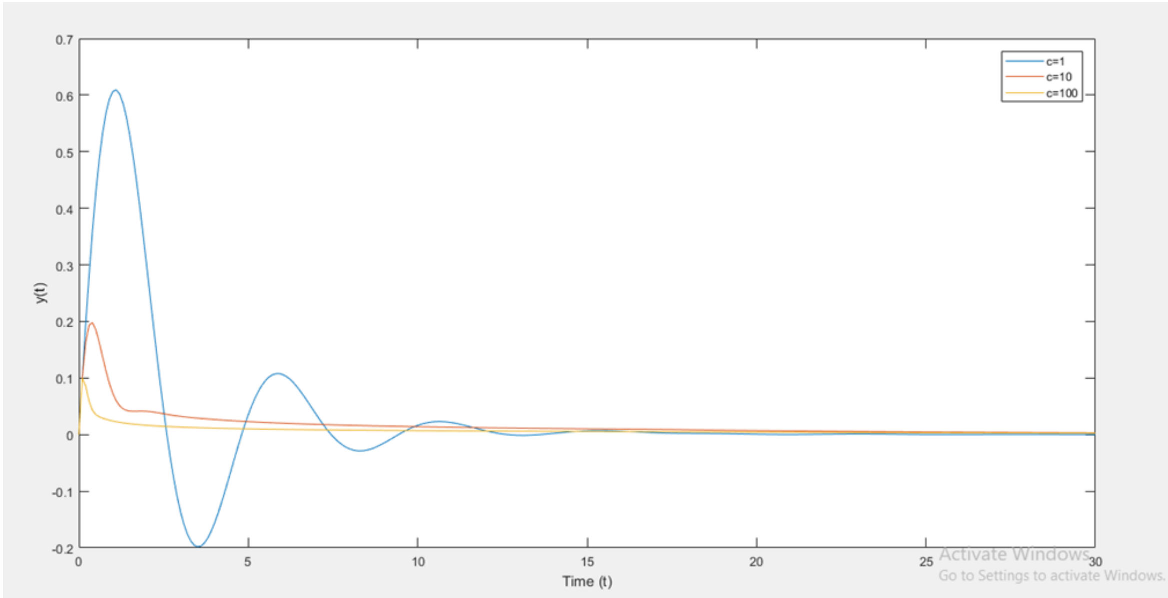

Fig. 3.4: Graph of free vibration response $y(t)$ vs. time t for large values of c=1, 10, 100.

- From the above table, we can say that this is the general pattern and there is no critical damping.

3.3. MATLAB code to solve the FDE using Laplace transform method for free vibration.

<u>MATLAB code 2: -</u>

```
%% mD^2x(t)+cD^ax(t)+kx(t)=F(t)
% Only for a=0.5
clc
clear all
m=1;
c=0.2;
k=1;
t=0:0.1:30;
p=[m,0,0,c,k];
r=(roots(p))';
diffp=[4*m,0,0,c];
dp=polyval(diffp,r);
y=0;
for i=1:4
```

```
y=y+(((r(i))/dp(i))*(exp((r(i)^2)*t)).*(1+erf(sym((
r(i))*((t).^(0.5))))));
end
plot(t,y);
y=double(y);
[Yp,tp]=findpeaks(y,t);
Td=diff(tp);
%% curve fitting X=At^C
a1=log(Yp');
a2=log(tp');
a3=length(a2);
a4=[ones(a3,1),a2];
phi=inv(a4'*a4)*a4'*a1;
A=exp(phi(1));
C=phi(2);
```

<u>Results of MATLAB code 2: -</u>

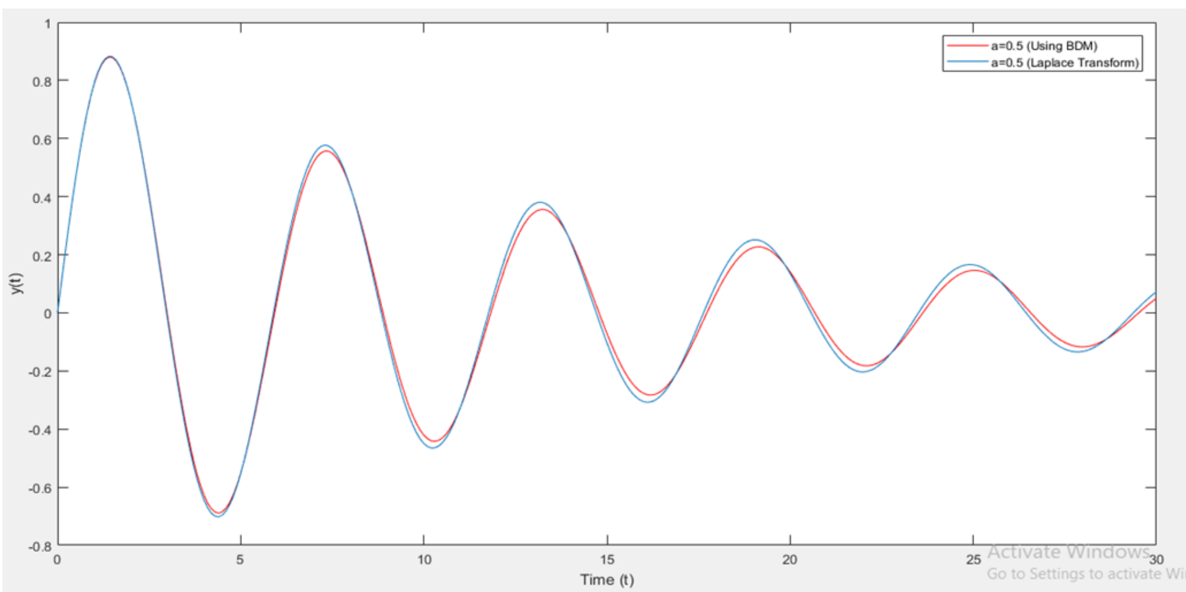

Fig. 3.5: Graph of free vibration response $y(t)$ vs. time t showing agreement between BDM and Laplace Method.

From the above graph, the Backward difference method is verified, as it agrees nicely with the Laplace transform method.

3.4. Power-law decay derivation for free vibration.

The fractional differential equation to be solved is given as:

$$m\ddot{x}+c\frac{d^{\alpha}x}{dt^{\alpha}}+kx=0 \quad (3.19)$$

Now, Laplace transform of fractional derivative based on Caputo and Grunwald-Letinkov's definition is given by:

$$L\left\{{}^{C/GL}D_0^{\alpha}\left(x(t)\right)\right\}=s^{\alpha}X(s)-s^{\alpha-1}x(0).......\alpha\in(0,1) \quad (3.20)$$

Here, Riemann-Liouville is not used because Laplace transform of fractional derivative based on Reimann-Liouville definition is given by:

$$L\{{}^{RL}_{o}D_t^{\alpha}(x(t))\}=s^{\alpha}X(s)-\sum_{k=0}^{n-1}s^k\left[{}_0D_t^{\alpha-k-1}(x(t))\right]_{t=0}.........(n-1\le\alpha<n) \quad (3.21)$$

As seen above, its practical applicability is limited by the absence of the physical interpretation of the limit values of fractional derivatives at the lower terminal t = 0. At the time of writing, such an interpretation is not known.

Therefore, by taking Laplace Transform based on Grunwald-Letinkov's definition on both sides of the fractional definition equation, we get

$$s^2X(s)-sx(0)-\dot{x}(0)+\frac{c}{m}\left\{s^{\alpha}X(s)-s^{\alpha-1}x(0)\right\}+\frac{k}{m}X(s)=0 \quad (3.22)$$

$$\therefore X(s)=\frac{sx(0)+\dot{x}(0)+\dfrac{x(0)c}{ms^{1-\alpha}}}{s^2+\dfrac{c}{m}s^{\alpha}+\dfrac{k}{m}} \quad (3.23)$$

For $\dot{x}(0)=0$, we get

$$X(s)=\frac{x(0)s+\dfrac{x(0)c}{ms^{1-\alpha}}}{s^2+\dfrac{c}{m}s^{\alpha}+\dfrac{k}{m}} \quad (3.24)$$

Taking limit $s\rightarrow 0$, we get

$$\lim_{s\to 0}X(s)=\lim_{s\to 0}\frac{x(0)s+\dfrac{x(0)c}{ms^{1-\alpha}}}{s^2+\dfrac{c}{m}s^{\alpha}+\dfrac{k}{m}}=\frac{x(0)c}{k}s^{\alpha-1} \quad (3.25)$$

By taking inverse Laplace transform, we have

$$x(t) \sim \frac{x(0)c}{k} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} ....(t \to \infty) \quad (3.26)$$

Similarly, for impulse response, we get following power-law decay,

$$x(t) = \frac{cm}{k^2} \frac{\alpha t^{-(\alpha+1)}}{\Gamma(1-\alpha)} ....(t \to \infty) \quad (3.27)$$

It is difficult to find fractional-order and damping coefficient from power-law decay as we would require to find the asymptotic solution of the given free vibration response.
So, forced vibration response is used to find fractional-order and damping.

CHAPTER 4

RESPONSE ANALYSIS (FORCED VIBRATION)

4.1. MATLAB code for Forced Vibration Response using Backward Difference Method (BDM).

<u>MATLAB code 3: -</u>

```
%% mD^2x(t)+cD^ax(t)+kx(t)=F(t)
clc;
clear all
%% Values of m, c, k, alpha, t0, T
m=1;
c=0.2;
k=1;
t0=0;
T=50;
n=500;
h=(T-t0)/n;
%% Forcing function
A=1;
w=1;
F=A*sin(w*(t0:h:T));
% Initial Conditions
y(1)=0;
y(2)=y(1)+(0*h);
a=0.5;
% for a=0.1:0.1:1
    warning('off','all');
        for j=1:n
            w(j)=((-
  1)^j)*((gamma(a+1))/((gamma(j+1))*(gamma(a-
  j+1))));
        end
        for i=3:n+1
            B=0;
            for j=2:i
                B=B+(w(j-1)*y(i+1-j));
            end
            y(i)=(((h^2*(F(i)-k*y(i-1)))+(m*(2*y(i-
      1)-     y(i-2)))-(c*h^(2-a)*B))/(m+(k*(h^(2-
      a)))));
        end
```

```
        figure;
        plot(t0:h:T,y);
        xlabel('Time (t)');
        ylabel('y(t)');
        % hold on
        %% finding timeperiod
        [X,t]=findpeaks(y,t0:h:T);
        Td=diff(t);
        %% fft
        Fs=1/T;
        L=size(y,2);
        Y=fft(y);
        P2 = abs(Y);
        P1 = P2(1:L/2+1);
        P1(2:end-1) = 2*P1(2:end-1);
        figure;
        f = 0:Fs:1/(2*h);
        plot(f,P1);
        title('Single-Sided Amplitude Spectrum of
    y(t)');
       xlabel('f (Hz)');
       ylabel('|A(f)|');
       % hold on;
% end
```

Results of MATLAB code 3: -

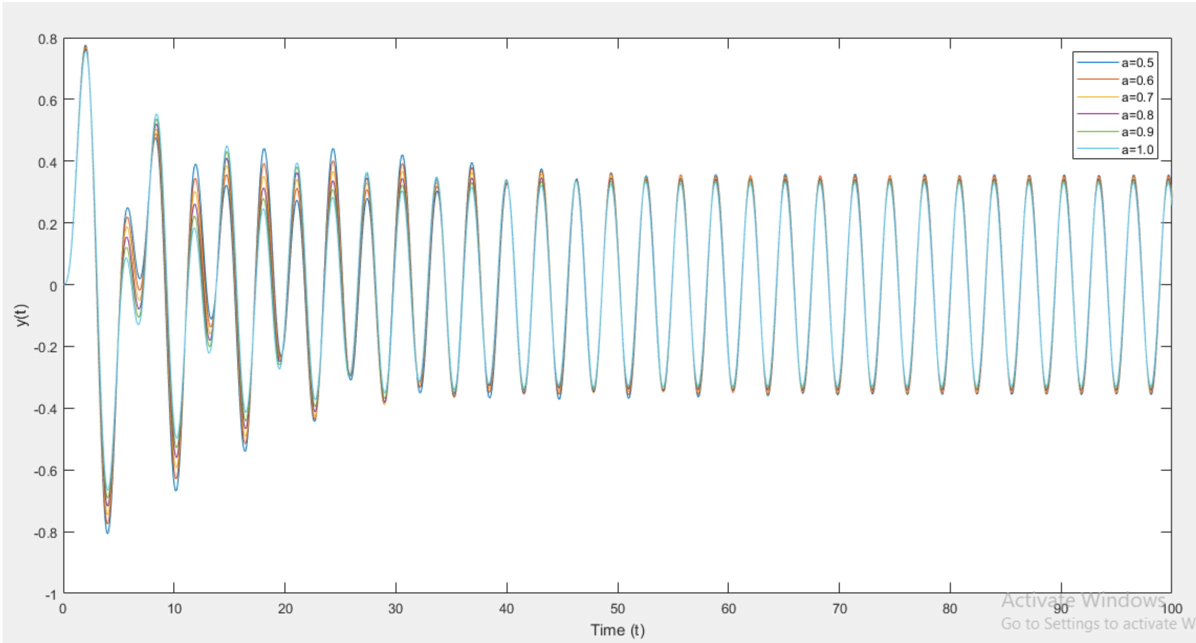

Fig. 4.1: Graph of forced vibration response $y(t)$ vs. time t for different values of α from 0.5 to 1.

From the above graph, we can say that the response follows forcing function after the initial transition period. We can also see that the amplitude of vibration decreases as fractional order value increases. To find the frequency components in the response, FFT is taken of the forced vibration response.

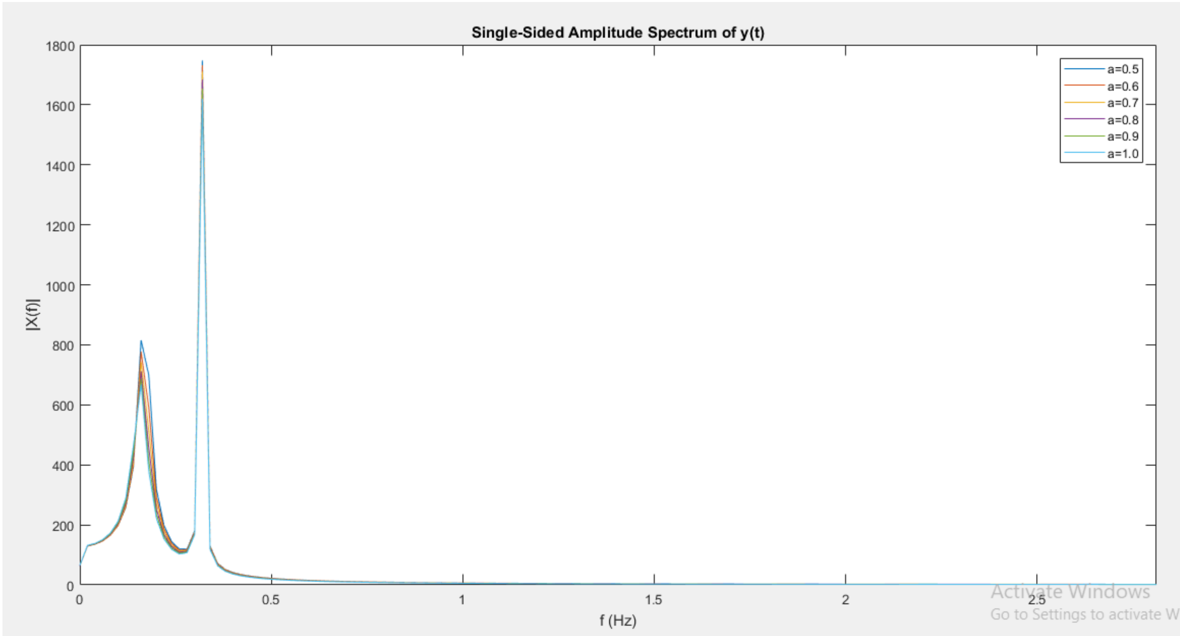

Fig. 4.2: Fast Fourier Transform of forced vibration response $y(t)$.

From the above FFT of the forced vibration response, we see that there are two frequency components. In which one frequency component is forcing function frequency and another frequency component is natural frequency. So, by knowing the forced vibration response of the fractional damped system we can find the natural frequency of the system by taking FFT of the response.

4.2. Relationship to estimate the fractional-order and damping coefficient using a forced vibration response.

Our given FDE is: $m\ddot{x} + cD^{\alpha}(x(t)) + kx = F(t)$ ……………………………………………..(4.1)

Let, $x(t) = \bar{X}e^{i\omega t}$ and $F(t) = Fe^{i\omega t}$ ……………………………………………………..(4.2)

Now, $D^{\alpha}(e^{i\omega t}) = (i\omega)^{\alpha} e^{i\omega t}$ …(for Grunwald-Letinkov Definition)………………………(4.3)

$D^{\alpha}(e^{i\omega t}) \sim (i\omega)^{\alpha} e^{i\omega t}$ ..$(t \to \infty)$ …(for RL and Caputo Definition)……………………..(4.4)

$\therefore \left\{-m\omega^2 + c(i\omega)^{\alpha} + k\right\} \bar{X}e^{i\omega t} = Fe^{i\omega t}$ ………………………………………………………...(4.5)

$$\therefore \bar{X} = \frac{F}{\left\{(k - m\omega^2) + c(i\omega)^\alpha\right\}} \quad \text{...(4.6)}$$

Now, $(i\omega)^\alpha = \omega^\alpha \left(e^{i\pi/2}\right)^\alpha = \omega^\alpha \left\{\cos(\tfrac{\pi\alpha}{2}) + i\sin(\tfrac{\pi\alpha}{2})\right\} = \omega^\alpha (a + ib)$ ....(4.7)

Where, $a = \cos(\tfrac{\pi\alpha}{2})$ and $b = \sin(\tfrac{\pi\alpha}{2})$ ....(4.8)

$$\therefore \bar{X} = \frac{F}{k - m\omega^2 + ac\omega^\alpha + ib\omega^\alpha} = Xe^{-i\phi} \quad \text{...(4.9)}$$

$$\therefore X = \frac{F}{\sqrt{\left(k - m\omega^2 + ac\omega^\alpha\right)^2 + (bc\omega^\alpha)^2}} \quad \text{and } \phi = \tan^{-1}\left(\frac{bc\omega^\alpha}{k - m\omega^2 + ac\omega^\alpha}\right) \quad \text{..................(4.10)}$$

$$\therefore X = \frac{F}{k}\frac{1}{\sqrt{(1 - (m\omega^2/k) + (ac\omega^\alpha/k))^2 + (bc\omega^\alpha/k)^2}} \quad \text{..(4.11)}$$

Now, $\dfrac{m\omega^2}{k} = \left(\dfrac{\omega}{\omega_n}\right)^2 = r^2$ and let $\dfrac{c\omega^\alpha}{k} = a_c$ ....(4.12)

$$\therefore X = \frac{F}{k}\frac{1}{\sqrt{(1 - r^2 + a * a_c)^2 + (b * a_c)^2}}$$

$$= \frac{F}{k}\frac{1}{\sqrt{(1 - r^2)^2 + (a * a_c)^2 + 2(1 - r^2)(a * a_c) + (b * a_c)^2}} \quad \text{..................(4.13)}$$

$$= \frac{F}{k}\frac{1}{\sqrt{(1 - r^2)^2 + 2(1 - r^2)(a * a_c) + a_c^2}} \quad \text{.................................}(a^2 + b^2 = 1)$$

Now, $a_c = \dfrac{c\omega^\alpha}{k} = \dfrac{c\omega_n^\alpha}{k}\left(\dfrac{\omega}{\omega_n}\right)^\alpha = \dfrac{2\xi\sqrt{km} * \omega_n^\alpha}{k} r^\alpha = 2\xi\omega_n^{\alpha-1} r^\alpha$........$\left(\xi = \dfrac{c}{2\sqrt{km}}\right)$ ....(4.14)

$$\therefore X = \frac{F}{k}\frac{1}{\sqrt{(1 - r^2)^2 + 4(1 - r^2)(a * \xi * \omega_n^{\alpha-1})(r^\alpha) + (2\xi\omega_n^{\alpha-1} r^\alpha)^2}} \quad \text{..............................(4.15)}$$

$$\therefore DAF = \frac{X}{F/k} = \frac{1}{\sqrt{(1 - r^2)^2 + 4(1 - r^2)(a * \xi * \omega_n^{\alpha-1})(r^\alpha) + (2\xi\omega_n^{\alpha-1} r^\alpha)^2}} \quad \text{.....................(4.16)}$$

$$\boxed{\therefore DAF_{r=1} = \frac{1}{2\xi\omega_n^{\alpha-1}} = \frac{\omega_n^{1-\alpha}}{2\xi} = \frac{\omega_n^{1-\alpha}\sqrt{km}}{c}} \quad \text{...(4.17)}$$

Finding DAF_{max} by using differentiation leads to a fractional polynomial that does not give an explicit function for 'r'.

So, we will use the concept of equivalent damping for finding DAF_{max}.

For equivalent damping, energy dissipated in one cycle should be also same.

$$\therefore E_d = \int_0^T F_d \dot{x} dt = \int_0^T cD^\alpha (x(t))\dot{x} dt \quad \text{.....(4.18)}$$

For, $x(t) = X \sin(\omega t - \phi)$ .....(4.19)

$D^\alpha (X \sin(\omega t - \phi)) = X \omega^\alpha \sin(\omega t + {}^{\alpha\pi}\!/_2 - \phi)$ ...(for GL definition).....(4.20)

$D^\alpha (X \sin(\omega t - \phi)) \sim Xw^\alpha \sin(\omega t + {}^{\alpha\pi}\!/_2 - \phi)...(t \to \infty)$ ...(for RL and Caputo definition)..(4.21)

$$\therefore E_d = \int_0^{2\pi/w} cX\omega^\alpha \sin(\omega t + \tfrac{\alpha\pi}{2} - \phi)(X\omega \cos(\omega t - \phi))dt \quad \text{.....(4.22)}$$

$$= \frac{cX^2\omega^{\alpha+1}}{2} \int_0^{2\pi/w} \{\sin(2\omega t - 2\phi + \tfrac{\alpha\pi}{2}) + \sin(\tfrac{\alpha\pi}{2})\} dt \quad \text{.....(4.23)}$$

$$= cX^2\omega^{\alpha+1} \frac{\pi}{\omega} \sin(\frac{\alpha\pi}{2}) = \pi c_{eq} \omega X^2 \quad \text{.....(4.44)}$$

$$\boxed{\therefore c_{eq} = c\omega^{\alpha-1} \sin({}^{\alpha\pi}\!/_2) \quad \text{.....(4.45)}}$$

Now, $X_{\max}$ for viscous damping is given by $X_{\max} = \dfrac{F}{c\omega_n}$ (for small damping)

$\therefore$ $X_{\max}$ for fractional derivative damping is given by $X_{\max} = \dfrac{F}{c_{eq}\omega_n}$ .....(4.46)

$$\therefore X_{\max} = \frac{F\omega_n}{c_{eq}\omega_n^2} = \frac{F}{k} \frac{m\omega_n}{c\omega_n^{\alpha-1} \sin(\tfrac{\alpha\pi}{2})} \quad \text{.....(4.47)}$$

$$\boxed{\therefore DAF_{\max} = \frac{X_{\max}}{F/k} = \frac{m\omega_n}{c\omega_n^{\alpha-1} \sin\left({}^{\alpha\pi}\!/_2\right)}} \quad \text{.....(4.48)}$$

From equation (4.17) and (4.48), we get

$$DAF_{\max} = \frac{m\omega_n}{\frac{\omega_n^{1-\alpha}\sqrt{km}}{(DAF_{r=1})}\sin(\frac{\alpha\pi}{2})} = \frac{m\omega_n(DAF_{r=1})}{\sqrt{km}\sin(\frac{\alpha\pi}{2})} \quad \text{.........(4.49)}$$

$$\boxed{\therefore \sin\left(\frac{\alpha\pi}{2}\right) = \frac{DAF_{r=1}}{DAF_{\max}} = \frac{X_{r=1}}{X_{\max}} \quad \text{.........(4.50)}}$$

This relationship gives the value of fractional order α if $DAF_{r=1}$ and DAF_{max} are known, which can be found through an experiment.

Then by using equation (4.17), we can find damping coefficient/factor after finding the value of fractional order α, as the natural frequency of the system is already known from FFT of forced vibration response.

4.3. Verification of the relationship using the numerically obtained solution.

6.3.1. MATLAB Code 4 to plot graph between Dynamic Amplification Factor (DAF) Vs. Frequency Ratio ($r=w/w_n$).

```
% mD^2x(t)+cD^ax(t)+kx(t)=F(t)
clc
clear all
% Values of m, c, k, alpha, t0, T
m=1;
c=0.2;
k=4;
wn=sqrt(k/m);
t0=0;
T=200;
n=10000;
h=(T-t0)/n;
t=t0:h:T;
% Forcing function
A=1;
% W=wn;
for W=0:0.001:5
    F=A*sin(W*(t0:h:T));
    % Initial Conditions
```

```
    y(1)=0;
    y(2)=y(1)+(0*h);
    a=0.5;
        for j=1:n
            w(j)=((-
1)^j)*((gamma(a+1))/((gamma(j+1))*(gamma(a-j+1))));
        end
        for i=3:n+1
            B=0;
            for j=2:i
                B=B+(w(j-1)*y(i+1-j));
            end
            y(i)=(((h^2*(F(i)-k*y(i-1)))+(m*(2*y(i-
1)-y(i-2)))-(c*h^(2-a)*B))/(m+(c*(h^(2-a)))));
        end
%          figure(1);
%          plot(t,y);
%          hold on;
    sum1=0.0; sum2=0.0;
    for i=8001:10001
        sum1=sum1+y(i)*cos(W*t(i));
        sum2=sum2+y(i)*sin(W*t(i));
    end
    a1=(2/2000)*sum1;
    b1=(2/2000)*sum2;

x(single((1000*W)+1))=((k/A)*sqrt((a1^2)+(b1^2)));
%      x=((k/A)*sqrt((a1^2)+(b1^2)))
end
r=((0:0.001:5)/wn);
figure(1);
plot(r,x);
hold on;
```

<u>Result of MATLAB Code 4:</u>

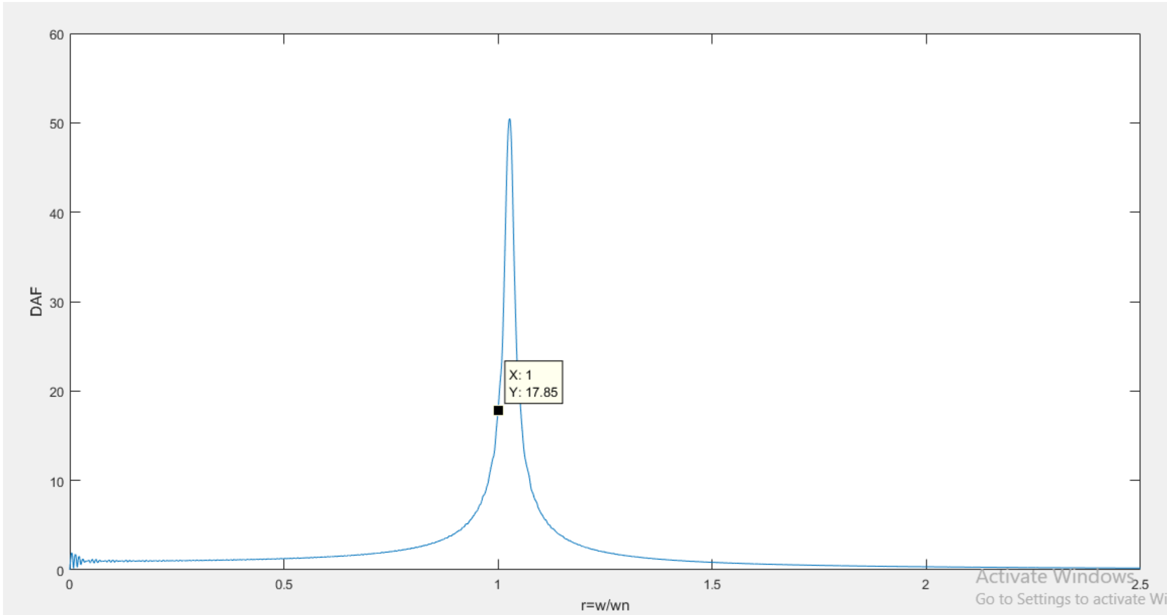

Fig. 4.3. The graph between Dynamic Amplification Factor (DAF) vs. Frequency Ratio (r)

Graph of r vs DAF is found out from numerically (Backward Difference Method) obtained solution of forced vibration response.

DAF at r=1 and maximum DAF is noted for each fractional order ‘α’ from the above graph and is given as below.

Table 4.1: Values of α assumed and α calculated from the derived relationship.

‘α’	**DAF at r=1**	**Max DAF**	**‘α’ calculated**
0.2	17.8500	50.4700	0.23
0.3	16.3317	34.0549	0.318
0.4	15.2202	24.9213	0.418
0.5	14.2244	19.5376	0.519
0.6	13.2928	16.0059	0.623
0.7	12.4173	13.6566	0.726
0.8	11.5962	11.9099	0.853
0.9	10.8274	10.8737	0.941
1.0	10.1078	10.1414	0.982

Then ‘α’ is calculated from derived formula and verified with ‘α’ assumed. As the table shows, ‘α’ calculated and ‘α’ assumed values are matching, which verifies the derived formula.

CHAPTER 5

TRANSMISSIBILITY FOR FRACTIONAL DAMPING (FRACTIONALLY DAMPED VIBRATION ISOLATOR)

5.1. Force Transmissibility derivation for Fractionally Damped Vibration Isolator.

Our given FDE is: $m\ddot{x}+cD^{\alpha}(x(t))+kx=F(t)$ ……………………………………..(5.1)

Let, $x(t)=\bar{X}e^{i\omega t}$ and $F(t)=Fe^{i\omega t}$ ………………………………………………….(5.2)

Now, $D^{\alpha}(e^{i\omega t})=(i\omega)^{\alpha}e^{i\omega t}$ …(for Grunwald-Letinkov Definition)………………………(5.3)

$D^{\alpha}(e^{i\omega t})\sim(i\omega)^{\alpha}e^{i\omega t}$ ..$(t\to\infty)$…(for RL and Caputo Definition)………………………..(5.4)

$\therefore\left\{-m\omega^2+c(i\omega)^{\alpha}+k\right\}\bar{X}e^{i\omega t}=F_0e^{i\omega t}$ …………………………………………………...(5.5)

Now, $(i\omega)^{\alpha}=\omega^{\alpha}\left(e^{i\pi/2}\right)^{\alpha}=\omega^{\alpha}\left\{\cos(\frac{\pi\alpha}{2})+i\sin(\frac{\pi\alpha}{2})\right\}=\omega^{\alpha}(a+ib)$ …………………………...(5.6)

Where, $a=\cos(\frac{\pi\alpha}{2})$ and $b=\sin(\frac{\pi\alpha}{2})$ ……………………………………………………(5.7)

$\therefore F_0e^{i\omega t}=Xe^{i\omega t}\left\{k-m\omega^2+ac\omega^{\alpha}+ibc\omega^{\alpha}\right\}$ ……………………………………………(5.8)

$$\begin{aligned}\therefore F_0&=|X|\sqrt{\left(k-m\omega^2+acw^{\alpha}\right)^2+\left(bc\omega^{\alpha}\right)^2}\\&=|X|\sqrt{(k-m\omega^2)^2+(ac\omega^{\alpha})^2+2(k-m\omega^2)(ac\omega^{\alpha})+(bc\omega^{\alpha})^2}\qquad\text{……………………...(5.9)}\\&=|X|\sqrt{(k-m\omega^2)^2+2(k-m\omega^2)(ac\omega^{\alpha})+(c\omega^{\alpha})^2}\text{..........}(\because a^2+b^2=1)\end{aligned}$$

Now, the force transmitted to the foundation is given by

$F_T=cD^{\alpha}(x(t))+kx(t)$ ……………………………………………………………..(5.10)

$\therefore F_T=\left\{c(i\omega)^{\alpha}+k\right\}Xe^{i\omega t}$ …………………………………………………………...(5.11)

$\therefore F_T=Xe^{i\omega t}\left\{k+ac\omega^{\alpha}+ibc\omega^{\alpha}\right\}$ …………………………………………………...(5.12)

$$\begin{aligned}\therefore|F_T|&=|X|\sqrt{(k+ac\omega^{\alpha})^2+(bc\omega^{\alpha})^2}\\&=|X|\sqrt{k^2+2kac\omega^{\alpha}+(c\omega^{\alpha})^2}\end{aligned}\qquad\text{……………………………………………………….(5.13)}$$

$$\therefore Transmissibility=\frac{|F_T|}{F_0}=\frac{\sqrt{k^2+2kac\omega^{\alpha}+(c\omega^{\alpha})^2}}{\sqrt{(k-m\omega^2)+2(k-m\omega^2)(ac\omega^{\alpha})+(c\omega^{\alpha})^2}}\text{ …………………(5.14)}$$

$$\therefore\frac{|F_T|}{F_0}=\frac{\sqrt{1+4a\zeta\omega_n^{\alpha-1}+4\zeta^2\omega_n^{2(\alpha-1)}r^{2\alpha}}}{\sqrt{(1-r^2)^2+4(1-r^2)a\zeta\omega_n^{\alpha-1}r^{\alpha}+4\zeta^2\omega_n^{2(\alpha-1)}r^{2\alpha}}}\text{......}(r=\frac{\omega}{\omega_n},\zeta=\frac{c}{2\sqrt{km}})\text{ ………(5.15)}$$

5.2. MATLAB Code 5 to plot transmissibility graph.

```
%% mD^2x(t)+cD^ax(t)+kx(t)=F(t)
clc
clear all
m=1;
k=2;
wn=sqrt(k/m);
% a=0.5;
c=0.1;
% for c=0.1:0.1:0.5
    warning('off','all');
    xe=c/(2*((k*m)^0.5));
    for a=0.5:0.1:1
        d=cos((pi*a)/2);
        for r=0:0.001:2

daf(single((1000*r)+1))=(sqrt(1+(4*d*xe*(wn^(a-
1))*(r^a))+(4*(xe^2)*(wn^(2*(a-
1)))*(r^(2*a))))./sqrt(((1-(r.^2)).^2)+((4*(1-
(r.^2))*d*xe.*(r.^a))/(wn^(1-
a)))+(((2*xe*(r.^a))/(wn^(1-a))).^2)));
        end

[X(single(10*a)),x(single(10*a))]=findpeaks(daf,(0:0.
001:2));
        figure(1);
        plot((0:0.001:2),daf);
        xlabel('r=w/wn');
        ylabel('transmissibility');
        hold on;
    end
    refline(0,1);
% end
```

Result of MATLAB Code 5

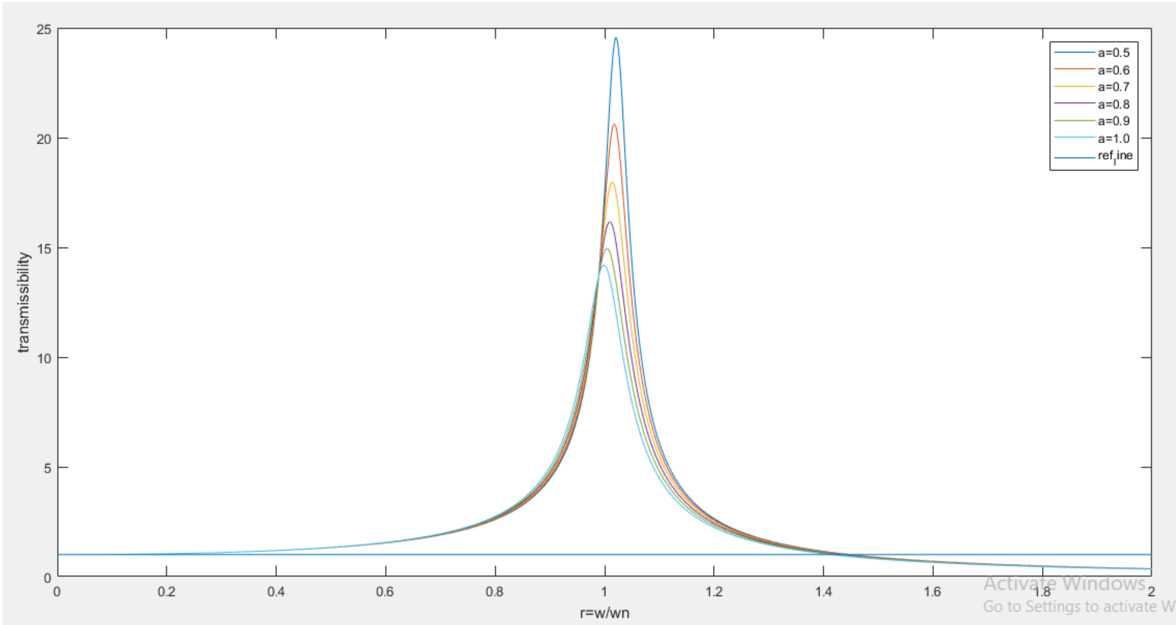

Fig. 5.1: Transmissibility plot for different values of fractional order from 0.5 to 1

The above graph shows that as fractional order is increasing force transmitted to the foundation is decreasing. This shows that viscous damping is better than fractional derivative damping for transmissibility.

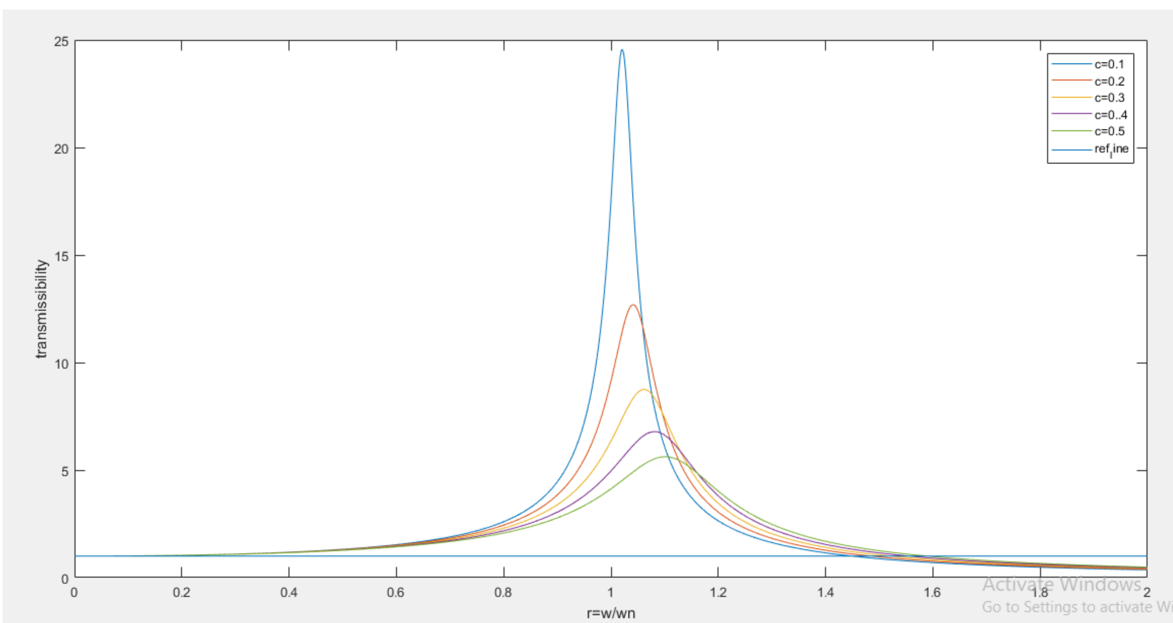

Fig. 5.2: Transmissibility plot for different values of the damping coefficient from 0.1 to 0.5

The above graph shows that as damping is increasing the force transmitted to the foundation is decreasing. Unlike the viscous damping, the curves do not intersect at $r = \sqrt{2}$.

CHAPTER 6

FRACTIONAL TUNED MASS DAMPER (FRACTIONAL DYNAMIC VIBRATION ABSORBER)

Fractional damping is considered for the dynamic vibration absorber and damping is considered zero for the main dynamic system as shown in the below figure.

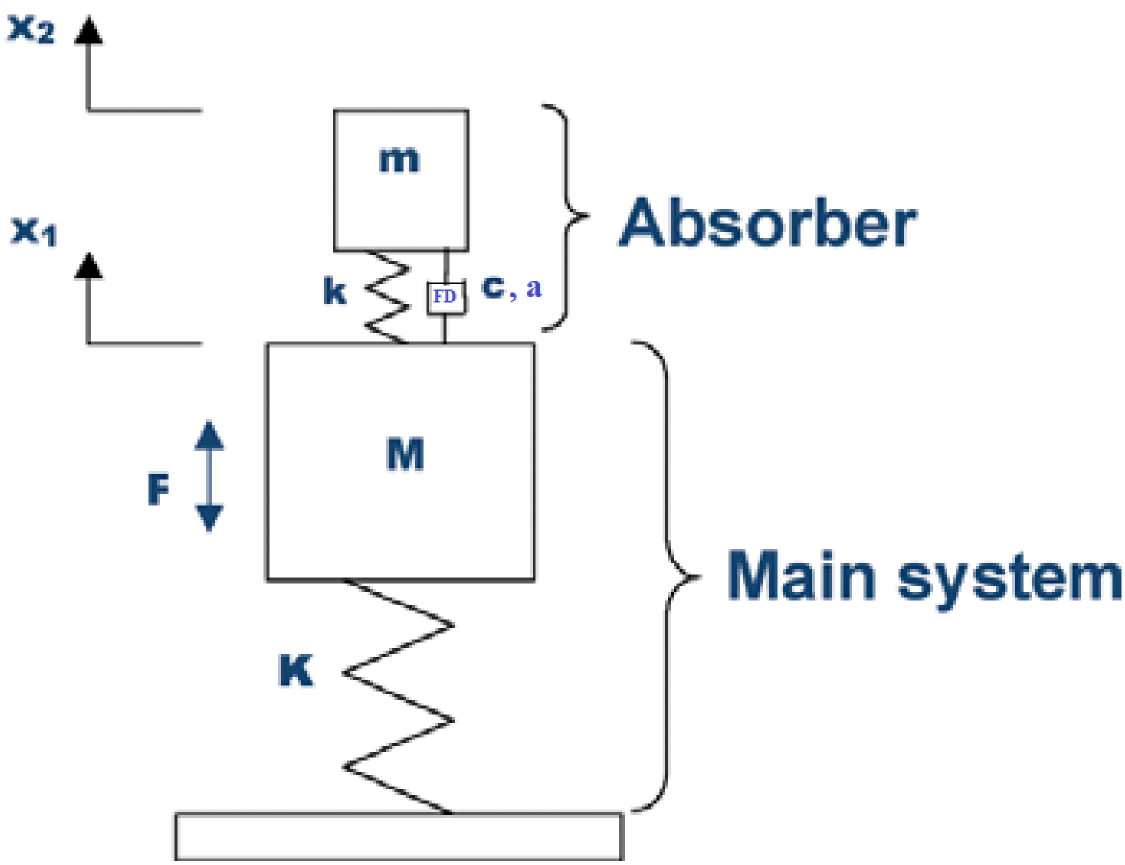

Fig. 6.1 Dynamic System with Fractional Dynamic Vibration Absorber.

6.1. Derivation for finding the amplitude of the main mass (X_1)

Newton's law applied to the mass M gives:

$$M\ddot{x}_1 + Kx_1 + k(x_1 - x_2) + cD^{\alpha}(x_1 - x_2) = F_0 \sin(\omega t) \qquad (6.1)$$

and applied to the small mass m

$$m\ddot{x}_2 + k(x_2 - x_1) + cD^{\alpha}(x_2 - x_1) = 0 \qquad (6.2)$$

The four terms on the left-hand side of (6.1) signify the "inertia force" of 'M', the main-spring force, the damper- spring force, and the dashpot force. We are interested in a solution for the forced vibrations only and do not consider the transient free vibration. Then both 'x_1' and 'x_2' are harmonic motions of the frequency 'ω' and can be represented by vectors. Any term in either (6.1) or (6.2) is representable by such a vector rotating with velocity 'ω'. The easiest manner of solving these equations is by writing the vectors as complex numbers. The equations then are:

$$-MX_1\omega^2 + KX_1 + k(X_1 - X_2) + c(i\omega)^\alpha (X_1 - X_2) = F_0 \quad \text{...(6.3)}$$

$$-mX_2\omega^2 + k(X_2 - X_1) + c(i\omega)^\alpha (X_2 - X_1) = 0 \quad \text{..(6.4)}$$

where 'X_1' and 'X_2' are (unknown) complex numbers, the other quantities being real.

These can be solved for 'X_1' and 'X_2'. We are primarily interested in the motion of the main mass 'X_1', and, to solve for it, we express 'X_2' in terms of 'X_1' using (6.4) and then substitute in the (6.1). This gives:

$$X_1 = F_0 \frac{\left\{\left(k - m\omega^2\right) + c(i\omega)^\alpha\right\}}{\left\{\left(K - M\omega^2\right)(k - m\omega^2 + c(i\omega)^\alpha) - m\omega^2 k - m\omega^2 c(i\omega)^\alpha\right\}} \quad \text{..............................(6.5)}$$

Now, $(i\omega)^\alpha = \omega^\alpha \left(e^{i\pi/2}\right)^\alpha = \omega^\alpha \left\{\cos(\tfrac{\pi\alpha}{2}) + i\sin(\tfrac{\pi\alpha}{2})\right\} = \omega^\alpha (a + ib)$ ....(6.6)

Where, $a = \cos(\tfrac{\pi\alpha}{2})$ and $b = \sin(\tfrac{\pi\alpha}{2})$

$$\therefore X_1 = F_0 \frac{\left\{\left(k - m\omega^2 + Ac\omega^\alpha\right) + iBc\omega^\alpha\right\}}{\left\{\left(\left(K - M\omega^2\right)\left(k - m\omega^2\right) + Ac\omega^\alpha (K - M\omega^2) - m\omega^2 k - m\omega^2 Ac\omega^\alpha\right) + iBc\omega^\alpha (K - M\omega^2 - m\omega^2)\right\}}$$

....(6.7)

The meaning which has to be attached to (6.7) is that in vector representation the displacement 'X_1' consists of two components, one in phase with the force F_0 and another a quarter turn ahead of it. Adding these two vectors geometrically, the magnitude of 'X_1' is expressed by:

$$\frac{X_1^2}{F_0^2}=\frac{\left\{\left(k-m\omega^2\right)+Ac\omega^\alpha\right\}^2+\left(Bc\omega^\alpha\right)^2}{\left\{\left(K-M\omega^2\right)\left(k-m\omega^2\right)-m\omega^2k+Ac\omega^\alpha\left(K-M\omega^2-m\omega^2\right)\right\}^2+\left\{Bc\omega^\alpha\left(K-M\omega^2-m\omega^2\right)\right\}^2}$$

....(6.8)

By expanding the square terms and by using the equation $A^2+B^2=1$, we get,

$$\frac{X_1^2}{F_0^2}=\frac{\left\{\left(k-mw^2\right)^2+2\left(k-mw^2\right)Acw^\alpha\right\}+\left(cw^\alpha\right)^2}{\left\{\left(\left(K-Mw^2\right)\left(k-mw^2\right)-mw^2k\right)^2+2\left(\left(K-Mw^2\right)\left(k-mw^2\right)-mw^2k\right)Acw^\alpha\left(K-Mw^2-mw^2\right)\right\}+\left\{cw^\alpha\left(K-Mw^2-mw^2\right)\right\}^2}$$

....(6.9)

which is the amplitude of the motion of the main mass M.

Thus, we are in a position to calculate the amplitude in all cases. In Eq. (6.9) 'X_1' is a function of eight variables: F_0, ω, c, K, k, M, m, and α. However, the number of variables can be reduced, as the following consideration shows. For example, if F_0 is doubled and everything else is kept the same, we should expect to see X_1 doubled, and there are several relations of this same character. To reveal them, it is useful to write Eq. (6.9) in a dimensionless form, for which purpose the following symbols are introduced:

μ = m/M = mass ratio = absorber mass/main mass

ω_a^2 = k/m = natural frequency of absorber

Ω_n^2 = K/M = natural frequency of main system

f = ω_a/Ω_n = frequency ratio (natural frequencies)

g = ω/Ω_n, = forced frequency ratio

$X_{st}=F_0/K$ = static deflection of system

c_c = 2mΩ_n = "critical" damping

After performing some algebra Eq. (6.9) is transformed into

$$\boxed{\frac{X_1}{X_{st}}=\sqrt{\frac{\left(f^2-g^2\right)^2+4A\xi g^\alpha\Omega_n^{\alpha-1}\left(f^2-g^2\right)+\left(2\xi g^\alpha\Omega_n^{\alpha-1}\right)^2}{\left\{\left(1-g^2\right)\left(f^2-g^2\right)-\mu f^2g^2\right\}^2+\left\{2\xi g^\alpha\Omega_n^{\alpha-1}\left(1-g^2-\mu g^2\right)\right\}^2+4A\xi g^\alpha\Omega_n^{\alpha-1}\left(1-g^2-\mu g^2\right)\left(\left(1-g^2\right)\left(f^2-g^2\right)-\mu f^2g^2\right)}}}$$

....(6.10)

This is the amplitude ratio X_1/X_{st} of the main mass as a function of the six essential variables μ, c/c_c=ξ, f, g, Ω_n, and α.

6.2. MATLAB Code 6 to plot the amplitude of the main mass (X_1).

```
clc
clear all
M=20;
K=30;
m=1;
k=1.5;
wn=sqrt(K/M);
wa=sqrt(k/m);
mu=m/M;
f=wa/wn;
% xe=0.1;
a=0.5;
% for a=0.1:0.1:1;
A=cos((a*pi)/2);
% c=0.32*2*m*wn;
% xe=c/2*m*wn;
for xe=0.01:0.01:0.1
    for g=0:0.001:2
        B=f^2-g^2;
        C=1-(g^2);
        D=2*xe*(g^a)*(wn^(a-1));

X(single((1000*g)+1))=sqrt((((B^2)+(2*A*D*B)+(D^2))/(
(((C*B)-(mu*(f*g)^2))^2)+((D*(C-
(mu*(g^2))))^2)+((2*A*D)*(C-(mu*(g^2)))*((C*B)-
(mu*((f*g)^2))))))));
    end
g=0:0.001:2;
plot(g,X);
xlabel('g=w/wn');
ylabel('X1/Xst');
hold on;
end
% end
```

<u>Result of MATLAB Code 6:</u>

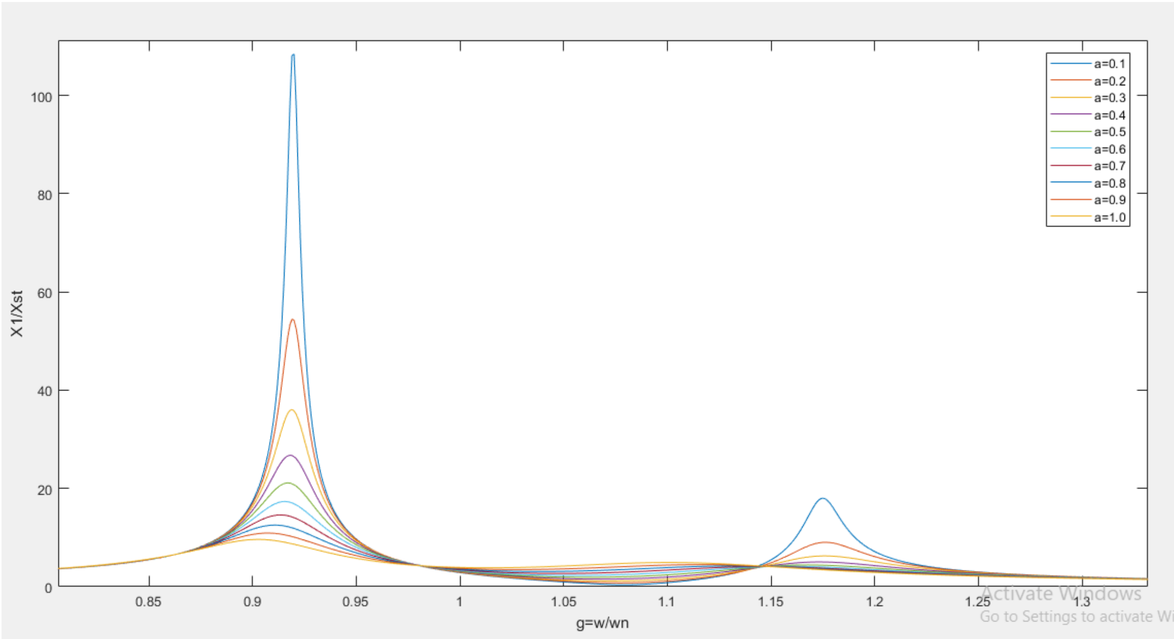

Fig. 6.2. Amplitudes of the main mass for various values of fractional-order from 0.1 to 1.

The above graph shows that as fractional order is increasing amplitude of vibration of the main mass is decreasing. This shows that viscous damping is better than fractional derivative damping for dynamic vibration absorber.

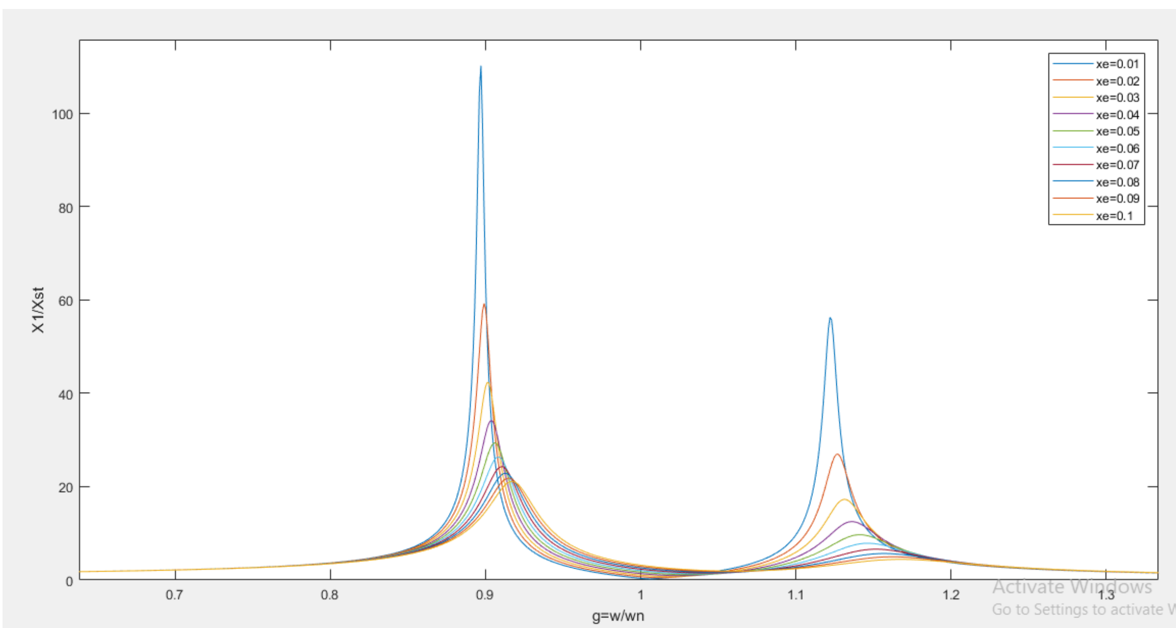

Fig. 6.3. Amplitudes of the main mass for various values of absorber damping from 0.01 to 0.1.

The above graph shows that as damping is increasing the amplitude of vibration of the main mass is decreasing. Unlike the viscous damping, the curves do not intersect at two fixed points.

CHAPTER 7
CONCLUSION AND FUTURE SCOPE

The damping force term in the fundamental equation of a dynamic system is replaced by a fractional derivative term. The resulting equation is solved and the solution is studied and compared with that of viscous damping case. A couple of differences are noted.

Unlike exponential decay in viscous damping, fractional derivative damping follows power-law decay. The damped time period decreases as damping increases. There is no critical damping. Two frequency components are seen in forced vibration response in which one frequency component is the natural frequency of the system. These results are just opposite to the behavior in the case of ordinary viscous damping.

By using the dynamic amplification factor of a given dynamic system with fractional derivative damping we can find both fractional-order and damping coefficient.

Also, it is seen that for both transmissibility and dynamic vibration absorber, viscous damping is better than fractional derivative damping for dynamic vibration absorber.

As fractional derivative considers the history of the system, it represents more closely to reality. The fractional derivative model is very well suited to model material damping.

A dynamic system with only a single degree of freedom is considered. It can be extended to a multi-degree of freedom or continuous system for accurate solution of the system.

Optimum damping value and fractional order can be found out for fractional dynamic vibration absorber.

This work is all theoretical, to do study practically we need a setup that will follow fractional derivative damping so that results can be verified.

In this work spring is considered linear, we can extend the work by considering the spring also nonlinear with fractional derivative damping.

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