An Infinite Family of S Invariant Theories on the Celestial Sphere

Raju Mandal

NISER,Bhubaneswar

July 6, 2024

Based on :
"All S invariant gluon OPEs on the celestial sphere"
arXiv: 2311.16796

Done in collaboration with
Shamik Banerjee(NISER), Sagnik Misra(NISER), Sudhakar Panda(CCSP) and Partha Paul(IISC).

Outline :

1. Motivation
2. Some elements of Celestial CFT ?
3. OPE and its importance
4. Celestial OPE
5. Brief discussion on null states
7. **Gluons and the S algebra**
8. **Building up S invariant OPEs**
9. **Knizhnik-Zamolodchikov (KZ) -type null states**

Motivation

- $\boxed{\textbf{Conformally soft theorems}}$ $\longrightarrow$ Gauge theories and Gravity in **4-D asymptotically flat spacetime** are enriched with **infinite** number of new **non-trivial symmetries**.

  (Guevara,Himwich,Pate and Strominger)

- $\boxed{\textbf{Gravity}}$ $\longrightarrow$ $\boxed{\textbf{wedge subalgebra of } w_{1+\infty} \textbf{ algebra}}$.

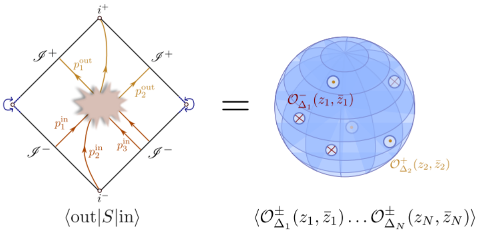
- $\boxed{\textbf{Gauge Theories}}$ $\longrightarrow$ $\boxed{S \textbf{ algebra}}$ 2103.03961

- Banerjee, Kulkarni and Paul have classified all w-invariant theories by computing the OPEs of such theories in case of gravity. 2301.13225; 2311.06485

- Our goal is to classify the theories which are invariant under **S algebra** and also to find the **KZ**-type null states of these theories. 2311.16796

(Banerjee, RM, Misra, Panda and Paul '23)

"2-D Celestial CFTs" are believed to be the holographic duals of the theories of QG in 4-D asymptotically flat spacetime.



[Courtesy: Laura Donnay]

- **Celestial correlation function** is obtained by taking the Mellin transform of the momentum space scattering amplitude.

(Pasterski,Shao,Strominger '17)

$$\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \prod_{i=1}^{n}\int_0^\infty d\omega_i\ \omega_i^{\Delta_i-1}\mathcal{S}_n(\{\omega_i,z_i,\bar{z}_i,\sigma_i,a_i\}) \qquad (1)$$

✧ Correlation functions thus obtained transform nicely under Lorentz transformations.

$$\mathcal{A}_n(\{z_i,\bar{z}_i,h_i,\bar{h}_i,a_i\}) = \prod_{i=1}^{n} \frac{1}{(cz_i+d)^{2h_i}} \frac{1}{(\bar{c}\bar{z}_i+\bar{d})^{2\bar{h}_i}} \mathcal{A}_n\left(\left\{\frac{az_i+b}{cz_i+d}, \frac{\bar{a}\bar{z}_i+\bar{b}}{\bar{c}\bar{z}_i+\bar{d}}, h_i, \bar{h}_i, a_i\right\}\right). \tag{2}$$

✧ Continuous Spectrum :

$$\Delta = 1 + i\mathbb{R}. \tag{3}$$

✧ 2, 3 and 4 -point celestial amplitudes have distributional supports on celestial sphere! $\xrightarrow[\text{std. correlators}]{\text{ways to get}}$ (2202.08288; 2302.10245)

✧ Celestial primary operators :

$$\mathcal{O}^{a,\epsilon}_{h,\bar{h}}(z,\bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1} A^a(\epsilon\omega, z, \bar{z}, \sigma) \tag{4}$$

✧ Conformally soft limits are defined at $\Delta = 1, 0, -1, \ldots$

✧ Leading Conformally Soft Theorem :

$$\left\langle R^{1,a}(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = -\sum_{k=1}^{n} \frac{T^a_k}{z-z_k} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle \tag{5}$$

$\Rightarrow$ Level zero Kac-Moody algebra. (Additional symmetries !)

✧ Subleading Conformally Soft Theorem :

$$\left\langle R^{0,a}(z,\bar{z}) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

$$= -\sum_{i=1}^{n} \frac{\epsilon_k}{z-z_k}(-2\bar{h}_k + 1 + (\bar{z}-\bar{z}_k)\bar{\partial}_k) T^a_k P^{-1}_k \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle \tag{6}$$

$\Rightarrow$ Current algebra. (More constraints !)

✧ No further soft factorization beyond subleading order for gauge theories.

✧ To get the algebra of infinite tower of soft gluons we have to consider OPE .

What is OPE and Why OPE ?

- **Product of two local operators** as a **sum of local operators at a single point**.

$$\mathcal{O}_i(z_1, \bar{z}_1)\mathcal{O}_j(z_2, \bar{z}_2) = \sum_k c_{ij}^k(z_{12}, \bar{z}_{12})\mathcal{O}_k(z_2) \tag{7}$$

- OPE is an **operator equation** :

$$\langle \mathcal{O}_i(z_1, \bar{z}_1)\mathcal{O}_j(z_2, \bar{z}_2)...\rangle = \sum_k c_{ij}^k(z_{12}, \bar{z}_{12})\langle \mathcal{O}_k(z_2)...\rangle \tag{8}$$

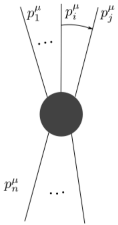
- $(N+1)$-point function $\xrightarrow{\text{OPE}}$ N-point function $\xrightarrow{\text{OPE}}$ ... $\xrightarrow{\text{OPE}}$ combination of OPE coefficients and 2-point function.

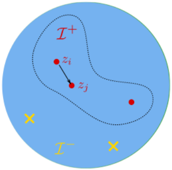
- Physical interpretation :

$\boxed{\textbf{OPE in CCFT} \Leftrightarrow \textbf{collinear limit in the bulk.}}$

How does one find OPE in CCFT ?

✧ OPE can be derived directly from the Celestial amplitude by taking the collinear limit of two gluons.





[Courtesy: Andrea Puhm]

✧ Example :

$$\mathcal{A}(1^{-a_1}, 2^{+a_2}, 3^{+a_3}, 4^{-a_4}) \xrightarrow[p_3 \cdot p_4 = 0]{z_{34} \to 0} -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1)\mathcal{A}(1^{-a_1}, 2^{+a_2}, 4^{-x}) + \text{subleading in } z_{34} + \ldots \tag{9}$$

✧ Leading OPE structure : 2011.00017

$$\mathcal{O}^{a_3}_{\Delta_3,+}(z_3,\bar{z}_3)\mathcal{O}^{a_3}_{\Delta_4,-}(z_4,\bar{z}_4) \sim -\frac{f^{a_3a_4x}}{z_{34}}B(\Delta_3-1,\Delta_4+1)\mathcal{O}^{x}_{\Delta_3+\Delta_4-1,-}(z_4,\bar{z}_4) \tag{10}$$

✧ Subleading,.... terms can be found similarly.

Celestial OPE from asymptotic symmetries

✧ Conformal soft theorems $\Rightarrow$ infinite dimensional asymptotic symmetries $\Rightarrow$ constraints on OPE coefficients.

✧ Celestial OPEs have been computed using these asymptotic symmetries.

(Pate, Raclariu, Strominger and Yuan '19)

(Banerjee, Ghosh, Paul '20)

Null states play an important role...

- Null states are the **primary descendants** of the algebra.
- In CCFT, null states are usually obtained using the OPE and its consistency with the soft factorization theorem.
- Null states inside correlation function $\longrightarrow$ Null decoupling equations (**BG equations**). (Banerjee and Ghosh '20)
- These PDEs have been solved to find scattering amplitude in few cases. (Fan,Fotopoulos,Stieberger,Taylor and Zhu '22)

  (Casali, Melton and Strominger'22)

  (Banerjee, RM, Akavoor and Paul '23)
- Theory of MHV-gravitons and mhv-gluons have been studied in detail.
- Null states will be used to distinguish different S-invariant theories on the Celestial sphere.

Gluons and the S algebra

The S algebra is obtained from the singular part of the OPE i.e.

Guevara, Himwich, Pate and Strominger '21

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)$$
$$\sim -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty}B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2). \tag{11}$$

Soft gluons :

$$R^{k,a}(z,\bar{z}) = \lim_{\Delta\to k}(\Delta-k)O^{a,+}_{\Delta}(z,\bar{z}), \qquad k=1,0,-1,\ldots \tag{12}$$

Holomorphic soft gluon currents :

$$R^{k,a}(z,\bar{z}) = \sum_{n=\frac{k-1}{2}}^{\frac{1-k}{2}}\frac{R^{k,a}_n(z)}{\bar{z}^{n+\frac{k-1}{2}}} \tag{13}$$

Modes of the Holomorphic currents :

$$R_n^{k,a}(z) = \sum_{\alpha \in \mathbb{Z} - \frac{k+1}{2}} \frac{R_{\alpha,n}^{k,a}}{z^{\alpha + \frac{k+1}{2}}} \tag{14}$$

Algebra :

$$[R_{\alpha,m}^{k,a}, R_{\beta,n}^{l,b}] = -if^{ab}{}_c \frac{(\frac{1-k}{2} - m + \frac{1-l}{2} - n)!}{(\frac{1-k}{2} - m)!(\frac{1-l}{2} - n)!} \frac{(\frac{1-k}{2} + m + \frac{1-l}{2} + n)!}{(\frac{1-k}{2} + m)!(\frac{1-l}{2} + n)!} R_{\alpha+\beta,m+n}^{k+l-1,c} \tag{15}$$

Redefinition :

$$S_{\alpha,m}^{q,a} = (q - m - 1)!(q + m - 1)! R_{\alpha,m}^{3-2q,a} \tag{16}$$

S Algebra :

$$[S_{\alpha,m}^{p,a}, S_{\beta,n}^{q,b}] = -if^{abc} S_{\alpha+\beta,m+n}^{p+q-1,c} \tag{17}$$

S algebra primaries

$$R^{k,a}_{p-\frac{k+1}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}(0,0)=0, \qquad p\geq 2 \tag{18}$$

and

$$\begin{aligned} & R^{k,a}_{p-\frac{k+1}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}(0,0) \\ & = -if^{abc}\frac{(-1)^{k+q+1}}{\Gamma(-k-q+2)}\frac{\Gamma(\Delta-1)}{\Gamma(\Delta+q+k-2)}\frac{\bar{\partial}^q}{q!}\mathcal{O}^{c,+}_{\Delta+k-1}(0,0) \end{aligned} \tag{19}$$

where $0\leq q\leq 1-k$, $k=1,0,-1,\ldots$

OPE of two positive helicity outgoing gluons

General structure of the OPE is

$$\begin{aligned}
&\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) \\
&= -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n \mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) \\
&\qquad + \sum_{p,q=0}^{\infty}\sum_{k=1}^{\tilde{n}_{p,q}} z_{12}^p \bar{z}_{12}^q C^k_{p,q}(\Delta_1,\Delta_2)\tilde{\mathcal{O}}^{ab}_{k,p,q}(z_2,\bar{z}_2).
\end{aligned} \tag{20}$$

Task is to determine:

- The OPE coefficients $C^k_{p,q}$ and
- the S-algebra descendants $\tilde{\mathcal{O}}^{ab}_{k,p,q}$ of a positive helicity soft gluon.

STRATEGY :

- ✍ We consider S invariant theories for all of which S-algebra is universal $\Rightarrow$ Existence of a Master OPE .
- ✍ We know that tree-level MHV sector of the pure YM theory is an example of such S invariant theories.
- ✍ This Master OPE inserted in a MHV gluon scattering amplitude $\Rightarrow$ known MHV OPE.
- ✍ $\boxed{\textbf{Master OPE} = \textbf{MHV-sector OPE} + \mathbf{R}}$
- ✍ R should vanish inside MHV scattering amplitude $\Rightarrow$ R is a lin. combination of MHV null states.
- ✍ R consists only non-singular terms.

Null states are important !

Using the above arguments we can rewrite (20) as,

$$\boxed{\begin{aligned}\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{Any Theory}} &= \mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{MHV}} \\ &+ \sum_{p,q=0}^{\infty} z_{12}^{p}\bar{z}_{12}^{q}\sum_{i=1}^{\tilde{n}_{p,q}} \tilde{C}^{k}_{p,q}(\Delta_1,\Delta_2) M^{a,b}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar{z}_2).\end{aligned}} \tag{21}$$

- ☆ $M^{a,b}_{k,p,q}$ are the MHV null states at $\mathcal{O}(z_{12}^{p}\bar{z}_{12}^{q})$.
- ☆ We perform the analysis at $\mathcal{O}(z_{12}^{0}\bar{z}_{12}^{0})$.

The general null state at $\mathcal{O}(1)$ in the MHV-sector is given by

$$\Psi_j^{ab}(\Delta) = R^{-j,a}_{\frac{j-1}{2},\frac{j+1}{2}} \mathcal{O}^b_{\Delta+j,+} - \frac{(-1)^j j}{\Gamma(j+2)} \frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-2)} R^{1,a}_{-1,0} \mathcal{O}^b_{\Delta-1,+} - \frac{(-1)^j}{\Gamma(j+1)} \frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)} R^{0,a}_{-1/2,1/2} \mathcal{O}^b_{\Delta,+} \tag{22}$$

where $j = 1, 2, 3, \ldots$

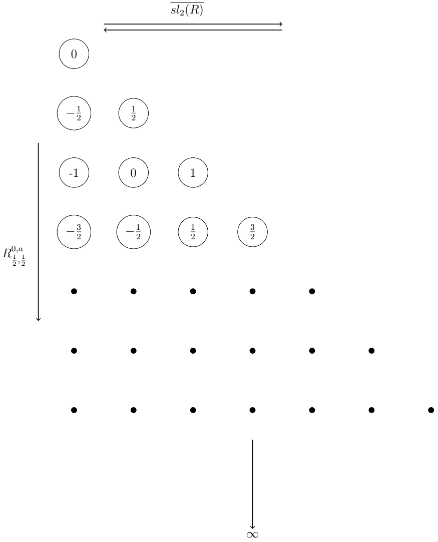
Let's consider the following basis :

$$M_k^{ab}(\Delta) = \sum_{i=1}^{k} \frac{1}{\Gamma(k-i+1)} \frac{\Gamma(\Delta+k-1)}{\Gamma(\Delta+i-1)} \Psi_i^{ab}(\Delta). \tag{23}$$

Also define,

$$M_k^a(\Delta) = f^{abc} M_k^{bc}. \tag{24}$$

✧ Focus only on the generators $(R^{1,a}_{n,0}, R^{0,a}_{\frac{1}{2},\frac{1}{2}}, H^0_{0,0}, H^0_{0,\pm 1})$ to study the action of S-algebra on the null states.



Action of the S algebra on the MHV null states

Action of the Leading Soft Gluon modes:

$$R^{1,a}_{0,0} M^{bc}_{k}(\Delta) = -if^{abd} M^{dc}_{k}(\Delta) - if^{acd} M^{bd}_{k} \tag{25}$$

$$R^{1,a}_{n,0} M^{bc}_{k}(\Delta) = 0, \, n > 0 \tag{26}$$

Action of the Subleading Soft Gluon mode :

$$\begin{aligned} &[R^{0,a}_{1/2,1/2}, M^{bc}_{k}(\Delta)] \\ &= -if^{abd}(k+2) M^{dc}_{k+1}(\Delta - 1) \\ &\qquad + (\Delta + k - 2)\left\{ if^{acd} M^{bd}_{k}(\Delta - 1) + if^{abd} M^{dc}_{k}(\Delta - 1) \right\}. \end{aligned} \tag{27}$$

Building up S invariant OPE

Observation :

- Consider the following set of null states,

$$M_k^{bc}(\Delta), k = 1, 2, 3, ..., n. \tag{28}$$

- Action of $R^{0,a}_{1/2,1/2}$ on the null states is closed if we set

$$M_{k+1}^{ab}(\Delta) = 0, k \geq n \geq 0. \tag{29}$$

Inference :

- We can get an S invariant OPE if we consider the finite set of null states (28).

S invariant OPEs at $\mathcal{O}(1)$:

$$\boxed{\begin{aligned} &\mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)|_{\mathcal{O}(1)} \\ &= \mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)\Big|^{MHV}_{\mathcal{O}(1)} + \sum_{k=1}^{n} B(\Delta_1+k,\Delta_2-1)M_k^{ab}(\Delta_1+\Delta_2)\end{aligned}} \tag{30}$$

Theory 1: MHV gluons

- $n = 0$ (trivial one)
- **OPE**

$$\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)\Big|^{\mathrm{MHV}}_{\mathcal{O}(1)}$$

$$= B(\Delta_1 - 1, \Delta_2 - 1)\left[\Delta_1 R^{1,a}_{-1,0}\mathcal{O}^{b,+}_{\Delta_1+\Delta_2-1}(0,0) + \frac{\Delta_1 - 1}{\Delta_1 + \Delta_2 - 2} R^{0,a}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{b,+}_{\Delta_1+\Delta_2}(0,0)\right] \tag{31}$$

Theory 2: SDYM

- $n = 1$
- **OPE**

$$\mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)|_{\mathcal{O}(1)}$$

$$= \mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)\Big|^{MHV}_{\mathcal{O}(1)} + B(\Delta_1 + 1, \Delta_2 - 1)M^{ab}_1(\Delta_1 + \Delta_2) \tag{32}$$

Action of $R^{0,a}_{1/2,1/2}$:

$$R^{0,x}_{\frac{1}{2},\frac{1}{2}}(\mathcal{O}^{a}_{\Delta_1,+}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2,+}(0,0))|_{\mathcal{O}(1)} - R^{0,x}_{\frac{1}{2},\frac{1}{2}}\Bigg[\mathcal{O}^{a}_{\Delta_1,+}(z,\bar{z})\mathcal{O}^{b}_{\Delta_2,+}(0,0)|^{MHV}_{\mathcal{O}(1)} + \sum_{k=1}^{n} B(\Delta_1+k,\Delta_2-1)M^{ab}_{k}(\Delta_1+\Delta_2)\Bigg]$$
$$= if^{xay}(n+2)B(\Delta_1+n,\Delta_2-1)M^{yb}_{n+1}(\Delta_1+\Delta_2-1) = 0. \tag{33}$$

- One can also verify that OPE (30) is invariant under the action of $R^{1,a}_{n,0}$ and $H^{0}_{0,1}$.
- Truncated OPE (30) is invariant under the S algebra.

Infinite family of S invariant theories :

- We have shown that the following set of equations are S invariant.

$$M^{ab}_{k+1}(\Delta) = 0, k \geq n \geq 0. \tag{34}$$

- We can truncate the OPE at $\mathcal{O}(1)$ at an arbitrary n in S invariant way.
- But the S invariance does not fix the value of integer n.
- Hence, different choices of the integer n give rise to a discrete infinite family of S-invariant OPEs.
- Each of these consistent OPEs correspond to a S invariant theory.
- We do not know the Lagrangian description of these theories except for for the MHV YM and the self-dual Yang-Mills theory.

- KZ-type null states involve the L_{-1} descendants on the CS^2.
- We obtain KZ-type null states by using OPE commutivity and taking different soft limits.

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) = \mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1). \tag{35}$$

- **KZ-type null states :**

$$K^a(\Delta) = \xi^a(\Delta) - i\sum_{k=1}^{n} M^a_k(\Delta+1), \tag{36}$$

where

$$\xi^a(\Delta) = C_A L_{-1}\mathcal{O}^{a,+}_{\Delta} - (\Delta+1)R^{1,b}_{-1,0}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta} - R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta+1}. \tag{37}$$

- These null states are also invariant under S algebra.

Outlook

- One can think of investigating the bulk theories for other values of $n \geq 2$.
- KZ-type null states are already found for all such theories, so they can be of some help investigating other theories.
- Only a finite number of descendants contribute to the subleading OPE $\Rightarrow$ need a reformulation of CCFT where Δ are discrete and bounded from below(?).
- Could there exist theories which are S invariant on CS^2 but not Lorentz invariant? Can we give any physical interpretations for those?
- Or can we rule out the S-invariant theories which do not have bulk-Lorentz invariance ?
- Could S-invariant non-Lorentz invariant theories arise from the SSB of Lorentz-invariant theories?

- S-invariance $\Rightarrow$ constraints on the Lagrangian formulations of these theories ?
- In celestial CFT the spectrum of operator dimensions is same for every S-inv theory $\Rightarrow$ different theories are not distinguished by their operator spectrum but by their null states $\Rightarrow$ Any lagrangian formulation of such CFT has to produce all the correct null states which might be useful to contrain the form of the Lagrangian.

Thank you for your attention !!