An Infinite Family of S Invariant Theories on the Celestial Sphere

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Based on
"All OPEs invariant under the infinite symmetry algebra for gluons on the celestial sphere"
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with
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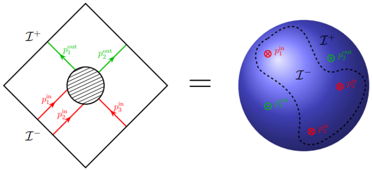
OUTLINE

- Celestial amplitude
- Factorizations of the celestial amplitude
- Introduction to the S algebra
- General structure of the gluon-gluon OPE
- Goal of the project
- Strategy
- Implementation of the idea
- Remarks

CELESTIAL HOLOGRAPHY:

$\boxed{\text{A quantum theory of gravity in (3+1)D AFS}} \equiv \boxed{\text{A 2D Celestial CFT on } CS^2 \text{ at null infinity}}$



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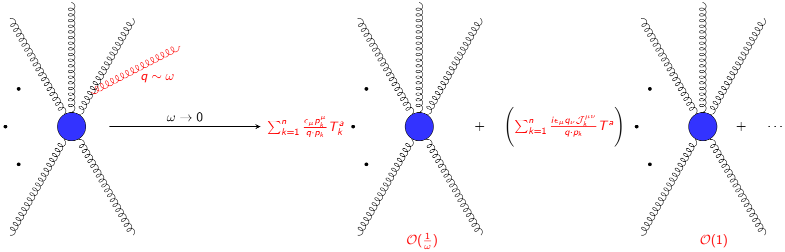
$$\langle out|\, S\, |in\rangle \qquad \langle \mathcal{O}^{\pm}_{\Delta_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{\pm}_{\Delta_n}(z_n,\bar{z}_n)\rangle \tag{1}$$

CELESTIAL AMPLITUDE/ CELESTIAL CORRELATION FUNCTION:

$$\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \prod_{i=1}^{n}\int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1}\boxed{\mathcal{S}_n(\{\omega_i, z_i, \bar{z}_i, \sigma_i, a_i\})} \tag{2}$$

SOFT FACTORIZATION THEOREM

tree level



(3)

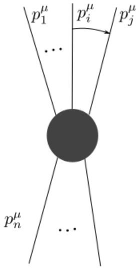
Mellin transform $\Downarrow$

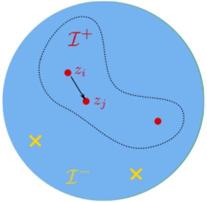
$$\boxed{\text{Ward identities of the conformally soft gluon currents on } CS^2}$$

$\Downarrow$ modes of the soft gluon currents

$$\boxed{\text{Holographic symmetry algebra}}$$

COLLINEAR FACTORIZATION AND CELESTIAL OPE





[image courtesy: Andrea Puhm]

$$\mathcal{A}(1^{-a_1}, 2^{+a_2}, 3^{+a_3}, 4^{-a_4}) \xrightarrow[p_3 \cdot p_4 = 0]{z_{34} \to 0} -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1)\mathcal{A}(1^{-a_1}, 2^{+a_2}, 4^{-x}) \quad (4)$$
$$+ \text{subleading in } z_{34} + \ldots$$

$$\downarrow$$

$$\mathcal{O}^{a_3}_{\Delta_3,+}(z_3, \bar{z}_3)\mathcal{O}^{a_3}_{\Delta_4,-}(z_4, \bar{z}_4) \sim -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1)\mathcal{O}^{x}_{\Delta_3+\Delta_4-1,-}(z_4, \bar{z}_4) \quad (5)$$

The S algebra is obtained from the singular part of the OPE between two **positive helicity outgoing gluons** on the celestial sphere :

Guevara, Himwich, Pate and Strominger '21

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2). \tag{6}$$

Infinite tower of soft gluons:

$$R^{k,a}(z,\bar{z}) = \lim_{\Delta\to k}(\Delta-k)O^{a,+}_{\Delta}(z,\bar{z}), \qquad k=1,0,-1,\dots \tag{7}$$

Holomorphic soft gluon currents:

$$R^{k,a}(z,\bar{z}) = \sum_{n=\frac{k-1}{2}}^{\frac{1-k}{2}} \frac{R^{k,a}_n(z)}{\bar{z}^{n+\frac{k-1}{2}}} \tag{8}$$

Modes of the Holomorphic currents :

$$R_n^{k,a}(z) = \sum_{\alpha \in \mathbb{Z} - \frac{k+1}{2}} \frac{R_{\alpha,n}^{k,a}}{z^{\alpha + \frac{k+1}{2}}} \tag{9}$$

Algebra :

$$[R_{\alpha,m}^{k,a}, R_{\beta,n}^{l,b}] = -if^{ab}{}_c \frac{(\frac{1-k}{2} - m + \frac{1-l}{2} - n)!}{(\frac{1-k}{2} - m)!(\frac{1-l}{2} - n)!} \frac{(\frac{1-k}{2} + m + \frac{1-l}{2} + n)!}{(\frac{1-k}{2} + m)!(\frac{1-l}{2} + n)!} R_{\alpha+\beta,m+n}^{k+l-1,c} \tag{10}$$

Redefinition :

$$S_{\alpha,m}^{q,a} = (q - m - 1)!(q + m - 1)! R_{\alpha,m}^{3-2q,a} \tag{11}$$

S Algebra :

$$[S_{\alpha,m}^{p,a}, S_{\beta,n}^{q,b}] = -if^{abc} S_{\alpha+\beta,m+n}^{p+q-1,c} \tag{12}$$

Guevara, Himwich, Pate and Strominger '21

GENERAL STRUCTURE OF THE OPE

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) = -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)$$
$$+\sum_{p,q=0}^{\infty}\sum_{k=1}^{\tilde{n}_{p,q}} z_{12}^p\bar{z}_{12}^q C^k_{p,q}(\Delta_1,\Delta_2)\tilde{\mathcal{O}}^{ab}_{k,p,q}(z_2,\bar{z}_2). \tag{13}$$

GOAL

1. To determine the OPE coefficients $C^k_{p,q}$
2. the S-algebra descendants $\tilde{\mathcal{O}}^{ab}_{k,p,q}$ of a positive helicity soft gluon at $\mathcal{O}(1)$
3. to argue that there exists a discrete infinite number of S invariant OPEs which correspond to infinite number of theories
4. show the invariance of the OPEs under the action of S algebra
5. Also to find the KZ-type null states of such theories.

STRATEGY

- We consider S invariant theories (OPEs) for all of which S-algebra is universal
  ➥ Existence of a Master OPE .
- We know that tree-level MHV sector of the pure YM theory is an example of such S invariant theories.
- This Master OPE inserted in a MHV gluon scattering amplitude $\Rightarrow$ known MHV OPE.
- $\boxed{\textbf{Master OPE} = \textbf{MHV-sector OPE} + \mathbf{R}}$
- R should vanish inside MHV scattering amplitude $\Rightarrow$ R is a lin. combination of MHV null states.

ROLE OF THE NULL STATES

Using the above arguments we can rewrite (13) as,

$$\boxed{\begin{aligned}\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{Any Theory}} &= \mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{MHV}} \\ &+ \sum_{p,q=0}^{\infty} z_{12}^{p}\bar{z}_{12}^{q}\sum_{i=1}^{\tilde{n}_{p,q}} \tilde{C}^{k}_{p,q}(\Delta_1,\Delta_2)M^{a,b}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar{z}_2).\end{aligned}} \tag{14}$$

- $M^{a,b}_{k,p,q}$ are the MHV null states at $\mathcal{O}(z_{12}^{p}\bar{z}_{12}^{q})$.
- We perform the analysis at $\mathcal{O}(z_{12}^{0}\bar{z}_{12}^{0})$.

STEP 1: FINDING THE NULL STATES

The general null state at $\mathcal{O}(1)$ in the MHV-sector is given by

$$\Psi_j^{ab}(\Delta) = R^{-j,a}_{\frac{j-1}{2},\frac{j+1}{2}}\mathcal{O}^b_{\Delta+j,+} - \frac{(-1)^j j}{\Gamma(j+2)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-2)}R^{1,a}_{-1,0}\mathcal{O}^b_{\Delta-1,+} - \frac{(-1)^j}{\Gamma(j+1)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)}R^{0,a}_{-1/2,1/2}\mathcal{O}^b_{\Delta,+} \tag{15}$$

where $j = 1, 2, 3, \dots$

Let's consider the following basis:

$$M_k^{ab}(\Delta) = \sum_{i=1}^{k}\frac{1}{\Gamma(k-i+1)}\frac{\Gamma(\Delta+k-1)}{\Gamma(\Delta+i-1)}\Psi_i^{ab}(\Delta). \tag{16}$$

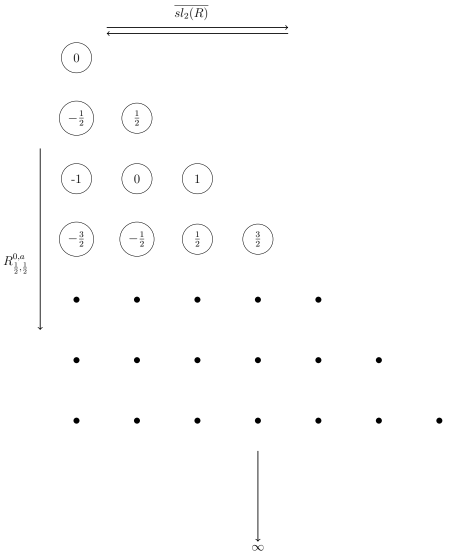
Also define,

$$M_k^a(\Delta) = f^{abc}M_k^{bc}. \tag{17}$$

STEP 2: WHICH GENERATORS? THERE ARE INFINITELY MANY!

To study the action of the S algebra, we need to focus on the action of the following set of generators only

$$\boxed{R^{1,a}_{n,0},\ R^{0,a}_{\frac{1}{2},\frac{1}{2}},\ H^{0}_{0,0},\ H^{0}_{0,\pm 1}}$$



STEP 3: ACTION OF THE S ALGEBRA ON THE MHV NULL STATES

Action of the Leading Soft Gluon mode:

$$R^{1,a}_{0,0} M^{bc}_k(\Delta) = -if^{abd} M^{dc}_k(\Delta) - if^{acd} M^{bd}_k \tag{18}$$

$$R^{1,a}_{n,0} M^{bc}_k(\Delta) = 0, n > 0 \tag{19}$$

Action of the Subleading Soft Gluon mode :

$$[R^{0,a}_{1/2,1/2}, M^{bc}_k(\Delta)]$$
$$= -if^{abd}(k+2)M^{dc}_{k+1}(\Delta - 1) + (\Delta + k - 2)\left\{ if^{acd} M^{bd}_k(\Delta - 1) + if^{abd} M^{dc}_k(\Delta - 1) \right\}. \tag{20}$$

STEP 4: BUILDING UP THE S INVARIANT OPEs

Observation:

- Consider the following set of null states,

$$M_k^{bc}(\Delta), k = 1, 2, 3, ..., n. \tag{21}$$

- Action of $R^{0,a}_{1/2,1/2}$ on the null states is closed if we set

$$M^{ab}_{k+1}(\Delta) = 0, k \geq n \geq 0. \tag{22}$$

Inference:

- We can get an S invariant OPE if we consider the finite set of null states (21).

S INVARIANT OPEs AT $\mathcal{O}(1)$

$$\boxed{\begin{aligned} &\mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)|_{\mathcal{O}(1)} \\ &= \mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)\Big|^{MHV}_{\mathcal{O}(1)} + \sum_{k=1}^{n} B(\Delta_1+k, \Delta_2-1) M_k^{ab}(\Delta_1+\Delta_2) \end{aligned}} \tag{23}$$

STEP 5: CHECKING THE S INVARIANCE OF THE OPEs

Action of $R^{0,a}_{1/2,1/2}$:

$$R^{0,x}_{\frac{1}{2},\frac{1}{2}}(\mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2,+}(0,0))|_{\mathcal{O}(1)} - R^{0,x}_{\frac{1}{2},\frac{1}{2}}\Bigg[\mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)|^{MHV}_{\mathcal{O}(1)} + \sum_{k=1}^{n} B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2)\Bigg]$$
$$= if^{xay}(n+2)B(\Delta_1+n,\Delta_2-1)M^{yb}_{n+1}(\Delta_1+\Delta_2-1) = 0. \tag{24}$$

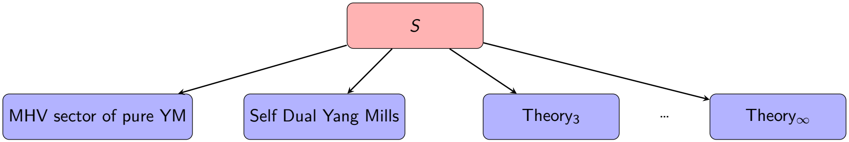
- One can also verify that OPE (23) is invariant under the action of $R^{1,a}_{n,0}$ and $H^0_{0,1}$.
- Truncated OPE (23) is invariant under the S algebra.

STEP 6: AN INFINITE FAMILY OF S INVARIANT THEORIES

- We have shown that the following set of equations are S invariant.

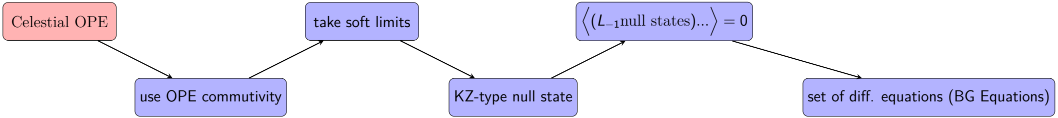
$$M^{ab}_{k+1}(\Delta) = 0, k \geq n \geq 0. \tag{25}$$

- We can truncate the OPE at $\mathcal{O}(1)$ at an arbitrary n in S invariant way.
- But the S invariance does not fix the value of integer n.
- Hence, different choices of the integer n give rise to a discrete infinite family of S-invariant OPEs.
- Each of these consistent OPEs correspond to a S invariant theory.
- Bulk theories?



STEP 7: KNIZHNIK-ZAMOLODCHIKOV(KZ) TYPE NULL STATES

- KZ-type null states involve the L_{-1} descendants on the CS^2.



- **KZ-type null states :**

$$K^a(\Delta) = \xi^a(\Delta) - i\sum_{k=1}^{n} M_k^a(\Delta+1), \tag{26}$$

where

$$\xi^a(\Delta) = C_A L_{-1}\mathcal{O}_{\Delta}^{a,+} - (\Delta+1)R_{-1,0}^{1,b}R_{0,0}^{1,b}\mathcal{O}_{\Delta}^{a,+} - R_{-\frac{1}{2},\frac{1}{2}}^{0,b}R_{0,0}^{1,b}\mathcal{O}_{\Delta+1}^{a,+}. \tag{27}$$

Banerjee, Ghosh 2021

- These null states would be helpful to find the other bulk theories.

REMARKS:

1. Only a finite number of the descendants contribute to the $\mathcal{O}(1)$ OPE even though $\Delta(= 1 + i\mathbb{R})$ is not bounded from below $\longrightarrow$ reformulation of CCFT?

2. To prove the S invariance, we assumed the Lorentz invariance of the bulk theory.
   - ➥ interpretation of the S invariant theories on CS^2 that lack the bulk Lorentz invariance?
   - ➥ can we find the reasons to rule out the S-invariant non-Lorentz-invariant theories on the celestial sphere?

3. Constraints on the Lagrangian formulation of the S invariant field theories?

4. In CCFT, the spectrum of the operator dimensions is the same for every theory. Different theories are distinguished by the null states, not by the operator spectrum.
   - ➥ Lagrangian formulation has to produce all the correct null states.

Thank you for your attention !!