An Infinite Family of S Invariant Theories on the Celestial Sphere

Shamik Banerjee [1,2] Raju Mandal [1,2] Sagnik Misra [1,2] Sudhakar Panda [3]
Partha Paul [1,4]

[1]HBNI, Mumbai [2]NISER, Bhubaneswar [3]CCSP, Gurugram [4]IMSC, Chennai

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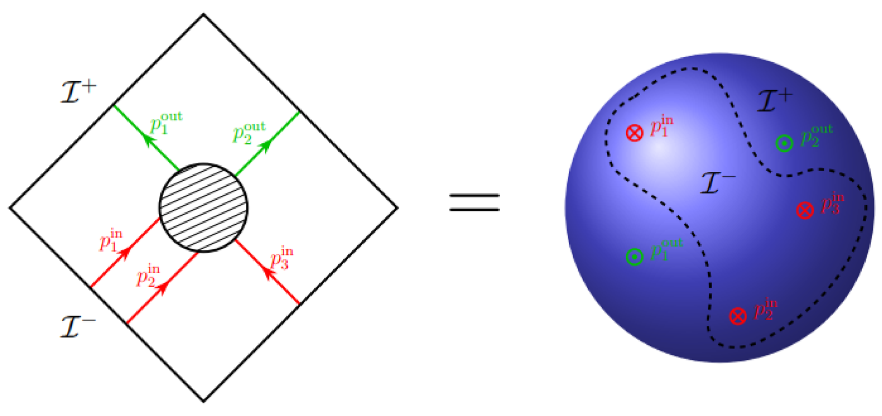
Celestial CFT : A Putative Dual for QG in AFS

A quantum theory of gravity in (3+1)-D AFS $\Longleftrightarrow$ A 2D CCFT on the Celestial sphere at null infinity.



$$\langle out|\, S\, |in\rangle \qquad \langle \mathcal{O}^{\pm}_{\Delta_1}(z_1,\bar z_1)...\mathcal{O}^{\pm}_{\Delta_n}(z_n,\bar z_n)\rangle \tag{1}$$

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Holographic Symmetry Algebras for Gravity and Gauge Theories in AFS

✧ Conformally soft theorems $\longrightarrow$ Infinite dimensional non-trivial symmetries in Gauge theories and Gravity in 4D AFS.

✧ Gravity $\longrightarrow$ wedge subalgebra of $w_{1+\infty}$ algebra. (Guevara,Himwich,Pate and Strominger '21)

✧ Gauge Theories $\longrightarrow$ S algebra

Goal and the Motivation

✧ Goal $\Rightarrow$ Our goal is to classify the theories which are invariant under **S algebra** and also to find the **KZ**-type null states of these theories. 2311.16796 (Banerjee, RM, Misra, Panda and Paul '23)

✧ Banerjee, Kulkarni and Paul '23 $\Rightarrow$ computed G^+G^+ OPE and KZ-type null states of any $w_{1+\infty}$ invariant theories 2301.13225; 2311.06485
$\Rightarrow$ existence of a discrete infinite family of $w_{1+\infty}$ invariant theories on the celestial sphere.

Gluons and the S Algebra

The S algebra is obtained from the singular part of the OPE i.e.
Guevara, Himwich, Pate and Strominger '21

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar z_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar z_2) \sim -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+n-1,\Delta_2-1)\frac{\bar z_{12}^n}{n!}\bar\partial_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar z_2). \tag{2}$$

Soft gluons :

$$R^{k,a}(z,\bar z) = \lim_{\Delta\to k}(\Delta-k)O^{a,+}_{\Delta}(z,\bar z), \quad k=1,0,-1,... \tag{3}$$

Holomorphic soft gluon currents :

$$R^{k,a}(z,\bar z) = \sum_{n=\frac{k-1}{2}}^{\frac{1-k}{2}} \frac{R^{k,a}_n(z)}{\bar z^{n+\frac{k-1}{2}}} \tag{4}$$

Modes of the Holomorphic currents :

$$R^{k,a}_n(z) = \sum_{\alpha\in\mathbb{Z}-\frac{k+1}{2}} \frac{R^{k,a}_{\alpha,n}}{z^{\alpha+\frac{k+1}{2}}} \tag{5}$$

Algebra :

$$[R^{k,a}_{\alpha,m}, R^{l,b}_{\beta,n}] = -if^{ab}{}_c\frac{(\frac{1-k}{2}-m+\frac{1-l}{2}-n)!}{(\frac{1-k}{2}-m)!(\frac{1-l}{2}-n)!}\frac{(\frac{1-k}{2}+m+\frac{1-l}{2}+n)!}{(\frac{1-k}{2}+m)!(\frac{1-l}{2}+n)!}R^{k+l-1,c}_{\alpha+\beta,m+n} \tag{6}$$

Redefinition :

$$S^{q,a}_{\alpha,m} = (q-m-1)!(q+m-1)!R^{3-2q,a}_{\alpha,m} \tag{7}$$

S Algebra :

$$[S^{p,a}_{\alpha,m}, S^{q,b}_{\beta,n}] = -if^{abc}S^{p+q-1,c}_{\alpha+\beta,m+n} \tag{8}$$

General Structure of the Gluon-Gluon OPE

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar z_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar z_2)$$
$$= -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty}B(\Delta_1+n-1,\Delta_2-1)\frac{\bar z_{12}^n}{n!}\bar\partial_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar z_2) + \sum_{p,q=0}^{\infty}\sum_{k=1}^{\tilde n_{p,q}} z_{12}^p\bar z_{12}^q C^k_{p,q}(\Delta_1,\Delta_2)\tilde{\mathcal{O}}^{ab}_{k,p,q}(z_2,\bar z_2). \tag{9}$$

Task is to determine:

✍ The OPE coefficients $C^k_{p,q}$ and

✍ the S-algebra descendants $\tilde{\mathcal{O}}^{ab}_{k,p,q}$ of a positive helicity soft gluon.

Strategy

✍ We consider S invariant theories for all of which S-algebra is universal $\Rightarrow$ Existence of a Master OPE .

✍ We know that tree-level MHV sector of the pure YM theory is an example of such S invariant theories.

✍ This Master OPE inserted in a MHV gluon scattering amplitude $\Rightarrow$ known MHV OPE.

✍ $\boxed{\text{Master OPE} = \text{MHV-sector OPE} + \text{R}}$

✍ R should vanish inside MHV scattering amplitude $\Rightarrow$ R is a lin. combination of MHV null states.

✍ R consists only non-singular terms.

OPEs in Terms of MHV Null States

Using the above arguments we can rewrite (9) as,

$$\boxed{\begin{aligned}\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar z_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar z_2)|_{\text{Any Theory}} &= \mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar z_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar z_2)|_{\text{MHV}} \\ &+ \sum_{p,q=0}^{\infty} z_{12}^p\bar z_{12}^q\sum_{i=1}^{\tilde n_{p,q}}\tilde C^k_{p,q}(\Delta_1,\Delta_2)M^{a,b}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar z_2).\end{aligned}} \tag{10}$$

MHV Null States at $\mathcal{O}(1)$

The general null state at $\mathcal{O}(1)$ in the MHV-sector is given by

$$\Psi^{ab}_j(\Delta) = R^{-j,a}_{\frac{j-1}{2},\frac{j+1}{2}}\mathcal{O}^b_{\Delta+j,+} - \frac{(-1)^j j}{\Gamma(j+2)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-2)}R^{1,a}_{-1,0}\mathcal{O}^b_{\Delta-1,+} - \frac{(-1)^j}{\Gamma(j+1)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)}R^{0,a}_{-1/2,1/2}\mathcal{O}^b_{\Delta,+} \tag{11}$$

where $j=1,2,3,...$
Let's consider the following basis :

$$M^{ab}_k(\Delta) = \sum_{i=1}^{k}\frac{1}{\Gamma(k-i+1)}\frac{\Gamma(\Delta+k-1)}{\Gamma(\Delta+i-1)}\Psi^{ab}_i(\Delta). \tag{12}$$

Action of the S algebra on the MHV Null States

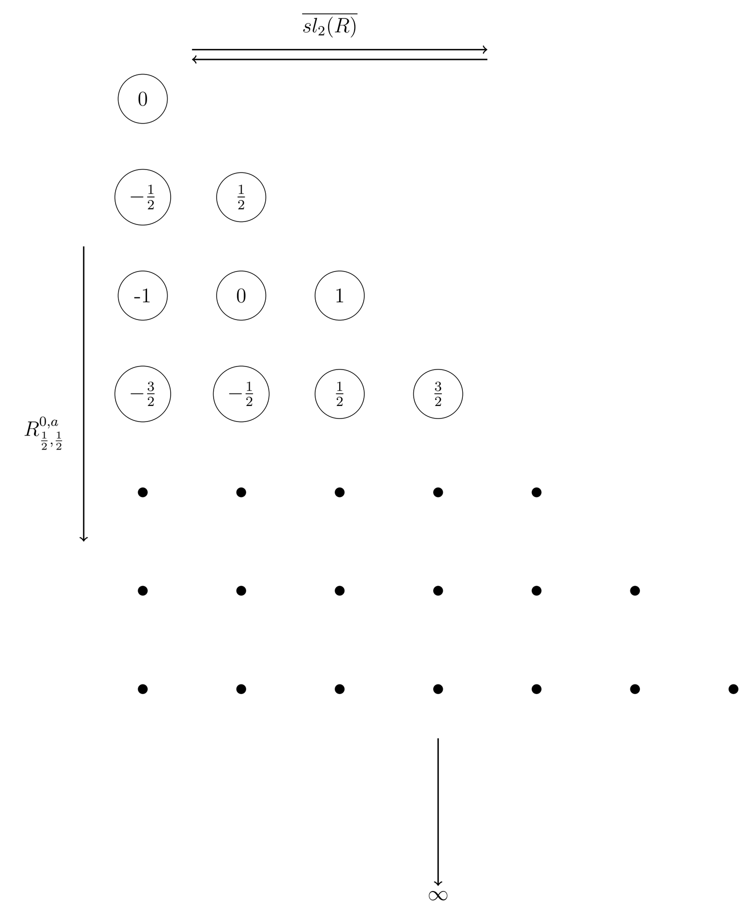
Action of the Leading Soft Gluon modes:

$$R^{1,a}_{0,0}M^{bc}_k(\Delta) = -if^{abd}M^{dc}_k(\Delta) - if^{acd}M^{bd}_k \tag{13}$$

$$R^{1,a}_{n,0}M^{bc}_k(\Delta) = 0, n>0 \tag{14}$$

Action of the Subleading Soft Gluon mode :

$$[R^{0,a}_{1/2,1/2}, M^{bc}_k(\Delta)] = -if^{abd}(k+2)M^{dc}_{k+1}(\Delta-1) + (\Delta+k-2)\left\{if^{acd}M^{bd}_k(\Delta-1) + if^{abd}M^{dc}_k(\Delta-1)\right\}. \tag{15}$$



Building up S invariant OPEs

Observation :

➠ Consider the following set of null states,

$$M^{bc}_k(\Delta), k=1,2,3,...,n. \tag{16}$$

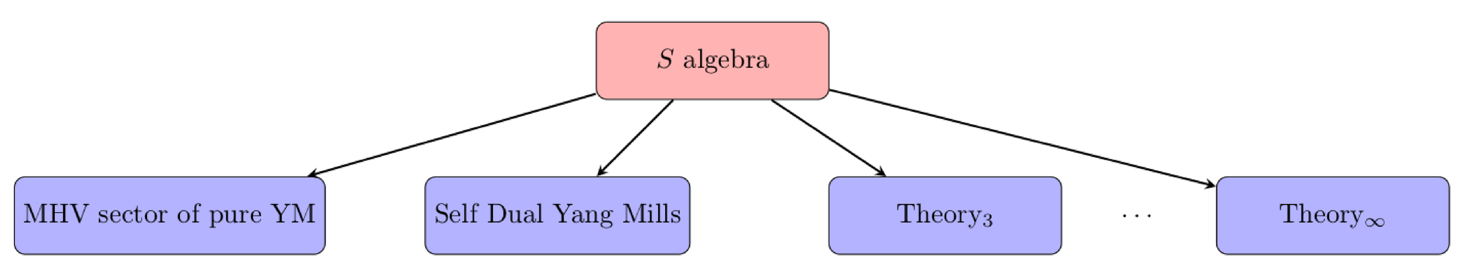
➠ Action of $R^{0,a}_{1/2,1/2}$ on the null states is closed if we set

$$M^{ab}_{k+1}(\Delta) = 0, k\geq n\geq 0. \tag{17}$$

Inference :

➠ We can get an S invariant OPE if we consider the finite set of null states (16).

$$\boxed{\mathcal{O}^a_{\Delta_1,+}(z,\bar z)\mathcal{O}^b_{\Delta_2,+}(0,0)|_{\mathcal{O}(1)} = \mathcal{O}^a_{\Delta_1,+}(z,\bar z)\mathcal{O}^b_{\Delta_2,+}(0,0)\Big|^{MHV}_{\mathcal{O}(1)} + \sum_{k=1}^{n}B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2)} \tag{18}$$



Knizhnik-Zamolodchikov Type Null States

$$\boxed{K^a(\Delta) = \xi^a(\Delta) - i\sum_{k=1}^{n}M^a_k(\Delta+1)}, \tag{19}$$

where

$$\xi^a(\Delta) = C_A L_{-1}\mathcal{O}^{a,+}_{\Delta} - (\Delta+1)R^{1,b}_{-1,0}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta} - R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta+1} \tag{20}$$

and

$$M^a_k(\Delta) = f^{abc}M^{bc}_k. \tag{21}$$

References

S. Banerjee, R. Mandal, S. Misra, S. Panda and P. Paul, "All OPEs invariant under the infinite symmetry algebra for gluons on the celestial sphere," [PhysRevD.110.026020]

Email : raju.mandal@niser.ac.in