Gauge Theories and The Aspects of Flatspace Holography

State-of-Art Seminar
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Outline :

- Motivations
- Introduction to the Flatspace Holography
- Soft Theorems $\longrightarrow$ Conformal Ward Identities
- Celestial OPE of Gluons
- Differential Equations for MHV Gluon Amplitudes
- HSA for $SU(N)$ Gauge Theories
- Discussion on the ongoing work...

Motivations

In search for the symmetries and where symmetries lead us to...

Asymptotic Symmetries of Flat Spacetime

- '60s : Poincare $\longrightarrow$ BMS $=$ Lorentz $\times$ Supertranslations

  Bondi,Burg,Metzner and Sachs

- '10s : BMS $\longrightarrow$ Virasoro $\ltimes$ Supertranslations (Gravity)
  (using asymptotic symmetry analysis and soft theorems.)

  Lorentz group $\longrightarrow$ Virasoro Kapec,Mitra,Strominger,...

  Another extension :

  Lorentz group $\longrightarrow$ $\overline{SL(2,\mathbb{C})}$ current algebra Banerjee,Ghosh,Paul '20

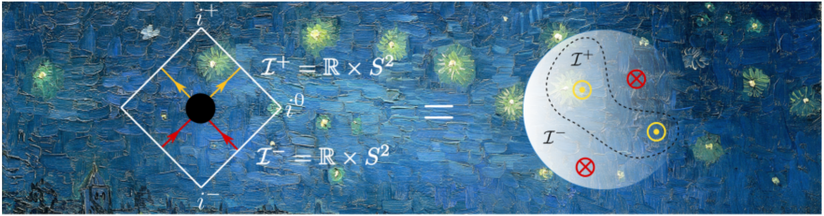
  $\downarrow$ [Guevara-Himwich-Pate-Strominger '21]

  wedge subalgebra of $w_{1+\infty}$ algebra (Gravity)

- **Gauge Theories** $\longrightarrow$ HSA $\times$ Poincare. (asymptotic symmetries)
- Implications of these infinite symmetries ...

Conjecture :

Theories of QG in (3+1)D AFS $\Longleftrightarrow$ 2D CCFTs on the Celestial sphere at null infinity.



[Courtesy: Andrea Puhm]

Null Infinity:

- $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
  $= -du^2 - 2dudr + r^2(d\theta^2 + sin^2\theta d\phi^2)$
  $= -du^2 - 2dudr + r^2 d\Omega_2^2$ where $u = (t - r)$

- In $(u, r, z, \bar{z})$ coordinates, Bondi Coordinates

  $ds^2 = -du^2 - 2dudr + r^2 \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$

  Where, $z = \frac{x^1+ix^2}{r+x^3} = e^{i\phi}\tan\frac{\theta}{2}$, $\bar{z} = c.c$

- We can go to future null-infinity by taking $r \longrightarrow \infty$ at fixed $(u, z, \bar{z})$

  $d\tilde{s}^2 = d\Omega_2^2$

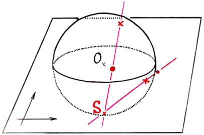
Action of 4D Lorentz group $\sim$ Action of global conformal group on $\mathbb{C}S^2$

$$SO(3,1)^{\uparrow} \cong SL(2,\mathbb{C})/\mathbb{Z}_2$$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \text{ with } det(\Lambda) = 1$$

$$x = \begin{pmatrix} x^0 - x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 + x^3 \end{pmatrix} \qquad z = \frac{x^1+ix^2}{r+x^3} \ , \ \bar{z} \longrightarrow c.c$$

$$x \longrightarrow x' = \Lambda x \Lambda^{\dagger} \qquad z \rightarrow z' = \frac{az+b}{cz+d} \ , \ c.c$$



[Courtesy : B. Oblak]

Celestial Amplitude : S-matrix in Boost Eigenbasis:

S-matrix in Momentum space $\xrightarrow[\text{[Pasterski,Shao]}]{\text{Mellin transform}}$ **Correlation function of 2D Celestial CFT**

$$\prod_{i=1}^{n}\int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1}\mathcal{S}_n(\{\omega_i, z_i, \bar{z}_i, \sigma_i, a_i\}) = \langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle$$

(Gauge theory)

massless $\quad p^\mu = \omega(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})$

conformal weights : $h_i = \frac{\Delta_i+\sigma_i}{2}$, $\bar{h}_i = \frac{\Delta_i-\sigma_i}{2}$ and
helicity : $\sigma_i = \pm 1$ (for gluons).

$$\Delta = 1 + i\mathbb{R} \text{ (Unitary principal series representation)}$$

Under Lorentz transformations, Scattering amplitudes transform as correlation functions of 2D CFT !

$$\mathcal{A}_n(\{z_i, \bar{z}_i, h_i, \bar{h}_i, a_i\}) \\ = \prod_{i=1}^{n} \frac{1}{(cz_i+d)^{2h_i}} \frac{1}{(\bar{c}\bar{z}_i+\bar{d})^{2\bar{h}_i}} \mathcal{A}_n\left(\left\{\frac{az_i+b}{cz_i+d}, \frac{\bar{a}\bar{z}_i+\bar{b}}{\bar{c}\bar{z}_i+\bar{d}}, h_i, \bar{h}_i, a_i\right\}\right).$$

Example : [S. Banerjee and S. Ghosh ; arXiv:2011.00017]

$$\mathcal{A}(1^{-a_1}, 2^{+a_2}, 4^{-x}) \sim f^{a_1 a_2 x} \frac{z_{14}^3}{z_{12}^3 z_{24}} \underbrace{\delta(\bar{z}_{14})\delta(\bar{z}_{24})} \prod_{i=1}^{3} \Theta(\epsilon_i \sigma_{i,1})$$

presence of delta distributions makes it look different from the usual 2d cft-amplitudes !

- break translation and scale invariances by coupling the theory with background field $\longrightarrow$ distribution functions are replaced by some smooth functions.

[Casali-Melton-Strominger '22] [Banerjee-Paul-Akavoor-RM '23]

Momentum space soft theorems:

At tree level

$$\mathcal{M}^a_{n+1}(p_1,...,p_n,q) \xrightarrow{\omega\to 0} (\underset{\mathcal{O}(\frac{1}{\omega})}{S^a_{(0)}} + \underset{\mathcal{O}(1)}{S^a_{(1)}})\mathcal{M}_n(p_1,...,p_n) + \mathcal{O}(q)$$

no more factorisation

$(q \sim \omega)$.

Leading term,

$$S^a_{(0)} \sim \sum_{k=1}^{n} \frac{\epsilon_\mu p_k^\mu}{q\cdot p_k} T^a_k$$

(universal)

Subleading term,

$$S^a_{(1)} = \sum_{k=1}^{n} i\frac{\epsilon_\mu q_\nu J_k^{\mu\nu}}{q\cdot p_k} T^a_k$$

where, $J_k^{\mu\nu}$ is the total angular momentum of the k'th particle.

Conformal Soft gluon theorems

✦ **Leading Conformal Soft Gluon Theorem**, $\Delta \to 1$

$$\left\langle j^a(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = -\sum_{k=1}^{n} \frac{T^a_k}{z - z_k} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle$$

where leading conformal soft gluon operator is defined as,

$$j^a(z) = \lim_{\Delta \to 1} (\Delta - 1) \mathcal{O}^a_{\Delta,+}(z,\bar{z})$$

and

$$T^a_k \mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = i f^{a a_i b} \mathcal{O}^b_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \delta_{ik}.$$

$\Rightarrow$ The modes j^a_n give level zero Kac-Moody algebra on celestial sphere.

[arXiv:2011.00017]

✦ Subleading Conformal Soft Gluon Theorem, $\Delta \to 0$

$$\left\langle S_1^{+a}(z,\bar{z})\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$$= -\sum_{i=1}^{n}\frac{\epsilon_k}{z-z_k}(-2\bar{h}_k+1+(\bar{z}-\bar{z}_k)\bar{\partial}_k)T_k^a P_k^{-1}\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

where subleading conformal soft gluon operator is defined as,

$$S_1^{+a}(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}^a_{\Delta,+}(z,\bar{z})$$

and

$$P_k^{-1}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = \mathcal{O}^{a_i}_{h_i-\frac{1}{2},\bar{h}_i-\frac{1}{2}}(z_i,\bar{z}_i)\delta_{ki}.$$

[Donnay,Puhm,Strominger;2018]

Where,

$$S_1^{+a}(z,\bar{z}) = J^a(z) + \bar{z}K^a(z).$$

Ward Identities of 2D conformal currents :

$$\left\langle J^a(z)\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = -\sum_{k=1}^{n}\frac{\epsilon_k}{z-z_k}(-2\bar{h}_k+1-\bar{z}_k\bar{\partial}_k)T^a_k P^{-1}_k\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

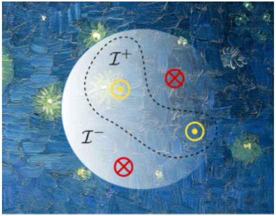
and

$$\left\langle K^a(z)\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = -\sum_{k=1}^{n}\frac{\epsilon_k}{z-z_k}\bar{\partial}_k T^a_k P^{-1}_k\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$\Rightarrow$ Current algebra of two currents $J^a(z)$ and $K^a(z)$.

[arXiv:2011.00017]

OPE structures :



1. From colliner limit :

$$\mathcal{A}(1^{-a_1}, 2^{+a_2}, 3^{+a_3}, 4^{-a_4}) \xrightarrow[p_3 \cdot p_4 = 0]{z_{34} \to 0} -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1)\mathcal{A}(1^{-a_1}, 2^{+a_2}, 4^{-x})$$
$$+ \text{ subleading in } z_{34} + \ldots$$

Leading OPE :

$$\mathcal{O}^{a_3}_{\Delta_3,+}(z_3, \bar{z}_3)\mathcal{O}^{a_3}_{\Delta_4,-}(z_4, \bar{z}_4) \sim -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1)\mathcal{O}^{x}_{\Delta_3+\Delta_4-1,-}(z_4, \bar{z}_4)$$

Subleading ... (similarly) [arXiv: 2011.00017]

2. From symmetry analysis : [ref: 2011.00017]

$$\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1)\mathcal{O}^{a_2}_{\Delta_2,\sigma_2}(z_2,\bar{z}_2) = -i z_{12}^{p}\bar{z}_{12}^{q} f^{a_1 a_2 x}\underbrace{C_{p,q}(\Delta_1,\Delta_2,\sigma_2)}\mathcal{O}^{x}_{\Delta,\sigma}(z_2,\bar{z}_2)+\ldots$$

Goal: To determine $C_{p,q}(\Delta_1,\Delta_2,\sigma_2)$ and values of p and q.

Step 1.
Null states of current algebra $\longrightarrow$ Null decoupling equations (BG equations)

Step 2.
BG equations $\longrightarrow$ a recursion relation

$$(\Delta_1+2p+1)C_{p,q}(\Delta_1,\Delta_2,\sigma_2) = (\Delta_2-\sigma_2+q-1)C_{p,q}(\Delta_1+1,\Delta_2-1,\sigma_2)$$

Step 3.
Global time translation $\longrightarrow$ another recursion relation

$$C_{p,q}(\Delta_1,\Delta_2,\sigma_2) = C_{p,q}(\Delta_1+1,\Delta_2,\sigma_2) + C_{p,q}(\Delta_1,\Delta_2+1,\sigma_2)$$

Step 4.
Above recursion relations lead to,

$$(\Delta_1+2p+1)C_{p,q}(\Delta_1,\Delta_2,\sigma_2)=(\Delta_1+\Delta_2+2p+q-\sigma_2+1)C_{p,q}(\Delta_1+1,\Delta_2,\sigma_2)$$

Step 5.
subleading soft gluon theorem $\longrightarrow$ following recursion relation

$$\frac{\Delta_1+q-1}{2p+\Delta_1+1}+\frac{\Delta_2-\sigma_2-1}{\Delta_2-\sigma_2+q-1}=\frac{2(\Delta_1+\Delta_2+2q-\sigma_2-1)}{\Delta_1+\Delta_2+2p+q-\sigma_2+1}$$

$$\Rightarrow p=-1, \quad q=0$$
$$\Rightarrow \Delta=\Delta_1+\Delta_2-1, \quad \sigma=\sigma_2$$

$$\mathcal{O}^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1)\mathcal{O}^{a_2}_{\Delta_2,\sigma_2}(z_2,\bar{z}_2)=\frac{-if^{a_1a_2x}}{z_{12}}\underbrace{C_{-1,0}(\Delta_1,\Delta_2,\sigma_2)}\mathcal{O}^{x}_{\Delta,\sigma}(z_2,\bar{z}_2)+\ldots$$

Step 6.

Using the values of p and q in the recursion relations one can get,

$$C_{-1,0}(\Delta_1, \Delta_2+1, \sigma_2) = \frac{\Delta_2 - \sigma_2}{\Delta_1 + \Delta_2 - \sigma_2 - 1} C_{-1,0}(\Delta_1, \Delta_2, \sigma_2).$$

Soln.

$$C_{-1,0}(\Delta_1, \Delta_2+1, \sigma_2) = \alpha B(\Delta_1 - 1, \Delta_2 - \sigma_2)$$

$$\alpha = 1 \text{ (comparing with Ward Identity)}$$

$$\mathcal{O}^{a_1}_{\Delta_1,+}(z_1, \bar{z}_1)\mathcal{O}^{a_2}_{\Delta_2,\sigma_2}(z_2, \bar{z}_2) = \frac{-if^{a_1 a_2 x}}{z_{12}} B(\Delta_1 - 1, \Delta_2 - \sigma_2)\mathcal{O}^{x}_{\Delta,\sigma}(z_2, \bar{z}_2) + \ldots$$

Similar analysis has been performed to determine the subleading OPE structure.

Null states of the current algebra → Correlation functions for descendants → Null decoupling equations.

$$\left[\frac{C_A}{2}\partial_{z_i} - h_i \sum_{j=1,j\neq i}^{n} \frac{T_i^a T_j^a}{z_i - z_j} + \frac{1}{2}\sum_{j=1,j\neq i}^{n} \frac{\epsilon_j(2\bar{h}_j - 1 - (\bar{z}_i - \bar{z}_j)\partial_{\bar{z}_j})}{z_i - z_j} T_j^a P_j^{-1} T_i^a P_{-1,-1}(i)\right] \times \langle \prod_{k=1}^{n} \mathcal{O}^{a_k}_{h_k,\bar{h}_k}(z_k,\bar{z}_k)\rangle_{MHV} = 0. \qquad i \in (1,2,\ldots,n-2)$$

Pure YM [S. Banerjee and S. Ghosh]

These equations have been solved to find the Scattering amplitudes !

- ✍ Fan,Fotopoulos,Stieberger,Taylor and Zhu in [arXiv: 2202.08288] have solved these equations for pure YM theory coupled with massless scalar background field.
- ✍ S. Banerjee,P. Paul,A. Manu and RM have recently solved BG equation for a pure YM theory coupled to a massive scalar background. [arXiv:2302.10245]

Holographic Symmetry Algebra (HSA)

- It is known that there is no soft factorisation beyond the subleading order (for gluons).
- Beta functions in the OPE have poles for $\Delta = 1, 0, -1, -2, ...$

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) = -\frac{if^{abc}}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2).$$

- **Soft gluon operator :**

$$R^{k,a}(z_1,\bar{z}_1) = \lim_{\Delta_1\to k}(\Delta_1-k)\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)$$

where $k = 1, 0, -1, ...$

[A. Guevara,E. Himwich,M. Pate and A. Strominger]

Mode Expansion :

$$R^{k,a}(z,\bar{z}) = \sum_{n=\frac{k-1}{2}}^{\frac{1-k}{2}} \frac{R^{k,a}_{\ n}(z)}{\bar{z}^{n+\frac{k-1}{2}}}$$

HSA :

[A. Guevara,E. Himwich,M. Pate and A. Strominger]

$$[R^{k,a}_{m,n}, R^{l,b}_{m',n'}]$$
$$= -if^{abc}\frac{(\frac{1-k}{2}-n+\frac{1-l}{2}-n')!}{(\frac{1-k}{2}-n)!(\frac{1-l}{2}-n')!}\frac{(\frac{1-k}{2}+n+\frac{1-l}{2}+n')!}{(\frac{1-k}{2}+n)!(\frac{1-l}{2}+n')!}R^{k+l-1,c}_{m+m',n+n'}$$

So, we now have infinite tower of soft gluons and their algebra !

This algebra is similar to $w_{1+\infty}$ algebra obtained in case of gravity.

General structure of OPE of the thories invariant under the above algebra for gluons

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)$$
$$= -\frac{if^{abc}}{z_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) - \frac{if^{abc}}{z_{12}}\sum_{n=1}^{\infty}B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}$$
$$+\sum_{m,n}\sum_{i=1}^{a(m,n)} z_{12}^m\bar{z}_{12}^n f^{abc}C^{(i)}_{m,n}\mathcal{O}^{(i),c}_{m,n}(z_2,\bar{z}_2)$$

Our Goals ?

1. **To determine the gluon-gluon OPE for the theories which are invariant under the HSA $\times$ Poincare.**
2. **To find the null states of such theories.**
3. **To classify all such dual theories which are invariant under this algebra.**

Motivated by:

An interesting recent work [arXiv:2301.13225] done in case of gravity by S. Banerjee,H. Kulkarni and P. Paul.

The authors have shown that there exists a discrete infinite family of wedge subalgebra of $w_{1+\infty}$ invariant dual theories whose bulk descriptions are not known except for the MHV and self-dual gravity.

Progress...

- ✔ We have already found a "subalgebra" which plays an important role to check the covariance of OPE under the full **HSA** $\times$ **Poincare**.
- ✔ We have also found the null states which furnish the representation of this subalgebra.
- ✔ We know how this algebra acts on the gluon primaries.
- ✔ General structure of OPE at $\mathcal{O}(1)$.

Remarks

- ✎ This "subalgebra" has $H^1_{-1/2,-1/2}$ which doesn't come from any soft theorem in case of gauge theories.
- ✎ In gravity $H^1_{-1/2,-1/2}$ was manifestly present there because of the leading soft graviton theorem.
- ✎ For $SU(N)$ gauge theories, ASG is (HSA $\times$ Poincare) which acts non-trivialy on the Hilbert space.

Thank You !!