Singularity structure of the four point celestial leaf amplitudes

Annual Review Talk
by

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Previous work?

- “All OPEs invariant under the infinite symmetry algebra for gluons on the celestial sphere” is published in Phys.Rev.D 110 (2024) 2, 026020.

  DOI: https://doi.org/10.1103/PhysRevD.110.026020

This talk

- based on “Singularity Structure of the Four Point Celestial Leaf Amplitudes” with
  Partha Paul(IMSC),
  Sagnik Misra(NISER) and
  Baishali Roy(RKMVU).

  arXiv: 2410.13969 [hep-th]

OUTLINE

① OUR GOAL	What do we want to do?
② MOTIVATION	Why do we want to do that?
③ CELESTIAL HOLOGRAPHY	QG $\leftrightarrow$ CCFT
④ BRIEF REVIEW:	What are these?
➥ The celestial amplitudes ➥ The Klein space ➥ Celestial leaf amplitudes	
⑤ ACHIEVING THE GOAL	How do we do it?
⑥ OUTLOOK	What do we expect now?

GOAL

① Singularity structure of the 4-point $\boxed{\text{celestial leaf amplitudes}}$

② Constraints on the leaf amplitudes coming from the decoupling of the null states

MOTIVATION

✎ Holographic principle in AdS spacetime is well understood.

✎ Celestial leaf amplitudes have a natural holographic interpretation given by AdS_3/CFT_2 correspondence

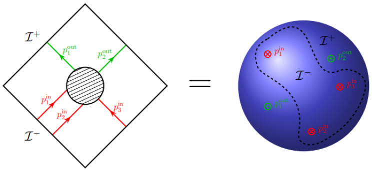
✎ Shiraz and collaborators' construction in 2311.03443,
Boundary correlator $\rightarrow$ S-matrix of a QFT $\rightarrow$ massless scattering in Minkowski spacetime $\rightarrow$ simple pole-type singularity in the 4-point correlators in cross-ratio space at $z = \bar{z}$

✎ In the AdS/CFT context, pole type singularity also appeared in the Lorentzian 4-point boundary correlators.

Gary,Giddings,Penedones 2009 Maldacena,Simons-Duffin,Zhiboedov 2017

QUICK REVIEW OF CELESTIAL CFT :

$\boxed{\text{A quantum theory of gravity in (3+1)D AFS}} \equiv \boxed{\text{A 2D Celestial CFT on } CS^2 \text{ at null infinity}}$



$$\langle out|\, S\, |in\rangle \qquad \langle \mathcal{O}^{\pm}_{\Delta_1}(z_1,\bar{z}_1)...\mathcal{O}^{\pm}_{\Delta_n}(z_n,\bar{z}_n)\rangle \tag{1}$$

☞ Null momentum in split signature $(-,-,+,+)$: Klein space

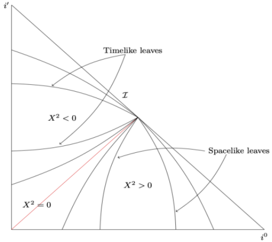
$$p^{\mu}_k = \epsilon_k \omega_k (1 - z_k \bar{z}_k, z_k + \bar{z}_k, 1 + z_k\bar{z}_k, z_k - \bar{z}_k) \tag{2}$$

KLEIN SPACE ($\mathbb{K}^{2,2}$)

Atanasov,Ball,Melton,Raclariu,Strominger 2021

$$SO(2,2) \simeq SL(2,\mathbb{R})_L \times SL(2,R)_R/\mathbb{Z}_2$$

$$\begin{aligned}
ds^2 &= -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + (dX^3)^2, \\
&\Big\downarrow X^0 + iX^1 = qe^{i\psi}, X^2 + iX^3 = re^{i\phi} \\
ds^2 &= -dq^2 - q^2 d\psi^2 + dr^2 + r^2 d\phi^2 \\
&\Big\downarrow q - r = \tan U, q + r = \tan V \\
ds^2 &= \frac{1}{cos^2 U cos^2 V}\left(- dU dV - \frac{1}{4} sin^2(V+U) d\psi^2 + \frac{1}{4} sin^2(V-U) d\phi^2 \right) \\
&\Big\downarrow \text{null infinity} \mathscr{I} \text{ at } V = \frac{\pi}{2} \\
\widetilde{ds}^2 &= -d\psi^2 + d\phi^2, \quad \psi \sim \psi + 2\pi, \phi \sim \phi + 2\pi \quad \text{Lorentzian torus}
\end{aligned} \tag{3}$$



FOLIATION OF THE KLEIN SPACE

- The lightcone $X^2 = 0$ divides the Klein space into two regions

$$\begin{aligned} &\text{Timelike:} \quad X^\mu = \tau \hat{x}_+^\mu, \;\; \hat{x}_+^2 = -1 \qquad \tau \in (0, \infty) \\ &\text{Spacelike:} \quad X^\mu = \tau \hat{x}_-^\mu, \;\; \hat{x}_-^2 = +1 \end{aligned} \tag{4}$$

In global coordinates,

$$\begin{aligned} &\text{Timelike wedge} \quad ds^2 = -d\tau^2 + \tau^2(-\cosh^2\rho d\psi^2 + \sinh^2\rho d\phi^2 + d\rho^2) \\ &\text{Spacelike wedge} \quad ds^2 = d\tau^2 - \tau^2(-\cosh^2\rho d\psi^2 + \sinh^2\rho d\phi^2 + d\rho^2) \end{aligned} \tag{5}$$

- each constant τ leaf gives a metric on $AdS_3/\mathbb{Z}$
- conformal boundary is reached by taking $\rho \to \infty$, which is a Celestial torus($\mathcal{CT}^2$)
- null coordinates(global coordinates), $\sigma = \frac{\psi+\phi}{2}$, $\bar{\sigma} = \frac{\psi-\phi}{2}$
- planar coordinates, $z = \tan\sigma, \bar{z} = \tan\bar{\sigma}$

CELESTIAL AMPLITUDE :

(Pasterski,Shao,Strominger '17)

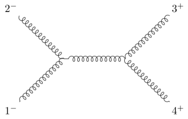
$$\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \prod_{i=1}^{n}\int_0^\infty d\omega_i\;\omega_i^{\Delta_i-1}\boxed{\mathcal{S}_n(\{\omega_i,z_i,\bar{z}_i,\sigma_i,a_i\})} \tag{6}$$

Mellin transform

Momentum space amplitude

MHV 4-GLUON CELESTIAL AMPLITUDE (COLOR STRIPPED) :

$\mathcal{A}(1^-2^-3^+4^+)$



$$= \prod_{j=1}^{4}\int_0^\infty d\omega_j\;\;\omega_j^{\Delta_j-1}\boxed{\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\delta\left(\sum_{i=1}^{4}p_i^\mu\right)}$$

Parke-Taylor Formula 1986

$$= \prod_{j=1}^{4}\int_0^\infty d\omega_j\;\;\omega_j^{\Delta_j-1}\mathcal{F}(\{\omega_i\})\delta(\omega_1-\omega_1^\star)\delta(\omega_2-\omega_2^\star)\delta(\omega_3-\omega_3^\star)\delta(z_{12}z_{34}\bar{z}_{13}\bar{z}_{24}-z_{13}z_{24}\bar{z}_{12}\bar{z}_{34}) \tag{7}$$

CELESTIAL LEAF AMPLITUDE

Melton,Sharma,Strominger '23

$$\mathcal{A}(1^-2^-3^+...n^+) = \prod_{j=1}^{n}\int_0^\infty d\omega_j \;\; \omega_j^{\Delta_j-1}\frac{\langle 12\rangle^3}{\langle 23\rangle\cdots\langle n1\rangle}\boxed{\int \frac{d^4X}{(2\pi)^4}e^{\sum_j \omega_j(i\hat{p}_j\cdot X-\epsilon)}}$$

$$= \frac{s_{12}^3}{s_{23}\cdots s_{n1}}\int \frac{d^4X}{(2\pi)^4}\prod_{i=1}^{n}\Phi_{2\bar{h}_i}(X,\hat{p}_i)$$

scalar conformal primary wavefunction

$\Big\downarrow$ break the integral over Klein space

$$= \frac{s_{12}^3}{s_{23}\cdots s_{n1}}\int_0^\infty d\tau\;\tau^3$$

$$s_{ij} := sin\sigma_{ij}$$

$$\times\left\{\int_{\hat{x}_+^2=-1} d^3\hat{x}_+\prod_{i=1}^{n}\Phi_{2\bar{h}_i}(\tau\hat{x}_+,\hat{p}_i) + \int_{\hat{x}_+^2=+1} d^3\hat{x}_-\prod_{i=1}^{n}\Phi_{2\bar{h}_i}(\tau\hat{x}_-,\hat{p}_i)\right\}$$

$\Big\downarrow$ τ integral

$$\mathcal{A}(1^-2^-3^+...n^+) = \frac{\delta(\beta)}{8\pi^3}\Big(\mathcal{L}(\sigma_i,\bar{\sigma}_i)+\mathcal{L}(\sigma_i,-\bar{\sigma}_i)\Big),$$

$$\beta = \sum_{k=1}^{n}(\Delta_k-1)$$

$\uparrow$ Timelike leaf $\qquad$ $\uparrow$ Spacelike leaf

LEAF AMPLITUDE:

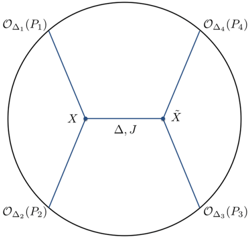
$$\mathcal{L}(\sigma_i, \bar{\sigma}_i) = \frac{s_{12}^3}{s_{23}\cdots s_{n1}} \int_{AdS_3/\mathbb{Z}} d^3\hat{x}_+ \prod_{i=1}^{n} \Phi_{2\bar{h}_i}(\hat{x}_+, \hat{p}_i) \tag{9}$$

INTERPRETATION:

$$\Phi_\Delta(X, \hat{p}) = \int_0^\infty d\omega \omega^{\Delta-1} e^{i\omega \hat{p}\cdot X - \epsilon\omega} = \frac{\Gamma(\Delta)}{(-i\hat{p}\cdot X + \epsilon)^\Delta}$$

$\big\downarrow$ hyperbolic coordinates

bulk-to-boundary propagator on Lorentzian $AdS_3/\mathbb{Z}$



PROPERTIES...

- Non-distributional, unlike celestial amplitudes
- Celestial amplitudes are obtained by adding the TL and SL leaf amplitudes
- Governed by the same infinite-dimensional soft "S algebra" $+$ Lorentz/conformal sym.

Melton, Sharma, Strominger '24

- 4-point massless scalar and gluon leaf amplitudes have a simple pole-type singularity in cross-ratio space

Paul, Roy, Misra, RM '24

- Constrained by the same BG equations

4-POINT MASSLESS SCALAR AMPLITUDE: ϕ^4 Theory

contact diagram:

$$A_4(p_1, p_2, p_3, p_4) = -i(2\pi)^4 \tilde{\lambda} \delta^4(p_1 + p_2 + p_3 + p_4) \tag{10}$$

4-point scalar leaf amplitudes:

$$\mathcal{L}_4^s(\sigma_i, \bar{\sigma}_i) = -i(2\pi)^4 \tilde{\lambda} \mathcal{C}_4(\sigma_i, \bar{\sigma}_i), \qquad \mathcal{L}_4^s(\sigma_i, -\bar{\sigma}_i) = -i(2\pi)^4 \tilde{\lambda} \mathcal{C}_4(\sigma_i, -\bar{\sigma}_i) \tag{11}$$

$$\mathcal{C}(\sigma_i, \bar{\sigma}_i) = \int_{\hat{x}_+^2 = -1} d^3\hat{x}_+ \prod_{j=1}^{4} \int_0^\infty d\omega_j \omega_j^{2\bar{h}_j - 1} e^{\sum_{k=1}^4 \epsilon_k \omega_k q_k \cdot \hat{x}_+ - \epsilon \omega_k}$$

$$\Big\downarrow \text{go to the planar coordinates} \qquad \lambda_i = \lambda, \forall i \;\; \text{(for simplicity)}$$

$$\mathcal{C}_4(z_i, \bar{z}_i) \tag{12}$$

$$\text{conformal transformations} \Big\downarrow z_1, \bar{z}_1 \to \infty; z_2 = \bar{z}_2 = 1; z_3 = z, \bar{z}_3 = \bar{z}; z_4 = \bar{z}_4 = 0$$

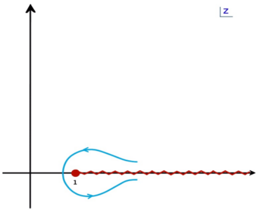
$$\mathcal{S}_4(z, \bar{z}) = \frac{i\pi}{2} \Gamma(1 + 2i\lambda) e^{2\pi\lambda} H(1 + i\lambda, 1 + i\lambda, 1 + 2i\lambda, 2 + 2i\lambda; u_+, v_+) - \frac{i\pi}{2} \Gamma(1 + 2i\lambda) e^{-2\pi\lambda} H(1 + i\lambda, 1 + i\lambda, 1 + 2i\lambda, 2 + 2i\lambda; u_-, v_-)$$

 Dolan, Osborn 2001

$$\boxed{\begin{aligned} &H(2\bar{h}_1, \bar{h}_1+\bar{h}_2-\bar{h}_3+\bar{h}_4, 1+\bar{h}_1-\bar{h}_2-\bar{h}_3+\bar{h}_4, 2\bar{h}_1+2\bar{h}_4; \tilde{u}, \tilde{v}) \\ &= \int_{-c-i\infty}^{-c+i\infty} \frac{ds}{2\pi i} \int_{-c'-i\infty}^{-c'+i\infty} \frac{dr}{2\pi i} \;\Gamma(-s)\Gamma(-r)\Gamma(2\bar{h}_1+r+s)\Gamma(-\bar{h}_1+\bar{h}_2+\bar{h}_3-\bar{h}_4-s) \\ &\qquad \times \Gamma(\bar{h}_1+\bar{h}_2-\bar{h}_3+\bar{h}_4+r+s)\Gamma(-\bar{h}_1-\bar{h}_2+\bar{h}_3+\bar{h}_4-r)\tilde{u}^r\tilde{v}^s \end{aligned}} \tag{13}$$

- scalar case

$$\begin{aligned} &\delta(\lambda)H(1+i\lambda, 1+i\lambda, 1+2i\lambda, 2+2i\lambda; u_\pm, v_\pm) \\ &= \frac{\delta(\lambda)}{\bar{z}-z\pm i\epsilon}\Big[\Big\{\log(z\mp i\epsilon)+\log(\bar{z}\pm i\epsilon)\Big\}\Big\{\log(1-\bar{z}\mp i\epsilon)+\log(1-z\pm i\epsilon)\Big\} \\ &\qquad - 2Li_2(z\mp i\epsilon)+2Li_2(\bar{z}\pm i\epsilon)\Big] \end{aligned} \tag{14}$$



CELESTIAL AMPLITUDE FROM LEAF AMPLITUDES:

- simple pole-type singularity as $z \to \bar{z}$ Timelike leaf

$$\delta(\lambda)\mathcal{S}_4(z,\bar{z}) = \frac{i\pi}{2}\frac{\delta(\lambda)}{\bar{z}-z+i\epsilon}\left[4\pi^2 + 2\pi i \ln\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right], \qquad z,\bar{z}>1 \tag{15}$$

- spacelike leaf

$$\delta(\lambda)\bar{\mathcal{S}}_4(z,\bar{z}) = \frac{i\pi}{2}\frac{\delta(\lambda)}{\bar{z}-z-i\epsilon}\left[-4\pi^2 - 2\pi i \ln\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right], \qquad z,\bar{z}>1 \tag{16}$$

- celestial amplitude is recovered by adding the timelike and spacelike leaf amplitudes

$$\begin{aligned}\widetilde{\mathcal{M}}_4(z,\bar{z}) &= ((-2\pi)^4\tilde{\lambda})\frac{\delta(\lambda)}{16}\Theta(z-1)\left[\frac{i}{\bar{z}-z+i\epsilon} - \frac{i}{\bar{z}-z-i\epsilon}\right] \\ &= \underset{\uparrow}{((-2\pi)^4\tilde{\lambda})\delta(\lambda)\frac{\pi}{8}\Theta(z-1)\delta(z-\bar{z})}\end{aligned} \tag{17}$$

4-point tree level scalar celestial amplitude

MHV GLUON LEAF AMPLITUDES

$$\begin{aligned}\mathcal{G}_4(z,\bar z) &= \lim_{z_1,\bar z_1\to\infty} z_1^{2h_1}\bar z_1^{2\bar h_1}\mathcal{L}_4^g(z_i,\bar z_i)\\ &= \frac{i\pi}{2}\Gamma(1+2i\lambda)e^{2\pi\lambda}\frac{z\bar z^2}{z-1}H(2+i\lambda,2+i\lambda,3+2i\lambda,4+2i\lambda;u_+,v_+)\\ &\quad-\frac{i\pi}{2}\Gamma(1+2i\lambda)e^{-2\pi\lambda}\frac{z\bar z^2}{z-1}H(2+i\lambda,2+i\lambda,3+2i\lambda,4+2i\lambda;u_-,v_-)\end{aligned} \tag{18}$$

$$\Big\downarrow \text{ using the identities of H functions}$$

$$= \mathcal{G}_4^{sing}(z,\bar z)+\mathcal{G}_4^{reg}(z,\bar z)$$

where,

$$\begin{aligned}\mathcal{G}_4^{sing}(z,\bar z) &= \frac{i\pi}{2}\Gamma(1+2i\lambda)e^{2\pi\lambda}\frac{\bar z}{z-1}\underbrace{H(1+i\lambda,1+i\lambda,1+2i\lambda,2+2i\lambda;u_+,v_+)}\\ &\quad-\frac{i\pi}{2}\Gamma(1+2i\lambda)e^{-2\pi\lambda}\frac{\bar z}{z-1}\underbrace{H(1+i\lambda,1+i\lambda,1+2i\lambda,2+2i\lambda;u_-,v_-)}\end{aligned} \tag{19}$$

same H function that appeared in scalar case

- $\mathcal{G}_4^{reg}(z,\bar z)$ is not singular at $z=\bar z$

MHV GLUON LEAF AMPLITUDES...

Timelike gluon leaf:

$$\delta(\lambda)\mathcal{G}_4(z,\bar{z}) = \frac{i\pi}{2}\frac{\bar{z}}{z-1}\frac{\delta(\lambda)}{\bar{z}-z+i\epsilon}\left[4\pi^2 + 2\pi i \ln\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right] + \delta(\lambda)\mathrm{Reg}^t(\mathrm{as}\ z\to\bar{z}) \tag{20}$$

Spacelike gluon leaf:

$$\delta(\lambda)\bar{\mathcal{G}}_4(z,\bar{z}) = \frac{i\pi}{2}\frac{\bar{z}}{z-1}\frac{\delta(\lambda)}{\bar{z}-z-i\epsilon}\left[-4\pi^2 - 2\pi i \ln\left(\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}\right)\right] + \delta(\lambda)\mathrm{Reg}^s(\mathrm{as}\ z\to\bar{z}) \tag{21}$$

MHV GLUON CELESTIAL AMPLITUDE

Adding them we get,

$$\widetilde{\mathcal{M}}_4^g(z,\bar{z}) = \delta(\lambda)\frac{\pi}{8}\Theta(z-1)\frac{z}{z-1}\delta(z-\bar{z}) + \delta(\lambda)\mathrm{Reg}(\mathrm{as}\ z\to\bar{z}). \tag{22}$$

$$\mathcal{L}_4^g(1_{\Delta_1}^{-,a_1}, 2_{\Delta_2}^{-,a_2}, 3_{\Delta_3}^{+,a_3}, 4_{\Delta_4}^{+,a_4}) = -g_{YM}^2 \frac{z_{12}^3}{z_{23}z_{34}z_{41}} \left[f^{a_1 a_2 x} f^{x a_3 a_4} - \frac{z_{12}z_{34}}{z_{13}z_{24}} f^{a_1 a_3 x} f^{x a_2 a_4} \right]$$

$$\int_{\hat{x}_+^2=+1} d^3x_+ \left(\prod_{j=1}^{2} \Phi_{\Delta_j+1,-}(\hat{x}_+, q_j) \right) \left(\prod_{j=3}^{4} \Phi_{\Delta_j-1,+}(\hat{x}_+, q_j) \right)$$

$$\Big\downarrow \text{collinear factorisation}$$

$$\mathcal{O}_{\Delta_3}^{+,a_3}(z_3, \bar{z}_3)\mathcal{O}_{\Delta_4}^{+,a_4}(z_4, \bar{z}_4)$$

$$= -\frac{g_{YM}}{2} \frac{1}{z_{34}} B(\Delta_3 - 1, \Delta_4 - 1) i f^{a_3 a_4 x} \mathcal{O}_{\Delta_3+\Delta_4-1}^{+,x}(z_4, \bar{z}_4)$$

$$+ \frac{g_{YM}}{2} \Big[B(\Delta_3, \Delta_4 - 1)(-i f^{x a_3 a_4}) L_{-1} \mathcal{O}_{\Delta_3+\Delta_4-1}^{+,x}(z_4, \bar{z}_4) + B(\Delta_3 - 1, \Delta_4 - 1)$$

$$\times \left[\frac{\Delta_4 - 1}{\Delta_3 + \Delta_4 - 2} R_{-1,0}^{1,a_3} \mathcal{O}_{\Delta_3+\Delta_4-1}^{+,a_4}(z_4, \bar{z}_4) \right] + \frac{\Delta_3 - 1}{\Delta_3 + \Delta_4 - 2} R_{-1,0}^{1,a_4} \mathcal{O}_{\Delta_3+\Delta_4-1}^{+,a_3}(z_4, \bar{z}_4) \Big] + \cdots \tag{23}$$

- Take sub-leading soft limit($\Delta \to 0$) in (23)
- check consistency with the sub-leading soft gluon theorem
- Obtain the $\mathcal{O}(1)$ null state

$$
\begin{aligned}
i f^{abc} L_{-1} \mathcal{O}^{+,c}_{\Delta-1,+}(z,\bar{z}) + R^{0,a}_{-\frac{1}{2},\frac{1}{2}} \mathcal{O}^{+,b}_{\Delta,+}(z,\bar{z}) - R^{1,b}_{-1,0} \mathcal{O}^{+,a}_{\Delta-1,+}(z,\bar{z}) \\
+ (\Delta-1) R^{1,a}_{-1,0} \mathcal{O}^{+,b}_{\Delta-1,+}(z,\bar{z}) = 0
\end{aligned}
\tag{24}
$$

- It is the same null state obtained for the celestial MHV amplitude
- Decoupling of these null states will give the same BG equations for leaf amplitudes. Banerjee, Ghosh 2021

OUTLOOK

- What happens to our conclusion without the constraints coming from bulk scale invariance?
- What about the singularity structure of the other bulk scattering processes?

Thank you for your attention !!