Massless Particles at Null Infinity

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Outline :

- Review of Lorentz Group $>$ idea of covering group
- Discussion on Little Group
- Construction of New Basis known as "Conformal Basis" or "Mellin Basis" Pasterski,Shao
- Unitary Representation of Poincare Group via "Method of Induced Representation" Wigner
- Quantum Field Theory at Null Infinity
- Some applications : Supertranslation,spacetime relalization etc.

$SL(2,\mathbb{R})$ is the Double Cover of $SO(2,1)$

Consider, $P = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \end{pmatrix} \longrightarrow P = \begin{pmatrix} P^0 - P^2 & P^1 \\ P^1 & P^0 + P^2 \end{pmatrix}$

det $P = -P^2 = -[-(P^0)^2 + (P^1)^2 + (P^2)^2]$

- $\eta =$ diag $(-1, 1, 1)$
- Let $\mathbb{V}$ be a 3 dimensional vector space of 2×2 real symmetric matrices.
- we can choose a basis :
  $\{e_0, e_1, e_2\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
- since $P \in \mathbb{V} \implies P = P^0 e_0 + P^1 e_1 + P^2 e_2$

Lorentz Group in (2+1) Dimensions

- $SL(2,\mathbb{R})$ Group: a group of 2×2 real matrices with Det $+1$.
  Group element $\Lambda \in SL(2,\mathbb{R})$ has the structure
  $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $(ad - bc) = 1$; where $a, b, c, d \in \mathbb{R}$
- We can take the representation of $SL(2,\mathbb{R})$ on the vector space $\mathbb{V}$.
- $R(\Lambda)P = \Lambda P \Lambda^T = P'$
- -Λ also gives same P'
- Det P = Det P' since Det $\Lambda = +1$
- components, $P'^{\mu} = R(\Lambda)^{\mu}{}_{\nu} P^{\nu}$, $\mu, \nu = 0, 1, 2$
- $\pm$ Λ induces LT of P vector
- $SO(2,1)^{\uparrow} \cong SL(2,\mathbb{R})/\mathbb{Z}_2$

Action on Null Momenta

- Define, $z = \frac{P^1}{P^0+P^2}$, where $[-(P^0)^2 + (P^1)^2 + (P^2)^2] = 0$
- under $P \longrightarrow P' = \Lambda P \Lambda^T$; $z \longrightarrow z' = \Lambda z = \frac{az+b}{cz+d}$
- variable z parametrizes the null momenta directions
- to see this, let's take the set of all null vectors of the form $k^\mu = (E, 0, E) \longrightarrow z = 0$
- make a rotation in 1-2 plane to get the most general form of null vector
  $k^\mu \longrightarrow k'^\mu = (E, Esin\theta, Ecos\theta)$
- $z \longrightarrow z' = tan\frac{\theta}{2}$
- z depends only on the direction θ of the null ray.

Standard Rotation and Boost Matrices

- $\underline{SL(2,\mathbb{R})}$ $\underline{SO(2,1)}$

$$R_{12}(\theta) =^{+}_{-} \begin{pmatrix} cos\frac{\theta}{2} & sin\frac{\theta}{2} \\ -sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{pmatrix} \qquad R_{12}(\theta)^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix}$$

$$B_1(\eta) =^{+}_{-} \begin{pmatrix} cosh(\frac{\eta}{2}) & -sinh(\frac{\eta}{2}) \\ -sinh(\frac{\eta}{2}) & cosh(\frac{\eta}{2}) \end{pmatrix} \qquad \Lambda_1(\eta)^{\mu}{}_{\nu} = \begin{pmatrix} cosh\eta & -sinh\eta & 0 \\ -sinh\eta & cosh\eta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_2(\zeta) =^{+}_{-} \begin{pmatrix} e^{\frac{\zeta}{2}} & 0 \\ 0 & e^{-\frac{\zeta}{2}} \end{pmatrix} \qquad \Lambda_2(\zeta)^{\mu}{}_{\nu} = \begin{pmatrix} cosh\eta & 0 & -sinh\eta \\ 0 & 1 & 0 \\ -sinh\eta & 0 & cosh\eta \end{pmatrix}$$

Generators and Lie Algebra

- We can easily find the geretaros J_0, K_1 and K_2. For example,

  $J_0 = -i \frac{d}{d\theta} R_{12}(\theta)|_{\theta=0}$

- Lie Algebra

  $$[J_0, K_1] = iK_2, \qquad [J_0, K_2] = -iK_1, \qquad [K_1, K_2] = -J_0$$

Little Group of A Null Momentum Direction

- Little group is a subgroup of the Lorentz group under which a a three vector remains invariant here in $(2+1)$ dimension.
- but here we will consider the little group of null direction instead.
- take a null vector $n^\mu = (1,0,1)$
- to find the little group , $\omega^{\mu\nu} n_\nu = 0$

$$\Rightarrow \omega^{\mu\nu} = -i\alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} = -i\alpha A^{\mu\nu}, where \;\; \Lambda = e^{-i\alpha A}$$

$$(A)^\mu{}_\nu = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix} = -(J_0)^\mu{}_\nu - (K_1)^\mu{}_\nu$$

Little Group of A Null Momentum Direction

- the little group, $D(z=0)$ of the ref null direction $\{(E,0,E), E>0|z=0\}$ is generated by two elements A and K_2
- where, $[K_2, A] = iA$
- Little group of the general null direction $\{k^\mu = (E, Esin\theta, Ecos\theta), E>0|z=tan\frac{\theta}{2}\}$, is given by $D(z) = R(z)D(z=0)R(z)^{-1}$

Hilbert Space Representation

- massless single particle quantum state is represented by $|p\rangle$ with $p^2 = 0$
- standard normalization of momentum states is given by, $\langle p_1|p_2\rangle = (2\pi)^2 2|\overrightarrow{p_1}|\delta^2(\overrightarrow{p_1} - \overrightarrow{p_2})$
- Unitary representation of arb. Lorentz transformation Λ in Hilbert Space is $U(\Lambda)$ and it acts on this state as,
  $$U(\Lambda)\,|p\rangle = |\Lambda p\rangle$$
- for example, if Λ is boost along 2 direction,then $U(\Lambda)\,|E,0,E\rangle = |e^{-\eta}E,0,e^{-\eta}E\rangle$, with rapidity η defined as $tanh\eta = v$
- state $|p\rangle$ is annihilated by the little group generator A $\implies A\,|E,0,E\rangle = 0$,$E > 0$

Change of Basis and Normalization

- $|\Delta, z=0\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\infty dE\ E^{\Delta-1} |E, 0, E\rangle\ ; \quad \Delta \in \mathbb{C}$
- $U(B_2(\eta)) |\Delta, z=0\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\infty dE\ E^{\Delta-1} |e^{-\eta}E, 0, e^{-\eta}E\rangle$
$$= e^{\eta\Delta} |\Delta, z=0\rangle$$
- Define, $|\Delta, z\rangle := \frac{1}{(1+z^2)^\Delta} U(R(z)) |\Delta, z=0\rangle$
$$= N_\Delta(z) U(R(z)) |\Delta, z=0\rangle$$
- $|\Delta, z\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{(1+z^2)^\Delta} \int_0^\infty dE\ E^{\Delta-1} |E, E sin\theta, E cos\theta\rangle$
- $\langle\Delta_2, z_2|\Delta_1, z_1\rangle = \delta(\lambda_2-\lambda_1)\delta(z_2-z_1)$, where $\Delta_i = \frac{1}{2} + i\lambda$, $\lambda \in \mathbb{R}$

Action of Arbitrary Lorentz Transformation

- $U(\Lambda)\left|\Delta, z\right\rangle = N_\Delta U(\Lambda)U(R(z))\left|\Delta, z=0\right\rangle$
  $= N_\Delta U(R(\Lambda z))U^{-1}(R(\Lambda z))U(\Lambda)U(R(z))\left|\Delta, z=0\right\rangle$
  $= N_\Delta U(R(\Lambda z))W(\Lambda, z)\left|\Delta, z=0\right\rangle$
- where,
  $W(\Lambda, z) = U^{-1}(R(\Lambda z))U(\Lambda)U(R(z)) = U(R^{-1}(\Lambda z)\Lambda R(z))$
- $W(\Lambda, z)\left|\Delta, z=0\right\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty dE\, E^{\Delta-1}W(\Lambda, z)\left|E, 0, E\right\rangle$
- $(R^{-1}(\Lambda z)\Lambda R(z))(1,0,1)$
  $= \left(\frac{(cz+d)^2+(az+b)^2}{1+z^2}, 0, \frac{(cz+d)^2+(az+b)^2}{1+z^2}\right)$

Action of Arbitrary Lorentz Transformation

- $$\begin{aligned}(R^{-1}(\Lambda z)\Lambda R(z))(E,0,E) &= B_2(\eta)(E,0,E)\\ &= (e^{-\eta}E,0,e^{-\eta}E), E>0\end{aligned}$$
  where, $e^{-\eta} = \frac{(cz+d)^2+(az+b)^2}{1+z^2} = (cz+d)^2\frac{1+(\Lambda z)^2}{1+z^2}$
  and $\Lambda z = \frac{az+b}{cz+d}$
- $U(\Lambda)\left|\Delta, z\right\rangle$
  $= N_\Delta U(R(\Lambda z))U^{-1}(R(\Lambda z))U(\Lambda)U(R(z))\left|\Delta, z=0\right\rangle$
  $= \frac{1}{(cz+d)^{2\Delta}}\left|\Delta, \Lambda z\right\rangle, \;\; \Lambda \in SL(2,\mathbb{R})$
- $\{\left|\Delta, z\right\rangle\}$ states with fixed Δ give the representaion of Lorentz group.
  $\Rightarrow U(\Lambda_2)U(\Lambda_1) = U(\Lambda_2\Lambda_1)$
  Unitarity: $(U(\Lambda)\left|\Delta', z'\right\rangle, U(\Lambda)\left|\Delta, z\right\rangle) = (\left|\Delta', z'\right\rangle, \left|\Delta, z\right\rangle)$
  $\Rightarrow$ "Unitary Principal Continuous Series Representation."

Representation on Wave Functions

- we can write the one-partice state for massless particle as,
  $$|\Psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda\, dz\ |\Delta, z\rangle\,\langle\Delta, z|\Psi\rangle$$
- inner product between two states,
  $$\langle\Phi|\Psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda\, dz\ \Phi^{\star}(\Delta, z)\Psi(\Delta, z)$$
- action of an arbitrary Lorentz transformation Λ on the wave function $\Rightarrow$
  $(U(\Lambda^{-1})\Psi)(\Delta, z) = \frac{1}{(cz+d)^{2\Delta^{\star}}}\Psi\big(\Delta, \frac{az+b}{cz+d}\big)$, where
  $\Delta^{\star} = \frac{1}{2} - i\lambda$

Lorentz Group in (3+1) Dimensions

- consider a four momentum $P^\mu = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}$
- we can associate a hermitian matrix with this four momentum P^μ as,

$$P = \begin{pmatrix} P^0 - P^3 & P^1 + iP^2 \\ P^1 - iP^2 & P^0 + P^3 \end{pmatrix}, \det P = -P^2$$

- In (3+1) dimension $SL(2,\mathbb{C})$ is double cover of $SO(3,1)$ and the group isomorphism is $SO(3,1)^{\uparrow} \cong SL(2,\mathbb{C})/\mathbb{Z}_2$
- $SL(2,\mathbb{C})$ acts on P as,

$$P \longrightarrow P' = \Lambda P \Lambda^{\dagger}, \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } (ad - bc) = 1$$

Lorentz Group in (3+1) Dimension

- In four dimesnions the space spanned by the null directions is a two-sphere. $(P^1)^2 + (P^2)^2 + (P^3)^2 = (P^0)^2 = const.$
- Define stereographic coordinate on two-sphere as, $z = \frac{P^1+iP^2}{P^0+P^3}$
- Under Lorentz transformation, $z \to z' = \frac{P^1+iP^2}{P^0+P^3} = \frac{az+b}{cz+d}$
- In spherical polar coordinate we can parametrize a null vector as,$P^\mu = \big(E, \quad Esin\theta cos\phi, \quad Esin\theta sin\phi, \quad Ecos\theta\big)$
- $\Rightarrow z = tan\frac{\theta}{2}e^{i\phi}$

Little Group of a Null Momentum Direction

- consider standard null direction : $\{(E, 0, 0, E), E > 0 | Z = 0\}$
- **little group** of this null direction is generated by (J_3, K_3, A, B) where $A = J_2 - K_1$ and $B = -J_1 - K_2$
- **Commutators**:
  $[A, B] = 0, [J_3, A] = iB, [J_3, B] = -iA, [J_3, K_3] = 0, [K_3, A] = iA, [K_3, B] = iB$

New Basis

- let's define the new state as,
  $|\lambda, \sigma, z = 0, \bar{z} = 0\rangle = \frac{1}{\sqrt{8\pi^4}} \int_0^\infty dE\; E^{i\lambda} \, |E, 0, 0, E; \sigma\rangle$,$\lambda \in \mathbb{R}$
  where σ is the helicity.
- std. normalization of the momentum states is,
  $\langle p_1, \sigma_1 | p_2, \sigma_2\rangle = (2\pi)^3 2 |\overrightarrow{p_1}| \delta^3(\overrightarrow{p_1} - \overrightarrow{p_2}) \delta_{\sigma_1, \sigma_2}$
- Now let's define,
  $$|\lambda, \sigma, z, \bar{z}\rangle = (\tfrac{1}{1+z\bar{z}})^{1+i\lambda} U(R(z, \bar{z}))\, |\lambda, \sigma, z = 0, \bar{z} = 0\rangle$$
  $$= N(z, \bar{z}) U(R(z, \bar{z}))\, |\lambda, \sigma, z = 0, \bar{z} = 0\rangle$$
  where,$U(R(z, \bar{z})) = e^{-i\phi J_3} e^{-i\theta J_2} e^{i\phi J_{J_3}}$,$z = tan\frac{\theta}{2} e^{i\phi}$
- $\langle \lambda_1, \sigma_1, z_1, \bar{z}_1 | \lambda_2, \sigma_2, z_2, \bar{z}_2\rangle = \delta(\lambda_1 - \lambda_2)\delta^2(z_1 - z_2)\delta_{\sigma_1, \sigma_2}$

Action of the Lorentz Group

- $U(\Lambda)\,|\lambda,\sigma,z,\bar{z}\rangle = N(z,\bar{z})U(\Lambda)U(R(z,\bar{z}))\,|\lambda,\sigma,z=0,\bar{z}=0\rangle$
  $=$
  $N(z,\bar{z})U(R(\Lambda z,\Lambda\bar{z}))U(R^{-1}(\Lambda z,\Lambda\bar{z})\Lambda R(z,\bar{z}))\,|\lambda,\sigma,z=0,\bar{z}=0\rangle$

  $= N(z,\bar{z})U(R(\Lambda z,\Lambda\bar{z}))U(W(\Lambda,z,\bar{z}))\,|\lambda,\sigma,z=0,\bar{z}=0\rangle$
- $W(\Lambda,z,\bar{z})$ belongs to the little group of the std. null ditection $\{(E,0,0,E)|z=0\}$.
- $W(\Lambda,z,\bar{z}) = \begin{pmatrix} e^{\frac{\alpha}{2}} & 0 \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} = D(\alpha)S(\beta)$

  wehre, $e^{\alpha} = \frac{1+|z|^2}{1+|\Lambda z|^2}\frac{1}{(cz+d)^2}$

- Now we can easily check that,
$U(D(\alpha)) = e^{\alpha L_0 + \bar{\alpha}\bar{L}_0}$
$D(\alpha)z = e^{\alpha} z$, where $L_0 = \frac{iK_3 - J_3}{2}$, $\bar{L}_0 = -L_0^{\dagger}$
- similarly,
$U(S(\beta)) = e^{\beta L_1 + \bar{\beta}\bar{L}_1}$
$S(\beta)z = \frac{z}{1-\beta z}$, where $L_1 = \frac{iK_1 - J_1}{2} + i\frac{iK_2 - J_2}{2}$, $\bar{L}_1 = -L_1^{\dagger}$
- We can also write the generator of the translation in z.
$U(T(\gamma)) = e^{\gamma L_{-1} + \bar{\gamma}\bar{L}_{-1}}$
$T(\gamma)z = z + \gamma$, $L_{-1} = \frac{J_1 - iK_1}{2} - i\frac{J_2 - iK_2}{2}$, $\bar{L}_{-1} = -L_{-1}^{\dagger}$
- An important point to notice,we can write
$L_1 = \frac{1}{2}(B - iA), \bar{L}_1 = -\frac{1}{2}(B + iA)$ with
$A\left|\lambda, \sigma, z = 0, \bar{z} = 0\right\rangle = B\left|\lambda, \sigma, z = 0, \bar{z} = 0\right\rangle = 0$

Lie ALgebra

The generators above obey the follwoing commutation relations:

- $[L_m, L_n] = (m-n)L_{m+n}, [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n},$
  $[L_m, \bar{L}_n] = 0$

- with,$L_n^\dagger = -\bar{L}_n, \bar{L}_n^\dagger = -L_n, n = -1, 0, 1$

With all these,we can now get,

$$
\begin{aligned}
U(\Lambda)\,|\lambda,\sigma,z,\bar{z}\rangle &= N(z,\bar{z})U(R(\Lambda z,\Lambda\bar{z}))U(W(\Lambda,z,\bar{z}))\,|\lambda,\sigma,0,0\rangle \\
&= N(z,\bar{z})U(R(\Lambda z,\Lambda\bar{z}))U(D(\alpha))U(S(\beta))\,|\lambda,\sigma,0,0\rangle \\
&= N(z,\bar{z})U(R(\Lambda z,\Lambda\bar{z}))U(D(\alpha))\,|\lambda,\sigma,0,0\rangle \\
&= \tfrac{1}{(cz+d)^{2h}}\tfrac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\,|\lambda,\sigma,\Lambda z,\Lambda\bar{z}\rangle
\end{aligned}
$$

where, $h = \frac{\Delta-\sigma}{2} = \frac{1+i\lambda-\sigma}{2}, \bar{h} = \frac{\Delta+\sigma}{2} = \frac{1+i\lambda+\sigma}{2}$

- $\therefore U(\Lambda)\,|h,\bar{h},z,\bar{z}\rangle = \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\,|h,\bar{h},\Lambda z,\Lambda\bar{z}\rangle$

  $$= \left(\frac{d\Lambda z}{dz}\right)^{h}\left(\frac{d\Lambda\bar{z}}{d\bar{z}}\right)^{\bar{h}}\,|h,\bar{h},\Lambda z,\Lambda\bar{z}\rangle$$

- Unitarity, $\left(U(\Lambda)\,|h',\bar{h}',z',\bar{z}'\rangle, U(\Lambda)\,|h,\bar{h},z,\bar{z}\rangle\right)$
  $= \left(|h',\bar{h}',z',\bar{z}'\rangle, |h,\bar{h},z,\bar{z}\rangle\right)$

Action of Spacetime Translation Operators

- $P^\mu = (P^0 = H, P^1, P^2, P^3)$
  Under Lorentz transformation, $U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$
- Define the time dependent state as,
  $$|h, \bar{h}, u, z, \bar{z}\rangle = e^{iHu} |h, \bar{h}, z, \bar{z}\rangle$$
  $$= (\tfrac{1}{1+z\bar{z}})^{1+i\lambda} e^{iHu} U(R(z,\bar{z})) |h, \bar{h}, 0, 0\rangle$$
- Under Lorentz transformation,
  $$U(\Lambda) |h, \bar{h}, u, z, \bar{z}\rangle =$$
  $$\frac{1}{(cz+d)^{2h}} \frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}} |h, \bar{h}, \frac{u(1+z\bar{z})}{|az+b|^2+|cz+d|^2}, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\rangle$$
- Under spacetime translation,
  $$e^{-il\cdot P} |h, \bar{h}, u, z, \bar{z}\rangle = |h, \bar{h}, u + f(z,\bar{z},l), z, \bar{z}\rangle$$
  where, $f(z,\bar{z},l) = \frac{(l^0-l^3)-(l^1-il^2)z-(l^1+il^2)\bar{z}+(l^0+l^3)z\bar{z}}{1+z\bar{z}}$

Null Infinity

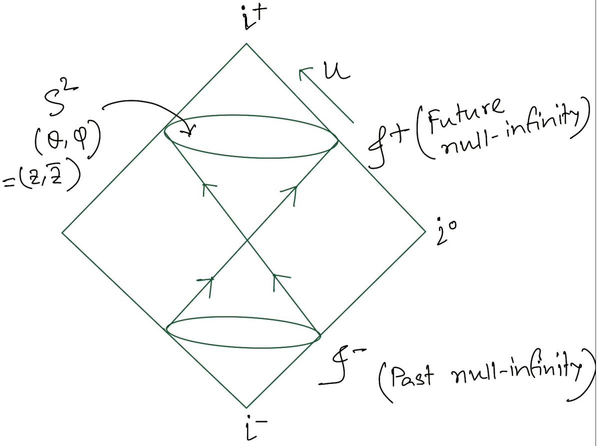
- $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
  $= -du^2 - 2dudr + r^2(d\theta^2 + sin^2\theta d\phi^2)$
  $= -du^2 - 2dudr + r^2 d\Omega_2^2$ where $u = (t - r)$
- In $(u, z, \bar{z})$ coordinates,

  $ds^2 = -du^2 - 2dudr + r^2 \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$

  Where, $z = \frac{x^1+ix^2}{r+x^3} = e^{i\phi} \tan\frac{\theta}{2}$
- We can go to future null-infinity by taking $r \longrightarrow \infty$ at fixed $(u, z, \bar{z})$

  $d\tilde{s}^2 = d\Omega_2^2$ [arXiv:1602.02653]

Null Infinity



Comments

- Poincare group action on the coordinate $(u, z, \bar{z})$ is,

$$\Lambda(u, z, \bar{z}) = \left(\frac{u(1 + z\bar{z})}{|az + b|^2 + |cz + d|^2}, \frac{az + b}{cz + d}, \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} \right) \text{ and}$$
$$T(l)(u, z, \bar{z}) = (u + f(z, \bar{z}, l), z, \bar{z})$$

- **The action of Poincare group on the $(u, z, \bar{z})$ space is same as the action of the Poincare group at null-infinity in Minkowski space if we consider $(u, z, \bar{z})$ with the Bondi coordinates.**
- **In this basis,massless particles can be thought of as living at null-infinity.**

Creation and Annhilation Fields

- Introduce Heisenberg-Picture creation operator, $A^{\dagger}_{\lambda,\sigma}(u,z,\bar{z})$ corresponding to the states $|\lambda,\sigma,u,z,\bar{z}\rangle$ such that,

$$U(\Lambda)A^{\dagger}_{\lambda,\sigma}(u,z,\bar{z})U(\Lambda)^{-1}$$
$$=\frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}A^{\dagger}_{\lambda,\sigma}\left(\frac{u(1+z\bar{z})}{|az+b|^2+|cz+d|^2},\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right)$$

  and

$$e^{-il\cdot P}A^{\dagger}_{\lambda,\sigma}e^{il\cdot P}=A^{\dagger}_{\lambda,\sigma}(u+f(z,\bar{z},l),z,\bar{z})$$

Action of Lorentz group on annihilation field is similar.

- We can write down these creation and annhilation operators in terms of standard creation and annhilation operator as,

  $A^{\dagger}_{\lambda,\sigma}(u,z,\bar{z}) = \frac{1}{\sqrt{8\pi^4}}(\frac{1}{1+z\bar{z}})^{1+i\lambda}\int_0^{\infty} dE E^{i\lambda} e^{iEu} a^{\dagger}(p,\sigma)$

  and

  $A_{\lambda,\sigma}(u,z,\bar{z}) = \frac{1}{\sqrt{8\pi^4}}(\frac{1}{1+z\bar{z}})^{1-i\lambda}\int_0^{\infty} dE E^{-i\lambda} e^{-iEu} a(p,\sigma)$

- The commutator is given by

  $[A_{\lambda,\sigma}(u,z,\bar{z}), A^{\dagger}_{\lambda,\sigma}(u',z',\bar{z}')]$

  $= \langle \lambda,\sigma,u,z,\bar{z}|\lambda',\sigma',u',z',\bar{z}'\rangle$

  $= \frac{\delta_{\sigma\sigma'}}{2\pi}\frac{\Gamma(i(\lambda'-\lambda))}{(1+z\bar{z})^{i(\lambda'-\lambda)}}\frac{\delta^2(z'-z)}{(-i(u'-u+i0_+))^{i(\lambda'-\lambda)}}$

Comments Again

- $A_{\lambda,\sigma}(u, z, \bar{z})$ and $A^{\dagger}_{\lambda,\sigma}(u, z, \bar{z})$ can be interpreted as the positive and negative frequency annihilation and creation fields living on null infinity of Minkowski space.

- we started with the coordinates $(z, \bar{z})$ in the momentum space but once we take the dynamics into account,$(z, \bar{z})$ with the time-like coordinate u transmutes into the null-infinity.

- Notice that,we **did not** arrive at the quantum theory on null-infinity by quantizing a classical theory on null-infinity.

Primary of $ISL(2,\mathbb{C})$

- We define a primary operator of $ISL(2,\mathbb{C})$ any Heisenberg picture operator $\phi_{h,\bar{h}}(u,z,\bar{z})$ tarnsforming as,

$$U(\Lambda)\phi_{h,\bar{h}}(u,z,\bar{z})U(\Lambda)^{-1}$$
$$=\frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\phi_{h,\bar{h}}\left(\frac{u(1+z\bar{z})}{|az+b|^2+|cz+d|^2},\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right)$$

and

$$e^{-il\cdot P}\phi_{h,\bar{h}}(u,z,\bar{z})e^{il\cdot P}=\phi_{h,\bar{h}}(u+f(z,\bar{z},l),z,\bar{z})$$

Symmetries of Massless Particles

- **Supertranslation:**
- Hamiltonian in standard momentum basis is given by,

  $H = \int d\mu(p)|\overrightarrow{p}|a^{\dagger}(p,\sigma)a(p,\sigma)$

  where,the Lorentz invariant measure is,
  $d\mu(p) = \frac{d^3\overrightarrow{p}}{(2\pi)^3 2|\overrightarrow{p}_1|}\delta^3(\overrightarrow{p}_1 - \overrightarrow{p}_2)$

- In terms of z a null vector p can be parametrized as ,

  $p = E(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}})$ An alternative parametrization is given as, $p = \omega(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})$

- with this parametrization,we can write,

  $H = \int d\mu(p) E a^\dagger(E, z, \bar{z}, \sigma) a(E, z, \bar{z}, \sigma)$

  $d\mu(p) = \frac{E^2 dE}{(2\pi)^3 2E} \frac{4d^2 z}{(1+z\bar{z})^2}$,where $d^2 z = dRe(z) dIm(z)$

- Let's now consider the charge defined as,
  $T_f = \int d\mu(p) E f(z, \bar{z}) a^\dagger(E, z, \bar{z}, \sigma) a(E, z, \bar{z}, \sigma) = T_f^\dagger$

  Where $f(z, \bar{z})$ is an arbitrary real smooth function on the 2-sphere.

Supertranslation

- This operator T_f has the following properties,

  (i) $[H, T_f] = 0 \quad \Rightarrow$ charges are conserved for any function f.

  (ii)$[T_f, T_{f'}] = 0$ for arbitrary f and f'

  (iii)$[T_f, a^\dagger(E, z, \bar{z}, \sigma)] = Ef(z, \bar{z}) a^\dagger(E, z, \bar{z}, \sigma)$ and

  $$[T_f, a(E, z, \bar{z}, \sigma)] = -Ef(z, \bar{z}) a(E, z, \bar{z}, \sigma)$$

- We can see the effect of a unitary transformation $U_f = e^{-iT_f}$ on $A^\dagger_{\lambda,\sigma}(u, z, \bar{z})$ as

$$e^{iT_f} A^\dagger_{\lambda,\sigma}(E, z, \bar{z}, \sigma) e^{-iT_f}$$
$$= \frac{1}{\sqrt{8\pi^4}} \left(\frac{1}{1+z\bar{z}}\right)^{1-i\lambda} \int_0^\infty dE E^{i\lambda} e^{iE(u+f(z,\bar{z}))} a^\dagger(p, \sigma)$$
$$= A^\dagger_{\lambda,\sigma}(u + f(z, \bar{z}), z, \bar{z}) \qquad (\textbf{Supertranslation})$$

- infinitesimal transformation, $f(z, \bar{z}) \frac{\partial A_{\lambda,\sigma}}{\partial u} = i[T_f, A_{\lambda,\sigma}]$

Important points to notice

- If we take $f(z, \bar{z})$ to be of the form,

  $f(z, \bar{z}, l) = \frac{(l^0 - l^3) - (l^1 - il^2)z - (l^1 + il^2)\bar{z} + (l^0 + l^3)z\bar{z}}{1 + z\bar{z}}$

  we get back global Minkowski spacetime translation.

- If we relax this condition then we get **Supertranslation** in the $(u, z, \bar{z})$ space.
- We can get the following result easily,
  $U(\Lambda)^{-1} T_f U(\Lambda) = T_{f'}, \;\; f'(z, \bar{z}) = \frac{|az+b|^2 + |cz+d|^2}{1 + z\bar{z}} f(\Lambda z, \Lambda \bar{z})$

- We can write the global translation generators in terms of T_f, $P^\mu = \int d\mu(p) p^\mu a^\dagger(E, z, \bar{z}, \sigma) a(E, z, \bar{z}, \sigma)$

  so we get,
  $P^0 = H = T_f, f = 1$
  $P^1 = T_f, f = \frac{z+\bar{z}}{1+z\bar{z}}$
  $P^2 = T_f, f = \frac{-i(z-\bar{z})}{1+z\bar{z}}$
  $P^2 = T_f, f = \frac{1-z\bar{z}}{1+z\bar{z}}$
- we can write down the following linear combinations,

  $P^0 + P^3 = T_{00}, f = \frac{2z^0\bar{z}^0}{1+z\bar{z}}$
  $P^0 - P^3 = T_{11}, f = \frac{2z^1\bar{z}^1}{1+z\bar{z}}$
  $P^1 + iP^2 = T_{10}, f = \frac{2z^1\bar{z}^0}{1+z\bar{z}}$
  $P^0 - iP^2 = T_{01}, f = \frac{2z^0\bar{z}^1}{1+z\bar{z}}$

Commutators again

- Define supertranslation generators T_{pq} corresponds to
  $F_{pq}(z,\bar{z}) = 2z^p\bar{z}^q$,
  $T_{pq} = \int d\mu(p)\frac{E}{1+z\bar{z}}2z^p\bar{z}^q a^\dagger(E,z,\bar{z},\sigma)a(E,z,\bar{z},\sigma)$

- Lorentz generators and T_{pq} satisfy the following commutation relations,

$$[L_n, T_{pq}] = (\tfrac{n+1}{2} - p)T_{p+n,q} \quad [\bar{L}_n, T_{pq}] = (\tfrac{n+1}{2} - p)T_{p,q+n}$$

$$[L_m, L_n] = (m-n)L_{m+n} \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}$$

$$[T_{pq}, T_{p'q'}] = 0 \qquad [L_m, \bar{L}_n] = 0$$

Space-time Realization

- Consider a massless scalar field in Minkowski space,
  $\phi(x) = \int d\mu(p)(e^{ip\cdot x}a(p) + e^{-ip\cdot x}a^{\dagger}(p))$
- unitry operator e^{iT_F} acts on this field as,
  $\phi_f(x) = e^{iT_f}\phi(x)e^{-iT_f}$
  $$= \int d\mu(p)(e^{ip\cdot x}e^{iE_p f(z,\bar{z})}a(p) + e^{-ip\cdot x}e^{-iE_p f(z,\bar{z})}a^{\dagger}(p))$$
- If we choose $f(z,\bar{z})$ of the the form,
  $f(z,\bar{z},l) = \frac{(l^0-l^3)-(l^1-il^2)z-(l^1+il^2)\bar{z}+(l^0+l^3)z\bar{z}}{1+z\bar{z}}$
- then, $\phi_f(x^{\mu}) = \phi(x^{\mu} + l^{\mu})$

  So,it is a space-time translation by four-vector l^{μ}.
- But there are many functions $f(z,\bar{z})$ for which there is no such simple geometric interpretation in terms of spacetime.

Thank you for listening !!