An Infinite Family of S Invariant Theories on the Celestial Sphere

Annual Review Talk
by
Raju Mandal

Supervisor : Prof. Shamik Banerjee

NISER,Bhubaneswar

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Last year ?

- Our work "MHV gluon scattering in the massive scalar background and celestial OPE" is published in JHEP.
DOI: https://doi.org/10.1007/JHEP10(2023)007

Current work

- This talk is based on our current work "All S invariant gluon OPEs on the celestial sphere" with Shamik Banerjee(NISER), Sagnik Misra(NISER), Sudhakar Panda(CCSP) and Partha Paul(IISC).
arXiv: 2311.16796

Outline :

1. Motivation
2. What is Celestial CFT ?
3. OPE and its importance
4. Celestial OPE
5. Brief discussion on null states
6. Gravity and $w_{1+\infty}$ algebra
7. **Gluons and S algebra**
8. **Building up S invariant OPEs**
9. **Knizhnik-Zamolodchikov (KZ) -type null states**

- $\boxed{\textbf{Conformally soft theorems}}$ $\longrightarrow$ Gauge theories and Gravity in **4-D asymptotically flat spacetime** are enriched with **infinite** number of new **non-trivial symmetries**.

(Guevara,Himwich,Pate and Strominger)

- $\boxed{\textbf{Gravity}}$ $\longrightarrow$ $\boxed{\textbf{wedge subalgebra of } w_{1+\infty} \textbf{ algebra}}$.

- $\boxed{\textbf{Gauge Theories}}$ $\longrightarrow$ $\boxed{S \textbf{ algebra}}$ 2103.03961

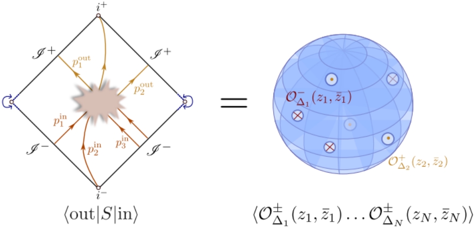
- Banerjee, Kulkarni and Paul have classified all w-invariant theories by computing the OPEs of such theories in case of gravity. 2301.13225

- Our goal is to classify the theories which are invariant under **S algebra** and also to find the **KZ**-type null states of these theories. 2311.16796

(Banerjee, RM, Misra, Panda and Paul '23)

Brief Review of Celestial CFT

"2-D Celestial CFTs" are believed to be the holographic duals of the theories of QG in 4-D asymptotically flat spacetime.



[Courtesy: Laura Donnay]

✧ **Celestial correlation function** is obtained by taking the Mellin transform of the momentum space scattering amplitude.

(Pasterski,Shao,Strominger '17)

$$\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle = \prod_{i=1}^{n}\int_0^\infty d\omega_i\ \omega_i^{\Delta_i-1}\mathcal{S}_n(\{\omega_i,z_i,\bar{z}_i,\sigma_i,a_i\}) \tag{1}$$

- Correlation functions thus obtained transform nicely under Lorentz transformations.

$$\mathcal{A}_n(\{z_i, \bar{z}_i, h_i, \bar{h}_i, a_i\}) = \prod_{i=1}^{n} \frac{1}{(cz_i + d)^{2h_i}} \frac{1}{(\bar{c}\bar{z}_i + \bar{d})^{2\bar{h}_i}} \mathcal{A}_n\left(\left\{\frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}, h_i, \bar{h}_i, a_i\right\}\right). \tag{2}$$

- Continuous Spectrum :

$$\Delta = 1 + i\mathbb{R}. \tag{3}$$

- 2, 3 and 4 -point celestial amplitudes have distributional supports on celestial sphere! $\xrightarrow[\text{std. correlators}]{\text{ways to get}}$ (2202.08288; 2302.10245)

- Celestial primary operators :

$$\mathcal{O}^{a,\epsilon}_{h,\bar{h}}(z, \bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1} A^a(\epsilon\omega, z, \bar{z}, \sigma) \tag{4}$$

- Conformally soft limits are defined at $\Delta = 1, 0, -1, \ldots$

✧ Leading Conformally Soft Theorem :

$$\left\langle R^{1,a}(z)\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle = -\sum_{k=1}^{n}\frac{T_k^a}{z-z_k}\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle \tag{5}$$

$\Rightarrow$ Level zero Kac-Moody algebra. (Additional symmetries !)

✧ Subleading Conformally Soft Theorem :

$$\left\langle R^{0,a}(z,\bar{z})\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle$$

$$= -\sum_{i=1}^{n}\frac{\epsilon_k}{z-z_k}(-2\bar{h}_k+1+(\bar{z}-\bar{z}_k)\bar{\partial}_k)T_k^aP_k^{-1}\left\langle\prod_{i=1}^{n}\mathcal{O}^{a_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle \tag{6}$$

$\Rightarrow$ Current algebra. (More constraints !)

✧ No further soft factorization beyond subleading order for gauge theories.

✧ To get the algebra of infinite tower of soft gluons we have to consider OPE .

What is OPE and Why OPE ?

- **Product of two local operators** as a **sum of local operators at a single point**.

$$\mathcal{O}_i(z_1,\bar{z}_1)\mathcal{O}_j(z_2,\bar{z}_2) = \sum_k c_{ij}^k(z_{12},\bar{z}_{12})\mathcal{O}_k(z_2) \tag{7}$$

- OPE is an **operator equation** :

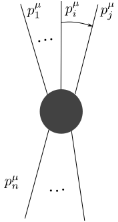
$$\langle\mathcal{O}_i(z_1,\bar{z}_1)\mathcal{O}_j(z_2,\bar{z}_2)...\rangle = \sum_k c_{ij}^k(z_{12},\bar{z}_{12})\langle\mathcal{O}_k(z_2)...\rangle \tag{8}$$

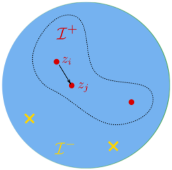
- $(N+1)$-point function $\xrightarrow{\text{OPE}}$ N-point function $\xrightarrow{\text{OPE}}$ ... $\xrightarrow{\text{OPE}}$ combination of OPE coefficients and 2-point function.

- Physical interpretation :

$\boxed{\textbf{OPE in CCFT} \Leftrightarrow \textbf{collinear limit in the bulk.}}$

- OPE can be derived directly from the Celestial amplitude by taking the collinear limit of two gluons.





[Courtesy: Andrea Puhm]

- Example :

$$\mathcal{A}(1^{-a_1}, 2^{+a_2}, 3^{+a_3}, 4^{-a_4}) \xrightarrow[p_3 \cdot p_4 = 0]{z_{34} \to 0} -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3 - 1, \Delta_4 + 1) \mathcal{A}(1^{-a_1}, 2^{+a_2}, 4^{-x})$$

$$+ \text{ subleading in } z_{34} + \ldots \tag{9}$$

✧ Leading OPE structure : 2011.00017

$$\mathcal{O}^{a_3}_{\Delta_3,+}(z_3,\bar{z}_3)\mathcal{O}^{a_3}_{\Delta_4,-}(z_4,\bar{z}_4) \sim -\frac{f^{a_3 a_4 x}}{z_{34}} B(\Delta_3-1,\Delta_4+1)\mathcal{O}^{x}_{\Delta_3+\Delta_4-1,-}(z_4,\bar{z}_4) \tag{10}$$

✧ Subleading,.... terms can be found similarly.

Celestial OPE from asymptotic symmetries

✧ Conformal soft theorems $\Rightarrow$ infinite dimensional asymptotic symmetries $\Rightarrow$ these symmetries put constraints on celestial OPE coefficients.

✧ Celestial OPEs have been computed using these asymptotic symmetries.

(Pate, Raclariu, Strominger and Yuan '19)

(Banerjee, Ghosh, Paul '20)

Null states in CCFT and its importance

- Null states are the **primary descendants** of the algebra.
- In CCFT null states are usually obtained using the OPE and its consistency with the soft factorization theorem.
- Null states inside correlation function $\longrightarrow$ Null decoupling equations (**BG equations**). (Banerjee and Ghosh '20)
- These PDEs have been solved to find scattering amplitude in few cases. (Fan,Fotopoulos,Stieberger,Taylor and Zhu '22)

  (Casali, Melton and Strominger'22)

  (Banerjee, RM, Akavoor and Paul '23)
- Theory of MHV-gravitons and mhv-gluons have been studied in detail.

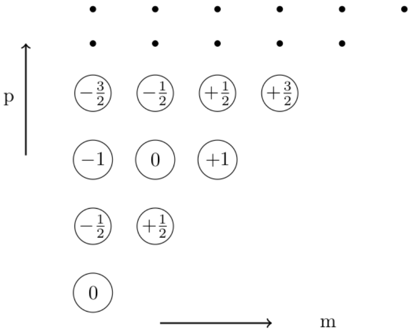
Gravity and w

✧ Asymptotic symmetry algebra : **wedge subalgebra** of $w_{1+\infty}$

(Guevara, Himwich, Pate, Strominger '21)

$$[w^{p}_{\alpha,m}, w^{q}_{\beta,n}] = [m(q-1) - n(p-1)]w^{p+q-2}_{\alpha+\beta,m+n} \tag{11}$$

where $p = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots,\ 1-p \leq m \leq p-1$ and $\Delta = -2p+4$.



- Recently all $w_{1+\infty}$ invariant theories have been classifed by determining their OPEs on the celestial sphere.

(Banerjee,Kulkarni and Paul '23)

$$\begin{aligned} G^{+}_{\Delta_1}(z,\bar{z})G^{+}_{\Delta_2}(0,0)\Big|_{\mathcal{O}(z^0\bar{z}^0)} &= G^{+}_{\Delta_1}(z,\bar{z})G^{+}_{\Delta_2}(0,0)\Big|^{MHV}_{\mathcal{O}(z^0\bar{z}^0)} \\ &+ \sum_{p=1}^{n} B(\Delta_1-1+p,\Delta_2-1)\Omega_p(\Delta_1+\Delta_2). \end{aligned} \tag{12}$$

and

$$\begin{aligned} G^{+}_{\Delta_1}(z,\bar{z})G^{+}_{\Delta_2}(0,0)\Big|_{\mathcal{O}(z^0\bar{z}^1)} &= G^{+}_{\Delta_1}(z,\bar{z})G^{+}_{\Delta_2}(0,0)\Big|^{MHV}_{\mathcal{O}(z^0\bar{z}^1)} \\ &+ \bar{z}\sum_{p=1}^{n} B(\Delta_1+p,\Delta_2-1)\Pi_p(\Delta_1+\Delta_2+1). \end{aligned} \tag{13}$$

- The integer n is not fixed by w-invariant.
- Different values of n correspond to different theories.

- Authors have also found the **KZ type null states** for all such theories.

$$\Xi_n(\Delta) = \xi(\Delta) + \sum_{k=1}^{n} \Pi_k(\Delta+1) = 0 \tag{14}$$

where

$$\xi(\Delta) = L_{-1}G^+_\Delta + H^0_{0,-1}H^1_{-\frac{3}{2},\frac{1}{2}}G^+_{\Delta-1} + H^0_{-1,0}G^+_\Delta + (\Delta-1)H^1_{-\frac{3}{2},-\frac{1}{2}}G^+_{\Delta-1}. \tag{15}$$

- MHV theory of gravity corresponds to $n = 0$.
- It has recently been checked explicity that $n = 4$ gives the OPE for quantum self-dual gravity. (Banerjee,Kulkarni,Paul '23)
- Bulk theories for other values of n ?

  Not known yet !

Soft Gluons and the S Algebra

Soft gluons :

$$R^{k,a}(z,\bar{z}) = \lim_{\Delta\to k}(\Delta - k)O^{a,+}_{\Delta}(z,\bar{z}), \qquad k = 1, 0, -1, \ldots \quad (16)$$

Holomorphic soft gluon currents :

$$R^{k,a}(z,\bar{z}) = \sum_{n=\frac{k-1}{2}}^{\frac{1-k}{2}} \frac{R^{k,a}_n(z)}{\bar{z}^{n+\frac{k-1}{2}}} \quad (17)$$

Modes of the Holomorphic currents :

$$R^{k,a}_n(z) = \sum_{\alpha\in\mathbb{Z}-\frac{k+1}{2}} \frac{R^{k,a}_{\alpha,n}}{z^{\alpha+\frac{k+1}{2}}} \quad (18)$$

The S algebra is obtained from the singular part of the OPE i.e.

Guevara, Himwich, Pate and Strominger '21

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n\mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2). \tag{19}$$

Algebra :

$$[R^{k,a}_{\alpha,m},R^{l,b}_{\beta,n}] = -if^{ab}{}_c\frac{(\frac{1-k}{2}-m+\frac{1-l}{2}-n)!}{(\frac{1-k}{2}-m)!(\frac{1-l}{2}-n)!}\frac{(\frac{1-k}{2}+m+\frac{1-l}{2}+n)!}{(\frac{1-k}{2}+m)!(\frac{1-l}{2}+n)!}R^{k+l-1,c}_{\alpha+\beta,m+n} \tag{20}$$

Redefinition :

$$S^{q,a}_{\alpha,m} = (q-m-1)!(q+m-1)!R^{3-2q,a}_{\alpha,m} \tag{21}$$

S Algebra :

$$[S^{p,a}_{\alpha,m},S^{q,b}_{\beta,n}] = -if^{abc}S^{p+q-1,c}_{\alpha+\beta,m+n} \tag{22}$$

S algebra primaries

$$R^{k,a}_{p-\frac{k+1}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}(0,0)=0, \qquad p\geq 2 \tag{23}$$

and

$$\begin{aligned}&R^{k,a}_{p-\frac{k+1}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}(0,0)\\&=-if^{abc}\frac{(-1)^{k+q+1}}{\Gamma(-k-q+2)}\frac{\Gamma(\Delta-1)}{\Gamma(\Delta+q+k-2)}\frac{\bar{\partial}^q}{q!}\mathcal{O}^{c,+}_{\Delta+k-1}(0,0)\end{aligned} \tag{24}$$

where $0\leq q\leq 1-k$, $k=1,0,-1,\ldots$

OPE of two positive helicity outgoing gluons

General structure of the OPE is

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)$$
$$= -\frac{if^{ab}{}_c}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1+n-1,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}_2^n \mathcal{O}^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)$$
$$+\sum_{p,q=0}^{\infty}\sum_{k=1}^{\tilde{n}_{p,q}} z_{12}^p \bar{z}_{12}^q C^k_{p,q}(\Delta_1,\Delta_2)\tilde{\mathcal{O}}^{ab}_{k,p,q}(z_2,\bar{z}_2). \tag{25}$$

Task is to determine:

- The OPE coefficients $C^k_{p,q}$ and
- the S-algebra descendants $\tilde{\mathcal{O}}^{ab}_{k,p,q}$ of a positive helicity soft gluon.

Strategy :

- We consider S invariant theories for all of which S-algebra is universal $\Rightarrow$ Existence of a Master OPE .
- We know that tree-level MHV sector of the pure YM theory is an example of such S invariant theories.
- This Master OPE inserted in a MHV gluon scattering amplitude $\Rightarrow$ known MHV OPE.
- $\boxed{\textbf{Master OPE} = \textbf{MHV-sector OPE} + \textbf{R}}$
- R should vanish inside MHV scattering amplitude $\Rightarrow$ R is a lin. combination of MHV null states.
- R consists only non-singular terms.

Null states are important !

Using the above arguments we can rewrite (25) as,

$$\boxed{\begin{aligned}\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{Any Theory}} &= \mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\big|_{\text{MHV}} \\ &+ \sum_{p,q=0}^{\infty} z_{12}^{p}\bar{z}_{12}^{q}\sum_{i=1}^{\tilde{n}_{p,q}} \tilde{C}^{k}_{p,q}(\Delta_1,\Delta_2) M^{a,b}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar{z}_2).\end{aligned}} \tag{26}$$

- ☆ $M^{a,b}_{k,p,q}$ are the MHV null states at $\mathcal{O}(z_{12}^{p}\bar{z}_{12}^{q})$.
- ☆ We perform the analysis at $\mathcal{O}(z_{12}^{0}\bar{z}_{12}^{0})$.

MHV NULL STATES AT $\mathcal{O}(1)$

The general null state at $\mathcal{O}(1)$ in the MHV-sector is given by

$$\Psi_j^{ab}(\Delta) = R^{-j,a}_{\frac{j-1}{2},\frac{j+1}{2}}\mathcal{O}^b_{\Delta+j,+} - \frac{(-1)^j j}{\Gamma(j+2)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-2)}R^{1,a}_{-1,0}\mathcal{O}^b_{\Delta-1,+} - \frac{(-1)^j}{\Gamma(j+1)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)}R^{0,a}_{-1/2,1/2}\mathcal{O}^b_{\Delta,+} \tag{27}$$

where $j = 1, 2, 3, \ldots$

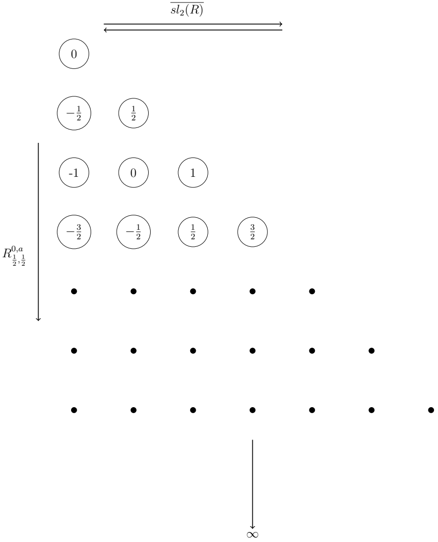
Now we will define the following basis of null states which is more convenient to work with

$$M_k^{ab}(\Delta) = \sum_{i=1}^{k}\frac{1}{\Gamma(k-i+1)}\frac{\Gamma(\Delta+k-1)}{\Gamma(\Delta+i-1)}\Psi_i^{ab}(\Delta). \tag{28}$$

Also define,

$$M_k^a(\Delta) = f^{abc}M_k^{bc}. \tag{29}$$

✧ Focus only on the generators $(R^{1,a}_{n,0}, R^{0,a}_{\frac{1}{2},\frac{1}{2}}, H^0_{0,0}, H^0_{0,\pm 1})$ to study the action of S-algebra on the null states.



Action of the S algebra on the MHV null states

Action of the Leading Soft Gluon modes:

$$R^{1,a}_{0,0} M^{bc}_k(\Delta) = -if^{abd} M^{dc}_k(\Delta) - if^{acd} M^{bd}_k \tag{30}$$

$$R^{1,a}_{n,0} M^{bc}_k(\Delta) = 0, n > 0 \tag{31}$$

Action of the Subleading Soft Gluon mode :

$$\begin{aligned}
&[R^{0,a}_{1/2,1/2}, M^{bc}_k(\Delta)] \\
&= -if^{abd}(k+2)M^{dc}_{k+1}(\Delta-1) \\
&\qquad + (\Delta+k-2)\left\{ if^{acd} M^{bd}_k(\Delta-1) + if^{abd} M^{dc}_k(\Delta-1) \right\}.
\end{aligned} \tag{32}$$

Building up S invariant OPE

Observation :

- Consider the following set of null states,

$$M_k^{bc}(\Delta), k = 1, 2, 3, ..., n. \quad (33)$$

- Action of $R^{0,a}_{1/2,1/2}$ on the null states is closed if we set

$$M_{k+1}^{ab}(\Delta) = 0, k \geq n \geq 0. \quad (34)$$

Inference :

- We can get an S invariant OPE if we consider the finite set of null states (33).

S invariant OPE at $\mathcal{O}(1)$:

$$\boxed{\begin{aligned} &\mathcal{O}^b_{\Delta_1,+}(z,\bar{z})\mathcal{O}^c_{\Delta_2,+}(0,0)|_{\mathcal{O}(1)} \\ &= \mathcal{O}^b_{\Delta_1,+}(z,\bar{z})\mathcal{O}^c_{\Delta_2,+}(0,0)\Big|^{MHV}_{\mathcal{O}(1)} + \sum_{k=1}^{n} B(\Delta_1+k, \Delta_2-1) M_k^{bc}(\Delta_1+\Delta_2) \end{aligned}}$$

(35)

S invariance of the OPE

Action of $R^{0,a}_{1/2,1/2}$:

$$
\begin{aligned}
&R^{0,x}_{\frac{1}{2},\frac{1}{2}}(\mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2,+}(0,0))|_{\mathcal{O}(1)} - R^{0,x}_{\frac{1}{2},\frac{1}{2}}\Bigg[\mathcal{O}^a_{\Delta_1,+}(z,\bar{z})\mathcal{O}^b_{\Delta_2,+}(0,0)|^{MHV}_{\mathcal{O}(1)} \\
&\qquad + \sum_{k=1}^{n} B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2)\Bigg] \\
&= if^{xay}(n+2)B(\Delta_1+n,\Delta_2-1)M^{yb}_{n+1}(\Delta_1+\Delta_2-1) = 0.
\end{aligned}
\tag{36}
$$

- One can also verify that OPE (35) is invariant under the action of $R^{1,a}_{n,0}$ and $H^0_{0,1}$.
- Truncated OPE (35) is invariant under the S algebra.

Infinite family of S invariant theories :

- We have shown that the following set of equations are S invariant.

$$M_{k+1}^{ab}(\Delta) = 0, k \geq n \geq 0. \tag{37}$$

- We can truncate the OPE at $\mathcal{O}(1)$ at an arbitrary n in S invariant way.
- But the S invariance does not fix the value of integer n.
- Hence, different choices of the integer n give rise to a discrete infinite family of S-invariant OPEs.
- Each of these consistent OPEs correspond to a S invariant theory.
- We do not know the Lagrangian description of these theories except for for the MHV YM and perhaps the self-dual Yang-Mills theory.

- KZ-type null states involve the L_{-1} descendants on the CS^2.
- We obtain KZ-type null states by using OPE commutivity and taking different soft limits.

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) = \mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1). \tag{38}$$

- **KZ-type null states :**

$$K^a(\Delta) = \xi^a(\Delta) - i\sum_{k=1}^{n} M^a_k(\Delta+1), \tag{39}$$

where

$$\xi^a(\Delta) = C_A L_{-1}\mathcal{O}^{a,+}_{\Delta} - (\Delta+1)R^{1,b}_{-1,0}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta} - R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta+1}. \tag{40}$$

- These null states are also invariant under S algebra.

Summary and Outlook

Summary :

- ✔ Discussed briefly about CCFT.
- ✔ Reviewed the results for the case of Gravity.
- ✔ Discussed the S invariant OPEs.

Outlook :

- ❍ One can think of investigating the theories for different values of n.
- ❍ KZ-type null states inside correlation function would give rise to the differential equations that could be solved to find the scattering amplitudes for different theories.

Thank you for your attention !!