# Holographic Symmetry Algebra for the MHV Sector Revisited


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## Celestial Amplitudes for Massless Particles

Parameterization of null momentum in $(-,+,+,+)$ signature:

$$p_k^\mu = \epsilon_k \omega_k (1 + z_k \bar{z}_k, z_k + \bar{z}_k, -i(z_k - \bar{z}_k), 1 - z_k \bar{z}_k), \quad p_k^2 = 0 \tag{1}$$

One particle state in boost eigenbasis: Pasterksi, Shao, Strominger

$$|\Delta, \sigma, z, \bar{z}\rangle = \int_0^\infty d\omega \omega^{\Delta - 1} |\omega, \sigma, z, \bar{z}\rangle \qquad \text{Mellin transform} \tag{2}$$

where $\Delta = 1 + i\lambda$ with $\lambda \in \mathbb{R}$, which ensures the following normalization

$$\langle \lambda_1, \sigma_1, z_1, \bar{z}_1 | \lambda_2, \sigma_2, z_2, \bar{z}_2 \rangle = \delta(\lambda_1 - \lambda_2) \delta^2(z_1 - z_2) \delta_{\sigma_1, \sigma_2} \tag{3}$$

and $\sigma$ is the helicity of the massless particle.

Celestial Amplitude:

$$\boxed{\mathcal{A}_n(\{z_i, \bar{z}_i, h_i, \bar{h}_i, a_i\}) = \langle \prod_{i=1}^n \mathcal{O}_{h_i, \bar{h}_i}^{a_i}(z_i, \bar{z}_i) \rangle = \prod_{i=1}^n \int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} \mathcal{S}_n(\{\omega_i, z_i, \bar{z}_i, \sigma_i, a_i\})} \tag{4}$$

where $a_i$ is the Lie algebra index of the $i$-th particle and scaling dimensions $(h_i, \bar{h}_i)$ are defined as

$$h_i = \frac{\Delta_i + \sigma_i}{2}, \quad \bar{h}_i = \frac{\Delta_i - \sigma_i}{2}. \tag{5}$$

Under Lorentz transformations, celestial amplitudes transform as:

$$\mathcal{A}_n(\{z_i, \bar{z}_i, h_i, \bar{h}_i, a_i\}) = \prod_{i=1}^n \frac{1}{(cz_i + d)^{2h_i}} \frac{1}{(\bar{c}\bar{z}_i + \bar{d})^{2\bar{h}_i}} \mathcal{A}_n\left(\left\{\frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}, h_i, \bar{h}_i, a_i\right\}\right). \tag{6}$$

Celestial MHV amplitudes:

$$\widetilde{\mathcal{M}}\left(1_{\Delta_1}^-(z_1, \bar{z}_1), 2_{\Delta_2}^-(z_2, \bar{z}_2), 3_{\Delta_3}^+(z_3, \bar{z}_3), \cdots, n_{\Delta_n}^+(z_n, \bar{z}_n)\right) \tag{7}$$

## Conformally Soft Theorems and Asymptotic Symmetries

(Positive helicity) leading conformally soft graviton theorem: (Tree level)

$$\langle H^1(z, \bar{z}) \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = \sum_{k=1}^n \left(\frac{\bar{z} - \bar{z}_k}{z - z_k}\right) H^1_{-\frac{1}{2}, -\frac{1}{2}}(k) \langle \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle \tag{8}$$

where the positive helicity leading conformally soft graviton $H^1(z, \bar{z})$ is defined as

$$H^1(z, \bar{z}) = \lim_{\Delta \to 1} (\Delta - 1) G^+_\Delta(z, \bar{z}) \tag{9}$$

and the operator $H^1_{-\frac{1}{2}, -\frac{1}{2}}(k)$ acts on a primary operator as

$$H_{-\frac{1}{2}, -\frac{1}{2}}(k) \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) = -\delta_{ik} \phi_{h_i + \frac{1}{2}, \bar{h}_i + \frac{1}{2}}(z_i, \bar{z}_i) \tag{10}$$

The $\bar{z}$ expansion of RHS of (8) leads us to define the following two currents

$$H^1(z, \bar{z}) = H^1_{\frac{1}{2}}(z) + \bar{z} H^1_{-\frac{1}{2}}(z). \tag{11}$$

Ward Identities: He, Lysov, Mitra, Strominger; Banerjee, Ghosh, Paul

$$\langle H^1_{\frac{1}{2}}(z) \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = -\sum_{k=1}^n \left(\frac{\bar{z}_k}{z - z_k}\right) H^1_{-\frac{1}{2}, -\frac{1}{2}}(k) \langle \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle \tag{12}$$

and

$$\langle H^1_{-\frac{1}{2}}(z) \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = \sum_{k=1}^n \left(\frac{1}{z - z_k}\right) H^1_{-\frac{1}{2}, -\frac{1}{2}}(k) \langle \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle \tag{13}$$

Supertranslation algebra:

$$[H^1_{\alpha, m}, H^1_{\beta, n}] = 0; \quad m, n = \pm\frac{1}{2}. \tag{14}$$

(Positive helicity) subleading conformally soft graviton theorem:

$$\langle H^0(z, \bar{z}) \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = \sum_{k=1}^n \frac{(\bar{z} - \bar{z}_k)^2}{z - z_k} \left[\frac{2\bar{h}_k}{\bar{z} - \bar{z}_k} - \bar{\partial}_k\right] \langle \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle \tag{15}$$

Positive helicity subleading soft graviton operators:

$$H^0(z, \bar{z}) = \lim_{\Delta \to 0} \Delta G^+_\Delta(z, \bar{z}) \tag{16}$$

Currents:

$$H^0(z, \bar{z}) = \sum_{n=-1}^1 \frac{H^0_n(z)}{\bar{z}^{n-1}} \tag{17}$$

## Conformally Soft Theorems and Asymptotic Symmetries (continued)

Ward Identities: Cachazo, Strominger; Banerjee, Ghosh, Paul

$$\langle H^0_1(z) \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = -\sum_{k=1}^n \frac{\bar{L}_1(k)}{z - z_k} \langle \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle \tag{18}$$

$$\langle H^0_0(z) \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = \sum_{k=1}^n \frac{2\bar{L}_0(k)}{z - z_k} \langle \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle \tag{19}$$

$$\langle H^0_{-1}(z) \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle = -\sum_{k=1}^n \frac{\bar{L}_{-1}(k)}{z - z_k} \langle \prod_{i=1}^n \phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i) \rangle \tag{20}$$

where the operators, $\bar{L}_1(k)$, $\bar{L}_0(k)$ and $\bar{L}_{-1}(k)$ are given by

$$\bar{L}_1(k) = \bar{z}_k^2 \bar{\partial}_k + 2\bar{h}_k \bar{z}_k, \quad \bar{L}_0(k) = \bar{z}_k \bar{\partial}_k + \bar{h}_k, \quad \bar{L}_{-1}(k) = \frac{\partial}{\partial \bar{z}_k} \tag{21}$$

which generate the anti-holomorphic Lorentz transformations or $\overline{SL(2, \mathbb{R})}$ transformations.

$$H^0_m(z) = \sum_{\alpha \in \mathbb{Z} - 1} \frac{H^0_{\alpha, m}}{z^{\alpha + 1}} \tag{22}$$

with $m = +1, 0, -1$ The modes $H^0_{\alpha, m}$ generate the $\overline{SL(2, \mathbb{R})}$ current algebra and with the following identifications

$$J^a_1 = -H^0_{a,1}, \quad J^a_0 = \frac{1}{2} H^0_{a,0}, \quad J^a_{-1} = -H^0_{a,-1} \tag{23}$$

$\hat{sl}_2(\mathbb{R})$ algebra:

$$[J^a_m, J^b_n] = (m - n) J^{a+b}_{m+n}. \tag{24}$$

where $m, n = 0, \pm 1$ and $a, b \in \mathbb{Z}$.

## Symmetry Algebra for Celestial MHV graviton Amplitudes and Null states (Found in 2020)

- Symmetry algebra generated by the conformally soft positive helicity gravitons in the MHV sector is
  $\hat{sl}_2(\mathbb{R}) \ltimes$ super-translations in $(2, 2)$ signature
- The representation of this algebra contains null states. They are the **primary descendants** of the representation.
- It also has **KZ-type null states**, which contain holomorphic translation descendant $(L_{-1} G^+_\Delta)$ of a positive helicity graviton Banerjee, Ghosh & Paul '20

$$\boxed{L_{-1} G^+_\Delta + H^0_{0,-1} H^1_{-\frac{3}{2}, \frac{1}{2}} G^+_{\Delta - 1} + H^0_{-1,0} G^+_\Delta + (\Delta - 1) H^1_{-\frac{3}{2}, -\frac{1}{2}} G^+_{\Delta - 1} = 0}. \tag{25}$$

- Decopuling of these null states gives rise to differential equations for the MHV graviton amplitudes.
- In the representation of the above symmetry algebra, there are only $(n - 2)$ such equations for an $n$ point MHV amplitude, corresponding to $(n - 2)$ positive helicity gravitons.
- The tree level MHV graviton scattering amplitudes can be completely determined by solving these equations.

## The Puzzle and Our Goal

- In WZW model, there are $n$ KZ equations for an $n$ point amplitude but in the MHV sector there are only $(n - 2)$ such equations.
- It was not possible to find the two missing equations using the above symmetry algebra.
- Our Goal is to find the two missing null states in the MHV sector of gravitons and gluons.

## REFERENCES


S. Banerjee, M. Maitra, R. Mandal and M. Patra, "Holographic symmetry algebra for the MHV sector revisited," [arXiv:2311.16796 [hep-th]] (accepted in JHEP)



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## Holographic Symmetry Algebra (HSA) for the MHV Graviton Sector (Found in 2021)

Singular terms in the OPE between two positive-helicity outgoing gravitons are given by Guevara, Himwich, Pate and Strominger '21

$$G^+_{\Delta_1}(z_1, \bar{z}_1) G^+_{\Delta_2}(z_2, \bar{z}_2) \sim -\frac{\bar{z}_{12}}{z_{12}} \sum_{n=0}^\infty B(\Delta_1 - 1 + n, \Delta_2 - 1) \frac{(\bar{z})^n_{12}}{n!} \bar{\partial}_2^n G^+_{\Delta_1 + \Delta_2}(z_2, \bar{z}_2) \tag{26}$$

Conformally soft positive-helicity gravitons, $H^k(z, \bar{z})$:

$$H^k(z, \bar{z}) = \lim_{\Delta \to k} (\Delta - k) G^+_\Delta(z, \bar{z}), \quad k = 2, 1, 0, -1, \cdots \tag{27}$$

Algebra of the modes:

$$\begin{aligned}\left[H^{k_1}_{\alpha_1, m_1}, H^{k_2}_{\alpha_2, m_2}\right] &= -\left[m_2(2 - k_1) - m_1(2 - k_2))\right] \\ &\times \frac{\left(\frac{2-k_1}{2} - m_1 + \frac{2-k_2}{2} - m_2 - 1\right)!}{\left(\frac{2-k_1}{2} - m_1\right)! \left(\frac{2-k_2}{2} - m_2\right)!} \frac{\left(\frac{2-k_1}{2} + m_1 + \frac{2-k_2}{2} + m_2 - 1\right)!}{\left(\frac{2-k_1}{2} + m_1\right)! \left(\frac{2-k_2}{2} + m_2\right)!} H^{k_1 + k_2}_{\alpha_1 + \alpha_2, m_1 + m_2}\end{aligned} \tag{28}$$

Light transformed operators:

$$w^p_m = \frac{1}{\kappa}(p - 1 - m)!(p - 1 + m)! H_m^{-2p+4} \tag{29}$$

where, $1 - p \leq m \leq p - 1$.

$$w_{1+\infty}: \qquad \boxed{[w^p_m, w^q_n] = [m(q - 1) - n(p - 1)] w^{p+q-2}_{m+n}} \tag{30}$$

- But, the puzzle remained unsolved.

## HSA for MHV Graviton Sector Revisited in 2025

- Mixed helicity OPE: Banerjee, Maitra, RM & Patra '25

$$G^-_{\Delta_1}(z_1, \bar{z}_1) G^+_{\Delta_2}(z_2, \bar{z}_2) \sim -\frac{\bar{z}_{12}}{z_{12}} \sum_{n=0}^\infty B(\Delta_1 + 3 + n, \Delta_2 - 1) \frac{(\bar{z})^n_{12}}{n!} \bar{\partial}_2^n G^-_{\Delta_1 + \Delta_2}(z_2, \bar{z}_2) \tag{31}$$

- OPE between two negative-helicity gravitons does not have a pole term in the MHV sector.

We define an infinite family of conformally soft negative-helicity gravitons $\bar{H}^k(z, \bar{z})$ as

$$\bar{H}^k(z, \bar{z}) = \lim_{\Delta \to k} (\Delta - k) G^-_\Delta(z, \bar{z}), \quad k = -2, -3, -4, -5, \cdots \tag{32}$$

Light transformed operators:

$$\bar{w}^q_n = \frac{1}{\kappa}(q - 1 - n)!(q - 1 + n)! \bar{H}_n^{-2q} \tag{33}$$

The complete symmetry algebra for MHV gravitons:

$$\boxed{\begin{gathered}[w^p_m, w^q_n] = [m(q - 1) - n(p - 1)] w^{p+q-2}_{m+n} \\ [w^p_m, \bar{w}^q_n] = [m(q - 1) - n(p - 1)] \bar{w}^{p+q-2}_{m+n} \\ [\bar{w}^p_m, \bar{w}^q_n] = 0\end{gathered}} \tag{34}$$

This extended symmetry algebra has additional null states. Decoupling of these null states give rise to two missing KZ equations

$$\boxed{L_{-1} G^-_\Delta + H^0_{-1,0} G^-_\Delta + (\Delta + 3) H^1_{-\frac{3}{2}, -\frac{1}{2}} G^-_{\Delta - 1} + H^0_{0,-1} H^1_{-\frac{3}{2}, \frac{1}{2}} G^-_{\Delta - 1} + \bar{H}^{-3}_{\frac{5}{2}, -\frac{1}{2}} G^+_{\Delta + 3} = 0}. \tag{35}$$

## Results for MHV Gluon Scattering Amplitudes (2025)

The complete symmetry algebra for MHV gluons: Banerjee, Maitra, RM & Patra '25

$$\boxed{\begin{gathered}[S^{p,a}_m, S^{q,b}_n] = -i f^{abc} S^{p+q-1,c}_{m+n} \\ [S^{p,a}_m, \bar{S}^{q,b}_n] = -i f^{abc} \bar{S}^{p+q-1,c}_{m+n} \\ [\bar{S}^{p,a}_m, \bar{S}^{q,b}_n] = 0\end{gathered}} \tag{36}$$

KZ type null states:

$$\boxed{C_A L_{-1} O^{a,-}_\Delta - (\Delta + 2) R^{1,b}_{-1,0} R^{1,b}_{0,0} O^{a,-}_\Delta - R^{0,b}_{-\frac{1}{2}, \frac{1}{2}} R^{1,b}_{0,0} O^{a,-}_{\Delta + 1} - \bar{R}^{-1,b}_{1,0} R^{1,b}_{0,0} O^{a,+}_{\Delta + 2} = 0.} \tag{37}$$

where $C_A$ is the quadratic Casimir of the adjoint representation.