Massless Particles at Null Infinity

Raju Mandal

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Abstract

In this review project I have studied about the construction of a new basis for massless one-particle state which is motivated by the work of Pasterski-Shao-Strominger. I have also studied the Lorentz transformation and spacetime translation properties of these new states. Lorentz transformations on these states have been studied by using Wigner's approach which also is known as "method of induced representation". The dynamics of the massless particles take place at null-infinity of the Minkowski space. The correlation function is Poincare invariant and the field operators in the correlation function can be thought of as they are inserted at various points $(z, \bar{z})$ on the 2-sphere at null-infinity. Then I have studied some applications of this new construction in case of supertranslation and spacetime relaization. This is a review work of the reference[1] and partly of [3].

Contents

1 Introduction

This work mainly focuses on the construction of the new basis for massless one-particle state (which is obtained by doing Mellin trasform of the momentum states) and doing quantum field theory in that basis and some applications also have been studied in this new basis.I have first discussed about the Lorentz group in $(2+1)$ dimesnions and have shown that $SL(2,\mathbb{R})$ is the double cover of the $SO(2,1)$.Then I have discussed about the little group of the null direction because little group plays an important role in the method of induced representation.Then I have discussed about infinite dimensional representation of the Lorentz group in Hilbert space.We also get to see that wavefunctions $\Psi(\Delta, z)$ in this construction transform as $\frac{1}{(cz+d)^{2\Delta^\star}}\Psi\big(\Delta, \frac{az+b}{cz+d}\big)$.The calculations for $(3+1)$ dimesions are similar.Here the main results are that the new states $|\lambda, \sigma, z, \bar{z}\rangle$ transorm under Lorentz transformation as $\frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\,|\lambda, \sigma, \Lambda z, \Lambda\bar{z}\rangle$.Now once we take the dynamics into account the states $|h, \bar{h}, u, z, \bar{z}\rangle = e^{iHu}\,|h, \bar{h}, z, \bar{z}\rangle$ transmutes to the null-infinity.We can think of these states as the asymptotic states for massless particles.Then the Poincare group action on these states have been studied.We also have constructed the creation and annhilation operators which can be thought of as living at the null-infinity.Then in the applcation part, I have discussed the effect of a unitary operator $U = e^{-iT_f}$ on the new creation and annhilation operators and we see that it generates supertranslation on 2-sphere.Then we see that if we choose the function $f(z, \bar{z})$ to be of a special form then we get back our global Minkowski translations.

2 Lorentz Group in (2+1) Dimensions

$SO(2,1)$ is the Lorentz group in $(2+1)$ dimensions and $SL(2,\mathbb{R})$ is the double cover of $SO(2,1)$.To see this, let's consider a 3-vector in $SO(2,1)$, $P = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \end{pmatrix}$

We can associate a real symmetric matrix with the above three vector as,

$$P = \begin{pmatrix} P^0 - P^2 & P^1 \\ P^1 & P^0 + P^2 \end{pmatrix}$$

Notice that the $detP = -P^2$,where $\eta = (-1, 1, 1)$

Now let's consider a 3 dimensional vector space $\mathbb{V}$ of 2×2 real symmetric matrices.We can choose a basis in this vector space $\mathbb{V}$,

$$\{e_0, e_1, e_2\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

So,we can write any arbitrary vector $P \in \mathbb{V}$ in this basis as,

$$P = P^0 e_0 + P^1 e_1 + P^2 e_2$$

Now, $SL(2, \mathbb{R})$ group is a group of 2×2 real matrices with $Det + 1$,

$$SL(2, \mathbb{R}) : \left\{ \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (ad - bc) = 1, a, b, c, d \in \mathbb{R} \right\}$$

The action of the $SL(2, \mathbb{R})$ group element on the elements of the vector space $\mathbb{V}$ is given by,

$$P' = \Lambda P \Lambda^T$$

It is very easy to see that, $DetP = DetP'$ because $Det\Lambda = +1$.So, this transformation preserves the norm.Since $\{^+_- \Lambda\}$ satisfy the above equation,so we see that this is a 2 to 1 mapping from $SL(2, \mathbb{R})$ to $SO(2, 1)$ or $SL(2, \mathbb{R})$ is the double cover of $SO(2, 1)$.The group isomorphism is given by $SO(2, 1)^{\uparrow} \cong SL(2, \mathbb{R})/\mathbb{Z}_2$.

2.1 Action on Null Momenta

Since we will do quantum field theory of massless particles in this project,so we are interested in null momenta.
Let's define a new variable

$$z = \frac{P^1}{P^0 + P^2}$$

with $[-(P^0)^2 + (P^1)^2 + (P^2)^2] = 0$,
Since under Lorentz transformation, $P' = \Lambda P \Lambda^T$, z goes to,

$$z \longrightarrow z' = \Lambda z = \frac{P'^1}{P'^0 + P'^2} = \frac{az + b}{cz + d}$$

Where Λ is

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The variable z has the physical interpretation that it parametrizes the space of null momentum directions.We can see this from the following explanation,
Let's take a set of null vectors of the form

$$k^\mu = (E, 0, E), z = 0$$

Now if we give a rotation in 1-2 plane,we get the most general form of null momentum vector in $(2 + 1)$ dimension,which is

$$k'^\mu = (E, Esin\theta, Ecos\theta), z = tan\frac{\theta}{2}$$

So, we can see that z depends only on the direction θ of the null ray.That's why z parametrizes the space of null directions which is circle in $(2 + 1)$ dimensions.Where θ can be thought of as the stereographic coordinate of the circle.

2.2 Standard Rotation and Boost Matrices

The standard rotation and boost matrices in my convention in $(2+1)$ dimensions are,

$\underline{SL(2,\mathbb{R})}$	$\underline{SO(2,1)}$
$R(z) =^{+}_{-} \begin{pmatrix} cos\frac{\theta}{2} & sin\frac{\theta}{2} \\ -sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{pmatrix}$	$R_{12}(\theta)^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix}$
$B_1(\eta) =^{+}_{-} \begin{pmatrix} cosh(\frac{\eta}{2}) & -sinh(\frac{\eta}{2}) \\ -sinh(\frac{\eta}{2}) & cosh(\frac{\eta}{2}) \end{pmatrix}$	$\Lambda_1(\eta)^{\mu}{}_{\nu} = \begin{pmatrix} cosh\eta & -sinh\eta & 0 \\ -sinh\eta & cosh\eta & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$B_2(\zeta) =^{+}_{-} \begin{pmatrix} e^{\frac{\zeta}{2}} & 0 \\ 0 & e^{-\frac{\zeta}{2}} \end{pmatrix}$	$\Lambda_2(\zeta)^{\mu}{}_{\nu} = \begin{pmatrix} cosh\eta & 0 & -sinh\eta \\ 0 & 1 & 0 \\ -sinh\eta & 0 & cosh\eta \end{pmatrix}$

We can get the $SO(2,1)$ matrices on the right if we know the $SL(2,\mathbb{R})$ matrices on the left from the following relation,

$$P'^{\mu} = R(\Lambda)^{\mu}{}_{\nu}P^{\nu} = \Lambda P \Lambda^{T}$$

Where Λ is $SL(2,\mathbb{R})$ matrix and $R(\Lambda)$ is the corresponding $SO(2,1)$ matrix which acts on three vector P^{ν}.

We will take the set $\{k^{\mu} = (E,0,E), E > 0\}$ as the reference null direction with $z = 0, \theta = 0$. Action of $R(z)$ on this reference null direction is the following,

$$\{k^{\mu} = (E,0,E) > 0 | z = 0\} \longrightarrow \{k'^{\mu} = (E, Esin\theta, Ecos\theta), z = tan\frac{\theta}{2}\}$$

We can now easily find the rotation and boost generators from infinitesimal transformations for both the $SL(2,\mathbb{R})$ and $SO(2,1)$ matrices,for example

$$J_0 = -i\frac{d}{d\theta}R_{12}(\theta)|_{\theta=0}$$

Once we get the generators in this way,we can see that these generators obey the following Lie Algebra,

$$[J_0, K_1] = iK_2, \qquad [J_0, K_2] = -iK_1, \qquad [K_1, K_2] = -J_0$$

2.3 Discussion on Little Group

We are interested in little group because when we will be finding the unitary representation of the Lorentz group on Hilbert space, we will follow Wigner's approach,what is also known as "Method of Induced Representation"[7],little group will play an important role there.

Little Group: Little group is defined w.r.t a 3-vector(in $(2+1)$ dimesnion),it's a subgroup of Lorentz group under which a 3-vector remains invariant, i.e,

$$\Lambda^{\mu}{}_{\nu}P^{\nu} = P^{\mu}$$

We can find the generator and the little group element by the following way,consider infinitesimal little group transformation so that,

$$\Lambda^{\mu}{}_{\nu}P^{\nu} = (\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu})P^{\nu} = P^{\mu} + \omega^{\mu}{}_{\nu}P^{\nu}$$

Since, Λ is the little group element,then

$$\omega^{\mu\nu}P_{\nu} = 0$$

If you take a null vector of the form $P^{\nu} = n^{\mu} = (1, 0, 1)$ then we get,

$$\Rightarrow \omega^{\mu\nu} = -i\alpha \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$= -i\alpha A^{\mu\nu}$$

We can see that little group in this case is a signle parameter group.Where A is the generator which can also be written in the follwoing form,

$$(A)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix} = -(J_0)^{\mu}{}_{\nu} - (K_1)^{\mu}{}_{\nu}$$

And the finite little group element is,

$$\Lambda = e^{-i\alpha A}$$

But here we are interested in the little group of the null direction instead of the null vector.In that case,the little group is given by the little group generator and the boost generator.
So, the little group $D(z = 0)$ of our reference null direction $\{k^{\mu} = (E, 0, E) > 0 | Z = 0\}$ is generated by the two elements A and K_2.
Where it is easy to see that,

$$[K_2, A] = iA$$

And the little group of the general null direction $\{k^{\mu} = (E, E\sin\theta, E\cos\theta) | z = \tan\frac{\theta}{2}\}$ is given by the following conjugation relation,

$$D(z) = R(z)D(z = 0)R(z)^{-1}$$

.

2.4 Hilbert Space Representation of the Lorentz Group

We represent a massless single particle state in momentum basis as $|p\rangle$ with $p^2 = 0$. Standard normalization of these states is given by,

$$\langle p_1 | p_2 \rangle = (2\pi)^2 2|\overrightarrow{p_1}|\delta^2(\overrightarrow{p_1} - \overrightarrow{p_2})$$

And unitary representation of an arbitrary Lorentz transformation Λ is given by,

$$U(\Lambda)\,|p\rangle = |\Lambda p\rangle$$

For example,if Λ is a boost along 2 direction,then

$$U(\Lambda)\left|E,0,E\right\rangle = \left|e^{-\eta}E,0,e^{-\eta}E\right\rangle$$

where η is the rapidity, defined as $\tanh\eta = v$. And the the state $|p\rangle$ is annihilated by the little group generator,i.e,

$$A\left|E,0,E\right\rangle = 0, E > 0$$

A annhilates this state because this little group generator gives zero acting on the corresponding momentum vector and here we are not considering any other quantum number or internal degrees of freedom of the particle.

2.4.1 Construction of New Basis

We are now going to define a new basis[4] corresponding to the reference null vector as,

$$\left|\Delta, z=0\right\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty dE\ E^{\Delta-1}\left|E,0,E\right\rangle\ ;\ \ \Delta\in\mathbb{C} \tag{2.1}$$

If we now apply the boost in 2-direction we get,

$$\begin{aligned} U(B_2(\eta))\left|\Delta, z=0\right\rangle &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty dE\ E^{\Delta-1}\left|e^{-\eta}E,0,e^{-\eta}E\right\rangle \\ &= e^{\eta\Delta}\left|\Delta, z=0\right\rangle \qquad \text{(By doing the change of variable,}E' = e^{-\eta}E) \end{aligned} \tag{2.2}$$

Now we will define a state in this basis corresponding to the most general null momentum by,

$$\begin{aligned} \left|\Delta, z\right\rangle &:= \frac{1}{(1+z^2)^{\Delta}}U(R(z))\left|\Delta, z=0\right\rangle \\ &= N_\Delta(z)U(R(z))\left|\Delta, z=0\right\rangle \end{aligned} \tag{2.3}$$

or,

$$\left|\Delta, z\right\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(1+z^2)^{\Delta}}\int_0^\infty dE\ E^{\Delta-1}\left|E, Esin\theta, Ecos\theta\right\rangle \tag{2.4}$$

The inner product of the states is,

$$\begin{aligned} \langle\Delta_2, z_2|\Delta_1, z_1\rangle = &\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(1+z_2^2)^{\Delta_2^\star}}\times\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(1+z_1^2)^{\Delta_1}}\times \\ &\int_0^\infty\int_0^\infty dE_2 dE_1 E_2^{\Delta_2^\star-1}E_1^{\Delta_1-1}\left\langle E_2, E_2sin\theta_2, E_2cos\theta_2|E_1, E_1sin\theta_1, E_1cos\theta_1\right\rangle \end{aligned} \tag{2.5}$$

Now we will write the bracket as,

$$
\begin{aligned}
&\langle E_2, E_2 sin\theta_2, E_2 cos\theta_2 | E_1, E_1 sin\theta_1, E_1 cos\theta_1 \rangle \\
&= \langle \bar{p_2} | \bar{p_1} \rangle \\
&= (2\pi)^2 2 |p_1| \delta^{(2)}(\bar{p_2} - \bar{p_1}) \\
&= (2\pi)^2 2E_1 \delta(p_{2x} - p_{1x}) \delta(p_{2y} - p_{1y}) \\
&= (2\pi)^2 2E_1 \delta(E_2 \sin\theta_2 - E_1 \sin\theta_1) \delta(E_2 \cos\theta_2 - E_1 \cos\theta_1) \\
&= (2\pi)^2 2E_1 \delta\left(\frac{2z_2 E_2}{1+z_2^2} - \frac{2z_1 E_1}{1+z_1^2} \right) \delta\left(\frac{E_2(1-z_2^2)}{1+z_2^2} - \frac{E_1(1-z_1^2)}{1+z_1^2} \right) \\
&= (2\pi)^2 2E_1 \delta(f_1(z_2, E_2)) \delta(f_2(z_2, E_2)) \\
&= (2\pi)^2 2E_1 \frac{(1+z_2^2)}{2E_2} \delta(z_2 - z_1) \delta(E_2 - E_1) \\
&= (2\pi)^2 (1+z_2^2) \delta(z_2 - z_1) \delta(E_2 - E_1)
\end{aligned}
$$

where in the last step i have used the following formula [8] for the delta function,

$$
\delta(f_1(x,y))\delta(f_2(x,y)) = \frac{\delta(x-x_0)\delta(y-y_0)}{|\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x}\frac{\partial f_1}{\partial y}|}
$$

with

$$
f_1(x_0, y_0) = 0 = f_2(x_0, y_0)
$$

So,we now get-

$$
\begin{aligned}
\langle \Delta_2, z_2 | \Delta_1, z_1 \rangle &= \frac{1}{(2\pi)^3} \frac{1}{(1+z_2^2)^{\Delta_2^\star}} \frac{1}{(1+z_1^2)^{\Delta_1}} \times \\
&\int_0^\infty \int_0^\infty dE_2 dE_1 E_2^{\Delta_2^\star - 1} E_1^{\Delta_1 - 1} \times (2\pi)^2 (1+z_2^2) \delta(z_2 - z_1) \delta(E_2 - E_1) \\
&= \frac{1}{2\pi} \frac{(1+z_2^2)\delta(z_2 - z_1)}{(1+z_2^2)^{\Delta_2^\star + \Delta_1}} \times \int_0^\infty dE_2 E^{\Delta_2^\star + \Delta_1 - 2} \\
&= \delta(z_2 - z_1)\delta(\lambda_2 - \lambda_1) \quad \text{for, } \Delta_i = \frac{1}{2} + i\lambda_i, \lambda_i \in \mathbb{R}
\end{aligned} \tag{2.6}
$$

Where I have used the following identity,

$$
2\pi\delta(\nu) = \int_0^\infty du \ \ u^{i\nu - 1}
$$

2.4.2 Action of Arbitrary Lorentz Transformation

We will now calculate the effect of arbitrary lorentz transformation on the state $|\Delta, z\rangle$,

$$
\begin{aligned}
U(\Lambda) |\Delta, z\rangle &= N_\Delta U(\Lambda) U(R(z)) |\Delta, z = 0\rangle \\
&= N_\Delta U(R(\Lambda z)) U^{-1}(R(\Lambda z)) U(\Lambda) U(R(z)) |\Delta, z = 0\rangle \\
&= N_\Delta U(R(\Lambda z)) W(\Lambda, z) |\Delta, z = 0\rangle
\end{aligned} \tag{2.7}
$$

where, we can write W as,

$$
W(\Lambda, z) = U^{-1}(R(\Lambda z)) U(\Lambda) U(R(z)) = U(R^{-1}(\Lambda z) \Lambda R(z))
$$

If we notice,$R^{-1}(\Lambda z)\Lambda R(z)$ belongs to the little group of the reference null vector.We can realize that from the following arguments:
(i)$R(z)$ takes $z=0$ to $z=tan\frac{\theta}{2}$
(ii)Λ takes z to $\Lambda z=\frac{az+b}{cz+d}$
(iii)Then in the last step,$R^{-1}(\Lambda z)$ takes Λz to $z=0$.

Now let's see the effect of $W(\Lambda,z)$ on the state $|\Delta,z=0\rangle$.Since,

$$(R^{-1}(\Lambda z)\Lambda R(z))(E,0,E)=(e^{-\eta},0,e^{-\eta}),E>0$$

with

$$e^{-\eta}=\frac{(cz+d)^2+(az+b)^2}{1+z^2}=(cz+d)^2\frac{1+(\Lambda z)^2}{1+z^2}$$

and

$$\Lambda z=\frac{az+b}{cz+d}$$

we can write,

$$\begin{aligned}W(\Lambda,z)\left|\Delta,z=0\right\rangle&=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty dE\ E^{\Delta-1}W(\Lambda,z)\left|E,0,E\right\rangle\\&=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^\infty dE\ E^{\Delta-1}\left|e^{-\eta}E,0,e^{-\eta}E\right\rangle\\&=e^{\eta\Delta}\left|\Delta,z=0\right\rangle\end{aligned}\tag{2.8}$$

Using eqn((2.3)) and eqn((2.8)) we can get,

$$\begin{aligned}U(\Lambda)\left|\Delta,z\right\rangle&=N_\Delta U(R(\Lambda z))U^{-1}(R(\Lambda z))U(\Lambda)U(R(z))\left|\Delta,z=0\right\rangle\\&=\frac{1}{(cz+d)^2}\left|\Delta,\Lambda z\right\rangle\end{aligned}\tag{2.9}$$

Using the above result we can say that $\{|\Delta,z\rangle\}$ states with fixed Δ give the representation of the Lorentz group.We can easily check that

$$U(\Lambda_2)U(\Lambda_1)=U(\Lambda_2\Lambda_1)$$

and also this represention is unitary i.e,

$$(U(\Lambda)\left|\Delta',z'\right\rangle,U(\Lambda\left|\Delta,z\right\rangle))=(\left|\Delta',z'\right\rangle,\left|\Delta,z\right\rangle)$$

This representation with fixed Δ is known as "Unitary principal continuous series Representation."

2.5 Representation of Lorentz Group on Wavefunctions

In this section we will see how the wavefunction transforms under Lorentz transformation.An arbitrary massless one-particle state $|\Psi\rangle$ can be written using completeness relation as,

$$|\Psi\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\lambda\ dz\ \left|\Delta,z\right\rangle\left\langle\Delta,z|\Psi\right\rangle\tag{2.10}$$

We can also write down the inner product of two such states as,

$$\langle\Phi|\Psi\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\lambda\ dz\ \Phi^\star(\Delta,z)\Psi(\Delta,z)\tag{2.11}$$

The action of arbitrary Lorentz transformation on the the wavefunction is given by,

$$\begin{aligned}
(U(\Lambda^{-1})\Psi)(\Delta, z) &= (|\Delta, z\rangle\,, U(\Lambda)^{-1}\Psi) \\
&= (U(\Lambda)\,|\Delta, z\rangle\,, |\Psi\rangle) \\
&= \frac{1}{(cz+d)^{2\Delta^\star}}\Psi(\Delta, \frac{az+b}{cz+d})
\end{aligned} \tag{2.12}$$

where $\Delta^\star = \frac{1}{2} - i\lambda$.

From this eqn((2.12)) we can also find out the differential form of the translation generator,special conformal transformation generator and the generator of the scale transformation. All we have to do to get that is we have to first write the exponential form of these unitary transformation and then expanding the wavefunction and comparing both sides we can get the differential form of the corresponding generators.

3 Lorentz Group in $(3+1)$ Dimensions

Now we will discuss about the Lorentz group in four dimensions.We will basically follow the same arguments as what we did in $(2+1)$ dimensions.But here our state will get an extra helicity index σ.

Consider a four momentum,

$$P^\mu = \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}$$

We can associate a hermitian matrix with this four-vector as,

$$P = \begin{pmatrix} P^0 - P^3 & P^1 + iP^2 \\ P^1 - iP^2 & P^0 + P^3 \end{pmatrix}$$

where, $detP = -P^2$.

In $(3+1)$ dimensions $SL(2,\mathbb{C})$ is double cover of $SO(3,1)$ and the group isomorphism is

$$SO(3,1)^\uparrow \cong SL(2,\mathbb{C})/\mathbb{Z}_2$$

The action of $SL(2,\mathbb{C})$ group element on P is given by,

$$P' = \Lambda P \Lambda^\dagger$$

where,

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad (ad - bc) = 1 \quad \text{and} \;\; a,,b,c,d \in \mathbb{C}$$

Now let's define the stereographic coordinate of the two sphere by,

$$z = \frac{P^1 + iP^2}{P^0 + P^3}$$

It can be checked that under Lorentz transformation this coordinate transforms as,

$$z \to z' = \frac{P'^1 + iP'^2}{P'^0 + P'^3} = \frac{az+b}{cz+d}$$

We can parametrize the most general null vector in spherical polar coordinate as,

$$P^\mu = \big(E, \quad Esin\theta cos\phi, \quad Esin\theta sin\phi, \quad Ecos\theta\big)$$

and

$$z = tan\frac{\theta}{2}e^{i\phi}$$

. In terms of z and $\bar{z}$,

$$P^\mu = E\left(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right)$$

Now again we will consider a reference null direction of the form,

$$\{(E,0,0,E), E>0|z=0\}$$

We can find the little group of this reference null direction by the same way we did in $(2+1)$ dimensions.
The Little group of the above null direction is given by (J_3, K_3, A, B), where $A = J_2 - K_1$ and $B = -J_1 - K_2$. And K and Js are the boost and rotation matrices in $(3+1)$ dimensions.

3.1 New Basis

We will define the new state in $(3+1)$ dimensions as,

$$|\lambda, \sigma, z=0, \bar{z}=0\rangle = \frac{1}{\sqrt{8\pi^4}}\int_0^\infty dE\ E^{i\lambda}\,|E,0,0,E;\sigma\rangle\,, \lambda \in \mathbb{R} \tag{3.1}$$

where σ is the helicity of the massless particle.
Standard normalization of momentum states is given by,

$$\langle p_1, \sigma_1|p_2, \sigma_2\rangle = (2\pi)^3 2|\overrightarrow{p_1}|\delta^3(\overrightarrow{p_1} - \overrightarrow{p_2})\delta_{\sigma_1,\sigma_2} \tag{3.2}$$

Now we will define the most general state in this basis as,

$$\begin{aligned}|\lambda, \sigma, z, \bar{z}\rangle &= (\frac{1}{1+z\bar{z}})^{1+i\lambda}U(R(z,\bar{z}))\,|\lambda,\sigma,z=0,\bar{z}=0\rangle \\ &= N(z,\bar{z})U(R(z,\bar{z}))\,|\lambda,\sigma,z=0,\bar{z}=0\rangle\end{aligned} \tag{3.3}$$

where,

$$U(R(z,\bar{z})) = e^{-i\phi J_3}e^{-i\theta J_2}e^{i\phi J_{J_3}}, z = tan\frac{\theta}{2}e^{i\phi}$$

These states are also delta function normalizable[6],i.e

$$\langle\lambda_1, \sigma_1, z_1, \bar{z}_1|\lambda_2, \sigma_2, z_2, \bar{z}_2\rangle = \delta(\lambda_1 - \lambda_2)\delta^2(z_1 - z_2)\delta_{\sigma_1,\sigma_2} \tag{3.4}$$

where,

$$\delta^2(z_1 - z_2) = \delta(Rez_1 - Rez_2)\delta(Imz_1 - Imz_2)$$

3.1.1 Action of Lorentz Group

The action of the little group on the the state $|\lambda, \sigma, z = 0, \bar{z} = 0\rangle$ is given by,

$$A|\lambda, \sigma, z = 0, \bar{z} = 0\rangle = B|\lambda, \sigma, z = 0, \bar{z} = 0\rangle = 0$$

These are set to zero on this state because of the fact that there are finite number of polarizations of the massless particles[9].

The effects of roration and boost on this state are the following ,

$$U(R_3(\phi))|\lambda, \sigma, z = 0, \bar{z} = 0\rangle = e^{i\sigma\phi}|\lambda, \sigma, z = 0, \bar{z} = 0\rangle$$

$$U(B_3(\eta))|\lambda, \sigma, z = 0, \bar{z} = 0\rangle = e^{\eta\Delta}|\lambda, \sigma, z = 0, \bar{z} = 0\rangle$$

where ϕ is the rotation around 3-axis and the η is the rapidity for the boost along 3-direction.

Now consider an arbitrary Lorentz transformation Λ,we would now like to calculte how the unitary transformation corresponding to this arbitrary Lorentz transformation acts on the state defined by eqn.(3.3).Here we will do by the same approach we did in $(2 + 1)$ dimesnions i.e, by follwoing Wigner's method of doing it.

$$\begin{aligned} U(\Lambda)|\lambda, \sigma, z, \bar{z}\rangle &= N(z, \bar{z})U(\Lambda)U(R(z, \bar{z}))|\lambda, \sigma, z = 0, \bar{z} = 0\rangle \\ &= N(z, \bar{z})U(R(\Lambda z, \Lambda\bar{z}))U(R^{-1}(\Lambda z, \Lambda\bar{z})\Lambda R(z, \bar{z}))|\lambda, \sigma, z = 0, \bar{z} = 0\rangle \\ &= N(z, \bar{z})U(R(\Lambda z, \Lambda\bar{z}))U(W(\Lambda, z, \bar{z}))|\lambda, \sigma, z = 0, \bar{z} = 0\rangle \end{aligned} \tag{3.5}$$

Again we can easily relaize that $W(\Lambda, z, \bar{z})$ belongs to the little group of the standard null direction $\{(E, 0, 0, E)|z = 0\}$

If we write this little group element explicitly then we can show that $W(\Lambda, z, \bar{z})$ can be decomposed as,

$$W(\Lambda, z, \bar{z}) = \begin{pmatrix} e^{\frac{\alpha}{2}} & 0 \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} = D(\alpha)S(\beta) \tag{3.6}$$

where $e^{\alpha} = \frac{1+|z|^2}{1+|\Lambda z|^2}\frac{1}{(cz+d)^2}$

We can easily find the $SO(3, 1)$ matrix corresponding to the little group element $W(\Lambda, z, \bar{z})$.We can also find the generators and the exponetial form of the $SO(2, 1)$ matrix.So we can also write our infinite dimensional operator $U(\Lambda)$ by the same exponential form,but in this is case the generators will have differential form which will also obey the same algebra.The differential form of the generators of infinite dimensional representation can be found using the eqn.((2.12)) but in $(3 + 1)$ dimesnions.So what we find is the following,

$$U(D(\alpha)) = e^{\alpha L_0 + \bar{\alpha}\bar{L}_0} \tag{3.7}$$

Where,

$$D(\alpha)z = e^{\alpha}z \tag{3.8}$$

and the generators can be written as,

$$L_0 = \frac{iK_3 - J_3}{2} \quad \text{with} \quad \bar{L}_0 = -L_0^{\dagger} \tag{3.9}$$

We also get,

$$U(S(\beta)) = e^{\beta L_1 + \bar{\beta}\bar{L}_1} \tag{3.10}$$

Where the effect of this transformation on z is teh following,

$$S(\beta)z = \frac{z}{1-\beta z} \tag{3.11}$$

where

$$L_1 = \frac{iK_1 - J_1}{2} + i\frac{iK_2 - J_2}{2} \quad \text{with} \quad \bar{L}_1 = -L_1^\dagger \tag{3.12}$$

Similarly we can also write for the translation of z,which is,

$$U(T(\gamma)) = e^{\gamma L_{-1} + \bar{\gamma}\bar{L}_{-1}} \tag{3.13}$$

and,

$$T(\gamma)z = z + \gamma \tag{3.14}$$

where,

$$\bar{L}_{-1} = -L_{-1}^\dagger \tag{3.15}$$

It is also important to notice that we can write the generators of translation as,

$$\begin{aligned} L_1 &= \frac{1}{2}(B - iA) \\ \bar{L}_1 &= -\frac{1}{2}(B + iA) \end{aligned} \tag{3.16}$$

Now it can easily be checked that the above mentioned generators will obey the following Lie algebra,

$$[L_m, L_n] = (m-n)L_{m+n}, [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}, \quad n = -1, 0, 1 \tag{3.17}$$

Now with all these information we can get the result[1] of how the arbitrary Lorentz transformation will act the general states.And the result is,

$$U(\Lambda)|\lambda, \sigma, z, \bar{z}\rangle = \frac{1}{(cz+d)^{2h}} \frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}} |\lambda, \sigma, \Lambda z, \Lambda\bar{z}\rangle \tag{3.18}$$

where $h = \frac{\Delta - \sigma}{2} = \frac{1+i\lambda-\sigma}{2}, \bar{h} = \frac{\Delta+\sigma}{2} = \frac{1+i\lambda+\sigma}{2}$
It can easily be checked that we can write eqn.(3.5) as,

$$U(\Lambda)|\lambda, \sigma, z, \bar{z}\rangle = \left(\frac{d\Lambda z}{dz}\right)^h \left(\frac{d\Lambda\bar{z}}{d\bar{z}}\right)^{\bar{h}} |h, \bar{h}, \Lambda z, \Lambda\bar{z}\rangle \tag{3.19}$$

Using these transformation laws one can show that this representation is indeed unitary,i.e,

$$(U(\Lambda)|h', \bar{h}', z', \bar{z}'\rangle, U(\Lambda)|h, \bar{h}, z, \bar{z}\rangle) = (|h', \bar{h}', z', \bar{z}'\rangle, |h, \bar{h}, z, \bar{z}\rangle) \tag{3.20}$$

4 Action of Spacetime Translation

Now we will discuss about the spacetime translation on the states $|h, \bar{h}, z, \bar{z}\rangle$.The four spacetime translation generators are $P^\mu = (P^0 = H, P^1, P^2, P^3)$.Under Lorentz transformation these generators transform as,

$$U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu \tag{4.1}$$

Let's define the time dependent states as,

$$|h, \bar{h}, u, z, \bar{z}\rangle = e^{iHu}|h, \bar{h}, z, \bar{z}\rangle = (\frac{1}{1+z\bar{z}})^{1+i\lambda} e^{iHu} U(R(z, \bar{z}))|h, \bar{h}, 0, 0\rangle \tag{4.2}$$

We can think of these states somewhat analogous to the Heisenberg picture state.
The effect of Lorentz transformation[5] and spacetime translations on this state is the following,

$$U(\Lambda)\left|h,\bar{h},u,z,\bar{z}\right\rangle = \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\left|h,\bar{h},\frac{u(1+z\bar{z})}{|az+b|^2+|cz+d|^2},\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right\rangle \tag{4.3}$$

$$e^{-il\cdot P}\left|h,\bar{h},u,z,\bar{z}\right\rangle = \left|h,\bar{h},u+f(z,\bar{z},l),z,\bar{z}\right\rangle \tag{4.4}$$

where,

$$f(z,\bar{z},l) = \frac{(l^0-l^3)-(l^1-il^2)z-(l^1+il^2)\bar{z}+(l^0+l^3)z\bar{z}}{1+z\bar{z}} \tag{4.5}$$

Before going further, let's discuss about null infinity here.We can write the Minkowski metric as,

$$\begin{aligned} ds^2 &= -dt^2+dx^2+dy^2+dz^2 \\ &= -du^2-2dudr+r^2(d\theta^2+sin^2\theta d\phi^2) \\ &= -du^2-2dudr+r^2d\Omega_2^2 \end{aligned} \tag{4.6}$$

In $(u,z,\bar{z})$ coordinates,we can write

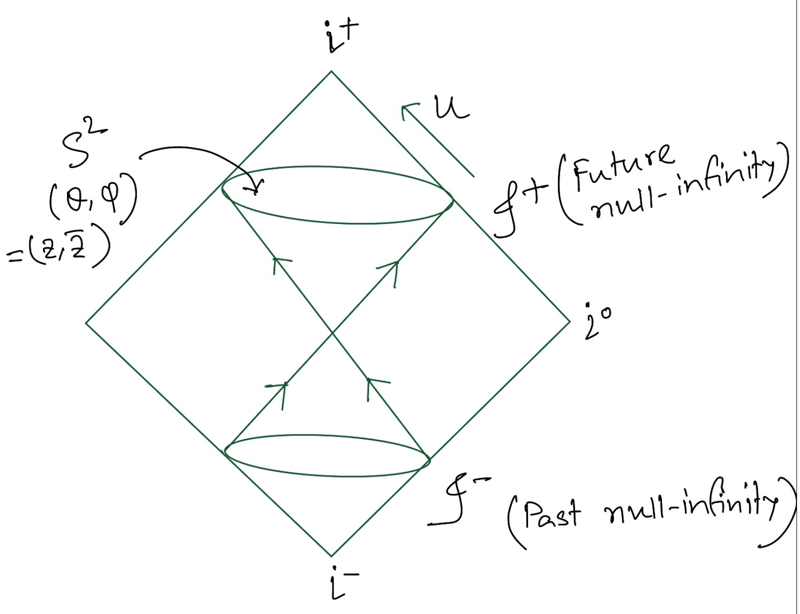
$$ds^2 = -du^2-2dudr+r^2\frac{4dzd\bar{z}}{(1+z\bar{z})^2} \tag{4.7}$$

where,

$$z = \frac{x^1+ix^2}{r+x^3} = e^{i\phi}\tan\frac{\theta}{2} \tag{4.8}$$

We can go to the null infinity by taking $r \longrightarrow \infty$ at fixed $(u,z,\bar{z})$.Actually the metric (37) cannot capture $r \longrightarrow \infty$ limit.So we have to do a scaling and have to take the limit to get the induced metric[2] which is the following,

$$d\tilde{s}^2 = d\Omega_2^2$$



In the above Penrose diagram of Minkowski spacetime, I have shown the future and past null-infinity.The 2-sphere S^2 in the above diagram is called "Celestial sphere".We can also have the same digram in momentum space. Now going back to our discussion,the Poincare group action on the coordinates $(u, z, \bar{z})$ is,

$$\begin{aligned}\Lambda(u, z, \bar{z}) = \left(\frac{u(1+z\bar{z})}{|az+b|^2 + |cz+d|^2}, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \right) \text{ and} \\ T(l)(u, z, \bar{z}) = (u + f(z, \bar{z}, l), z, \bar{z})\end{aligned} \tag{4.9}$$

So,"The action of Poincare group on the $(u, z, \bar{z})$ space is same as the action of the Poincare group at null-infinity in Minkowski space if we consider $(u, z, \bar{z})$ with the Bondi coordinates."

In this new basis massless single particles can be thought of as living at null-infinity.

5 Creation and Annhilation Fields

let's now introduce Heisenberg-Picture creation operator $A^\dagger_{\lambda,\sigma}(u, z, \bar{z})$ corresponding to the states $|\lambda, \sigma, u, z, \bar{z}\rangle$ such that,

$$\begin{aligned}U(\Lambda)A^\dagger_{\lambda,\sigma}(u, z, \bar{z})U(\Lambda)^{-1} = \frac{1}{(cz+d)^{2h}} \frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}} A^\dagger_{\lambda,\sigma}\left(\frac{u(1+z\bar{z})}{|az+b|^2 + |cz+d|^2}, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \right), \\ e^{-il\cdot P}A^\dagger_{\lambda,\sigma}e^{il\cdot P} = A^\dagger_{\lambda,\sigma}(u + f(z, \bar{z}, l), z, \bar{z})\end{aligned} \tag{5.1}$$

The above transformation can be checked by the same way we check the following relation in standard qft,

$$U(\Lambda)a_p^\dagger U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\overrightarrow{p}}}{E_{\overrightarrow{p}}}}a^\dagger_{\Lambda\overrightarrow{p}}$$

We can write these creation and annihilation fields in terms of the momentum space creation and annihilation operators as,

$$\begin{aligned} A^\dagger_{\lambda,\sigma}(u,z,\bar{z}) &= \frac{1}{\sqrt{8\pi^4}}(\frac{1}{1+z\bar{z}})^{1+i\lambda}\int_0^\infty dE E^{i\lambda}e^{iEu}a^\dagger(p,\sigma), \\ A_{\lambda,\sigma}(u,z,\bar{z}) &= \frac{1}{\sqrt{8\pi^4}}(\frac{1}{1+z\bar{z}})^{1-i\lambda}\int_0^\infty dE E^{-i\lambda}e^{-iEu}a(p,\sigma) \end{aligned} \tag{5.2}$$

With all these,we can check the following commutation and anti-commutation relation which is,

$$\begin{aligned} [A_{\lambda,\sigma}(u,z,\bar{z}), A^\dagger_{\lambda,\sigma}(u',z',\bar{z}')]_{\frac{+}{-}} &== \langle\lambda,\sigma,u,z,\bar{z}|\lambda',\sigma',u',z',\bar{z}'\rangle \\ &= \frac{\delta_{\sigma\sigma'}}{2\pi}\frac{\Gamma(i(\lambda'-\lambda))}{(1+z\bar{z})^{i(\lambda'-\lambda)}}\frac{\delta^2(z'-z)}{(-i(u'-u+i0_+))^{i(\lambda'-\lambda)}} \end{aligned} \tag{5.3}$$

So, $A_{\lambda,\sigma}(u,z,\bar{z})$ and $A^\dagger_{\lambda,\sigma}(u,z,\bar{z})$ can be interpreted as the positive and negative frequency annihilation and creation fields living on null infinity of Minkowski space.Another important point to notice is that,we started with the coordinates $(z,\bar{z})$ in the momentum space but once we take the dynamics into account,$(z,\bar{z})$ with the time-like coordinate u transmutes into the null-infinity.Also notice that,we **did not** arrive at the quantum field theory on null-infinity by quantizing a classical theory on null-infinity.

6 Primary of $ISL(2,\mathbb{C})$

Here we will define primary operator of $ISL(2,\mathbb{C})$ as the Heisenberg-picture operators which transform as,

$$\begin{aligned} U(\Lambda)\phi_{h,\bar{h}}(u,z,\bar{z})U(\Lambda)^{-1} &= \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\phi_{h,\bar{h}}\left(\frac{u(1+z\bar{z})}{|az+b|^2+|cz+d|^2}, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right), \\ e^{-il\cdot P}\phi_{h,\bar{h}}(u,z,\bar{z})e^{il\cdot P} &= \phi_{h,\bar{h}}(u+f(z,\bar{z},l),z,\bar{z}) \end{aligned} \tag{6.1}$$

The correlation functions are invariant under $ISL(2,\mathbb{C})$ transformations,i.e,

$$\langle\Omega|\prod_{i=1}^n U(l,\Lambda)\phi_i(P_i)U^{-1}(l,\Lambda)|\Omega\rangle = \langle\Omega|\prod_{i=1}^n \phi_i(P_i)|\Omega\rangle \tag{6.2}$$

where $\phi_i(P_i)$ is some $ISL(2,\mathbb{C})$ operator which is inserted at $P_i = (u,z,\bar{z})$ and $|\Omega\rangle$ is the Poincare invariant vacuum.

7 Applications

7.1 Supertranslation

Hamiltonian in standard qft is written as,

$$H = \int d\mu(p)|\overrightarrow{p}|a^\dagger(p,\sigma)a(p,\sigma) \tag{7.1}$$

where the Lorentz invariant measure $d\mu(p)$ given by,

$$d\mu(p) = \frac{d^3\overrightarrow{p}}{(2\pi)^3 2|\overrightarrow{p}_1|}\delta^3(\overrightarrow{p}_1 - \overrightarrow{p}_2) \tag{7.2}$$

In terms of z, a null vector p can be parametrized as

$$\begin{aligned} p = E(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}) \quad \text{or} \\ p = \omega(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}) \end{aligned} \tag{7.3}$$

With the above parametrization of p Hamiltonian takes the form,

$$H = \int d\mu(p) E a^\dagger(E, z, \bar{z}, \sigma) a(E, z, \bar{z}, \sigma) \tag{7.4}$$

where,

$$d\mu(p) = \frac{E^2 dE}{(2\pi)^3 2E}\frac{4d^2z}{(1+z\bar{z})^2}, d^2z = dRe(z)dIm(z) \tag{7.5}$$

Let's now consider the following charge defined as,

$$T_f = \int d\mu(p) E f(z,\bar{z}) a^\dagger(E, z, \bar{z}, \sigma) a(E, z, \bar{z}, \sigma) = T_f^\dagger \tag{7.6}$$

where $f(z, \bar{z})$ is an arbitrary real smooth function on the 2-sphere.

By doing simple calculations one can show that the operator T_f satifies the following commutation relations,
(i)

$$[H, T_f] = 0 \tag{7.7}$$

(ii)

$$[T_f, T_{f'}] = 0 \tag{7.8}$$

(iii)

$$\begin{aligned} [T_f, a^\dagger(E, z, \bar{z}, \sigma)] = E f(z,\bar{z}) a^\dagger(E, z, \bar{z}, \sigma) \\ [T_f, a(E, z, \bar{z}, \sigma)] = -E f(z,\bar{z}) a(E, z, \bar{z}, \sigma) \end{aligned} \tag{7.9}$$

Supertranslation operator in $(u, z, \bar{z})$ space is given by $U = e^{-iH_f}$.

We will now consider a unitary transformation $U_f = e^{-iT_f}$,which acts on $A^\dagger_{\lambda,\sigma}(u, z, \bar{z})$ as[3],

$$\begin{aligned} e^{iT_f} A^\dagger_{\lambda,\sigma}(E, z, \bar{z}, \sigma) e^{-iT_f} &= \frac{1}{\sqrt{8\pi^4}}(\frac{1}{1+z\bar{z}})^{1-i\lambda}\int_0^\infty dE E^{i\lambda} e^{iE(u+f(z,\bar{z}))} a^\dagger(p, \sigma) \\ &= A^\dagger_{\lambda,\sigma}(u + f(z,\bar{z}), z, \bar{z}) \end{aligned} \tag{7.10}$$

And for infinitesimal transformamtion we get,

$$f(z,\bar{z})\frac{\partial A_{\lambda,\sigma}}{\partial u} = i[T_f, A_{\lambda,\sigma}] \tag{7.11}$$

So,we can say that $U_f = e^{-iT_f}$ generates point transformation in $(u, z, \bar{z})$ space which is given by,

$$(u, z, \bar{z}) \longrightarrow (u + f(z,\bar{z}), z, \bar{z}) \tag{7.12}$$

here $f(z,\bar{z})$ is arbitrary real smooth function of z and $\bar{z}$.If we choose $f(z,\bar{z})$ of the following form,

$$f(z,\bar{z},l) = \frac{(l^0 - l^3) - (l^1 - il^2)z - (l^1 + il^2)\bar{z} + (l^0 + l^3)z\bar{z}}{1 + z\bar{z}} \tag{7.13}$$

then we get back our global Minkowski spacetime translations.But if we relax this condition and consider an arbitrary $f(z,\bar{z})$,we get **Supertranslation**.It is called supertranslation because at each point on the 2-sphere we now get different amount of translation depending on z and $\bar{z}$ and since there are infinitely many such translations,we call it supertranslation.

Under Lorentz transformation the supertranslation charges transform as,

$$U^{-1}(\Lambda)T_f U(\Lambda) = T_{f'}, f'(z,\bar{z}) = \frac{|az+b|^2 + |cz+d|^2}{1+z\bar{z}} f(\Lambda z, \Lambda\bar{z}) \tag{7.14}$$

Now we will write our global translation generators P^μs in terms of T_f.
P^μ in terms of creation and annihilation operator is written as,

$$P^\mu = \int d\mu(p) p^\mu a^\dagger(E,z,\bar{z},\sigma) a(E,z,\bar{z},\sigma) \tag{7.15}$$

Now if we use the prametrization for P^μ given in eqn.((7.3)),and comparing with the definition of T_f given in eqn.((7.6)), we can find the functions $f(z,\bar{z})$ corresponding to each P^μ. If we do that we will get,

$$P^0 = H = T_f, f = 1$$

$$P^1 = T_f, f = \frac{z+\bar{z}}{1+z\bar{z}}$$

$$P^2 = T_f, f = \frac{-i(z-\bar{z})}{1+z\bar{z}}$$

$$P^2 = T_f, f = \frac{1-z\bar{z}}{1+z\bar{z}}$$

But, we would like to write this generators in a different basis by taking the following linear combinations,

$$P^0 + P^3 = T_{00}, f = \frac{2z^0\bar{z}^0}{1+z\bar{z}}$$

$$P^0 - P^3 = T_{11}, f = \frac{2z^1\bar{z}^1}{1+z\bar{z}}$$

$$P^1 + iP^2 = T_{10}, f = \frac{2z^1\bar{z}^0}{1+z\bar{z}}$$

$$P^0 - iP^2 = T_{01}, f = \frac{2z^0\bar{z}^1}{1+z\bar{z}}$$

Let's now consider the charges defined as,

$$T_f = T_{pq}, f = \frac{2z^p\bar{z}^q}{1+z\bar{z}} \tag{7.16}$$

where, $p, q \in \mathbb{Z}$.
We will redefine $f(z,\bar{z})$ as,

$$F(z,\bar{z}) = (1+z\bar{z})f(z,\bar{z}) \tag{7.17}$$

With this definition we can write the charges as,

$$T_f = \int d\mu(p)\frac{E}{1+z\bar{z}}F(z,\bar{z})a^\dagger(E,z,\bar{z},\sigma)a(E,z,\bar{z},\sigma) \tag{7.18}$$

so we can write,

$$T_{pq} = \int d\mu(p)\frac{E}{1+z\bar{z}}2z^p\bar{z}^q a^\dagger(E,z,\bar{z},\sigma)a(E,z,\bar{z},\sigma) \tag{7.19}$$

From these we can now easily find the algebra of Lorentz generators and the supertranslator generators, which are the followings

$$\begin{aligned}
[L_n, T_{pq}] &= (\frac{n+1}{2} - p)T_{p+n,q} \\
[\bar{L}_n, T_{pq}] &= (\frac{n+1}{2} - p)T_{p,q+n} \\
[L_m, L_n] &= (m-n)L_{m+n} \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} \\
[T_{pq}, T_{p'q'}] &= 0 \\
[L_m, \bar{L}_n] &= 0
\end{aligned} \tag{7.20}$$

7.2 Spacetime Realization

Now let's consider a massless scalar field in Minkowski space,

$$\phi(x) = \int d\mu(p)(e^{ip\cdot x}a(p) + e^{-ip\cdot x}a^\dagger(p)) \tag{7.21}$$

We now want to see the effect of unitary transformation $U = e^{-iT_f}$ on this field.
It is easy to see that the field ϕ transforms as,

$$\begin{aligned}
\phi_f(x) &= e^{iT_f}\phi(x)e^{-iT_f} \\
&= \int d\mu(p)(e^{ip\cdot x}e^{iE_p f(z,\bar{z})}a(p) + e^{-ip\cdot x}e^{-iE_p f(z,\bar{z})}a^\dagger(p))
\end{aligned} \tag{7.22}$$

now if we take $f(z,\bar{z})$ to be of the form,

$$f(z,\bar{z},l) = \frac{(l^0 - l^3) - (l^1 - il^2)z - (l^1 + il^2)\bar{z} + (l^0 + l^3)z\bar{z}}{1+z\bar{z}}$$

then our field transform as,

$$\phi_f(x^\mu) = \phi(x^\mu + l^\mu) \tag{7.23}$$

which is a spacetime translation by four vector l^μ.But as $f(z,\bar{z})$ can be arbitrary,so there are infinitely many functions for which we don't have such simple geometric explanation.

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