Horner's Rule to Evaluate a Polynomial

Horner's rule is an efficient algorithm for computing the value of a polynomial.

Consider the polynomial $p(x) = x^2 - x - 1$.

Suppose you want to evaluate $p(x)$ at $x = 3$.

The brute force approach is to compute

$$p(3) = 3 \cdot 3 - 3 - 1 = 5$$

Horner's rule writes this computation from the highest exponent to the lowest.

$$p(3) = ((3 \cdot 1) - 1) \cdot 3 - 1$$

I find this representation somewhat confusing, so let's reformulate it.

Horner's Rule

To evaluate $p(x) = 6x^3 - 2x^2 + 7x + 5$ at $x = 4$, list the coefficients (leaving sufficient space for computations)

Horner's Rule $x = 4$			
6	−2	7	5

Next, bring down the leading coefficient 6

Horner's Rule $x = 4$			
6	−2	7	5
6			

Horner's Rule

Continuing the computation on the previous slide.

Multiply the 6 by 4 and place the product 24 under the next coefficient

Horner's Rule $x = 4$			
6	-2	7	5
	24		
6			

Add the values in the second column.

Horner's Rule $x = 4$			
6	-2	7	5
	24		
6	22		

Horner's Rule

Continuing the computation on the previous slides. Multiply the 22 by 4 and place the product 88 it under the next coefficient

Horner's Rule $x = 4$			
6	-2	7	5
	24	88	
6	22		

Add the values in the third column.

Horner's Rule $x = 4$			
6	-2	7	5
	24	88	
6	22	95	

Horner's Rule

Continuing the computation on the previous slides. Multiply the 95 by 4 and place the product 380 it under the next coefficient

Horner's Rule $x = 4$			
6	-2	7	5
	24	88	380
6	22	95	

Add the values in the fourth column.

Horner's Rule $x = 4$			
6	-2	7	5
	24	88	380
6	22	95	$\boxed{385}$

Another Example of Horner's Rule

Therefore, $p(x) = 6x^3 - 2x^2 + 7x + 5 = 385$ when $x = 4$,
Here is another complete example.
Evaluate $p(x) = x^3 - 6x^2 + 11x - 6$ at $x = 2$.

Horner's Rule $x = 2$			
1	-6	11	-6
	2	-8	6
1	-4	3	$\boxed{0}$

Therefore, $p(2) = 0$.

Missing Powers in a Polynomial

Horner's rule is <u>positional</u> meaning missing powers have to be accounted for using a 0 place-holder.
Evaluate $p(x) = 5x^4 - 3x^2 + 12$ at $x = -4$.

Horner's Rule $x = -4$				
5	0	-3	0	12
	-20	80	-308	1232
5	-20	77	-308	$\boxed{1244}$

Notice 0's were included for the missing cubic and linear terms.

The Complexity of Horner's Rule

It is interesting to count the number of multiplications and additions when Horner's rule is executed.

For the linear case: $p(x) = a_1 \cdot x + a_0$ there is 1 multiply and 1 addition.

For the quadratic case: $p(x) = (a_2 \cdot x + a_1) \cdot x + a_0$ there are 2 multiples and 2 additions.

For the cubic case: $p(x) = ((a_3 \cdot x + a_2) \cdot x + a_1) \cdot x + a_0$ there are 3 multiples and 3 additions.

An inductive proof would show Horner's rule executes n multiples and n additions when computing the value of a polynomial of degree n.

Synthetic Division

In an algebra class, Horner's rule is often taught as synthetic division

For instance, let $p(x) = 5x^4 + 3x^3 - 2x^2 + 4x - 6$. Then, at $x = -2$, Horner's rule computes

Horner's Rule $x = -2$				
5	3	−2	4	−6
	−10	14	−24	40
5	−7	12	−20	$\boxed{36}$

You can conclude that

$$5x^4 + 3x^3 - 2x^2 + 4x - 6 = (5x^3 - 7x^2 + 12x - 20)(x + 2) + 36$$

Iterating Horner's Rule

Consider what happens when Horner's rule is iterated.

For instance, let $p(x) = x^3 - 6x^2 + 11x - 6$ at $x = 4$.

Iterated Horner's Rule

$$\begin{array}{rrrr}
1 & -6 & 11 & -6 \\
 & 4 & -8 & 12 \\
\hline
1 & -2 & 3 & \boxed{6} \\
 & 4 & 8 & \\
\hline
1 & 2 & \boxed{11} & \\
 & 4 & & \\
\hline
1 & \boxed{6} & & \\
\boxed{1} & & & \\
\hline
\end{array}$$

Taylor Polynomials

You can show the values: 6, 11, 6 and 1 computed on the last slide are the values of $p(4)$ and its derivatives.

- $6 = p(4)$ (the value of $p(x)$ at $x = 4$).
- $11 = p'(4)$ (the derivative of $p(x)$ at $x = 4$).
- $6 = p''(4)/2!$ (second derivative $p''(4)$ divided by $2!$).
- $1 = p'''(4)/3!$ (third derivative $p'''(4)$ divided by $3!$).

That is, these values are the coefficients of the Taylor polynomial for $p(x)$ at $x = 4$.

$$p(x) = x^3 - 6x^2 + 11x - 6 = (x-4)^3 + 6(x-4)^2 + 11(x-4) + 6$$

A topic beyond the scope of this course.