MATH 100: Differential Calculus

Final Exam Review

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Week 1

Introduction to limits, graphical limits

Limits

Definition 2.1.1: Informal Definition of Limits

Let $f(x)$ be a function that exists at all points in some open interval containing a real value a except possibly at a itself. We say that the **limit** of $f(x)$ as x approaches a is equal to a real number L if the function values $f(x)$ become arbitrarily close to L for all x sufficiently close to a. This is denoted,

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

If the values of f do **not** become arbitrarily close to any real number L as x becomes arbitrarily close to a, we say that the limit of f as x approaches a **does not exist** and we write,

$$\lim_{x \to a} f(x) = DNE.$$

Definition 2.2.1: Left & Right Hand Limits

Let $f : D \to \mathbb{R}$ be a function that exists at all points in some open interval containing a real value a except possibly at a itself. We say that the **left hand limit of** $f(x)$ **as** x **approaches** a is equal to some real value L_1 if the values of $f(x)$ become arbitrarily close to L_1 for all x sufficiently close to a with $x < a$. This is denoted by,

$$\lim_{x \to a^-} f(x) = L_1 \quad \text{or} \quad f(x) \to L_1 \text{ as } x \to a^-$$

Similarly, the **right hand limit of** $f(x)$ **as** x **approaches** a is equal to L_2 if the values of $f(x)$ become arbitrarily close to L_2 for all x sufficiently close to a with $x > a$. This is denoted by,

$$\lim_{x \to a^+} f(x) = L_2 \quad \text{or} \quad f(x) \to L_2 \text{ as } x \to a^+$$

Limits

Definition 2.2.2: (Slightly More) Formal Limit

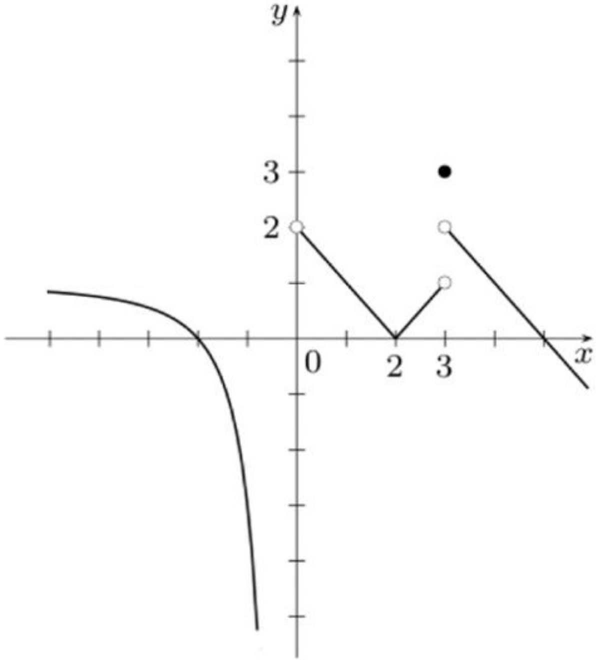
Let $f : D \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a real number. Suppose that the left and right hand limits of f at $x = a$ exist and that $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = L$. Then, we define **the limit of** $f(x)$ **as** x **approaches** a to be L as well and we write

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, then $\lim\limits_{x \to a} f(x) = DNE$.

Example

Use the graph of the function $f(x)$ to answer each question.
Use ∞, $-\infty$ or DNE where appropriate.



(a) $f(2) =$

(b) $f(0) =$

(c) $f(3) =$

(d) $\lim_{x \to 0^-} f(x) =$

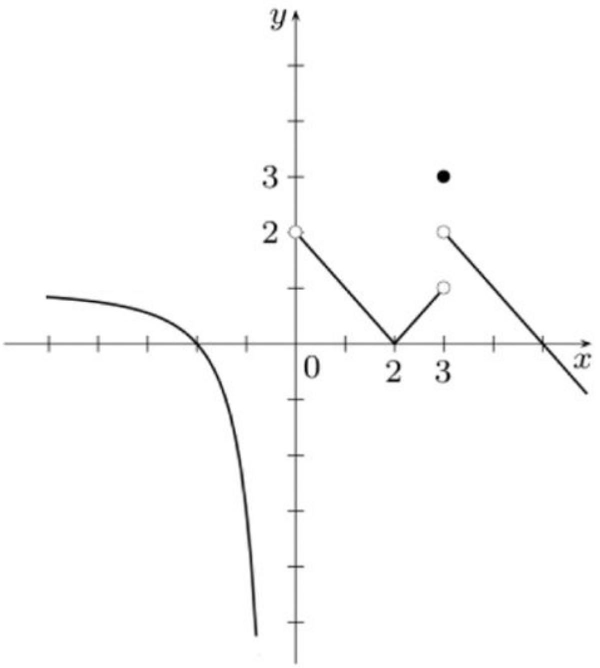
(e) $\lim_{x \to 0^+} f(x) =$

(f) $\lim_{x \to 0} f(x) =$

(g) $\lim_{x \to 2} f(x) =$

(h) $\lim_{x \to -5^-} f(x) =$

Use the graph of the function $f(x)$ to answer each question.
Use ∞, $-\infty$ or DNE where appropriate.



(a) $f(2) =$ 0

(b) $f(0) =$ DNE

(c) $f(3) =$ 3

(d) $\lim_{x \to 0^-} f(x) =$ -INF

(e) $\lim_{x \to 0^+} f(x) =$ 2

(f) $\lim_{x \to 0} f(x) =$ DNE

(g) $\lim_{x \to 2} f(x) =$ 0

(h) $\lim_{x \to -5^-} f(x) =$ DNE

Week 2

Vertical Asymptotes, Algebraic Techniques for Limits

Vertical Asymptote

Definition 2.3.2: Vertical Asymptotes

The vertical line $x = a$ is a **vertical asymptote** for a function f if *at least one* of the following are true,

$$\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = \infty,$$
$$\lim_{x\to a} f(x) = -\infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = -\infty.$$

Algebraic Technique for Limits

Theorem 2.5.1: Limit Laws

Let f and g be functions, let $a \in \mathbb{R}$ be a real number, and let c be a constant. Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are not infinite. Then, all of the following equations hold.

1. **Sum Law.** $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. **Constant Law.** $\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x)$
3. **Product Law.** $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
4. **Quotient Law.** $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided that $\lim_{x\to a} g(x) \neq 0$
5. **Power Law.** $\lim_{x\to a} f^n(x) = \lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ for all positive integers n
6. **Root Law.** $\lim_{x\to a} \sqrt[n]{f(x)} = \lim_{x\to a} [(f(x)]^{1/n} = \left[\lim_{x\to a} f(x)\right]^{1/n} = \sqrt[n]{\lim_{x\to a} f(x)}$ for all positive integers n. If n is even, then we assume $\lim_{x\to a} f(x) > 0$

Moreover, all of these laws hold for left and right hand limits as well.

Calculating HA

Solution Strategy 5: Calculating Infinite Limit

If $p(x)$ and $q(x)$ are polynomials and you need to calculate $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ or $\lim_{x\to-\infty} \frac{p(x)}{q(x)}$, do the following.

1. Identify which of $p(x)$ or $q(x)$ have smaller degrees; denote the smaller degree by m.
2. Multiply both top and bottom by $\frac{1}{x^m}$.
3. All terms other than the leading two on the top and bottom then become negligible as $x \to \pm\infty$, so the limit of the new function becomes the limit of the ratio of the leading terms. Heuristically, the values of these limits can be remembered as follows:

   i If the degree of p and q are equal, then the limit of $\frac{p(x)}{q(x)}$ as $x \to \pm\infty$ is the ratio of the leading coefficients of each.

   ii If the degree of q is larger than the degree of p, then $\frac{p(x)}{q(x)} \to 0$ as $x \to \pm\infty$.

   iii If the degree of p is larger than the degree of q, then $\frac{p(x)}{q(x)}$ approaches a signed infinite as $x \to \pm\infty$, the sign of which can be determined by looking at the ratio of the leading terms and the degree of the polynomial.

Examples

1) $\lim\limits_{x \to 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$

2) $\lim\limits_{x \to -3} \frac{\frac{1}{x+2} + 1}{x + 3}$

3) $\lim\limits_{x \to 9} \frac{x - 9}{\sqrt{x} - 3}$

Examples

1) $$\lim_{x \to 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$$

Plug in the value $x = 1$

$$\lim_{x \to 1}\left(\frac{x^3 + 3x^2 + 5}{4 - 7x}\right) = \frac{1^3 + 3 \cdot 1^2 + 5}{4 - 7 \cdot 1} = -3$$

2) $$\lim_{x \to -3} \frac{\frac{1}{x+2} + 1}{x + 3}$$

Simplify $\frac{\frac{1}{x+2} + 1}{x+3}$: $\quad \frac{1}{x+2}$

$$\lim_{x \to -3}\left(\frac{\frac{1}{x+2} + 1}{x + 3}\right) = \lim_{x \to -3}\left(\frac{1}{x+2}\right) = \frac{1}{-3+2} = -1$$

3) $$\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3}$$

Rationalize denominator $\frac{x-9}{\sqrt{x} - 3}$: $\quad \sqrt{x} + 3$

$$\lim_{x \to 9}\left(\frac{x - 9}{\sqrt{x} - 3}\right) = \lim_{x \to 9}\left(\sqrt{x} + 3\right) = \sqrt{9} + 3 = 6$$

Week 3

Limits at Infinity, continuity, IVT

Limits at Infinity

Definition 2.7.1: Limits at Infinity & Horizontal Asymptotes

Let f be a function defined on an interval (a, ∞) where $a \in \mathbb{R}$. The expression

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

means that values of $f(x)$ can be made arbitrarily close to L for all x sufficiently large in the positive direction. Similarly, the expression

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to -\infty$$

means that values of $f(x)$ can be made arbitrarily close to L for all x sufficiently large in the negative direction.

If $\lim_{x \to \infty} f(x) = L$, then the horizontal line $y = L$ is called a **horizontal asymptote** at ∞. If $\lim_{x \to -\infty} f(x) = L$, then $y = L$ is called a **horizontal asymptote** at $-\infty$.

If $f(x)$ is getting arbitrarily large in the positive or negative direction as x gets arbitrarily large in the positive or negative direction, we write,

$$\lim_{x \to \pm\infty} f(x) = \pm\infty.$$

Let $r > 0$ be a rational number. Then,

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

Example

Find all vertical and horizontal asymptotes of the function.

$$g(x) = \frac{2x^2 + x - 1}{x^2 + 4x + 3}$$

V.A : $\dfrac{\text{non-zero}}{\text{zero}}$

$x^2 + 4x + 3 = 0$

$(x+3)(x+1) = 0$

$x = -3 \quad x = -1$

Substituting $x = -3$ in $g(x)$: $g(x) = \dfrac{14}{0}$

" " $x = -1$ in $g(x)$: $g(x) = \dfrac{0}{0}$

$\therefore$ V.A : $x = -3$

$$g(x) = \frac{2x^2 + x - 1}{x^2 + 4x + 3}$$

deg = 2

deg = 2

$$g(x) = \frac{\frac{2x^2 + x - 1}{x^2}}{\frac{x^2 + 4x + 3}{x^2}}$$

(Dividing numerator and denominator by x^2)

$$g(x) = \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{4}{x} + \frac{3}{x^2}}$$

$$\lim_{x \to \infty} \frac{2 + \cancel{\frac{1}{x}} - \cancel{\frac{1}{x^2}}}{1 + \cancel{\frac{4}{x}} + \cancel{\frac{3}{x^2}}}$$

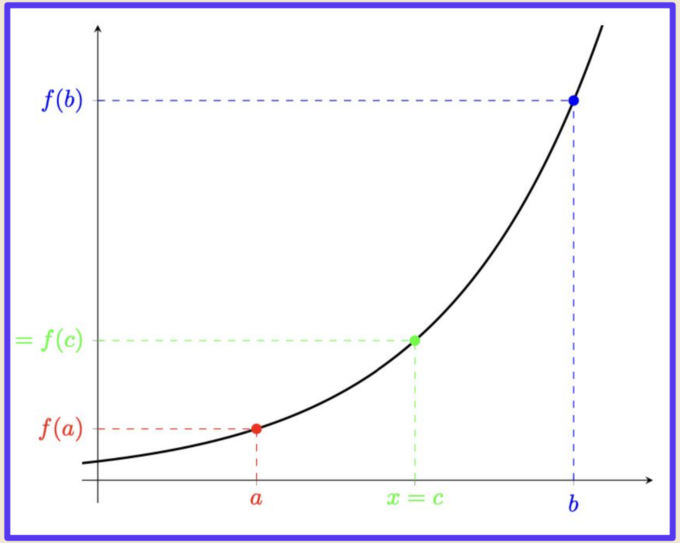
$$\lim_{x \to \infty} \frac{2}{1} = 2$$

$$\boxed{y = 2}$$

Intermediate Value Theorem

Theorem 2.6.3: Intermediate Value Theorem

Let $f(x)$ be a function continuous on a closed interval $[a, b]$ where $f(a) \neq f(b)$. Let $N \in (f(a), f(b))$; that is, N is a number between a and b. Then, there exists another number $c \in (a, b)$ such that $f(c) = N$.



Continuity

Definition 2.6.1: Continuity

Let $f : D \to \mathbb{R}$ be a function and let $a \in D$. f is said to be **continuous** at $x = a$ if,

$$\lim_{x \to a} f(x) = f(a).$$

If f is not continuous at $x = a$, then f is called **discontinuous at** a and a is called a **point of discontinuity**. If f is continuous at all points in some subset A of its domain, we say that f is continuous on A. In particular, if f is continuous on its entire domain, then we say that f is **continuous** on its domain.

Definition 2.6.2: Types of Discontinuity

Let $f(x)$ be a function and suppose that $f(x)$ is discontinuous at $x = a$.

i) If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$, then $x = a$ is called a **removable discontinuity** of $f(x)$.

ii) If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $x = a$ is called a **jump discontinuity**.

iii) If $f(x)$ has a vertical asymptote at $x = a$, then $x = a$ is called an **infinite discontinuity** of $f(x)$.

Theorem 2.3.1: Continuity of Elementary Functions

All of the elementary functions are continuous on their domains. These include:

- **Polynomials**: continuous on all of $\mathbb{R}$.
- **Rational functions**: continuous at all x-values that are not roots of the denominator.
- **Root functions**: continuous at all x-values for which the roots are defined and for which the denominator does not become zero.
- **Exponential functions** a^x: continuous on all of $\mathbb{R}$.
- **Logarithmic functions** $\log_a(x)$: continuous on all $x > 0$.
- **Trigonometric functions**:
  - $\sin(x)$ and $\cos(x)$: continuous on $\mathbb{R}$.

Week 4

Tangent Line, definition of derivative

Derivatives and Tangent

Definition 3.1.4: Derivatives

Let $f(x)$ be a function that is defined on some open interval that contains the value $x = a$. Then, the **derivative** of $f(x)$ at $x = a$ is equal to the slope of the tangent line to $f(x)$ at $x = a$; this is given by,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the limit defining $f'(a)$ exists as a finite number, then $f(x)$ is called **differentiable** at $x = a$.

Using the **limit definition** of a derivative, calculate $f'(2)$ for $f(x) = 2 - x^2$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$a = 2$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$f(x) = 2 - x^2$

$$f(2+h) = 2 - (2+h)^2$$
$$= 2 - (4 + h^2 + 4h)$$
$$= 2 - 4 - h^2 - 4h$$
$$= -2 - h^2 - 4h$$

$$f(2) = 2 - (2)^2$$
$$= 2 - 4$$
$$= -2$$

$$f'(2) = \lim_{h \to 0} \frac{-2 - h^2 - 4h - (-2)}{h}$$
$$= \lim_{h \to 0} \frac{-h^2 - 4h}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(-h-4)}{\cancel{h}}$$
$$= \lim_{h \to 0} -h - 4$$
$$= 0 - 4 = -4$$

$f'(2) = -4$

Week 5

Derivatives of polynomials, exponential functions, power and product rules

Psst:

- *I have more practice questions in my SL session slides! You can access them from [here](#).*
- *Some questions ahead adapted from [Paul's Online Notes](#). Great Resource btw!*

Linearity Rules of Differentiation

Constant Rule:

If f(x) = c*g(x), where g(x) is differentiable,
Then f′(x) = c*g′(x)

Sum and Difference Rule:

If f(x) = g(x) + h(x),
Then f′(x) = g′(x) + h′(x)

Similarly,
If f(x) = g(x) - h(x),
Then f′(x) = g′(x) + h′(x)

(Assuming that g(x) and h(x) are both differentiable)

Power Rule

If $f(x) = x^n$, where n is a constant,
Then $f'(x) = n*x^{n - 1}$
This works for n = *any* constant - positive, negative, zero, or fractional.

<u>Examples:</u>

Q: *Find the differentiation of* $f(x) = 3x^{27} + 40.5x^2 + 81.$

A: $f'(x) = 3*27*x^{27 - 1} + 40.5*2*x^{2 - 1} + 81*0*x^{0 - 1}$
$= 81x^{26} + 81x + 0$
$= 81x^{26} + 81x$
$= 81x(x^{25} + 1)$

Q: *Find the differentiation of* $f(x) = x^2 - 4/x.$

A: $f(x) = x^2 - 4x^{-1}$
So $f'(x) = 2*x^{2 -1} - 4*(-1)*x^{-1 - 1}$
$= 2x - (-4x^{-2})$
$= 2x + 4/x^2$

Exponential Functions

Exponential Functions are those where the power is a function of x.
E.g., $f(x) = 2^x$, $g(x) = e^x$

Differential of exponential functions $f(x) = a^x$:
$f'(x) = \ln(a)*a^x$

Where ln(a) is the log of *a* with base *e*.
$Y = e^x$ is also called *THE* Exponential Function.
Note: $\ln(e) = 1$, so $f'(e^x) = e^x$!

<u>Example:</u>

Q: *Find the differentiation of* $f(x) = x^2 + 3e^x + e^2 5^x$.

A: $f'(x) = 2x + 3e^x + e^2 * \ln(5) * 5^x$

Product Rule

If $f(x) = g(x)*h(x)$, where $g(x)$ and $h(x)$ are both differentiable,
Then $f'(x) = g'(x)*h(x) + g(x)*h'(x)$

<u>Examples:</u>

Q: *Find the differentiation of* $f(x) = x^5e^x$.

A: *If* $g(x) = x^5$ *and* $h(x) = e^x$, *then* $f(x) = g(x)*h(x)$

$$\begin{aligned} \text{So } f'(x) &= g'(x)*h(x) + g(x)*h(x) \\ &= 5x^4*e^x + x^5*e^x \\ &= e^x(5x^4 + x^5) \end{aligned}$$

Q: *Find the differentiation of* $f(x) = (x^{1/3} - 7x)(x^{1/2} + 3x^4)$

A: *If* $g(x) = (x^{1/3} - 7x)$ *and* $h(x) = (x^{1/2} + 3x^4)$, *then* $\mathbf{f(x) = g(x)*h(x)}$

$$\begin{aligned} \text{So } f'(x) &= g'(x)*h(x) + g(x)*h(x) \\ &= (x^{-2/3}/3 - 7)(x^{1/2} + 3x^4) + (x^{1/3} - 7x)(x^{-1/2}/2 + 12x^3) \end{aligned}$$

Week 6

Quotient rule, Trigonometric derivatives, Chain rule

Quotient Rule

If f(x) = g(x)/h(x), where g(x) and h(x) are differentiable,
Then $f'(x) = [\, g'(x)*h(x) = g(x)*h'(x)\,]/h(x)^2$

<u>Examples:</u>

Q: *Find the differentiation of* $f(x) = (x^2 - x + 2)/(x^3 + 4x)$.

A: *If* $g(x) = (x^2 - x + 2)$ *and* $h(x) = (x^3 + 4x)$, *then* $f(x) = g(x)/h(x)$

So $f'(x) = [\, g'(x)*h(x) = g(x)*h'(x)\,]/h(x)^2$

$$= \frac{\mathbf{d/dx}(x^2 - x + 2)*(x^3 + 4x) - (x^2 - x + 2)*\mathbf{d/dx}(x^3 + 4x)}{(x^3 + 4x)^2}$$

$$= \frac{-x^4 + 2x^3 - 2x^2 - 8}{(x^3 + 4x)^2}$$

Q: *What is the slope of the tangent line on* $y = [\, 9x^2 + 14x + 4\,]/x^3$ *at* $x = 5$?

A: *To find the slope of the tangent line at x = 5, we need to find the derivative of the function and plug in x = 5.*

If $g(x) = (9x^2 + 14x + 4)$ *and* $h(x) = x^3$, *then* $\boldsymbol{f(x) = g(x)/h(x)}$, *so* $f'(x) = [\, g'(x)*h(x) = g(x)*h'(x)\,]/h(x)^2$

$$f'(x) = \frac{(18x + 14)(x^3) - (9x^2 + 14x + 4)(3x^2)}{x^6}$$

So, the slope at x = 5 is $f'(5) = -377/625$

Trigonometric Derivative Rules

- $f(x) = \sin(x)$ $f'(x) = \cos(x)$
- $f(x) = \cos(x)$ $f'(x) = -\sin(x)$
- $f(x) = \tan(x)$ $f'(x) = \sec^2(x)$
- $f(x) = \sec(x)$ $f'(x) = \sec(x)\tan(x)$
- $f(x) = \cot(x)$ $f'(x) = -\mathrm{cosec}^2(x)$
- $f(x) = \mathrm{cosec}(x)$ $f'(x) = -\cot(x)\mathrm{cosec}(x)$

Pattern: if the trig function starts with "c", its differential will be negative!

<u>Example:</u>

Q: *Find the differentiation of* $f(x) = 7\tan(x) + 5\sin(x)\cos(x)$

A: *If* $g(x) = \tan(x)$ *and* $h(x) = \sin(x)\cos(x)$*, then* $f(x) = 7g(x) + 5h(x)$.

$g'(x) = \sec^2(x)$ *and* $h'(x) = \cos^2(x) - \sin^2(x)$ *[by product rule]*

So $f'(x) = 7\sec^2(x) + 5(\cos^2(x) - \sin^2(x))$

Chain Rule

The chain rule can be applied to **composite functions** - those functions where the output of a function h(x) is taken as the input of another function g(x). I.e., f(x) = g(h(x)) = goh(x)

According to chain rule, if f(x) = g(h(x)), where g(x) and h(x) are both differentiable,
Then f’(x) = g’(h(x))*h’(x)

<u>Example:</u>

Q: *Find the differentiation of* $f(x) = e^{2x} + \tan^2(x + 1)$.

A: *If* $g(x) = e^{2x}$ *and* $h(x) = \tan^2(x)$, *then* $\mathbf{f(x) = g(x) + h(x)}$.
If $g_1(x) = 2x$ *and* $g_2(x) = e^x$, *then* $g(x) = g_1(g_2(x))$ *[composite function]*
If $h_1(x) = x^2$, $h_2(x) = \tan(x)$, *and* $h_3(x) = x + 1$, *then* $h(x) = h_1(h_2(h_3(x)))$ *[composite function]*

Therefore, $f'(x) = g'(x) + h'(x)$
$g'(x) = \mathbf{d/dx}(e^{2x}) * \mathbf{d/dx}(2x) = e^{2x}*2$
$\quad = 2e^{2x}$
$h'(x) = \mathbf{d/dx}(\tan^2(x + 1)) * \mathbf{d/dx}(\tan(x + 1)) * \mathbf{d/dx}(x + 1)$
$\quad = 2\tan(x + 1)*\sec^2(x + 1)*(1 + 0) = 2\tan(x+1)\sec^2(x+1)$
So $f'(x) = 2e^{2x} + 2\tan(x+1)\sec^2(x+1)$

Week 7

Logarithmic differentiation, implicit differentiation, derivatives of inverse trigonometric functions, L'Hôpital's rule

Differentiation of Inverse Trigonometric Functions

Inverse trigonometric functions are the opposite of trigonometric functions. For example, if tan(45°) gives 1, then arctan(1) gives 45°.
I.e., which angle should we plug into the tan() function to so that we get an answer of 1?

- $f(x) = \arcsin(x)$ $\quad f'(x) = 1/\sqrt{(1 - x^2)}$
- $f(x) = \arccos(x)$ $\quad f'(x) = - 1/\sqrt{(1 - x^2)}$
- $f(x) = \arctan(x)$ $\quad f'(x) = 1/(1 + x^2)$
- $f(x) = \text{arccot}(x)$ $\quad f'(x) = - 1/(1 + x^2)$

<u>Examples:</u>

Q: *Find the differentiation of $f(w) = \sin(w) + w^2\tan^{-1}(w)$.*

A: $f'(w) = \cos(w) + 2w*\tan^{-1}(w) + \dfrac{w^2}{1 + w^2}$

Q: *Find the differentiation of $h(x) = \dfrac{\sin^{-1}(1)*\sin^{-1}(x)}{x + 1}$*

A: $\sin^{-1}(1) = \Pi/2$. So $h(x) = (\Pi/2)*\dfrac{\sin^{-1}(x)}{x + 1}$

$$h'(x) = (\Pi/2)*\dfrac{\left[\dfrac{1 + x}{\sqrt{(1 - x^2)}} - \sin^{-1}(x)\right]}{(x + 1)^2}$$

Logarithmic Differentiation

- Differentiation of $a^x = \ln(a)*a^x$
- Differentiation of $e^x = e^x$
- Differentiation of $\ln(x) = 1/x$
- Differentiation of $\log_a(x) = 1/[\ \ln(a)*x\ ]$

<u>Examples:</u>

Q: *Find the differentiation of* $f(x) = x^x$.

A: *Since we cannot apply any traditional diff. rules, let's start by rewriting* $f(x) = x^x$ *as* $e^{\ln(x\hat{}x)} = e^{x\ln(x)}$.
Differentiating now, $f'(x) = e^{x\ln(x)} * \mathbf{d/dx}(x\ln(x))$ [chain rule]
$f'(x) = e^{x\ln(x)}[\ 1*\ln(x) + x/x\ ] = e^{x\ln(x)}[\ \ln(x) + 1\ ] = x^x[\ \ln(x) + 1\ ]$ $[\ e^{x\ln(x)} = x^x\ ]$

Q: *Differentiate* $f(x) = (2x - e^{8x})^{\sin(2x)}$.

A: *Rewriting* $f(x)$ *as* $e^{\ln[\ (2x - e\hat{}8x)\hat{}\sin(2x)\ ]} = e^{\sin(2x)*\ln(2x - e\hat{}8x)}$. *Let* $y = \sin(2x)*\ln(2x - e^{8x})$.
So $f'(x) = e^y*y' = (2x - e^{8x})^{\sin(2x)}*[\ 2\cos(2x)*\ln(2x - e^{8x}) + \dfrac{\sin(2x)*(2 - 8e^{8x})}{2x - e^{8x}}\ ]$ $[\ e^y = f(x)\ ]$

Try: *Find the differentiation of* $y = \sin(3z + z^2)/(6 - z^4)^3$.
Can you solve this using (a) logarithmic differentiation *and* (b) quotient rule*?*

Implicit Differentiation

It's the same as regular differentiation, just that we don't have a "y" variable isolated on the left side of the equation.

Just remember that if you're differentiating a **function of x** such as y with respect to (w.r.t) x, you will end up with $y' = dy/dx$ (duh).
If you are differentiating a **function of y** such as y^2 w.r.t x, **you will have to apply the chain rule** and will end up with something like $2y*y' = 2y*dy/dx$.

At the end of the question, make sure to isolate the required variable (most likely y').

Examples:

Q: *Find the differentiation of the implicit function* $x^2 + y^3 = 4$.

A: *Differentiation gives* $2x + 3y^2*y' = 0$
So $y' = -2x/3y^2$

Q: *Assume* $x = x(t)$, $y = y(t)$, $z = z(t)$ *and differentiate* $x^2*\cos(y) = \sin(y^3 + 4z)$ *with respect to t.*

A: *Differentiation gives* $2xx'*\cos(y) - x^2*\sin(y)y' = (3y^2y' + 4z')*\cos(y^3 + 4z)$
Note that the question does not ask for a particular derivative or variable so we can leave it as is.

L'Hôpital's Rule

Let's get back to solving limits.
Basically, we can use L'Hôpital's Rule to solve limit functions when direct substitution gives a division-based indeterminate form.
In that case, he rule says we can individually replace the numerator and denominator with their derivatives and find the limit of the new function. The answer would be the same.

Indeterminate forms: 0/0, ∞/∞, 0^0, ∞^0, ∞^∞, 1^∞, ∞ - ∞, etc.
L'Hôpital's Rule only works with the first two forms.
There may be times when you need to convert a different type of indeterminate limit into a division format.

- For power-based limits: *use* $f(x) = e^{\ln(f(x))}$. *Find limit of ln(f(x)) and plug in its answer at the end as power of e to get the answer to the question.*
- For multiplication-based limits: *use f(x)*g(x) = f(x)/[1/g(x)]* *[e.g.* $xe^x = e^x/(1/x) = x/(1/e^x)$ *]*

<u>Examples:</u>
Q: *Evaluate:* $\lim_{x\to\infty} e^x/x^2$

A: *Direct Sub. gives ∞/∞. So we apply L'Hôpital's Rule to get* $\lim_{x\to\infty} e^x/2x$*. But this is still indeterminate, so we apply the rule* ***again****. We get* $\lim_{x\to\infty} e^x/2$*. This is not indeterminate. The limit value is ∞/2 = ∞.*

Q: Find $\lim_{x\to 0^+}(1-\sin(2x))^{1/x}$.

A: Attempting direct substitution results in the indeterminate form 1^∞. We rewrite the function as an exponential:

$$(1-\sin(2x))^{1/x} = e^{(1/x)\ln(1-\sin(2x))}$$

Using L'Hôpital's Rule, we find $\lim_{x\to 0^+}\frac{\ln(1-\sin(2x))}{x} = -2$. Therefore, the limit is e^{-2}. $\square$

Week 8

Related rates problems, linear approximations and differentials

Linearization

- We know that differentiation of $f(x)$ at $x=a$ **=** slope of tangent line to $f(x)$ at $x=a$.
- The equation of the tangent line would be $y - f(a) = f'(a)(x - a)$
- So we have a linear line that touches/goes near the actual value of the function at $x=a$.
- Let's use this line to find approximations of the $f(x)$ value near $x=a$ and call the tangent line equation $\boldsymbol{L(a) = f'(a)(x - a) + f(a)}$.
- That's it.

<u>Example:</u>

Q: *Find the linearization of* $f(x) = x^2 - 5x$ *at* $x = 1$ *and use it to approximate the value of* $f(1.1)$.

A: $f(1) = 1^2 - 5(1) = 1 - 5 = -4$

$f'(x) = 2x - 5$

So $f'(a) = 2(1) - 5 = 2 - 5 = -3$

So $L(1) = -3(x - 1) - 4$. *(That's the linearization of* $f(x)$ *at* $x = 1$*)*

Therefore $f(1.1) \approx L(1.1)$

$= -3(1.1 - 1) - 4 = -3(0.1) - 4 = -0.3 - 4 = -4.3$ *[Actual value:* $f(1.1) = -4.29$*]*

Differentials

Differentials can be considered the change in linear approximation of the function going from x to ($x + \Delta x$). You can also write this as dy.

- Finding differentials numerically is literally just as easy as differentiating a function and multiplying both sides by **dx** to get an isolated **dy** on the left hand side of the equation.

Examples:

Q: *Write the differential of $f(s) = s^2 - \sec(s)$.*

A: *$dy/ds = f'(s) = 2s - \sec(s)\tan(s)$*
So, differential $dy = (2s - \sec(s)\tan(s))*ds$

Q: *Compute dy and* Δy for $y = e^{x\wedge 2}$ as x changes from 3 to 3.01.

A: *First, actual change $\Delta y = e^{3.01\wedge 2} - e^{3\wedge 2} = 501.927$*
Now, $dy = 2xe^{x\wedge 2}*dx$
$\Delta x = 3.01 - 3 = 0.01 \approx dx$
Then, $dy = 2(3)e^{3\wedge 2}*(0.01) = 486.185$

Related Rates

Important Topic!

We can take relevant information out of word problems, and use quantity_1, quantity_2, with rate of change_1 to find rate of change_2.

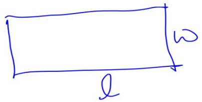
Steps:

1. Draw a diagram and label it with all given values (with units!)
2. Find the variable whos rate of change we need
3. Write the formula for that variable using the given quantities
4. Differentiate both sides of the formula with respect to time (to get rate of change function)
5. Plug in all known values
6. Isolate the required rate of change
7. Et voila!

Related Rates - Examples

Q: *The length of a rectangle is increasing at a rate of 8cm/s and its width is increasing at a rate of 3cm/s. When the length is 20cm and the width is 10cm, how fast is the area increasing?*

A:



$\frac{dl}{dt} = 8 \qquad \frac{dw}{dt} = 3$

$A = lw \Rightarrow \frac{dA}{dt} = \frac{dl}{dt} \cdot w + l \cdot \frac{dw}{dt}$

$\Rightarrow \frac{dA}{dt} = 8 \cdot 10 + 20 \cdot 3$

$\Rightarrow \frac{dA}{dt} = 80 + 60 = \boxed{140 \text{ cm/s}}$

Q: *Suppose that we have two resistors connected in parallel with resistances R_1 and R_2 measured in ohms (Ω). The total resistance, R, is then given by $1/R = 1/R_1 + 1/R_2$. Suppose that R_1 is increasing at a rate of 0.4 Ω/min and R_2 is decreasing at a rate of 0.7Ω/min. At what rate is R changing when $R_1 = 80Ω$ and $R_2 = 105Ω$?*

A:

First, let's note that we're looking for R' and that we know $R'_1 = 0.4$ and $R'_2 = -0.7$. Be careful with the signs here.

Also, since we'll eventually need it let's determine R at the time we're interested in.

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{105} = \frac{37}{1680} \qquad \Rightarrow \qquad R = \frac{1680}{37} = 45.4054\,\Omega$$

Next, we need to differentiate the equation given in the problem statement.

$$-\frac{1}{R^2}R' = -\frac{1}{(R_1)^2}R'_1 - \frac{1}{(R_2)^2}R'_2$$
$$R' = R^2\left(\frac{1}{(R_1)^2}R'_1 + \frac{1}{(R_2)^2}R'_2\right)$$

Finally, all we need to do is plug into this and do some quick computations.

$$R' = (45.4054)^2\left(\frac{1}{80^2}(0.4) + \frac{1}{105^2}(-0.7)\right)$$

So, it looks like R is decreasing at a rate of 0.002045Ω/min.

Week 9

Local/global extrema, Fermat’s theorem,
Intervals increase/decrease, First Derivative Test

Local/Global Extremes

What is a local extreme?
How is it different from a global extreme?

Is there anything a local extreme needs that a global extreme doesn't?

Is it possible to have a global extreme but no local extremes?

Here are some key questions you can use to double check your understanding!

Try writing a definition that makes sense to you! Then we'll show you and explain ours.

Fermat's Theorem

First Derivative Test

Build yourself a list of steps or a checklist for how to approach the first derivative test!

1. Find $f'(x)$
2. Find your critical values ($f'(x) = 0$)
3. Draw number line with critical points
4. Interval Test to find intervals of increase/decrease

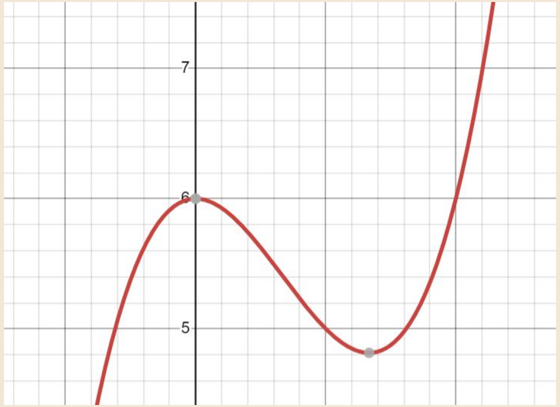
Examples!

Q: Find the intervals of increase/decrease of $f(x) = x^3 - 12x + 5$

Week 10

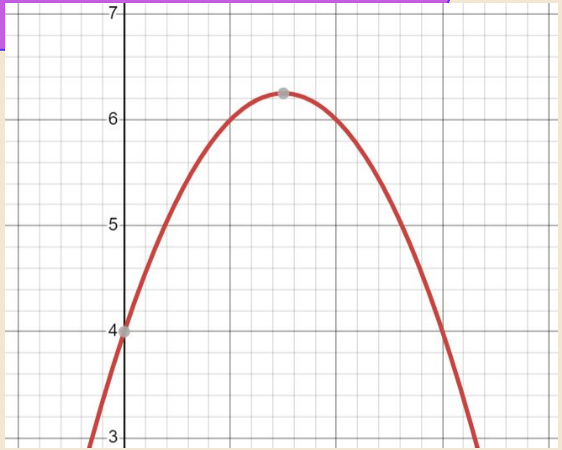
Intervals of concavity, inflection points, second derivative test, curve sketching

Critical Points vs Inflection points



Critical Points

These are for FIRST derivative test only. They show you where the graph changes from increasing/decreasing and vice versa



Inflection Points

Only for SECOND derivative tests. Shows you where concavity changes.

Second Derivative Test

Build another checklist that works for you!

1. Find $f''(x)$
2. Find your points of inflection ($f''(x) = 0$)
3. Draw number line with inflection points
4. Interval Test to find intervals of concave up/down

Examples!

Q: Find the intervals of concavity of
$f(x) = x^3 - 12x + 5$

Curve Sketching — Putting it all together

1. Check for intercepts

For both x and y, then label them on the graph

2. Find asymptotes

Make sure you know the difference between finding horizontal and vertical asymptotes!

3. First Derivative Test

Use the checklist you made earlier! Mark your critical points

4. Second Derivative Test

Again, use your checklist and label your inflection points

5. Draw your graph!

yay connect the dots

Examples!

Q: sketch the graph of $f(x) = \frac{x^2}{1+x^2}$

1. find intercepts

$$f(x) = \frac{x^2}{1+x^2} = 0$$

$$x^2 = 0$$

$$x = 0$$

x-int at $x=0$

$$f(0) = \frac{0^2}{1+0^2} = \frac{0}{1} = 0$$

y-int and x-int at $(0,0)$

Examples!

2. Find asymptotes

$$f(x) = \frac{x^2}{1+x^2}$$

can $1+x^2 = 0$?

$$\boxed{\text{No, so no V.A.'s}}$$

$$\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}}{\cancel{\frac{1}{x^2}} + \frac{x^2}{x^2}} = \lim_{x\to\infty} \frac{1}{1} = 1$$

$$\lim_{x\to-\infty} \frac{x^2}{1+x^2} = 1$$

$$\boxed{\therefore \text{H.A. @ } 1}$$

Examples!

3. 1st Derivative Test

$$f(x) = \frac{x^2}{1+x^2}$$

$$f'(x) = \frac{(1+x^2)(2x) - (x^2)(2x)}{(1+x^2)^2}$$

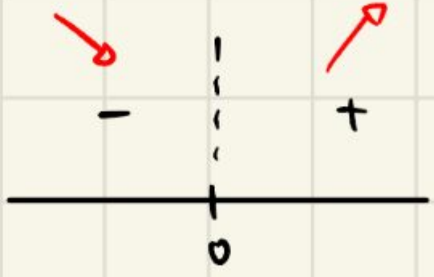
$$= \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$$\frac{2x}{(1+x^2)^2} = 0$$

$$2x = 0$$

$$x = 0$$



2x − +

Examples!

4. 2nd Derivative Test

$$f'(x) = \frac{2x}{(1+x^2)^2}$$

$$f''(x) = \frac{(1+x^2)^{\cancel{2}1}(2) - (2x)(2)\cancel{(1+x^2)}(2x)}{(1+x^2)^{\cancel{4}}}$$

$$= \frac{(1+x^2)(2) - 8x^2}{(1+x^2)^3} = \frac{2+2x^2-8x^2}{(1+x^2)^3}$$

$$= \frac{2-6x^2}{(1+x^2)^3}$$

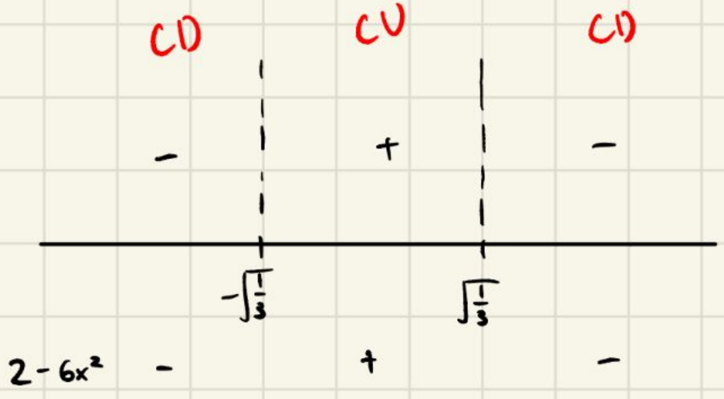
$$\frac{2-6x^2}{(1+x^2)^3} = 0$$

$$2-6x^2 = 0$$

$$6x^2 = 2$$

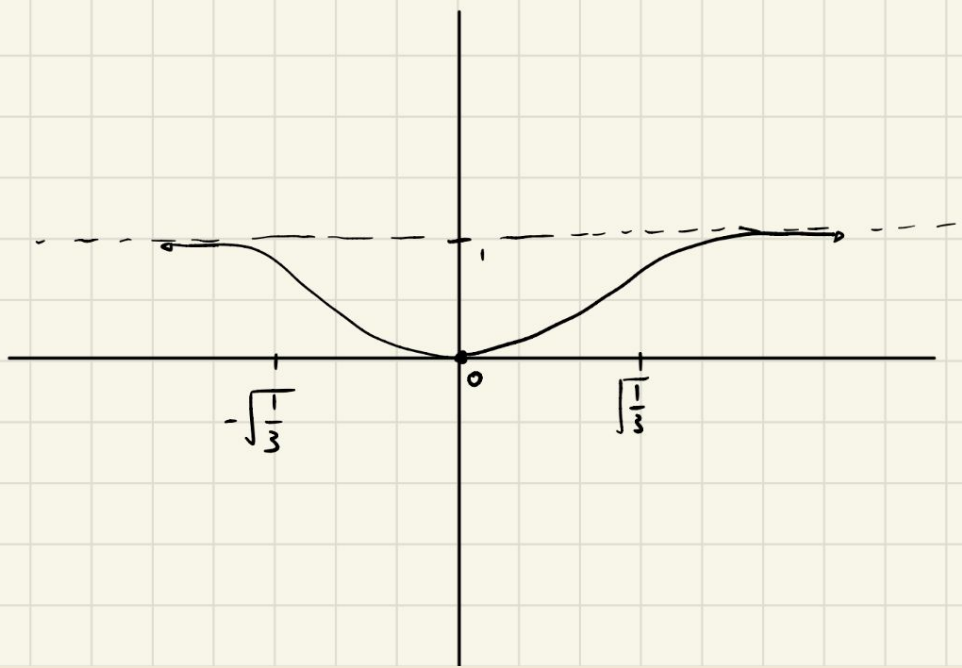
$$x^2 = \frac{2}{6} = \frac{1}{3}$$

$$x = \pm\sqrt{\frac{1}{3}}$$



Examples!

5. connect the dots!



Week 11

Closed interval method, extreme value theorem, optimization

Closed Interval Method

Basically, you want to find all the critical values between a closed interval [a, b]

Then you plug in your crit. values and interval endpoints into the original function

You wanna find global max/min so just pick your smallest and biggest values!

Examples

Find the absolute max/min values of $f(x) = (x^2 - 3)(e^x)$ on $[-2, 2]$

$f(x) = (x^2 - 3)(e^x)$
$f'(x) = (x^2 - 3)(e^x) + (e^x)(2x)$
$= x^2e^x - 3e^x + 2xe^x$
$= e^x(x^2 + 2x - 3)$
$= e^x(x + 3)(x - 1) = 0$
$x = -3, 1$

$f(-3) = (9 - 3)(e^3) = 6e^3$
$f(1) = (1 - 3)(e^1) = -2e$
$f(-2) = (4 - 3)(e^{-2}) = 1/e^2$
$f(2) = (4 - 3)(e^2) = e^2$

$6e^3$ is our biggest, and $-2e$ is our smallest, so...

Absolute max at $(-3, 6e^3)$
Absolute min at $(1, -2e)$

Extreme Value Theorem

If “f” is continuous on a closed interval, then it HAS to have an absolute max/min in that interval at either the crit. points or at the interval endpoints.

Optimization Tips!

Optimization basically has 3 main steps:

1. List given values and DRAW

Typically given just one number. Make sure to draw a diagram!

2. Build your equations

You usually end up with 2 equations. One of them ends up differentiated. The other is used to sub in values for the first equation.

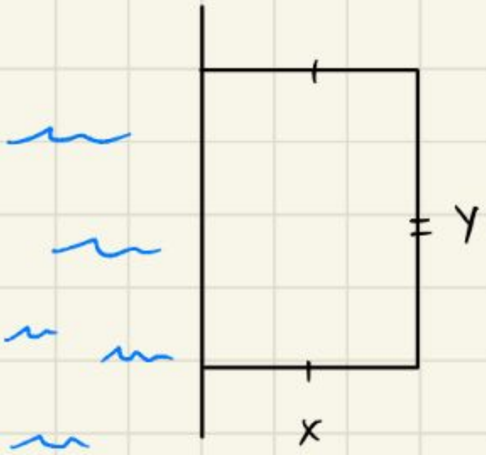
3. Solve your equations/closed interval test

Depending if the question wants max/min, look for the biggest/smallest y-value like in closed interval.

Examples

Q: A farmer wants to enclose a rectangular field against a straight river bank. They have a total of 500m of fencing. What are the dimensions of the biggest possible enclosure?

$P = 500m$



let length = y
let width = x

$A = xy$

$A(x) = x(500 - 2x)$

$\boxed{= 500x - 2x^2}$

$P = 2x + y$

$500 = 2x + y$

$y = 500 - 2x$

Examples

$$A(x) = 500x - 2x^2$$

$$A'(x) = 500 - 4x = 0$$

$$4x = 500$$

$$x = \frac{500}{4}$$

$$\boxed{x = 125}$$

$$y = 500 - 2x$$

$$= 500 - 2(125)$$

$$= 500 - 250$$

$$\boxed{y = 250}$$

Examples

$A(0) = 500(0) - 2(0)^2 = 0$

$\boxed{A(125) = 250}$

$A(250) = 500(250) - 2(250)^2 = 0$

A(125) gives biggest value of 250,

so maximum dimensions are length 250

and width 125.

Week 12

Rolle’s Theorem, Mean Value Theorem

Rolle's Theorem

Rolle's theorem has 3 conditions:

As long as these conditions are satisfied, Rolle's theorem says that there will be a value "c" where $f'(c) = 0$

1. $f(x)$ must be continuous on $[a, b]$

2. $f(x)$ must be differentiable on (a, b)

3. $f(a) = f(b)$

Examples

Find all values of “c” that satisfy Rolle’s theorem for $f(x) = x^2 - 5x + 3$ on [0, 5]

1. f(x) is a polynomial, so continuous everywhere
2. f(x) is differentiable on (0, 5)
3. f(0) = f(5)

$f(0) = 0 - 5(0) + 3 = 3$
$f(5) = 25 - 25 + 3 = 3$
$f(0) = f(5)$

To find values of “c”, we need to find the crit. values of f(x).

$f(x) = x^2 - 5x + 3$
$f'(x) = 2x - 5 = 0$
$2x = 5$
$x = 5/2$

Here, 5/2 is the only crit. value and is within [0, 5], so here c = 5/2

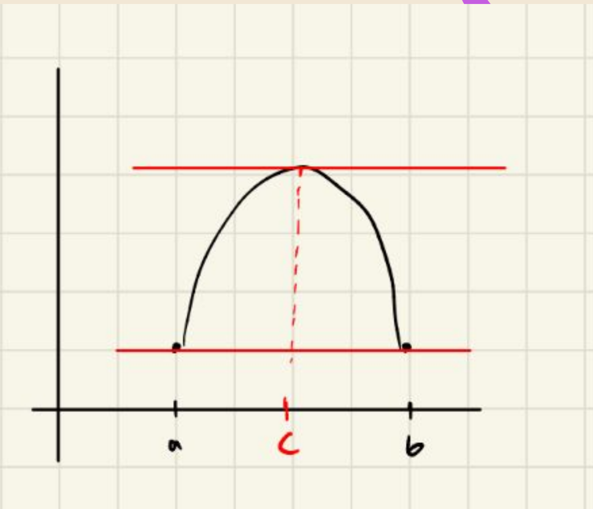
Mean Value Theorem

MVT has almost the same conditions as Rolle's theorem except for the third one:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In MVT, you draw a line between the two ends of the function and then draw a tangent line that's parallel to it

The point where the tangent line = 0 is where "c" is gonna be



Examples!

Find all values of "c" that satisfy MVT for $f(x) = x^2 - 4x + 1$ on [1, 5]

1. f(x) is a polynomial, so continuous everywhere
2. f(x) is differentiable on (1, 5)

3. $f(x) = x^2 - 4x + 1$

$f'(x) = 2x - 4 = f(5) - f(1)/5 - 1$

$2x - 4 = (6 + 2)/4$

$2x - 4 = 8/4$

$2x - 4 = 2$

$2x = 6$

$x = 3$

Here, x = 3 is well within the interval of [1, 5] and all the conditions for MVT are satisfied, so c = 3

All The Best!