THE UNIVERSITY OF BRITISH COLUMBIA

Department of Computer Science, Mathematics, Physics and Statistics
Okanagan Campus

MATH 100 Midterm 2

Winter 2021, Term 1

Irving K. Barber Faculty of Science

University of British Columbia - Okanagan

Instructor: Dr. Paul D. Lee   Thursday, November 18, 2021

This test has 7 questions for a total of 28 points.

READ THE QUESTIONS CAREFULLY

In order to get full credit, you must SHOW ALL OF YOUR WORK

NO CALCULATORS ARE PERMITTED ON THIS EXAMINATION

1. Each candidate must be prepared to produce, upon request, a UBC card for identification.
2. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
   - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   - speaking or communicating with other candidates;
   - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
3. Candidates must not destroy or mutilate any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the invigilator.
4. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

NAME (LAST, FIRST): SODHI, RAJVEER   STUDENT NUMBER: 93589539

SIGNATURE: ______   LAB SECTION: ~~003~~ 28

Question:	1	2	3	4	5	6	7	Total
Points:	3	10	3	3	3	3	3	28
Score:	3	10	3	3	3	3	2.5	30.5

+3

3/3 [3] 1. Find the equation of the tangent line to the curve $y = \frac{x}{x^2-3}$ at the point $(2,2)$

The equation for the tangent line of a curve at (x_1, y_1) is $(y-y_1) = m_t(x-x_1)$, where m_t = slope of the tangent.

Also, at this point, $m_t = y'(x_1)$

$\therefore (y-y_1) = [y'(x_1)](x-x_1)$

Putting in values, $(y-2) = [y'(2)](x-2)$ — 1

Now, finding y':

$$y = \frac{x}{x^2-3}$$

$$\therefore y' = \frac{(x^2-3)\times\frac{d}{dx}(x) - x\times\frac{d}{dx}(x^2-3)}{(x^2-3)^2} \quad [\because \text{quotient rule}]$$

$$= \frac{(x^2-3)(1) - (x)(2x)}{(x^2-3)^2}$$

$$= \frac{x^2-3-2x^2}{(x^2-3)^2}$$

$$= \frac{-x^2-3}{(x^2-3)^2}$$

$$\therefore \text{At } x_1 = 2,\ y' = \frac{-(2)^2-3}{[(2)^2-3]^2} = \frac{-4-3}{(4-3)^2} = \frac{-7}{(1)^2} = -7$$

Putting this value in 1, we get:

$$y-2 = (-7)(x-2)$$
$$= -7x+14$$

$$\boxed{y = -7x+16}$$

The equation of the tangent line to the curve $y = \frac{x}{x^2-3}$ at the point $(2,2)$ is $y = -7x+16$.

10 2. Calculate derivatives for the following functions. You do not need to simplify your answers.

(a) $y(x) = \frac{1}{\sqrt[3]{x^2}} - 5x^4 + \sin(x) + \pi^3$ $\Rightarrow y(x) = x^{-2/3} - 5x^4 + \sin(x) + \pi^3$

$y'(x) = -\frac{2}{3} x^{-5/3} - 20x^3 + \cos(x)$

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$\Rightarrow y'(x) = \frac{-2}{3\sqrt[3]{x^5}} - 20x^3 + \cos(x)$

(b) $r(\theta) = \sin^2(\theta)e^{-2\theta}$

$r'(\theta) = 2\sin(\theta)\cdot\cos(\theta)\cdot e^{-2\theta} + \sin^2(\theta)\cdot -2e^{-2\theta}$

$\Rightarrow r'(\theta) = 2e^{-2\theta}\left[\sin(\theta)\cos(\theta) - \sin^2(\theta)\right]$

(c) $G(t) = \frac{\cos(t) + t^2}{e^t - 2t + 1}$

$G'(t) = \frac{(e^t - 2t + 1) \times \frac{d}{dt}(\cos(t) + t^2) + (\cos(t) + t^2) \times \frac{d}{dt}(e^t - 2t + 1)}{(e^t - 2t + 1)^2}$

$= \frac{(e^t - 2t + 1)\cdot(-\sin(t) + 2t) + (\cos(t) + t^2)\cdot(e^t - 2)}{(e^t - 2t + 1)^2}$

(d) $p(s) = 5^{\tan(s)}$

Taking ln on both sides,

$\ln(p(s)) = \tan(s) \cdot \ln(5)$

Differentiating w.r.t s,

$\frac{1}{p(s)} \cdot p'(s) = \sec^2(s) \cdot \ln(5)$

$\therefore \; p'(s) = 5^{\tan(s)} \cdot \ln(5) \cdot \sec^2(s)$

(e) $f(x) = \sqrt{\ln(x^2 + x + 1)}$

$f'(x) = \frac{1}{2\sqrt{\ln(x^2+x+1)}} \times \frac{1}{(x^2+x+1)} \times (2x+1)$

$\Rightarrow f'(x) = \frac{2x+1}{(2)(x^2+x+1)\left(\sqrt{\ln(x^2+x+1)}\right)}$

[3] 5. Find all values of x where the following function is differentiable:

$$f(x) = \begin{cases} x^2 + x & \text{if } x \leq 1 \\ 2 + \ln(x^2) & \text{if } x > 1 \end{cases}$$

Full work and explanation must be shown to receive full marks (just writing an answer is not sufficient).

The only point where the function may not be differentiable is when the definition of the function changes. I.e., at $x = 1$. We have to find if $\lim_{x\to 1} f'(x)$ exists.

Now, when $x \leq 1$,
$f(x) = x^2 + x$
$\boxed{\therefore f'(x) = 2x+1}$ ✓

when $x > 1$,
$f(x) = 2 + \ln(x^2) = 2 + 2\ln(x)$
$\boxed{\therefore f'(x) = \frac{2}{x}}$ ✓

Now,
$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 2x+1 = 2(1)+1 = 3$

$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} \frac{2}{x} = \frac{2}{1} = 2$ ✓

Since $\lim_{x\to 1^-} f'(x) \neq \lim_{x\to 1^+} f'(x)$, $\lim_{x\to 1} f'(x)$ DNE.

$\therefore$ The function is differentiable everywhere but $x = 1$.

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[3] 6. Use any **limit definition of the derivative** to find $f'(3)$ for the function

$$f(x) = -2x^2 - x + 1$$

Points will not be awarded for using the power rule to differentiate.

To find derivative using limit derivation of $f(x)$, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$\Rightarrow f'(x) = \lim_{h\to 0} \frac{[-2(x+h)^2 - (x+h) + 1] - [-2x^2 - x + 1]}{h}$

$= \lim_{h\to 0} \frac{(\cancel{-2x^2} - 2h^2 - 4xh \cancel{-x} - h \cancel{+1} + \cancel{2x^2} \cancel{+x} \cancel{-1})}{h}$

$= \lim_{h\to 0} \frac{-2h^2 - 4xh - h}{h}$

$= \lim_{h\to 0} -2h - 4x - 1$

$= -4x - 1$

$\therefore$ For the function $f(x) = -2x^2 - x + 1$, the differentiation $f'(x)$ is $-(4x+1)$

$\therefore$ At $x = 3$, $f'(x) = -(4(3)+1) = -(12+1) = -13$.

$\boxed{\therefore f'(3) = -13}$

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$\boxed{3}$ 7. (a) The displacement of a particle after t seconds is given by the equation

$$\boxed{s(t) = t^2 \ln(t)}$$

Find the velocity and acceleration after $t = 3$ seconds. Leave your answer as an exact value.

We know that the differentiation of the displacement function w.r.t. t is nothing but the velocity function $v(t)$. Moreover, the second derivative of $s(t)$ is equal to the accelaration function $a(t)$. $a(t)$ is also equal to the first derivative of $v(t)$.

$\therefore s(t) = t^2 \cdot \ln(t)$

$\Rightarrow s'(t) = v(t) = 2t \cdot \ln(t) + \frac{t^2}{t} = 2t \cdot \ln(t) + t$ [$\because$ product rule] ✓

And $a(t) = v'(t) = \frac{2t}{t} + 2\ln(t) + 1 = 2\ln(t) + 3$ [$\because$ product rule]

$\therefore$ At $t = 3s$, $v(t) = 2(3) \cdot \ln(3) + 3 = 6\ln(3) + 3$ units/second

and $a(t) = 2 \cdot \ln(3) + 3$ units/second/second ✓

$\frac{2.5}{3}$

(b) **Bonus +3:** Use implicit differentiation to prove

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}}$$

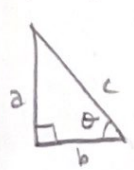
Let there be an angle θ such that $\sec^{-1}(x) = \theta$. This implies $\sec\theta = x$. — (1)

Now, differentiating both sides of (1) w.r.t. x, we get:

$\sec\theta \cdot \tan\theta \cdot \frac{d\theta}{dx} = 1$

$\Rightarrow \frac{d\theta}{dx} = \frac{1}{\sec\theta \cdot \tan\theta}$ — (2).

Also, in this right-angled triangle, $\sec\theta = x/1$, which means $c = x$, $b = 1$.



Then, by pythagoras' thm.,

$a = \sqrt{c^2 - b^2} = \sqrt{x^2 - 1}$.

This implies $\sec^{-1}(x/1) = \tan^{-1}(a/1)$

$= \tan^{-1}(\sqrt{x^2-1}/1)$ — (3)

✓

$\therefore$ (2) $= \frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{\sec\theta \cdot \tan\theta}$

($\because \theta = \sec^{-1}(x)$, as shown in (1))

$\Rightarrow \frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{\sec(\sec^{-1}(x)) \times \tan(\sec^{-1}(x))}$

$\Rightarrow \frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{\sec(\sec^{-1}(x)) \times \tan(\tan^{-1}(\sqrt{x^2-1}))}$

($\because$ (3))

$\Rightarrow \frac{d}{dx}(\operatorname{arcsec}) = \frac{1}{(x)(\sqrt{x^2-1})}$

Hence proved. +3

Formulas

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Derivatives:

1. $\dfrac{d}{dx} \arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx} \arccos(x) = \dfrac{-1}{\sqrt{1-x^2}}$
3. $\dfrac{d}{dx} \arctan(x) = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx} \csc(x) = -\csc(x)\cot(x)$
5. $\dfrac{d}{dx} \sec(x) = \sec(x)\tan(x)$
6. $\dfrac{d}{dx} \cot(x) = -\csc^2(x)$

Logarithm Rules:

1. $\log_b(AB) = \log_b(A) + \log_b(B)$
2. $\log_b\left(\dfrac{A}{B}\right) = \log_b(A) - \log_b(B)$
3. $\log_b(A^C) = C\log_b(A)$

Trig Identities:

1. $\csc(x) = \dfrac{1}{\sin(x)}$
2. $\sec(x) = \dfrac{1}{\cos(x)}$
3. $\cot(x) = \dfrac{1}{\tan(x)}$