MATH 100: Differential Calculus

Supplemental Learning

Quiz 3!

Differentiation

Friday, October 20th

Differentiation Rules

Trigonometric and Inverse Trig Derivatives

Differentiability

Logarithmic Differentiation

Implicit Differentiation

L'Hopital's Rule

Some formulas you may be given on the test (depending on your prof):

Algebra Identities

- $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$
- $a^{xy} = (a^x)^y = (a^y)^x$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
- $ax^2 + bx + c = 0 \implies x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $a^{x+y} = a^x a^y$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x^n) = n\log_a(x)$

$2x^2yy' + y^2 = 2$

Differentiation Rules

Constant Rule

Power Rule

Sum and Difference Rule

Product Rule

Quotient Rule

Chain Rule

Find the derivative of $y = 6x^{100} - x^{55} + x + 90$

Answer: $600x^{99} - 55x^{54} + 1$

Find the derivative of $y = \frac{x + 1}{x}$

Answer: $\frac{-1}{x^2}$

Find the derivative of $y = (x - 1)(x - 2)$

Answer: $2x - 3$

For the function:

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1.$$

Show that f'(1) = 100*f'(0).

Answer:

1. $f'(x) = x^{99} + x^{98} + \ldots x^0 + 0$
2. $f'(1) = 1^{99} + 1^{98} + \ldots + 1^0 = 100$
3. $f'(0) = 0^{99} + 0^{98} + \ldots + 0^0 = 1$
4. Therefore, f'(1) = 100 = 100*f'(0)

For some constants a and b, find the derivative of the function $y = (ax^2 + b)^2$

Answer: $4ax(ax^2 + b)$

Let $f(x)$ be a function satisfying $f(1) = 1$ and $f'(1) = -2$. Let $g(x) = \frac{f(x)}{x}$. Find the equation of the tangent line to $g(x)$ at $x = 1$

$$g'(x) = \frac{f'(x) \cdot x - f(x) \cdot 1}{x^2}$$

$$\Rightarrow g'(1) = \frac{f'(1) \cdot 1 - f(1)}{1^2} = \frac{-2-1}{1} = \boxed{-3}$$

$$g(1) = \frac{f(1)}{1} = \frac{1}{1} = \boxed{1}$$

$(y = m(x - x_0) + y_0)$ or $(y = mx + b)$

$$\boxed{y = -3(x-1) + 1} \Rightarrow \boxed{y = -3x + 4}$$

Calculate y'' for $y = xe^x$

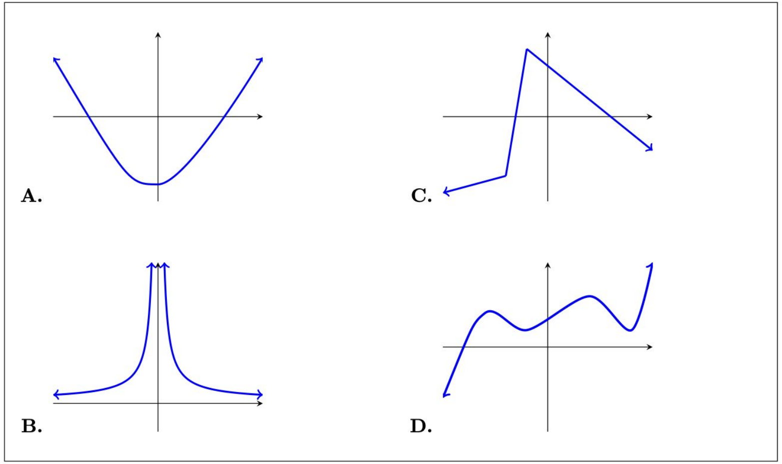
$$y' = 1 \cdot e^x + x \cdot e^x = e^x + xe^x$$

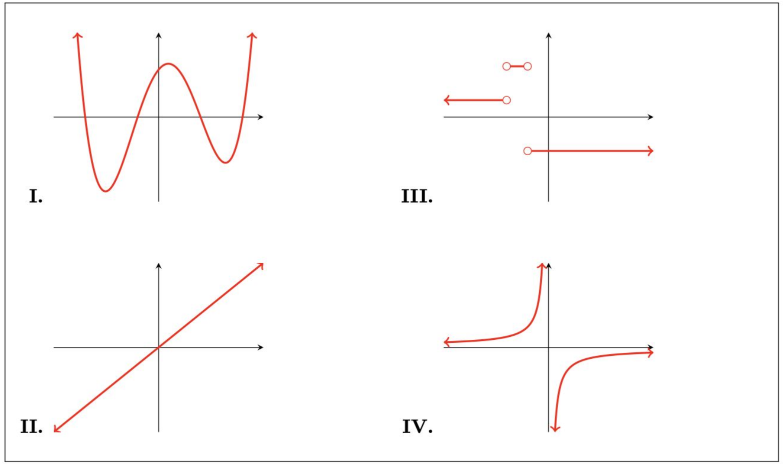
$$y'' = [e^x + xe^x]'$$

$$= e^x + 1 \cdot e^x + xe^x$$

$$= \boxed{2e^x + xe^x}$$

Match the following graphs with their derivatives.





Differentiation and Trigonometry

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}$$

Note!

1. These formulas should be given to you. You are not expected to memorize them.
2. The prefix “arc” to any tangent function is the same as saying the inverse of that function.
   E.g. - $\arctan(x) = \tan^{-1}(x)$

$2x^2yy' + y^2 = 2$

Compute the derivative of $f(x) = \sin^2 x$.

Answer: $2\sin x\cos x$ or $\sin 2x$

Compute the derivative of $f(x) = \sin 2x$.

Answer: $2\cos 2x$ or $2(\cos^2 x - \sin^2 x)$

Compute the derivative of $f(x) = 2\tan x - 7\sec x$.

Answer: $2\sec^2 x – 7\sec x \tan x$

Make a question with all the concepts we've learned till now?
Challenge accepted.

Compute the derivative of $f(x) = 4(x^2 + 1)^3 + \frac{\sin x}{e^x} + \tan x * \arctan(2x) + 12.$

Answer: $24(x^2 + 1)x + \frac{e^x \cos x - e^x \sin x}{e^{2x}} + \frac{2 \tan x}{1 + 4x^2} + \sec^2 x \tan^{-1} 2x$

Calculate the derivative of the function

$$f(x) = \frac{x^2 \sin(x)}{2x^2 - 1}$$

$T = x^2 \sin(x)$

$B = 2x^2 - 1$

$T' = 2x \sin(x) + x^2 \cos(x)$

$B' = 4x$

$$f'(x) = \frac{T'B - TB'}{[B]^2} = \frac{(2x\sin(x) + x^2\cos(x))(2x^2-1) - x^2\sin(x)(4x)}{(2x^2-1)^2}$$

Differentiability!!!

The definition of differentiability is expressed as follows:

- f is differentiable on an open interval (a,b) if $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists for every c in (a,b).
- f is differentiable, meaning $f'(c)$ exists, then f is continuous at c.

Prove that $f(x) = |x+2|$ is not differentiable at $x = -2$

$$|x+2| = \begin{cases} +(x+2) & \text{if } x \geq -2 \\ -(x+2) & \text{if } x < -2 \end{cases}$$

$$\lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0^+} \frac{+(-2+h+2) - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = \boxed{1}$$

$$\lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0^-} \frac{-(-2+h+2) - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \boxed{-1}$$

$\therefore \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$ does not exist

$\therefore f$ is <u>not</u> differentiable at $x = -2$

Using the **limit definition** of a derivative, show that $f(x) = \begin{cases} x & \text{if } x < 1 \\ \dfrac{1}{x} & \text{if } x \geq 1 \end{cases}$ is **not** differentiable at the point $x = 1$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h) - 1}{h} = \lim_{h \to 0^-} \frac{h}{h} = \boxed{1}$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{\frac{1}{1+h} - 1}{h} \cdot \frac{1+h}{1+h}$$

$$= \lim_{h \to 0^+} \frac{1 - (1+h)}{h(1+h)}$$

$$= \lim_{h \to 0^+} \frac{-h}{h(1+h)}$$

$$= \lim_{h \to 0} \frac{-1}{1+h}$$

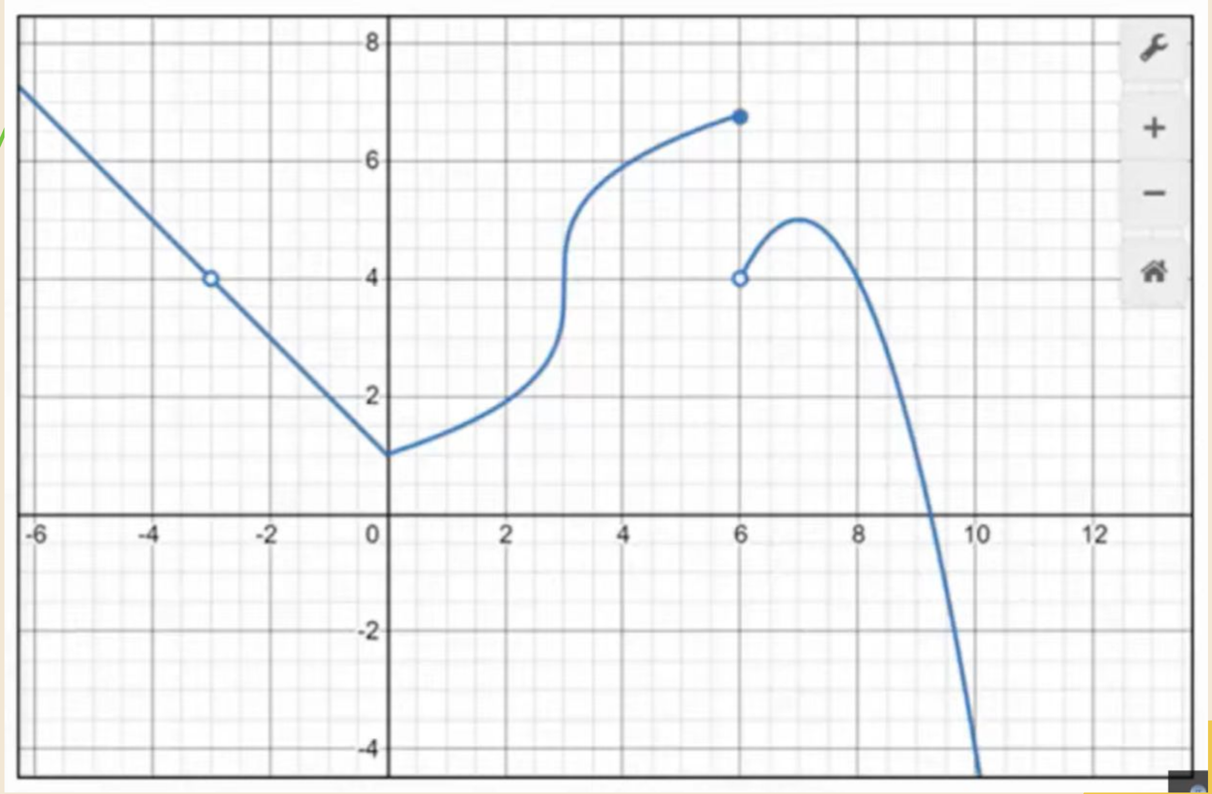
$$= \boxed{-1}$$

$$\therefore \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \neq \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$\therefore$ f is not differentiable at $x = 1$

Using the graph of $f(x)$ below, state the x-values (a) where f is **not** continuous and (b) where f is **not** differentiable. Explain your answers.

[Note: there is no vertical asymptote near $x = 10$]



(a) $\underline{x=-3}$: removable discontinuity [$f(-3)$ is undefined]

$\underline{x=6}$: jump discontinuity $\left[\lim\limits_{x\to 6^-} f(x) \neq \lim\limits_{x\to 6^+} f(x)\right]$

$(\approx 6.8) \qquad (=4)$

(b) $\underline{x=-3 \text{ \& } x=6}$: Discontinuities are automatically not differentiable

$\underline{x=0}$: $\underbrace{\lim\limits_{x\to 0^-} f'(x)}_{=-1} \neq \underbrace{\lim\limits_{x\to 0^+} f'(x)}_{\approx 1/2}$

$\underline{x=3}$: vertical tangent at $x=3$

A few more differentiability questions:

Consider the function,

$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 3x & \text{if } 1 \leq x < 2 \\ \dfrac{1}{x-4} & \text{if } x \geq 2 \end{cases}$$

Determine all values at which $f(x)$ is **not** differentiable. Explain your answer.

Consider the function,

$$f(x) = \begin{cases} 2 - x & \text{if } x < 1 \\ x & \text{if } x > 1 \end{cases}$$

Using the limit definition of the derivative, show why $f(x)$ is **not** differentiable at $x = 1$.

Use the limit definition of the derivative to show that $f(x) = x^2 + |x+1|$ is **not** differentiable at $x = -1$.

Let f be the function defined below, where c and d are constants. If f is differentiable at $x=-1$, what is the value of $c-d$?

$$f(x)=\begin{cases} x^2+(2c+1)x-d, x\geq -1 \\ e^{2x+2}+cx+3d, x<-1 \end{cases}$$

A. -2 B. 0 C. 2 D. 3 E. 4

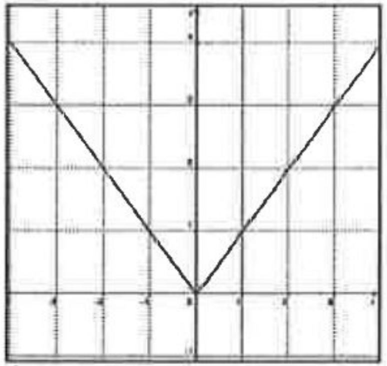
E. Continuity: $\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^-} f(x) \Rightarrow 1-(2c+1)-d=1-c-3d \Rightarrow c+4d=-1$

$$f'(x)=\begin{cases} 2x+2c+1, x\geq -1 \\ 2e^{2x+2}+c, x<-1 \end{cases}$$

Differentiability: $\lim\limits_{x\to -1^+} f'(x)=\lim\limits_{x\to -1^-} f'(x) \Rightarrow -2+2c+1=2+c \Rightarrow c=3, d=-1$

$c-d=3+1=4$

. The graph of $f(x)=\sqrt{x^2+0.0001}-0.01$ is shown in the graph to the right. Which of the following statements are true?



I. $\lim_{x\to 0} f(x)=0$.

II. f is continuous at $x=0$.

III. f is differentiable at $x=0$.

A. I only   B. II only   C. I and II only   D. I, II, and III   E. None are true

D. A trap problem. This looks like $|x|$ which is not differentiable at $x=0$.

But the function is given and $f'(x)=\dfrac{x}{\sqrt{x^2+0.0001}}$ and $f'(0)=0$.

$2x^2yy'+y^2=2$

Let $f(x)$ be given by the function below. What values of a, b, and c do **NOT** make $f(x)$ differentiable?

$$f(x)=\begin{cases} a\cos x, x\le 0 \\ b\sin(x+c\pi), x>0 \end{cases}$$

A. $a=0,\ b=0,\ c=0$

B. $a=0,\ b=0,\ c=100$

C. $a=5,\ b=5,\ c=0.5$

D. $a=-1,\ b=1,\ c=1$

E. $a=-8,\ b=8,\ c=-2.5$

D. For continuity, $a\cos 0=b\sin c\pi \Rightarrow a=b\sin c\pi$

$$f'(x)=\begin{cases} -a\sin x, x\le 0 \\ b\cos(x+c\pi), x>0 \end{cases}$$

For differentiability, $0=b\cos c\pi$

If $b=0, a=0$ so choices A and B are true.

If $c=0.5$ or $c=-2.5$, b can take on any value.

if $c=0.5$ and $b=5$, then $a=5\sin .5\pi=5$ so choice C is true.

if $c=-2.5$ and $b=8$, then $a=8\sin 2.5\pi=-8$ so choice E is true.

if $c=1$ and $b=1$, then f is not differentiable so choice D is false.

. Let f be the function defined below. Which of the following statements about f are **NOT** true?

$$f(x)=\begin{cases}\dfrac{x^3-1}{x-1}, x\neq 1\\ 3x, x=1\end{cases}$$

I. f has a limit at $x=1$.

II. f is continuous at $x=1$.

III. f is differentiable at $x=1$.

IV. The derivative of f' is continuous at $x=1$.

A. IV only

B. III and IV only

C. II, III, and IV only

D. I, II, III, and IV

E. All statements are true

A. $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\left(x^2+x+1\right)=3$, so limit exists

Since $f(1)=3$, f is continuous

$$f'(x)=\begin{cases}2x+1, x\neq 1\\ 3, x=1\end{cases}$$

f is differentiable at $x=1$ as $\lim\limits_{x\to 1^-} f'(x)=\lim\limits_{x\to 1^+} f'(x)=3=f'(1)$

But since $\lim\limits_{x\to 1} f''(1)=2$ and $f''(1)=0$, the derivative of f' is not continuous at $x=1$.

Let $f(x)=\begin{cases}-6x^2+14x-8, x\le 1\\ \sin(2x-2), x>1\end{cases}$. Show that $f(x)$ is differentiable.

Both a parabola and a sine curve are continuous. $f(x)$ is continuous at $x=1$ because

$\lim_{x\to 1^-} f(x)=-6+14-8=0 \qquad \lim_{x\to 1^+} f(x)=\sin 0=0$

$$f'(x)=\begin{cases}-12x+14, x<1\\ 2\cos(2x-2), x\ge 1\end{cases}$$

$f(x)$ is differentiable at $x=1$ because

$\lim_{x\to 1^-} f'(x)=-12+14=2 \qquad \lim_{x\to 1^+} f'(x)=2\cos 0=2$

Find the values of a and b that make the following function differentiable:

$$y = \begin{cases} ax^3 - 2, x \le 3 \\ b(x-2)^2 + 10, x > 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = 27a - 2 \qquad \lim_{x\to 3^+} f(x) = b + 10 \Rightarrow \text{For continuity}: 27a = b + 12$$

$$f'(x) = \begin{cases} 3ax^2, x \le 3 \\ 2b(x-2), x > 3 \end{cases}$$

$$\lim_{x\to 3^-} f'(x) = 27a \qquad \lim_{x\to 3^+} f'(x) = 2b \Rightarrow \text{For differentiability}: 27a = 2b$$

$$2b = b + 12 \Rightarrow \boxed{b = 12} \qquad 27a = 12 + 12 \Rightarrow \boxed{a = \frac{8}{9}}$$

Logarithmic Differentiation

- Differentiation of $e^x = e^x$
- Differentiation of $\ln(x) = 1/x$

Calculate the following.

i) The derivative of $f(x) = 5x^3 - 3x^{-2/3} + 2x^{\ln(2)-1}$

ii) An equation of the tangent line to $g(t) = \dfrac{2t^e - 5t^2 + 4}{t^{4/5}}$ at $t = 1$

iii) The second derivative of $k(p) = p^2 2^p$

iv) Suppose the distant of a particle at a time t is given by $F(t) = \dfrac{\sqrt{t}e^t}{2^t}$ nanometres where $t > 0$ is measured in seconds. Calculate the velocity of the particle at time $t = 1$.

v) An equation of the tangent line to $f(x) = \dfrac{x^{-1/2} + e^x}{x^2 + x + 1}$ at $x = 0$

Important Questions from CLP (as per profs):

§2.3 - Interpretations of the Derivative. 1, 3, 5.

§2.4 - Arithmetic of Derivatives. 1 - 3, 5 - 11, 13, 16

§2.6 - Using the Arithmetic of Derivatives. 1, 2, 3 - 16, 18 - 22

§2.7 - Derivative of Exponential Functions. 1, 2, 3, 5 - 11, 13.

§2.8 - Derivatives of Trigonometric Functions. 3 - 13, 14, 15, 17, 18, 22, 23, 24

§2.9 - One More Tool - The Chain Rule. 2 - 30 (omit 28)

§2.10 - The Natural Logarithm. 3 - 30.

§2.12 - Inverse Trigonometric Functions. 6, 7 (don't worry about the domains, just take the derivatives), 12, 13, 14.

From APEX:

- Section 2.1: # 7-14, 19, 21, 27
- Section 2.3: # 11-25 (any), 28, 29, 31, 33, 36, 37
- Section 2.4: # 15-47 (odd), 49, 51

From OpenStax:

- Section 3.1 # 13, 15, 16, 18, 19, 23, 26, 27, 28
- Section 3.2 # 58, 62, 63, 75, 76, 77, 79, 96

Any Questions?????

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