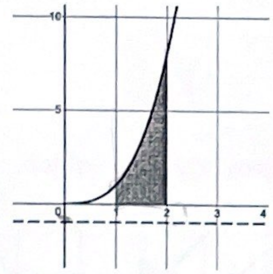
ANSWERS ONLY: For questions 1 – 3, only the answer is required. You do not need to show any work.



1. $\boxed{\text{1 mark}}$ The region between $y = x^3$, $y = 0$, $x = 1$ and $x = 2$ is shown to the right. If this region is rotated about the line $y = -1$, which one of the following expressions gives the volume of the resulting solid?

Answer (just write the letter of your choice): A

A. $\pi \int_1^2 \left[(x^3+1)^2 - 1\right] dx$

B. $\pi \int_1^2 (x^3+1)^2 \, dx$

C. $\pi \int_1^2 x^6 - 1 \, dx$

D. $\pi \int_1^2 (x^3-1)^2 - 1 \, dx$

E. $\pi \int_1^2 (x^3-1)^2 \, dx$

2. $\boxed{\text{1 mark}}$ Which one of the following substitutions for the integral $\int \tan^5(x)\sec^5(x)\,dx$ is correct?

$\sec^4 x \cdot \tan^5 x \cdot (\tan^2 x+1)^2$
let $u = \tan x$
$\frac{du}{dx} = \sec^2 u$

$(1-\sec^2 x)^2 \sec^5 x \cdot \tan x$
$u = \sec x$, $du/dx = \sec x \tan x$
$(1-u^2)^2 \cdot u^4 \cdot du$

Answer (just write the letter of your choice): E

A. $u = \tan(x) \Longrightarrow \int u^5(u^2+1)^{5/2}\,du$

B. $u = \tan(x) \Longrightarrow \int u^4(u^2+1)^2\,du$

C. $u = \tan(x) \Longrightarrow \int u^5(u^2+1)^3\,du$

D. $u = \sec(x) \Longrightarrow \int (u^2-1)^{5/2}u^5\,du$

E. $u = \sec(x) \Longrightarrow \int (u^2-1)^2u^4\,du$

F. $u = \sec(x) \Longrightarrow \int (u^2-1)^2u^3\,du$

3. $\boxed{\text{1 mark}}$ Which one of the following formulas results after using integration by parts on

$$\int x^4 (\ln(x))^2 \, dx?$$

$u = [\ln(x)]^2$, $du = \frac{2\ln(x)}{x} \cdot dx$
$dv = x^4 \cdot dx$, $v = x^5/5$
$uv - \int v \cdot du = \frac{x^5(\ln(x))^2}{5} - \frac{2}{5}\int \frac{x^5 \cdot \ln(x)}{x} \cdot dx$
$\Rightarrow \frac{x^5}{5}[\ln(x)]^2 - \frac{2}{5}\int x^4 \cdot \ln(x) \cdot dx$

Answer (just write the letter of your choice): C

A. $x^4(\ln(x))^2 - 8\int x^3 \ln(x)\,dx$

B. $\frac{1}{5}x^5(\ln(x))^2 - \frac{1}{5}\int x^5(\ln(x))^2\,dx$

C. $\frac{1}{5}x^5(\ln(x))^2 - \frac{2}{5}\int x^4 \ln(x)\,dx$

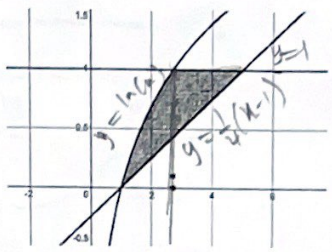
D. $\frac{1}{5}x^5(\ln(x))^2 - \frac{2}{5}\int x^5 \ln(x)\,dx$

For all of the remaining questions, write complete solutions and make sure to show all necessary work unless indicated otherwise. Partial marks may be awarded.

Formulas:

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x)) \qquad \sin^2(x) = \frac{1}{2}(1-\cos(2x)) \qquad \sin(2x) = 2\sin(x)\cos(x)$$

4. $\boxed{\text{4 marks}}$ Consider the region under the line $y = 1$, to the right of the graph of $y = \ln(x)$, and above the line $y = \frac{1}{4}(x-1)$, as shown in the diagram to the right.



Find the area of this region (in exact form).
[Hint: choose your variable of integration carefully.]

First, we must convert the functions to in terms of x.

$\therefore \; y = \frac{1}{4}(x-1) \Rightarrow x = 4y+1$

$y = \ln(x) \Rightarrow x = e^y$

And now the bounds of the area are $(y=0)$ and $(y=1)$.

$\therefore$ Area $= \int_0^1 |f(y) - g(y)| \cdot dy$. Since $x = 4y+1$ is the function on top,

we can take that as $f(y)$ and $x = e^y$ as $g(y)$.

$\therefore$ Area $= \int_0^1 [(4y+1) - (e^y)] \cdot dy = \int_0^1 (4y+1-e^y) \cdot dy = \left[\frac{4y^2}{2} + y - e^y\right]_0^1$

$\Rightarrow \left[\frac{4(1)^2}{2} + 1 - e^1\right] - \left[\frac{4(0)^2}{2} + 0 - e^0\right]$

$\Rightarrow (2+1-e) - (-1)$

$\Rightarrow 2+1-e+1$

$\Rightarrow 4-e$

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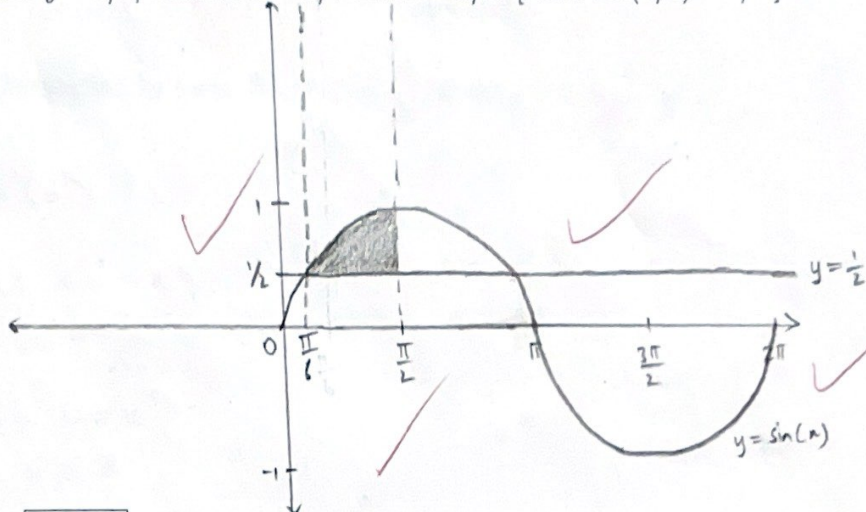
Area of this region is $(4-e)$ units.

5. (a) $\boxed{\text{2 marks}}$ Sketch and shade in the region beneath the graph of $y = \sin(x)$ and above the line $y = 1/2$, between $x = \pi/6$ and $x = \pi/2$. [Note: $\sin(\pi/6) = 1/2$.]



(b) $\boxed{\text{3 marks}}$ The region in part (a) is rotated around the x-axis and the volume of the resulting solid is given by

$$\int_{\pi/6}^{\pi/2} \pi \left(\sin^2(x) - \frac{1}{4} \right) dx$$

Determine the value of this integral to find the volume of the solid. Give your answer simplified as much as possible.

$$\pi \int_{\pi/6}^{\pi/2} \left(\sin^2(x) - \frac{1}{4} \right) . dx$$

Using half-angle formula,

$$\pi \int_{\pi/6}^{\pi/2} \left[\frac{1 - \cos(2x)}{2} - \frac{1}{4} \right] . dx$$

$$\Rightarrow \pi \int_{\pi/6}^{\pi/2} \left[\frac{1}{2} - \frac{1}{2}\cos(2x) - \frac{1}{4} \right] . dx$$

$$\Rightarrow \pi \left[\frac{x}{2} - \frac{1}{2} \cdot \frac{\sin(2x)}{2} - \frac{x}{4} \right]_{\pi/6}^{\pi/2}$$

$$\Rightarrow \pi \left[\frac{\pi}{4} - \frac{1}{4} \cdot \sin(\pi) - \frac{\pi}{8} \right] - \pi \left[\frac{\pi}{12} - \frac{1}{4} . \sin(\pi/3) - \frac{\pi}{24} \right]$$

$$\Rightarrow \pi \left(\frac{\pi}{8} - 0 \right) - \pi \left(\frac{\pi}{24} - \frac{1}{4} \cdot \frac{\sqrt{3}}{2} \right)$$

continuing from below,

$$\text{Volume} = \frac{\pi^2}{8} - \pi \left(\frac{\pi}{24} - \frac{\sqrt{3}}{8} \right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{24} - \frac{\sqrt{3}\pi}{8}$$

$$\Rightarrow \frac{3\pi^2 - \pi^2 - 3\sqrt{3}\pi}{24}$$

$$\Rightarrow \frac{2\pi^2 - 3\sqrt{3}\pi}{24}$$

Volume of the resulting solid is

$$\frac{2\pi^2 - 3\sqrt{3}\pi}{24} \text{ or } \frac{\pi^2}{12} - \frac{\sqrt{3}\pi}{8}$$

6. [6 marks] Find

LIATE

$$\int \arcsin(x)\,dx$$

Hint: Integration by parts. Recall that $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$

$I = \int 1 \cdot \sin^{-1}(x) \cdot dx$

Let $u = \sin^{-1}(x)$ ✓

Then, $du = \frac{1}{\sqrt{1-x^2}} \cdot dx$

Let $dv = 1 \cdot dx$

Then, $v = x$

Using IBP, $I = uv - \int v \cdot du$

$= x\sin^{-1}(x) - \underbrace{\int x \cdot \frac{1}{\sqrt{1-x^2}} \cdot dx}_{(i)}$ ✓

↳ let this integral be i.

Solving (i), $\int \frac{x}{\sqrt{1-x^2}} \cdot dx$

Let $1 - x^2 = t$ ✓

$\therefore \frac{dt}{dx} = -2x$

$\Rightarrow dx = \frac{dt}{-2x}$. Substituting dx with $\frac{dt}{-2x}$ in (i), we get:

$i = \int \frac{x}{\sqrt{t}} \cdot \frac{dt}{-2x} = \frac{-1}{2}\int \frac{dt}{\sqrt{t}}$ ✓

$\Rightarrow \frac{-1}{2}(2\sqrt{t}) + c \Rightarrow -\sqrt{t} + c = -\sqrt{1-x^2} + c$ ✓

Putting value of (i) back in I, $I = x\sin^{-1}(x) - (-\sqrt{1-x^2}) + C$ general constant, including constant from (i).

$\boxed{I = x\sin^{-1}(x) + \sqrt{1-x^2} + c}$ ✓ 6/6

7. $\boxed{\text{6 marks}}$ Find $\displaystyle\int_{-\pi/2}^{\pi/2} \left|\sin^3(x)\cos^2(x)\right| dx$. Give the exact answer.

let $I = \int \sin^3(x) \cdot \cos^2(x) \cdot dx$ Then,

$I = \int \sin^2(x) \cdot \sin(x) \cdot \cos^2(x) \cdot dx$

$\Rightarrow \int [1-\cos^2(x)] \cdot \cos^2(x) \cdot \sin(x) \cdot dx$

Let $\cos(x) = u$

$\therefore \frac{du}{dx} = -\sin(x)$

$\Rightarrow dx = \frac{du}{-\sin(x)}$ Substituting value into I,

$I = \int (1-u^2) \cdot (u^2) \cdot \sin(x) \cdot \frac{du}{-\sin(x)}$

$\Rightarrow -\int (1-u^2) \cdot (u^2) \cdot du$

$\Rightarrow -\int (u^2-1)(u^2) \cdot du$

$\Rightarrow \int (u^4 - u^2) \cdot du$

$\Rightarrow \left[\frac{u^5}{5} - \frac{u^3}{3}\right]$ 1

$\Rightarrow \left[\frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3}\right]$ 0.5

why? explain

To find $\int_{-\pi/2}^{\pi/2} |\sin^3(x) \cdot \cos^2(x)| \cdot dx$, we can simply do $2\int_0^{\pi/2} (\sin^3(x) \cdot \cos^2(x)) \cdot dx$. 1

0.5

$\Rightarrow 2\left[\frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3}\right]_0^{\pi/2} = 2\left[\frac{\cos^5(\pi/2)}{5} - \frac{\cos^3(\pi/2)}{3} - \frac{\cos^5(0)}{5} + \frac{\cos^3(0)}{3}\right]$

1.5

$\Rightarrow 2\left[0 - 0 - \frac{1}{5} + \frac{1}{3}\right] \Rightarrow 2\left(\frac{1}{3} - \frac{1}{5}\right) = 2\left(\frac{2}{15}\right) = \frac{4}{15}$