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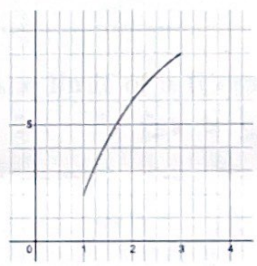
MATH 101 Midterm 1 Version A

Instructor: Dr. Wayne Broughton **February 11, 2022**

This test has 7 questions worth 24 marks

ANSWERS ONLY: For questions 1 – 3, only the answer is required. You do not need to show any work.

1. [1 mark] Consider the graph of $y = f(x)$ for $1 \le x \le 3$ as shown. Which one of the following is true about the right-endpoint approximation R_5 of $\displaystyle\int_1^3 f(x)\,dx$?



$\Delta x = \frac{b-a}{n} = \frac{2}{n} = \frac{2}{5} = 0.4$

A. $R_5 = 0.4\,(f(1.4) + f(1.8) + f(2.2) + f(2.6) + f(3))$ and it is an overestimate

B. $R_5 = 0.4\,(f(1.4) + f(1.8) + f(2.2) + f(2.6) + f(3))$ and it is an underestimate

C. $R_5 = 0.5\,(f(1) + f(1.5) + f(2) + f(2.5) + f(3))$ and it is an overestimate

D. $R_5 = 0.5\,(f(1) + f(1.5) + f(2) + f(2.5) + f(3))$ and it is an underestimate

Answer (just write the letter of your choice): A

2. [1 mark] If $\displaystyle\int_1^9 f(x)\,dx = 12$, $\displaystyle\int_1^4 f(x)\,dx = 7$, and $\displaystyle\int_6^9 f(x)\,dx = 8$, then what is the value of $\displaystyle\int_4^6 f(x)\,dx$?

$\int_4^9 f(u)\,du = 5$ || $-\int_9^6 f(u)\,du = -8$ || $\int_4^6 f(u)\,du = -8+5=-3$

Answer only: -3

3. [3 marks] If $F(x) = \displaystyle\int_{x^4}^{e} \ln\left(1+t^2\right)\,dt$, then what is the derivative $F'(x)$? $-\frac{d}{dx}\left[\int_e^{x^4} \ln(1+t^2)\,dt\right]$

Answer only (but part marks may be given): $-\ln(1+x^8) \times 4x^3$

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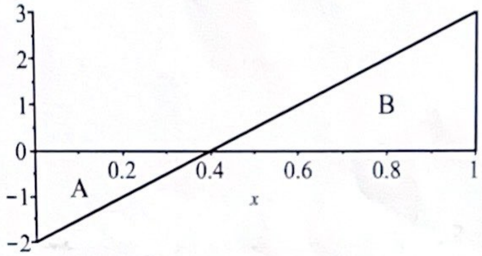
For all of the remaining questions, write complete solutions and make sure to show all necessary work unless indicated otherwise. Partial marks may be awarded.

Formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

4. [5 marks] The graph of $f(x) = 5x - 2$ is shown to the right, with two triangular regions labelled A and B. What does the integral $\int_0^1 (5x-2)\,dx$ equal in terms of area(A) and area(B)?



Answer only: area(B) − area(A)

Find the value of $\int_0^1 (5x-2)\,dx$ **by calculating the limit of a Riemann sum.** No marks for doing it another way.

Here, $a=0$, $b=1$. $\therefore \Delta x = \frac{b-a}{n} = \frac{1}{n}$

For Right Riemann Sum,

$$x_i = a + \Delta x\, i = 0 + \frac{i}{n} = \frac{i}{n}$$

$$\therefore f(x_i) = f(i/n) = \left[\frac{5i}{n} - 2\right]$$

$$\therefore R(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5i}{n} - 2\right)\left(\frac{1}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5i}{n^2} - \frac{2}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty} \frac{5}{n^2}\sum_{i=1}^{n} i - \frac{2}{n}\sum_{i=1}^{n} 1$$

$$\Rightarrow \lim_{n\to\infty} \left[\left(\frac{5}{n^2}\right)\left(\frac{n(n+1)}{2}\right) - \left(\frac{2}{n}\right)(n)\right]$$

$$\Rightarrow \lim_{n\to\infty} \left[\left(\frac{5}{2}\right)\left(\frac{\cancel{n^2}}{\cancel{n^2}} + \frac{\cancel{n}}{n^2}\right) - 2\right]$$

$$\Rightarrow \lim_{n\to\infty} \left[\left(\frac{5}{2}\right)\left(1 + n^{-1}\right) - 2\right]$$

when $n \to \infty$, $n^{-m} = 0$.

$$\therefore \Rightarrow \left(\frac{5}{2}\right) - 2$$

$$\Rightarrow \boxed{\frac{1}{2}}$$

The limit of the Right Reimann Sum of $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5i}{n} - 2\right)\cdot\frac{1}{n}$ is $\frac{1}{2}$

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5. Find the following indefinite integrals.

(a) [3 marks] $\int x^{1/3}(x^2-1)\,dx$

$\Rightarrow \int \left(x^{2+\frac{1}{3}} - x^{1/3} \right) \cdot dx$

$\Rightarrow \int x^{7/3} \cdot dx - \int x^{1/3} \cdot dx$

$\Rightarrow \boxed{\dfrac{x^{10/3}}{10/3} - \dfrac{x^{4/3}}{4/3} + C}$ ✓

(b) [4 marks] $\int \frac{2+x^2}{6x+x^3}\,dx$

Let $u = 6x + x^3$. Differentiating both sides w.r.t. x,

$\frac{du}{dx} = 6 + 3x^2$

$\Rightarrow dx = \frac{du}{3(2+x^2)}$. Substituting into integral, we get:

$\frac{1}{3} \int \frac{2+x^2}{u} \cdot \frac{du}{2+x^2}$

$\Rightarrow \frac{1}{3} \int \frac{du}{u}$

$\Rightarrow \frac{1}{3} \ln(u) + C$ ✓

$\Rightarrow \boxed{\frac{1}{3}(6x+x^3) + C}$

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6. [3 marks] Suppose that $T(t)$ is the temperature in °C of some food that was removed from the freezer t hours ago, and that its rate of warming up is given by $T'(t) = 6e^{-(1/5)t}$ (in °C/hour). Assuming that $T(0) = -6$, find a formula for $T(t)$ for $t > 0$.

If rate $T'(t) = 6e^{-(1/5)t}$, then the total change in temperature is given by the integral:

$$T(t) = \int 6e^{-(1/5)t} \cdot dt$$

$$\Rightarrow 6\int e^{(-\frac{1}{5})t} \cdot dt$$

$$\Rightarrow 6\left[\frac{e^{-t/5}}{-1/5}\right] + C$$

$$\Rightarrow -30e^{-t/5} + C$$

$$\therefore T(t) = -30e^{-t/5} + C$$

Now, $T(0) = -6 = -30e^{-0/5} + C$

$$\Rightarrow -6 = -30 + C$$

$$\Rightarrow C = 24$$

$$\boxed{\therefore T(t) = -30e^{-t/5} + 24 \ (t>0)}$$

3/3

7. [4 marks] Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\,dx$. [You can work out the indefinite integral first if you want.] Give the exact answer but try to simplify it as much as possible.

Let $\cos(x) = u$. Differentiating both sides w.r.t x,

$$\frac{du}{dx} = -\sin(x)$$

$$\Rightarrow dx = -\frac{du}{\sin(x)}$$

For changing limits, $\frac{\pi}{2}$ becomes $\cos(\frac{\pi}{2})$ $\Rightarrow 0$

0 becomes $\cos(0)$ $\Rightarrow 1$

Substituting values in integral,

$$\int_1^0 \frac{\sin(x)}{1+u^2} \times \frac{du}{-\sin(x)}$$

$$\Rightarrow -\int_1^0 \frac{du}{1+u^2}$$

$$\Rightarrow \int_0^1 \frac{du}{1+u^2} \quad \text{(property of definite integrals)}$$

$$\Rightarrow \left[\tan^{-1}(u)\right]_0^1$$

$$\Rightarrow \tan^{-1}(1) - \tan^{-1}(0)$$

$$\Rightarrow \frac{\pi}{4} - 0$$

$$\Rightarrow \boxed{\frac{\pi}{4}}$$

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