MATH 101

Integral Calculus

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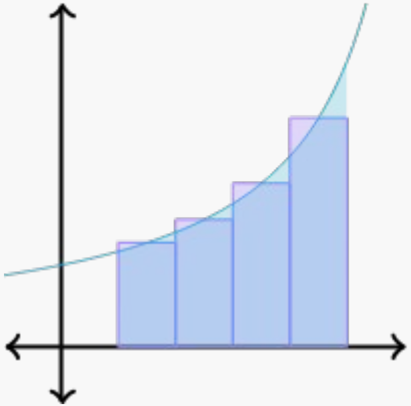
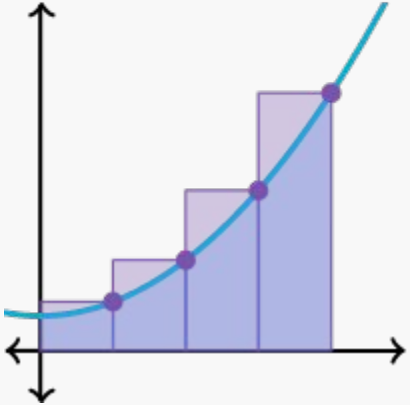
01

Riemann Sums

Rajveer





Left Reimann Sum

- Overestimate
- $L_n = \sum_{k=0}^{n-1} f(x_k)\Delta x$
- $\Delta x = \frac{b-a}{n}$

Midpoint Reimann Sum

- Best Estimate
- $M_n = \sum_{k=1}^{n} f(\bar{x}_k)\Delta x$
- $\bar{x}_k = \frac{x_k + x_{k-1}}{2}$

Right Reimann Sum

- Underestimate
- $R_n = \sum_{k=1}^{n} f(x_k)\Delta x$
- $\Delta x = \frac{b-a}{n}$

Example

$$f(x) = 3x - 1 \text{ on } [-1, 2];\ R_8$$

$\int\limits_a^b f(x)\,dx \approx \Delta x \left(f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n)\right)$

where $\Delta x = \frac{b-a}{n}$.

We have that $f(x) = 3x - 1$, $a = -1$, $b = 2$, and $n = 8$.

Therefore, $\Delta x = \frac{2-(-1)}{8} = \frac{3}{8}$.

Divide the interval $[-1, 2]$ into $n = 8$ subintervals of the length $\Delta x = \frac{3}{8}$ with the following endpoints: $a = -1, -\frac{5}{8}, -\frac{1}{4}, \frac{1}{8}, \frac{1}{2}, \frac{7}{8}, \frac{5}{4}, \frac{13}{8}, 2 = b$.

Now, just evaluate the function at the right endpoints of the subintervals.

$f(x_1) = f\left(-\frac{5}{8}\right) = -\frac{23}{8} = -2.875$

$f(x_2) = f\left(-\frac{1}{4}\right) = -\frac{7}{4} = -1.75$

$f(x_3) = f\left(\frac{1}{8}\right) = -\frac{5}{8} = -0.625$

$f(x_4) = f\left(\frac{1}{2}\right) = \frac{1}{2} = 0.5$

$f(x_5) = f\left(\frac{7}{8}\right) = \frac{13}{8} = 1.625$

$f(x_6) = f\left(\frac{5}{4}\right) = \frac{11}{4} = 2.75$

$f(x_7) = f\left(\frac{13}{8}\right) = \frac{31}{8} = 3.875$

$f(x_8) = f(2) = 5$

Finally, just sum up the above values and multiply by $\Delta x = \frac{3}{8}$:
$\frac{3}{8}(-2.875 - 1.75 - 0.625 + 0.5 + 1.625 + 2.75 + 3.875 + 5) = 3.1875.$

ANSWER

$\int\limits_{-1}^{2} (3x - 1)\,dx \approx 3.1875$ A

02

Definite Integrals

Sanjith

Definite Integrals as Limits of Riemann Sums

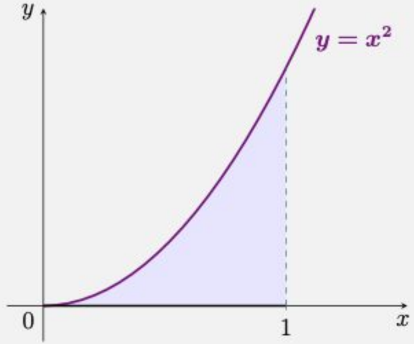
In the limit definition of definite integral, x_i is an arbitrary point in the ith subinterval. To simplify the computation, we often take the x_i's to be the right endpoints. This simplifies the definition of a definite integral as follows.

Theorem 1.2.1. *Let f be a function. Suppose that f is integrable on an interval $[a, b]$. Then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and

$$x_i = a + i\Delta x, \qquad 1 \le i \le n \tag{1.2.1}$$



The exact area of the shaded region above is given by the definite integral $\int_0^1 x^2\,dx$.

Express the following definite integral as limits of Riemann sums.

$$\int_5^{13} x^4\,dx$$

$$\Delta x = \frac{b-a}{n} = \frac{13-5}{n} = \frac{8}{n} \qquad \text{and} \qquad x_i = a + i\Delta x = 5 + \frac{8i}{n}$$

Using Theorem 1.2.1, we have

$$\begin{aligned}
\int_5^{13} x^4\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} (x_i)^4 \cdot \frac{8}{n} && f(x_i) = (x_i)^4 \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(5 + \frac{8i}{n}\right)^4 \cdot \frac{8}{n} && x_i = 5 + \frac{8i}{n}
\end{aligned}$$

Express each limit as a definite integral on the given interval.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\ln(x_i)}{x_i} \Delta x, \quad [1, 3]$$

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\ln(x_i)}{x_i} \Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \Delta x \\ &= \int_1^3 f(x)\,dx \\ &= \boxed{\int_1^3 \frac{\ln x}{x}\,dx} \end{aligned}$$

$$f(x_i) = \frac{\ln(x_i)}{x_i}$$

$$f(x) = \frac{\ln x}{x}, \quad [a, b] = [1, 3]$$

03

Antiderivatives

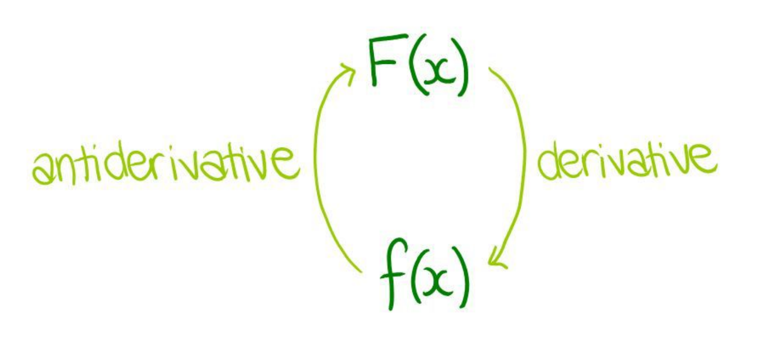
Sanjith

Antiderivatives

Let f be a function. A function F is called an **antiderivative** of f on an interval I if the derivative of $F(x)$ is equal to $f(x)$; that is,

$$F'(x) = f(x) \qquad \text{for all } x \text{ in I}$$

For example, the function $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ because $F'(x) = \frac{d}{dx}[x^2] = 2x$. So



Determine whether $F(x)$ is an antiderivative of $f(x)$.

$$f(x) = \frac{2x}{\sqrt{1+x^2}}, \qquad F(x) = \sqrt{1+x^2}$$

We need to find the derivative of $F(x) = \sqrt{1+x^2}$ and see if it is equal to $f(x) = \frac{2x}{\sqrt{1+x^2}}$.

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[\sqrt{1+x^2}\right] = \frac{d}{dx}\left[(1+x^2)^{\frac{1}{2}}\right] \\ &= \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}\frac{d}{dx}[x^2] \\ &= \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x) \\ &= \frac{1}{2}\frac{1}{(1+x^2)^{\frac{1}{2}}}(2x) \\ &= \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{d}{dx}[(g(x))^n] = n(g(x))^{n-1}\frac{d}{dx}[g(x)]$$

$$\frac{a}{b} - c = \frac{a-bc}{b}, \quad \frac{d}{dx}[x^n] = nx^{n-1}$$

$$x^{-m} = \frac{1}{x^m}$$

Since $F'(x)$ is not equal to $f(x)$, it follows that $F(x)$ is not an antiderivative of $f(x)$.

04

Fundamental Theorem of Calculus

Rajveer

FTC 1

Let f (x) be continuous on [a, b]. Then, the area function

$$g(x) = \int_a^x f(t)\, dt$$

is continuous for all x ∈ [a, b] and g ′(x) = f (x) for all x ∈ (a, b).

FTC 2

Let f (x) be continuous on [a, b] and let F (x) be any antiderivative for f (x)
on [a, b]. Then,

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Example

$$\int_0^{\pi} \sin(x)\ dx$$

$$= \left[-\cos(x)\right]_0^{\pi}$$

$$= -\cos(\pi) - (-\cos(0))$$

$$= 1 + 1$$

$$= 2$$

05

Net Change Theorem

Sanjith

The Net Change Theorem

If $y = F(x)$ is a function, the **net change** in y as x goes from a to b is $F(b) - F(a)$. If the rate of change of y, $F'(x)$, is given, then the total change in $F(x)$ as x changes from a to b is given by the integral $\int_a^b F'(x)\,dx$; that is,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In words, this says that the integral of the rate of change is the net change.

find the net change in $F(x)$ as x goes from a to b.

$$F'(x) = x^3, \quad a = 1, \quad b = 3$$

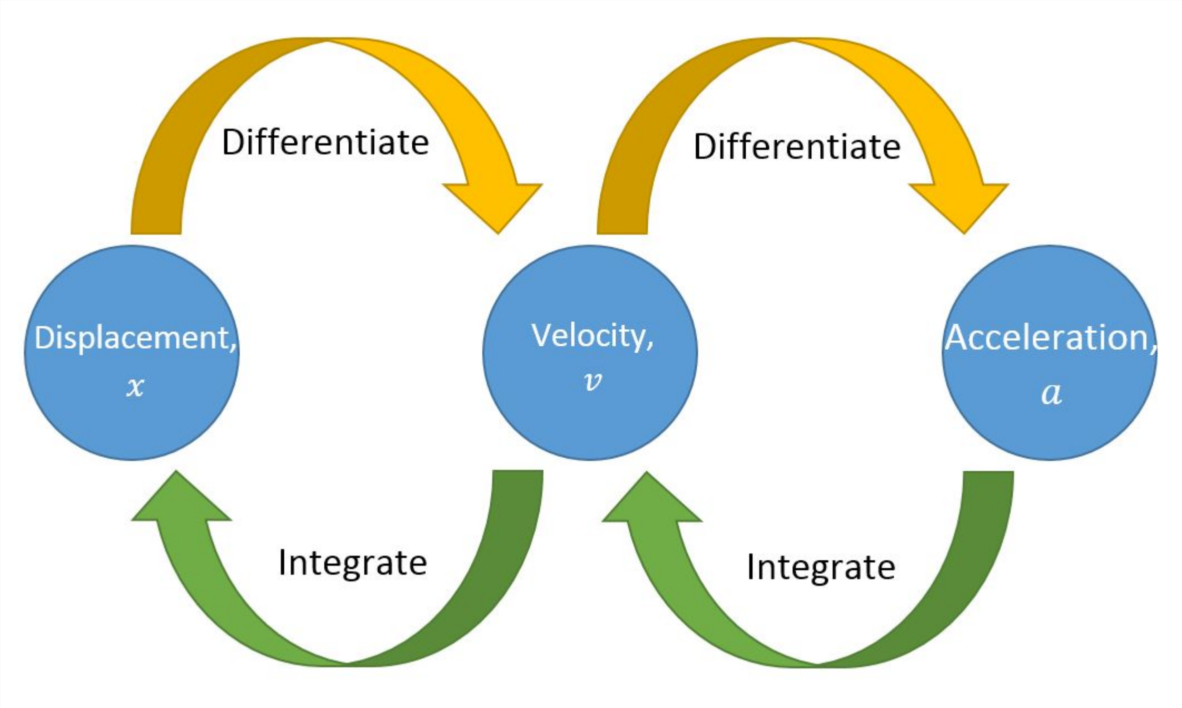
The net change in $F(x)$ is

$$\begin{aligned}
\int_1^3 F'(x)\,dx &= \int_1^3 x^3\,dx \\
&= \left.\frac{x^4}{4}\right|_1^3 \\
&= \frac{3^4}{4} - \frac{1^4}{4} \\
&= \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = \boxed{20}
\end{aligned}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$F(x)\Big|_a^b = F(b) - F(a)$$

Displacement, Velocity and Acceleration Relation



06

u-Substitution

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- Used when you're trying to find the antiderivative of a function that involves a composition.
- *u* should ideally be the inner function of the composition. You would usually find *u'* somewhere in the question as well (which will get cancelled out).
- All x terms must cancel!

$$\int f(g(x))\, g'(x)\, dx$$

Here, take u = g(x)

$$\int \cos(x^2)\, 2x\, dx$$

Here, take u = x^2

Example:

$$\int 3x\sqrt{1-x^2}\,dx$$

Take the constant out: $\int a\cdot f(x)dx = a\cdot\int f(x)dx$

$$= 3\cdot\int x\sqrt{1-x^2}\,dx$$

Apply u – substitution: $\int -u^2 du$

$$= 3\cdot\int -u^2 du$$

Take the constant out: $\int a\cdot f(x)dx = a\cdot\int f(x)dx$

$$= 3\left(-\int u^2 du\right)$$

Apply the Power Rule: $\frac{u^3}{3}$

$$= 3\left(-\frac{u^3}{3}\right)$$

Substitute back $u = \sqrt{1-x^2}$

$$= 3\left(-\frac{\left(\sqrt{1-x^2}\right)^3}{3}\right)$$

Simplify $3\left(-\frac{\left(\sqrt{1-x^2}\right)^3}{3}\right)$: $-\left(1-x^2\right)^{\frac{3}{2}}$

$$= -\left(1-x^2\right)^{\frac{3}{2}}$$

Add a constant to the solution

$$= -\left(1-x^2\right)^{\frac{3}{2}} + C$$

07

Areas Between Curves

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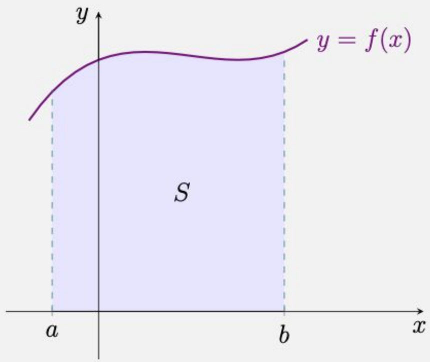
Area Under a Curve

Let f be a continuous function such that $f(x) \geq 0$ for every x on the interval $[a, b]$. Let S be the region under the curve $y = f(x)$ and above the interval $[a, b]$ as illustrated in the following figure.



Then, the area of S is given by the following definite integral:

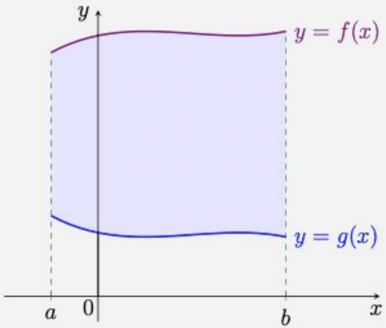
$$\text{Area}(S) = \int_a^b f(x)\,dx$$

The Area Between Two Curves

Let f and g be continuous functions such that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Then, the area between the curves $y = f(x)$ and $y = g(x)$ from a to b is given by

$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx$$

The condition $f(x) \geq g(x)$ on $[a, b]$ means the graph of f is above the graph of g on the interval $[a, b]$.



The area of the shaded region is equal to the area under the curve $y = f(x)$ (given by $\int_a^b f(x)\, dx$) minus the area under the curve $y = g(x)$ (given by $\int_a^b g(x)\, dx$). That is,

$$\text{Area} = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

find the area of the region bounded by the given curves.

$$y = x^2 - 8 \text{ and } y = -x^2 + 10$$

Let $f(x) = x^2 - 8$ and $g(x) = -x^2 + 10$.

Finding the interval. Set $f(x) = g(x)$. Then,

$$\begin{aligned} x^2 - 8 &= -x^2 + 10 \\ 2x^2 - 8 &= 10 \\ 2x^2 &= 18 \\ x^2 &= 9 \\ x &= \pm\sqrt{9} \\ x &= \pm 3 \end{aligned}$$

so the x-coordinates of the intersection between f and g are -3 and 3. The interval is thus $[-3, 3]$.

We now need to *determine whether f is above g*. Let's choose an arbitrary number in the interval $[-3, 3]$, for example 0. Substituting this into the equations for f and g, we get

$$f(0) = 0^2 - 8 = -8 \qquad \text{and} \qquad g(0) = -(0)^2 + 10 = 10$$

Since $g(0) > f(0)$, the graph of $y = g(x)$ is above the graph of $y = f(x)$.

So, the *area* between the curves $y = x^2 - 8$ and $y = -x^2 + 10$ is

$$\begin{aligned}
\text{Area} &= \int_{-3}^{3} [g(x) - f(x)]\, dx \\
&= \int_{-3}^{3} [(-x^2 + 10) - (x^2 - 8)]\, dx \\
&= \int_{-3}^{3} (-2x^2 + 18)\, dx \\
&= \left(-2\frac{x^3}{3} + 18x\right)\Bigg|_{-3}^{3} \\
&= \left(-\frac{2}{3}(3)^3 + 18(3)\right) - \left(-\frac{2}{3}(-3)^3 + 18(-3)\right) \\
&= \left(-\frac{54}{3} + 54\right) - \left(\frac{54}{3} - 54\right) \\
&= (-18 + 54) - (18 - 54) \\
&= (36) - (-36) \\
&= 36 + 36 = 72
\end{aligned}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad \int k\, dx = kx + C, \quad \int_a^b f(x)\, dx = F(x)\Big|_a^b$$

$$F(x)\Big|_a^b = F(b) - F(a)$$

08

Rotating Volumes

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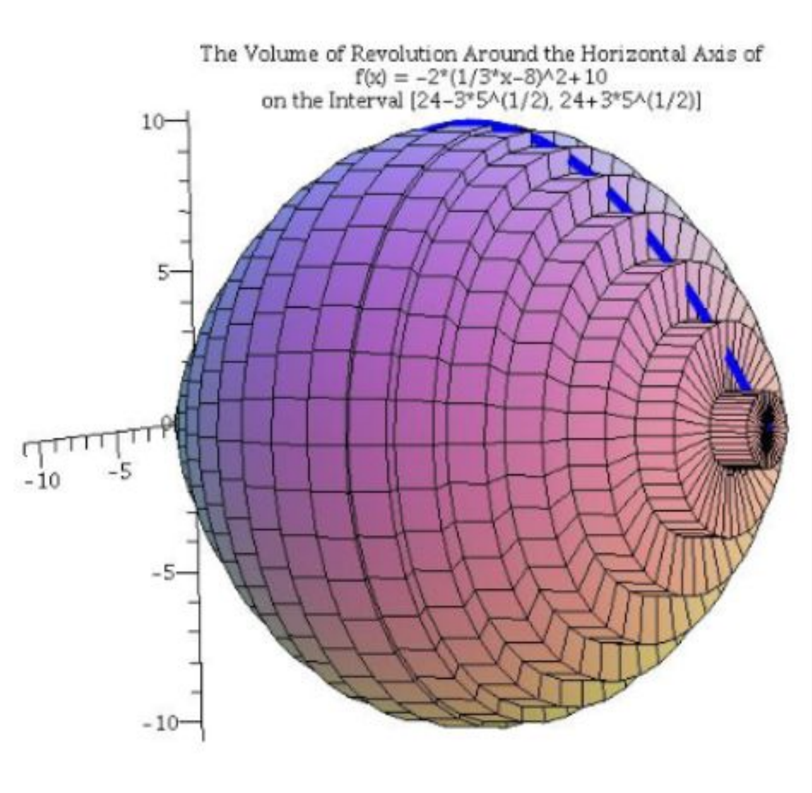
Volumes of curves rotating around an axis can be found. If the area of a circle is πr^2, and we consider f(x) to be the radius, then the overall volume of the rotating solid is:

$$\sum_{i=1}^{n} \pi f^2(x_i^*)\Delta x.$$

On Integration, we get:

$$\pi \int_a^b f^2(x)\ dx.$$



Example

Calculate the volume of the solid obtained by rotating the area bounded by $f(x) = x^2$ and the x-axis over the interval $[0, 2]$ around the x-axis.

$$\pi \cdot \int_0^2 x^4 dx$$

$$\int_0^2 x^4 dx = \frac{32}{5}$$

$$= \pi \frac{32}{5}$$

$$\pi \frac{32}{5} = \frac{32\pi}{5}$$

$$= \frac{32\pi}{5}$$

09

Integration by Parts

Integration by Parts

Note. Suppose we want to take the integral $\int f(x)g(x)\,dx$ using integration parts. Then, there are two possibilities:

- Either we choose $u = f(x)$ and $dv = g(x)dx$, or
- we choose $u = g(x)$ and $dv = f(x)dx$.

$$\int u\,dv = uv - \int v\,du$$

Choose u in this order: **LIATE**

Logs
Inverse
Algebraic
Trig
Exponential

Evaluate the following integral

$$\int 2xe^x \, dx$$

Choose $u = 2x$ and $dv = e^x dx$

Then, $du = 2dx$ and $v = \int dv = \int e^x \, dx = e^x$

$$\int 2xe^x \, dx = \int u \, dv \qquad \text{Since } u = 2x \text{ and } dv = e^x dx$$

$$= uv - \int v \, du \qquad \int u \, dv = uv - \int v \, du$$

$$= (2x)e^x - \int e^x (2dx) \qquad \text{Substitute } u \text{ and } v$$

$$= 2xe^x - 2\int e^x \, dx \qquad \int k f(x) \, dx = k \int f(x) \, dx$$

$$= 2xe^x - 2e^x + C \qquad \int e^x \, dx = e^x + C$$

$$= (2x - 2)e^x + C$$

5

Minute Break!

10

Trigonometric Integrals

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Properties

$\int \cos(x)\ dx = \sin(x) + C$	$\int \sin(x)\ dx = -\cos(x) + C$
$\int \sec^2(x)\ dx = \tan(x) + C$	$\int \sec(x)\tan(x)\ dx = \sec(x) + C$
$\int \csc^2(x)\ dx = -\cot(x) + C$	$\int \csc(x)\cot(x)\ dx = -\csc(x) + C$
$\int \frac{1}{1+x^2}\ dx = \arctan(x) + C$	$\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin(x) + C$

$\int \sin^n(x)\cos^m(x)\ dx$, one of m or n is odd

- Suppose the power on $\sin(x)$ is odd.
- Split off a single factor of $\sin(x)$. You'll be left with an integral of the form $\int \sin(x)\sin^{2k}(x)\cos^m(x)\ dx$ where $2k$ is some even number.
- Rewrite $\sin^{2k}(x)$ using the identity $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^{2k}(x) = \left(\sin^2(x)\right)^k = \left(1 - \cos^2(x)\right)^k$$

- Substitute into the integral:

$$\int \sin(x)\sin^{2k}(x)\cos^m(x)\ dx = \int \sin(x)(1-\cos^2(x))^k\cos^m(x)\ dx$$

- You can not compute the integral by doing a substitution of $u = \cos(x)$ and integrating the resulting polynomial.
- The strategy for $\cos(x)$ having an odd power is essentially the same, except you use the identity $\cos^2(x) = 1 - \sin^2(x)$ and you make the substitution $u = \sin(x)$.

$\int \sin^n(x)\cos^m(x)\ dx$, both n and m is even

- If both of the exponents on $\sin(x)$ and $\cos(x)$ are even, then you proceed using the following identities:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

- These ones can be tricky because, depending on the integral, you may need to use these identities multiple times.

$\int \tan^m(x)\sec^{2k}(x)\ dx$ with $k \geq 1$

- Start by splitting off a factor of $\sec^2(x)$:

$$\int \tan^m(x)\sec^{2k}(x)\ dx = \int \sec^2(x)\tan^m(x)\sec^{2k-2}(x)\ dx$$

- Use the identity $\sec^2(x) = \tan^2(x) + 1$ to simplify,

$$\sec^{2k-2}(x) = \sec^{2(k-1)}(x) = \left(\sec^2(x)\right)^{k-1} = \left(\tan^2(x) + 1\right)^{k-1}$$

- Substitute into the integral,

$$\int \sec^2(x)\tan^m(x)\sec^{2k-2}(x)\ dx = \int \sec^2(x)\tan^m(x)\left(\tan^2(x) + 1\right)^{k-1}\ dx$$

- Now substitute $u = \tan(x)$ and integrate the resulting polynomial.

$\int \tan^m(x)\sec^n(x)\ dx$ with m odd, $n, m \geq 1$

- Start by splitting off a factor of $\sec(x)\tan(x)$:

$$\int \tan^m(x)\sec^n(x)\ dx = \int \sec(x)\tan(x)\tan^{2k}(x)\sec^{n-1}(x)\ dx$$

  where $2k = m - 1$.

- Use the identity $\tan^2(x) = \sec^2(x) - 1$ to simplify,

$$\tan^{2k}(x) = \left(\tan^2(x)\right)^k = \left(\sec^2(x) - 1\right)^k$$

- Substitute into the integral,

$$\int \sec(x)\tan(x)\tan^{2k}(x)\sec^{n-1}(x)\ dx = \int \sec(x)\tan(x)\left(\sec^2(x)-1\right)^k(x)\sec^{n-1}(x)\ dx$$

- Now substitute $u = \sec(x)$ and integrate the resulting polynomial.

Example:

$$\int \sin^7(x)\cos^{184}(x)\ dx$$

- Separate one of the odd sin(x) terms to get $\sin(x).\sin^6(x).\cos^{184}(x)$
- Rewrite $\sin^6(x)$ as $(\sin^2(x))^3$ to get $\sin(x).(\sin^2(x))^3.\cos^{184}(x)$
- Rewrite $\sin^2(x)$ as $1 - \cos^2(x)$ to get $\sin(x).(1 - \cos^2(x))^3.\cos^{184}(x)$
- Take $u = \cos(x)$ to get $(1 - u^2)^3.u^{184}$
- Expand to get integral of $-u^{184} + 3u^{186} - 3u^{188} + u^{190}du$
- Replace $u = \cos(x)$ at the end to get:

$$-\frac{\cos^{185}(x)}{185} + \frac{3\cos^{187}(x)}{187} - \frac{\cos^{189}(x)}{63} + \frac{\cos^{191}(x)}{191} + C$$

11

Trigonometric Substitution

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- $\sqrt{a^2 - x^2}$; substitute $x = a\sin(\theta)$
- $\sqrt{a^2 + x^2}$; substitute $x = a\tan(\theta)$
- $\sqrt{x^2 - a^2}$; substitute $x = a\sec(\theta)$

Example: $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

- Let x = sin(u) to get $\int \sin^2(u).du$
- Rewrite with trig identities to get ½∫(1 - cos(2u)).du
- Expand to get ½u - ¼sin(2u) + C
- Put u = $\sin^{-1}(x)$ to get

$$= \frac{1}{2}\left(\arcsin(x) - \frac{1}{2}\sin\left(2\arcsin(x)\right)\right) + C$$

12

Improper Integrals

Sanjith

Improper Integrals of the Form $\int_a^{\infty} f(x)\,dx$

Let a be a real number, and let f be a function. The improper integral $\displaystyle\int_a^{\infty} f(x)dx$ is defined to be the limit of $\displaystyle\int_a^{t} f(x)dx$ as t approaches ∞.

That is,

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^{t} f(x)dx$$

If the limit exists (as a finite number), the improper integral $\displaystyle\int_a^{\infty} f(x)dx$ is said to be **convergent**. Otherwise, it is said to be **divergent**.

Improper Integrals of the Form $\int_{-\infty}^{b} f(x)\,dx$

Let b be a real number, and let f be a function. The improper integral $\int_{-\infty}^{b} f(x)dx$ is defined to be the limit of $\int_t^{b} f(x)dx$ as t approaches $-\infty$.

That is,

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^{b} f(x)dx$$

If the limit exists (that is, it is a finite number), the improper integral $\int_{-\infty}^{b} f(x)dx$ is said to be **convergent**. Otherwise, it is said to be **divergent**.

Improper Integrals of the Form $\int_{-\infty}^{\infty} f(x)\,dx$

Let f be a function. The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined to be the sum of improper integrals $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{\infty} f(x)dx$, where a is any real number. That is,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx.$$

We say that $\int_{-\infty}^{\infty} f(x)dx$ **converges** if both $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{\infty} f(x)dx$ are convergent, and **diverges** otherwise.

Improper Integrals of the form $\int_a^b f(x)\,dx$ with an Infinite Discontinuity at b

Let f be a function. Suppose that f is continuous on $[a, b)$ and has an *infinite discontinuity at b* (this means that the line $x = b$ is a vertical asymptote of f). Then the improper integral $\int_a^b f(x)dx$ is defined to be the limit of $\int_a^t f(x)dx$ as t approaches b from the left. That is,

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

This is *convergent* if the limit is finite, and *divergent* otherwise.

Improper Integrals of the form $\int_a^b f(x)\,dx$ with an Infinite Discontinuity at a

Let f be a function. Suppose that f is continuous on $(a, b]$ and has an *infinite discontinuity at a* (that is, f has a vertical asymptote at $x = a$). Then, the improper integral $\int_a^b f(x)dx$ is defined to be the limit of $\int_t^b f(x)dx$ as t approaches a from the right. That is,

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

This is *convergent* if the limit is finite, and divergent otherwise.

Improper Integral of the form $\int_a^b f(x)\,dx$ with an Infinite Discontinuity in (a,b)

Let f be a function. Suppose that f has an infinite discontinuity at some point c between a and b ($a < c < b$). Then, the improper integral $\int_a^b f(x)dx$ is defined to be the sum of the improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$. That is,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{3.6.13}$$

The integral $\int_a^b f(x)dx$ is *convergent* if both integrals on the righthand side of (3.6.13) are convergent, and divergent otherwise. In other words, if one of $\int_a^c f(x)dx$ or $\int_c^b f(x)$ diverges, then the integral $\int_a^b f(x)dx$ is divergent.

Determine whether improper integral is convergent. Find the value if converges.

$$\int_{-\infty}^{\infty} x^3 e^{-x^4}\, dx$$

It is convenient to choose $a = 0$. We can then write the integral as

$$\int_{-\infty}^{\infty} x^3 e^{-x^4}\, dx = \int_{-\infty}^{0} x^3 e^{-x^4}\, dx + \int_{0}^{\infty} x^3 e^{-x^4}\, dx$$

Determining whether $\int_{-\infty}^{0} x^3 e^{-x^4}\, dx$ converges or diverges, we find

$$\int_{-\infty}^{0} x^3 e^{-x^4}\, dx$$

$$= \lim_{t\to-\infty} \int_{t}^{0} x^3 e^{-x^4} dx$$

$$= \lim_{t\to-\infty} \int_{-t^4}^{0} e^u \left(\frac{du}{-4}\right)$$

Use substitution: Let $u = -x^4$. Then $du = -4x^3 dx$, so that $\frac{du}{-4} = x^3 dx$. And change the bounds

$$= \lim_{t\to-\infty} \left(-\frac{1}{4}\int_{-t^4}^{0} e^u\, du\right)$$

$$\int kf(x)\, dx = k\int f(x)\, dx$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{4}e^u\Big|_{-t^4}^{0}\right)$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{4}e^0 + \frac{1}{4}e^{-t^4}\right)$$

$$F(x)\Big|_a^b = F(b) - F(a)$$

$$= -\frac{1}{4} + \frac{1}{4}(0)$$

Since $\lim\limits_{t\to-\infty} (-t^4) = -\infty$, it follows that $\lim\limits_{t\to-\infty} e^{-t^4} = 0$

$$= -\frac{1}{4} + 0 = -\frac{1}{4}$$

Since the limit is a finite number, it follows that the improper integral $\int_{-\infty}^{0} x^3 e^{-x^4}\, dx$ converges.

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Differential Equations

Rajveer

Separable Differential Equations:

Format: y' = f(x).g(y)

- Differential equations involve x, y(x), and derivatives of y.
- To solve, isolate the x and y terms and integrate both sides.
- Isolate y(x) to one side. *You may be given an initial value to solve for C.*

Example: $y' = \dfrac{\cos(x)}{\sin(y)}$

=> sin(y).dy = cos(x).dx

Integrating both sides, we get:

=> -cos(y) = sin(x) + C

=> sin(x) = -cos(y) + C

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Sequences, Geometric Series

Rajveer

Sequences: Convergence

Definition. A sequence is a list of numbers enumerated using the natural numbers: $a_1, a_2, a_3, \ldots$ where a_1 is the first number in the sequence, a_2 is the second, and so on. We denote a sequence by $\{a_n\}_{n=1}^{\infty}$ and refer to a_n as the "general term" of the sequence.

If a sequence converges to a real number L as n goes to infinity, L is the limit of the sequence. If it doesn't have a limit, we say the sequence diverves.

If $\{a_n\}_{n=1}^{\infty}$ converges to L, $\lim_{n\to\infty} a_n = L$

Sequences: Divergence

- If a sequence keeps growing positively larger as n goes to infinity:

$$\lim_{n \to \infty} a_n = \infty$$

- If a sequence keeps growing negatively larger:

$$\lim_{n \to \infty} a_n = -\infty$$

- If a sequence does not become either positive or negative, we simply say it diverges.

Sequences: Theorem

Theorem 7.1.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and suppose that $f(x)$ is a function such that $f(n) = a_n$ for all integers $n \geq 1$. Then,

1. If $\lim\limits_{x\to\infty} f(x) = L$ where $L \in \mathbb{R}$, then $\lim\limits_{n\to\infty} a_n = L$
2. If $\lim\limits_{x\to\infty} f(x) = \pm\infty$, then $\lim\limits_{n\to\infty} a_n = \pm\infty$

Example: Determine divergence for $\{ne^{-n}\}_{n=1}^{\infty}$

$\lim\limits_{n\to\infty} ne^{-n}$

Use L'Hopital's rule to find limit value = $1/\infty = 0$. Since this converges, the sequence also converges.

Geometric Series

Let $a, r \in \mathbb{R}$ A **geometric series** is a series of the form,

$$\sum_{n=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \ldots +$$

$$r = \frac{ar^n}{ar^{n-1}}$$

Theorem 7.2.1. $\sum_{n=0}^{\infty} ar^n$ converges for all values of $r \in (-1, 1) \iff |r| < 1$ and diverges otherwise. In the case that it converges,

$$\sum_{n=0}^{\infty} ar^n = \frac{1}{1-r}, \quad |r| < 1$$

If the geometric series starts at $n = 1$, then the formula is slightly different:

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r} - a = \frac{ar}{1-r}, \quad |r| < 1$$

Example

Determine if the series is convergent or divergent: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

r can be found to be 0.5, which is < 1. Hence, we can say that this series will converge.

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Limit, Integral, p-Series, Comparison Test

Rajveer

Limit Test for Divergence

Theorem 7.2.2: Limit Test for Divergence. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $\lim\limits_{n\to\infty} a_n \neq 0$. Then, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Example:

Determine whether or not the series $\sum\limits_{n=1}^{\infty} \frac{8 \cdot 4^n - 7n(n+1)}{4^n n(n+1)}$ is convergent or divergent.

We need to find $\lim\limits_{x\to\infty} \frac{8 \cdot 4^n - 7n(n+1)}{4^n n(n+1)}$

- Numerator can be broken down into two
- Both parts converge to 0, so total limit converges to 0
- This does not tell us anything about the divergence of the series.

Integral Test

Integral Test. Let $f(x)$ be a continuous, positive, decreasing function on $[1, \infty)$. Suppose that $f(n) = a_n$ for all integers $n \geq 1$. Then,

1. If $\displaystyle\int_1^\infty f(x)\,dx$ converges, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges.
2. If $\displaystyle\int_1^\infty f(x)\,dx$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ also diverges.

Example: Determine if the series is convergent or divergent: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Taking integral of $1/(n^2 + 1)$ we get $\tan^{-1}(n) + C$. As n approaches infinity, this value converges to $\pi/2$. Hence, the series also converges.

p-Series Test

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Series Comparison Test

The Series Comparison Test. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series and suppose that $a_n, b_n \geq 0$ and that $a_n \leq b_n$ for all $n \geq 1$. Then,

1. If $\sum_{n=1}^{\infty} a_n$ diverges, then so too does $\sum_{n=1}^{\infty} b_n$: "If the smaller sum diverges, the bigger one does as well."
2. If $\sum_{n=1}^{\infty} b_n$ converges, then so too does $\sum_{n=1}^{\infty} a_n$: "If the bigger sum converges, the smaller one does as well."

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Ratio, Root, Alternating Series Test

Sanjith

The Ratio Test

The following theorem is very useful to determine whether a series is absolutely convergent. It is also useful in testing series invloving factorials and/or a constant raised to the power of n.

Theorem 5.6.1. [The Ratio Test] *Let* $\sum a_n$ *be a series.*

(a) *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, *then the series* $\sum a_n$ *is absolutely convergent (and therefore convergent by Theorem 5.5.2)*

(b) *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, *the test is inconclusive.*

(c) *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *or* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, *the series* $\sum a_n$ *is divergent.*

Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^2}{8^n}$$

Let $a_n = \frac{n^2}{8^n}$. For every n, $a_n > 0$. So $|a_n| = a_n$ for all n and we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{8^{n+1}}}{\frac{n^2}{8^n}}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^2}{8^{n+1}} \right) \left(\frac{8^n}{n^2} \right) \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^2}{8^n(8)} \right) \left(\frac{8^n}{n^2} \right) \qquad a^{n+1} = a^n(a)$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \left(\frac{1}{8} \right) \qquad \text{Simplify}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \left(\frac{1}{8} \right) \qquad \frac{a^k}{b^k} = \left(\frac{a}{b}\right)^k$$

$$= \left(\lim_{n\to\infty} \frac{n+1}{n} \right)^2 \left(\frac{1}{8} \right) \qquad \lim_{n\to\infty} [b_n]^k = \left[\lim_{n\to\infty} b_n\right]^k$$

$$= (1)^2 \frac{1}{8} = \frac{1}{8} \qquad \lim_{n\to\infty} \left(\frac{n+1}{n} \right) = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) = 1 + 0 = 1$$

Since this limit is less than 1, it follows that the given series is absolutely convergent (by the Ratio Test), and therefore convergent .

The Root Test

The following test may be useful for series of the form $\sum (b_n)^{g(n)}$.

Theorem 5.6.2. [The Root Test] *Let* $\sum a_n$ *be a series.*

(a) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, *then the series* $\sum a_n$ *is absolutely convergent (and hence convergent).*

(b) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, *then the test is inconclusive.*

(c) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, *then the series* $\sum a_n$ *is divergent.*

Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty}\left(\frac{4n+3}{5n+1}\right)^n$$

Let $a_n = \left(\frac{4n+3}{5n+1}\right)^n$. Since a_n is of the form $(b_n)^{g(n)}$ we can try the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{4n+3}{5n+1}\right)^n\right|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{4n+3}{5n+1}\right)^n} = \lim_{n\to\infty}\frac{4n+3}{5n+1} = \frac{4}{5}$$

Since $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, it follows that the given series is convergent by the Root Test.

Alternating Series Test

The following theorem is stated for alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$. Note that the statement also holds for alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n}b_n$.

Let $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *be an alternating series. Suppose that the following two conditions are satisfied.*

(1) $b_{n+1} \leq b_n$ *for all* $n \geq 1$.

(2) $\lim_{n\to\infty} b_n = 0$

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is convergent.*

Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Since it is an alternating series, with $b_n = \frac{1}{n}$, we will use the Alternating Series Test.

- Checking condition (1). We have $b_n = \frac{1}{n}$ and $b_{n+1} = \frac{1}{n+1}$. Since $n+1 \geq n$, it follows that $\frac{1}{n+1} \leq \frac{1}{n}$ for all n. So $b_{n+1} \leq b_n$ for all n.
- Checking condition (2). Clearly, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Since (1) and (2) are satisfied, by the Alternating Series Test, it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent .

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Absolute Convergence

Sanjith

Absolutely Convergent Series

A series $\sum a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
So, to determine whether a series $\sum a_n$ is absolutely convergent,

- we first need to consider the series $\sum |a_n|$ whose nth term is the absolutely value of a_n, then
- determine whether it is convergent.

Useful fact: for every integer m, the absolute value of $(-1)^m$ equals 1; that is, $|(-1)^m| = 1$.

Determine whether the series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$$

- First, the series of absolute values is:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{\frac{3}{2}}} \right| = \sum_{n=1}^{\infty} \frac{|(-1)^n|}{\left| n^{\frac{3}{2}} \right|} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

- This latter series is a p-series with $p = \frac{3}{2}$. Since $p > 1$, it converges.

Conclusion: because the series of absolute values is convergent, we can conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$ is absolutely convergent .

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Power Series

Rajveer

Basically, an infinite series with an x^n term in it.

$$\sum_{n=0}^{\infty} a_n x^n$$

→ Centered on x

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

→ Centered on c

Convergence

Theorem 7.1.1. There are only three possibilities for the nature of convergence of a power series $\mathcal{P}(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.

1. $\mathcal{P}(x)$ converges only at $x = x_0$; in this case, we set $R = 0$.
2. $\mathcal{P}(x)$ converges for all real values $x \in \mathbb{R}$; in this case we set $R = +\infty$.
3. There exists some $R > 0$ such that $\mathcal{P}(x)$ converges for all x that satisfy $|x - x_0| < R$ and diverges for all x such that $|x - x_0| > R$.

Representation

- Sometimes it is possible to calculate a **power series representation** of a function $f(x)$ on an interval on which the power series converges.
- That is, given some $f(x)$, we are interested in finding a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ with radius of convergence I such that, $f(a) = \sum_{n=0}^{\infty} a_n(a - x_0)^n$ for all $a \in I$.
- For example,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad \underbrace{|x| < 1}_{\text{Interval of Convergence}}$$

Remember: sum of geometric series $ar^n = a/(1-r)$. In the above example, a = 1, r = x.

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Taylor, MacLaurin Series

Sanjith

Taylor Series

Let $y = f(x)$ be a function, and let a be a number.

- Let $n \geq 0$ be an integer. We write $f^{(n)}(a)$ for the nth derivative of $f(x)$ evaluated at a.
  - If $n = 0$, $f^{(0)}(a) = f(a)$. This is just $f(x)$ evaluated at a.
  - If $n = 1$, $f^{(1)}(a) = f'(a)$. This is the first derivative of $f(x)$ at a.
  - If $n = 2$, $f^{(2)}(a) = f''(a)$. This is the second derivative of $f(x)$ evaluated at a.
  - Etc.
- Let $n \geq 1$ be an integer. The factorial of n, denoted $n!$, is defined by

  $$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

  By convention, $0! = 1$. For example, $4! = 4(3)(2)(1) = 24$.
- If $f(x)$ has the derivative of all orders at a, the **Taylor series** of f at a (or about a or centered at a) is the series

  $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

  If the Taylor series converges to $f(x)$ for every x in an open interval $(a-R, a+R)$, we write

  $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{for} \qquad |x-a| < R$$

Maclaurin Series

The **Maclaurin series** of a function f is the Taylor series of f centered at $a = 0$; that is, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Use the definition of the Taylor series to find the first four nonzero terms of the series of $f(x)$ at a.

$$f(x) = e^{-4x}, \qquad a = 0$$

To find the first four nonzero terms of the Taylor series of $f(x) = e^{-4x}$ at $a = 0$, we first need to find the successive derivatives of $f(x)$. Recall the chain rule for the natural exponential function.

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}\frac{d}{dx}[g(x)]$$

$$f(x) = e^{-4x} \qquad f(0) = 1$$

$$f'(x) = -4e^{-4x} \qquad f'(0) = -4$$

$$f''(x) = 16e^{-4x} \qquad f''(0) = 16$$

$$f'''(x) = -64e^{-4x} \qquad f'''(0) = -64$$

So, the first four nonzero terms of the Taylor series of $f(x)$ at $a = 0$ are

$$\begin{aligned}
&f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\
&= 1 - 4(x-0) + \frac{16}{2!}(x-0)^2 + \frac{-64}{3!}(x-0)^3 \\
&= 1 - 4x + \frac{16}{2(1)}(x)^2 + \frac{-64}{3(2)(1)}(x)^3 \\
&= \boxed{1 - 4x + 8x^2 - \frac{32}{3}x^3}
\end{aligned}$$

Thank You!

All the best for finals!

For Questions:

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Note: this is not guaranteed to be an exhaustive list of topics or concepts. Use the textbook!
Some snippets in this slides are taken from Paul Tsopmene Workbook