# MATH 100: Differential Calculus

# Supplemental Learning

# MATH 100 2021 Midterm 2



3/[3] 1. Find the equation of the tangent line to the curve $y = \frac{x}{x^2-3}$ at the point $(2,2)$

The equation for the tangent line of a curve at $(x_1, y_1)$ is $(y - y_1) = m_t(x - x_1)$, where $m_t$ = slope of the tangent.

Also, at this point, $m_t = y'(x_1)$

$\therefore (y - y_1) = [y'(x_1)](x - x_1)$

Putting in values, $(y-2) = [y'(2)](x-2)$ — ①

Now, finding $y'$:

$$y = \frac{x}{x^2-3}$$

$$\therefore y' = \frac{(x^2-3)\times\frac{d}{dx}(x) - x \times \frac{d}{dx}(x^2-3)}{(x^2-3)^2} \quad [\because \text{quotient rule}]$$

$$= \frac{(x^2-3)(1) - (x)(2x)}{(x^2-3)^2}$$

$$= \frac{x^2-3-2x^2}{(x^2-3)^2}$$

$$= \frac{-x^2-3}{(x^2-3)^2}$$

$$\therefore \text{At } x_1 = 2,\ y' = \frac{-(2)^2-3}{[(2)^2-3]^2} = \frac{-4-3}{(4-3)^2} = \frac{-7}{(1)^2} = -7$$

Putting this value in ①, we get:

$$y - 2 = (-7)(x-2)$$
$$= -7x + 14$$

$$\boxed{y = -7x + 16}$$

The equation of the tangent line to the curve $y = \frac{x}{x^2-3}$ at the point $(2,2)$ is $y = -7x + 16$.



Up on my GitHub repo!

XX %

# Quiz 5

Friday, November 24th *(MATH 100-001)*

# Topics on the Quiz

L'Hôpital's Rule

Related Rates

Linear Approximation

Differentials

$\boxed{\text{Extreme Values/ Critical Values}}$

Bonus!
(at least in Broughton's class)

# Given Algebraic Identities on the Quiz

**Algebra Identities**

- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$
- $a^{x+y} = a^x a^y$
- $a^{xy} = (a^x)^y = (a^y)^x$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
- $\log_a(x^n) = n\log_a(x)$
- $ax^2 + bx + c = 0 \implies x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

# L’Hôpital’s Rule

# L'Hôpital's Rule:
# Topic - Indeterminate Forms with Powers

Such as: $0^0$, $\text{inf}^{\text{inf}}$, $1^{\text{inf}}$
Note that $0^{\text{inf}} = 0$, and so it is not an indeterminate form.

This happens when: $\lim_{x\to a} f(x)^{g(x)}$

- $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
- $\lim_{x\to a} f(x) = \text{inf}$ and $\lim_{x\to a} g(x) = 0$
- $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\text{inf}$

To fix this, we take the ln() of the entire function and raise *e* to that power.
i.e., $\lim_{x\to a} f(x)^{g(x)} = e^{\lim x \to a \ln(f(x)^\wedge g(x))} = e^{\lim x \to a\, g(x)\ln(f(x))}$
And we can use L'Hôpital's Rule to calculate $\lim_{x \to a} g(x)\ln(f(x))$.

Dominic Huang

# L'Hôpital's Rule: Topic - Indeterminate Forms with Powers

Find $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$

Trying direct substitution gives an indeterminate form $0^0$. To apply L'Hôpital's Rule, we first rewrite the function in exponential form:

$$x^{\sqrt{x}} = e^{\sqrt{x}\ln(x)}$$

Now, we focus on finding $\lim_{x\to 0^+} \sqrt{x}\ln(x)$. As $x \to 0^+$, $\sqrt{x} \to 0$ and $\ln(x) \to -\infty$, making it another indeterminate form $(0 \times -\infty)$. To deal with this, rewrite it as:

$$\sqrt{x}\ln(x) = \frac{\ln(x)}{1/\sqrt{x}}$$

Applying L'Hôpital's Rule to this expression, we find:

$$\lim_{x\to 0^+} \sqrt{x}\ln(x) = 0$$

Therefore, $\lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = 1.$ $\square$

"hence proved"

Dominic Huang

# L'Hôpital's Rule:
# Topic - Indeterminate Forms with Powers

Find $\lim_{x\to 0^+}(1-\sin(2x))^{1/x}$.

Attempting direct substitution results in the indeterminate form $1^\infty$. We rewrite the function as an exponential:

$$(1-\sin(2x))^{1/x} = e^{(1/x)\ln(1-\sin(2x))}$$

Using L'Hôpital's Rule, we find $\lim_{x\to 0^+}\frac{\ln(1-\sin(2x))}{x} = -2$. Therefore, the limit is $e^{-2}$. $\square$

Dominic Huang

# L'Hôpital's Rule:
# Topic - Indeterminate Forms with Powers

Find $\lim_{x\to-\infty}(x^2+1)e^x$.

This function approaches the indeterminate form $\infty^0$. We rewrite the function as:

$$(x^2+1)e^x = e^{e^x \ln(x^2+1)}$$

Applying L'Hôpital's Rule, we find $\lim_{x\to-\infty} \frac{e^x}{\ln(x^2+1)} = 0$. Thus, $\lim_{x\to-\infty}(x^2+1)e^x = e^0 = 1$. $\square$

Dominic Huang

# L'Hôpital's Rule:
# Topic - Indeterminate Forms with Powers

Showing that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ is a real number.

Consider rewriting $\left(1+\frac{1}{x}\right)^x$ in exponential form as follows:

$$\left(1+\frac{1}{x}\right)^x = e^{x\ln\left(1+\frac{1}{x}\right)}$$

To find the limit, we focus on the exponent $x\ln\left(1+\frac{1}{x}\right)$ as $x\to\infty$.

Now, $x\to\infty$ and $\ln\left(1+\frac{1}{x}\right)\to 0$, resulting in an indeterminate form $\infty\times 0$. Rewrite it as:

$$x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

Applying L'Hôpital's Rule, we get:

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$$

Thus, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e.$ $\square$

Paul Lee

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

$$\lim_{\theta\to 0} \frac{e^{\sin(\theta)} - 1}{2\theta}$$

$$\lim_{x\to 0} \frac{\arcsin(x)}{x}$$

$$\lim_{t\to 1^+} \ln(t)\tan(\pi t/2)$$

$$\lim_{\theta\to 0} \frac{e^{\sin(\theta)} - 1}{2\theta} \overset{L}{=} \lim_{\theta\to 0} \frac{e^{\sin(\theta)}\cos(\theta)}{2} = \boxed{\frac{1}{2}}$$

$$\lim_{x\to 0} \frac{\arcsin(x)}{x} \overset{L}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = \frac{1}{1} = \boxed{1}$$

$$\begin{aligned}
&= \lim_{t\to 1^+} \frac{\ln(t)\sin(\pi t/2)}{\cos(\pi t/2)} \\
&\overset{L}{=} \lim_{t\to 1^+} \frac{(1/t)\sin(\pi t/2) + \ln(t)\cos(\pi t/2)\cdot(\pi/2)}{-\sin(\pi t/2)\cdot(\pi/2)} \\
&= \frac{(1/1)\sin(\pi/2) + \ln(1)\cos(\pi/2)\cdot(\pi/2)}{-\sin(\pi/2)\cdot(\pi/2)} \\
&= \frac{1+0}{-1\cdot(\pi/2)} \\
&= \boxed{-\frac{2}{\pi}}
\end{aligned}$$

Paul Lee

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

Same q as before, written differently

$$\lim_{x\to 0^+} x^{\sqrt{x}}$$

$$\lim_{x\to\infty} x\sin(\pi/x)$$

$$\lim_{x\to\infty} (x - \ln(x))$$

Let $y = x^{\sqrt{x}}$ so that $\ln(y) = \ln(x^{\sqrt{x}}) = \sqrt{x}\ln(x)$.

$$\begin{aligned}
\lim_{x\to 0^+} \sqrt{x}\ln(x) &= \lim_{x\to 0^+} \frac{\ln(x)}{1/\sqrt{x}} \\
&\overset{L}{=} \lim_{x\to 0^+} \frac{1/x}{1/(2x^{3/2})} \\
&= \lim_{x\to 0^+} 2x^{1/2} \\
&= \boxed{0}
\end{aligned}$$

So now we have

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln(x^{\sqrt{x}})} = e^{\left(\lim_{x\to 0^+} \ln\left(x^{\sqrt{x}}\right)\right)} = e^0 = \boxed{1}$$

$$\begin{aligned}
\lim_{x\to\infty} x\sin(\pi/x) &= \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x} \\
&\overset{L}{=} \lim_{x\to\infty} \frac{\cos(\pi/x)\cdot(-\cancel{1}\pi/x^2)}{-1/x^2} \\
&= \lim_{x\to\infty} \cos(\pi/x)\cdot(\pi) \\
&= \cos\left(\lim_{x\to\infty} \pi/x\right)\cdot\pi \\
&= \cos(0)\cdot\pi \\
&\cancel{= \boxed{1}} \quad = \boxed{\pi}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} (x - \ln(x)) &= \lim_{x\to\infty} (x - \ln(x)) \cdot \frac{x+\ln(x)}{x+\ln(x)} \\
&= \lim_{x\to\infty} \frac{x^2 - [\ln(x)]^2}{x + \ln(x)} \\
&\overset{L}{=} \lim_{x\to\infty} \frac{2x - 2\ln(x)\cdot(1/x)}{1 + (1/x)} \\
&= \lim_{x\to\infty} \frac{2x^2 - 2\ln(x)}{x+1} \\
&\overset{L}{=} \lim_{x\to\infty} \frac{4x - (2/x)}{1} \\
&= \lim_{x\to\infty} \frac{4x^2 - 2}{x} \\
&\overset{L}{=} \lim_{x\to\infty} \frac{8x}{1} \\
&= \infty
\end{aligned}$$

Paul Lee

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

$$\lim_{x\to 0}(1-2x)^{1/x}$$

$$\lim_{x\to 0}\left(\cot(x)-\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\sqrt{x}e^{-x/2}$$

Let $y=(1-2x)^{1/x}$ so that $\ln(y)=(1/x)\ln(1-2x)=\frac{\ln(1-2x)}{x}$

$$\lim_{x\to 0}\frac{\ln(1-2x)}{x}\overset{L}{=}\lim_{x\to 0}\frac{\frac{-2}{1-2x}}{1}=\boxed{-2}$$

Now,

$$\begin{aligned}\lim_{x\to 0}(1-2x)^{1/x}&=\lim_{x\to 0}e^{\ln\left((1-2x)^{1/x}\right)}\\&=e^{\lim_{x\to 0}(1-2x)^{1/x}}\\&=\boxed{e^{-2}}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\left(\cot(x)-\frac{1}{x}\right)&=\lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)}-\frac{1}{x}\right)\\&=\lim_{x\to 0}\frac{x\cos(x)-\sin(x)}{x\sin(x)}\\&\overset{L}{=}\lim_{x\to 0}\frac{1\cdot\cos(x)-x\sin(x)-\cos(x)}{1\cdot\sin(x)+x\cos(x)}\\&=\lim_{x\to 0}\frac{-x\sin(x)}{\sin(x)+x\cos(x)}\\&\overset{L}{=}\lim_{x\to 0}\frac{-1\cdot\sin(x)+(-x)\cos(x)}{\cos(x)+1\cdot\cos(x)+x(-\sin(x))}\\&=\frac{0+0}{1+1+0}\\&=\boxed{0}\end{aligned}$$

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x}e^{-x/2}&=\lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}}\\&\overset{L}{=}\lim_{x\to\infty}\frac{1/(2\sqrt{x})}{e^{x/2}\cdot(1/2)}\\&=\lim_{x\to\infty}\frac{1}{\sqrt{x}e^{x/2}}\\&=\boxed{0}\end{aligned}$$

Paul Lee

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

$$\lim_{x\to 0^+} (\tan(2x))^x$$

$$\lim_{x\to\infty} x^{1/x}$$

$$\lim_{x\to\infty} (e^x + x)^{1/x}$$

Let $y = (\tan(2x))^x$ so that $\ln(y) = x\ln(\tan(2x))$.

$$\begin{aligned}
\lim_{x\to 0^+} x\ln(\tan(2x)) &= \lim_{x\to 0^+} \frac{\ln(\tan(2x))}{1/x} \\
&\overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{\tan(2x)}\cdot \sec^2(2x)\cdot 2}{-1/x^2} \\
&= \lim_{x\to 0^+} \frac{\frac{\cos(2x)}{\sin(2x)}\cdot \frac{1}{\cos^2(2x)}\cdot 2}{-1/x^2} \\
&= \lim_{x\to 0^+} \frac{-2x^2}{\sin(2x)\cos(2x)} \\
&\overset{L}{=} \lim_{x\to 0^+} \frac{-4x}{\cos(2x)\cdot 2\cdot\cos(2x) + \sin(2x)\cdot(-\sin(2x))\cdot 2} \\
&= \frac{0}{2+0} \\
&= \boxed{0}
\end{aligned}$$

Then $\lim_{x\to 0^+} (\tan(2x))^x = e^0 = \boxed{1}$

Let $y = x^{1/x}$ so that $\ln(y) = (1/x)\ln(x) = \dfrac{\ln(x)}{x}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)}{x} &\overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \\
&= \lim_{x\to\infty} \frac{1}{x} \\
&= \boxed{0}
\end{aligned}$$

Then $\lim_{x\to\infty} x^{1/x} = e^0 = \boxed{1}$

Let $y = (e^x + x)^{1/x}$ so that $\ln(y) = (1/x)\ln(e^x + x) = \dfrac{\ln(e^x+x)}{x}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(e^x+x)}{x} &\overset{L}{=} \lim_{x\to\infty} \frac{\frac{e^x+1}{e^x+x}}{1} \\
&= \lim_{x\to\infty} \frac{e^x+1}{e^x+x} \\
&\overset{L}{=} \lim_{x\to\infty} \frac{e^x}{e^x+1} \\
&\overset{L}{=} \lim_{x\to\infty} \frac{e^x}{e^x} \\
&= \lim_{x\to\infty} 1 \\
&= 1
\end{aligned}$$

Then $\lim_{x\to\infty} (e^x + x)^{1/x} = e^1 = \boxed{e}$

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$$

$e^x - e^{-x} - 2x \to 1 - 1 - 0 = 0$ as $x \to 0$

$x - \sin(x) \to 0 - 0 = 0$ as $x \to 0$

so we can use L'Hôpital.

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos(x)}$$

Again, $e^x + e^{-x} - 2 \to 1 + 1 - 2 = 0$ as $x \to 0$

$1 - \cos(x) \to 1 - 1 = 0$ as $x \to 0$

so use L'Hôpital again

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos(x)} \overset{L}{=} \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(x)}$$

yet again: $e^x - e^{-x} \to 0$ as $x \to 0$

$\sin(x) \to 0$

so use L'Hôpital one final time.

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(x)} \overset{L}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos(x)} = 2$$

Thus, $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)} = 2$

Chad Davis

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

$$\lim_{x\to\infty} \frac{\ln\left(\ln(x)\right)}{x}$$

Here $\ln(x) \to \infty$ as $x \to \infty$

Thus $\dfrac{\ln(\ln(x))}{x} \to \dfrac{\infty}{\infty}$ as $x \to \infty$

So we use L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{\ln(\ln(x))}{x} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln(x)} \cdot \left(\frac{1}{x}\right)}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x\ln(x)}$$

$= 0$ as $x\ln(x) \to \infty$ as $x \to \infty$

Chad Davis

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

$$\lim_{x\to 0^+} \cot(2x)\sin(6x)$$

Note $\cot(2x) \to \infty$ as $x \to 0^+$

Thus $\cot(2x)\sin(6x) \to 0 \cdot \infty$ as $x \to 0^+$

We can use L'Hôpital but we need to rewrite the function first:

$$\cot(2x)\sin(6x) = \frac{\sin(6x)}{1/\cot(2x)} = \frac{\sin(6x)}{\tan(2x)}$$

and here, $\dfrac{\sin(6x)}{\tan(2x)} \to \dfrac{0}{0}$ as $x \to 0^+$

so we can apply L'Hôpital to this representation.

$$\lim_{x\to 0^+} \cot(2x)\sin(6x) = \lim_{x\to 0^+} \frac{\sin(6x)}{\tan(2x)}$$

$$\overset{L}{=} \lim_{x\to 0^+} \frac{6\cos(6x)}{2\sec^2(2x)}$$

$$= \lim_{x\to 0^+} 3\cos(6x)\cos^2(2x) = 3.$$

so $\displaystyle\lim_{x\to 0^+} \cot(2x)\sin(6x) = 3$

# L'Hôpital's Rule

***Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.***

$$\lim_{x\to 0^+} \left(\cos(x)\right)^{1/x^2}$$

First we need to rewrite the function.

$$(\cos(x))^{1/x^2} = e^{\ln(\cos(x)^{1/x^2})}$$

$$= e^{\frac{1}{x^2}\ln(\cos(x))}$$

Now we calculate $\lim_{x\to 0^+} \frac{\ln(\cos(x))}{x^2}$

Here $\frac{\ln(\cos(x))}{x^2} \to \frac{\ln(1)}{0} = \frac{0}{0}$ as $x \to 0^+$

so we can use L'Hôpital's Rule.

$$\Rightarrow \lim_{x\to 0^+} \frac{\ln(\cos(x))}{x^2} \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{\cos(x)}\cdot(-\sin(x))}{2x}$$

$$= \lim_{x\to 0^+} \frac{-\tan(x)}{2x}$$

Now, $\frac{\tan(x)}{2x} \to \frac{0}{0}$ as $x \to 0^+$ so use L'Hôpital again:

$$-\lim_{x\to 0^+} \frac{\tan(x)}{2x} \overset{L}{=} -\lim_{x\to 0^+} \frac{\sec^2(x)}{2} = -\frac{1}{2}$$

Thus $\lim_{x\to 0^+} \frac{\ln(\cos(x))}{x^2} = -\frac{1}{2}$

so: $\lim_{x\to 0^+} (\cos(x))^{1/x^2}$

$$= \lim_{x\to 0^+} e^{\frac{1}{x^2}\ln(\cos(x))}$$

$$= e^{\lim_{x\to 0^+} \frac{1}{x^2}\ln(\cos(x))}$$

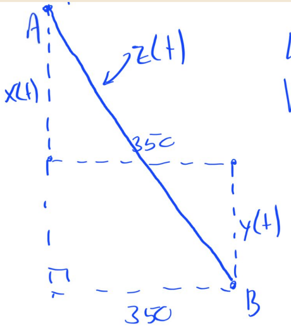
$$= e^{-1/2}$$

# Related Rates

Chad Davis

# Related Rates

Two people on bikes are separated by 350 meters directly east and west from each other. Person A starts riding north at a rate of 5 m/s and 7 minutes later Person B starts riding south at 3 m/s. At what rate is the distance separating the two people changing 25 minutes after Person A starts riding?




Let $x(t)$ be the distance bike A has gone after $t$ minutes.
Let $y(t)$ be the distance bike B has gone after $t$ minutes.
Let $z(t)$ be the distance b/w bike A and bike B

From the above diagram, we see that

$$(x(t)+y(t))^2+350^2=z^2(t)$$

(Pythagoras)

Differentiate:

$$2(x(t)+y(t))(x'(t)+y'(t)) = 2z(t)z'(t)$$

We are given $x'(t)=5$, $y'(t)=3$.
Thus, after 25 minutes, bike A has travelled

$$5\cdot(25)(60) = 7500 \text{ metres.}$$

Since bike B started 7 minutes after bike A, after 25 minutes, bike B has only been moving for $25-7=18$ minutes. Thus, at the 25 minute mark, bike B has gone

$$18(3)(60) = 3240 \text{ metres.}$$

Thus, after 25 minutes, we have:
Moreover, the actual distance b/w the bikes at the 25 minute point is,

$$(7500+3240)^2+350^2 = z^2(t)$$

$$\Rightarrow z(t) = \sqrt{(7500+3240)^2+350^2}$$

Thus,

$$z'(t) = \frac{\cancel{2}(x(t)+y(t))(x'(t)+y'(t))}{\cancel{2}z(t)}$$

so at 25 minutes:

$$z'(t) = \frac{(7500+3240)(5+3)}{\sqrt{(7500+3240)^2+350^2}}$$

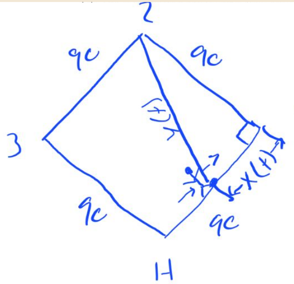
$$\approx 7.9956 \text{ m/s.}$$

Chad Davis

# Related Rates

A standard Major League Baseball diamond is a square with side lengths of 90 feet. On the season opener of the 2021 season, Bo Bichette of the Toronto Blue Jays hits a line drive down the third base line on the first pitch he sees and starts running towards first base. Suppose that he is running to first base at a rate of 29 feet per second.

(a) At what rate is his distance from second base decreasing when he is halfway to first base?




Let $x(t)$ be the distance from Bo to first base

Let $y(t)$ " " second base

Then: $x^2(t) + 90^2 = y^2(t)$

We need to find $y'(t)$, so differentiate

$$\Rightarrow 2x(t)x'(t) = 2y(t)y'(t)$$

$$\Rightarrow y'(t) = \frac{x(t)x'(t)}{y(t)}$$

We are given that $x(t) = 45$ as Bo is halfway to first. Also, $x'(t) = -29$ (neg. b/c the distance is decreasing)

Also, when $x(t)$ is 45, then

$$45^2 + 90^2 = y^2(t)$$

$$\Rightarrow y(t) = \sqrt{45^2 + 90^2}$$

Thus, when Bo is halfway to first,

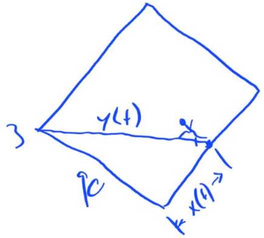
$$y'(t) = \frac{45(-29)}{\sqrt{45^2+90^2}} \approx -12.9692$$

$\Rightarrow$ The distance b/w Bo & second is decreasing at a rate of 12.9692 ft/sec.

Chad Davis

# Related Rates

(b) At what rate is his distance from third base increasing at the same moment?




Let $x(t)$ be the distance from home plate to BC

Let $y(t)$ " " third base to BC

$$\Rightarrow x^2(t) + 90^2 = y^2(t)$$

$$\Rightarrow 2x(t)x'(t) = 2y(t)y'(t)$$

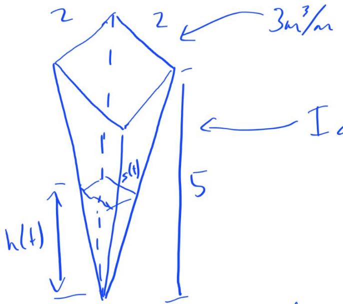
$$\Rightarrow y'(t) = \frac{x(t)x'(t)}{y(t)}$$

By the same argument, $y'(t) \approx 12.97$ ft/sec when BC is half way to first.

Chad Davis

# Related Rates

A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m × 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is (1/3)(height)(area of base).




I am a very good artist...

Let $h(t)$ be the height of water at time $t$.

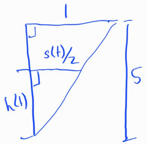
Let $V(t)$ " volume "

Let $s(t)$ be the side length of base of water at time $t$.

Then: $V(t) = \frac{1}{3} h(t) s^2(t)$

We need to relate $s(t)$ to $h(t)$.

Look at a cross-section:




$\Rightarrow \frac{s(t)/2}{h(t)} = \frac{1}{5}$

$\Rightarrow s(t) = \frac{2h(t)}{5}$

$\Rightarrow V(t) = \frac{1}{3} h(t) \left(\frac{2h(t)}{5}\right)^2$

$= \frac{4}{75} h^3(t)$

$\Rightarrow V'(t) = \frac{4}{25} h^2(t) h'(t)$

Given $V'(t) = 3$ and we need to find $h'(t)$ when $h(t) = 4$:

$V'(t) = \frac{4}{25} h^2(t) h'(t)$

$\Rightarrow 3 = \frac{4}{25}(4)^2 h'(t)$

$\Rightarrow h'(t) = \frac{75}{64}$

$\Rightarrow$ height of water rising at $\frac{75}{64}$ m/min when water is 4 m deep.

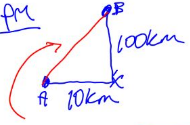
Paul Lee

# Related Rates

At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

Noon


4PM




$(35\ \text{km/h})\cdot(4\ \text{hrs}) = 140\ \text{km}$

$(25\ \text{km/h})\cdot(4\ \text{hrs}) = 100\ \text{km}$

$$r = \sqrt{10^2 + 100^2} = \sqrt{100 + 10000} = \sqrt{10{,}100}$$


$$x^2 + y^2 = r^2 \Rightarrow 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2r \cdot \frac{dr}{dt}$$

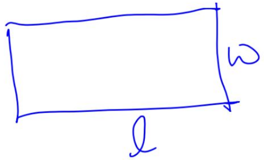
$$\Rightarrow \frac{dr}{dt} = \frac{x}{r} \cdot \frac{dx}{dt} + \frac{y}{r} \cdot \frac{dy}{dt}$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{10}{\sqrt{10{,}100}} \cdot (-35) + \frac{100}{\sqrt{10{,}100}} \cdot (25)\ \text{km/h}}$$

Paul Lee

# Related Rates

The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?




$$\frac{dl}{dt} = 8 \qquad \frac{dw}{dt} = 3$$

$$A = lw \Rightarrow \frac{dA}{dt} = \frac{dl}{dt} \cdot w + l \cdot \frac{dw}{dt}$$

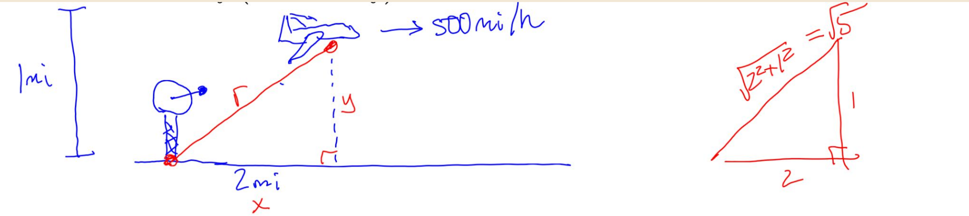
$$\Rightarrow \frac{dA}{dt} = 8 \cdot 10 + 20 \cdot 3$$

$$\Rightarrow \frac{dA}{dt} = 80 + 60 = \boxed{140 \text{ cm/s}}$$

Paul Lee

# Related Rates

A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away (horizontally) from the station.




$$x^2+y^2=r^2 \Rightarrow 2x\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=2r\cdot\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt}=\frac{x}{r}\cdot\frac{dx}{dt}+\frac{y}{r}\frac{dy}{dt}$$

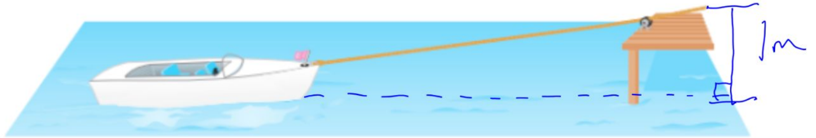
$$\Rightarrow \frac{dr}{dt}=\frac{2}{\sqrt{5}}\cdot(500)+\frac{1}{\sqrt{5}}(0)$$

$$\Rightarrow \boxed{\frac{dr}{dt}=\frac{1000}{\sqrt{5}}\ \text{mi/h}}$$

Paul Lee

# Related Rates

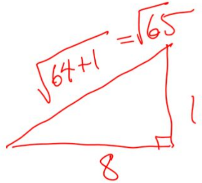
A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?




$$\frac{dr}{dt} = -1 \text{ m/s}$$

$$\frac{dx}{dt} = ?$$

Same as Q5/Q6: $2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2r \cdot \frac{dr}{dt}$




$$\Rightarrow \frac{dx}{dt} = \frac{r}{x} \cdot \frac{dr}{dt} - \frac{y}{x} \cdot \frac{dy}{dt}$$

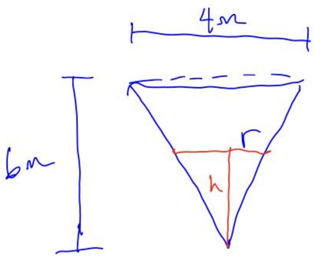
$$\Rightarrow \frac{dx}{dt} = \frac{\sqrt{65}}{8} \cdot (-1) - \frac{1}{8} \cdot (0)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = \frac{-\sqrt{65}}{8} \text{ m/s}}$$

Paul Lee

# Related Rates

Water is leaking out of an inverted conical tank at a rate of 10,000 cm/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$




$$\frac{dh}{dt} = 20 \text{ cm/min}$$

$$\frac{dV}{dt} = +C - 10{,}000$$

Similar triangles:

$$\frac{h}{6} = \frac{r}{2} \Rightarrow \boxed{r = \frac{1}{3}h}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{3}\cdot\frac{dh}{dt}}$$

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3$$

Then $\frac{dV}{dt} = \frac{1}{9}\pi h^2 \cdot \frac{dh}{dt} \Rightarrow C - 10{,}000 = \frac{1}{9}\pi(200)^2(20) = \frac{800{,}000\pi}{9}$

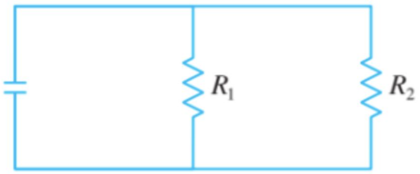
$$\Rightarrow \boxed{C = 10{,}000 + \frac{800{,}000\pi}{9} \text{ cm}^3/\text{min}}$$

Paul Lee

# Related Rates

If two resistors with resistances $R_1$ and $R_2$ are connected in parallel, as in the figure, then the total resistance $R$, measured in ohms ($\Omega$), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$




If $R_1$ and $R_2$ are increasing at rates of $0.3\,\Omega/\text{s}$ and $0.2\,\Omega/\text{s}$, respectively, how fast is $R$ changing when $R_1 = 80\,\Omega$ and $R_2 = 100\,\Omega$?

$$-\frac{1}{R^2}\cdot\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\cdot\frac{dR_2}{dt}$$

$$\Rightarrow \frac{dR}{dt} = \frac{R^2}{R_1^2}\cdot\frac{dR_1}{dt} + \frac{R^2}{R_2^2}\cdot\frac{dR_2}{dt} \Rightarrow$$

$$\boxed{\frac{dR}{dt} = \frac{\left(\frac{400}{9}\right)^2}{80^2}\cdot(0.3) + \frac{\left(\frac{400}{9}\right)^2}{100^2}(0.2) \quad \Omega/s}$$

Wayne Broughton

# Related Rates

**An oil spill is spreading out in a circle. The radius increases at 1 m/s. How fast is the area of the oil spill changing when radius is 30m?**


$$A = \pi r^2$$

$$\Longrightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

When $r = 30$, $\frac{dr}{dt} = 1$

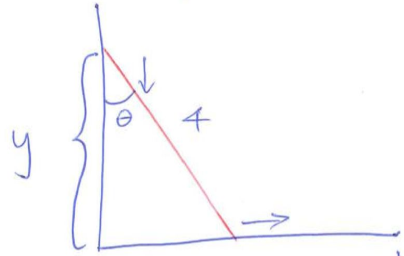
$$\Longrightarrow \frac{dA}{dt} = 2\pi(30)\cdot 1 = 60\pi$$

$$\approx 188.5 \ m^2/sec$$

Wayne Broughton

# Related Rates

**A ladder is leaning on a wall and slides down. When the angle between the ladder and wall is 45 deg (or pi/4 rad), it is growing at 1.5 rad/sec. The ladder is 4 meters long. How fast is the top of the ladder moving down at that point?**




$\theta$ = angle

$y$ = vertical distance from top of ladder to ground

We know: $\frac{d\theta}{dt} = 1.5$ when $\theta = \frac{\pi}{4}$.

Want: $\frac{dy}{dt}$.

But $\frac{y}{4} = \cos(\theta) \implies y = 4\cos(\theta)$

$\left[\text{or } \theta = \cos^{-1}\left(\frac{y}{4}\right) = \arccos\left(\frac{y}{4}\right)\right]$.

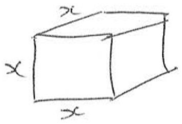
So $\frac{dy}{dt} = -4\sin(\theta)\frac{d\theta}{dt}$.

At $\theta = \frac{\pi}{4}$, $\frac{dy}{dt} = -4\sin\left(\frac{\pi}{4}\right)(1.5) = -4\left(\frac{1}{\sqrt{2}}\right)(1.5) = -4.24$ m/s.

Wayne Broughton

# Related Rates

**A crystal in the shape of a cube is growing such that its surface area increases at 2 mm$^2$/minute. How fast is its volume changing when its side length is 3mm?**




$$V = x^3$$

$$A = 6x^2$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\frac{dA}{dt} = 12x \frac{dx}{dt}.$$

Know: $\frac{dA}{dt} = 2$, $x = 3$.

So $2 = 12(3)\frac{dx}{dt}$

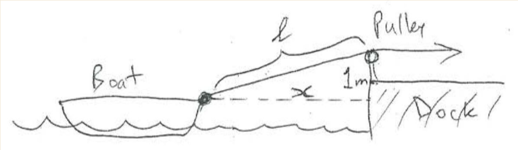
$$\Rightarrow \frac{dx}{dt} = \frac{2}{36} = \frac{1}{18} \quad (\text{mm}/\text{min}).$$

Then $\frac{dV}{dt} = 3(3)^2\left(\frac{1}{18}\right) = 1.5 \text{ mm}^3/\text{min}.$

Wayne Broughton

# Related Rates

**A boat is pulled toward a dock by a rope tied to the front of the boat and going over a pulley 1 meter above the front of the boat. If the rope is being pulled in at 0.5 m/s when the boat is 4 meters from the dock, how fast is the boat moving towards the dock at that point?**




$\frac{d\ell}{dt} = -0.5\,m/s$ when $x = 4\,m$.

Want: $\frac{dx}{dt}$

How are $\ell$ and $x$ related?

Pythagorean theorem: $\ell^2 = x^2 + 1$.

$\left[\text{or } \ell = \sqrt{x^2+1}\right]$.

Differentiate with respect to time $t$.

$$\frac{d}{dt}(\ell^2) = \frac{d}{dt}(x^2+1)$$

$$\cancel{2}\ell \cdot \frac{d\ell}{dt} = \cancel{2}x\frac{dx}{dt}$$

$$\Longrightarrow \frac{dx}{dt} = \frac{\ell\left(\frac{d\ell}{dt}\right)}{x}$$

When $x = 4$, $\frac{d\ell}{dt} = -0.5$,

and $\ell = \sqrt{x^2+1} = \sqrt{4^2+1} = \sqrt{17}$

Then $\frac{dx}{dt} = \frac{(\sqrt{17})(-0.5)}{4} = -0.515\,m/s$.

So the boat is moving at 0.515 m/s toward the dock at that point.

# Linearization & Differentials

Chad Davis

# Linearization and Differentials

Let $f(x) = \ln(3x+1)$

(a) Calculate the linearization of $f(x)$ at $x = 0$.

$f(0) = \ln(3(0)+1) = \ln(1) = 0$

$f'(x) = \frac{1}{3x+1}(3) \Rightarrow f'(0) = 3$

$\Rightarrow L(x) = f(0) + f'(0)(x-0)$

$= 3x$

Chad Davis

# Linearization and Differentials

(b) Use part (a) to approximate the value $\ln(0.97)$.

$$\ln(0.97) = \ln(3(-0.01)+1)$$
$$= f(-0.01)$$
$$\approx L_0(-0.01)$$
$$= 3(-0.01)$$
$$= -0.03$$

Chad Davis

# Linearization and Differentials

(c) Calculate the differential of $f(x)$ at $x = 0$.

$$dy = f'(0)dx$$

$$\Rightarrow dy = 3dx$$

Chad Davis

# Linearization and Differentials

(d) Use part(c) to approximate the total change in $f(x)$ as $x$ varies from 0 to 0.02.

$$\Delta x = 0.02 - 0 = 0.02.$$

$$\Rightarrow \Delta y \approx dy(0.02) = 3(0.02) = 0.06$$

Chad Davis

# Linearization and Differentials

Let $f(x) = \sqrt{x+3}$.

(a) Use a linearization of $f(x)$ to approximate $\sqrt{4.02}$.

We need to pick the $a$-value.

Note: $\sqrt{4.02} = \sqrt{1.02+3}$ suggests we use $a=1$

Then $f(1) = \sqrt{1+3} = \sqrt{4} = 2$

and $f'(x) = \frac{1}{2}(x+3)^{-1/2} \Rightarrow f'(1) = \frac{1}{4}$

$\Rightarrow L_1(x) = 2 + \frac{1}{4}(x-1)$

Then $\sqrt{4.02} = (1.02+3)^{1/2}$

$= f(1.02)$

$\approx L_1(1.02)$

$= 2 + \frac{1}{4}(1.02-1)$

$= 2 + \frac{(0.02)}{4} = 2.005$

Chad Davis

# Linearization and Differentials

(b) Use a differential to approximate the value $4 - \sqrt{15.98}$

**Hint.** For these questions, you have to determine the value $x = a$ where you need to calculate the linearization and differential.

$4 = \sqrt{16} = \sqrt{13+3}$ ← calculate differential at $x = 13$

$f'(x) = \frac{1}{2}(x+3)^{-1/2} \Rightarrow f'(13) = \frac{1}{2}(13+3)^{-1/2} = \frac{1}{8}$

Differential @ $x = 13$ is
$dy = f'(13)dx \Rightarrow dy = \frac{1}{8}dx$

Now $4 - \sqrt{15.98} = \sqrt{16} - \sqrt{15.98}$
$= \sqrt{13+3} - \sqrt{12.98+3}$
$= f(13) - f(12.98)$

so $\Delta x = 13 - 12.98 = 0.02$

$\Rightarrow 4 - \sqrt{15.98} \approx dy(0.02) = \frac{1}{8}(0.02) = 0.0025$

actual value: $4 - \sqrt{15.98} \approx 0.002500782$

Paul Lee

# Linearization

Find the linearization of $f(x) = \sin(x)$ at $a = 0$

**Solution:** $L(x) = f'(a)(x - a) + f(a)$

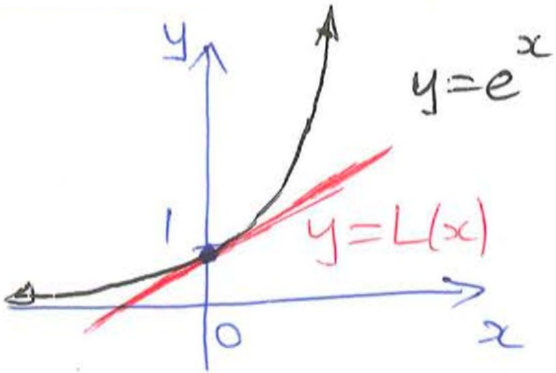
$\Rightarrow L(x) = \cos(0)(x - 0) + \sin(0)$

$\Rightarrow \boxed{L(x) = x}$

So for $x$-values near $x = 0$, $\sin(x) \approx x$

# Linearization

**What is the linear approximation of of f(x) = e$^x$ at x = 0?**




$$f(0)=e^0=1$$

$$f'(x)=e^x \Longrightarrow f'(0)=1.$$

$$\text{So } L(x)=f(0)+f'(0)(x-0)$$
$$=1+x.$$

$e^x \approx 1+x$ if $x$ is close to 0.

Wayne Broughton

# Linearization

**What is the linear approximation of of ycos(πy) = x for y near x = -1.**

$L(x) = f(-1) + f'(-1)(x-(-1))$

$f(-1) = 1$. Need $f'(x)$

Implicit diff: $y\cos(\pi y) = x$

Product rule:

$$\frac{dy}{dx} \cdot \cos(\pi y) + y \cdot \frac{d}{dx}\left[\cos(\pi y)\right] = \frac{d}{dx}(x) = 1$$

$$y'\cos(\pi y) + y \cdot \underbrace{\left[-\sin(\pi y)\right] \cdot \pi y'}_{\text{Chain rule}} = 1$$

$$\Longrightarrow y'\left[\cos(\pi y) - \pi y \sin(\pi y)\right] = 1$$

$$\Longrightarrow y' = f'(x) = \frac{1}{\cos(\pi y) - \pi y \sin(\pi y)}.$$

At $(-1, 1)$: $f'(-1) = \dfrac{1}{\cos(\pi) - \pi \cdot \sin(\pi)}$

$$= \frac{1}{-1-0} = -1.$$

So $L(x) = 1 + (-1)\cdot(x-(-1)) = 1-(x+1)$

$= -x$.

So $y \approx -x$ near $(-1, 1)$.

Dominic Huang

# Linearization

Use Linear Approximation of the function $f(x) = \sqrt{x}$ at $x = 7$ to estimate $\sqrt{49.1}$

We know that $\sqrt{49} = 7$. We can use linear approximation to estimate $\sqrt{49.1}$.

1. Identify the function: $f(x) = \sqrt{x}$
2. Find the derivative: $f'(x) = \frac{1}{2\sqrt{x}}$
3. Choose the point: $a = 49$
4. Compute $f(a) = \sqrt{49} = 7$ and $f'(a) = \frac{1}{2\sqrt{49}} = \frac{1}{14}$
5. Apply the linear approximation formula:

$$L(x) = 7 + \frac{1}{14}(x - 49)$$

6. Estimate $\sqrt{49.1}$ using $L(49.1)$:

$$L(49.1) = 7 + \frac{1}{14}(49.1 - 49) = 7 + \frac{1}{14}(0.1) \approx 7.0071$$

Thus, $\sqrt{49.1}$ is approximately 7.0071. $\square$

Dominic Huang

# Differentials

Given $y = x^2$, find $dy$.

1. First, find the derivative $f'(x)$:

$$f'(x) = 2x$$

2. Next, calculate $dy$:

$$dy = 2xdx$$

Paul Lee

# Differentials

Compute $\Delta y$ and $dy$ for the function $f(x) = \dfrac{2}{x}$ at the value $x = 4$ and $\Delta x = dx = 1$

$$\begin{aligned}
\Delta y &= f(x + \Delta x) - f(x) \\
&= f(4+1) - f(4) \\
&= f(5) - f(4) \\
&= \frac{2}{5} - \frac{2}{4} \\
&= \frac{4}{10} - \frac{5}{10} \\
&= \boxed{-\frac{1}{10}}
\end{aligned}$$

$$\begin{aligned}
dy &= f'(x)\,dx \\
&= \frac{-2}{x^2}\,dx \\
&= \frac{-2}{4^2} \cdot (1) \\
&= \boxed{-\frac{1}{8}}
\end{aligned}$$

Dominic Huang

# Differentials

A circle has a radius of 10 cm which is measured with a possible error of $\pm 0.1$ cm. Use differentials to estimate the error in the calculated area of the circle.

1. The area $A$ of a circle with radius $r$ is given by $A = \pi r^2$.
2. Differentiate both sides with respect to $r$:
$$dA = 2\pi r\, dr$$
3. Plug in $r = 10$ cm and $dr = 0.1$ cm:
$$dA = 2\pi \times 10 \times 0.1 = 2\pi$$
4. Interpret the result: The error in the calculated area is approximately $2\pi\,\text{cm}^2$ or around $6.28\,\text{cm}^2$.

$\square$

# Optimization

Wayne Broughton

# Optimization

**Find the absolute max and min values of $f(x) = x^3 - 6x^2 + 5$ on [-3, 5], given f is continuous.**

1) $f'(x) = 3x^2 - 12x$. This exists at all $x$.

So solve for $f'(x) = 0$.

$$3x^2 - 12x = 0$$

$$\iff x(3x-12) = 3x(x-4) = 0.$$

So $x = 0$ or $x = 4$.

These are both in $[-3, 5]$.

$$f(0) = 0^3 - 6 \cdot 0^2 + 5 = 5$$

$$f(4) = 4^3 - 6 \cdot 4^2 + 5 = -27.$$

2) Endpoints: $f(-3) = (-3)^3 - 6(-3)^2 + 5 = -76$

$$f(5) = 5^3 - 6 \cdot 5^2 + 5 = -20.$$

3) The absolute max. of $f(x)$ is $\boxed{f(0) = 5}$.

" " min. " " " $\boxed{f(-3) = -76}$

Dominic Huang

# Optimization

Find the critical points of $f(x) = x^2 - 4x + 4$.

To find the critical points of $f(x) = x^2 - 4x + 4$, we first find its derivative, $f'(x)$.

$$f'(x) = 2x - 4$$

To find the critical points, we set $f'(x) = 0$ and solve for $x$.

$$\begin{aligned} 2x - 4 &= 0 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Thus, $x = 2$ is a critical point of $f(x)$. $\square$

Dominic Huang

# Optimization

Find the critical points of $f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ 2x & \text{if } x > 0. \end{cases}$

To find the critical points of the given piecewise function, we first find its derivative for each piece, $f'(x)$.

For $x < 0$:

$$f'(x) = 2x$$

For $x > 0$:

$$f'(x) = 2$$

The derivative changes abruptly at $x = 0$ and the function is discontinuous at this point. Therefore, $x = 0$ is a critical point of $f(x)$ where $f'(x)$ is undefined due to the discontinuity. $\square$

# Any Questions?????

# Leave me some feedback!

Completely anonymous and not affiliated with the university. Click here.