MATH 100: Differential Calculus

Supplemental Learning

MATH 100 2021 Midterm 2

3/[3] 1. Find the equation of the tangent line to the curve $y = \frac{x}{x^2-3}$ at the point $(2, 2)$

The equation for the tangent line of a curve at (x_1, y_1) is $(y - y_1) = m_t(x - x_1)$, where m_t = slope of the tangent.

Also, at this point, $m_t = y'(x_1)$

$$\therefore (y - y_1) = [y'(x_1)](x - x_1)$$

Putting in values, $(y-2) = [y'(2)](x-2)$ — (1)

Now, finding y':

$$y = \frac{x}{x^2-3}$$

$$\therefore y' = \frac{(x^2-3) \times \frac{d}{dx}(x) - x \times \frac{d}{dx}(x^2-3)}{(x^2-3)^2} \quad [\because \text{quotient rule}]$$

$$= \frac{(x^2-3)(1) - (x)(2x)}{(x^2-3)^2}$$

$$= \frac{x^2-3-2x^2}{(x^2-3)^2}$$

$$= \frac{-x^2-3}{(x^2-3)^2}$$

$$\therefore \text{At } x_1 = 2,\ y' = \frac{-(2)^2-3}{[(2)^2-3]^2} = \frac{-4-3}{(4-3)^2} = \frac{-7}{(1)^2} = -7$$

Putting this value in (1), we get:

$$y - 2 = (-7)(x-2)$$
$$= -7x + 14$$

$$\boxed{y = -7x + 16}$$

The equation of the tangent line to the curve $y = \frac{x}{x^2-3}$ at the point $(2,2)$ is $y = -7x + 16$.

[Up on my GitHub repo!]

XX %

Quiz 5

Friday, November 24th *(MATH 100-001)*

Topics on the Quiz

L'Hôpital's Rule

Related Rates

Linear Approximation

Differentials

Extreme Values/ Critical Values

Bonus!
(at least in Broughton's class)

Given Algebraic Identities on the Quiz

Algebra Identities

- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$
- $a^{x+y} = a^x a^y$
- $a^{xy} = (a^x)^y = (a^y)^x$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
- $\log_a(x^n) = n\log_a(x)$
- $ax^2 + bx + c = 0 \implies x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

L'Hôpital's Rule:
Topic - Indeterminate Forms with Powers

Such as: 0^0, $\inf^{\inf}$, $1^{\inf}$
Note that $0^{\inf} = 0$, and so it is not an indeterminate form.

This happens when: $\lim_{x\to a} f(x)^{g(x)}$

- $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
- $\lim_{x\to a} f(x) = \inf$ and $\lim_{x\to a} g(x) = 0$
- $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\inf$

To fix this, we take the ln() of the entire function and raise *e* to that power.
i.e., $\lim_{x\to a} f(x)^{g(x)} = e^{\lim x \to a \ln(f(x)^\wedge g(x))} = e^{\lim x \to a\, g(x)\ln(f(x))}$
And we can use L'Hôpital's Rule to calculate $\lim_{x \to a} g(x)\ln(f(x))$.

L’Hôpital’s Rule

Dominic Huang

L'Hôpital's Rule:
Topic - Indeterminate Forms with Powers

$$\text{Find} \lim_{x\to 0^+} x^{\sqrt{x}}$$

Trying direct substitution gives an indeterminate form 0^0. To apply L'Hôpital's Rule, we first rewrite the function in exponential form:

$$x^{\sqrt{x}} = e^{\sqrt{x}\ln(x)}$$

Now, we focus on finding $\lim_{x\to 0^+} \sqrt{x}\ln(x)$. As $x \to 0^+$, $\sqrt{x} \to 0$ and $\ln(x) \to -\infty$, making it another indeterminate form $(0 \times -\infty)$. To deal with this, rewrite it as:

$$\sqrt{x}\ln(x) = \frac{\ln(x)}{1/\sqrt{x}}$$

Applying L'Hôpital's Rule to this expression, we find:

$$\lim_{x\to 0^+} \sqrt{x}\ln(x) = 0$$

Therefore, $\lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = 1$. $\square$

"hence proved"

Dominic Huang

L'Hôpital's Rule:
Topic - Indeterminate Forms with Powers

Find $\lim_{x\to 0^+}(1-\sin(2x))^{1/x}$.

Attempting direct substitution results in the indeterminate form 1^∞. We rewrite the function as an exponential:

$$(1-\sin(2x))^{1/x} = e^{(1/x)\ln(1-\sin(2x))}$$

Using L'Hôpital's Rule, we find $\lim_{x\to 0^+}\frac{\ln(1-\sin(2x))}{x} = -2$. Therefore, the limit is e^{-2}. $\square$

Dominic Huang

L'Hôpital's Rule:
Topic - Indeterminate Forms with Powers

Find $\lim_{x \to -\infty} (x^2 + 1)e^x$.

This function approaches the indeterminate form ∞^0. We rewrite the function as:

$$(x^2 + 1)e^x = e^{e^x \ln(x^2+1)}$$

Applying L'Hôpital's Rule, we find $\lim_{x \to -\infty} \frac{e^x}{\ln(x^2+1)} = 0$. Thus, $\lim_{x \to -\infty} (x^2 + 1)e^x = e^0 = 1$. $\square$

Dominic Huang

L'Hôpital's Rule:
Topic - Indeterminate Forms with Powers

Showing that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ is a real number.

Consider rewriting $\left(1+\frac{1}{x}\right)^x$ in exponential form as follows:

$$\left(1+\frac{1}{x}\right)^x = e^{x\ln\left(1+\frac{1}{x}\right)}$$

To find the limit, we focus on the exponent $x\ln\left(1+\frac{1}{x}\right)$ as $x\to\infty$.

Now, $x\to\infty$ and $\ln\left(1+\frac{1}{x}\right)\to 0$, resulting in an indeterminate form $\infty\times 0$. Rewrite it as:

$$x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

Applying L'Hôpital's Rule, we get:

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$$

Thus, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e.$ $\square$

Paul Lee

L'Hôpital's Rule

Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.

$$\lim_{\theta \to 0} \frac{e^{\sin(\theta)} - 1}{2\theta}$$

$$\lim_{x \to 0} \frac{\arcsin(x)}{x}$$

$$\lim_{t \to 1^+} \ln(t) \tan(\pi t/2)$$

$$\lim_{\theta \to 0} \frac{e^{\sin(\theta)} - 1}{2\theta} \overset{L}{=} \lim_{\theta \to 0} \frac{e^{\sin(\theta)} \cos(\theta)}{2} = \boxed{\frac{1}{2}}$$

$$\lim_{x \to 0} \frac{\arcsin(x)}{x} \overset{L}{=} \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = \frac{1}{1} = \boxed{1}$$

$$\begin{aligned} &= \lim_{t \to 1^+} \frac{\ln(t)\sin(\pi t/2)}{\cos(\pi t/2)} \\ &\overset{L}{=} \lim_{t \to 1^+} \frac{(1/t)\sin(\pi t/2) + \ln(t)\cos(\pi t/2)\cdot(\pi/2)}{-\sin(\pi t/2)\cdot(\pi/2)} \\ &= \frac{(1/1)\sin(\pi/2) + \ln(1)\cos(\pi/2)\cdot(\pi/2)}{-\sin(\pi/2)\cdot(\pi/2)} \\ &= \frac{1+0}{-1\cdot(\pi/2)} \\ &= \boxed{-\frac{2}{\pi}} \end{aligned}$$

Paul Lee

L'Hôpital's Rule

Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.

Same q as before, written differently

$$\lim_{x\to 0^+} x^{\sqrt{x}}$$

$$\lim_{x\to\infty} x\sin(\pi/x)$$

$$\lim_{x\to\infty} (x-\ln(x))$$

Let $y = x^{\sqrt{x}}$ so that $\ln(y) = \ln(x^{\sqrt{x}}) = \sqrt{x}\ln(x)$.

$$\begin{aligned}\lim_{x\to 0^+}\sqrt{x}\ln(x) &= \lim_{x\to 0^+}\frac{\ln(x)}{1/\sqrt{x}}\\ &\overset{L}{=} \lim_{x\to 0^+}\frac{1/x}{1/(2x^{3/2})}\\ &= \lim_{x\to 0^+} 2x^{1/2}\\ &= \boxed{0}\end{aligned}$$

So now we have

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln(x^{\sqrt{x}})} = e^{\left(\lim_{x\to 0^+}\ln\left(x^{\sqrt{x}}\right)\right)} = e^0 = \boxed{1}$$

$$\begin{aligned}\lim_{x\to\infty} x\sin(\pi/x) &= \lim_{x\to\infty}\frac{\sin(\pi/x)}{1/x}\\ &\overset{L}{=} \lim_{x\to\infty}\frac{\cos(\pi/x)\cdot(-\cancel{1}\pi/x^2)}{-1/x^2}\\ &= \lim_{x\to\infty}\cos(\pi/x)\cdot(\pi)\\ &= \cos\left(\lim_{x\to\infty}\pi/x\right)\cdot\pi\\ &= \cos(0)\cdot\pi\\ &\cancel{= \boxed{1}} \quad = \boxed{\pi}\end{aligned}$$

$$\begin{aligned}\lim_{x\to\infty}(x-\ln(x)) &= \lim_{x\to\infty}(x-\ln(x))\cdot\frac{x+\ln(x)}{x+\ln(x)}\\ &= \lim_{x\to\infty}\frac{x^2-[\ln(x)]^2}{x+\ln(x)}\\ &\overset{L}{=} \lim_{x\to\infty}\frac{2x-2\ln(x)\cdot(1/x)}{1+(1/x)}\\ &= \lim_{x\to\infty}\frac{2x^2-2\ln(x)}{x+1}\\ &\overset{L}{=} \lim_{x\to\infty}\frac{4x-(2/x)}{1}\\ &= \lim_{x\to\infty}\frac{4x^2-2}{x}\\ &\overset{L}{=} \lim_{x\to\infty}\frac{8x}{1}\\ &= \infty\end{aligned}$$

Paul Lee

L'Hôpital's Rule

Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.

$$\lim_{x\to 0}(1-2x)^{1/x}$$

$$\lim_{x\to 0}\left(\cot(x)-\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\sqrt{x}e^{-x/2}$$

Let $y=(1-2x)^{1/x}$ so that $\ln(y)=(1/x)\ln(1-2x)=\dfrac{\ln(1-2x)}{x}$

$$\lim_{x\to 0}\frac{\ln(1-2x)}{x}\overset{L}{=}\lim_{x\to 0}\frac{\frac{-2}{1-2x}}{1}=\boxed{-2}$$

Now,

$$\begin{aligned}\lim_{x\to 0}(1-2x)^{1/x}&=\lim_{x\to 0}e^{\ln\left((1-2x)^{1/x}\right)}\\&=e^{\lim_{x\to 0}(1-2x)^{1/x}}\\&=\boxed{e^{-2}}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\left(\cot(x)-\frac{1}{x}\right)&=\lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)}-\frac{1}{x}\right)\\&=\lim_{x\to 0}\frac{x\cos(x)-\sin(x)}{x\sin(x)}\\&\overset{L}{=}\lim_{x\to 0}\frac{1\cdot\cos(x)-x\sin(x)-\cos(x)}{1\cdot\sin(x)+x\cos(x)}\\&=\lim_{x\to 0}\frac{-x\sin(x)}{\sin(x)+x\cos(x)}\\&\overset{L}{=}\lim_{x\to 0}\frac{-1\cdot\sin(x)+(-x)\cos(x)}{\cos(x)+1\cdot\cos(x)+x(-\sin(x))}\\&=\frac{0+0}{1+1+0}\\&=\boxed{0}\end{aligned}$$

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x}e^{-x/2}&=\lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}}\\&\overset{L}{=}\lim_{x\to\infty}\frac{1/(2\sqrt{x})}{e^{x/2}\cdot(1/2)}\\&=\lim_{x\to\infty}\frac{1}{\sqrt{x}e^{x/2}}\\&=\boxed{0}\end{aligned}$$

Paul Lee

L'Hôpital's Rule

Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.

$$\lim_{x\to 0^+} (\tan(2x))^x$$

$$\lim_{x\to\infty} x^{1/x}$$

$$\lim_{x\to\infty} (e^x + x)^{1/x}$$

Let $y = (\tan(2x))^x$ so that $\ln(y) = x\ln(\tan(2x))$.

$$\begin{aligned}
\lim_{x\to 0^+} x\ln(\tan(2x)) &= \lim_{x\to 0^+} \frac{\ln(\tan(2x))}{1/x} \\
&\stackrel{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{\tan(2x)}\cdot \sec^2(2x)\cdot 2}{-1/x^2} \\
&= \lim_{x\to 0^+} \frac{\frac{\cos(2x)}{\sin(2x)}\cdot \frac{1}{\cos^2(2x)}\cdot 2}{-1/x^2} \\
&= \lim_{x\to 0^+} \frac{-2x^2}{\sin(2x)\cos(2x)} \\
&\stackrel{L}{=} \lim_{x\to 0^+} \frac{-4x}{\cos(2x)\cdot 2\cdot \cos(2x) + \sin(2x)\cdot(-\sin(2x))\cdot 2} \\
&= \frac{0}{2+0} \\
&= \boxed{0}
\end{aligned}$$

Then $\lim_{x\to 0^+} (\tan(2x))^x = e^0 = \boxed{1}$

Let $y = x^{1/x}$ so that $\ln(y) = (1/x)\ln(x) = \dfrac{\ln(x)}{x}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)}{x} &\stackrel{L}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \\
&= \lim_{x\to\infty} \frac{1}{x} \\
&= \boxed{0}
\end{aligned}$$

Then $\lim_{x\to\infty} x^{1/x} = e^0 = \boxed{1}$

Let $y = (e^x + x)^{1/x}$ so that $\ln(y) = (1/x)\ln(e^x + x) = \dfrac{\ln(e^x + x)}{x}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(e^x + x)}{x} &\stackrel{L}{=} \lim_{x\to\infty} \frac{\frac{e^x+1}{e^x+x}}{1} \\
&= \lim_{x\to\infty} \frac{e^x+1}{e^x+x} \\
&\stackrel{L}{=} \lim_{x\to\infty} \frac{e^x}{e^x+1} \\
&\stackrel{L}{=} \lim_{x\to\infty} \frac{e^x}{e^x} \\
&= \lim_{x\to\infty} 1 \\
&= 1
\end{aligned}$$

Then $\lim_{x\to\infty} (e^x + x)^{1/x} = e^1 = \boxed{e}$

Chad Davis

L'Hôpital's Rule

Find the following limits. If you are applying L'Hopital's rule, you must indicate this clearly.

(a) $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$

(b) $\displaystyle \lim_{x \to \infty} \frac{\ln\big(\ln(x)\big)}{x}$

(c) $\displaystyle \lim_{x \to 0} \cot(2x)\sin(6x)$

(d) $\displaystyle \lim_{x \to 0^+} \big(\cos(x)\big)^{1/x^2}$

Related Rates

Chad Davis

Related Rates

Two people on bikes are separated by 350 meters directly east and west from each other. Person A starts riding north at a rate of 5 m/s and 7 minutes later Person B starts riding south at 3 m/s. At what rate is the distance separating the two people changing 25 minutes after Person A starts riding?

Chad Davis

Related Rates

A standard Major League Baseball diamond is a square with side lengths of 90 feet. On the season opener of the 2021 season, Bo Bichette of the Toronto Blue Jays hits a line drive down the third base line on the first pitch he sees and starts running towards first base. Suppose that he is running to first base at a rate of 29 feet per second.

(a) At what rate is his distance from second base decreasing when he is halfway to first base?

(b) At what rate is his distance from third base increasing at the same moment?

Chad Davis

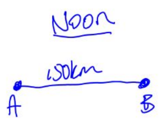
Related Rates

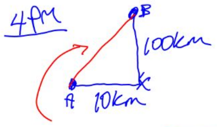
A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m $\times$ 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is (1/3)(height)(area of base).

Paul Lee

Related Rates

At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?





$(35\text{ km/h})\cdot(4\text{ hrs}) = 140\text{ km}$

$(25\text{ km/h})\cdot(4\text{ hrs}) = 100\text{ km}$

$$r = \sqrt{10^2+100^2} = \sqrt{100+10000} = \sqrt{10{,}100}$$

$$x^2+y^2=r^2 \Rightarrow 2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} = 2r\cdot\frac{dr}{dt}$$

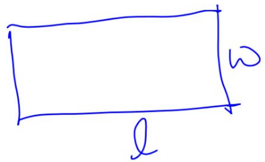
$$\Rightarrow \frac{dr}{dt} = \frac{x}{r}\cdot\frac{dx}{dt} + \frac{y}{r}\cdot\frac{dy}{dt}$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{10}{\sqrt{10{,}100}}\cdot(-35) + \frac{100}{\sqrt{10{,}100}}\cdot(25)\ \text{km/h}}$$

Paul Lee

Related Rates

The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?



$$\frac{dl}{dt} = 8 \qquad \frac{dw}{dt} = 3$$

$$A = lw \Rightarrow \frac{dA}{dt} = \frac{dl}{dt} \cdot w + l \cdot \frac{dw}{dt}$$

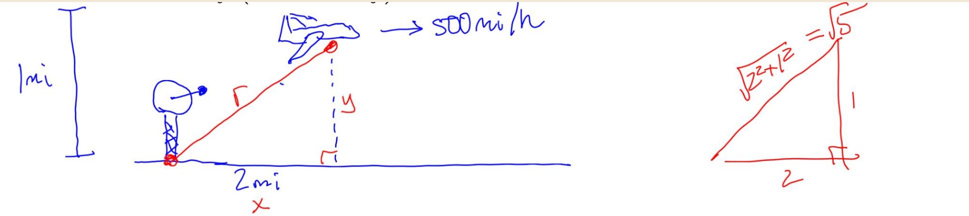
$$\Rightarrow \frac{dA}{dt} = 8 \cdot 10 + 20 \cdot 3$$

$$\Rightarrow \frac{dA}{dt} = 80 + 60 = \boxed{140 \text{ cm/s}}$$

Paul Lee

Related Rates

A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away (horizontally) from the station.



$$x^2+y^2=r^2 \Rightarrow 2x\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} = 2r\cdot\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{x}{r}\cdot\frac{dx}{dt} + \frac{y}{r}\frac{dy}{dt}$$

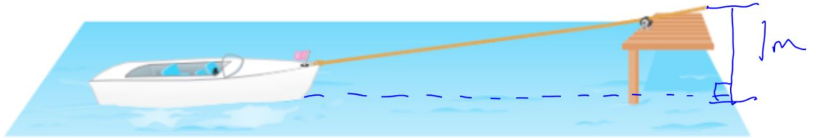
$$\Rightarrow \frac{dr}{dt} = \frac{2}{\sqrt{5}}\cdot(500) + \frac{1}{\sqrt{5}}(0)$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{1000}{\sqrt{5}} \text{ mi/h}}$$

Paul Lee

Related Rates

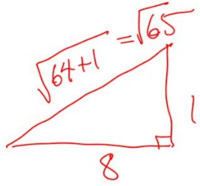
A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



$$\frac{dr}{dt} = -1 \text{ m/s}$$

$$\frac{dx}{dt} = ?$$

Same as Q5/Q6: $2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2r \cdot \frac{dr}{dt}$



$$\Rightarrow \frac{dx}{dt} = \frac{r}{x} \cdot \frac{dr}{dt} - \frac{y}{x} \cdot \frac{dy}{dt}$$

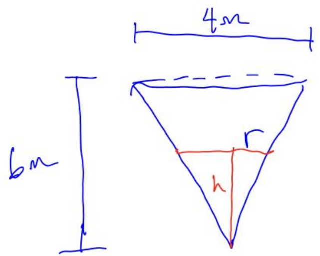
$$\Rightarrow \frac{dx}{dt} = \frac{\sqrt{65}}{8} \cdot (-1) - \frac{1}{8} \cdot (0)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = \frac{-\sqrt{65}}{8} \text{ m/s}}$$

Paul Lee

Related Rates

Water is leaking out of an inverted conical tank at a rate of 10,000 cm/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$



$$\frac{dh}{dt} = 20\,\text{cm/min}$$

$$\frac{dV}{dt} = +C - 10{,}000$$

Similar triangles:

$$\frac{h}{6} = \frac{r}{2} \Rightarrow \boxed{r = \frac{1}{3}h}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{3}\cdot\frac{dh}{dt}}$$

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3$$

Then $\frac{dV}{dt} = \frac{1}{9}\pi h^2 \cdot \frac{dh}{dt} \Rightarrow C - 10{,}000 = \frac{1}{9}\pi(200)^2(20) = \frac{800{,}000\pi}{9}$

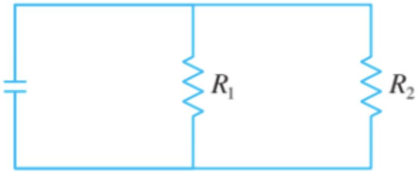
$$\Rightarrow \boxed{C = 10{,}000 + \frac{800{,}000\pi}{9}\ \text{cm}^3/\text{min}}$$

Paul Lee

Related Rates

If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$



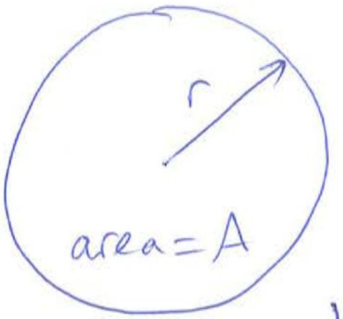
If R_1 and R_2 are increasing at rates of $0.3\,\Omega/\text{s}$ and $0.2\,\Omega/\text{s}$, respectively, how fast is R changing when $R_1 = 80\,\Omega$ and $R_2 = 100\,\Omega$?

$$-\frac{1}{R^2}\cdot\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\cdot\frac{dR_2}{dt}$$

$$\Rightarrow \frac{dR}{dt} = \frac{R^2}{R_1^2}\cdot\frac{dR_1}{dt} + \frac{R^2}{R_2^2}\cdot\frac{dR_2}{dt} \Rightarrow \boxed{\frac{dR}{dt} = \frac{\left(\frac{400}{9}\right)^2}{80^2}\cdot(0.3) + \frac{\left(\frac{400}{9}\right)^2}{100^2}(0.2) \quad \Omega/s}$$

Related Rates

An oil spill is spreading out in a circle. The radius increases at 1 m/s. How fast is the area of the oil spill changing when radius is 30m?



$$A = \pi r^2$$

$$\Longrightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

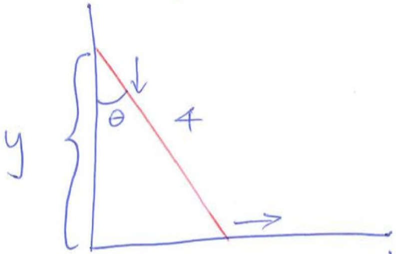
When $r = 30$, $\frac{dr}{dt} = 1$

$$\Longrightarrow \frac{dA}{dt} = 2\pi(30)\cdot 1 = 60\pi$$

$$\approx 188.5 \ m^2/sec$$

Related Rates

A ladder is leaning on a wall and slides down. When the angle between the ladder and wall is 45 deg (or pi/4 rad), it is growing at 1.5 rad/sec. The ladder is 4 meters long. How fast is the top of the ladder moving down at that point?



θ = angle

y = vertical distance from top of ladder to ground

We know: $\frac{d\theta}{dt} = 1.5$ when $\theta = \frac{\pi}{4}$.

Want: $\frac{dy}{dt}$.

But $\frac{y}{4} = \cos(\theta) \Rightarrow y = 4\cos(\theta)$

$\left[\text{or } \theta = \cos^{-1}\left(\frac{y}{4}\right) = \arccos\left(\frac{y}{4}\right)\right]$.

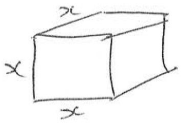
So $\frac{dy}{dt} = -4\sin(\theta)\frac{d\theta}{dt}$.

At $\theta = \frac{\pi}{4}$, $\frac{dy}{dt} = -4\sin\left(\frac{\pi}{4}\right)(1.5) = -4\left(\frac{1}{\sqrt{2}}\right)(1.5) = -4.24$ m/s.

Wayne Broughton

Related Rates

A crystal in the shape of a cube is growing such that its surface area increases at 2 mm²/minute. How fast is its volume changing when its side length is 3mm?



$$V = x^3$$

$$A = 6x^2$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\frac{dA}{dt} = 12x \frac{dx}{dt}.$$

Know: $\frac{dA}{dt} = 2, \quad x = 3.$

So $2 = 12(3)\frac{dx}{dt}$

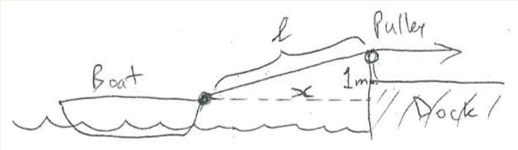
$$\Rightarrow \frac{dx}{dt} = \frac{2}{36} = \frac{1}{18} \quad (mm/min).$$

Then $\frac{dV}{dt} = 3(3)^2\left(\frac{1}{18}\right) = 1.5 \ mm^3/min.$

Wayne Broughton

Related Rates

A boat is pulled toward a dock by a rope tied to the front of the boat and going over a pulley 1 meter above the front of the boat. If the rope is being pulled in at 0.5 m/s when the boat is 4 meters from the dock, how fast is the boat moving towards the dock at that point?



$\frac{d\ell}{dt} = -0.5\,m/s$ when $x = 4\,m$.

Want: $\frac{dx}{dt}$

How are ℓ and x related?

Pythagorean theorem: $\ell^2 = x^2 + 1$.

$$\left[\text{or } \ell = \sqrt{x^2+1}\right].$$

Differentiate with respect to time t.

$$\frac{d}{dt}(\ell^2) = \frac{d}{dt}(x^2+1)$$

$$\cancel{2}\ell \cdot \frac{d\ell}{dt} = \cancel{2}x\frac{dx}{dt}$$

$$\Longrightarrow \frac{dx}{dt} = \frac{\ell\left(\frac{d\ell}{dt}\right)}{x}$$

When $x = 4$, $\frac{d\ell}{dt} = -0.5$,

and $\ell = \sqrt{x^2+1} = \sqrt{4^2+1} = \sqrt{17}$

Then $\frac{dx}{dt} = \frac{(\sqrt{17})(-0.5)}{4} = -0.515\,m/s$.

So the boat is moving at 0.515 m/s toward the dock at that point.

Linearization & Differentials

Chad Davis

Linearization and Differentials

Let $f(x) = \ln(3x+1)$

(a) Calculate the linearization of $f(x)$ at $x = 0$.

(b) Use part (a) to approximate the value $\ln(0.97)$.

(c) Calculate the differential of $f(x)$ at $x = 0$.

(d) Use part(c) to approximate the total change in $f(x)$ as x varies from 0 to 0.02.

Chad Davis

Linearization and Differentials

Let $f(x) = \sqrt{x+3}$.

(a) Use a linearization of $f(x)$ to approximate $\sqrt{4.02}$.

(b) Use a differential to approximate the value $4 - \sqrt{15.98}$

Hint. For these questions, you have to determine the value $x = a$ where you need to calculate the linearization and differential.

Paul Lee

Linearization

Find the linearization of $f(x) = \sin(x)$ at $a = 0$

Solution: $L(x) = f'(a)(x - a) + f(a)$

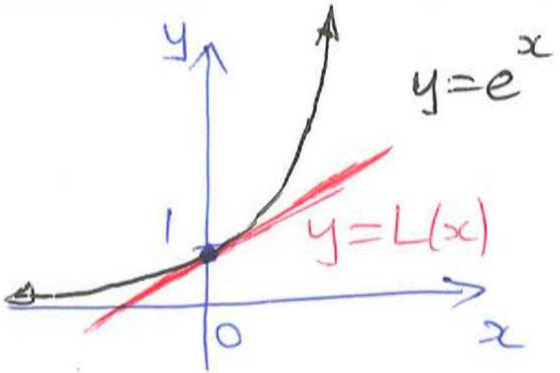
$\Rightarrow L(x) = \cos(0)(x - 0) + \sin(0)$

$\Rightarrow \boxed{L(x) = x}$

So for x-values near $x = 0$, $\sin(x) \approx x$

Linearization

What is the linear approximation of of f(x) = e^x at x = 0?



$$f(0)=e^0=1$$

$$f'(x)=e^x \Longrightarrow f'(0)=1.$$

$$\text{So } L(x)=f(0)+f'(0)(x-0)$$
$$=1+x.$$

$e^x \approx 1+x$ if x is close to 0.

Wayne Broughton

Linearization

What is the linear approximation of of ycos(πy) = x for y near x = -1.

$$L(x) = f(-1) + f'(-1)(x-(-1))$$

$f(-1) = 1$. Need $f'(x)$

<u>Implicit diff</u>: $y\cos(\pi y) = x$

Product rule:

$$\frac{dy}{dx}\cdot\cos(\pi y) + y\cdot\frac{d}{dx}\left[\cos(\pi y)\right] = \frac{d}{dx}(x) = 1$$

$$y'\cos(\pi y) + \underbrace{y\cdot\left[-\sin(\pi y)\right]\cdot\pi y'}_{\text{Chain rule}} = 1$$

$$\Rightarrow y'\left[\cos(\pi y) - \pi y\sin(\pi y)\right] = 1$$

$$\Rightarrow y' = f'(x) = \frac{1}{\cos(\pi y) - \pi y\sin(\pi y)}.$$

At $(-1, 1)$: $f'(-1) = \dfrac{1}{\cos(\pi) - \pi\cdot\sin(\pi)}$

$$= \frac{1}{-1-0} = -1.$$

So $L(x) = 1 + (-1)\cdot(x-(-1)) = 1-(x+1)$

$$= -x$$

So $y \approx -x$ near $(-1,1)$.

Dominic Huang

Linearization

Use Linear Approximation of the function $f(x) = \sqrt{x}$ at $x = 7$ to estimate $\sqrt{49.1}$

We know that $\sqrt{49} = 7$. We can use linear approximation to estimate $\sqrt{49.1}$.

1. Identify the function: $f(x) = \sqrt{x}$
2. Find the derivative: $f'(x) = \frac{1}{2\sqrt{x}}$
3. Choose the point: $a = 49$
4. Compute $f(a) = \sqrt{49} = 7$ and $f'(a) = \frac{1}{2\sqrt{49}} = \frac{1}{14}$
5. Apply the linear approximation formula:

$$L(x) = 7 + \frac{1}{14}(x - 49)$$

6. Estimate $\sqrt{49.1}$ using $L(49.1)$:

$$L(49.1) = 7 + \frac{1}{14}(49.1 - 49) = 7 + \frac{1}{14}(0.1) \approx 7.0071$$

Thus, $\sqrt{49.1}$ is approximately 7.0071. $\square$

Dominic Huang

Differentials

Given $y = x^2$, find dy.

1. First, find the derivative $f'(x)$:

$$f'(x) = 2x$$

2. Next, calculate dy:

$$dy = 2x dx$$

Paul Lee

Differentials

Compute Δy and dy for the function $f(x) = \dfrac{2}{x}$ at the value $x = 4$ and $\Delta x = dx = 1$

$$\begin{aligned}
\Delta y &= f(x + \Delta x) - f(x) \\
&= f(4+1) - f(4) \\
&= f(5) - f(4) \\
&= \frac{2}{5} - \frac{2}{4} \\
&= \frac{4}{10} - \frac{5}{10} \\
&= \boxed{-\frac{1}{10}}
\end{aligned}$$

$$\begin{aligned}
dy &= f'(x)\,dx \\
&= \frac{-2}{x^2}\,dx \\
&= \frac{-2}{4^2} \cdot (1) \\
&= \boxed{-\frac{1}{8}}
\end{aligned}$$

Dominic Huang

Differentials

A circle has a radius of 10 cm which is measured with a possible error of ± 0.1 cm. Use differentials to estimate the error in the calculated area of the circle.

1. The area A of a circle with radius r is given by $A = \pi r^2$.
2. Differentiate both sides with respect to r:

$$dA = 2\pi r\, dr$$

3. Plug in $r = 10\,\text{cm}$ and $dr = 0.1\,\text{cm}$:

$$dA = 2\pi \times 10 \times 0.1 = 2\pi$$

4. Interpret the result: The error in the calculated area is approximately $2\pi\,\text{cm}^2$ or around $6.28\,\text{cm}^2$.

$\square$

Optimization

Wayne Broughton

Optimization

Find the absolute max and min values of $f(x) = x^3 - 6x^2 + 5$ on [-3, 5], given f is continuous.

1) $f'(x) = 3x^2 - 12x$. This exists at all x.

So solve for $f'(x) = 0$.

$$3x^2 - 12x = 0$$

$$\Longleftrightarrow x(3x-12) = 3x(x-4) = 0.$$

So $x = 0$ or $x = 4$.

These are both in $[-3, 5]$.

$$f(0) = 0^3 - 6 \cdot 0^2 + 5 = 5$$

$$f(4) = 4^3 - 6 \cdot 4^2 + 5 = -27.$$

2) Endpoints: $f(-3) = (-3)^3 - 6(-3)^2 + 5 = -76$

$$f(5) = 5^3 - 6 \cdot 5^2 + 5 = -20.$$

3) The absolute max. of $f(x)$ is $\boxed{f(0) = 5}$

" " min. " " " $\boxed{f(-3) = -76}$

Dominic Huang

Optimization

Find the critical points of $f(x) = x^2 - 4x + 4$.

To find the critical points of $f(x) = x^2 - 4x + 4$, we first find its derivative, $f'(x)$.

$$f'(x) = 2x - 4$$

To find the critical points, we set $f'(x) = 0$ and solve for x.

$$\begin{aligned} 2x - 4 &= 0 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Thus, $x = 2$ is a critical point of $f(x)$. $\square$

Dominic Huang

Optimization

Find the critical points of $f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ 2x & \text{if } x > 0. \end{cases}$

To find the critical points of the given piecewise function, we first find its derivative for each piece, $f'(x)$.

For $x < 0$:

$$f'(x) = 2x$$

For $x > 0$:

$$f'(x) = 2$$

The derivative changes abruptly at $x = 0$ and the function is discontinuous at this point. Therefore, $x = 0$ is a critical point of $f(x)$ where $f'(x)$ is undefined due to the discontinuity. $\square$

Any Questions?????

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