Four Wheel Steering (4WS) Control

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A. Bike Dynamics Equation

$$\begin{aligned}(I_{bz} + I_{fz})\ddot{\phi_b} = (I_{fx} - I_{fy})\{\Omega^2 cos(\phi_w)^2 sin(\phi_b)cos(\phi_b) \\ -\Omega \cos(\phi_w)(\cos(2\phi_b) + I_{fz})\dot{\phi_w}\} \\ +(m_b h_b + m_f h_f) g \sin(\phi_b)\end{aligned} \tag{1}$$

$$\begin{aligned}X &= \begin{bmatrix}\phi_b \\ \dot{\phi_b} \\ \phi_w\end{bmatrix} \\ u &= \dot{\phi_w}\end{aligned} \tag{2}$$

$$\dot{X} = f(X) + g(X)u \tag{3}$$

$$\begin{aligned}\begin{bmatrix}\dot{\phi_b} \\ \ddot{\phi_b} \\ \dot{\phi_w}\end{bmatrix} = \begin{bmatrix}\dot{\phi_b} \\ \\ \frac{(I_{fx}-I_{fy})}{(I_{bz}+I_{fz})}\{\Omega^2 cos(\phi_w)^2 sin(\phi_b)cos(\phi_b)\}+ \\ \frac{1}{(I_{bz}+I_{fz})}(m_b h_b + m_f h_f) g \sin(\phi_b) \\ \\ 0\end{bmatrix} + \\ \begin{bmatrix}0 \\ \\ \frac{(I_{fx}-I_{fy})}{(I_{bz}+I_{fz})}\{-\Omega \cos(\phi_w)(\cos(2\phi_b) + I_{fz})\dot{\phi_w}\} \\ \\ 1\end{bmatrix}\dot{\phi_w}\end{aligned} \tag{4}$$

B. Control Barrier Equations

Requirement is:

$$\begin{aligned} |\phi_b| &\leq \phi_{bMax} \\ |\dot{\phi}_b| &\leq \dot{\phi}_{bMax} \\ |\phi_w| &\leq \phi_{wMax} \end{aligned} \tag{5}$$

So CBFs are defined as:

$$\begin{aligned} |\phi_b| &\leq \phi_{bMax} \\ |\dot{\phi}_b| &\leq \dot{\phi}_{bMax} \\ |\phi_w| &\leq \phi_{wMax} \end{aligned} \tag{6}$$

This leads to siz CBF equations:

$$\begin{aligned} h_1(X): &\quad 0 \leq \phi_{bMax} - \phi_b \\ h_2(X): &\quad 0 \leq \phi_{bMax} + \phi_b \\ h_3(X): &\quad 0 \leq \dot{\phi}_{bMax} - \dot{\phi}_b \\ h_4(X): &\quad 0 \leq \dot{\phi}_{bMax} + \dot{\phi}_b \\ h_5(X): &\quad 0 \leq \phi_{wMax} - \phi_w \\ h_6(X): &\quad 0 \leq \phi_{wMax} + \phi_w \end{aligned} \tag{7}$$

C. QP Solution

$$\begin{aligned} u^* = \arg\min_{u \epsilon U} \frac{1}{2} ||u_{des} - u_{asif}||^2 \\ s.t.\ L_f h_i + L_g h_i u + \alpha h_i \geq 0 \end{aligned} \tag{8}$$

The Lie derivative of a scalar function such as h (as opposed to a vector function) is given by:

$$\begin{aligned} Lfh &= \nabla h f(X) \\ &= \Big\langle [\frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}, \frac{\delta}{\delta x_3}], h \Big\rangle f(X) \\ &= \Big\langle [\frac{\delta}{\delta \phi_b}, \frac{\delta}{\delta \dot{\phi}_b}, \frac{\delta}{\delta \phi_w}], h \Big\rangle f(X) \end{aligned} \tag{9}$$

Thus, the Lie derivative inequalities provided above in the squared error optimizer are:

$$\begin{aligned} & L_f h_1 + L_g h_1 u + \alpha h_1 \geq 0 \\ & \Big\langle [-1,\ 0,\ 0], h_1(X) \Big\rangle f(X) + \Big\langle [-1,\ 0,\ 0], h_1(X) \Big\rangle g(X) u + \alpha h_1 \geq 0 \\ & - f_1(X) + 0 + \alpha h_6 \geq 0 \\ & - \dot{\phi}_b + 0 + \alpha(\phi_{bMax} - \phi_b) \geq 0 \end{aligned} \tag{10}$$

$$\begin{aligned} & L_f h_2 + L_g h_2 u + \alpha h_2 \geq 0 \\ & \Big\langle [+1,\ 0,\ 0], h_2(X) \Big\rangle f(X) + \Big\langle [+1,\ 0,\ 0], h_2(X) \Big\rangle g(X) u + \alpha h_2 \geq 0 \\ & + f_1(X) + 0 + \alpha h_6 \geq 0 \\ & + \dot{\phi}_b + 0 + \alpha(\phi_{bMax} + \phi_b) \geq 0 \end{aligned} \tag{11}$$

$$
\begin{aligned}
&L_f h_3 + L_g h_3 u + \alpha h_3 \geq 0\\
&\Big\langle [0,\ -1,\ 0], h_3(X) \Big\rangle f(X) + \Big\langle [0,\ -1,\ 0], h_3(X) \Big\rangle g(X)u + \alpha h_3 \geq 0\\
&- f_2(X) - g_2(X)u + \alpha h_3 \geq 0\\
&\\
&- \left[\frac{(I_{fx} - I_{fy})}{(I_{bz} + I_{fz})} \{ \Omega^2 cos(\phi_w)^2 sin(\phi_b) cos(\phi_b) \} + \right.\\
&\left. \frac{1}{(I_{bz} + I_{fz})} (m_b h_b + m_f h_f) g \sin(\phi_b) \right]\\
&- \left[\frac{(I_{fx} - I_{fy})}{(I_{bz} + I_{fz})} \{ -\Omega \cos(\phi_w)(\cos(2\phi_b) + I_{fz}) \dot{\phi_w} \} \right]\\
&+ \alpha(\dot{\phi}_{bMax} - \dot{\phi_b}) \geq 0
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&L_f h_3 + L_g h_3 u + \alpha h_3 \geq 0\\
&\Big\langle [0,\ -1,\ 0], h_3(X) \Big\rangle f(X) + \Big\langle [0,\ -1,\ 0], h_3(X) \Big\rangle g(X)u + \alpha h_3 \geq 0\\
&- f_2(X) - g_2(X)u + \alpha h_3 \geq 0\\
&\\
&+ \left[\frac{(I_{fx} - I_{fy})}{(I_{bz} + I_{fz})} \{ \Omega^2 cos(\phi_w)^2 sin(\phi_b) cos(\phi_b) \} + \right.\\
&\left. \frac{1}{(I_{bz} + I_{fz})} (m_b h_b + m_f h_f) g \sin(\phi_b) \right]\\
&+ \left[\frac{(I_{fx} - I_{fy})}{(I_{bz} + I_{fz})} \{ -\Omega \cos(\phi_w)(\cos(2\phi_b) + I_{fz}) \dot{\phi_w} \} \right]\\
&+ \alpha(\dot{\phi}_{bMax} + \dot{\phi_b}) \geq 0
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&L_f h_5 + L_g h_5 u + \alpha h_5 \geq 0\\
&\Big\langle [0,\ 0,\ -1], h_5(X) \Big\rangle f(X) + \Big\langle [0,\ 0,\ -1], h_5(X) \Big\rangle g(X)u + \alpha h_5 \geq 0\\
&0 - g_3(X)u + \alpha h_6 \geq 0\\
&0 - \dot{\phi_w} + \alpha(\phi_{wMax} - \phi_w) \geq 0
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&L_f h_6 + L_g h_6 u + \alpha h_6 \geq 0\\
&\Big\langle [0,\ 0,\ -1], h_6(X) \Big\rangle f(X) + \Big\langle [0,\ 0,\ -1], h_6(X) \Big\rangle g(X)u + \alpha h_6 \geq 0\\
&0 + g_3(X)u + \alpha h_6 \geq 0\\
&0 + \dot{\phi_w} + \alpha(\phi_{wMax} + \phi_w) \geq 0
\end{aligned} \tag{15}
$$

D. Calculation of α in QP Equations

α is a class Kappa function defined as:

$$
\begin{aligned}
&\alpha(r) \triangleq - \min_{\substack{x \epsilon A(r)\\ u \epsilon U}} \left(\min_{i \epsilon [i,n]} L_f h_i + L_g h_i u \right)\\
&\text{where,}\\
&A(r) \triangleq \{ x\ \epsilon\ S \mid \exists\ i\ \epsilon\ [1,n],\ 0 \leq h_i(X) \leq r \} \subseteq S\\
&\text{where, S is not just the safety set, but is the VIABLE safety set.}
\end{aligned} \tag{16}
$$

In its expanded form this means: α is a class Kappa function defined as:

$$\alpha(r) \triangleq - \min_{\substack{x \epsilon A(r) \\ u \epsilon U}} \left(\min \begin{bmatrix} L_f h_1 + L_g h_1 u \\ L_f h_2 + L_g h_2 u \\ L_f h_3 + L_g h_3 u \\ L_f h_4 + L_g h_4 u \\ L_f h_5 + L_g h_5 u \\ L_f h_6 + L_g h_6 u \end{bmatrix} \right)$$

with the constraints placed by the $A(r)$ definition/inequality

$$\begin{aligned} 0 \leq h_1 \leq r \\ 0 \leq h_2 \leq r \\ 0 \leq h_3 \leq r \\ 0 \leq h_4 \leq r \\ 0 \leq h_5 \leq r \\ 0 \leq h_6 \leq r \end{aligned} \tag{17}$$

The inequalities $0 \leq h_i \leq r$ place the additional constraint/relation between the α Kappa function's r and the CBF state constraints of (obtained by summing up similar CBF equations and simplifying):

$$\begin{aligned} -\frac{r}{2} \leq \phi_b \leq +\frac{r}{2} \\ -\frac{r}{2} \leq \dot{\phi}_b \leq +\frac{r}{2} \\ -\frac{r}{2} \leq \phi_w \leq +\frac{r}{2} \end{aligned} \tag{18}$$

So CBF max state values chosen should be within the overall $r/2$ bounds placed by the class Kappa function.

When we put the $\alpha(r)$ equation in words, it effectively means:

- Create an empty 2D table of size $x \times u$ to represent all combos of states and controls
- The number of states itself that can be created/checked is defined by all states within the viable safety set which satisfy $0 \leq h_i \leq r$ given in the $A(r)$ equation, and expanded in (17)'s inequalities.
- For each of the entries in the 2D matrix, calculate a $n \times 1$ vector corresponding to each of the CBF equations.
- Now for each of the vectors in the 2D matrix, calculate and store in-place the minimum of that vector
- Then, calculate the min over the whole 2D matrix
- Negate this value to obtain the value of α

Ofcourse, all these steps can be vectorized and handled more efficiently using an optimizaiton toolbox like fmincon() in MATLAB.