Modeling ECG Waveform Using Optimal Smoothing Bézier-Bernstein Curves

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***Abstract*—This paper considers a problem for modeling mathematically waveform of some Electrocardiogram (ECG) which is the process of measuring and recording the electrical activity of the heart over a period of time. Such a modeling is done by using the smoothing Bézier-Bernstein curves. A concise representation for designing the optimal curves with high precision is derived, which has the additional merit of lending itself to the development of computational procedures in a straightforward manner. The usefulness and effectiveness are demonstrated by some experimental studies.**

I. Introduction

Electrocardiogram (ECG) has been widely used as a clinical tool for detecting and diagnosing on some cardiac conditions (see e.g. [1]). Such a ECG signal is measured and recorded as the electrical activity of the heart by using the ECG machine ‘Electrocardiography’. Even when we use such a ECG machine, we often face some difficulties to understand the clinical significance of uncommon findings on ECG with pathological condition.

For solving the above difficulties, a natural idea is to improve the knowledge of ECG machines for uncommon case by using computer-based simulations [2]. Therein, the so-called mathematical modeling on ECG waveform may play a key role in order to link the models on the electrical active of heart to the ECG waveform with high precision. In addition, the low complexity computation may be required for implementing the ECG simulation. Therefore, such a mathematical ECG modeling has been studied by several researchers.

For example, Desyoo et al. in [3] have developed mathematical model by employing an Kernel approach. Therein, ECG waveform is modeled as some curves using some kernel functions. Similar works have been exhibited in Chutchavong et al’s work [4] and Kubicek et al’s work [5], but they respectively have used Bernstein polynomial and Fourier series. In these works, the central issue is to determine a series of weight coefficients on some functions (e.g. Kernel function, Bernstein polynomial and Fourier series) so that the constituted curves can approximate the ECG waveform well. However, the series of weight has been determined by hand or simple optimization techniques – such as discrete least square technique, in [3], [4], [5]. Hence, the above methods will not work well in a case where ECG signal is corrupted by some observation noises due to the digitalization on electrical data, etc.

In this paper, we considers a problem for modeling mathematically ECG waveform by Bézier-Bernstein polynomial. This basic idea for the modeling is similar to Chutchavong et al’s work [4], but the big difference with their work is to develop a modeling method by using the theory of smoothing [6], [7]. Thus, our proposed modeling method will be robust against some noises of ECG data. We here derive a concise representation for designing the optimal smoothing curves with high precision, which has the additional merit of lending itself to the development of computational procedures in a straightforward manner. The usefulness and effectiveness are demonstrated by some experimental studies.

II. Problem Statement

We first give a problem statement on mathematical modeling of ECG waveform using Bézier-Bernstein curves [4].

Figure 1 illustrates a one-cycle ECG waveform [8] which is measured and stored at a sampling rate by using electrodes placed on some examinee’s skin, where we note that the trend of ECG waveform has been removed by an appropriate method. Such a ECG waveform may be basically represented by three types of waves, i.e. P wave, QRS wave and T wave (see Figure 1)[1]. What we want to do here is to mathematically model each of three waves with high precision as well as low computational complexity.

Now, suppose that a set of one-cycle ECG data (i.e. voltage [mV]) $\mathcal{D}_k,\ k \in \{\mathrm{P}, \mathrm{QRS}, \mathrm{T}\}$ consisting of P wave, QRS wave and T wave is given as

$$\mathcal{D}_k = \left\{ (s_i, d_i) : s_i = \frac{i}{N_k} \in [0,1], \; d_i \in \mathbf{R},\ i = 1, \cdots, N_k \right\}. \quad (1)$$

[1]Note here that U wave is not considered since the corresponding amplitude is generally low and it is overlapped with T wave.

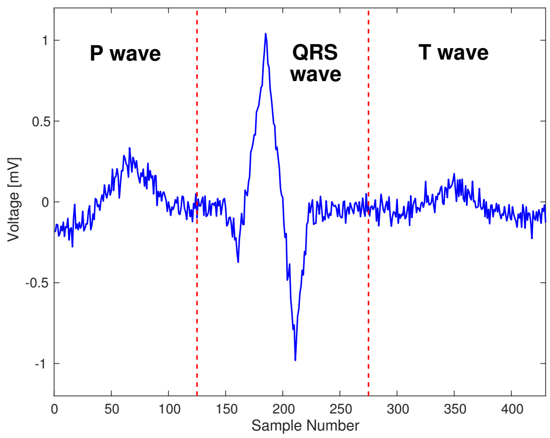

Fig. 1. Measured ECG data.

We then consider to approximate a set of ECG data $\mathcal{D}_k$ by Bézier-Bernstein curves $x_k(t) \in \mathbf{R},\ t \in [0,1]$ which is constituted as linear combination of Bernstein basis polynomials,

$$x_k(t) = \sum_{i=0}^{n} p_i B_i^n(t). \tag{2}$$

Here, p_i's in (2) are 'control points' called as Bernstein coefficients. Also, $B_i^n(\cdot)$ is Bernstein polynomials of degree n defined by

$$B_i^n(t) = \left(\begin{array}{c} n \\ i \end{array} \right) t^i (1-t)^{n-i},\ \ i = 1, 2, \cdots, n \tag{3}$$

where $\left(\begin{array}{c} n \\ i \end{array} \right)$ is a binomial coefficient, i.e.

$$\left(\begin{array}{c} n \\ i \end{array} \right) = \frac{n!}{i!(n-i)!}.$$

For example, cubic Bernstein polynomials, i.e. $B_i^n(t)$ for the case of $n = 3$ is given as

$$B_i^3(t) = \begin{cases} (1-t)^3 & i = 0 \\ 3t(1-t)^2 & i = 1 \\ 3t^2(1-t) & i = 2 \\ t^3 & i = 3 \\ 0 & \mathrm{otherwise.} \end{cases} \tag{4}$$

Choosing the control points p_is' in (2) appropriately, a class of Bézier-Bernstein curves $x_k(t)$ of degree n can be generated on the interval $[0,1]$ for t. Then, the problem of modeling the ECG waveform can be stated as follows:

Problem 1 (ECG Waveform Modeling): Suppose that a set of ECG data $\mathcal{D}_k,\ k \in \{\mathrm{P}, \mathrm{QRS}, \mathrm{T}\}$ is given. Then, find a sequence of control points $p_i,\ i = 0, 1, \cdots, n$ in (2) so that $x_k(t)$ approximates the data set $\mathcal{D}_k$ well.

III. Mathematical Modeling using Optimal Smoothing Bézier-Bernstein Curves

As shown in Figure 1, we see that ECG data set $\mathcal{D}_k$ in (1) is often corrupted by some observation noises due to the digitalization on electrical data, etc. Then, a natural way for solving Problem 1 will be to use the theory of smoothing [6], [7], which enables us to constitute $x_k(t)$ not only close to the given data $\mathcal{D}_k$ in (1) but also sufficiently smooth. Here, we develop a method for designing optimal smoothing Bézier-Bernstein curves in the sequel. Then, the optimal solution is derived in Section III-A. In Section III-B, we finally summarize the procedures for modeling ECG waveform.

Now suppose that we are given a set of data $\mathcal{D}_k$ in (1). Also, let $p \in \mathbf{R}^M$ $(M = n+1)$ be a vector consisting of control points p_i's defined as

$$p = [p_0\ p_1\ \cdots\ p_n]^T. \tag{5}$$

Then, a problem of optimal smoothing Bézier-Bernstein curves is stated as follows:

Problem 2 (Optimal Smoothing Bézier-Bernstein Curves): Find $x(t)$, or equivalently the vector p in (5), such that

$$\min_{p \in \mathbf{R}^M} J(p) = \lambda \int_0^1 \left(x^{(2)}(t) \right)^2 dt + \sum_{i=1}^{N_k} w_i \left(x(s_i) - d_i \right)^2. \tag{6}$$

Here, $\lambda (> 0)$ denotes a smoothing parameter, $w_i (0 < w_i \leq 1)\ \forall i$ are the weights for approximation errors.

A. Optimal Solution

Optimal solution of Problem 2 can be obtained as follows: Letting $b(t) \in \mathbf{R}^M$ be a vector consisting of Bernstein polynomial with degree n as

$$b(t) = \left[B_0^n(t)\ B_1^n(t)\ \cdots\ B_n^n(t) \right]^T, \tag{7}$$

then Bézier-Bernstein curves $x_k(t)$ in (2) is expressed by

$$x_k(t) = b^T(t) p. \tag{8}$$

Then, the integral term in the cost (6) can be written as

$$\int_0^1 \left(x^{(2)}(t) \right)^2 dt = p^T Q p. \tag{9}$$

Here $Q \in \mathbf{R}^{M \times M}$ is Gramian matrix defined by

$$Q = \int_0^1 \left(\frac{d^2 b(t)}{dt^2} \right) \left(\frac{d^2 b(t)}{dt^2} \right)^T dt. \tag{10}$$

This Gramian Q can be computed readily by using the Bernstein polynomial in (3).

On the other hand, the second term is written as

$$\sum_{i=1}^{N_k} w_i \left(x(s_i) - d_i \right)^2 = \left(B^T p - d^k \right)^T W_k \left(B^T p - d^k \right), \tag{11}$$

where $B \in \mathbf{R}^{M \times N_k}$, $W_k \in \mathbf{R}^{N_k \times N_k}$ and $d_k \in \mathbf{R}^{N_k}$ are defined by

$$B = [b(s_1)\ b(s_2)\ \cdots\ b(s_{N_k})] \tag{12}$$

$$W_k = \mathrm{diag}\{w_1, w_2, w_{N_k}\} \tag{13}$$

$$d^k = [d_1\ d_2\ \cdots\ d_{N_k}]^T. \tag{14}$$

Then, the cost function $J(p)$ in (6) is expressed as

$$J(p) = p^T\left(\lambda Q + BWB^T\right)p - 2p^T BWd^k + \mathrm{const.} \tag{15}$$

Hence, we see that the optimal solution of Problem 2 is obtained as a solution of

$$\left(\lambda Q + BWB^T\right)p = BWd^k. \tag{16}$$

Note that the equation (16) has at least one solution, since in general the relation

$$\mathrm{rank}[S + UU^T] = \mathrm{rank}[S + UU^T, Uv]$$

holds for any matrices $S = S^T \geq 0$, U and vector v of compatible dimensions. Obviously, the solution of Problem 2 is a unique if and only if $\lambda Q + BWB^T > 0$ (see e.g. [9]).

B. Procedure for Mathematical ECG Modeling

We are now in the position to summarize a procedure for mathematical ECG modeling using the optimal smoothing Bézier-Bernstein curves. Then, the procedure can be summarized as follows.

Procedure 1 (Procedure for Mathematical ECG Modeling): The following procedures (P1)-(P5) for modeling ECG waveform is carried out for each of data set $\mathcal{D}_k, k = \{\mathrm{P}, \mathrm{QRS}, \mathrm{T}\}$.

(P1) Set up degree $n(\geq 3)$ of Bernstein polynomial.
(P2) Specify the design parameter λ and W in (16).
(P3) Compute the matrices Q and B.
(P4) Find p by solving the linear algebraic equation (16).
(P5) Construct a mathematical model as the Bézier-Bernstein curves $x_k(t)$ by (2).

One of our concern in Procedure 1 may be to select an optimal degree $n(\geq 3)$. Modeling of ECG waveform with such a optimal selection of degree n is achieved by following procedure.

Procedure 2 (Mathematical ECG Modeling with Optimal Selection of degree n): Let $p(\geq 3) \in \mathbf{N}$ and $\epsilon(> 0) \in \mathbf{R}$ be a positive integer and a threshold respectively. Then the following procedures (S1)-(S5) is carried out for each of data set $\mathcal{D}_k, k = \{\mathrm{P}, \mathrm{QRS}, \mathrm{T}\}$.

(S1) Set up $n = 3$, p and ϵ.
(S2) Construct a mathematical model by Procedure 1.
(S3) Compute the error function $E(n)$ between data set $\mathcal{D}_k$ and $x_k(s_i)$, i.e.

$$E(n) = \frac{1}{N_k}\sum_{i=1}^{N_k}\left(x_k(s_i) - d_i^k\right)^2. \tag{17}$$

(S4) If $E(n) > \epsilon$ holds, then go to (S2) with $n = n + 1$.
(S5) If $E(n) \leq \epsilon$ or $n \geq p$ holds, then we stop this procedure and get the ECG model in (S2).

IV. Experimental Studies

We demonstrate the performance of our proposed method by experimental studies. As an example, we here apply our modeling method in Section III to the ECG data in Figure 1. Then, a data set $\mathcal{D}_k$ on P wave ($k = \mathrm{P}$), QRS wave ($k = \mathrm{QRS}$) and T wave ($k = \mathrm{T}$) are given by the method in Section II, and the corresponding numbers N_k are $N_\mathrm{P} = 126$, $N_\mathrm{QRS} = 150$ and $N_\mathrm{P} = 155$ respectively. Then we constitute a Bézier-Bernstein curves for their ECG data by Procedure 2 and then get the models of ECG waveform. Here, we set the design parameters λ and W in (16) as $\lambda = 10^{-9}$ and $W = I_{N_k}$. Figure 2 (a)-(c) illustrates the results on error function $E(n)$ for P, QRS and T waves, which are computed by (17). Also, we here set a threshold ϵ as $\epsilon = 2.6 \times 10^{-3}$, $\epsilon = 6.4 \times 10^{-3}$ and $\epsilon = 2.2 \times 10^{-3}$ for P, QRS and T waves respectively. Then the corresponding optimal degree $n^\star$ was selected as $n^\star = 6$, $n^\star = 13$ and $n^\star = 8$. Also, the curves constituted by (2) are illustrated in Figure 3 (a)-(c) and the corresponding mathematical models are given as

$$\begin{aligned} x_P(t) &= -62.1016t^6 + 192.8810t^5 - 217.0541t^4 \\ &\quad +104.4438t^3 - 19.4674t^2 + 1.4411t - 0.1920 \end{aligned} \tag{18}$$

$$\begin{aligned} x_{QRS}(t) &= -2161500t^{13} + 13411000t^{12} - 36325000t^{11} \\ &\quad +56309000t^{10} - 55003000t^9 + 35129000t^8 \\ &\quad -14730000t^7 + 3967100t^6 - 651850t^5 + 59449t^4 \\ &\quad -2505.5t^3 + 31.8703t^2 - 0.6130t + 0.0014 \end{aligned} \tag{19}$$

$$\begin{aligned} x_T(t) &= 325.0174t^8 - 1223.7t^7 + 1805.8t^6 - 1300t^5 \\ &\quad +455.1791t^4 - 63.2424t^3 + 0.6160t^2 + 0.2663t \\ &\quad -0.0521. \end{aligned} \tag{20}$$

The blue and red lines in Figure 3 (a)-(c) denote ECG data and the constituted curves by the mathematical ECG model in (18)-(20). In addition, the reconstructed one-cycle ECG waveform by merging the mathematical ECG models in (18)-(20) is plotted in Figure 4. From these results, we see that our proposed modeling method works quite well. In particular, we may observe that the obtained model enables us to implement the ECG simulation with not only high precision but also lower complexity computation.

V. Concluding Remarks

In this paper, we developed a method for modeling mathematically ECG waveform by employing the smoothing Bézier-Bernstein curves. The central issue was how to determine the control points of Bézier-Bernstein for a given set of ECG data. We here derived a concise representation for designing the optimal smoothing curves with high precision is derived, which has the additional merit of lending itself to the development of computational procedures in a straightforward manner. The usefulness and effectiveness were demonstrated by some experimental studies. In particular, it is shown that the proposed modeling method is robust against the corrupt noises of ECG data.

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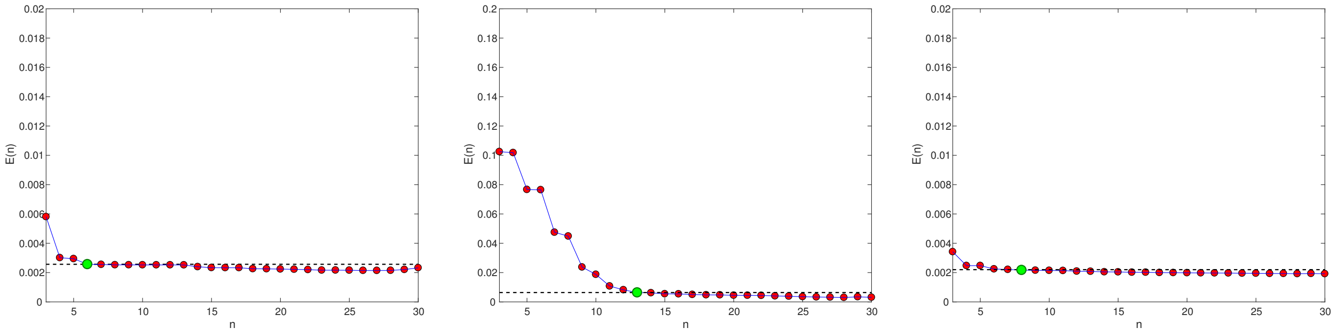
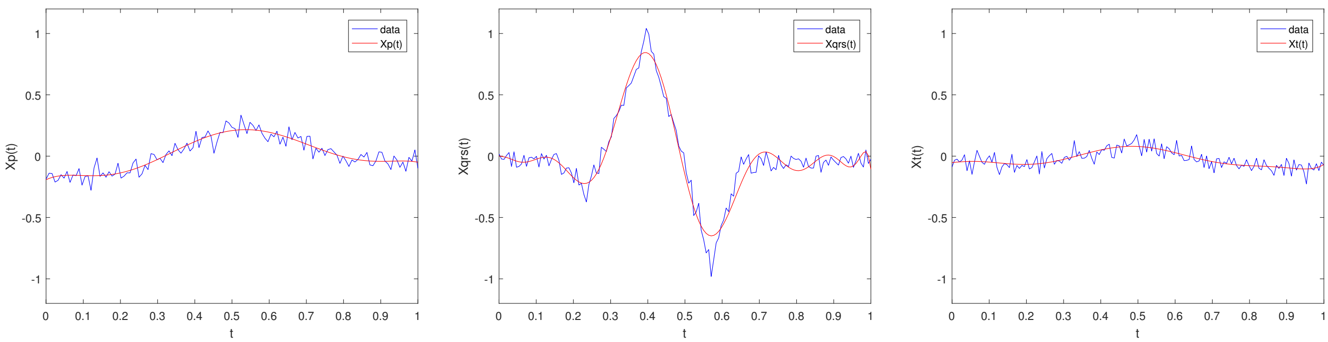


(a) P wave (b) QRS wave (c) T wave

Fig. 2. Computed error function $E(n)$ (red circles) and the optimal selection (green circle).



(a) P wave (b) QRS wave (c) T wave

Fig. 3. Smoothing Bézier-Bernstein curves.

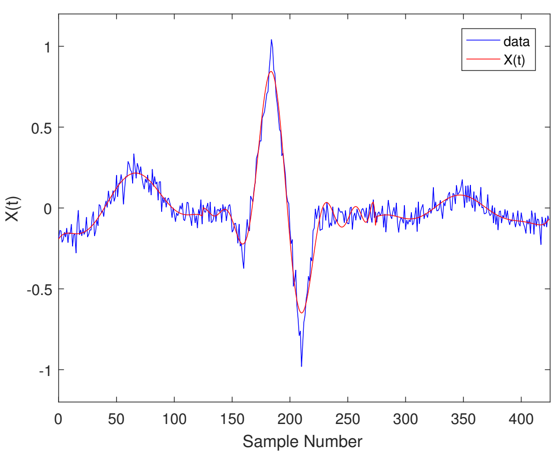


Fig. 4. Reconstructed ECG models by merging P, QRS and T waves in Figure 3.

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