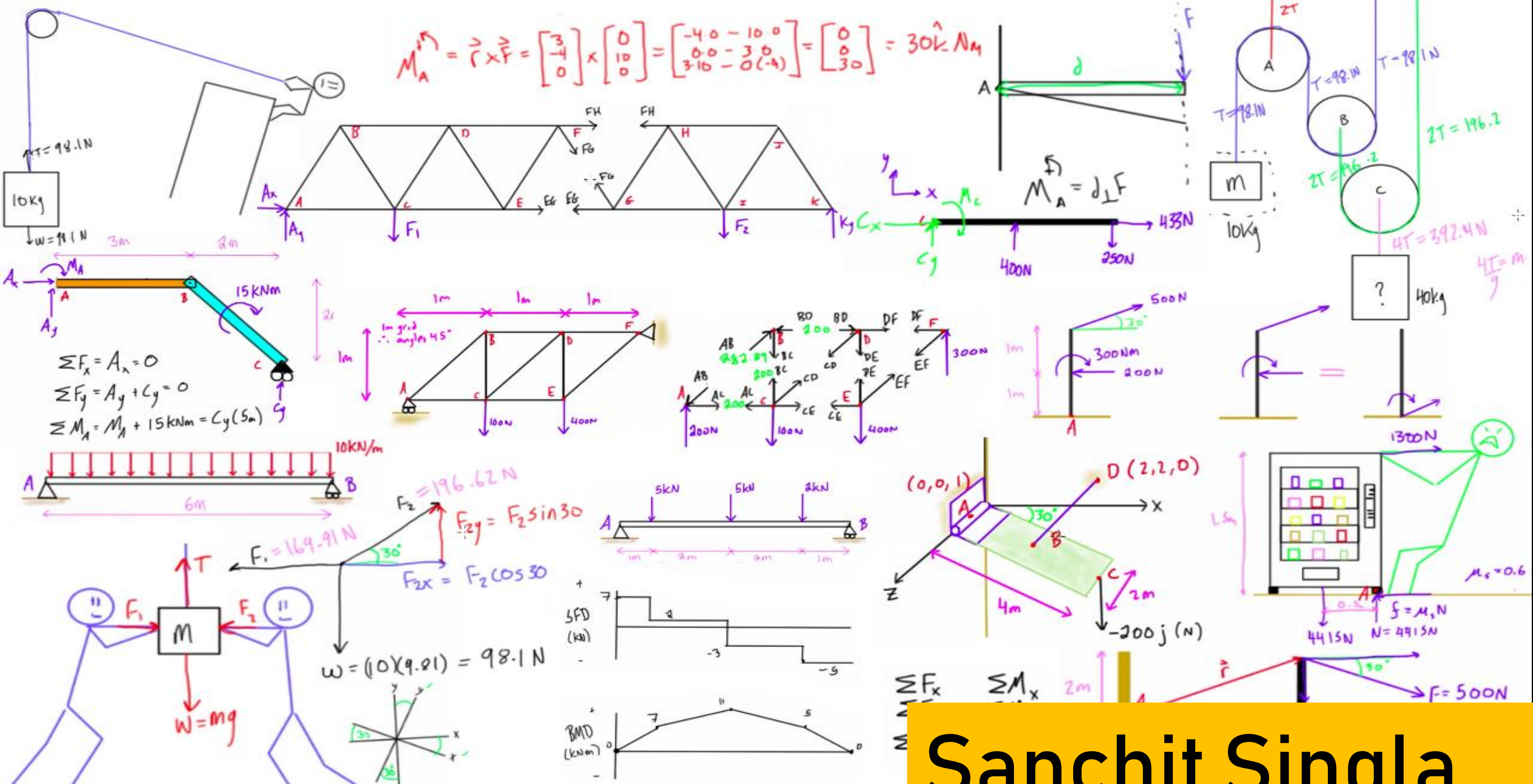


Sanchit Singla
Assistant Professor

MEC107
Basic Engineering Mechanics

Learning Outcomes

After this lecture, you will be able to

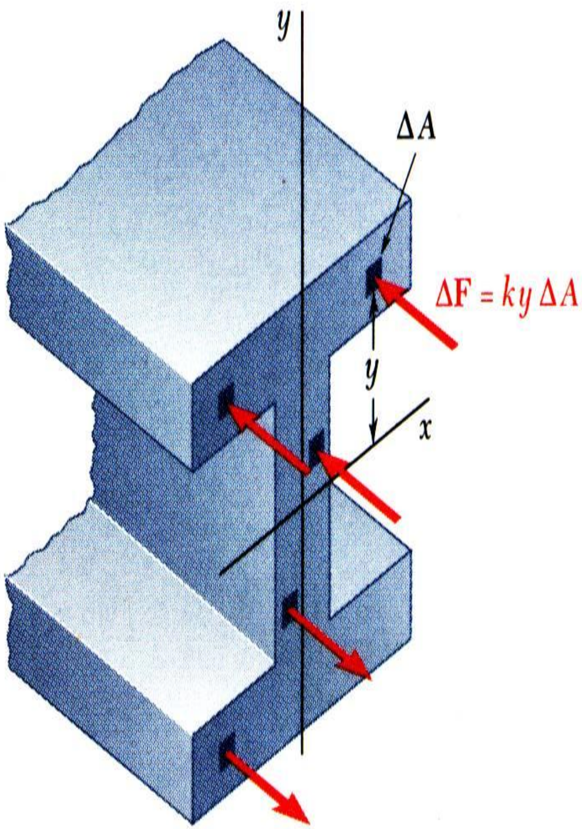
- ✓ learn about Moment of Inetia of a n area.
- ✓ understand Moment of Inertia of an Area by Integration.
- ✓ know about Polar Moment of Inertia.
- ✓ understand about Radius of Gyration.
- ✓ learn Parallel Axis Theorem.
- ✓ understand about oments of Inertia of Composite Areas.

INTRODUCTION

- Previously considered distributed forces which were proportional to the area or volume over which they act.
  - The resultant was obtained by summing or integrating over the areas or volumes.
  - The moment of the resultant about any axis was determined by computing the first moments of the areas or volumes about that axis.

- Will now consider forces which are proportional to the area or volume over which they act but also vary linearly with distance from a given axis.
  - It will be shown that the magnitude of the resultant depends on the first moment of the force distribution with respect to the axis.
  - The point of application of the resultant depends on the second moment of the distribution with respect to the axis.

Moment of Inertia of an Area



- Consider distributed forces $\Delta\vec{F}$ whose magnitudes are proportional to the elemental areas ΔA on which they act and also vary linearly with the distance of ΔA from a given axis.

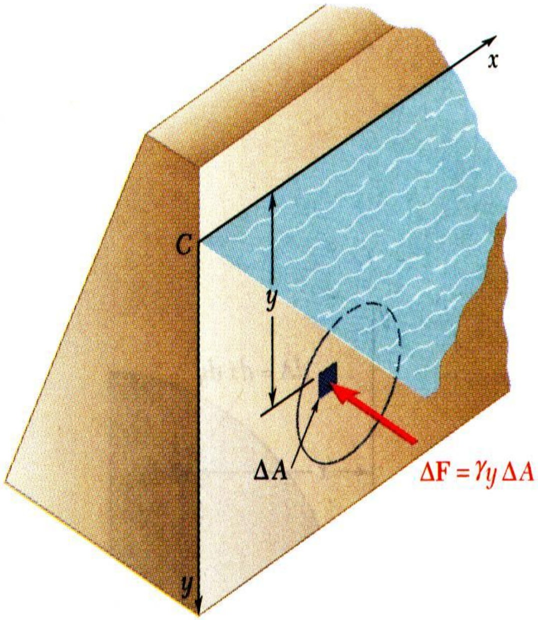
- Example: Consider a beam subjected to pure bending. Internal forces vary linearly with distance from the neutral axis which passes through the section centroid.

$$\Delta\vec{F} = ky\Delta A$$

$$R = k\int y\,dA = 0 \qquad \int y\,dA = Q_x = \text{first moment}$$

$$M = k\int y^2 dA \qquad \int y^2 dA = \text{second moment}$$

Moment of Inertia of an Area



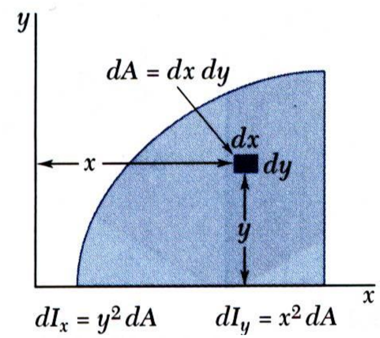
- Example: Consider the net hydrostatic force on a submerged circular gate.

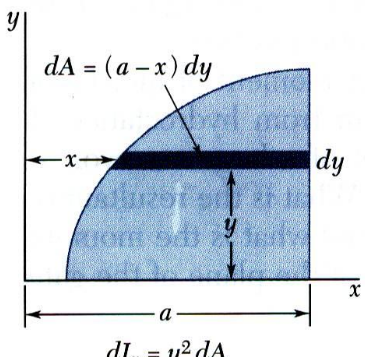
$$\Delta F = p\Delta A = \gamma y \Delta A$$

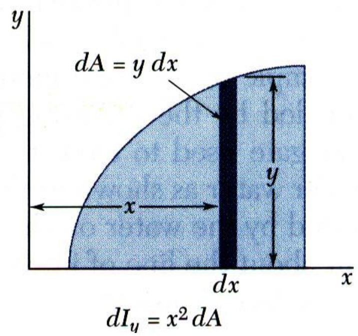
$$R = \gamma \int y \, dA$$

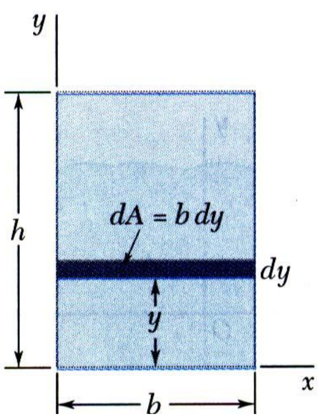
$$M_x = \gamma \int y^2 dA$$

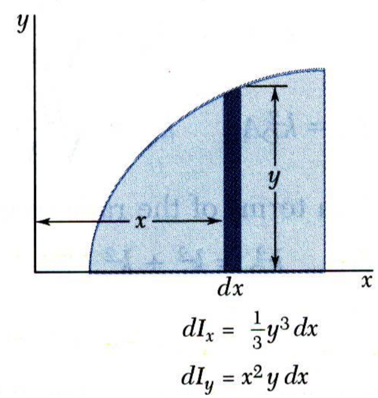
Moment of Inertia of An Area By Integration











- *Second moments* or *moments of inertia* of an area with respect to the *x* and *y* axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing *dA* to be a thin strip parallel to one of the coordinate axes.

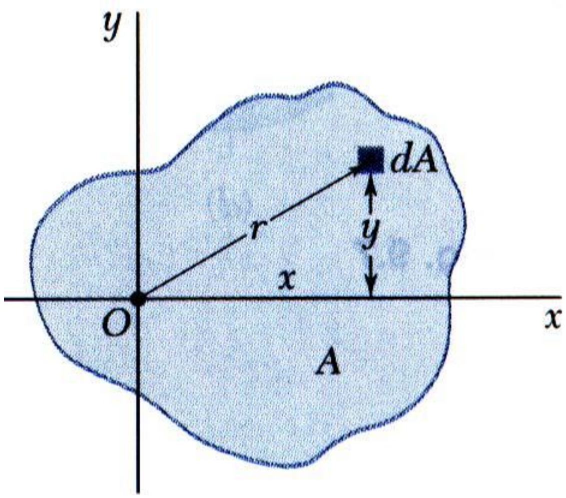
- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b dy = \tfrac{1}{3} bh^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\,dx$$

Polar Moment of Inertia



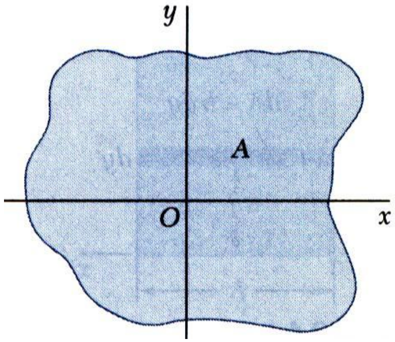
- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

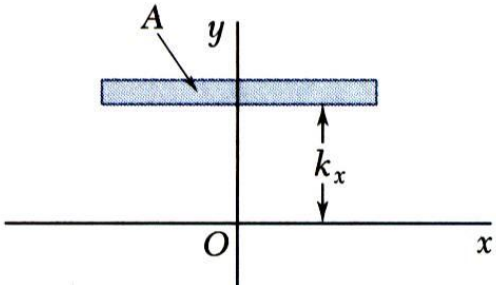
$$J_0 = \int r^2 dA$$

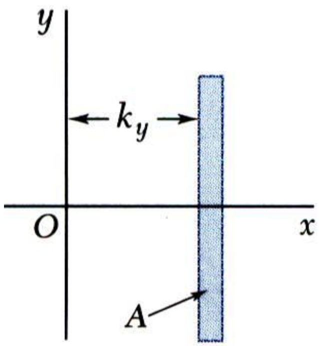
- The polar moment of inertia is related to the rectangular moments of inertia,

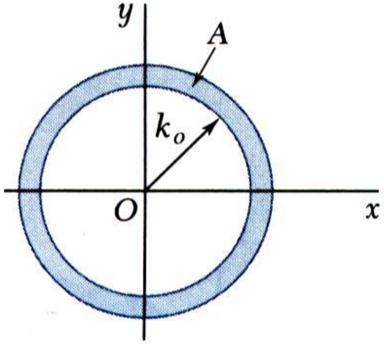
$$J_0 = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

Radius of Gyration of An Area









- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

= *radius of gyration* with respect to the x axis

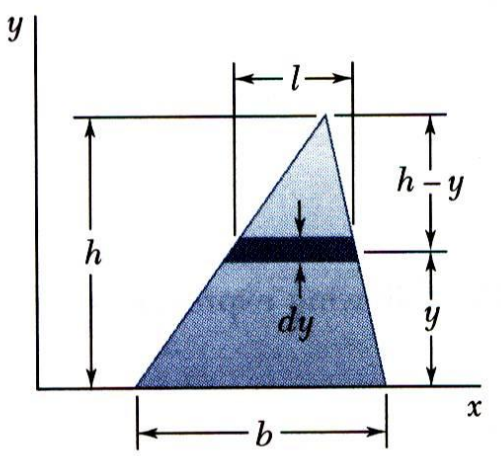
- Similarly,

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \qquad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$

Numerical: 1



Determine the moment of inertia of a triangle with respect to its base.

SOLUTION:

- A differential strip parallel to the *x* axis is chosen for *dA.*

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

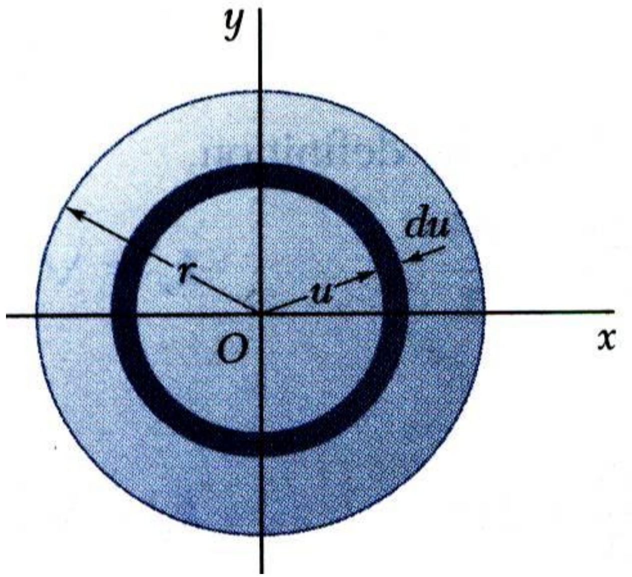
- Integrating dI_x from $y = 0$ to $y = h$,

$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$\boxed{I_x = \frac{bh^3}{12}}$$

Numerical: 2



a) Determine the centroidal polar moment of inertia of a circular area by direct integration.

b) Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter.

SOLUTION:

- An annular differential area element is chosen,

$$dJ_O = u^2 dA \qquad dA = 2\pi u\, du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u\, du) = 2\pi \int_0^r u^3 du$$

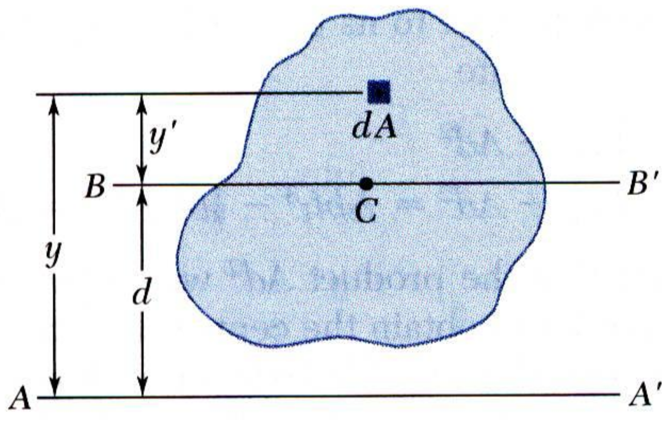
$$\boxed{J_O = \frac{\pi}{2} r^4}$$

- From symmetry, $I_x = I_y$,

$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2} r^4 = 2I_x$$

$$\boxed{I_{diameter} = I_x = \frac{\pi}{4} r^4}$$

Parallel Axis Theorem



- Consider moment of inertia *I* of an area *A* with respect to the axis *AA'*

$$I = \int y^2 dA$$

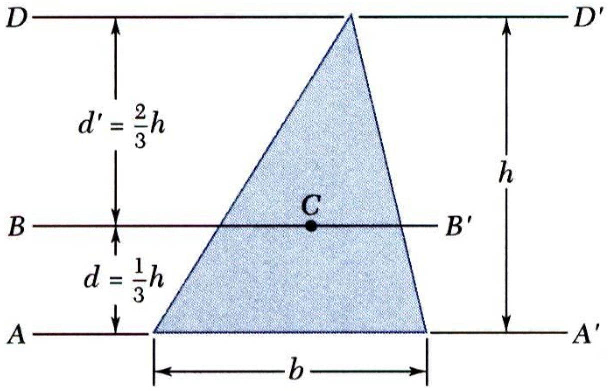
- The axis *BB'* passes through the area centroid and is called a *centroidal axis*.

$$I = \int y^2 dA = \int (y' + d)^2 dA$$

$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$I = \bar{I} + Ad^2 \qquad \textit{parallel axis theorem}$$

Numerical: 3



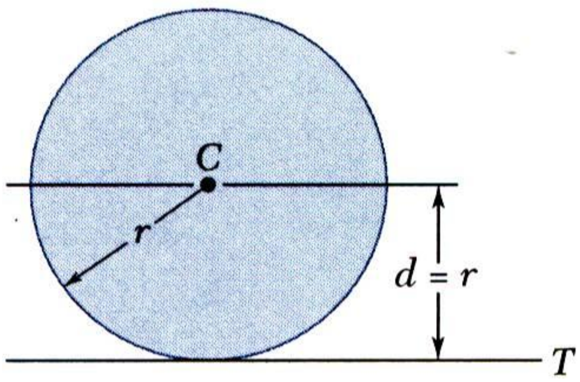
- Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$

$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$

$$= \tfrac{1}{36}bh^3$$

Numerical: 4



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$

$$= \tfrac{5}{4}\pi r^4$$

Perpendicular Axis Theorem

PERPENDICULAR AXIS THEOREM:

The moment of inertia of a plane area about an axis normal to the plane is equal to the sum of the moments of inertia about any two mutually perpendicular axes lying in the plane and passing through the given axis.

Perpendicular Axis Theorem

7.9. THEOREM OF PERPENDICULAR AXIS

It states, *If I_{XX} and I_{YY} be the moments of inertia of a plane section about two perpendicular axis meeting at O, the moment of inertia I_{ZZ} about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:*

$$I_{ZZ} = I_{XX} + I_{YY}$$

Proof :

Consider a small lamina (P) of area da having co-ordinates as x and y along OX and OY two mutually perpendicular axes on a plane section as shown in Fig. 7.4.

Now consider a plane OZ perpendicular to OX and OY. Let (r) be the distance of the lamina (P) from Z-Z axis such that $OP = r$.

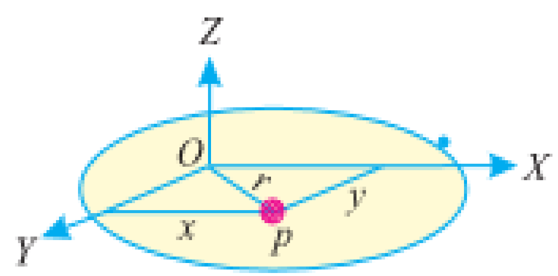


Fig. 7.4. Theorem of perpendicular axis.

From the geometry of the figure, we find that

$$r^2 = x^2 + y^2$$

We know that the moment of inertia of the lamina P about X-X axis,

$$I_{XX} = da.\ y^2 \qquad \text{...}[\because I = \text{Area} \times (\text{Distance})^2]$$

Similarly, $$I_{YY} = da.\ x^2$$

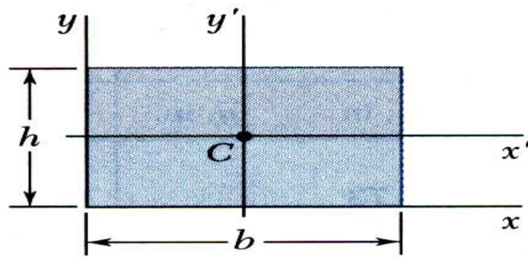
and $$I_{ZZ} = da.\ r^2 = da\ (x^2 + y^2) \qquad \text{...}(\because r^2 = x^2 + y^2)$$

$$= da.\ x^2 + da.\ y^2 = I_{YY} + I_{XX}$$

Moments of Inertia of Composite Areas

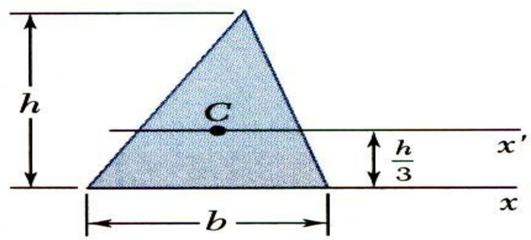
- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

Rectangle



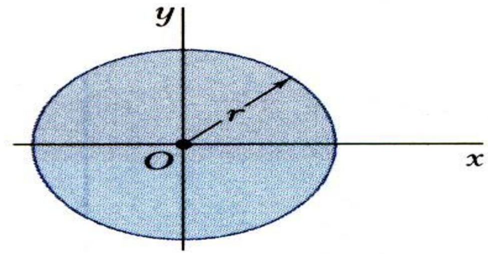
$$\bar{I}_{x'} = \frac{1}{12} bh^3$$
$$\bar{I}_{y'} = \frac{1}{12} b^3h$$
$$I_x = \frac{1}{3} bh^3$$
$$I_y = \frac{1}{3} b^3h$$
$$J_C = \frac{1}{12} bh(b^2 + h^2)$$

Triangle



$$\bar{I}_{x'} = \frac{1}{36} bh^3$$
$$I_x = \frac{1}{12} bh^3$$

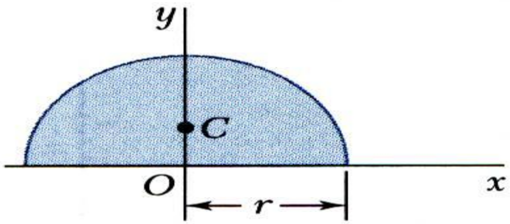
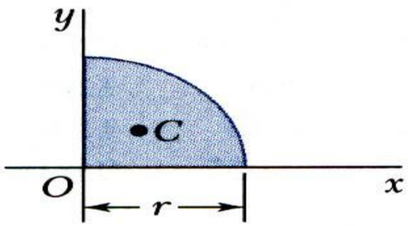
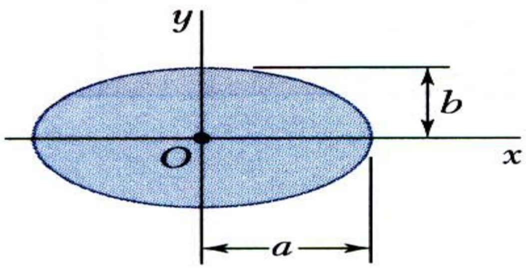
Circle



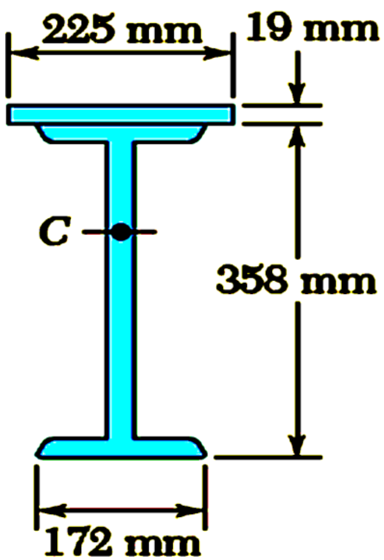
$$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$$
$$J_O = \frac{1}{2} \pi r^4$$

Moments of Inertia of Composite Areas

- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi a b^3$ $\bar{I}_y = \frac{1}{4}\pi a^3 b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Numerical: 5



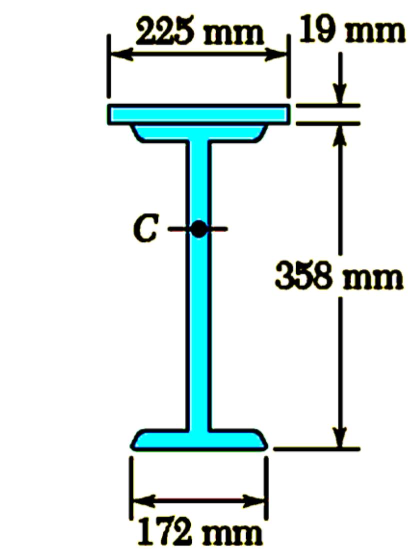
The strength of a 360x57.8 rolled steel beam is increased by attaching a plate to its upper flange.

Determine the moment of inertia and radius of gyration with respect to an axis which is parallel to the plate and passes through the centroid of the section.

SOLUTION:

- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.
- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.
- Calculate the radius of gyration from the moment of inertia of the composite section.

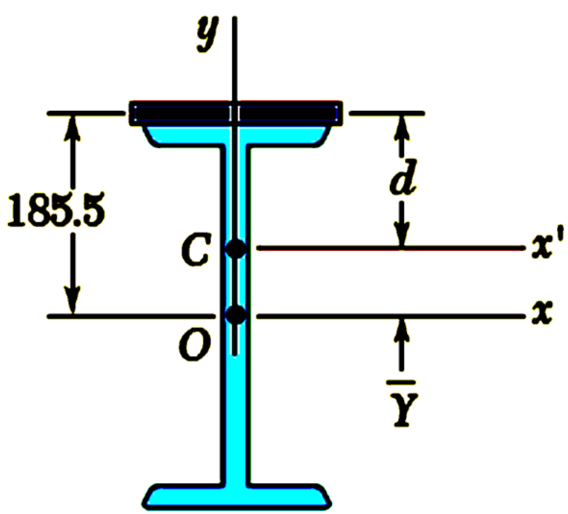
Numerical: 5



SOLUTION:

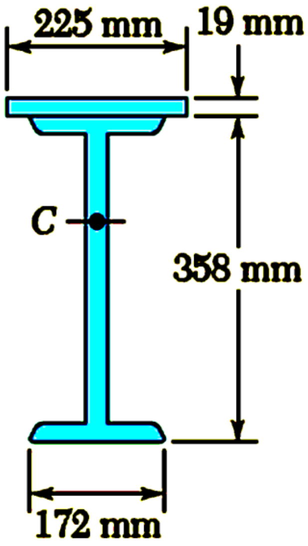
- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.

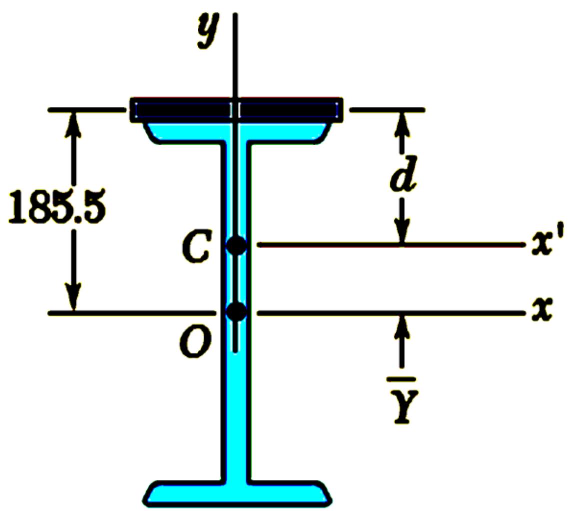
Section	$Area$, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
Plate	4275	188.5	805837.5
Beam Section	7230	0	0
	$\sum A = 11505$		$\sum \bar{y}A = 805837.5$



$$\bar{Y}\sum A = \sum \bar{y}A \qquad \bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{805837.5\text{mm}^3}{11505\text{mm}^2} = 70.04\,\text{mm}$$

Numerical: 5





- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.

$$I_{x',\text{beam section}} = \bar{I}_x + A\bar{Y}^2 = \left(160\times10^6\,\text{mm}^4\right)+\left(7230\text{mm}^2\right)\left(70.04\text{mm}^2\right)$$
$$= 195.47\times10^6\ \text{mm}^4$$
$$I_{x',\text{plate}} = \bar{I}_x + Ad^2 = \tfrac{1}{12}(225\text{mm})(19\text{mm})^3 + \left(4275\text{mm}^2\right)(188.5-70.0)^2$$
$$= 60.12\times10^6\ \text{mm}^4$$

$$I_{x'} = I_{x',\text{beam section}} + I_{x',\text{plate}} = (195.47+60.12)\times10^6\ \text{mm}^4$$

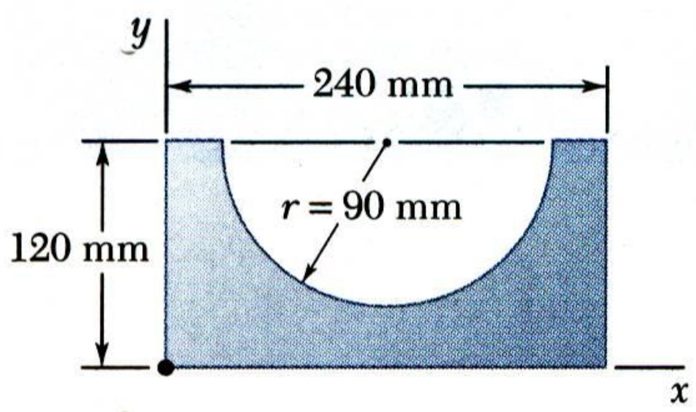
$$\boxed{I_{x'} = 256\times10^6\ \text{mm}^4}$$

- Calculate the radius of gyration from the moment of inertia of the composite section.

$$k_{x'} = \sqrt{\frac{I_{x'}}{A}} = \frac{255.6\times10^6\ \text{mm}^4}{11505\text{mm}^2}$$

$$\boxed{k_{x'} = 149.1\,\text{mm}}$$

Numerical: 6

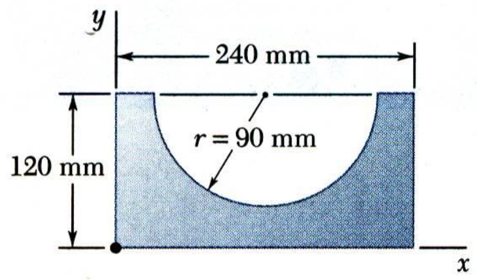


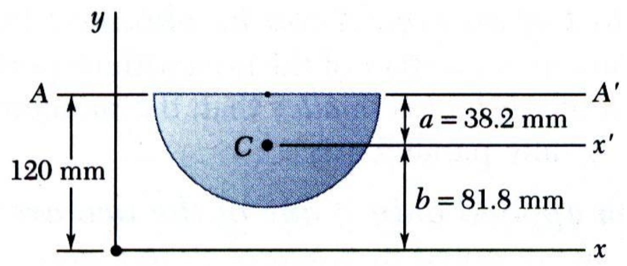
Determine the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

Numerical: 6





$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\ \text{mm}$$

$$\text{b} = 120 - \text{a} = 81.8\ \text{mm}$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$

$$= 12.72 \times 10^3\ \text{mm}^2$$

<u>SOLUTION</u>:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120) = 138.2 \times 10^6\ \text{mm}^4$$

Half-circle:

moment of inertia with respect to *AA'*,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76 \times 10^6\ \text{mm}^4$$

moment of inertia with respect to *x'*,

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = \left(25.76 \times 10^6\right)\left(12.72 \times 10^3\right)$$

$$= 7.20 \times 10^6\ \text{mm}^4$$

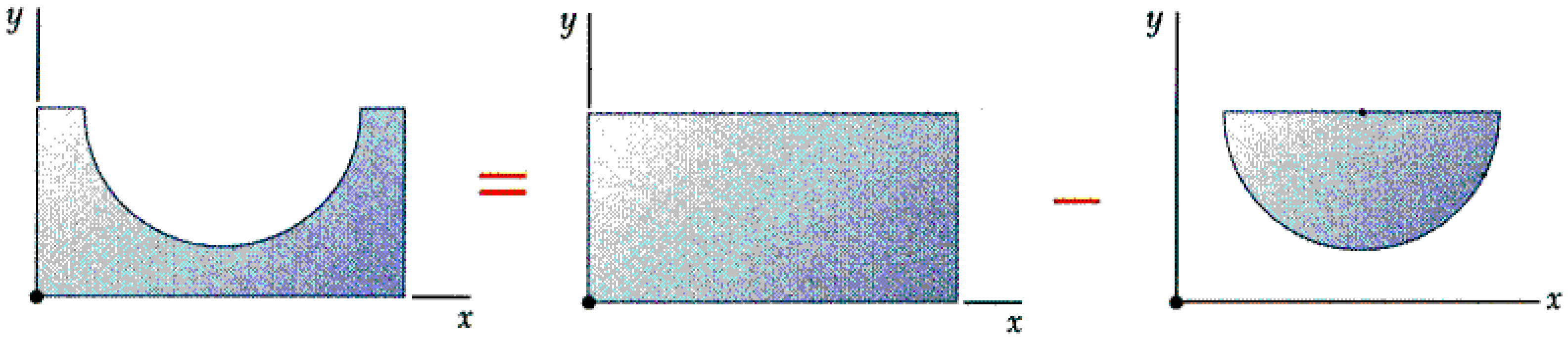
moment of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 + \left(12.72 \times 10^3\right)(81.8)^2$$

$$= 92.3 \times 10^6\ \text{mm}^4$$

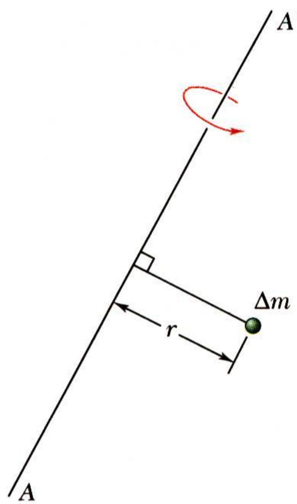
Numerical: 6

- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.



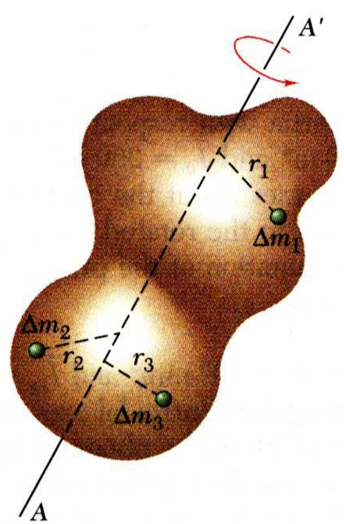
$$\boxed{I_x = 45.9 \times 10^6 \text{mm}^4}$$

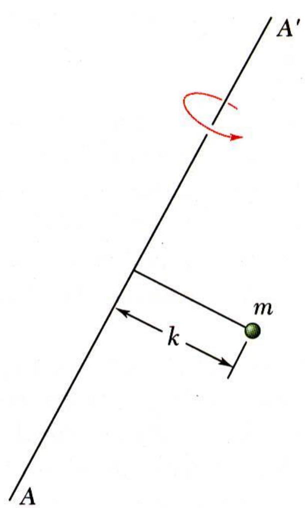
Moment of Inertia of A Mass



- Angular acceleration about the axis *AA'* of the small mass Δm due to the application of a couple is proportional to $r^2\Delta m$.

  $r^2\Delta m$ = *moment of inertia* of the mass Δm with respect to the axis *AA'*





- For a body of mass *m* the resistance to rotation about the axis *AA' is*

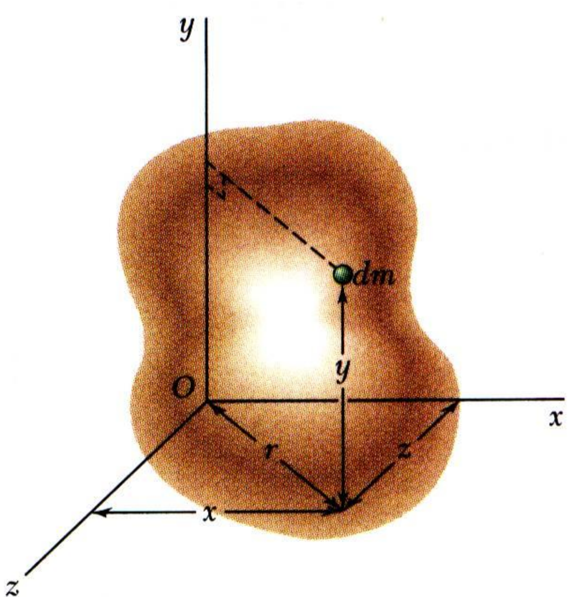
$$I = r_1^2\Delta m + r_2^2\Delta m + r_3^2\Delta m + \cdots$$

$$= \int r^2 dm = mass\ moment\ of\ \ inertia$$

- The radius of gyration for a concentrated mass with equivalent mass moment of inertia is

$$I = k^2 m \qquad k = \sqrt{\frac{I}{m}}$$

Moment of Inertia of A Mass



- Moment of inertia with respect to the *y* coordinate axis is

$$I_y = \int r^2 dm = \int \left(z^2 + x^2\right) dm$$

- Similarly, for the moment of inertia with respect to the *x* and *z* axes,

$$I_x = \int \left(y^2 + z^2\right) dm$$

$$I_z = \int \left(x^2 + y^2\right) dm$$

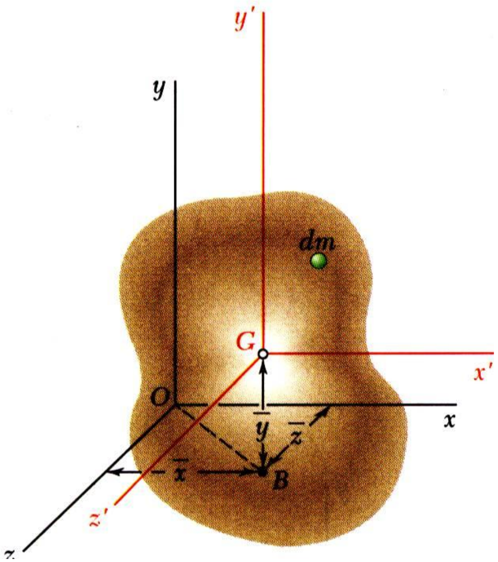
- In SI units,

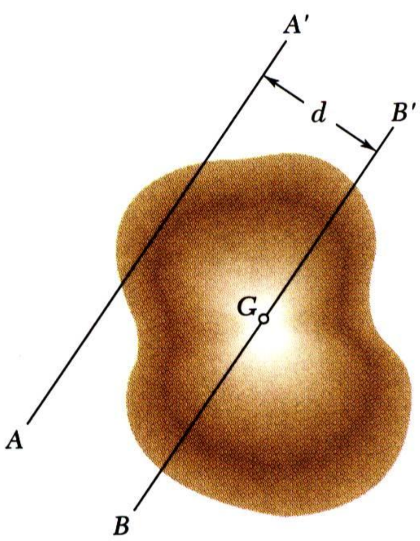
$$I = \int r^2 dm = \left(\mathrm{kg} \cdot \mathrm{m}^2\right)$$

In U.S. customary units,

$$I = \left(slug \cdot ft^2\right) = \left(\frac{lb \cdot s^2}{ft}\, ft^2\right) = \left(lb \cdot ft \cdot s^2\right)$$

Moment of Inertia of A Mass





- For the rectangular axes with origin at *O* and parallel centroidal axes,

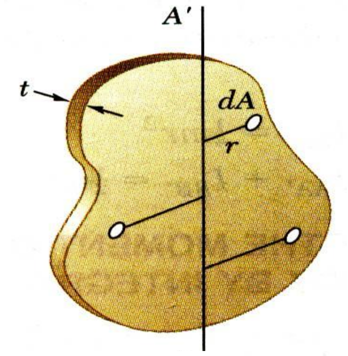
$$I_x = \int \left(y^2 + z^2\right) dm = \int \left[(y' + \bar{y})^2 + (z' + \bar{z})^2\right] dm$$
$$= \int \left(y'^2 + z'^2\right) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + \left(\bar{y}^2 + \bar{z}^2\right) \int dm$$

$$I_x = \bar{I}_{x'} + m\left(\bar{y}^2 + \bar{z}^2\right)$$
$$I_y = \bar{I}_{y'} + m\left(\bar{z}^2 + \bar{x}^2\right)$$
$$I_z = \bar{I}_{z'} + m\left(\bar{x}^2 + \bar{y}^2\right)$$

- Generalizing for any axis *AA'* and a parallel centroidal axis,

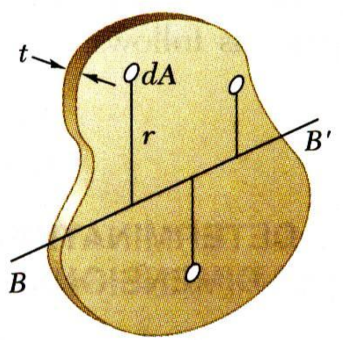
$$I = \bar{I} + md^2$$

Moments of Inertia of Thin Plates



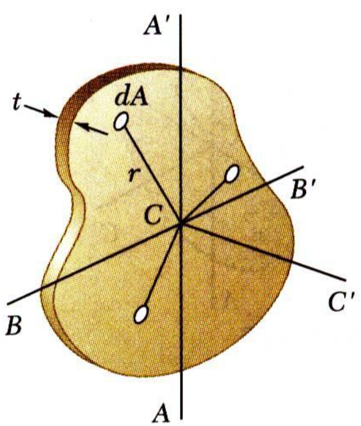
- For a thin plate of uniform thickness *t* and homogeneous material of density ρ, the mass moment of inertia with respect to axis *AA′* contained in the plate is

$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$
$$= \rho t\, I_{AA',area}$$



- Similarly, for perpendicular axis *BB′* which is also contained in the plate,

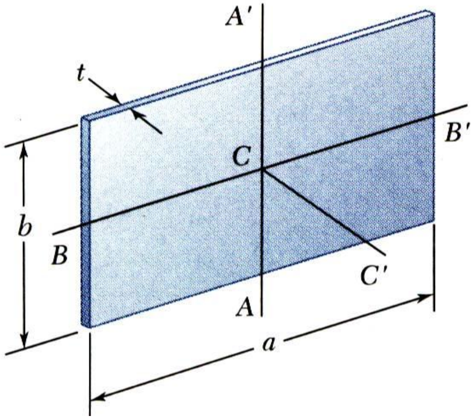
$$I_{BB'} = \rho t\, I_{BB',area}$$



- For the axis *CC′* which is perpendicular to the plate,

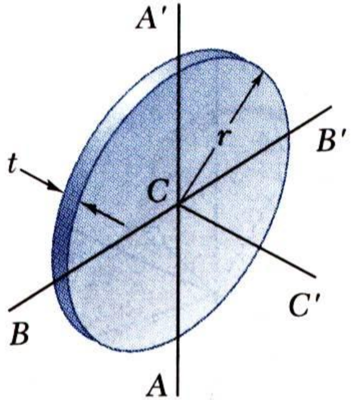
$$I_{CC'} = \rho t\, J_{C,area} = \rho t \left(I_{AA',area} + I_{BB',area} \right)$$
$$= I_{AA'} + I_{BB'}$$

Moments of Inertia of Thin Plates



- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t \, I_{AA',area} = \rho t \left(\tfrac{1}{12} a^3 b\right) = \tfrac{1}{12} m a^2$$



- For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t \, I_{AA',area} = \rho t \left(\tfrac{1}{4} \pi r^4\right) = \tfrac{1}{4} m r^2$$