Lecture 21

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Equivalence Relation

(A,R) , R is s.t.b. equivalence relation,

(i) Reflexive (ii) Symmetric (iii) Transitive

Q9.

Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a \text{ and } b \text{ are the same age}\}$

b) $\{(a, b) \mid a \text{ and } b \text{ have the same parents}\}$

(a) Reflexive:- a and a are of same age ✓

$(a,a) \in R$

Symmetric, $(a,b) \in R$ a and b are of same age

b and a are of same age

$(b,a) \in R$

Yes Equivalence Relation

Transitive a and b are of same age ⇒ a and c are of same age.

b and c are of same age

(b) Equivalence relation

Q10. Check for Congruence Modulo m, whether this relation on the set of integers is equivalence relation, where $m > 1$ be a positive integer.

$(a,b) \in R$, $a \equiv b \bmod m$, $m \mid (a-b)$

(a) Reflexive, $a \equiv a \bmod m$, $m \mid (a-a)$. $m \mid 0$ ✓

(a) Reflexive, $a \equiv a \bmod m$, $m \mid (a-a)$, $m \mid 0$ ✓

(b) Symmetric $a \equiv b \bmod m$, $m \mid (a-b)$, $a-b = km$

$b-a = (-k)m$

$m \mid b-a$

$b \equiv a \bmod m$

Yes Equivalence relation

(c) Transitive

$a \equiv b \bmod m$, $b \equiv c \bmod m$

$m \mid (a-b)$, $m \mid (b-c)$

$(a-b) = k_1 m$, $(b-c) = k_2 m$

Add $a-c = (k_1+k_2)m$, $m \mid (a-c)$,

$a \equiv c \bmod m$

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* $(\mathbb{Z}^+, \mid)$ Check is it equivalence relation.

$(a,a) \in R$, $a \mid a$ ✓ Yes Reflexive

Check is it POSET

$(a,b) \in R$ $a \mid b$ $2 \mid 4$, $4 \nmid 2$ Not symmetric

$(a,b),(b,c) \in R$, $a \mid b$, $b \mid c$ Yes Transitive

$b = k_1 a$, $c = k_2 b \Rightarrow c = k_1 k_2 a$

$a \mid c$

Partial Ordering

R is s.t.b. partial ordering if R is reflexive

R is antisymmetric

R is transitive

(A, R) → POSET

Partially ordered set:

Q12.

2. Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

  a) {(0, 0), (2, 2), (3, 3)}

  b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}

  c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)}

  d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}

  e) {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)}

	(a)	(b)	(d)
Reflexive	✗ $(1,1) \notin R$	✓	✓
Antisymmetric	✓	✓	✓
Transitive	✓	✓	✗ $(1,3),(3,0) \in R$ $(1,0) \notin R$

Q13.

7. Determine whether the relations represented by these zero–one matrices are partial orders.

  a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → Reflexive ✓ Antisymmetric ✓ Transitive ✓

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$$

  c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$$

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reflexive ✓

Not antisymmetric

Not transitive

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \boxed{0} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \boxed{1} \\ & & \end{bmatrix}$$

Reflexive; $m_{ii} = 1 \ \forall i$

Irreflexive; $m_{ii} = 0, \forall i$

Symmetric, $M_R^T = M_R$, Antisymmetric; $m_{ij} m_{ji} \neq 1$ for $i \neq j$

Asymmetric $m_{ij} m_{ji} \neq 1$ for $i \neq j$

$m_{ii} = 0$

Dual of POSET

$$(A, R) \xrightarrow{\text{Dual}} (A, R^{-1})$$

$R : A \to B$ $B = A$

$R^{-1} : B \to A$

Q14.

Find the duals of these posets.

a) $(\{0, 1, 2\}, \leq)$ b) $(Z, \geq)$ $\Longrightarrow (Z, \leq)$

$R = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,2)\}$

Dual.

R^{-1} $\{(0,0), (1,0), (2,0), (1,1), (2,1), (2,2)\}$

$R^{-1} = \{(0,0), (1,0), (2,0), (1,1), (2,1), (2,2)\}$

$\text{Dual} = (\{0,1,2\}, \geq)$

Definition

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

$a, b \in S$ $\quad aRb$ or bRa Comparable

$(A, \preccurlyeq)$

Q15.

Which of these pairs of elements are comparable in the poset $(\mathbf{Z}^+, |)$?

a) 5, 15 ✓ b) 6, 9 ✗ c) 8, 16 ✓ d) 7, 7 ✓

$5 \mid 15, \; 15 \nmid 5$ $\quad 6 \nmid 9, \; 9 \nmid 6$

Incomparable

Definition

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

$(\mathbb{Z}^+, |)$ $\quad (\mathbb{Z}^+, \leq)$

Totally ordered.