Surface Area and Surface Integral

Surface Area

Let $z = f(x,y)$ be the equation of the surface, then the surface area is given by

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

where R is the orthogonal projection of the surface on x-y plane.

Similarly, if $x = g(y,z)$, then

$$A = \iint_R \sqrt{1 + g_y^2 + g_z^2}\, dy\, dz$$

where R is the projection of the surface on y-z plane

and in case of the surface $y = h(x,z)$

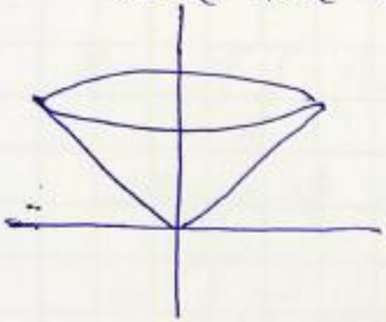
$$A = \iint_R \sqrt{1 + h_x^2 + h_z^2}\, dx\, dz$$

where R is the projection of the surface on x-z plane.

<u>Example</u> (1) Find the surface area of the surface $z^2 = x^2 + y^2$, $0 \le z \le 4$

Sol:-

First we need to understand the surface $z^2 = x^2 + y^2$, this is an infinite cone, but have the limit of z set between 0 to 4.



Its projection in x-y plane is the circle $x^2+y^2 = 4^2$

$$z = \sqrt{x^2+y^2}$$

$$A = \iint_R \sqrt{1 + z_x^2 + z_y^2}\, dx\,dy$$

$z = \sqrt{x^2+y^2}$, $z_x = \dfrac{1}{2\sqrt{x^2+y^2}} 2x = \dfrac{x}{\sqrt{x^2+y^2}}$

$$z_y = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$A = \iint_R \left(1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right)^{1/2} dx\,dy$$

$= \iint_R \sqrt{2}\, dx\,dy$, we change it to polar co-ordinates

$$= \int_0^{2\pi}\int_0^4 \sqrt{2}\, r\,dr\,d\theta = \sqrt{2}\int_0^{2\pi} \frac{r^2}{2}\Big|_0^4 d\theta$$

$$= \sqrt{2}\int_0^{2\pi} 8\, d\theta = 8\sqrt{2}\int_0^{2\pi} d\theta = 16\sqrt{2}\,\pi$$

Example(2) Find the surface area of the given surface $z = x^2+y^2$, $0 \le z \le 9$

Sol: $A = \iint\limits_R (1+f_x^2+f_y^2)^{1/2}\, dx\,dy$

$z = x^2+y^2$, $z_x = 2x$, $z_y = 2y$

$$A = \iint\limits_R (1+4x^2+4y^2)^{1/2}\, dx\,dy = \iint\limits_R \sqrt{1+4(x^2+y^2)}\, dx\,dy$$

change it to polar $x = r\cos\theta$, $y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^{3} \sqrt{1+4r^2}\; r\,dr\,d\theta = \frac{1}{8}\int_0^{2\pi}\int_0^{3} \sqrt{1+4r^2}\; 8r\,dr\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3}(1+4r^2)^{3/2}\Big|_0^3\, d\theta$$

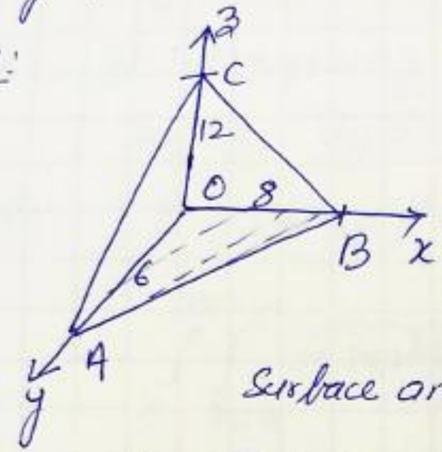
$$= \frac{1}{12}\int_0^{2\pi} \left((37)^{3/2} - 1^{3/2}\right) d\theta$$

$$= \frac{1}{12}(37\sqrt{37}-1)\;\theta\Big|_0^{2\pi}$$

$$= \frac{1}{6}\pi(37\sqrt{37}-1)$$

Example (3) Find the surface area of the portion of the plane $3x+4y+2z=24$ bounded by the co-ordinates planes in the first octant.

Sol:



$$3x+4y+2z=24$$

$$\Rightarrow \frac{x}{8}+\frac{y}{6}+\frac{z}{12}=1$$

The surface ABC will have the projection OAB in the xy-plane

$$\text{Surface area} = \iint_R (1+f_x^2+f_y^2)^{1/2}\,dx\,dy$$

$$z=(24-3x-4y)/2, \quad z_x=-3/2, \quad z_y=-4/2$$

$$=\int_0^8 \int_{y=0}^{\frac{24-3x}{4}} \left(1+\frac{9}{4}+\frac{16}{4}\right)^{1/2} dx\,dy$$

$$= \frac{\sqrt{29}}{\sqrt{4}} \int_0^8 \int_0^{\frac{24-3x}{4}} dy\,dx = \frac{\sqrt{29}}{\sqrt{4}} \int_0^8 [y]_0^{\frac{24-3x}{4}} dx$$

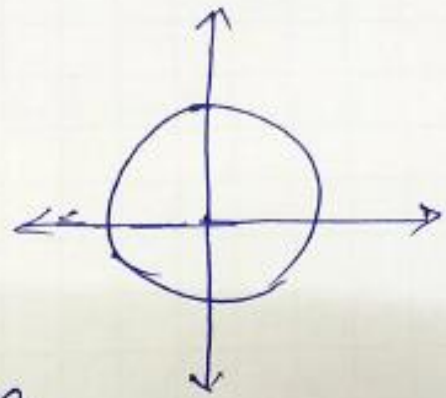
$$= \frac{\sqrt{29}}{\sqrt{4}} \int_0^8 \frac{24-3x}{4}\,dx = \frac{\sqrt{29}}{2} \times \frac{1}{4}\left[24x-\frac{3}{2}x^2\right]_0^8$$

$$= \frac{\sqrt{29}}{8}(192-96) = 12\sqrt{29}$$

Example Determine the surface area of the part of $z = xy$ that lies in the cylinder $x^2+y^2 = 1$

Sol: $z = f(x,y) = xy$

$z_x = y, \quad z_y = x$



$$A = \iint_R (1+x^2+y^2)^{1/2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_{r=0}^{1} \sqrt{(1+r^2)}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}(1+r^2)^{3/2}\Big|_0^1 d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} (2^{3/2}-1) = \frac{2\pi}{3}(2\sqrt{2}-1)$$

Example Surface area of sphere $x^2+y^2+z^2 = a^2$

Surface Integral

$$\int_S \vec{F}\cdot\hat{n}\,ds \quad \text{or} \int_S \vec{F}\cdot d\vec{S}$$

Here $\vec{n}$ is the unit outward normal to the surface.

Here the surface S is Orientable surface. (We may say that S is an orientable surface if it has two sides, which may be painted in two different colours)

$$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$, where α, β, γ are the angles which unit normal $\vec{n}$ makes with the positive directions of x, y and z-axis, respectively.

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_S (f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma)\,ds$$

$$= \int_S f_1\cos\alpha\,ds + f_2\cos\beta\,ds + f_3\cos\gamma\,ds.$$

$$= \int_S f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy$$

where $(\cos\alpha)ds = dy\,dz$

or $(n\cdot\hat{i})ds = dy\,dz$ or $ds = \dfrac{dy\,dz}{n\cdot\hat{i}}$ —(1)

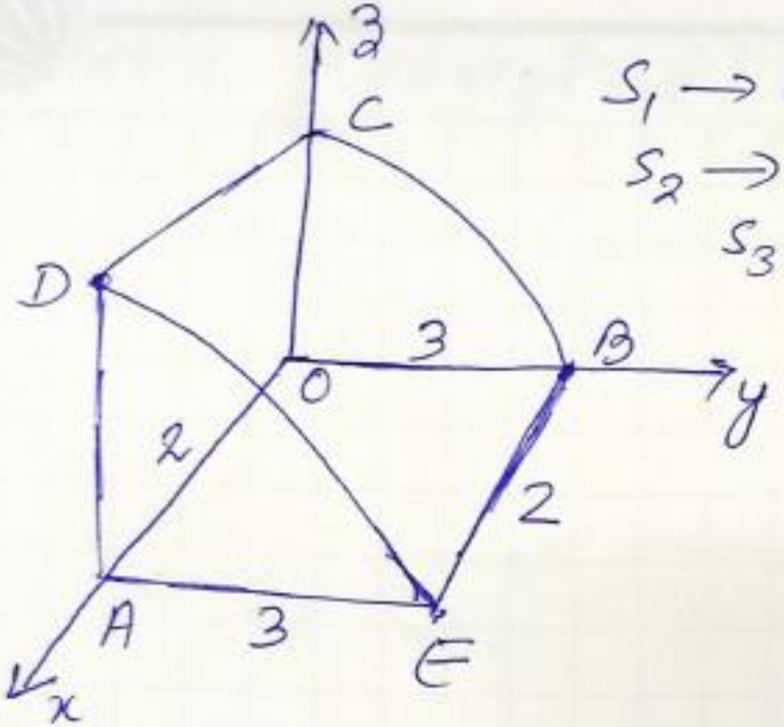
similarly $ds = \dfrac{dz\,dx}{\hat{n}\cdot\hat{j}}$ —(2)

$ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$ —(3)

(1)–(3) give expressions for the element surface area in terms of its projection on the co-ordinate planes.

Q(1) Evaluate $\int_S \vec{F}\cdot\hat{n}\, ds$, where

$\vec{F} = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $x=2$, $y=0$ and $z=0$.



$S_1 \rightarrow OAEB$

$S_2 \rightarrow OADC$

$S_3 \rightarrow OBC$

$S_4 \rightarrow AED$

$S_5 \rightarrow BCDE$

S is piecewise smooth and contains S_1, S_2, S_3
S_4 and S_5.

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_{S_1} + \int_{S_2} + \int_{S_3} + \int_{S_4} + \int_{S_5}$$

$$\int_{S_1} \vec{F}\cdot\hat{n}\, ds = \int_{S_1} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})(-\hat{k})\, ds$$

$$= -4\int_{S_1} xz^2\, ds = 0 \ (\text{as } z=0)$$

$$\int_{S_2} \vec{F}\cdot\hat{n}\, ds = \int_{S_2} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})(-\hat{j})\, ds$$

$$= \int_{S_2} y^2\, ds = 0 \ (\text{as } y=0)$$

$$\int_{S_3} \vec{F}\cdot\hat{n}\, ds = \int_{S_3} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})(-\hat{i})\, ds$$

$$= -\int_{S_3} 2x^2y\, ds = 0 \ (\text{as } x=0)$$

$$\int_{S_4} \vec{F}\cdot\hat{n}\, ds = \int_{S_4} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot\hat{i}\, ds$$

$$= \int_{S_4} 2x^2y\, ds = \iint_{S_4} 8y\,\frac{dy\,dz}{\hat{i}\cdot\hat{i}}$$

$$= \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\, dy\, dz = 4\int_0^3 (9-z^2)\, dz$$

$$= 4\left(9z - z^3/3\right)\Big|_0^3 = 4(27-9) = 72$$

$$\int_S \vec{F}\cdot\hat{n}\,ds$$

$$\nabla(y^2+z^2) = 2y\hat{j}+2z\hat{k}$$

$$\hat{n} = \frac{2y\hat{j}+2z\hat{k}}{\sqrt{4y^2+4z^2}} = \frac{2y\hat{j}+2z\hat{k}}{6} = \frac{y\hat{j}+z\hat{k}}{3}$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{z/3}.$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_0^3\int_0^2 \left(\frac{-y^3+4xz^3}{3}\right)\frac{dx\,dy}{(z/3)}$$

$$= \int_0^2\int_0^3\left(-\frac{y^3}{z}+4xz^2\right)dy\,dx$$

Put $y = 3\sin\theta,\ z = 3\cos\theta$

$dy = 3\cos\theta\,d\theta$

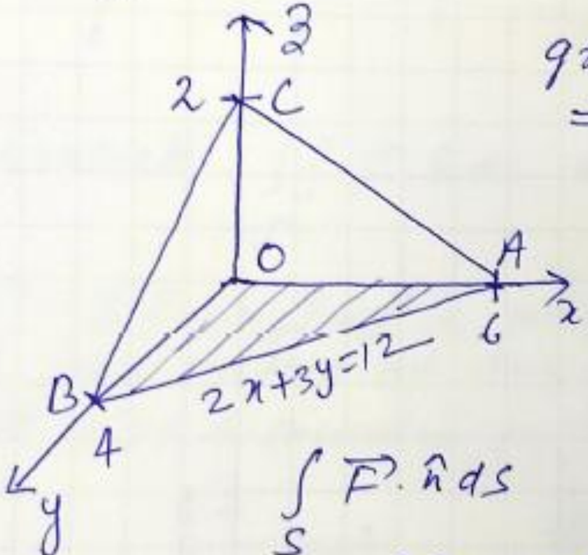
$$= \int_0^2\int_0^{\pi/2}\left[\frac{-27\sin^3\theta}{3\cos\theta}+4x\,9\cos^2\theta\right]3\cos\theta\,d\theta\,dx$$

$= 108$

$$\int_S \vec{F}\cdot\hat{n}\,ds = 0+0+0+72+108 = 180$$

Q) Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$ and S is the portion of the plane $2x+3y+6z=12$ in the first octant.

<u>Sol:</u>



$$\text{grad}\,(2x+3y+6z-12) = 2\hat{i}+3\hat{j}+6\hat{k}$$

$$\hat{n} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \iint_{\text{over } OAB} (6z\hat{i}-4\hat{j}+y\hat{k})\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$= \iint_{\text{over } OAB} \frac{1}{7}(12z-12+6y)\,\frac{dx\,dy}{\frac{6}{7}}$$

$$= \frac{1}{6}\int_0^6 \int_0^{\frac{12-2x}{3}} \left(2(12-2x-3y)-12+6y\right)dx\,dy$$

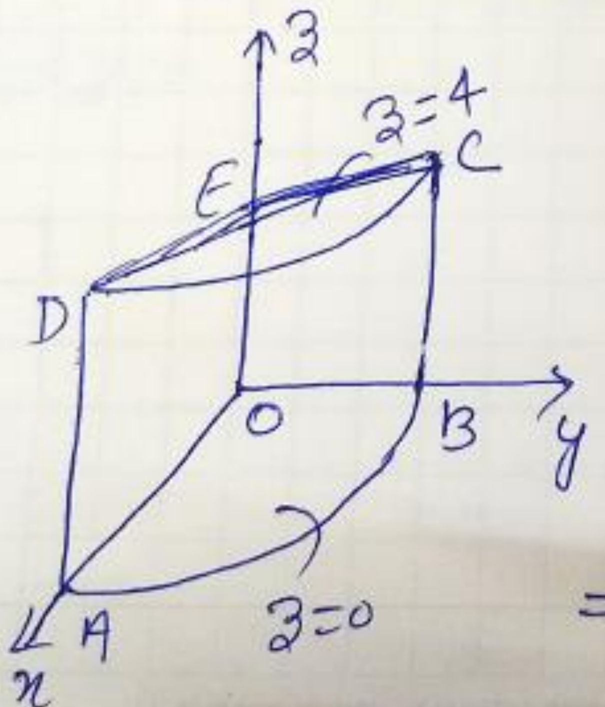
$$= \frac{1}{6}\int_0^6 \int_0^{\frac{12-2x}{3}} (12-4x)\,dy\,dx$$

$$= \frac{1}{6}\int_0^6 (12-4x)\left(\frac{12-2x}{3}\right)dx = \boxed{\quad}$$

Q) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where

$\vec{F} = z^2\hat{i} + xy\hat{j} - y^2\hat{k}$, where S is the portion of the surface of the cylinder $x^2+y^2=36$, $0 \le z \le 4$ included in the first octant.

Sol:



ABCD is the required surface

$$\nabla(x^2+y^2-36) = 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4\times 36}} = \frac{x\hat{i}+y\hat{j}}{6}$$

From the figure it is clear that, we can project the surface either on x-z plane or on y-z plane.

We take the projection on y-z plane OBCE

$$ds = \frac{dy\,dz}{\hat{n}\cdot\hat{i}} = \frac{dy\,dz}{x/6}$$

$$\vec{F}\cdot\hat{n} = \frac{1}{6}xz^2 + \frac{1}{6}y\cdot xy = \frac{1}{6}(z^2x + xy^2)$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{4}\int_{y=0}^{6} \frac{1}{\cancel{6}}(z^2x + xy^2)\frac{dy\,dz}{\frac{x}{\cancel{6}}}$$

$$= \int_{z=0}^{4}\int_{y=0}^{6}(y^2+z^2)\,dy\,dz = \boxed{}$$