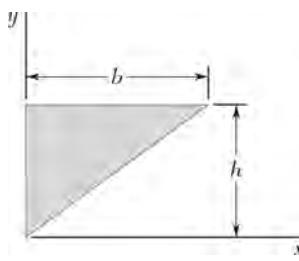


CHAPTER 9



PROBLEM 9.1

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION

By observation

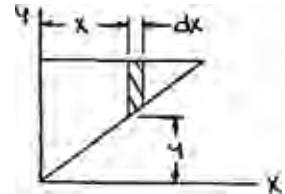
$$y = \frac{h}{b}x$$

Now

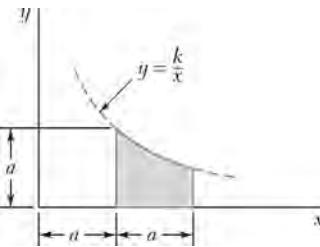
$$\begin{aligned} dI_y &= x^2 \quad dA = x^2[(h-y)dx] \\ &= hx^2 \left(1 - \frac{x}{b}\right) dx \end{aligned}$$

Then

$$\begin{aligned} I_y &= \int dI_y = \int_0^b hx^2 \left(1 - \frac{x}{b}\right) dx \\ &= h \left[\frac{1}{3}x^3 - \frac{x^4}{4b} \right]_0^b \end{aligned}$$



$$\text{or } I_y = \frac{1}{12}b^3h \blacktriangleleft$$



PROBLEM 9.2

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION

At $x = a$, $y = a$:

$$a = \frac{k}{a} \quad \text{or} \quad k = a^2$$

Then

$$y = \frac{a^2}{x}$$

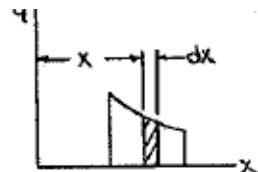
Now

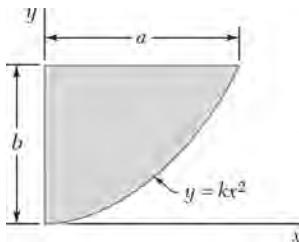
$$\begin{aligned} dI_y &= x^2 \quad dA = x^2(y \, dx) \\ &= x^2 \left(\frac{a^2}{x} \, dx \right) = a^2 x \, dx \end{aligned}$$

Then

$$I_y = \int dI_y = \int_a^{2a} a^2 x \, dx = a^2 \left[\frac{1}{2} x^2 \right]_a^{2a} = \frac{a^2}{2} [(2a)^2 - (a)^2]$$

$$\text{or } I_y = \frac{3}{2} a^4 \blacktriangleleft$$





PROBLEM 9.3

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION

$$y = kx^2$$

For $x = a$:

$$b = ka^2$$

$$k = \frac{b}{a^2}$$

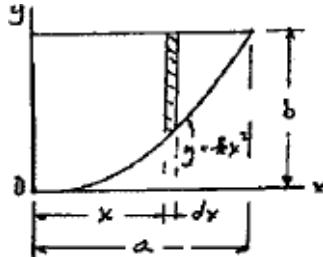
Thus:

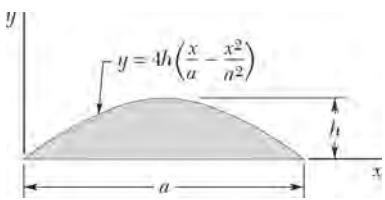
$$y = \frac{b}{a^2} x^2$$

$$dA = (b - y)dx$$

$$dI_y = x^2 dA = x^2 (b - y)dx = x^2 \left(b - \frac{b}{a^2} x^2 \right) dx$$

$$I_y = \int dI_y = \int_0^a \left(bx^2 - \frac{b}{a^2} x^4 \right) dx = \frac{1}{3} a^3 b - \frac{1}{5} a^3 b \quad I_y = \frac{2}{15} a^3 b \blacktriangleleft$$

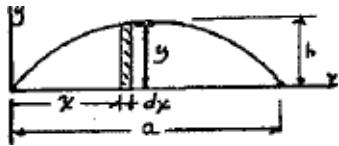




PROBLEM 9.4

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



$$y = 4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)$$

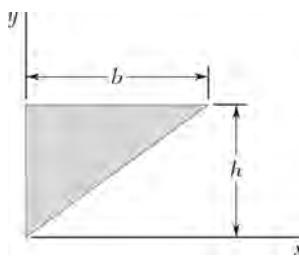
$$dA = y dx$$

$$dI_y = x^2 dA = 4hx^2\left(\frac{x}{a} - \frac{x^2}{a^2}\right)dx$$

$$I_y = 4h \int_0^a \left(\frac{x^3}{a} - \frac{x^4}{a^2} \right) dx$$

$$I_y = 4h \left[\frac{x^4}{4a} - \frac{x^5}{5a^2} \right]_0^a = 4h \left(\frac{a^3}{4} - \frac{a^3}{5} \right)$$

$$I_y = \frac{1}{5}ha^3 \blacktriangleleft$$



PROBLEM 9.5

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

By observation

$$y = \frac{h}{b}x$$

or

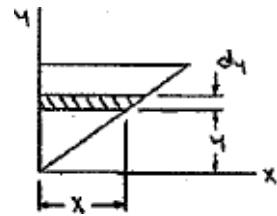
$$x = \frac{b}{h}y$$

Now

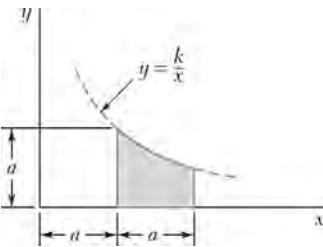
$$\begin{aligned} dI_x &= y^2 dA = y^2(x dy) \\ &= y^2 \left(\frac{b}{h} y dy \right) \\ &= \frac{b}{h} y^3 dy \end{aligned}$$

Then

$$\begin{aligned} I_x &= \int dI_x = \int_0^h \frac{b}{h} y^3 dy \\ &= \frac{b}{h} \left[\frac{1}{4} y^4 \right]_0^h \end{aligned}$$



$$\text{or } I_x = \frac{1}{4} bh^3 \blacktriangleleft$$



PROBLEM 9.6

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

At $x = a$, $y = a$:

$$a = \frac{k}{a} \quad \text{or} \quad k = a^2$$

Then

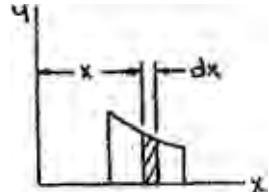
$$y = \frac{a^2}{x}$$

Now

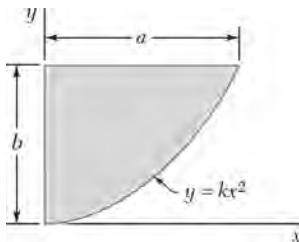
$$\begin{aligned} dI_x &= \frac{1}{3} y^3 dx = \frac{1}{3} \left(\frac{a^2}{x} \right)^3 dx \\ &= \frac{1}{3} \frac{a^6}{x^3} dx \end{aligned}$$

Then

$$\begin{aligned} I_x &= \int dI_x = \int_a^{2a} \frac{1}{3} \frac{a^6}{x^3} dx = \frac{1}{3} a^6 \left[-\frac{1}{2} \frac{1}{x^2} \right]_a^{2a} \\ &= -\frac{1}{6} a^6 \left[\frac{1}{(2a)^2} - \frac{1}{(a)^2} \right] \end{aligned}$$



$$\text{or } I_x = \frac{1}{8} a^4 \blacktriangleleft$$

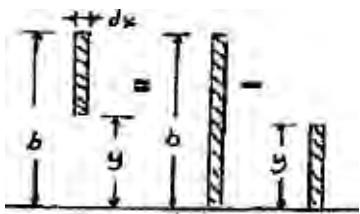


PROBLEM 9.7

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

See figure of solution of Problem 9.3.



$$y = \frac{b}{a^2} x$$

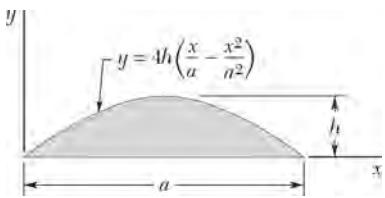
$$dI_x = \frac{1}{3} b^3 dx - \frac{1}{3} y^3 dx = \frac{1}{3} b^3 dx - \frac{1}{3} \frac{b^3}{a^6} x^6 dx$$

$$I_x = \int dI_x = \int_0^a \left(\frac{1}{3} b^3 - \frac{1}{3} \frac{b^3}{a^6} x^6 \right) dx$$

$$= \frac{1}{3} b^3 a - \frac{1}{3} \frac{b^3}{a^6} \frac{a^7}{7} = \left(\frac{1}{3} - \frac{1}{21} \right) ab^3$$

$$= \left(\frac{7}{21} - \frac{1}{21} \right) ab^3 = \frac{6}{21} ab^3$$

$$I_x = \frac{2}{7} ab^3 \blacktriangleleft$$



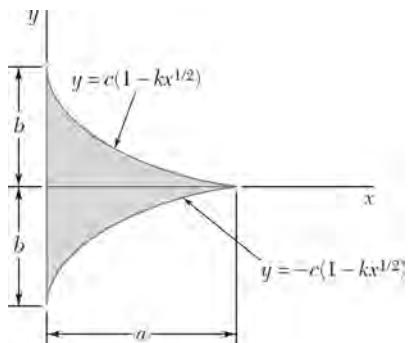
PROBLEM 9.8

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

See figure of solution of Problem 9.4.

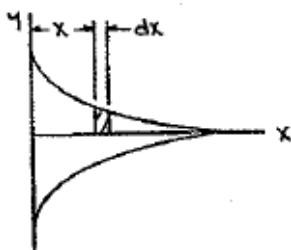
$$\begin{aligned}
 y &= h_1 + (h_2 - h_1) \frac{x}{a} \\
 dI_x &= \frac{1}{3} y^3 dx \\
 I_x &= \int dI_y = \frac{1}{3} \int_0^a \left[h_1 + (h_2 - h_1) \frac{x}{a} \right]^3 dx \\
 &= \frac{1}{12} \left[\left[h_1 + (h_2 - h_1) \frac{x}{a} \right]^4 \left(\frac{a}{h_2 - h_1} \right) \right]_0^a \\
 &= \frac{a}{12(h_2 - h_1)} (h_2^4 - h_1^4) = \frac{a}{12} \cdot \frac{(h_2^2 + h_1^2)(h_2 + h_1)(h_2 - h_1)}{h_2 - h_1} \\
 I_x &= \frac{a}{12} (h_1^2 + h_2^2)(h_1 + h_2) \blacktriangleleft
 \end{aligned}$$



PROBLEM 9.9

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



$$\text{At } x = 0, y = b:$$

$$b = c(1 - 0) \quad \text{or} \quad c = b$$

$$\text{At } x = a, y = 0:$$

$$0 = b(1 - ka^{1/2}) \quad \text{or} \quad k = \frac{1}{a^{1/2}}$$

Then

$$y = b \left(1 - \frac{x^{1/2}}{a^{1/2}} \right)$$

Now

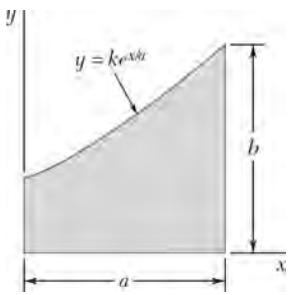
$$dI_x = \frac{1}{3} y^3 dx = \frac{1}{3} \left[b \left(1 - \frac{x^{1/2}}{a^{1/2}} \right) \right]^3 dx$$

or

$$dI_x = \frac{1}{3} b^3 \left(1 - 3 \frac{x^{1/2}}{a^{1/2}} + 3 \frac{x}{a} - \frac{x^{3/2}}{a^{3/2}} \right) dx$$

Then

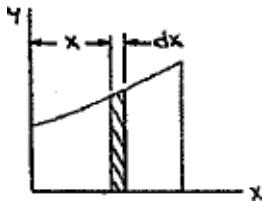
$$\begin{aligned} I_x &= \int dI_x = 2 \int_0^a \frac{1}{3} b^3 \left(1 - 3 \frac{x^{1/2}}{a^{1/2}} + 3 \frac{x}{a} - \frac{x^{3/2}}{a^{3/2}} \right) dx \\ &= \frac{2}{3} b^3 \left[x - 2 \frac{x^{3/2}}{a^{1/2}} + \frac{3}{2} \frac{x^2}{a} - \frac{2}{5} \frac{x^{5/2}}{a^{3/2}} \right]_0^a \end{aligned} \quad \text{or} \quad I_x = \frac{1}{15} ab^3 \blacksquare$$



PROBLEM 9.10

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



$$\text{At } x = a, y = b: \quad b = k e^{a/a} \quad \text{or} \quad k = \frac{b}{e}$$

Then

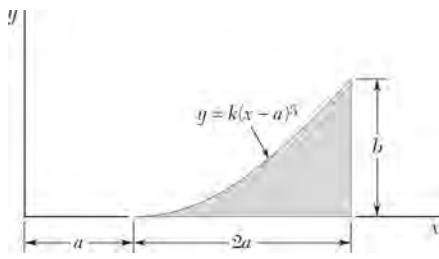
$$y = \frac{b}{e} e^{x/a} = b e^{x/a-1}$$

Now

$$\begin{aligned} dI_x &= \frac{1}{3} y^3 dx = \frac{1}{3} (b e^{x/a-1})^3 dx \\ &= \frac{1}{3} b^3 e^{3(x/a-1)} dx \end{aligned}$$

Then

$$\begin{aligned} I_x &= \int dI_x = \int_0^a \frac{1}{3} b^3 e^{3(x/a-1)} dx = \frac{b^3}{3} \left[\frac{a}{3} e^{3(x/a-1)} \right]_0^a \\ &= \frac{1}{9} ab^3 (1 - e^{-3}) \quad \text{or} \quad I_x = 0.1056ab^3 \blacksquare \end{aligned}$$



PROBLEM 9.11

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

At $x = 3a$, $y = b$:

$$b = k(3a - a)^3$$

or

$$k = \frac{b}{8a^3}$$

Then

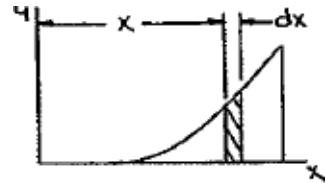
$$y = \frac{b}{8a^3}(x - a)^3$$

Now

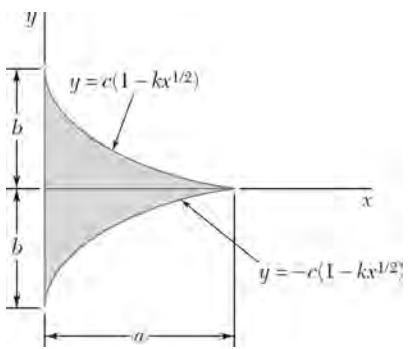
$$\begin{aligned} dI_x &= \frac{1}{3} y^3 dx \\ &= \frac{1}{3} \left[\frac{b}{8a^3} (x - a)^3 \right]^3 dx \\ &= \frac{b^3}{1536a^9} (x - a)^9 dx \end{aligned}$$

Then

$$\begin{aligned} I_x &= \int dI_x = \int_a^{3a} \frac{b^3}{1536a^9} (x - a)^9 dx = \frac{b^3}{1536a^9} \left[\frac{1}{10} (x - a)^{10} \right]_a^{3a} \\ &= \frac{b^3}{15,360a^9} [(3a - a)^{10} - 0] \end{aligned}$$



$$\text{or } I_x = \frac{1}{15} ab^3 \blacktriangleleft$$



PROBLEM 9.12

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

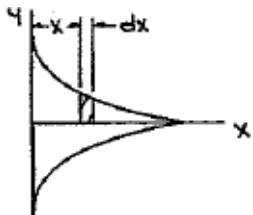
SOLUTION

$$\text{At } x=0, \quad y=b:$$

$$b = c(1-0) \quad \text{or} \quad c = b$$

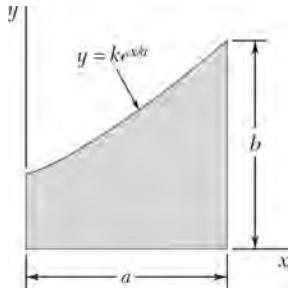
$$x=a, \quad y=0:$$

$$0 = c(1-ka^{1/2}) \quad \text{or} \quad k = \frac{1}{a^{1/2}}$$



Then

$$y = b \left(1 - \frac{x^{1/2}}{a^{1/2}} \right) \blacktriangleleft$$



PROBLEM 9.13

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION

At $x = a$, $y = b$:

$$b = ke^{a/a}$$

or

$$k = \frac{b}{e}$$

Then

$$y = \frac{b}{e} e^{x/a} = b e^{x/a-1}$$

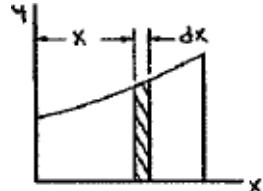
Now

$$dI_y = x^2 dA = x^2 (y dx)$$

$$= x^2 (be^{x/a-1} dx)$$

Then

$$I_y = \int dI_y = \int_0^a bx^2 e^{x/a-1} dx$$



Now use integration by parts with

$$u = x^2 \quad dv = e^{x/a-1} dx$$

$$du = 2x dx \quad v = ae^{x/a-1}$$

Then

$$\int_0^a x^2 e^{x/a-1} dx = \left[x^2 ae^{x/a-1} \right]_0^a - \int_0^a (ae^{x/a-1}) 2x dx$$

$$= a^3 - 2a \int_0^a x e^{x/a-1} dx$$

Using integration by parts with

$$u = x \quad dv = e^{x/a-1} dx$$

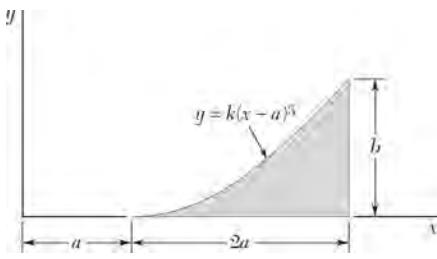
$$du = dx \quad v = ae^{x/a-1}$$

Then

$$I_y = b \left\{ a^3 - 2a \left[(xae^{x/a-1}) \Big|_0^a - \int_0^a (ae^{x/a-1}) dx \right] \right\}$$

$$= b \left\{ a^3 - 2a \left[a^2 - (a^2 e^{x/a-1}) \Big|_0^a \right] \right\}$$

$$= b \left\{ a^3 - 2a \left[a^2 - (a^2 - a^2 e^{-1}) \right] \right\} \quad \text{or } I_y = 0.264a^3b \blacktriangleleft$$



PROBLEM 9.14

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION

At $x = 3a$, $y = b$:

$$b = k(3a - a)^3$$

or

$$k = \frac{b}{8a^3}$$

Then

$$y = \frac{b}{8a^3}(x - a)^3$$

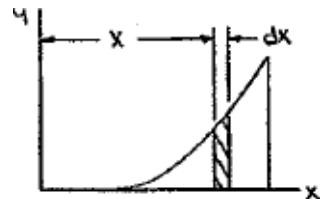
Now

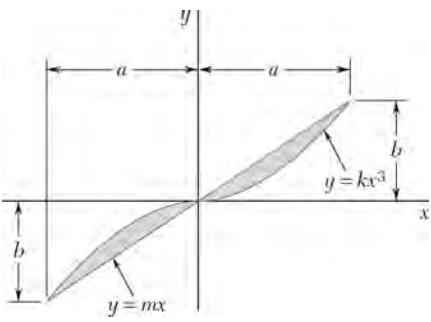
$$\begin{aligned} dI_y &= x^2 dA = x^2 (y dx) = x^2 \left[\frac{b}{8a^3}(x - a)^3 dx \right] \\ &= \frac{b}{8a^3} x^2 (x^3 - 3x^2 a + 3xa^2 - a^3) dx \end{aligned}$$

Then

$$\begin{aligned} I_y &= \int dI_y = \int_a^{3a} \frac{b}{8a^3} (x^5 - 3x^4 a + 3x^3 a^2 - a^3 x^2) dx \\ &= \frac{b}{8a^3} \left[\frac{1}{6}x^6 - \frac{3}{5}ax^5 + \frac{3}{4}a^2x^4 - \frac{1}{3}a^3x^3 \right]_a^{3a} \\ &= \frac{b}{8a^3} \left\{ \left[\frac{1}{6}(3a)^6 - \frac{3}{5}a(3a)^5 + \frac{3}{4}a^2(3a)^4 - \frac{1}{3}a^3(3a)^3 \right] \right. \\ &\quad \left. - \left[\frac{1}{6}(a)^6 - \frac{3}{5}a(a)^5 + \frac{3}{4}a^2(a)^4 - \frac{1}{3}a^3(a)^3 \right] \right\} \end{aligned}$$

$$\text{or } I_y = 3.43a^3b \blacktriangleleft$$





PROBLEM 9.15

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION

At $x = a$, $y_1 = y_2 = b$:

$$y_1: \quad b = ma \quad \text{or} \quad m = \frac{b}{a}$$

$$y_2: \quad b = ka^3 \quad \text{or} \quad k = \frac{b}{a^3}$$

Then

$$y_1 = \frac{b}{a}x$$

or

$$x_1 = \frac{a}{b}y$$

and

$$y_2 = \frac{b}{a^3}x^3$$

or

$$x_2 = \frac{a}{b^{1/3}}y^{1/3}$$

Now

$$dA = (x_2 - x_1)dy$$

$$= \left(\frac{a}{b^{1/3}}y^{1/3} - \frac{a}{b}y \right) dy$$

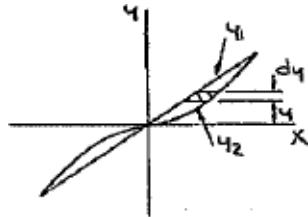
Then

$$A = \int dA = 2 \int_0^b a \left(\frac{y^{1/3}}{b^{1/3}} - \frac{1}{b}y \right) dy$$

$$= 2a \left[\frac{3}{4} \frac{1}{b^{1/3}} y^{4/3} - \frac{1}{2b} y^2 \right]_0^b = \frac{1}{2}ab$$

Now

$$dI_x = y^2 dA = y^2 \left[\left(\frac{a}{b^{1/3}}y^{1/3} - \frac{a}{b}y \right) dy \right]$$



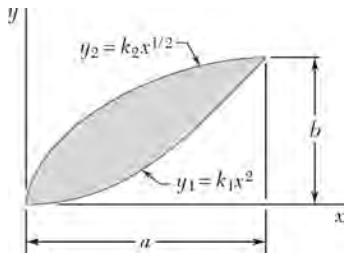
PROBLEM 9.15 (Continued)

Then

$$\begin{aligned} I_x &= 2 \int_0^b a \left(\frac{1}{b^{1/3}} y^{7/3} - \frac{1}{b} y^3 \right) dy \\ &= 2a \left[\frac{3}{10} \frac{1}{b^{1/3}} y^{10/3} - \frac{1}{4b} y^4 \right]_0^b \end{aligned} \quad \text{or } I_x = \frac{1}{10} ab^3 \blacktriangleleft$$

and

$$k_x^2 = \frac{I_x}{A} = \frac{\frac{1}{10} ab^3}{\frac{1}{2} ab} = \frac{1}{5} b^2 \quad \text{or } k_x = \frac{b}{\sqrt{5}} \blacktriangleleft$$



PROBLEM 9.16

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION

For

$$y_1 = k_1 x^2 \quad y_2 = k_2 x^{1/2}$$

$$x = 0 \quad \text{and} \quad y_1 = y_2 = b$$

$$b = k_1 a^2 \quad b = k_2 a^{1/2}$$

$$k_1 = \frac{b}{a^2} \quad k_2 = \frac{b}{a^{1/2}}$$

Thus,

$$y_1 = \frac{b}{a^2} x^2 \quad y_2 = \frac{b}{a^{1/2}} x^{1/2}$$

$$dA = (y_2 - y_1) dx$$

$$A = \int_0^a \left[\frac{b}{a^{1/2}} x^{1/2} - \frac{b}{a^2} x^2 \right] dx$$

$$A = \frac{2}{3} \frac{ba^{3/2}}{a^{1/2}} - \frac{ba^3}{3a^2}$$

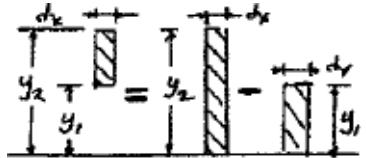
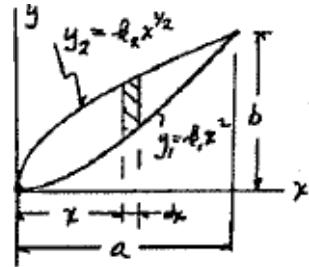
$$A = \frac{1}{3} ab$$

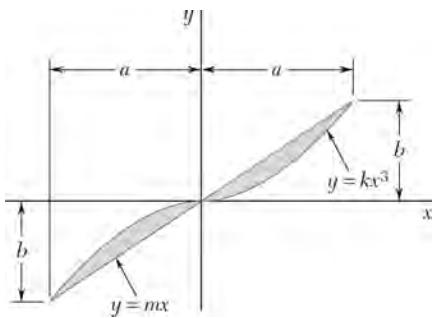
$$\begin{aligned} dI_x &= \frac{1}{3} y_2^3 dx - \frac{1}{3} y_1^3 dx \\ &= \frac{1}{3} \frac{b^3}{a^{3/2}} x^{3/2} dx - \frac{1}{3} \frac{b^3}{a^6} x^6 dx \end{aligned}$$

$$\begin{aligned} I_x &= \int dI_x = \frac{b^3}{3a^{3/2}} \int_0^a x^{3/2} dx - \frac{b^3}{3a^6} \int_0^a x^6 dx \\ &= \frac{b^3}{3a^{3/2}} \frac{a^{5/2}}{\left(\frac{5}{2}\right)} - \frac{b^3}{3a^6} \frac{a^7}{7} = \left(\frac{2}{15} - \frac{1}{21}\right) ab^3 \quad I_x = \frac{3}{35} ab^3 \end{aligned}$$

$$k_x^2 = \frac{I_x}{A} = \frac{\left(\frac{3}{35} ab^3\right)}{\frac{ab}{b}}$$

$$k_x = b \sqrt{\frac{9}{35}}$$





PROBLEM 9.17

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION

At $x = a$, $y_1 = y_2 = b$:

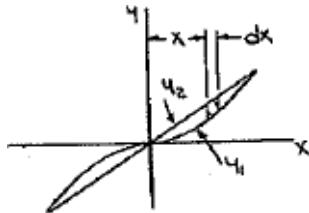
$$y_1: b = ka^3 \quad \text{or} \quad k = \frac{b}{a^3}$$

$$y_2: b = ma \quad \text{or} \quad m = \frac{b}{a}$$

Then

$$y_1 = \frac{b}{a^3} x^3$$

$$y_2 = \frac{b}{a} x$$



Now

$$dA = (y_2 - y_1)dx = \left(\frac{b}{a}x - \frac{b}{a^3}x^3 \right)dx$$

Then

$$A = \int dA = 2 \int_0^a \frac{b}{a} \left(x - \frac{1}{a^2}x^3 \right) dx$$

$$= 2 \frac{b}{a} \left[\frac{1}{2}x^2 - \frac{1}{4a^2}x^4 \right]_0^a = \frac{1}{2}ab$$

Now

$$dI_y = x^2 dA = x^2 \left[\left(\frac{b}{a}x - \frac{b}{a^3}x^3 \right) dx \right]$$

Then

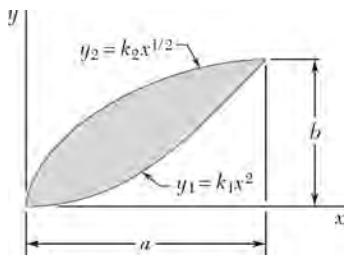
$$I_y = \int dI_y = 2 \int_0^a \frac{b}{a} x^2 \left(x - \frac{1}{a^2}x^3 \right) dx$$

$$= 2 \frac{b}{a} \left[\frac{1}{4}x^4 - \frac{1}{6} \frac{1}{a^2}x^6 \right]_0^a \quad \text{or} \quad I_y = \frac{1}{6}a^3b \blacktriangleleft$$

and

$$k_y^2 = \frac{I_y}{A} = \frac{\frac{1}{6}a^3b}{\frac{1}{2}ab} = \frac{1}{3}a^2$$

$$\text{or} \quad k_y = \frac{a}{\sqrt{3}} \blacktriangleleft$$



PROBLEM 9.18

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION

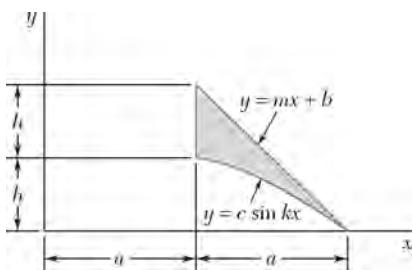
See figure of solution on Problem 9.16.

$$A = \frac{1}{3}ab \quad dI_y = x^2 dA = x^2(y_2 - y_1)dx$$

$$I_y = \int_0^a x^2 \left(\frac{b}{a^{1/2}} x^{1/2} - \frac{b}{a^2} x^2 \right) dx = \frac{b}{a^{1/2}} \int_0^a x^{5/2} dx - \frac{b}{a^2} \int_0^a x^4 dx$$

$$I_y = \frac{b}{a^{1/2}} \cdot \frac{b^{7/2}}{\left(\frac{7}{2}\right)} - \frac{b}{a^2} \cdot \frac{a^5}{5} = \left(\frac{2}{7} - \frac{1}{5}\right) a^3 b \quad I_y = \frac{3}{35} a^3 b \blacktriangleleft$$

$$k_y^2 = \frac{I_y}{A} = \frac{\left(\frac{3}{35} a^3 b\right)}{\frac{ab}{3}} \quad k_y = a \sqrt{\frac{9}{35}} \blacktriangleleft$$



PROBLEM 9.19

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION

$$y_1: \text{At } x = 2a, \quad y = 0: \quad 0 = c \sin k(2a)$$

$$2ak = \pi \quad \text{or} \quad k = \frac{\pi}{2a}$$

$$\text{At } x = a, \quad y = h: \quad h = c \sin \frac{\pi}{2a}(a) \quad \text{or} \quad c = h$$

$$y_2: \text{At } x = a, \quad y = 2h: \quad 2h = ma - b$$

$$\text{At } x = 2a, \quad y = 0: \quad 0 = m(2a) + b$$

Solving yields

$$m = -\frac{2h}{a}, \quad b = 4h$$

Then

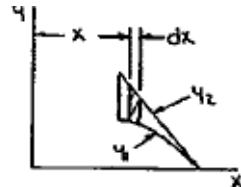
$$\begin{aligned} y_1 &= h \sin \frac{\pi}{2a} x & y_2 &= -\frac{2h}{a} x + 4h \\ &= \frac{2h}{a} (-x + 2a) \end{aligned}$$

Now

$$dA = (y_2 - y_1)dx = \left[\frac{2h}{a}(-x + 2a) - h \sin \frac{\pi}{2a} x \right] dx$$

Then

$$\begin{aligned} A &= \int dA = \int_a^{2a} h \left[\frac{2}{a}(-x + 2a) - \sin \frac{\pi}{2a} x \right] dx \\ &= h \left[-\frac{1}{a}(-x + 2a)^2 + \frac{2a}{\pi} \cos \frac{\pi}{2a} x \right]_a^{2a} \\ &= h \left[\left(-\frac{2a}{\pi} \right) + \frac{1}{a}(-a + 2a)^2 \right] = ah \left(1 - \frac{2}{\pi} \right) \\ &= 0.36338ah \end{aligned}$$



PROBLEM 9.19 (Continued)

Find: I_x and k_x

We have

$$dI_x = \left(\frac{1}{3}y_2 - \frac{1}{3}y_1 \right) dx = \frac{1}{3} \left\{ \left[\frac{2h}{a}(-x+2a) \right]^3 - \left(h \sin \frac{\pi}{2a} x \right)^3 \right\} dx \\ = \frac{h^3}{3} \left[\frac{8}{a^3}(-x+2a)^3 - \sin^3 \frac{\pi}{2a} x \right]$$

Then

$$I_x = \int dI_x = \int_a^{2a} \frac{h^3}{3} \left[\frac{8}{a^3}(-x+2a)^3 - \sin^3 \frac{\pi}{2a} x \right] dx$$

Now

$$\sin^3 \theta = \sin \theta (1 - \cos^2 \theta) = \sin \theta - \sin \theta \cos^2 \theta$$

Then

$$I_x = \frac{h^3}{3} \int_a^{2a} \left[\frac{8}{a^3}(-x+2a)^3 - \left(\sin \frac{\pi}{2a} x - \sin \frac{\pi}{2a} x \cos^2 \frac{\pi}{2a} x \right) \right] dx \\ = \frac{h^3}{3} \left[-\frac{2}{a^3}(-x+2a)^4 + \frac{2a}{\pi} \cos \frac{\pi}{2a} x - \frac{2a}{3\pi} \cos^3 \frac{\pi}{2a} x \right]_a^{2a} \\ = \frac{h^3}{3} \left[\left(-\frac{2a}{\pi} + \frac{2a}{3\pi} \right) + \frac{2}{a^3}(-a+2a)^4 \right] \\ = \frac{2}{3}ah^3 \left(1 - \frac{2}{3\pi} \right)$$

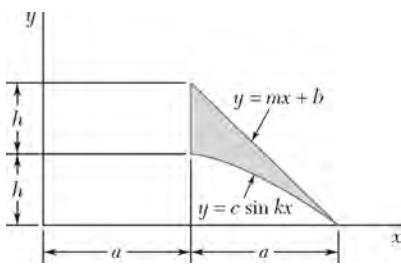
$$I_x = 0.52520ah^3$$

$$\text{or } I_x = 0.525ah^3 \blacktriangleleft$$

and

$$k_x^2 = \frac{I_x}{A} = \frac{0.52520ah^3}{0.36338ah}$$

$$\text{or } k_x = 1.202h \blacktriangleleft$$



PROBLEM 9.20

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION

$$y_1: \text{At } x = 2a, y = 0: \quad 0 = c \sin k(2a)$$

$$2ak = \pi \quad \text{or} \quad k = \frac{\pi}{2a}$$

$$\text{At } x = a, y = h: \quad h = c \sin \frac{\pi}{2a}(a) \quad \text{or} \quad c = h$$

$$y_2: \text{At } x = a, y = 2h: \quad 2h = ma - b$$

$$\text{At } x = 2a, y = 0: \quad 0 = m(2a) + b$$

Solving yields

$$m = -\frac{2h}{a}, \quad b = 4h$$

Then

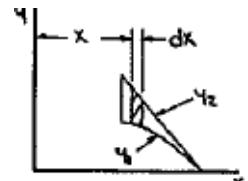
$$y_1 = h \sin \frac{\pi}{2a} x \quad y_2 = -\frac{2h}{a} x + 4h \\ = \frac{2h}{a} (-x + 2a)$$

Now

$$dA = (y_2 - y_1)dx = \left[\frac{2h}{a}(-x + 2a) - h \sin \frac{\pi}{2a} x \right] dx$$

Then

$$A = \int dA = \int_a^{2a} h \left[\frac{2}{a}(-x + 2a) - \sin \frac{\pi}{2a} x \right] dx \\ = h \left[-\frac{1}{a}(-x + 2a)^2 + \frac{2a}{\pi} \cos \frac{\pi}{2a} x \right]_a^{2a} \\ = h \left[\left(-\frac{2a}{\pi} \right) + \frac{1}{a}(-a + 2a)^2 \right] = ah \left(1 - \frac{2}{\pi} \right) \\ = 0.36338ah$$



PROBLEM 9.20 (Continued)

Find: I_y and k_y

We have

$$dI_y = x^2 dA = x^2 \left\{ \left[\frac{2h}{a} (-x + 2a) - h \sin \frac{\pi}{2a} x \right] dx \right\}$$

$$= h \left[\frac{2}{a} (-x^3 + 2ax^2) - x^2 \sin \frac{\pi}{2a} x \right] dx$$

Then

$$I_y = \int dI_y = \int_a^{2a} h \left[\frac{2}{a} (-x^3 + 2ax^2) - x^2 \sin \frac{\pi}{2a} x \right] dx$$

Now using integration by parts with

$$u = x^2 \quad dv = \sin \frac{\pi}{2a} x dx$$

$$du = 2x dx \quad v = -\frac{2a}{\pi} \cos \frac{\pi}{2a} x$$

Then

$$\int x^2 \sin \frac{\pi}{2a} x dx = x^2 \left(-\frac{2a}{\pi} \cos \frac{\pi}{2a} x \right) - \int \left(-\frac{2a}{\pi} \cos \frac{\pi}{2a} x \right) (2x dx)$$

Now let

$$u = x \quad dv = \cos \frac{\pi}{2a} x dx$$

$$du = dx \quad v = \frac{2a}{\pi} \sin \frac{\pi}{2a} x$$

Then

$$\int x^2 \sin \frac{\pi}{2a} x dx = -\frac{2a}{\pi} x^2 \cos \frac{\pi}{2a} x + \frac{4a}{\pi} \left[x \left(\frac{2a}{\pi} \sin \frac{\pi}{2a} x \right) - \int \left(\frac{2a}{\pi} \sin \frac{\pi}{2a} x \right) dx \right]$$

$$I_y = h \left[\frac{2}{a} \left(-\frac{1}{4} x^4 + \frac{2}{3} a x^3 \right) \right.$$

$$\left. - \left(-\frac{2a}{\pi} x^2 \cos \frac{\pi}{2a} x + \frac{8a^2}{\pi^2} x \sin \frac{\pi}{2a} x + \frac{16a^3}{\pi^3} \cos \frac{\pi}{2a} x \right) \right]_a^{2a}$$

$$= h \left\{ \frac{2}{a} \left[-\frac{1}{4} (2a)^4 + \frac{2}{3} a (2a)^3 \right] - \frac{2a}{\pi} (2a)^2 + \frac{16a^3}{\pi^3} \right\}$$

$$- h \left\{ \frac{2}{a} \left[-\frac{1}{4} (a)^4 + \frac{2}{3} a (a)^3 \right] - \frac{8a^2}{\pi^2} (a) \right\}$$

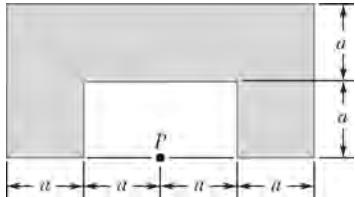
$$= 0.61345a^3 h$$

$$\text{or } I_y = 0.613a^3 h \blacktriangleleft$$

and

$$k_y^2 = \frac{I_y}{A} = \frac{0.61345a^3 h}{0.36338ah}$$

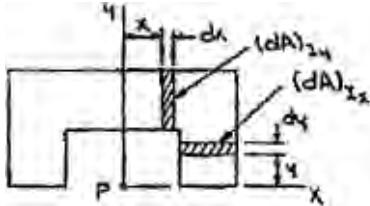
$$\text{or } k_y = 1.299a \blacktriangleleft$$



PROBLEM 9.21

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



We have

$$dI_x = y^2 dA = y^2 [(x_2 - x_1) dy]$$

Then

$$\begin{aligned} I_x &= 2 \left[\int_2^a y^2 (2a - a) dy + \int_a^{2a} y^2 (2a - 0) dy \right] \\ &= 2 \left\{ a \left[\frac{1}{3} y^3 \right]_0^a + 2a \left[\frac{1}{3} y^3 \right]_a^{2a} \right\} \\ &= 2 \left\{ a \left[\frac{1}{3} (a)^3 \right] + \frac{2}{3} a \left[(2a)^3 - (a)^3 \right] \right\} \\ &= 10a^4 \end{aligned}$$

Also

$$dI_y = x^2 dA = x^2 [(y_2 - y_1) dx]$$

Then

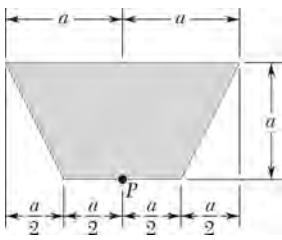
$$\begin{aligned} I_y &= 2 \left[\int_0^a x^2 (2a - a) dx + \int_a^{2a} x^2 (2a - 0) dx \right] \\ &= 10a^4 \end{aligned}$$

Now

$$J_P = I_x + I_y = 10a^4 + 10a^4 \quad \text{or} \quad J_P = 20a^4 \blacktriangleleft$$

and

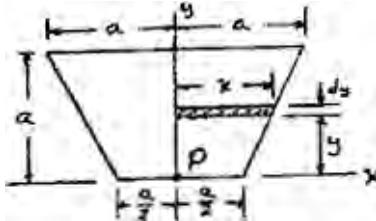
$$\begin{aligned} k_P^2 &= \frac{J_P}{A} = \frac{20a^4}{(4a)(2a) - (2a)(a)} \\ &= \frac{10}{3} a^2 \quad \text{or} \quad k_P = 1.826a \blacktriangleleft \end{aligned}$$



PROBLEM 9.22

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



$$x = \frac{a}{2} + \frac{a}{2} \frac{y}{a} = \frac{1}{2}(a + y)$$

$$dA = x dy = \frac{1}{2}(a + y) dy$$

$$\frac{1}{2}A = \int_0^a dA = \int_0^a \frac{1}{2}(a + y) dy = \frac{1}{2} \left[ay + \frac{y^2}{2} \right]_0^a = \frac{1}{2} \left(a^2 + \frac{a^2}{2} \right) = \frac{3}{4}a^2 \quad A = \frac{3}{2}a^2 \quad \square$$

$$\begin{aligned} \frac{1}{2}I_x &= \int y^2 dA = \int_0^a y^2 \frac{1}{2}(a + y) dy = \frac{1}{2} \int_0^a (ay^2 + y^3) dy \\ &= \frac{1}{2} \left[a \frac{y^3}{3} + \frac{y^4}{4} \right]_0^a = \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4} \right) a^4 = \frac{1}{2} \frac{7}{12} a^4 \quad I_x = \frac{7}{12} a^4 \quad \square \end{aligned}$$

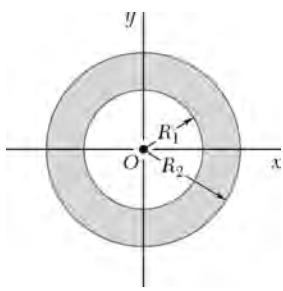
$$\frac{1}{2}I_y = \int_0^a \frac{1}{3}x^3 dy = \frac{1}{3} \int_0^a \left(\frac{1}{2}(a + y) \right)^3 dy$$

$$\frac{1}{2}I_y = \frac{1}{24} \int_0^a (a + y)^3 dy = \frac{1}{24} \left[\frac{1}{4}(a + y)^4 \right]_0^a = \frac{1}{96} [(2a)^4 - a^4] = \frac{15}{96} a^4$$

$$\frac{1}{2}I_y = \frac{5}{32} a^4 \quad I_y = \frac{5}{16} a^4 \quad \square$$

$$\text{From Eq. (9.4):} \quad J_O = I_x + I_y = \frac{7}{12} a^4 + \frac{5}{16} a^4 = \left(\frac{28+15}{48} \right) a^4 \quad J_O = \frac{43}{48} a^4 \quad \blacktriangleleft$$

$$J_O = k_O^2 A \quad k_O^2 = \frac{J_O}{A} = \frac{\frac{43}{48} a^4}{\frac{3}{2} a^2} = \frac{43}{72} a^2 \quad k_O = a \sqrt{\frac{43}{72}} \quad k_O = 0.773a \quad \blacktriangleleft$$



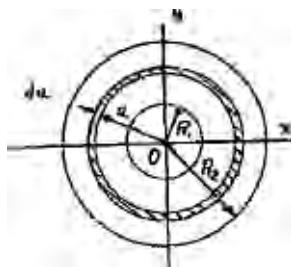
PROBLEM 9.23

(a) Determine by direct integration the polar moment of inertia of the annular area shown with respect to Point O . (b) Using the result of Part a, determine the moment of inertia of the given area with respect to the x axis.

SOLUTION

(a)

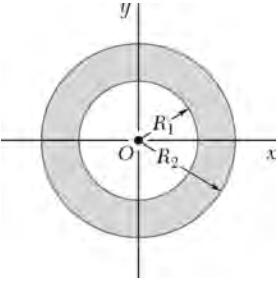
$$\begin{aligned} dA &= 2\pi u \, du \\ dJ_O &= u^2 dA = u^2 (2\pi u \, du) \\ &= 2\pi u^3 \, du \\ J_O &= \int dJ_O = 2\pi \int_{R_1}^{R_2} u^3 \, du \\ &= 2\pi \left[\frac{1}{4} u^4 \right]_{R_1}^{R_2} \end{aligned}$$



$$J_O = \frac{\pi}{2} [R_2^4 - R_1^4] \blacktriangleleft$$

(b) From Eq. (9.4): (Note by symmetry.) $I_x = I_y$

$$\begin{aligned} J_O &= I_x + I_y = 2I_x \\ I_x &= \frac{1}{2} J_O = \frac{\pi}{4} [R_2^4 - R_1^4] \\ I_x &= \frac{\pi}{4} [R_2^4 - R_1^4] \blacktriangleleft \end{aligned}$$



PROBLEM 9.24

(a) Show that the polar radius of gyration k_O of the annular area shown is approximately equal to the mean radius $R_m = (R_1 + R_2)/2$ for small values of the thickness $t = R_2 - R_1$. (b) Determine the percentage error introduced by using R_m in place of k_O for the following values of t/R_m : $1, \frac{1}{2}$, and $\frac{1}{10}$.

SOLUTION

(a) From Problem 9.23:

$$\begin{aligned} J_O &= \frac{\pi}{2} [R_2^4 - R_1^4] \\ k_O^2 &= \frac{J_O}{A} = \frac{\frac{\pi}{2} [R_2^4 - R_1^4]}{\pi (R_2^2 - R_1^2)} \\ k_O^2 &= \frac{1}{2} [R_2^2 + R_1^2] \end{aligned}$$

Thickness: $t = R_2 - R_1$

Mean radius: $R_m = \frac{1}{2}(R_1 + R_2)$

Thus, $R_1 = R_m - \frac{1}{2}t$ and $R_2 = R_m + \frac{1}{2}t$

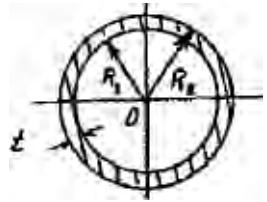
$$k_O^2 = \frac{1}{2} \left[\left(R_m + \frac{1}{2}t \right)^2 + \left(R_m - \frac{1}{2}t \right)^2 \right] = R_m^2 + \frac{1}{4}t^2$$

For t small compared to R_m : $k_O^2 \approx R_m^2$

$$k_O \approx R_m \blacktriangleleft$$

(b) Percentage error = $\frac{(\text{exact value}) - (\text{approximation value})}{(\text{exact value})} 100$

$$\text{P.E.} = -\frac{\sqrt{R_m^2 + \frac{1}{4}t^2} - R_m}{\sqrt{R_m^2 + \frac{1}{4}t^2}} (100) = -\frac{\sqrt{1 + \frac{1}{4}\left(\frac{t}{R_m}\right)^2} - 1}{\sqrt{1 + \frac{1}{4}\left(\frac{t}{R_m}\right)^2}} 100$$



PROBLEM 9.24 (Continued)

For $\frac{t}{R_m} = 1$:

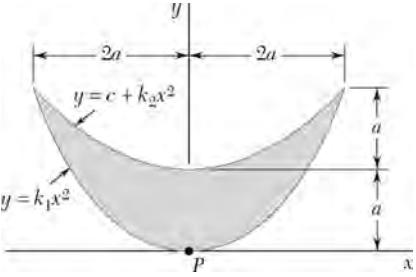
$$\text{P.E.} = \frac{\sqrt{1 + \frac{1}{4}} - 1}{\sqrt{1 + \frac{1}{4}}} (100) = -10.56\% \quad \blacktriangleleft$$

For $\frac{t}{R_m} = \frac{1}{2}$:

$$\text{P.E.} = -\frac{\sqrt{1 + \frac{1}{4} \left(\frac{1}{2}\right)^2} - 1}{\sqrt{1 + \frac{1}{4} \left(\frac{1}{2}\right)^2}} (100) = -2.99\% \quad \blacktriangleleft$$

For $\frac{t}{R_m} = \frac{1}{10}$:

$$\text{P.E.} = \frac{\sqrt{1 + \frac{1}{4} \left(\frac{1}{10}\right)^2} - 1}{\sqrt{1 + \frac{1}{4} \left(\frac{1}{10}\right)^2}} (100) = -0.125\% \quad \blacktriangleleft$$



PROBLEM 9.25

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point *P*.

SOLUTION

$$y_1: \text{At } x = 2a, \quad y = 2a;$$

$$2a = k_1(2a)^2 \quad \text{or} \quad k_1 = \frac{1}{2a}$$

$$y_2: \text{At } x = 0, \quad y = a; \quad a = c$$

$$\text{At } x = 2a, \quad y = 2a;$$

$$2a = a + k_2(2a)^2 \quad \text{or} \quad k_2 = \frac{1}{4a}$$

Then

$$\begin{aligned} y_1 &= \frac{1}{2a}x^2 & y_2 &= a + \frac{1}{4a}x^2 \\ &= \frac{1}{4a}(4a^2 + x^2) \end{aligned}$$

Now

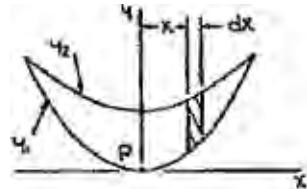
$$\begin{aligned} dA &= (y_2 - y_1)dx = \left[\frac{1}{4a}(4a^2 + x^2) - \frac{1}{2a}x^2 \right] dx \\ &= \frac{1}{4a}(4a^2 - x^2)dx \end{aligned}$$

Then

$$A = \int dA = 2 \int_0^{2a} \frac{1}{4a}(4a^2 - x^2)dx = \frac{1}{2a} \left[4a^2x - \frac{1}{3}x^3 \right]_0^{2a} = \frac{8}{3}a^2$$

Now

$$\begin{aligned} dI_x &= \left(\frac{1}{3}y_2^3 - \frac{1}{3}y_1^3 \right) dx = \frac{1}{3} \left\{ \left[\frac{1}{4a}(4a^2 + x^2) \right]^3 - \left[\frac{1}{2a}x^2 \right]^3 \right\} dx \\ &= \frac{1}{3} \left[\frac{1}{64a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 + x^6) - \frac{1}{8a^3}x^6 \right] dx \\ &= \frac{1}{192a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6)dx \end{aligned}$$



PROBLEM 9.25 (Continued)

Then

$$\begin{aligned}
 I_x &= \int dI_x = 2 \int_0^{2a} \frac{1}{192a^3} (64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6) dx \\
 &= \frac{1}{96a^3} \left[64a^6x + 16a^4x^3 + \frac{12}{5}a^2x^5 - x^7 \right]_0^{2a} \\
 &= \frac{1}{96a^3} \left[64a^6(2a) + 16a^4(2a)^3 + \frac{12}{5}a^2(2a)^5 - (2a)^7 \right] \\
 &= \frac{1}{96}a^4 \left(128 + 128 + \frac{12}{5} \times 32 - 128 \right) = \frac{32}{15}a^4
 \end{aligned}$$

Also

$$dI_y = x^2 dA = x^2 \left[\frac{1}{4a} (4a^2 - x^2) dx \right]$$

Then

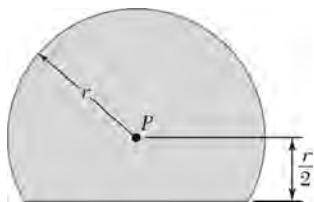
$$\begin{aligned}
 I_y &= \int dI_y = 2 \int_0^{2a} \frac{1}{4a} x^2 (4a^2 - x^2) dx = \frac{1}{2a} \left[\frac{4}{3}a^2x^3 - \frac{1}{5}x^5 \right]_0^{2a} \\
 &= \frac{1}{2a} \left[\frac{4}{3}a^2(2a)^3 - \frac{1}{5}(2a)^5 \right] = \frac{32}{2}a^4 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{32}{15}a^4
 \end{aligned}$$

Now

$$J_P = I_x + I_y = \frac{32}{15}a^4 + \frac{32}{15}a^4 \quad \text{or} \quad J_P = \frac{64}{15}a^4 \blacktriangleleft$$

and

$$k_P^2 = \frac{J_P}{A} = \frac{\frac{64}{15}a^4}{\frac{8}{3}a^2} = \frac{8}{5}a^2 \quad \text{or} \quad k_P = 1.265a \blacktriangleleft$$



PROBLEM 9.26

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION

The equation of the circle is

$$x^2 + y^2 = r^2$$

So that

$$x = \sqrt{r^2 - y^2}$$

Now

$$dA = x dy = \sqrt{r^2 - y^2} dy$$

Then

$$A = \int dA = 2 \int_{-r/2}^r \sqrt{r^2 - y^2} dy$$

Let

$$y = r \sin \theta; \quad dy = r \cos \theta d\theta$$

Then

$$\begin{aligned} A &= 2 \int_{-\pi/6}^{\pi/2} \sqrt{r^2 - (r \sin \theta)^2} r \cos \theta d\theta \\ &= 2 \int_{-\pi/6}^{\pi/2} r^2 \cos^2 \theta d\theta = 2r^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{-\pi/6}^{\pi/2} \\ &= 2r^2 \left[\frac{\pi}{2} - \left(\frac{-\pi}{6} + \frac{\sin -\pi/3}{4} \right) \right] = 2r^2 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{8} \right) \\ &= 2.5274r^2 \end{aligned}$$

Now

$$dI_x = y^2 dA = y^2 \left(\sqrt{r^2 - y^2} dy \right)$$

Then

$$I_x = \int dI_x = 2 \int_{-r/2}^r y^2 \sqrt{r^2 - y^2} dy$$

Let

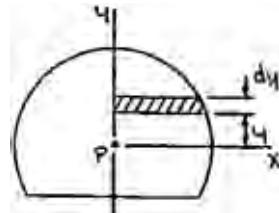
$$y = r \sin \theta; \quad dy = r \cos \theta d\theta$$

Then

$$\begin{aligned} I_x &= 2 \int_{-\pi/6}^{\pi/2} (r \sin \theta)^2 \sqrt{r^2 - (r \sin \theta)^2} r \cos \theta d\theta \\ &= 2 \int_{-\pi/6}^{\pi/2} r^2 \sin^2 \theta (r \cos \theta) r \cos \theta d\theta \end{aligned}$$

Now

$$\sin 2\theta = 2 \sin \theta \cos \theta \Rightarrow \sin^2 \theta \cos^2 \theta = \frac{1}{4} \sin 2\theta$$



PROBLEM 9.26 (Continued)

Then

$$\begin{aligned}
 I_x &= 2 \int_{-\pi/6}^{\pi/2} r^4 \left(\frac{1}{4} \sin^2 2\theta \right) d\theta = \frac{r^4}{2} \left[\frac{\theta}{2} - \frac{\sin 4\theta}{8} \right]_{-\pi/6}^{\pi/2} \\
 &= \frac{r^4}{2} \left[\frac{\frac{\pi}{2}}{2} - \left(\frac{\frac{\pi}{6}}{2} - \frac{\sin -\frac{2\pi}{3}}{8} \right) \right] \\
 &= \frac{r^4}{2} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{16} \right)
 \end{aligned}$$

Also

$$dI_y = \frac{1}{3} x^3 dy = \frac{1}{3} \left(\sqrt{r^2 - y^2} \right)^3 dy$$

Then

$$I_y = \int dI_y = 2 \int_{-r/2}^r \frac{1}{3} (r^2 - y^2)^{3/2} dy$$

Let

$$y = r \sin \theta; \quad dy = r \cos \theta d\theta$$

Then

$$\begin{aligned}
 I_y &= \frac{2}{3} \int_{-\pi/6}^{\pi/2} [r^2 - (r \sin \theta)^2]^{3/2} r \cos \theta d\theta \\
 I_y &= \frac{2}{3} \int_{-\pi/6}^{\pi/2} (r^3 \cos^3 \theta) r \cos \theta d\theta
 \end{aligned}$$

Now

$$\cos^4 \theta = \cos^2 \theta (1 - \sin^2 \theta) = \cos^2 \theta - \frac{1}{4} \sin^2 2\theta$$

Then

$$\begin{aligned}
 I_y &= \frac{2}{3} \int_{-\pi/6}^{\pi/2} r^4 \left(\cos^2 \theta - \frac{1}{4} \sin^2 2\theta \right) d\theta \\
 &= \frac{2}{3} r^4 \left[\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) - \frac{1}{4} \left(\frac{\theta}{2} - \frac{\sin 4\theta}{8} \right) \right]_{-\pi/6}^{\pi/2} \\
 &= \frac{2}{3} r^4 \left\{ \left[\frac{\frac{\pi}{2}}{2} - \frac{1}{4} \left(\frac{\pi}{2} \right) \right] - \left[\frac{-\frac{\pi}{6}}{2} + \frac{\sin -\frac{\pi}{3}}{4} - \frac{1}{4} \left(\frac{-\frac{\pi}{6}}{2} - \frac{\sin -\frac{2\pi}{3}}{8} \right) \right] \right\} \\
 &= \frac{2}{3} r^4 \left[\frac{\pi}{4} - \frac{\pi}{16} + \frac{\pi}{12} + \frac{1}{4} \left(\frac{\sqrt{3}}{2} \right) - \frac{\pi}{48} + \frac{1}{32} \left(\frac{\sqrt{3}}{2} \right) \right] \\
 &= \frac{2}{3} r^4 \left(\frac{\pi}{4} + \frac{9\sqrt{3}}{64} \right)
 \end{aligned}$$

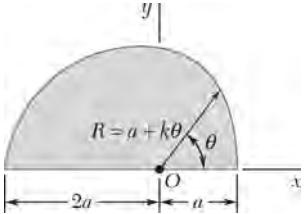
PROBLEM 9.26 (Continued)

Now

$$J_P = I_x + I_y = \frac{r^4}{2} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{16} \right) + \frac{2}{3} r^4 \left(\frac{\pi}{4} + \frac{9\sqrt{3}}{64} \right)$$
$$= r^4 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{16} \right) = 1.15545r^4 \quad \text{or } J_P = 1.155r^4 \blacktriangleleft$$

and

$$k_P^2 = \frac{J_P}{A} = \frac{1.15545r^4}{2.5274r^2} \quad \text{or } k_P = 0.676r \blacktriangleleft$$



PROBLEM 9.27

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point O .

SOLUTION

At $\theta = \pi$, $R = 2a$:

$$2a = a + k(\pi)$$

or

$$k = \frac{a}{\pi}$$

Then

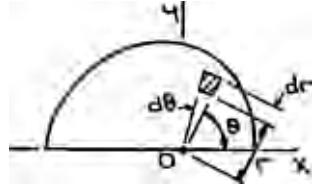
$$R = a + \frac{a}{\pi} \theta = a \left(1 + \frac{\theta}{\pi}\right)$$

Now

$$\begin{aligned} dA &= (dr)(r d\theta) \\ &= r dr d\theta \end{aligned}$$

Then

$$A = \int dA = \int_0^\pi \int_0^{a(1+\theta/\pi)} r dr d\theta = \int_0^\pi \left[\frac{1}{2} r^2 \right]_0^{a(1+\theta/\pi)} d\theta$$



$$\begin{aligned} A &= \int_0^\pi \frac{1}{2} a^2 \left(1 + \frac{\theta}{\pi}\right)^2 d\theta = \frac{1}{2} a^2 \left[\frac{\pi}{3} \left(1 + \frac{\theta}{\pi}\right)^3 \right]_0^\pi \\ &= \frac{1}{6} \pi a^2 \left[\left(1 + \frac{\pi}{\pi}\right)^3 - (1)^3 \right] = \frac{7}{6} \pi a^2 \end{aligned}$$

Now

$$dJ_O = r^2 dA = r^2 (r dr d\theta)$$

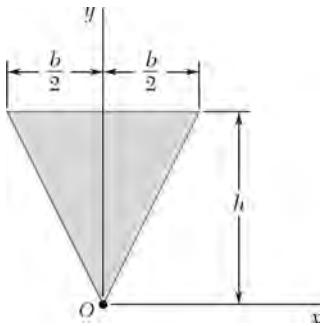
Then

$$\begin{aligned} J_O &= \int dJ_O = \int_0^\pi \int_0^{a(1+\theta/\pi)} r^3 dr d\theta \\ &= \int_0^\pi \left[\frac{1}{4} r^4 \right]_0^{a(1+\theta/\pi)} d\theta = \int_0^\pi \frac{1}{4} a^4 \left(1 + \frac{\theta}{\pi}\right)^4 d\theta \\ &= \frac{1}{4} a^4 \left[\frac{\pi}{5} \left(1 + \frac{\theta}{\pi}\right)^5 \right]_0^\pi = \frac{1}{20} \pi a^4 \left[\left(1 + \frac{\pi}{\pi}\right)^5 - (1)^5 \right] \quad \text{or } J_O = \frac{31}{20} \pi a^4 \blacktriangleleft \end{aligned}$$

and

$$k_O^2 = \frac{J_O}{A} = \frac{\frac{31}{20} \pi a^4}{\frac{7}{6} \pi a^2} = \frac{93}{70} a^2 \quad \text{or } k_O = 1.153a \blacktriangleleft$$

PROBLEM 9.28



Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point O .

SOLUTION

By observation:

$$y = \frac{h}{\frac{b}{2}} x$$

or

$$x = \frac{b}{2h} y$$

Now

$$dA = x dy = \left(\frac{b}{2h} y \right) dy$$

and

$$dI_x = y^2 dA = \frac{b}{2h} y^3 dy$$

Then

$$\begin{aligned} I_x &= \int dI_x = 2 \int_0^h \frac{b}{2h} y^3 dy \\ &= \frac{b}{h} \frac{y^4}{4} \Big|_0^h = \frac{1}{4} b h^3 \end{aligned}$$

From above:

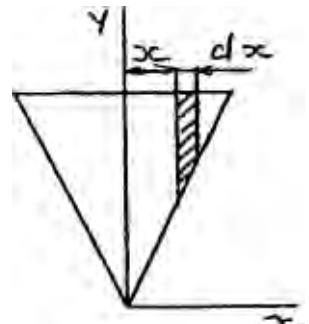
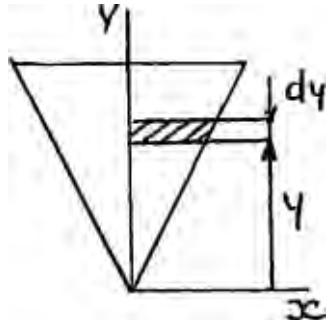
$$y = \frac{2h}{b} x$$

Now

$$\begin{aligned} dA &= (h - y) dx = \left(h - \frac{2h}{b} x \right) dx \\ &= \frac{h}{b} (b - 2x) dx \end{aligned}$$

and

$$dI_y = x^2 dA = x^2 \frac{h}{b} (b - 2x) dx$$



PROBLEM 9.28 (Continued)

Then

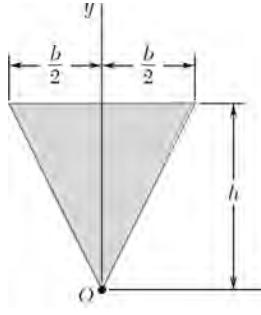
$$\begin{aligned}
 I_y &= \int dI_y = 2 \int_0^{b/2} \frac{h}{b} x^2 (b - 2x) dx \\
 &= 2 \frac{h}{b} \left[\frac{1}{3} b x^3 - \frac{1}{2} x^4 \right]_0^{b/2} \\
 &= 2 \frac{h}{b} \left[\frac{b}{3} \left(\frac{b}{2} \right)^3 - \frac{1}{2} \left(\frac{b}{2} \right)^4 \right] = \frac{1}{48} b^3 h
 \end{aligned}$$

Now

$$J_O = I_x + I_y = \frac{1}{4} b h^3 + \frac{1}{48} b^3 h \quad \text{or} \quad J_O = \frac{b h}{48} (12h^2 + b^2) \blacktriangleleft$$

and

$$k_O^2 = \frac{J_O}{A} = \frac{\frac{b h}{48} (12h^2 + b^2)}{\frac{1}{2} b h} = \frac{1}{24} (12h^2 + b^2) \quad \text{or} \quad k_O = \frac{\sqrt{12h^2 + b^2}}{24} \blacktriangleleft$$



PROBLEM 9.29*

Using the polar moment of inertia of the isosceles triangle of Problem 9.28, show that the centroidal polar moment of inertia of a circular area of radius r is $\pi r^4/2$. (Hint: As a circular area is divided into an increasing number of equal circular sectors, what is the approximate shape of each circular sector?)

PROBLEM 9.28 Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point O .

SOLUTION

First the circular area is divided into an increasing number of identical circular sectors. The sectors can be approximated by isosceles triangles. For a large number of sectors the approximate dimensions of one of the isosceles triangles are as shown.

For an isosceles triangle (see Problem 9.28):

$$J_O = \frac{bh}{48}(12h^2 + b^2)$$

Then with

$$b = r\Delta\theta \quad \text{and} \quad h = r$$

$$(\Delta J_O)_{\text{sector}} \approx \frac{1}{48}(r\Delta\theta)(r)[12r^2 + (r\Delta\theta)^2]$$

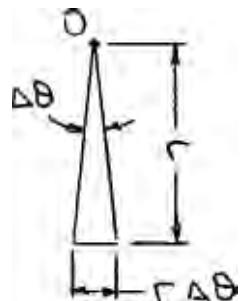
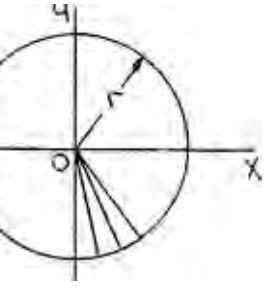
$$= \frac{1}{48}r^4\Delta\theta[(12 + (\Delta\theta)^2)]$$

Now

$$\begin{aligned} \frac{dJ_O}{d\theta} &= \lim_{\Delta\theta \rightarrow 0} \left(\frac{\Delta J_O}{\Delta\theta} \right) = \lim_{\Delta\theta \rightarrow 0} \left\{ \frac{1}{48}r^4[12 + (\Delta\theta)^2] \right\} \\ &= \frac{1}{4}r^4 \end{aligned}$$

Then

$$(J_O)_{\text{circle}} = \int dJ_O = \int_0^{2\pi} \frac{1}{4}r^4 d\theta = \frac{1}{4}r^4 [\theta]_0^{2\pi}$$



$$\text{or } (J_O)_{\text{circle}} = \frac{\pi}{2}r^4 \blacktriangleleft$$

PROBLEM 9.30*

Prove that the centroidal polar moment of inertia of a given area A cannot be smaller than $A^2/2\pi$. (Hint: Compare the moment of inertia of the given area with the moment of inertia of a circle that has the same area and the same centroid.)

SOLUTION

From the solution to sample Problem 9.2, the centroidal polar moment of inertia of a circular area is

$$(J_C)_{\text{cir}} = \frac{\pi}{2} r^4$$

The area of the circle is

$$A_{\text{cir}} = \pi r^2$$

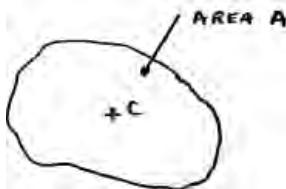
So that

$$[J_C(A)]_{\text{cir}} = \frac{A^2}{2\pi}$$

Two methods of solution will be presented. However, both methods depend upon the observation that as a given element of area dA is moved closer to some Point C , the value of J_C will be decreased ($J_C = \int r^2 dA$; as r decreases, so must J_C).

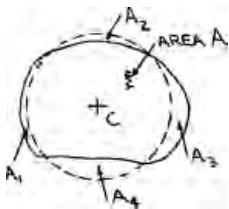
Solution 1

Imagine taking the area A and drawing it into a thin strip of negligible width and of sufficient length so that its area is equal to A . To minimize the value of $(J_C)_A$, the area would have to be distributed as closely as possible about C . This is accomplished by winding the strip into a tightly wound roll with C as its center; any voids in the roll would place the corresponding area farther from C than is necessary, thus increasing the value of $(J_C)_A$. (The process is analogous to rewinding a length of tape back into a roll.) Since the shape of the roll is circular, with the centroid of its area at C , it follows that



$$(J_C)_A \geq \frac{A^2}{2\pi} \quad \text{Q.E.D.} \blacktriangleleft$$

where the equality applies when the original area is circular.



PROBLEM 9.30* (Continued)

Solution 2

Consider an area A with its centroid at Point C and a circular area of area A with its center (and centroid) at Point C . Without loss of generality, assume that

$$A_1 = A_2 \quad A_3 = A_4$$

It then follows that

$$(J_C)_A = (J_C)_{\text{cir}} + [J_C(A_1) - J_C(A_2) + J_C(A_3) - J_C(A_4)]$$

Now observe that

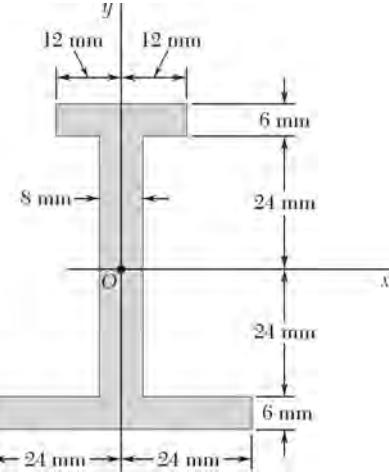
$$J_C(A_1) - J_C(A_2) \geq 0$$

$$J_C(A_3) - J_C(A_4) \geq 0$$

since as a given area is moved farther away from C its polar moment of inertia with respect to C must increase.

$$(J_C)_A \geq (J_C)_{\text{cir}}$$

or $(J_C)_A \geq \frac{A^2}{2\pi}$ Q.E.D. ◀



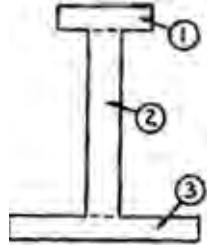
PROBLEM 9.31

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the x axis.

SOLUTION

First note that

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= [(24)(6) + (8)(48) + (48)(6)] \text{ mm}^2 \\ &= (144 + 384 + 288) \text{ mm}^2 \\ &= 816 \text{ mm}^2 \end{aligned}$$



Now

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

where

$$\begin{aligned} (I_x)_1 &= \frac{1}{12}(24 \text{ mm})(6 \text{ mm})^3 + (144 \text{ mm}^2)(27 \text{ mm})^2 \\ &= (432 + 104,976) \text{ mm}^4 \\ &= 105,408 \text{ mm}^4 \end{aligned}$$

$$(I_x)_2 = \frac{1}{12}(8 \text{ mm})(48 \text{ mm})^3 = 73,728 \text{ mm}^4$$

$$\begin{aligned} (I_x)_3 &= \frac{1}{12}(48 \text{ mm})(6 \text{ mm})^3 + (288 \text{ mm}^2)(27 \text{ mm})^2 \\ &= (864 + 209,952) \text{ mm}^4 = 210,816 \text{ mm}^4 \end{aligned}$$

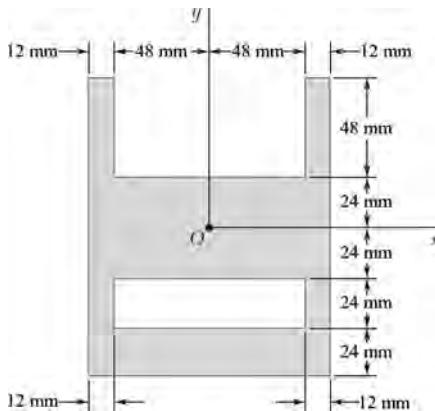
Then

$$I_x = (105,408 + 73,728 + 210,816) \text{ mm}^4$$

$$= 389,952 \text{ mm}^4 \quad \text{or } I_x = 390 \times 10^3 \text{ mm}^4 \blacktriangleleft$$

and

$$k_x^2 = \frac{I_x}{A} = \frac{389,952 \text{ mm}^4}{816 \text{ mm}^2} \quad \text{or } k_x = 21.9 \text{ mm} \blacktriangleleft$$



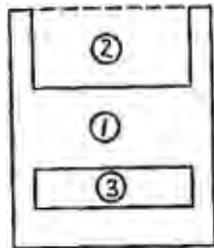
PROBLEM 9.32

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the x axis.

SOLUTION

First note that

$$\begin{aligned} A &= A_1 - A_2 - A_3 \\ &= [(120)(144) - (48)(96) - (96)(24)] \text{ mm}^2 \\ &= (17280 - 4608 - 2304) \text{ mm}^2 \\ &= 10368 \text{ mm}^2 \end{aligned}$$



Now

$$I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$$

where

$$(I_x)_1 = \frac{1}{12}(120 \text{ mm})(144 \text{ mm})^3 = 29.8594 \times 10^6 \text{ mm}^4$$

$$\begin{aligned} (I_x)_2 &= \frac{1}{12}(96 \text{ mm})(48 \text{ mm})^3 + (4608 \text{ mm})^2(48 \text{ mm})^2 \\ &= 11.501568 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\begin{aligned} (I_x)_3 &= \frac{1}{12}(96 \text{ mm})(24 \text{ mm})^3 + (2304 \text{ mm})^2(36 \text{ mm})^2 \\ &= 3.096576 \times 10^6 \text{ mm}^4 \end{aligned}$$

Then

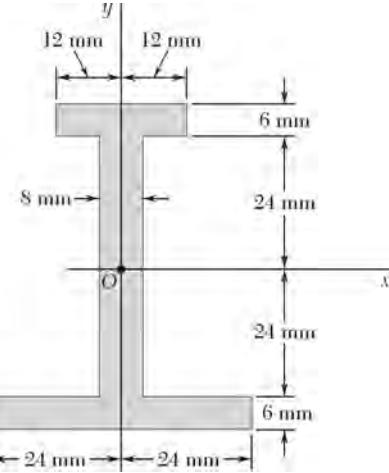
$$I_x = (29.8594 - 11.501568 - 3.096576) \times 10^6 \text{ mm}^4$$

$$= 15.261256 \times 10^6 \text{ mm}^4 \quad \text{or} \quad I_x = 15.26 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$k_x^2 = \frac{I_x}{A} = \frac{15.261256 \times 10^6 \text{ mm}^4}{10368 \text{ mm}^2}$$

$$\text{or} \quad k_x = 38.4 \text{ mm} \blacktriangleleft$$



PROBLEM 9.33

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the y axis.

SOLUTION

First note that

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= [(24 \times 6) + (8)(48) + (48)(6)] \text{ mm}^2 \\ &= (144 + 384 + 288) \text{ mm}^2 \\ &= 816 \text{ mm}^2 \end{aligned}$$

Now

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

where

$$\begin{aligned} (I_y)_1 &= \frac{1}{12}(6 \text{ mm})(24 \text{ mm})^3 = 6912 \text{ mm}^4 \\ (I_y)_2 &= \frac{1}{12}(48 \text{ mm})(8 \text{ mm})^3 = 2048 \text{ mm}^4 \\ (I_y)_3 &= \frac{1}{12}(6 \text{ mm})(48 \text{ mm})^3 = 55,296 \text{ mm}^4 \end{aligned}$$

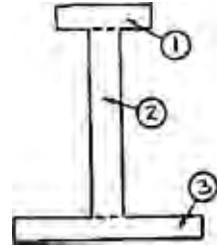
Then

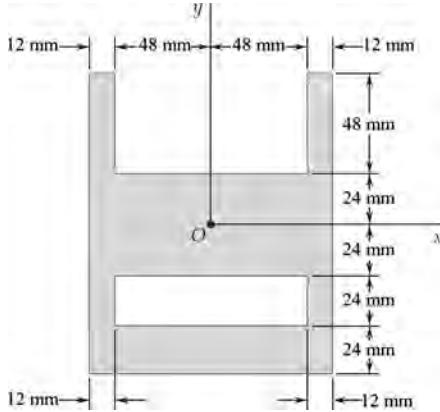
$$I_y = (6912 + 2048 + 55,296) \text{ mm}^4 = 64,256 \text{ mm}^4$$

$$\text{or } I_y = 64.3 \times 10^3 \text{ mm}^4 \blacktriangleleft$$

and

$$k_y^2 = \frac{I_y}{A} = \frac{64,256 \text{ mm}^4}{816 \text{ mm}^2} \quad \text{or} \quad k_y = 8.87 \text{ mm} \blacktriangleleft$$





PROBLEM 9.34

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the y axis.

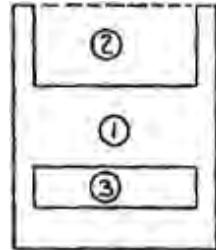
SOLUTION

First note that

$$\begin{aligned} A &= A_1 - A_2 - A_3 \\ &= [(120)(144) - (48)(96) - (96)(24)] \text{ mm}^2 \\ &= (17280 - 4608 - 2304) \text{ mm}^2 \\ &= 10368 \text{ mm}^2 \end{aligned}$$

Now

$$I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$$



where

$$(I_y)_1 = \frac{1}{12}(144 \text{ mm})(120 \text{ mm})^3 = 20.736 \times 10^6 \text{ mm}^4$$

$$(I_y)_2 = \frac{1}{12}(48 \text{ mm})(96 \text{ mm})^3 = 3.538944 \times 10^6 \text{ mm}^4$$

$$(I_y)_3 = \frac{1}{12}(24 \text{ mm})(96 \text{ mm})^3 = 1.769472 \times 10^6 \text{ mm}^4$$

Then

$$I_y = (20.736 - 3.53844 - 1.769472) \times 10^6 \text{ mm}^4$$

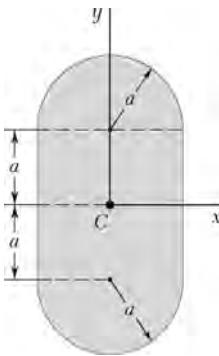
$$= 15.428088 \times 10^6$$

$$\text{or } I_y = 15.43 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$k_y^2 = \frac{I_y}{A} = \frac{15.428088 \times 10^6 \text{ mm}^4}{10368 \text{ mm}^2}$$

$$\text{or } k_y = 38.6 \text{ mm} \blacktriangleleft$$



PROBLEM 9.35

Determine the moments of inertia of the shaded area shown with respect to the x and y axes when $a = 20 \text{ mm}$.

SOLUTION

We have

$$I_x = (I_x)_1 + 2(I_x)_2$$

where

$$(I_x)_1 = \frac{1}{12}(40 \text{ mm})(40 \text{ mm})^3$$

$$= 213.33 \times 10^3 \text{ mm}^4$$



$$\begin{aligned} (I_x)_2 &= \left[\frac{\pi}{8}(20 \text{ mm})^4 - \frac{\pi}{2}(20 \text{ mm})^2 \left(\frac{4 \times 20}{3\pi} \text{ mm} \right)^2 \right] \\ &\quad + \frac{\pi}{2}(20 \text{ mm})^2 \left[\left(\frac{4 \times 20}{3\pi} + 20 \right) \text{ mm} \right]^2 \\ &= 527.49 \times 10^3 \text{ mm}^4 \end{aligned}$$

Then

$$I_x = [213.33 + 2(527.49)] \times 10^3 \text{ mm}^4$$

$$\text{or } I_x = 1.268 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

Also

$$I_y = (I_y)_1 + 2(I_y)_2$$

where

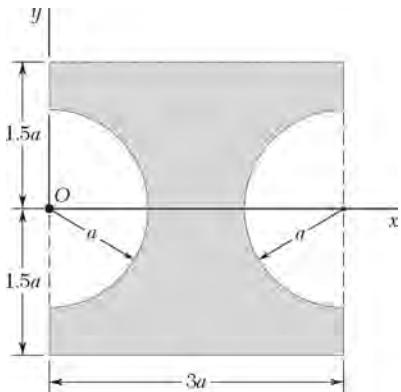
$$(I_y)_1 = \frac{1}{12}(40 \text{ mm})(40 \text{ mm})^3 = 213.33 \times 10^3 \text{ mm}^4$$

$$(I_y)_2 = \frac{\pi}{8}(20 \text{ mm})^4 = 62.83 \times 10^3 \text{ mm}^4$$

Then

$$I_y = [213.33 + 2(62.83)] \times 10^3 \text{ mm}^4$$

$$\text{or } I_y = 339 \times 10^3 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.36

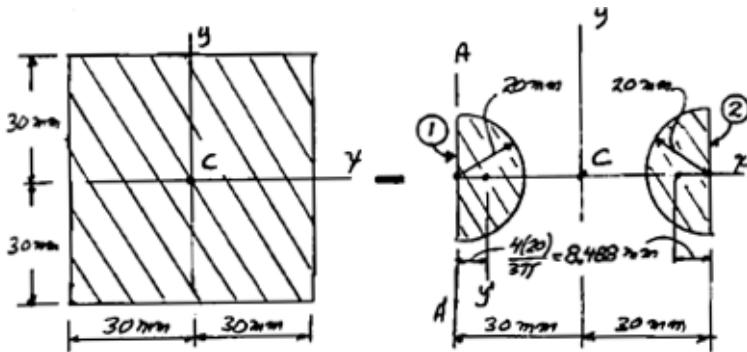
Determine the moments of inertia of the shaded area shown with respect to the x and y axes when $a = 20 \text{ mm}$.

SOLUTION

Given:

$$\text{Area} = \text{Square} - 2(\text{Semicircles})$$

For $a = 20 \text{ mm}$, we have



Square:

$$I_x = I_y = \frac{1}{12}(60)^4 = 1080 \times 10^3 \text{ mm}^4$$

Semicircle ①:

$$I_x = \frac{\pi}{8}(20)^4 = 62.83 \times 10^3 \text{ mm}^4$$

$$I_{AA'} = \bar{I}_{y'} + Ad^2; \quad \frac{\pi}{8}(20)^4 = \bar{I}_{y'} + \left(\frac{\pi}{2}\right)(20)^2(8.488)^2$$

$$\bar{I}_{y'} = 62.83 \times 10^3 - 45.27 \times 10^3$$

$$\bar{I}_{y'} = 17.56 \times 10^3 \text{ mm}^4$$

$$I_y = \bar{I}_{y'} + A(30 - 8.488)^2 = 17.56 \times 10^3 + \frac{\pi}{2}(20)^2(21.512)^2$$

$$I_y = 308.3 \times 10^3 \text{ mm}^4$$

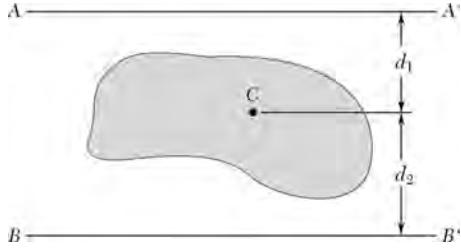
PROBLEM 9.36 (Continued)

Semicircle ②: Same as semicircle ①.

Entire Area:

$$\begin{aligned} I_x &= 1080 \times 10^3 - 2(62.83 \times 10^3) \\ &= 954.3 \times 10^3 \text{ mm}^4 \end{aligned} \quad I_x = 954 \times 10^3 \text{ mm}^4 \blacktriangleleft$$

$$\begin{aligned} I_y &= 1080 \times 10^3 - 2(308.3 \times 10^3) \\ &= 463.3 \times 10^3 \text{ mm}^4 \end{aligned} \quad I_y = 463 \times 10^3 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.37

For the 4000-mm² shaded area shown, determine the distance d_2 and the moment of inertia with respect to the centroidal axis parallel to AA' knowing that the moments of inertia with respect to AA' and BB' are 12×10^6 mm⁴ and 23.9×10^6 mm⁴, respectively, and that $d_1 = 25$ mm.

SOLUTION

We have

$$I_{AA'} = \bar{I} + Ad_1^2 \quad (1)$$

and

$$I_{BB'} = \bar{I} + Ad_2^2$$

Subtracting

$$I_{AA'} - I_{BB'} = A(d_1^2 - d_2^2)$$

or

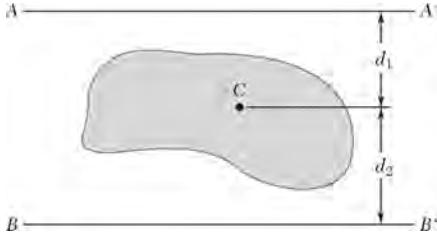
$$\begin{aligned} d_2^2 &= d_1^2 - \frac{I_{AA'} - I_{BB'}}{A} \\ &= (25 \text{ mm})^2 - \frac{(12 - 23.9)10^6 \text{ mm}^4}{4000 \text{ mm}^2} \\ &= 3600 \text{ mm}^2 \end{aligned}$$

$$\text{or } d_2 = 60.0 \text{ mm}$$

Using Eq. (1):

$$\bar{I} = 12 \times 10^6 \text{ mm}^4 - (4000 \text{ mm}^2)(25 \text{ mm})^2$$

$$\text{or } \bar{I} = 9.50 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.38

Determine for the shaded region the area and the moment of inertia with respect to the centroidal axis parallel to BB' , knowing that $d_1 = 25 \text{ mm}$ and $d_2 = 15 \text{ mm}$ and that the moments of inertia with respect to AA' and BB' are $7.84 \times 10^6 \text{ mm}^4$ and $5.20 \times 10^6 \text{ mm}^4$, respectively.

SOLUTION

We have

$$I_{AA'} = \bar{I} + Ad_1^2 \quad (1)$$

and

$$I_{BB'} = \bar{I} + Ad_2^2$$

Subtracting

$$I_{AA'} - I_{BB'} = A(d_1^2 - d_2^2)$$

or

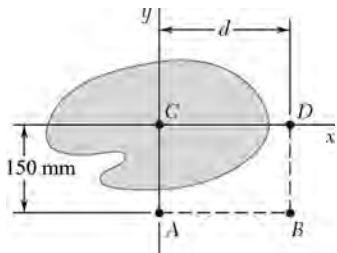
$$A = \frac{I_{AA'} - I_{BB'}}{d_1^2 - d_2^2} = \frac{(7.84 - 5.20)10^6 \text{ mm}^4}{(25 \text{ mm})^2 - (15 \text{ mm})^2}$$

or $A = 6600 \text{ mm}^2 \blacktriangleleft$

Using Eq. (1):

$$\begin{aligned} \bar{I} &= 7.84 \times 10^6 \text{ mm}^4 - (6600 \text{ mm}^2)(25 \text{ mm})^2 \\ &= 3.715 \times 10^6 \text{ mm}^4 \end{aligned}$$

or $\bar{I} = 3.72 \times 10^6 \text{ mm}^4 \blacktriangleleft$



PROBLEM 9.39

The shaded area is equal to $3.125 \times 10^4 \text{ mm}^2$. Determine its centroidal moments of inertia \bar{I}_x and \bar{I}_y , knowing that $\bar{I}_y = 2\bar{I}_x$ and that the polar moment of inertia of the area about Point A is $J_A = 878.9 \times 10^6 \text{ mm}^4$.

SOLUTION

$$\text{Given: } A = 3.125 \times 10^4 \text{ mm}^2 \quad \bar{I}_y = 2\bar{I}_x, \quad J_A = 878.9 \times 10^6 \text{ mm}^4$$

$$J_A = \bar{J}_C + A(150 \text{ mm})^2$$

$$878.9 \times 10^6 \text{ mm}^4 = \bar{J}_C + (3.125 \times 10^4 \text{ mm}^2)(150 \text{ mm})^2$$

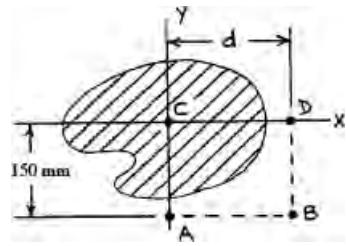
$$\bar{J}_C = 175.775 \times 10^6 \text{ mm}^4$$

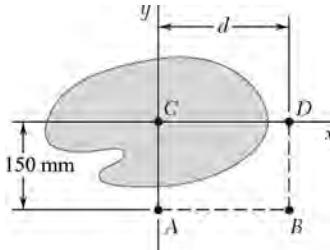
$$\bar{J}_C = \bar{I}_x + \bar{I}_y \quad \text{with} \quad \bar{I}_y = 2\bar{I}_x$$

$$175.775 \times 10^6 \text{ mm}^4 = \bar{I}_x + 2\bar{I}_x$$

$$\bar{I}_x = 58.6 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{I}_y = 2\bar{I}_x = 117.2 \times 10^6 \text{ mm}^4 \blacktriangleleft$$





PROBLEM 9.40

The polar moments of inertia of the shaded area with respect to Points A, B, and D are, respectively, $J_A = 1125 \times 10^6 \text{ mm}^4$, $J_B = 2625 \times 10^6 \text{ mm}^4$, and $J_D = 1781.25 \times 10^6 \text{ mm}^4$. Determine the shaded area, its centroidal moment of inertia \bar{J}_C , and the distance d from C to D.

SOLUTION

See figure at solution of Problem 9.39.

Given: $J_A = 1125 \times 10^6 \text{ mm}^4$, $J_B = 2625 \times 10^6 \text{ mm}^4$, $J_D = 1781.25 \times 10^6 \text{ mm}^4$

$$J_B = \bar{J}_C + A(CB)^2; \quad 2625 \times 10^6 \text{ mm}^4 = \bar{J}_C + A(150^2 + d^2) \quad (1)$$

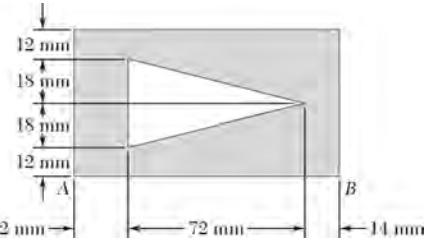
$$J_D = \bar{J}_C + A(CD)^2; \quad 1781.25 \times 10^6 \text{ mm}^4 = \bar{J}_C + Ad^2 \quad (2)$$

$$\text{Eq. (1) subtracted by Eq. (2): } J_B - J_D = 843.75 \times 10^6 \text{ mm}^4 = A(150)^2 \quad A = 37500 \text{ mm}^2 \blacktriangleleft$$

$$J_A = \bar{J}_C + A(AC)^2; \quad 1125 \times 10^6 \text{ mm}^4 = \bar{J}_C + (37500 \text{ mm}^2)(150 \text{ mm})^2$$

$$J_C = 281.25 \times 10^6 \text{ mm}^4 \quad \bar{J}_C = 281 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

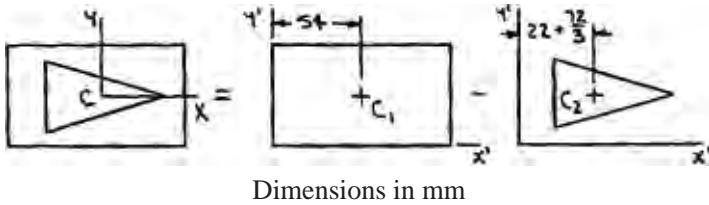
$$\text{Eq. (2): } 1781.25 \times 10^6 \text{ mm}^4 = 281.25 \times 10^6 \text{ mm}^4 + (37500 \text{ mm}^2)d^2 \quad d = 200 \text{ mm} \blacktriangleleft$$



PROBLEM 9.41

Determine the moments of inertia \bar{I}_x and \bar{I}_y of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



Dimensions in mm

First locate centroid C of the area.

Symmetry implies $\bar{Y} = 30$ mm.

	A, mm^2	\bar{x}, mm	$\bar{x}A, \text{mm}^3$
1	$108 \times 60 = 6480$	54	349,920
2	$-\frac{1}{2} \times 72 \times 36 = -1296$	46	-59,616
Σ	5184		290,304

Then

$$\bar{X} \Sigma A = \Sigma \bar{x}A : \quad \bar{X}(5184 \text{ mm}^2) = 290,304 \text{ mm}^3$$

or

$$\bar{X} = 56.0 \text{ mm}$$

Now

$$\bar{I}_x = (I_x)_1 - (I_x)_2$$

where

$$(I_x)_1 = \frac{1}{12}(108 \text{ mm})(60 \text{ mm})^3 = 1.944 \times 10^6 \text{ mm}^4$$

$$(I_x)_2 = 2 \left[\frac{1}{36}(72 \text{ mm})(18 \text{ mm})^3 + \left(\frac{1}{2} \times 72 \text{ mm} \times 18 \text{ mm} \right)(6 \text{ mm})^2 \right] \\ = 2(11,664 + 23,328) \text{ mm}^4 = 69.984 \times 10^3 \text{ mm}^4$$

$(I_x)_2$ is obtained by dividing A_2 into \Rightarrow

Then

$$\bar{I}_x = (1.944 - 0.069984) \times 10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_x = 1.874 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

PROBLEM 9.41 (Continued)

Also

$$\bar{I}_y = (I_y)_1 - (I_y)_2$$

where

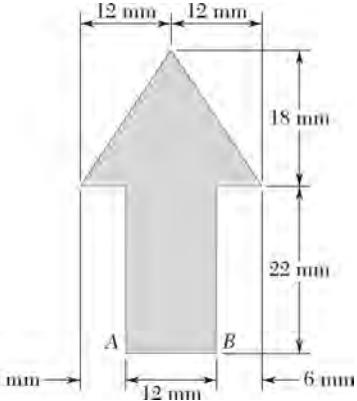
$$(I_y)_1 = \frac{1}{12}(60 \text{ mm})(108 \text{ mm})^3 + (6480 \text{ mm}^2)[(56.54) \text{ mm}]^2 \\ = (6,298,560 + 25,920) \text{ mm}^4 = 6.324 \times 10^6 \text{ mm}^4$$

$$(I_y)_2 = \frac{1}{36}(36 \text{ mm})(72 \text{ mm})^3 + (1296 \text{ mm}^2)[(56 - 46) \text{ mm}]^2 \\ = (373,248 + 129,600) \text{ mm}^4 = 0.502 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_y = (6.324 - 0.502)10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_y = 5.82 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.44

Determine the moments of inertia \bar{I}_x and \bar{I}_y of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION

First locate C of the area:

Symmetry implies $\bar{X} = 12 \text{ mm}$.

	A, mm^2	\bar{y}, mm	$\bar{y}A, \text{mm}^3$
1	$12 \times 22 = 264$	11	2904
2	$\frac{1}{2}(24)(18) = 216$	28	6048
Σ	480		8952

Then

$$\bar{Y} \Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(480 \text{ mm}^2) = 8952 \text{ mm}^3$$

$$\bar{Y} = 18.65 \text{ mm}$$

Now

$$\bar{I}_x = (I_x)_1 + (I_x)_2$$

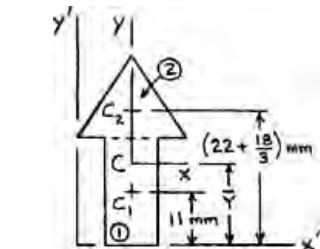
where

$$(I_x)_1 = \frac{1}{12}(12 \text{ mm})(22 \text{ mm})^3 + (264 \text{ mm}^2)[(18.65 - 11) \text{ mm}]^2 \\ = 26,098 \text{ mm}^4$$

$$(I_x)_2 = \frac{1}{36}(24 \text{ mm})(18 \text{ mm})^3 + (216 \text{ mm}^2)[(28 - 18.65) \text{ mm}]^2 \\ = 22,771 \text{ mm}^4$$

Then

$$\bar{I}_x = (26,098 + 22,771) \times 10^3 \text{ mm}^4$$



$$\text{or } \bar{I}_x = 48.9 \times 10^3 \text{ mm}^4 \blacktriangleleft$$

PROBLEM 9.42 (Continued)

Also

$$\bar{I}_y = (I_y)_1 + (I_y)_2$$

where

$$(I_y)_1 = \frac{1}{12}(22 \text{ mm})(12 \text{ mm})^3 = 3168 \text{ mm}^4$$

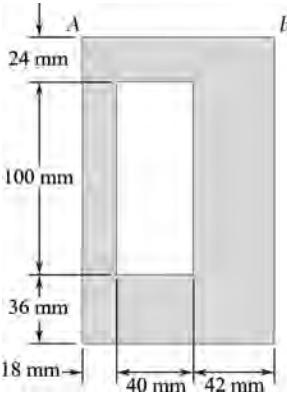
$$(I_y)_2 = 2 \left[\frac{1}{36}(18 \text{ mm})(12 \text{ mm})^3 + \left(\frac{1}{2} \times 18 \text{ mm} \times 12 \text{ mm} \right) (4 \text{ mm})^2 \right] \\ = 5184 \text{ mm}^4$$

$(I_y)_2$ is obtained by dividing A_2 into Δ]

Then

$$\bar{I}_y = (3168 + 5184) \text{ mm}^4$$

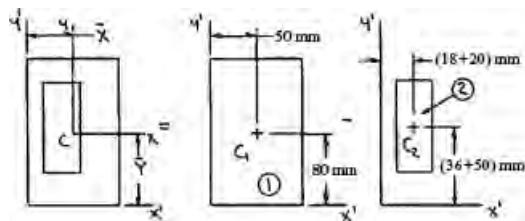
$$\text{or } \bar{I}_y = 8.35 \times 10^3 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.43

Determine the moments of inertia \bar{I}_x and \bar{I}_y of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



First locate centroid C of the area.

	A , mm^2	\bar{x} , mm	\bar{y} , mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$100 \times 160 = 16 \times 10^3$	50	80	800×10^3	1280×10^3
2	$-100 \times 40 = -4 \times 10^3$	38	86	-152×10^3	-344×10^3
Σ	12×10^3			648×10^3	936×10^3

Then

$$\bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(12 \times 10^3 \text{ mm}^2) = 648 \times 10^3 \text{ mm}^3$$

or

$$\bar{X} = 54.0 \text{ mm}$$

and

$$\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(12 \times 10^3 \text{ mm}^2) = 936 \times 10^3 \text{ mm}^3$$

or

$$\bar{Y} = 78.0 \text{ mm}$$

Now

$$\bar{I}_x = (I_x)_1 - (I_x)_2$$

where

$$(I_x)_1 = \frac{1}{12}(100 \text{ mm})(160 \text{ mm})^3 + (16 \times 10^3 \text{ mm}^2)[(80 - 78) \text{ mm}]^2$$

$$= (34.133 + 0.064) \times 10^6 \text{ mm}^4$$

$$= 34.19733 \times 10^6 \text{ mm}^4$$

$$(I_x)_2 = \frac{1}{12}(40 \text{ mm})(100 \text{ mm})^3 + (4 \times 10^3 \text{ mm}^2)[(86 - 78) \text{ mm}]^2$$

$$= (3.3333 + 0.256) \times 10^6 = (3.58933 \times 10^6) \text{ mm}^4$$

PROBLEM 9.43 (Continued)

Then

$$\begin{aligned}\bar{I}_x &= (34.19733 - 3.58933) \times 10^6 \text{ mm}^4 \\ &= 30.608 \times 10^6 \text{ mm}^4\end{aligned}\quad \text{or } \bar{I}_x = 30.6 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

Also

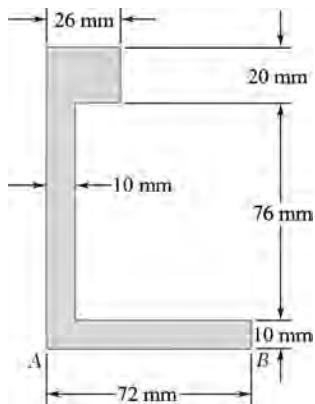
$$\bar{I}_y = (I_y)_1 - (I_y)_2$$

where

$$\begin{aligned}(I_y)_1 &= \frac{1}{12} (160 \text{ mm}) (100 \text{ mm})^3 + (16 \times 10^3 \text{ mm}^2) [(54 - 50) \text{ mm}]^2 \\ &= (13.33333 + 0.256) \times 10^6 \text{ mm}^4 = 13.58933 \text{ mm}^4 \\ (I_y)_2 &= \frac{1}{12} (100 \text{ mm}) (40 \text{ mm})^3 + (4 \times 10^3 \text{ mm}^2) [(54 - 38) \text{ mm}]^2 \\ &= (0.533333 + 1.024) \times 10^6 \text{ mm}^4 = 1.55733 \times 10^6 \text{ mm}^4\end{aligned}$$

Then

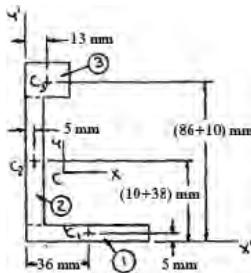
$$\begin{aligned}\bar{I}_y &= (13.58933 - 1.55733) \times 10^6 \text{ mm}^4 \\ &= 12.032 \times 10^6 \text{ mm}^4\end{aligned}\quad \text{or } \bar{I}_y = 12.03 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.44

Determine the moments of inertia \bar{I}_x and \bar{I}_y of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



First locate centroid C of the area.

	A , mm 2	\bar{x} , mm	\bar{y} , mm	$\bar{x}A$, mm 3	$\bar{y}A$, mm 3
1	$72 \times 10 = 720$	36	5	25.92×10^3	3.6×10^3
2	$10 \times 76 = 760$	5 mm	48 mm	3.8×10^3	36.48×10^3
3	$26 \times 20 = 520$	13 mm	96 mm	676×10^3	49.92×10^3
Σ	2000			36.48×10^3	90×10^3

Then

$$\bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(2000 \text{ mm}^2) = 36.48 \times 10^3 \text{ mm}^3$$

or

$$\bar{X} = 18.24 \text{ mm}$$

and

$$\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(2000 \text{ mm}^2) = 90 \times 10^3 \text{ mm}^3$$

or

$$\bar{Y} = 45 \text{ mm}$$

PROBLEM 9.44 (Continued)

Now

$$\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

where

$$\begin{aligned}(I_x)_1 &= \frac{1}{12}(72 \text{ mm})(10 \text{ mm})^3 + (720 \text{ mm}^2)[(45 - 5) \text{ mm}]^2 \\ &= (6 + 1152) \times 10^3 \text{ mm}^4 = 1158 \times 10^3 \text{ mm}^4\end{aligned}$$

$$\begin{aligned}(I_x)_2 &= \frac{1}{12}(10 \text{ mm})(76 \text{ mm})^3 + (760 \text{ mm}^2)[(48 - 45) \text{ mm}]^2 \\ &= (365.8133 + 6.84) \times 10^3 \text{ mm}^4 = (372.6533) \times 10^3 \text{ mm}^4\end{aligned}$$

$$\begin{aligned}(I_x)_3 &= \frac{1}{12}(26 \text{ mm})(20 \text{ mm})^3 + (520 \text{ mm}^2)[(96 - 45) \text{ mm}]^2 \\ &= (17.3333) + 1352.52) \times 10^3 \text{ mm}^4 = 1369.8533 \times 10^3 \text{ mm}^4\end{aligned}$$

Then

$$\bar{I}_x = (1158 + 372.6533 + 1369.8533) \times 10^3 \text{ mm}^4 = 2900.5066 \times 10^3 \text{ mm}^4$$

$$\text{or } \bar{I}_x = 2.900 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

Also

$$\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

where

$$\begin{aligned}(I_y)_1 &= \frac{1}{12}(10 \text{ mm})(72 \text{ mm})^3 + (720 \text{ mm}^2)[(36 - 18.24) \text{ mm}]^2 \\ &= (311.04 + 227.1) \times 10^3 \text{ mm}^4 = 538.14 \times 10^3 \text{ mm}^4\end{aligned}$$

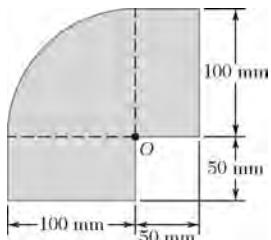
$$\begin{aligned}(I_y)_2 &= \frac{1}{12}(76 \text{ mm})(10 \text{ mm})^3 + (760 \text{ mm}^2)[(18.24 - 5) \text{ mm}]^2 \\ &= (6.3333 + 133.2262) \times 10^3 \text{ mm}^4 = 139.5595 \times 10^3 \text{ mm}^4\end{aligned}$$

$$\begin{aligned}(I_y)_3 &= \frac{1}{12}(20 \text{ mm})(26 \text{ mm})^3 + (520 \text{ mm}^2)[(18.24 - 13) \text{ mm}]^2 \\ &= (29.2933 + 14.2780) \times 10^3 \text{ mm}^4 = 43.5713 \times 10^3 \text{ mm}^4\end{aligned}$$

Then

$$\bar{I}_y = (538.14 + 139.5595 + 43.5713) \times 10^3 \text{ mm}^4 = 721.2708 \times 10^3 \text{ mm}^4$$

$$\text{or } \bar{I}_y = 0.721 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

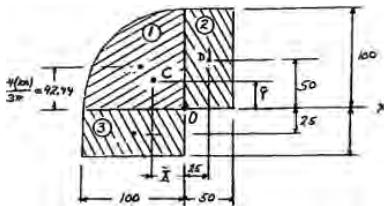


PROBLEM 9.45

Determine the polar moment of inertia of the area shown with respect to (a) Point O , (b) the centroid of the area.

SOLUTION

Dimensions in mm Symmetry: $\bar{X} = \bar{Y}$



Determination of centroid, C , of entire section.

Section	Area, mm^2	\bar{y} , mm	$\bar{y}A$, mm^3
1	$\frac{\pi}{4}(100)^2 = 7.854 \times 10^3$	42.44	333.3×10^3
2	$(50)(100) = 5 \times 10^3$	50	250×10^3
3	$(100)(50) = 5 \times 10^3$	-25	-125×10^3
Σ	17.854×10^3		458.3×10^3

$$\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(17.854 \times 10^3 \text{ mm}^2) = 458.3 \times 10^3 \text{ mm}^3$$

$$\bar{Y} = 25.67 \text{ mm} \quad \bar{X} = \bar{Y} = 25.67 \text{ mm}$$

$$\text{Distance } O \text{ to } C: \quad \overline{OC} = \sqrt{2\bar{Y}} = \sqrt{2}(25.67) = 36.30 \text{ mm}$$

$$(a) \quad \underline{\text{Section 1:}} \quad J_O = \frac{\pi}{8}(100)^4 = 39.27 \times 10^6 \text{ mm}^4$$

$$\underline{\text{Section 2:}} \quad J_O = J + A(\overline{OD})^2 = \frac{1}{12}(50)(100)[50^2 + 100^2] + (50)(100) \left[\left(\frac{50}{2} \right)^2 + \left(\frac{100}{2} \right)^2 \right]$$

$$J_O = 5.208 \times 10^6 + 15.625 \times 10^6 = 20.83 \times 10^6 \text{ mm}^4$$

PROBLEM 9.45 (Continued)

Section 3: Same as Section 2; $J_O = 20.83 \times 10^6 \text{ mm}^4$

Entire section:

$$\begin{aligned} J_O &= 39.27 \times 10^6 + 2(20.83 \times 10^6) \\ &= 80.94 \times 10^6 \quad J_O = 80.9 \times 10^6 \text{ mm}^4 \blacktriangleleft \end{aligned}$$

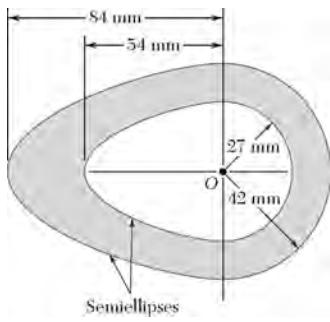
(b) Recall that,

$$\overline{OC} = 36.30 \text{ mm} \quad \text{and} \quad A = 17.854 \times 10^3 \text{ mm}^2$$

$$J_O = \bar{J}_C + A(\overline{OC})^2$$

$$80.94 \times 10^6 \text{ mm}^4 = \bar{J}_C + (17.854 \times 10^3 \text{ mm}^2)(36.30 \text{ mm})^2$$

$$\bar{J}_C = 57.41 \times 10^6 \text{ mm}^4 \quad \bar{J}_C = 57.4 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



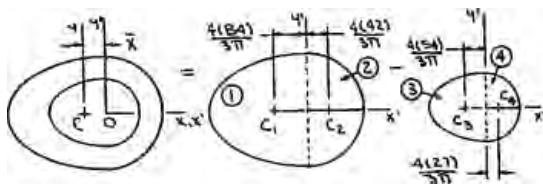
PROBLEM 9.46

Determine the polar moment of inertia of the area shown with respect to (a) Point O , (b) the centroid of the area.

SOLUTION

First locate centroid C of the figure.

Note that symmetry implies $\bar{Y} = 0$.



Dimensions in mm

	A, mm^2	\bar{X}, mm	$\bar{X}A, \text{mm}^3$
1	$\frac{\pi}{2}(84)(42) = 5541.77$	$-\frac{112}{\pi} = -35.6507$	-197,568
2	$\frac{\pi}{2}(42)^2 = 2770.88$	$\frac{56}{\pi} = 17.8254$	49,392
3	$-\frac{\pi}{2}(54)(27) = -2290.22$	$-\frac{72}{\pi} = -22.9183$	52,488
4	$-\frac{\pi}{2}(27)^2 = -1145.11$	$\frac{36}{\pi} = 11.4592$	-13,122
Σ	4877.32		-108,810

Then

$$\bar{X} \Sigma A = \Sigma \bar{x}A: \quad \bar{X}(4877.32 \text{ mm}^2) = -108,810 \text{ mm}^3$$

or

$$\bar{X} = -22.3094$$

(a)

$$J_O = (J_O)_1 + (J_O)_2 - (J_O)_3 - (J_O)_4$$

where

$$(J_O)_1 = \frac{\pi}{8}(84 \text{ mm})(42 \text{ mm})[(84 \text{ mm})^2 + (42 \text{ mm})^2]$$

$$= 12.21960 \times 10^6 \text{ mm}^4$$

PROBLEM 9.46 (Continued)

$$(J_O)_2 = \frac{\pi}{4}(42 \text{ mm})^4 \\ = 2.44392 \times 10^6 \text{ mm}^4$$

$$(J_O)_3 = \frac{\pi}{8}(54 \text{ mm})(27 \text{ mm})[(54 \text{ mm})^2 + (27 \text{ mm})^2] \\ = 2.08696 \times 10^6 \text{ mm}^4$$

$$(J_O)_4 = \frac{\pi}{4}(27 \text{ mm})^4 \\ = 0.41739 \times 10^6 \text{ mm}^4$$

Then

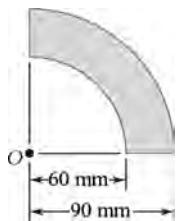
$$J_O = (12.21960 + 2.44392 - 2.08696 - 0.41739) \times 10^6 \text{ mm}^4 \\ = 12.15917 \times 10^6 \text{ mm}^4$$

or $J_O = 12.16 \times 10^6 \text{ mm}^4$ ◀

(b) $J_O = \bar{J}_C + A\bar{X}^2$

or $\bar{J}_C = 12.15917 \times 10^6 \text{ mm}^4 - (4877.32 \text{ mm}^2)(-22.3094 \text{ mm})^2$

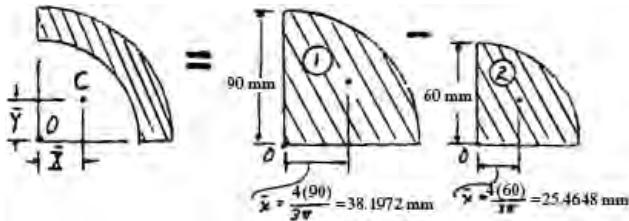
or $\bar{J}_C = 9.73 \times 10^6 \text{ mm}^4$ ◀



PROBLEM 9.47

Determine the polar moment of inertia of the area shown with respect to (a) Point O , (b) the centroid of the area.

SOLUTION



Section	Area, mm^2	\bar{x} , mm	$\bar{x}A$, mm^3
1	$\frac{\pi}{4}(90)^2 = 6.3617 \times 10^3$	38.1972	243.000×10^3
2	$-\frac{\pi}{4}(60)^2 = -2.8274 \times 10^3$	25.4648	-72.0000×10^3
Σ	3.5343×10^3		171×10^3

Then

$$\bar{X}A = \Sigma \bar{x}A: \quad \bar{X}(3.5343 \times 10^3 \text{ mm}^2) = 171 \times 10^3 \text{ mm}^3$$

$$\bar{X} = 48.3830 \text{ mm}$$

Then

$$J_O = \frac{\pi}{8}(90 \text{ mm})^4 - \frac{\pi}{8}(60 \text{ mm})^4 = 20.6756 \times 10^6 \text{ mm}^4$$

$$J_O = 20.7 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

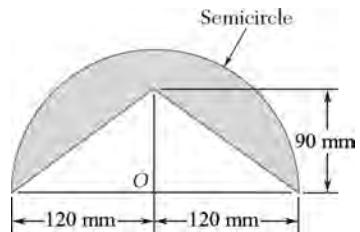
$$\overline{OC} = \sqrt{2}\bar{X} = \sqrt{2}(48.383 \text{ mm}) = 68.4239 \text{ mm}$$

$$J_O = \bar{J}_C + A(\overline{OC})^2:$$

$$20.6756 \times 10^6 \text{ mm}^4 = \bar{J}_C + (3.5343 \text{ mm}^2)(68.4239 \text{ mm})^2$$

$$\bar{J}_C = 4.1286 \times 10^6 \text{ mm}^4$$

$$\bar{J}_C = 4.13 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

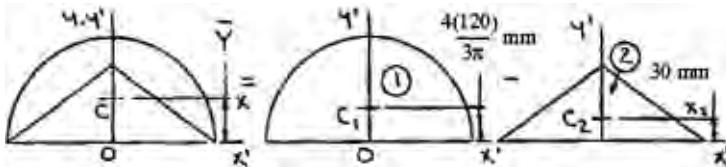


PROBLEM 9.48

Determine the polar moment of inertia of the area shown with respect to
(a) Point O , (b) the centroid of the area.

SOLUTION

First locate centroid C of the figure.



	A, mm^2	\bar{y}, mm	$\bar{y}A, \text{mm}^3$
1	$\frac{\pi}{2}(120)^2 = 22.61947 \times 10^3$	$\frac{160}{\pi} = 50.9296$	1152×10^3
2	$-\frac{1}{2}(240)(90) = -10.8 \times 10^3$	30	-324×10^3
Σ	11.81947×10^3		828×10^3

Then

$$\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(11.81947 \times 10^3 \text{ mm}^2) = 828 \times 10^3 \text{ mm}^3$$

or

$$\bar{Y} = 70.0539 \text{ mm}$$

(a)

$$J_O = (J_O)_1 - (J_O)_2$$

where

$$(J_O)_1 = \frac{\pi}{4}(120 \text{ mm})^4 = 162.86016 \times 10^6 \text{ mm}^4$$

$$(J_O)_2 = (I_{x'})_2 + (I_{y'})_2$$

Now

$$(I_{x'})_2 = \frac{1}{12}(240 \text{ mm})(90 \text{ mm})^3 = 14.58 \times 10^6 \text{ mm}^4$$

and

$$(I_{y'})_2 = 2 \left[\frac{1}{12}(90 \text{ mm})(120 \text{ mm})^3 \right] = 25.92 \times 10^6 \text{ mm}^4$$

[Note: $(I_{y'})_2$ is obtained using

PROBLEM 9.48 (Continued)

Then

$$\begin{aligned}(J_O)_2 &= (14.58 + 25.92) \times 10^6 \text{ mm}^4 \\ &= 40.5 \times 10^6 \text{ mm}^4\end{aligned}$$

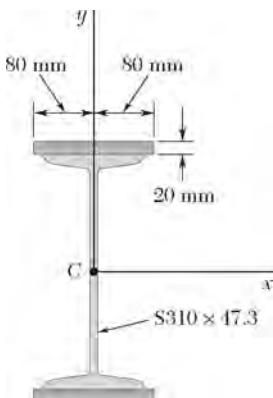
Finally

$$\begin{aligned}J_O &= (162.86016 - 40.5) \times 10^6 \text{ mm}^4 = 122.36016 \times 10^6 \text{ mm}^4 \\ \text{or} \quad J_O &= 122.4 \times 10^6 \text{ mm}^4 \blacktriangleleft\end{aligned}$$

(b)

or

$$\begin{aligned}J_O &= \bar{J}_C + A\bar{Y}^2 \\ J_C &= 122.36016 \times 10^6 \text{ mm}^4 - (11.81947 \times 10^3 \text{ mm}^2)(70.0539 \text{ mm})^2 \\ &= 64.3555 \times 10^6 \text{ mm}^4 \quad \text{or} \quad \bar{J}_C = 64.4 \times 10^6 \text{ mm}^4 \blacktriangleleft\end{aligned}$$



PROBLEM 9.49

Two 20-mm steel plates are welded to a rolled S section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION

S section:

$$A = 6010 \text{ mm}^2$$

$$\bar{I}_x = 90.3 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 3.88 \times 10^6 \text{ mm}^4$$

Note:

$$\begin{aligned} A_{\text{total}} &= A_S + 2A_{\text{plate}} \\ &= 6010 \text{ mm}^2 + 2(160 \text{ mm})(20 \text{ mm}) \\ &= 12,410 \text{ mm}^2 \end{aligned}$$

Now

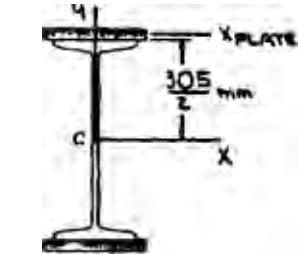
$$\bar{I}_x = (\bar{I}_x)_S + 2(I_x)_{\text{plate}}$$

where

$$\begin{aligned} (I_x)_{\text{plate}} &= \bar{I}_{x_{\text{plate}}} + Ad^2 \\ &= \frac{1}{12}(160 \text{ mm})(20 \text{ mm})^3 + (3200 \text{ mm}^2)[(152.5 + 10) \text{ mm}]^2 \\ &= 84.6067 \times 10^6 \text{ mm}^4 \end{aligned}$$

Then

$$\begin{aligned} \bar{I}_x &= (90.3 + 2 \times 84.6067) \times 10^6 \text{ mm}^4 \\ &= 259.5134 \times 10^6 \text{ mm}^4 \end{aligned}$$



$$\text{or } \bar{I}_x = 260 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{259.5134 \times 10^6 \text{ mm}^4}{12410 \text{ mm}^2} \quad \text{or} \quad \bar{k}_x = 144.6 \text{ mm} \blacktriangleleft$$

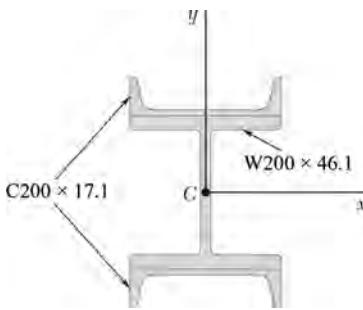
PROBLEM 9.49 (Continued)

Also

$$\begin{aligned}\bar{I}_y &= (\bar{I}_y)_S + 2(\bar{I}_y)_{\text{plate}} \\ &= 3.88 \times 10^6 \text{ mm}^4 + 2 \left[\frac{1}{12} (20 \text{ mm}) (160 \text{ mm})^3 \right] \\ &= 17.5333 \times 10^6 \text{ mm}^4\end{aligned}\quad \text{or } \bar{I}_y = 17.53 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{17.5333 \times 10^6 \text{ mm}^4}{12,410 \text{ mm}^2} \quad \text{or} \quad \bar{k}_y = 37.6 \text{ mm} \blacktriangleleft$$



PROBLEM 9.50

Two channels are welded to a rolled W section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION

W section: $A = 5880 \text{ mm}^2$

$$\bar{I}_x = 45.8 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 15.4 \times 10^6 \text{ mm}^4$$

Channel: $A = 2170 \text{ mm}^2$

$$\bar{I}_x = 13.5 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.545 \times 10^6 \text{ mm}^4$$

$$\begin{aligned} A_{\text{total}} &= A_W + 2A_{\text{chan}} \\ &= 5880 + 2(2170) = 10220 \text{ mm}^2 \end{aligned}$$

Now

$$\bar{I}_x = (\bar{I}_x)_W + 2(\bar{I}_x)_{\text{chan}}$$

where

$$\begin{aligned} (\bar{I}_x)_{\text{chan}} &= \bar{I}_{x_{\text{chan}}} + Ad^2 \\ &= 13.5 \times 10^6 \text{ mm}^4 + (2170 \text{ mm}^2)(114.5 \text{ mm})^2 = 41.9492 \times 10^6 \text{ mm}^4 \end{aligned}$$

Then

$$\bar{I}_x = [(45.8 + 2 \times 41.9492)] \times 10^6 \text{ mm}^4 = 129.6984 \times 10^6 \text{ mm}^4 \quad \bar{I}_x = 129.7 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

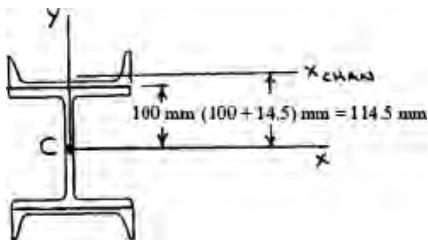
$$\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{129.6984 \times 10^6 \text{ mm}^4}{10220 \text{ mm}^2} \quad \bar{k}_x = 112.7 \text{ mm} \blacktriangleleft$$

Also

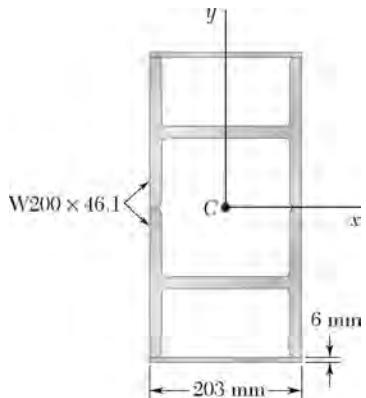
$$\begin{aligned} \bar{I}_y &= (\bar{I}_y)_W + 2(\bar{I}_y)_{\text{chan}} \\ &= [15.4 + 2(0.545)] \times 10^6 \text{ mm}^4 = 16.49 \times 10^6 \text{ mm}^4 \quad \bar{I}_y = 16.49 \times 10^6 \text{ mm}^4 \blacktriangleleft \end{aligned}$$

and

$$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{16.49 \times 10^6 \text{ mm}^4}{10220 \text{ mm}^2} \quad \bar{k}_y = 40.2 \text{ mm} \blacktriangleleft$$



PROBLEM 9.51



To form a reinforced box section, two rolled W sections and two plates are welded together. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal axes shown.

SOLUTION

(Note: x and y interchanged as w.r.t. Fig. (19.3A)

W section:

$$A = 5880 \text{ mm}^2$$

$$\bar{I}_x = 15.4 \times 10^6 \text{ mm}^4$$

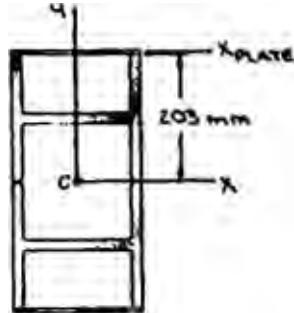
$$\bar{I}_y = 45.8 \times 10^6 \text{ mm}^4$$

Note:

$$A_{\text{total}} = 2A_W + 2A_{\text{plate}}$$

$$= 2(5880 \text{ mm}^2) + 2(203 \text{ mm})(6 \text{ mm})$$

$$= 14,196 \text{ mm}^2$$



Now

$$\bar{I}_x = 2(I_x)_W + 2(I_x)_{\text{plate}}$$

where

$$(I_x)_W = \bar{I}_x + Ad^2 = 15.4 \times 10^6 \text{ mm}^4 + (5880 \text{ mm}^2) \left(\frac{203}{2} \text{ mm} \right)^2$$

$$= 75.9772 \times 10^6 \text{ mm}^4$$

$$(I_x)_{\text{plate}} = \bar{I}_{x_{\text{plate}}} + Ad^2$$

$$= \frac{1}{12}(203 \text{ mm})(6 \text{ mm})^3 + (1218 \text{ mm}^2)[(203 + 3) \text{ mm}]^2$$

$$= 51.6907 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_x = [2(75.9772) + 2(51.6907)] \times 10^6 \text{ mm}^4$$

$$= 255.336 \times 10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_x = 255 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{255.336 \times 10^6 \text{ mm}^4}{14,196 \text{ mm}^2}$$

$$\text{or } \bar{k}_x = 134.1 \text{ mm} \blacktriangleleft$$

PROBLEM 9.51 (Continued)

also

$$\bar{I}_y = 2(I_y)_W + 2(I_y)_{\text{plate}}$$

where

$$(I_y)_W = I_y$$

$$(I_y)_{\text{plate}} = \frac{1}{12}(6 \text{ mm})(203 \text{ mm})^3 = 4.1827 \times 10^6 \text{ mm}^4$$

Then

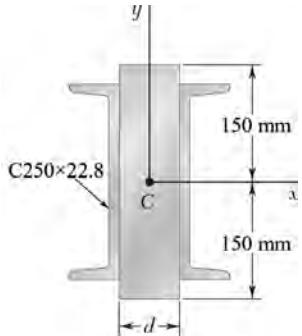
$$\begin{aligned}\bar{I}_y &= [2(45.8) + 2(4.1827)] \times 10^6 \text{ mm}^4 \\ &= 99.9654 \times 10^6 \text{ mm}^4\end{aligned}$$

$$\text{or } \bar{I}_y = 100.0 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{99.9654 \times 10^6 \text{ mm}^4}{14,196 \text{ mm}^2}$$

$$\text{or } \bar{k}_y = 83.9 \text{ mm} \blacktriangleleft$$



PROBLEM 9.52

Two channels are welded to a $d \times 300$ mm steel plate as shown. Determine the width d for which the ratio \bar{I}_x/\bar{I}_y of the centroidal moments of inertia of the section is 16.

SOLUTION

Channel:

$$A = 2890 \text{ mm}^2$$

$$\bar{I}_x = 28.0 \times 10^6 \text{ mm}^4$$

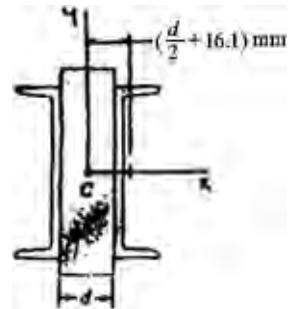
$$\bar{I}_y = 0.945 \times 10^6 \text{ mm}^4$$

Now

$$\bar{I}_x = 2(\bar{I}_x)_C + (\bar{I}_x)_{\text{plate}}$$

$$= 2(28.0 \times 10^6 \text{ mm}^4) + \frac{1}{12} d(300 \text{ mm})^3$$

$$= (56 \times 10^6 \text{ mm}^4 + 2.25 \times 10^6 d)$$



and

$$\bar{I}_y = 2(\bar{I}_y)_C + (\bar{I}_y)_{\text{plate}}$$

$$(\bar{I}_y)_C = \bar{I}_y + Ad^2$$

$$= 0.945 \times 10^6 \text{ mm}^4 + (2890 \text{ mm}^2) \left(\frac{d}{2} + 16.1 \text{ mm} \right)^2$$

$$= (0.945 \times 10^6 + (2890) \left(\frac{d^2}{4} + 16.1d + 16.1^2 \right))$$

$$= (722.5d^2 + 46529d + 1.69412 \times 10^6)$$

$$(\bar{I}_y)_{\text{plate}} = \frac{1}{12}(300 \text{ mm})d^3 = 25d^3$$

$$\bar{I}_y = 25d^3 + 2(722.5d^2 + 46529d + 1.69412 \times 10^6)$$

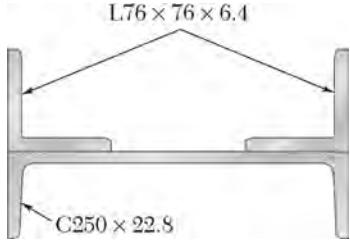
$$= (25d^3 + 1445d^2 + 93058d + 3388240)$$

$$\bar{I}_x = 16\bar{I}_y \Rightarrow 56 \times 10^6 + 2.25 \times 10^6 d = 16(25d^3 + 1445d^2 + 93058d + 3.38824 \times 10^6)$$

$$400d^3 + 23120d^2 - 761.072 \times 10^3 d - 1.78816 \times 10^6 = 0$$

Solving yields $d = 25.0999 \text{ mm}$ (the other roots are negative)

$$d = 25.10 \text{ mm} \blacktriangleleft$$



PROBLEM 9.53

Two L $76 \times 76 \times 6.4$ -mm angles are welded to a C 250×22.8 channel. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the web of the channel.

SOLUTION

Angle:

$$A = 929 \text{ mm}^2$$

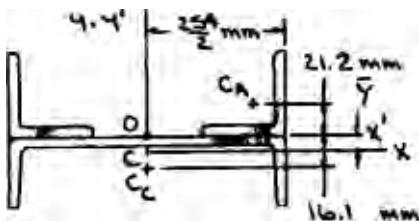
$$\bar{I}_x = \bar{I}_y = 0.512 \times 10^6 \text{ mm}^4$$

Channel:

$$A = 2890 \text{ mm}^2$$

$$\bar{I}_x = 0.945 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 28.0 \times 10^6 \text{ mm}^4$$



First locate centroid C of the section

	$A, \text{ mm}^2$	$\bar{y}, \text{ mm}$	$\bar{y}A, \text{ mm}^3$
Angle	$2(929) = 1858$	21.2	39,389.6
Channel	2890	-16.1	-46,529
Σ	4748		-7139.4

Then

$$\bar{Y} \Sigma A = \Sigma \bar{y} A: \quad \bar{Y}(4748 \text{ mm}^2) = -7139.4 \text{ mm}^3$$

or

$$\bar{Y} = -1.50366 \text{ mm}$$

Now

$$\bar{I}_x = 2(I_x)_L + (I_x)_C$$

where

$$(I_x)_L = \bar{I}_x + Ad^2 = 0.512 \times 10^6 \text{ mm}^4 + (929 \text{ mm}^2)[(21.2 + 1.50366) \text{ mm}]^2 \\ = 0.990859 \times 10^6 \text{ mm}^4$$

$$(I_x)_C = \bar{I}_x + Ad^2 = 0.949 \times 10^6 \text{ mm}^4 + (2890 \text{ mm}^2)[(16.1 - 1.50366) \text{ mm}]^2 \\ = 1.56472 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_x = [2(0.990859) + 1.56472] \times 10^6 \text{ mm}^4$$

$$\text{or} \quad \bar{I}_x = 3.55 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

PROBLEM 9.53 (Continued)

Also

$$\bar{I}_y = 2(I_y)_L + (I_y)_C$$

where

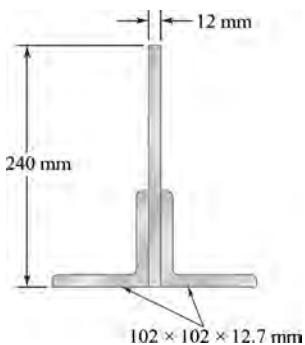
$$(I_y)_L = \bar{I}_y + Ad^2 = 0.512 \times 10^6 \text{ mm}^4 + (929 \text{ mm}^2)[(127 - 21.2) \text{ mm}]^2 \\ = 10.9109 \times 10^6 \text{ mm}^4$$

$$(I_y)_C = \bar{I}_y$$

Then

$$\bar{I}_y = [2(10.9109) + 28.0] \times 10^6 \text{ mm}^4$$

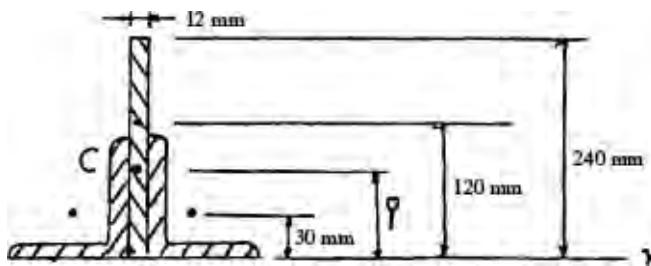
or $\bar{I}_y = 49.8 \times 10^6 \text{ mm}^4$ ◀



PROBLEM 9.54

Two L102 × 102 × 12.7-mm angles are welded to a steel plate as shown. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the plate.

SOLUTION



For 102×102×12.7 mm

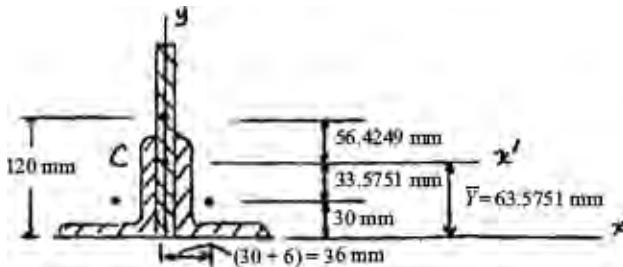
$$A = 2420 \text{ mm}^2, \quad \bar{I}_x = \bar{I}_y = 2.30 \times 10^6 \text{ mm}^4$$

Section	Area, mm ²	\bar{y} , mm	$\bar{y}A$, mm ³
Plate	(12)(240) = 2880	120	345.6×10^3
Two angles	$2(2420) = 4840$	30	145.2×10^3
Σ	7720		490.8×10^3

$$\bar{Y}A = \Sigma \bar{y} A$$

$$\bar{Y}(7720 \text{ mm}^2) = 490.8 \times 10^3 \text{ mm}^3$$

$$\bar{Y} = 63.5751 \text{ mm}$$

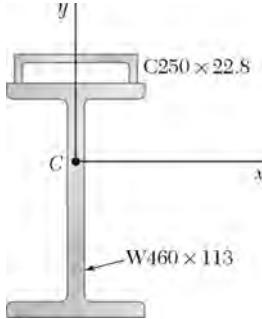


PROBLEM 9.54 (Continued)

Entire section:

$$\begin{aligned}\bar{I}_x &= \Sigma(\bar{I}_{x'} + Ad^2) = \left[\frac{1}{12}(12)(240)^3 + (12)(240)(56.4249)^2 \right] + 2[2.30 \times 10^6 + (2420)(33.5751)^2] \\ &= 13.824 \times 10^6 + 9.1693 \times 10^6 + 4.6 \times 10^6 + 2.7280 \times 10^6 \\ &= 30.3213 \times 10^6 \text{ mm}^4 \quad \bar{I}_x = 30.3 \times 10^6 \text{ mm}^4 \blacktriangleleft\end{aligned}$$

$$\begin{aligned}\bar{I}_y &= \frac{1}{12}(240)(12)^3 + 2[2.30 \times 10^6 + (2420)(36)^2] \\ &= (34.56 \times 10^3 + 4.6 \times 10^6 + 6.27264 \times 10^6) \\ &= 10.9072 \times 10^6 \text{ mm}^4 \quad \bar{I}_y = 10.91 \times 10^6 \text{ mm}^4 \blacktriangleleft\end{aligned}$$



PROBLEM 9.55

The strength of the rolled W section shown is increased by welding a channel to its upper flange. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

SOLUTION

W Section:

$$A = 14,400 \text{ mm}^2$$

$$\bar{I}_x = 554 \times 10^6 \text{ mm}^4$$

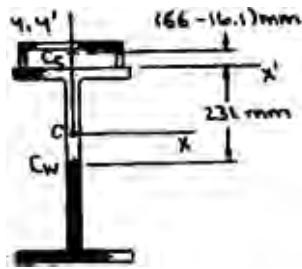
$$\bar{I}_y = 63.3 \times 10^6 \text{ mm}^4$$

Channel:

$$A = 2890 \text{ mm}^2$$

$$\bar{I}_x = 0.945 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 28.0 \times 10^6 \text{ mm}^4$$



First locate centroid C of the section.

	$A, \text{ mm}^2$	$\bar{y}, \text{ mm}$	$\bar{y}A, \text{ mm}^3$
W Section	14400	-231	-3326400
Channel	2890	49.9	144211
Σ	17290		-3182189

Then

$$\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(17,290 \text{ mm}^2) = -3,182,189 \text{ mm}^3$$

or

$$\bar{Y} = -184.047 \text{ mm}$$

Now

$$\bar{I}_x = (I_x)_W + (I_x)_C$$

where

$$(I_x)_W = \bar{I}_x + Ad^2$$

$$= 554 \times 10^6 \text{ mm}^4 + (14,400 \text{ mm}^2)(231 - 184.047)^2 \text{ mm}^2$$

$$= 585.75 \times 10^6 \text{ mm}^4$$

$$(I_x)_C = \bar{I}_x - Ad^2$$

$$= 0.945 \times 10^6 \text{ mm}^4 + (2890 \text{ mm}^2)(49.9 + 184.047)^2 \text{ mm}^2$$

$$= 159.12 \times 10^6 \text{ mm}^4$$

PROBLEM 9.55 (Continued)

Then

$$\bar{I}_x = (585.75 + 159.12) \times 10^6 \text{ mm}^4$$

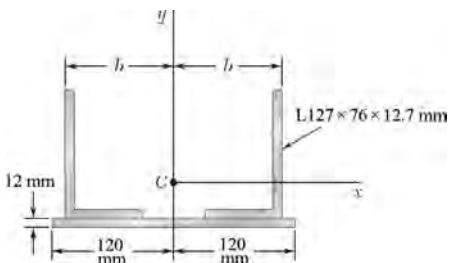
$$\text{or } \bar{I}_x = 745 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

also

$$\bar{I}_y = (I_y)_W + (I_y)_C$$

$$= (63.3 + 28.0) \times 10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_y = 91.3 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.56

Two L127 × 76 × 12.7-mm angles are welded to a 12 mm steel plate. Determine the distance b and the centroidal moments of inertia \bar{I}_x and \bar{I}_y of the combined section, knowing that $\bar{I}_y = 4\bar{I}_x$.

SOLUTION

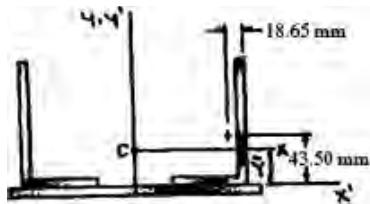
Angle:

$$A = 93.75 \text{ mm}^2$$

$$\bar{I}_x = 235.75 \text{ mm}^4$$

$$\bar{I}_y = 63.75 \text{ mm}^4$$

First locate centroid C of the section.



	Area, mm ²	\bar{y} , mm	$\bar{y}A$, mm ³
Angle	$2(93.75) = 187.50$	43.50 / 326.25	13.05
Plate	$(10)(12.5) = 125$	-6.25	-31.25
Σ	312.50		29.50

Then

$$\bar{Y} \Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(312 \text{ mm}^2) = 29.50 \text{ mm}^3$$

or

$$\bar{Y} = 23.6 \text{ mm}$$

Now

$$\bar{I}_x = 2(I_x)_{\text{angle}} + (I_x)_{\text{plate}}$$

where

$$(I_x)_{\text{angle}} = I_x + Ad^2 = 235.75 \text{ mm}^4 + (93.75 \text{ mm}^2)[(43.50 - 23.6) \text{ mm}]^2 = 295.15 \text{ mm}^4$$

$$(I_x)_{\text{plate}} = \bar{I}_x + Ad^2 = \frac{1}{12}(250 \text{ mm})(12.5 \text{ mm})^3 + (125 \text{ mm}^2)[(6.25 + 23.6) \text{ mm}]^2 = 180.8 \text{ mm}^4$$

Then

$$\bar{I}_x = [2(295.15) + 180.8] \text{ mm}^4 = 771.1125 \text{ mm}^4$$

or $\bar{I}_x = 771.1 \text{ mm}^4$ ◀

PROBLEM 9.56 (Continued)

We have

$$\bar{I}_y = 4\bar{I}_x = 4(771.1125 \text{ mm}^4) = 3084.45 \text{ mm}^4$$

$$\text{or } \bar{I}_y = 3084.5 \text{ mm}^4 \blacktriangleleft$$

Now

$$\bar{I}_y = 2(I_y)_{\text{angle}} + (\bar{I}_y)_{\text{plate}}$$

where

$$(\bar{I}_y)_{\text{angle}} = \bar{I}_y + Ad^2 = 63.75 \text{ mm}^4 + (93.75 \text{ mm}^2)[(b - 18.65) \text{ mm}]^2$$

$$(\bar{I}_y)_{\text{plate}} = \frac{1}{12}(12.5 \text{ mm})(250 \text{ mm})^3 = 1041.6675 \text{ mm}^4$$

Then

$$3084.45 \text{ mm}^4 = 2[63.75 + 93.75(b - 18.65)^2] \text{ mm}^4 + 1041.6675 \text{ mm}^4$$

$$\text{or } b = 98.5 \text{ mm} \blacktriangleleft$$



PROBLEM 9.57

The panel shown forms the end of a trough that is filled with water to the line AA' . Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

From Section 9.2:

$$R = \gamma \int y dA, \quad M_{AA'} = \gamma \int y^2 dA$$

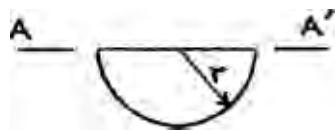
Let y_P = distance of center of pressure from AA' . We must have

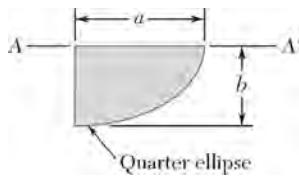
$$Ry_P = M_{AA'} : \quad y_P = \frac{M_{AA'}}{R} = \frac{\gamma \int y^2 dA}{\gamma \int y dA} = \frac{I_{AA'}}{\bar{y} A} \quad (1)$$

For semicircular panel:

$$I_{AA'} = \frac{\pi}{8} r^4 \quad \bar{y} = \frac{4r}{3\pi} \quad A = \frac{\pi}{2} r^2$$

$$y_P = \frac{I_{AA'}}{\bar{y} A} = \frac{\frac{\pi}{8} r^4}{\left(\frac{4r}{3\pi}\right) \frac{\pi}{2} r^2} = \frac{3\pi}{16} r \quad \blacktriangleleft$$



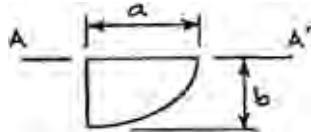


PROBLEM 9.58

The panel shown forms the end of a trough that is filled with water to the line AA' . Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

See solution of Problem 9.57 for derivation of Eq. (1):

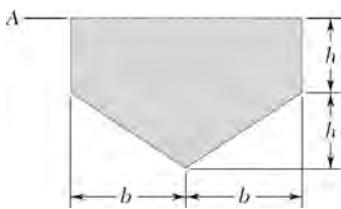


$$y_P = \frac{I_{AA'}}{\bar{y}A} \quad (1)$$

For quarter ellipse panel:

$$I_{AA'} = \frac{\pi}{16} ab^3 \quad \bar{y} = \frac{4b}{3\pi} \quad A = \frac{\pi}{4} ab$$

$$y_P = \frac{I_{AA'}}{\bar{y}A} = \frac{\frac{\pi}{16} ab^3}{\left(\frac{4b}{3\pi}\right)\left(\frac{\pi}{4} ab\right)} \quad y_P = \frac{3\pi}{16} b \blacktriangleleft$$



PROBLEM 9.59

The panel shown forms the end of a trough that is filled with water to the line AA' . Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 491 of the text:

$$y_P = \frac{I_{AA'}}{\bar{y}A}$$

Now

$$\bar{y}A = \sum \bar{y}A$$

$$\begin{aligned} &= \frac{h}{2}(2b \times h) + \frac{4}{3}h\left(\frac{1}{2} \times 2b \times h\right) \\ &= \frac{7}{3}bh^2 \end{aligned}$$

and

$$I_{AA'} = (I_{AA'})_1 + (I_{AA'})_2$$

where

$$(I_{AA'})_1 = \frac{1}{3}(2b)(h)^3 = \frac{2}{3}bh^3$$

$$\begin{aligned} (I_{AA'})_2 &= \bar{I}_x + Ad^2 = \frac{1}{36}(2b)(h)^3 + \left(\frac{1}{2} \times 2b \times h\right)\left(\frac{4}{3}h\right)^2 \\ &= \frac{11}{6}bh^3 \end{aligned}$$

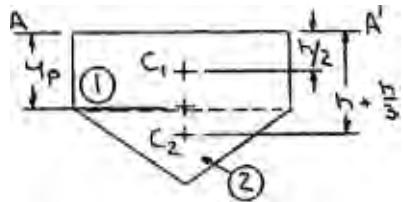
Then

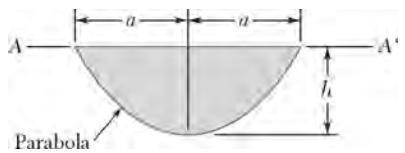
$$I_{AA'} = \frac{2}{3}bh^3 + \frac{11}{6}bh^3 = \frac{5}{2}bh^3$$

Finally,

$$y_P = \frac{\frac{5}{2}bh^3}{\frac{7}{3}bh^2}$$

$$\text{or } y_P = \frac{15}{14}h \blacktriangleleft$$





PROBLEM 9.60*

The panel shown forms the end of a trough that is filled with water to the line AA' . Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 491 of the text:

$$y_p = \frac{I_{AA'}}{\bar{y}A}$$

For a parabola:

$$\bar{y} = \frac{2}{5}h \quad A = \frac{4}{3}ah$$

Now

$$dI_{AA'} = \frac{1}{3}(h-y)^3 dx$$

By observation

$$y = \frac{h}{a^2}x^2$$

So that

$$\begin{aligned} dI_{AA'} &= \frac{1}{3} \left(h - \frac{h}{a^2}x^2 \right)^3 dx = \frac{1}{3} \frac{h^3}{a^6} (a^2 - x^2)^3 dx \\ &= \frac{1}{3} \frac{h}{a^6} (a^6 - 3a^4 x^2 + 3a^2 x^4 - x^6) dx \end{aligned}$$

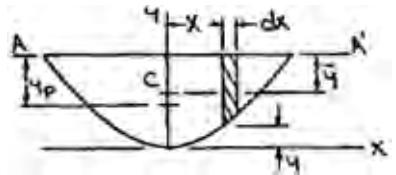
Then

$$\begin{aligned} I_{AA'} &= 2 \int_0^a \frac{1}{3} \frac{h^3}{a^6} (a^6 - 3a^4 x^2 + 3a^2 x^4 - x^6) dx \\ &= \frac{2}{3} \frac{h^3}{a^6} \left[a^6 x - a^4 x^3 + \frac{3}{5} a^2 x^5 - \frac{1}{7} x^7 \right]_0^a \\ &= \frac{32}{105} ah^3 \end{aligned}$$

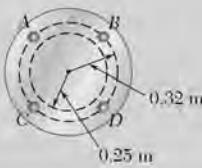
Finally,

$$y_p = \frac{\frac{32}{105} ah^3}{\frac{2}{5} h \times \frac{4}{3} ah}$$

$$\text{or } y_p = \frac{4}{7}h \blacktriangleleft$$



PROBLEM 9.61



The cover for a 0.5 m-diameter access hole in a water storage tank is attached to the tank with four equally spaced bolts as shown. Determine the additional force on each bolt due to the water pressure when the center of the cover is located 1.4 m below the water surface.

SOLUTION

From Section 9.2:

$$R = \gamma \bar{y}A \quad y_P = \frac{I_{AA'}}{\bar{y}A}$$

where R is the resultant of the hydrostatic forces acting on the cover

and y_P is the depth to the point of application of R .

Recalling that $\gamma = \rho g$, we have

$$\begin{aligned} R &= (10^3 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2)(1.4 \text{ m})[\pi(0.25 \text{ m})^2] \\ &= 2696.67 \text{ N} \end{aligned}$$

Also

$$\begin{aligned} I_{AA'} &= \bar{I}_x + A\bar{y}^2 = \frac{\pi}{4}(0.25 \text{ m})^4 + [\pi(0.25 \text{ m})^2](1.4 \text{ m})^2 \\ &= 0.387913 \text{ m}^4 \end{aligned}$$

Then

$$y_P = \frac{0.387913 \text{ m}^4}{(1.4 \text{ m})[\pi(0.25 \text{ m})^2]} = 1.41116 \text{ m}$$

Now note that symmetry implies

$$F_A = F_B \quad F_C = F_D$$

Next consider the free-body of the cover.

We have

$$\begin{aligned} \text{At } C: \sum M_{CD} &= 0: [2(0.32 \text{ m}) \sin 45^\circ](2F_A) \\ &\quad - [0.32 \sin 45^\circ - (1.41116 - 1.4)] \text{ m} \\ &\quad \times (2696.67 \text{ N}) = 0 \end{aligned}$$

or

$$F_A = 640.92 \text{ N}$$

Then

$$\text{At } C: \sum F_z = 0: 2(640.92 \text{ N}) + 2F_C - 2696.67 \text{ N} = 0$$

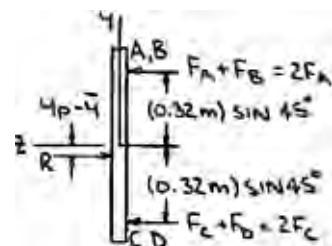
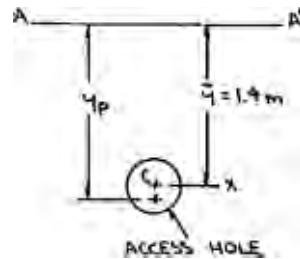
or

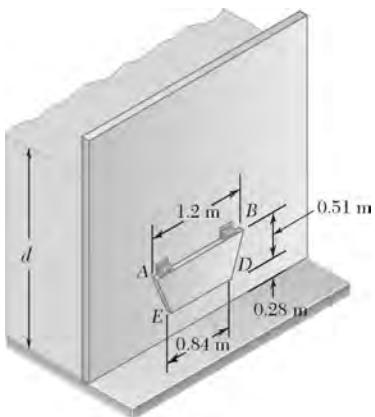
$$F_C = 707.42 \text{ N}$$

and

$$F_A = F_B = 641 \text{ N} \quad \blacktriangleleft$$

$$F_C = F_D = 707 \text{ N} \quad \blacktriangleleft$$





PROBLEM 9.62

A vertical trapezoidal gate that is used as an automatic valve is held shut by two springs attached to hinges located along edge AB . Knowing that each spring exerts a couple of magnitude $1470 \text{ N} \cdot \text{m}$, determine the depth d of water for which the gate will open.

SOLUTION

From Section 9.2:

$$R = \gamma \bar{y}A \quad y_P = \frac{I_{SS'}}{\bar{y}A}$$

where R is the resultant of the hydrostatic forces acting on the gate

and y_P is the depth to the point of application of R . Now

$$\begin{aligned} \bar{y}A &= \Sigma \bar{y}A = [(h + 0.17) \text{ m}] \left(\frac{1}{2} \times 1.2 \text{ m} \times 0.51 \text{ m} \right) + [(h + 0.34) \text{ m}] \left(\frac{1}{2} \times 0.84 \times 0.51 \text{ m} \right) \\ &= (0.5202h + 0.124848) \text{ m}^3 \end{aligned}$$

Recalling that $\gamma = \rho g$, we have

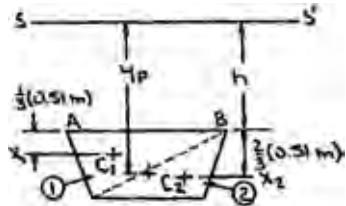
$$\begin{aligned} R &= (10^3 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2)(0.5202h + 0.124848) \text{ m}^3 \\ &= 5103.162(h + 0.24) \text{ N} \end{aligned}$$

Also

$$I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2$$

where

$$\begin{aligned} (I_{SS'})_1 &= I_x + Ad^2 \\ &= \frac{1}{36}(1.2 \text{ m})(0.51 \text{ m})^3 + \left(\frac{1}{2} \times 1.2 \text{ m} \times 0.51 \text{ m} \right) [(h + 0.17) \text{ m}]^2 \\ &= [0.0044217 + 0.306(h + 0.17)^2] \text{ m}^4 \\ &= (0.306h^2 + 0.10404h + 0.0132651) \text{ m}^4 \\ (I_{SS'})_2 &= \bar{I}_{X_2} + Ad^2 \\ &= \frac{1}{36}(0.84 \text{ m})(0.51 \text{ m})^3 + \left(\frac{1}{2} \times 0.84 \text{ m} \times 0.51 \text{ m} \right) [(h + 0.34) \text{ m}]^2 \\ &= [0.0030952 + 0.2142(h + 0.34)^2] \text{ m}^4 \\ &= (0.2142h^2 + 0.145656h + 0.0278567) \text{ m}^4 \end{aligned}$$



PROBLEM 9.62 (Continued)

Then

$$I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2 \\ = (0.5202h^2 + 0.249696h + 0.0411218) \text{ m}^4$$

and

$$y_P = \frac{(0.5202h^2 + 0.244696h + 0.0411218) \text{ m}^4}{(0.5202h + 0.124848) \text{ m}^3} \\ = \frac{h^2 + 0.48h + 0.07905}{h + 0.24} \text{ m}$$

For the gate to open, require that

$$\Sigma M_{AB}: \quad M_{\text{open}} = (y_P - h)R$$

Substituting

$$2940 \text{ N} \cdot \text{m} = \left(\frac{h^2 + 0.48h + 0.07905}{h + 0.24} - h \right) \text{ m} \times 5103.162(h + 0.24) \text{ N}$$

or

$$5103.162(0.24h + 0.07905) = 2940$$

or

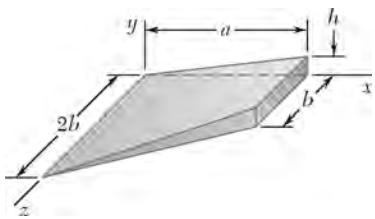
$$h = 2.0711 \text{ m}$$

Then

$$d = (2.0711 + 0.79) \text{ m}$$

or

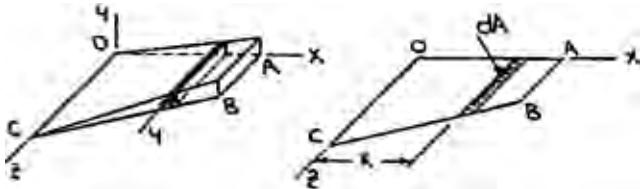
$$d = 2.86 \text{ m} \blacktriangleleft$$



PROBLEM 9.63*

Determine the x coordinate of the centroid of the volume shown.
(Hint: The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION



First note that

$$y = \frac{h}{a}x$$

Now

$$\bar{x} \int dV = \int \bar{x}_{EL} dV$$

where

$$\bar{x}_{EL} = x \quad dV = y dA = \left(\frac{h}{a}x \right) dA$$

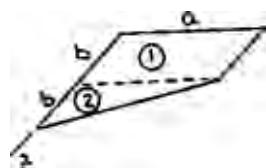
Then

$$\bar{x} = \frac{\int x \left(\frac{h}{a}x dA \right)}{\int \frac{h}{a}x dA} = \frac{\int x^2 dA}{\int x dA} = \frac{(I_z)_A}{(\bar{x}A)_A}$$

where $(I_z)_A$ and $(\bar{x}A)_A$ pertain to area.

$OABC$: $(I_z)_A$ is the moment of inertia of the area with respect to the z axis, \bar{x}_A is the x coordinate of the centroid of the area, and A is the area of $OABC$. Then

$$\begin{aligned} (I_z)_A &= (I_z)_{A_1} + (I_z)_{A_2} \\ &= \frac{1}{3}(b)(a)^3 + \frac{1}{12}(b)(a)^3 \\ &= \frac{5}{12}a^3b \end{aligned}$$



and

$$\begin{aligned} (\bar{x}A)_A &= \Sigma \bar{x}A \\ &= \left[\left(\frac{a}{2} \right) (a \times b) \right] + \left[\left(\frac{a}{3} \right) \left(\frac{1}{2} \times a \times b \right) \right] \\ &= \frac{2}{3}a^2b \end{aligned}$$

PROBLEM 9.63* (Continued)

Finally,

$$\bar{x} = \frac{\frac{5}{12}a^3b}{\frac{2}{3}a^2b}$$

$$\text{or } \bar{x} = \frac{5}{8}a \blacktriangleleft$$

Analogy with hydrostatic pressure on a submerged plate:

Recalling that $P = \gamma y$, it follows that the following analogies can be established.

$$\text{Height } y \sim P$$

$$dV = y dA \sim p dA = dF$$

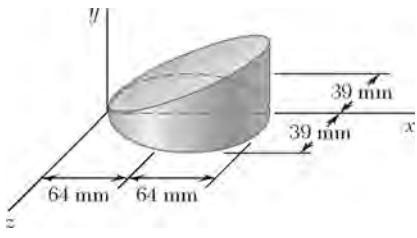
$$x dV = x(y dA) \sim y dF = dM$$

Recalling that

$$y_P = \frac{M_x}{R} \left(= \frac{\int dM}{\int dF} \right)$$

It can then be concluded that

$$x \sim y_P$$



PROBLEM 9.64*

Determine the x coordinate of the centroid of the volume shown; this volume was obtained by intersecting an elliptic cylinder with an oblique plane. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION

Following the “Hint,” it can be shown that (see solution to Problem 9.63)

$$\bar{x} = \frac{(I_z)_A}{(\bar{x}A)_A} x$$

where $(I_z)_A$ and $(\bar{x}A)_A$ are the moment of inertia and the first moment of the area, respectively, of the elliptical area of the base of the volume with respect to the z axis. Then

$$\begin{aligned}(I_z)_A &= \bar{I}_z + Ad^2 \\ &= \frac{\pi}{4}(39 \text{ mm})(64 \text{ mm})^3 + [\pi(64 \text{ mm})(39 \text{ mm})](64 \text{ mm})^2 \\ &= 12.779520\pi \times 10^6 \text{ mm}^4 \\ (\bar{x}A)_A &= (64 \text{ mm})[\pi(64 \text{ mm})(39 \text{ mm})] \\ &= 0.159744\pi \times 10^6 \text{ mm}^3\end{aligned}$$

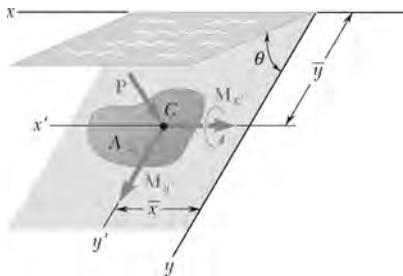
Finally

$$\bar{x} = \frac{12.779520\pi \times 10^6 \text{ mm}^4}{0.159744\pi \times 10^6 \text{ mm}^4}$$

or $\bar{x} = 80.0 \text{ mm}$ ◀

PROBLEM 9.65*

Show that the system of hydrostatic forces acting on a submerged plane area A can be reduced to a force \mathbf{P} at the centroid C of the area and two couples. The force \mathbf{P} is perpendicular to the area and is of magnitude $P = \gamma g A \bar{y} \sin \theta$, where γ is the density of the liquid, and the couples are $\mathbf{M}_{x'} = (\gamma g \bar{I}_{x'} \sin \theta) \mathbf{i}$ and $\mathbf{M}_{y'} = (\gamma g \bar{I}_{y'} \sin \theta) \mathbf{j}$, where $\bar{I}_{x'y'} = \int x'y' dA$ (see Section 9.8). Note that the couples are independent of the depth at which the area is submerged.



SOLUTION

Let γ be the specific weight = ρg

The pressure p at an arbitrary depth ($y \sin \theta$) is

$$p = \gamma(y \sin \theta)$$

so that the hydrostatic force dF exerted on an infinitesimal area dA is

$$dF = (\gamma y \sin \theta) dA$$

Equivalence of the force \mathbf{P} and the system of infinitesimal forces dF requires

$$\Sigma F: \quad P = \int dF = \gamma y \sin \theta dA = \gamma \sin \theta \int y dA$$

or $P = \gamma A \bar{y} \sin \theta \blacktriangleleft$

Equivalence of the force and couple (\mathbf{P} , $\mathbf{M}_{x'} + \mathbf{M}_{y'}$) and the system of infinitesimal hydrostatic forces requires

$$\Sigma M_x: \quad -\bar{y}P - M_{x'} = \int (-y dF)$$

Now

$$\begin{aligned} -\int y dF &= -\int y(\gamma y \sin \theta) dA = -\gamma \sin \theta \int y^2 dA \\ &= -(\gamma \sin \theta) I_x \end{aligned}$$

Then

$$-\bar{y}P - M_{x'} = -(\gamma \sin \theta) I_x$$

or

$$\begin{aligned} M_{x'} &= (\gamma \sin \theta) I_x - \bar{y}(\gamma A \bar{y} \sin \theta) \\ &= \gamma \sin \theta (I_x - A \bar{y}^2) \end{aligned}$$

or $M_{x'} = \gamma \bar{I}_{x'} \sin \theta \blacktriangleleft$

PROBLEM 9.65 (Continued)

$$\Sigma M_y: \bar{x}P + M_{y'} = \int x dF$$

Now

$$\begin{aligned}\int x dF &= \int x(\gamma y \sin \theta) dA = \gamma \sin \theta \int xy dA \\ &= (\gamma \sin \theta) I_{xy}\end{aligned}$$

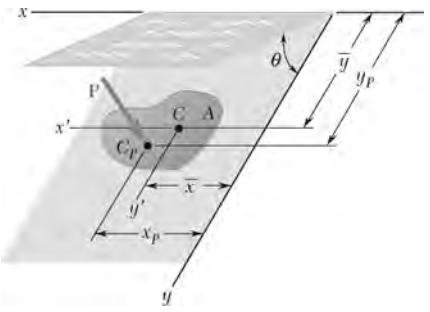
Then

$$\bar{x}P + M_{y'} = (\gamma \sin \theta) I_{xy}$$

or

$$\begin{aligned}M_{y'} &= (\gamma \sin \theta) I_{xy} - \bar{x}(\gamma A \bar{y} \sin \theta) \\ &= \gamma \sin \theta (I_{xy} - A \bar{x} \bar{y})\end{aligned}$$

$$\text{or } M_{y'} = \gamma \bar{I}_{x'y'} \sin \theta \blacktriangleleft$$



PROBLEM 9.66*

Show that the resultant of the hydrostatic forces acting on a submerged plane area A is a force \mathbf{P} perpendicular to the area and of magnitude $P = \gamma g A \bar{y} \sin \theta = \bar{p}A$, where γ is the density of the liquid and \bar{p} is the pressure at the centroid C of the area. Show that \mathbf{P} is applied at a point C_P , called the center of pressure, whose coordinates are $x_P = I_{xy}/A\bar{y}$ and $y_P = I_x/A\bar{y}$, where $I_{xy} = \int xy dA$ (see Section 9.8). Show also that the difference of ordinates $y_P - \bar{y}$ is equal to $k_x'^2/\bar{y}$ and thus depends upon the depth at which the area is submerged.

SOLUTION

Let γ be the specific weight = ρg

The pressure P at an arbitrary depth ($y \sin \theta$) is

$$P = \gamma(y \sin \theta)$$

so that the hydrostatic force dP exerted on an infinitesimal area dA is

$$dP = (\gamma y \sin \theta) dA$$

The magnitude \mathbf{P} of the resultant force acting on the plane area is then

$$\begin{aligned} P &= \int dP = \int \gamma y \sin \theta dA = \gamma \sin \theta \int y dA \\ &= \gamma \sin \theta (\bar{y} A) \end{aligned}$$

Now

$$\bar{p} = \gamma \bar{y} \sin \theta$$

$$P = \bar{p}A \quad \blacktriangleleft$$

Next observe that the resultant \mathbf{P} is equivalent to the system of infinitesimal forces $d\mathbf{P}$. Equivalence then requires

$$\Sigma M_x: -y_P P = - \int y dP$$

Now

$$\begin{aligned} \int y dP &= \int y (\gamma y \sin \theta) dA = \gamma \sin \theta \int y^2 dA \\ &= (\gamma \sin \theta) I_x \end{aligned}$$

Then

$$y_P P = (\gamma \sin \theta) I_x$$

or

$$y_P = \frac{(\gamma \sin \theta) I_x}{\gamma \sin \theta (\bar{y} A)}$$

$$\text{or } y_P = \frac{I_x}{A\bar{y}} \quad \blacktriangleleft$$

PROBLEM 9.66* (Continued)

$$\Sigma M_y: x_P P = \int x dP$$

Now

$$\begin{aligned} \int x dP &= \int x(\gamma y \sin \theta) dA = \gamma \sin \theta \int xy dA \\ &= (\gamma \sin \theta) I_{xy} \end{aligned} \quad (\text{Equation 9.12})$$

Then

$$x_P P = (\gamma \sin \theta) I_{xy}$$

or

$$x_P = \frac{(\gamma \sin \theta) I_{xy}}{\gamma \sin \theta (\bar{y} A)}$$

$$\text{or } x_P = \frac{I_{xy}}{A \bar{y}} \blacktriangleleft$$

Now

$$I_x = \bar{I}_{x'} + A \bar{y}^2$$

From above

$$I_x = (A \bar{y}) y_P$$

By definition

$$\bar{I}_{x'} = \bar{k}_{x'}^2 A$$

Substituting

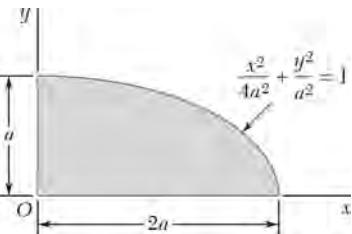
$$(A \bar{y}) y_P = \bar{k}_{x'}^2 A + A \bar{y}^2$$

Rearranging yields

$$y_P - \bar{y} = \frac{\bar{k}_{x'}^2}{\bar{y}} \blacktriangleleft$$

Although $\bar{k}_{x'}$ is not a function of the depth of the area (it depends only on the shape of A), \bar{y} is dependent on the depth.

$$(y_P - \bar{y}) = f(\text{depth})$$



PROBLEM 9.67

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION

First note

$$y = a\sqrt{1 - \frac{x^2}{4a^2}}$$

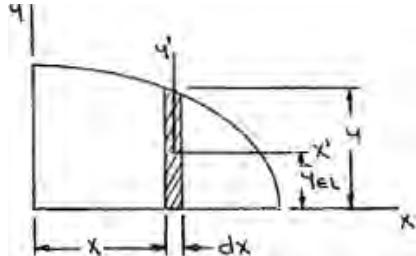
$$= \frac{1}{2}\sqrt{4a^2 - x^2}$$

We have

$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA$$

where

$$d\bar{I}_{x'y'} = 0 \quad (\text{symmetry}) \quad \bar{x}_{EL} = x$$



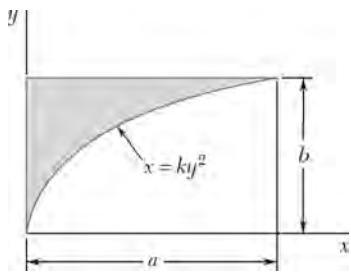
Then

$$I_{xy} = \int dI_{xy} = \int_0^{2a} x \left(\frac{1}{4}\sqrt{4a^2 - x^2} \right) \left(\frac{1}{2}\sqrt{4a^2 - x^2} \right) dx$$

$$= \frac{1}{8} \int_0^{2a} (4a^2x - x^3) dx = \frac{1}{8} \left[2a^2x^2 - \frac{1}{4}x^4 \right]_0^{2a}$$

$$= \frac{a^4}{8} \left[2(2)^2 - \frac{1}{4}(2)^4 \right]$$

$$\text{or} \quad I_{xy} = \frac{1}{2}a^4 \blacktriangleleft$$



PROBLEM 9.68

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION

At

$$x = a, \quad y = b; \quad a = kb^2$$

or

$$k = \frac{a}{b^2}$$

Then

$$x = \frac{a}{b^2} y^2$$

We have

$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL} \bar{y}_{EL} dA$$

where

$$d\bar{I}_{x'y'} = 0 \quad (\text{symmetry}) \quad \bar{x}_{EL} = \frac{1}{2}x = \frac{a}{2b^2} y^2$$

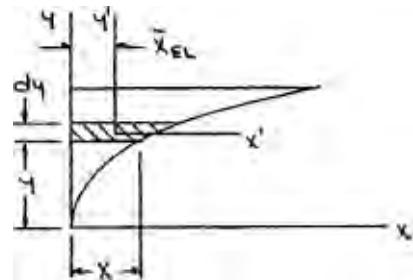
$$\bar{y}_{EL} = y \quad dA = x dy = \frac{a}{b^2} y^2 dy$$

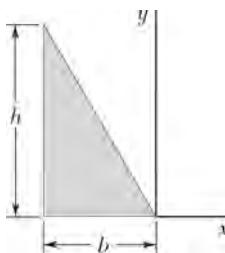
Then

$$I_{xy} = \int dI_{xy} = \int_0^b \left(\frac{a}{2b^2} y^2 \right) (y) \left(\frac{a}{b^2} y^2 dy \right)$$

$$= \frac{a^2}{2b^4} \int_0^b y^5 dy = \frac{a^2}{2b^4} \left[\frac{1}{6} y^6 \right]_0^b$$

$$\text{or} \quad I_{xy} = \frac{1}{12} a^2 b^2 \quad \blacktriangleleft$$





PROBLEM 9.69

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION

First note that

$$y = -\frac{h}{b}x$$

Now

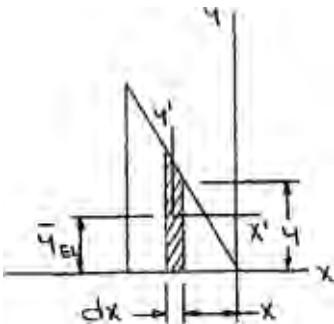
$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA$$

where

$$d\bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

$$\bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{1}{2}y = -\frac{1}{2}\frac{h}{b}x$$

$$dA = y dx = -\frac{h}{b}x dx$$

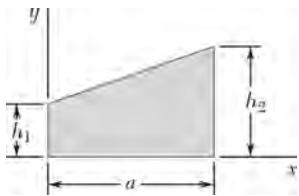


Then

$$I_{xy} = \int dI_{xy} = \int_{-b}^0 x \left(-\frac{1}{2} \frac{h}{b} x \right) \left(-\frac{h}{b} x dx \right)$$

$$= \frac{1}{2} \frac{h^2}{b^2} \int_{-b}^0 x^3 dx = \frac{1}{2} \frac{h^2}{b^2} \left[\frac{1}{4} x^4 \right]_{-b}^0$$

$$\text{or} \quad I_{xy} = -\frac{1}{8} b^2 h^2 \blacktriangleleft$$



PROBLEM 9.70

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

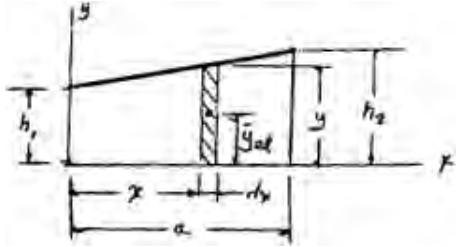
SOLUTION

$$y = h_1 + (h_2 - h_1) \frac{x}{a}$$

$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL} \bar{y}_{EL} dA$$

$d\bar{I}_{x'y'} = 0$ (by symmetry)

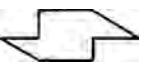
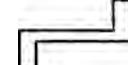
$$\bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{1}{2} y \quad dA = y dx$$

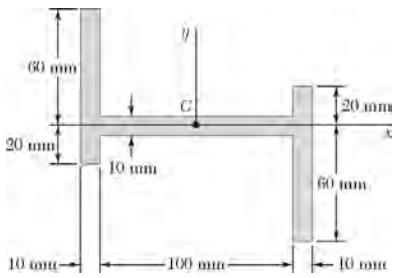


$$\begin{aligned}
 I_{xy} &= \int dI_{xy} - \int_0^a x \left(\frac{1}{2} y \right) y dx = \frac{1}{2} \int_0^a x y^2 dx \\
 &= \frac{1}{2} \int_0^a x \left[h_1 + (h_2 - h_1) \frac{x}{a} \right]^2 dx \\
 &= \frac{1}{2} \int_0^a \left[h_1^2 x + 2h_1(h_2 - h_1) \frac{x^2}{a} + (h_2 - h_1)^2 \frac{x^3}{a^2} \right] dx \\
 &= \frac{1}{2} \left[h_1^2 \frac{a^2}{2} + 2h_1(h_2 - h_1) \frac{a^3}{3a} + (h_2 - h_1)^2 \frac{a^4}{4a^2} \right] \\
 &= \frac{1}{2} \left[h_1^2 \frac{a^2}{2} + \frac{2}{3} h_1 h_2 a^2 - \frac{2}{3} h_1^2 a^2 + \frac{1}{4} h_2^2 a^2 - \frac{1}{2} h_2 h_1 a + \frac{1}{4} h_1^2 a^2 \right] \\
 &= \frac{a^2}{2} \left[h_1^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + h_1 h_2 \left(\frac{2}{3} - \frac{1}{2} \right) + h_2^2 \left(\frac{1}{4} \right) \right] \\
 &= \frac{a^2}{2} \left[h_1^2 \left(\frac{1}{12} \right) + h_1 h_2 \left(\frac{1}{6} \right) + h_2^2 \left(\frac{1}{4} \right) \right] \\
 I_{xy} &= \frac{a^2}{24} (h_1^2 + 2h_1 h_2 + 3h_2^2) \blacksquare
 \end{aligned}$$

PROBLEM 9.70 (Continued)

The following table is provided for the convenience of the instructor, as many problems in this and the next lesson are related.

Type of Problem						
Compute I_x and I_y	Figure 9.12			Figure 9.13B		Figure 9.13A
Compute I_{xy}	9.67	9.72	9.73	9.74	9.75	9.78
$I_{x'}$, $I_{y'}$, $I_{x'y'}$ by equations	9.79	9.80	9.81	9.83	9.82	9.84
Principal axes by equations	9.85	9.86	9.87	9.89	9.88	9.90
$I_{x'}$, $I_{y'}$, $I_{x'y'}$ by Mohr's circle	9.91	9.92	9.93	9.95	9.94	9.96
Principal axes by Mohr's circle	9.97	9.98	9.100	9.101	9.103	9.106



PROBLEM 9.71

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have

$$\bar{I}_{xy} = (I_{xy})_1 + (\bar{I}_{xy})_2 + (I_{xy})_3$$

Now symmetry implies

$$(\bar{I}_{xy})_2 = 0$$

and for the other rectangles

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

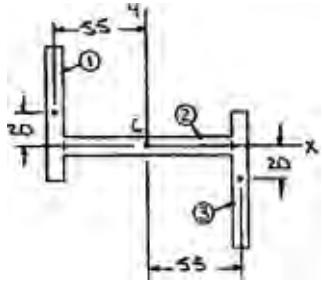
where

$$I_{x'y'} = 0 \quad (\text{symmetry})$$

Thus

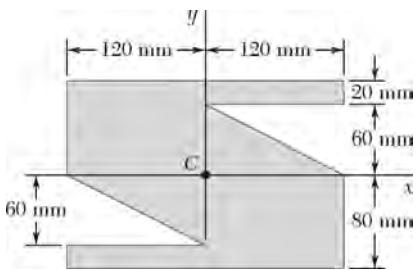
$$\bar{I}_{xy} = (\bar{x}\bar{y}A)_1 + (\bar{x}\bar{y}A)_3$$

Dimensions in mm



	$A, \text{ mm}^2$	$\bar{x}, \text{ mm}$	$\bar{y}, \text{ mm}$	$\bar{x}\bar{y}A, \text{ mm}^4$
1	$10 \times 80 = 800$	-55	20	-880,000
3	$10 \times 80 = 800$	55	-20	-880,000
Σ				-1,760,000

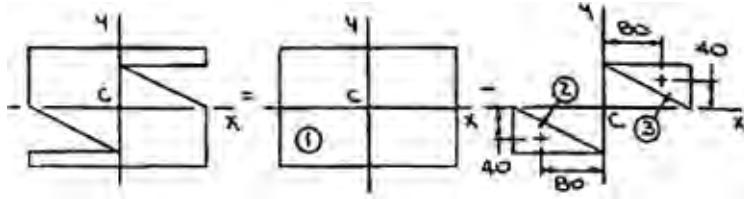
$$\bar{I}_{xy} = -1.760 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.72

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Dimensions in mm

We have

$$\bar{I}_{xy} = (\bar{I}_{xy})_1 - (\bar{I}_{xy})_2 - (\bar{I}_{xy})_3$$

Now symmetry implies

$$(\bar{I}_{xy})_1 = 0$$

and for each triangle

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

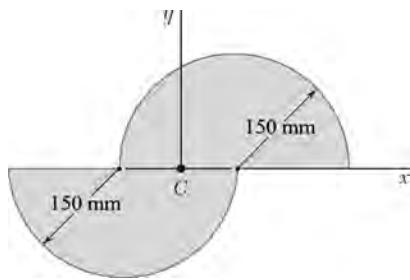
where, using the results of Sample Problem 9.6, $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$ for both triangles. Note that the sign of $\bar{I}_{x'y'}$ is unchanged because the angles of rotation are 0° and 180° for triangles 2 and 3, respectively. Now

	$A, \text{ mm}^2$	$\bar{x}, \text{ mm}$	$\bar{y}, \text{ mm}$	$\bar{x}\bar{y}A, \text{ mm}^4$
2	$\frac{1}{2}(120)(60) = 3600$	-80	-40	11.520×10^6
3	$\frac{1}{2}(120)(60) = 3600$	80	40	11.520×10^6
Σ				23.040×10^6

Then

$$\bar{I}_{xy} = -\left\{ 2\left[-\frac{1}{72}(120 \text{ mm})^2(60 \text{ mm})^2 \right] + 23.040 \times 10^6 \text{ mm}^4 \right\}$$

$$\text{or } \bar{I}_{xy} = -21.6 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.73

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have

$$\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$$

For each semicircle

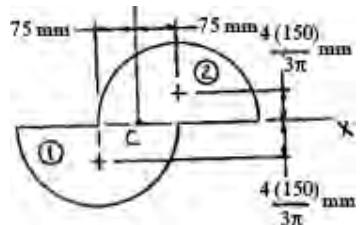
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

and

$$I_{x'y'} = 0 \quad (\text{symmetry})$$

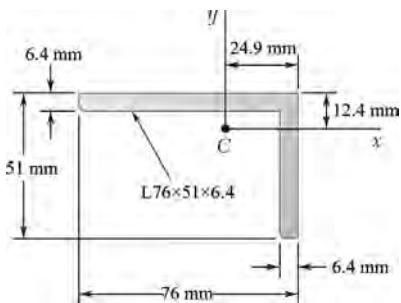
Thus

$$\bar{I}_{xy} = \Sigma \bar{x}\bar{y}A$$



	$A, \text{ mm}^2$	$\bar{x}, \text{ mm}$	$\bar{y}, \text{ mm}$	$\bar{x}\bar{y}A, \text{ mm}^4$
1	$\frac{\pi}{2}(150)^2 = 11250\pi$	-75	$-\frac{200}{\pi}$	168.75×10^6
2	$\frac{\pi}{2}(150)^2 = 11250\pi$	75	$\frac{200}{\pi}$	168.75×10^6
Σ				337.5×10^6

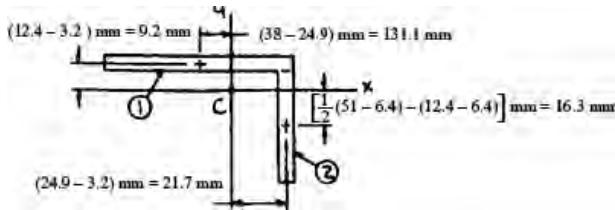
$$\bar{I}_{xy} = 338 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.74

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have

$$\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$$

For each rectangle

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

and

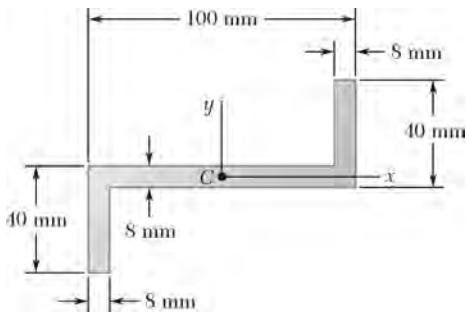
$$\bar{I}_{x'y'} = 0 \quad (\text{symmetry})$$

Thus

$$\bar{I}_{xy} = \sum \bar{x}\bar{y}A$$

	$A, \text{ mm}^2$	$\bar{x}, \text{ mm}$	$\bar{y}, \text{ mm}$	$\bar{x}\bar{y}A, \text{ mm}^4$
1	$7.6 \times 6.4 = 486.4$	-13.7	9.2	-58620.928
2	$6.4 \times 44.6 = 285.44$	21.7	-16.3	-100962.982
Σ				-159583.91

$$\bar{I}_{xy} = 159.6 \times 10^3 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.75

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have

$$I_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$

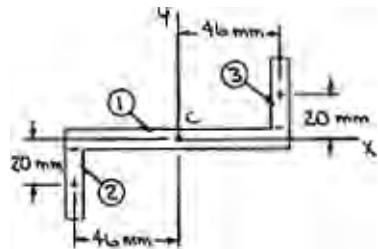
Now symmetry implies $(\bar{I}_{xy})_1 = 0$

and for the other rectangles

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

where $\bar{I}_{x'y'} = 0$ (symmetry)

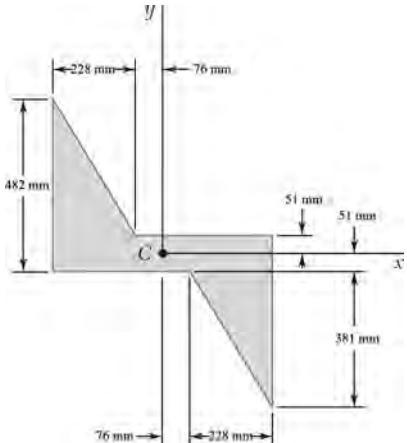
$$\text{Thus } \bar{I}_{xy} = (\bar{x}\bar{y}A)_2 + (\bar{x}\bar{y}A)_3$$



	A, mm^2	\bar{X}, mm	\bar{y}, mm	$\bar{x}\bar{y}A, \text{mm}^4$
2	$8 \times 32 = 256$	-46	-20	235,520
3	$8 \times 32 = 256$	46	20	235,520
Σ				471,040

$$\bar{I}_{xy} = 471 \times 10^3 \text{ mm}^4 \blacktriangleleft$$

PROBLEM 9.76



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have

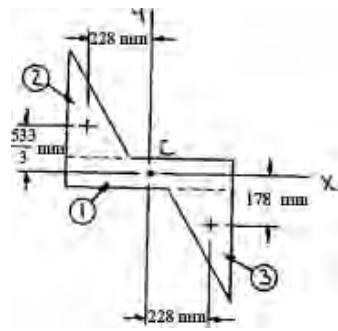
$$\bar{I}_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$

Now, symmetry implies

$$(\bar{I}_{xy})_1 = 0$$

and for each triangle

$$I_{xy} = \bar{I}_{x'y'} + \bar{x} \bar{y} A$$



where, using the results of Sample Problem 9.6, $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$ for both triangles. Note that the sign of $\bar{I}_{x'y'}$ is unchanged because the angles of rotation are 0° and 180° for triangles 2 and 3, respectively.

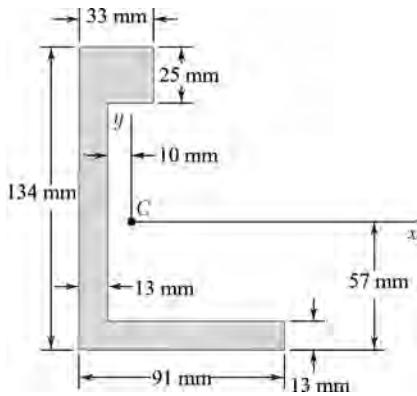
Now

	A, mm^2	\bar{x}, mm	\bar{y}, mm	$\bar{x} \bar{y} A, \text{mm}^4$
2	$\frac{1}{2}(228)(380) = 43.32 \times 10^3$	-228	$\frac{533}{3}$	1.75481×10^9
3	$\frac{1}{2}(228)(381) = (43.434 \times 10^3)$	228	178	1.76273×10^9
				3.51754×10^9

Then

$$\begin{aligned} \bar{I}_{xy} &= \left\{ -\frac{1}{72} \times (228)^2 \times (380)^2 \right\} - \left\{ \frac{1}{72} (228)^2 (381)^2 \right\} - 3.51754 \times 10^9 \\ &= -3.7266 \times 10^9 \end{aligned}$$

or $\bar{I}_{xy} = -3.73 \times 10^9 \text{ mm}^4$ \blacktriangleleft



PROBLEM 9.77

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have

$$\bar{I}_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$

For each rectangle

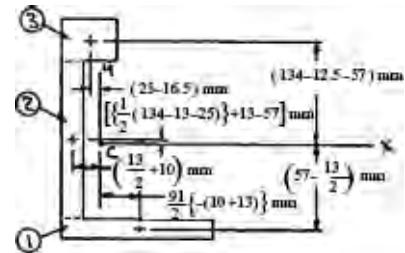
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

and

$$\bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

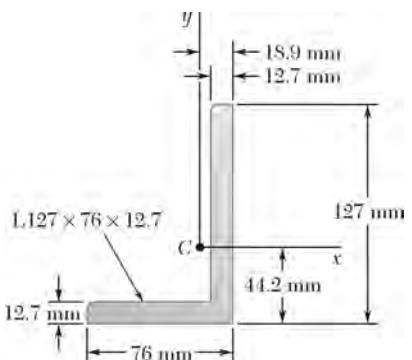
Thus

$$I_{xy} = \Sigma \bar{x}\bar{y}A$$



	A, mm^2	\bar{x}, mm	\bar{y}, mm	$\bar{x}\bar{y}A, \text{mm}^4$
1	$91 \times 13 = 1183$	22.5	-50.5	-1.3442×10^6
2	$13 \times 25 = 1248$	-16.5	4	-82.368×10^3
3	$33 \times 10 = 330$	-6.5	64.5	-345.88125×10^3
Σ				-1.7724×10^6

$$\bar{I}_{xy} = -1.772 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.78

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have

$$\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$$

For each rectangle

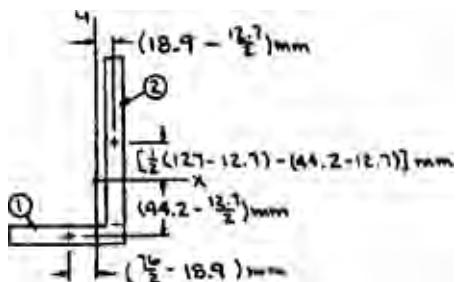
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

and

$$\bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

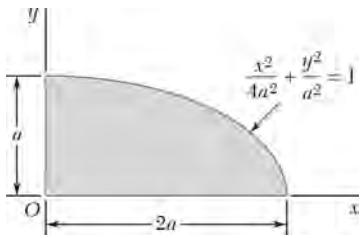
Thus

$$I_{xy} = \sum \bar{x} \bar{y} A$$



	A, mm^2	\bar{x}, mm	\bar{y}, mm	$\bar{x}\bar{y}A, \text{mm}^4$
1	$76 \times 12.7 = 965.2$	-19.1	-37.85	697,777
2	$12.7 \times (127 - 12.7) = 1451.61$	12.55	25.65	467,284
Σ				1,165,061

$$\bar{I}_{xy} = 1.165 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.79

Determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (a) through 45° counterclockwise, (b) through 30° clockwise.

SOLUTION

From Figure 9.12:

$$I_x = \frac{\pi}{16}(2a)(a)^3$$

$$= \frac{\pi}{8}a^4$$

$$I_y = \frac{\pi}{16}(2a)^3(a)$$

$$= \frac{\pi}{2}a^4$$

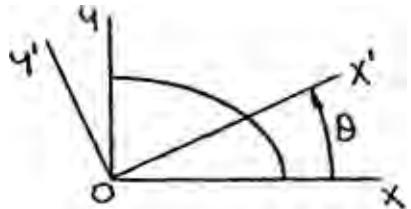
From Problem 9.67:

$$I_{xy} = \frac{1}{2}a^4$$

First note

$$\frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = \frac{5}{16}\pi a^4$$

$$\frac{1}{2}(I_x - I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right) = -\frac{3}{16}\pi a^4$$



Now use Equations (9.18), (9.19), and (9.20).

Equation (9.18):

$$\begin{aligned} I_{x'} &= \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy} \sin 2\theta \\ &= \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 2\theta - \frac{1}{2}a^4 \sin 2\theta \end{aligned}$$

Equation (9.19):

$$\begin{aligned} I_{y'} &= \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta + I_{xy} \sin 2\theta \\ &= \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos 2\theta + \frac{1}{2}a^4 \sin 2\theta \end{aligned}$$

Equation (9.20):

$$\begin{aligned} I_{x'y'} &= \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy} \cos 2\theta \\ &= -\frac{3}{16}\pi a^4 \sin 2\theta + \frac{1}{2}a^4 \cos 2\theta \end{aligned}$$

PROBLEM 9.79 (Continued)

$$(a) \quad \theta = +45^\circ: \quad I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 90^\circ - \frac{1}{2}a^4 \sin 90^\circ$$

or $I_{x'} = 0.482a^4 \blacktriangleleft$

$$I_{y'} = \frac{5}{16}\pi + \frac{3}{16}\pi a^4 \cos 90^\circ + \frac{1}{2}a^4$$

or $I_{y'} = 1.482a^4 \blacktriangleleft$

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin 90^\circ + \frac{1}{2}a^4 \cos 90^\circ$$

or $I_{x'y'} = -0.589a^4 \blacktriangleleft$

$$(b) \quad \theta = -30^\circ: \quad I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos(-60^\circ) - \frac{1}{2}a^4 \sin(-60^\circ)$$

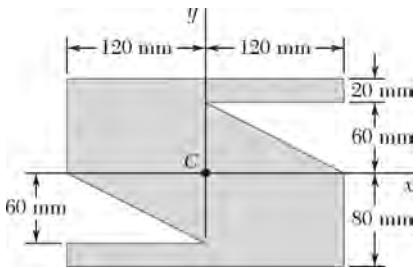
or $I_{x'} = 1.120a^4 \blacktriangleleft$

$$I_{y'} = \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos(-60^\circ) + \frac{1}{2}a^4 \sin(-60^\circ)$$

or $I_{y'} = 0.843a^4 \blacktriangleleft$

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin(-60^\circ) + \frac{1}{2}a^4 \cos(-60^\circ)$$

or $I_{x'y'} = 0.760a^4 \blacktriangleleft$



PROBLEM 9.80

Determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the x and y axes 30° counterclockwise.

SOLUTION

From Problem 9.72:

$$\bar{I}_{xy} = -21.6 \times 10^6 \text{ mm}^4$$

Now

$$\bar{I}_x = (\bar{I}_x)_1 - (I_x)_2 - (I_x)_3$$

where

$$\begin{aligned} (\bar{I}_x)_1 &= \frac{1}{12}(240 \text{ mm})(160 \text{ mm})^3 \\ &= 81.920 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\begin{aligned} (I_x)_2 &= (I_x)_3 = \frac{1}{36}(120 \text{ mm})(60 \text{ mm})^3 \\ &\quad + \left[\frac{1}{2}(120 \text{ mm})(60 \text{ mm}) \right] (40 \text{ mm})^2 \\ &= 6.480 \times 10^6 \text{ mm}^4 \end{aligned}$$

Then

$$\bar{I}_x = [81.920 - 2(6.480)] \times 10^6 \text{ mm}^4 = 68.96 \times 10^6 \text{ mm}^4$$

Also

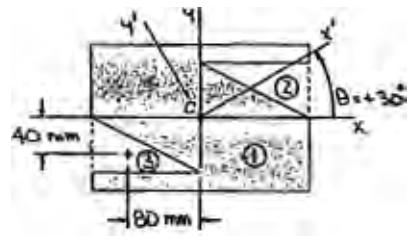
$$\bar{I}_y = (\bar{I}_y)_1 - (I_y)_2 - (I_y)_3$$

where

$$\begin{aligned} (\bar{I}_y)_1 &= \frac{1}{12}(160 \text{ mm})(240 \text{ mm})^3 = 184.320 \times 10^6 \text{ mm}^4 \\ (I_y)_2 &= (I_y)_3 = \frac{1}{36}(60 \text{ mm})(120 \text{ mm})^3 \\ &\quad + \left[\frac{1}{2}(120 \text{ mm})(60 \text{ mm}) \right] (80 \text{ mm})^2 \\ &= 25.920 \times 10^6 \text{ mm}^4 \end{aligned}$$

Then

$$\bar{I}_y = [184.320 - 2(25.920)] \times 10^6 \text{ mm}^4 = 132.48 \times 10^6 \text{ mm}^4$$



PROBLEM 9.80 (Continued)

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(68.96 + 132.48) \times 10^6 = 100.72 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(68.96 - 132.48) \times 10^6 = -31.76 \times 10^6 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

$$\begin{aligned}\text{Eq. (9.18): } I_x &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy} \sin 2\theta \\ &= [100.72 + (-31.76)\cos 60^\circ - (-21.6)\sin 60^\circ] \times 10^6 \text{ mm}^4\end{aligned}$$

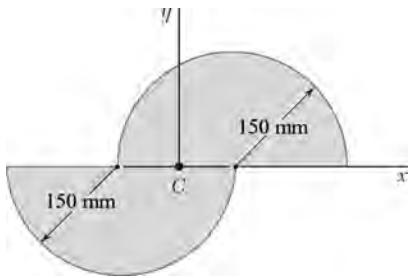
$$\text{or } \bar{I}_{x'} = 103.5 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\begin{aligned}\text{Eq. (9.19): } \bar{I}_{y'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy} \sin 2\theta \\ &= [100.72 - (-31.76)\cos 60^\circ + (-21.6)\sin 60^\circ] \times 10^6 \text{ mm}^4\end{aligned}$$

$$\text{or } \bar{I}_{y'} = 97.9 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\begin{aligned}\text{Eq. (9.20): } \bar{I}_{x'y'} &= \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy} \sin 2\theta \\ &= [-31.76\sin 60^\circ + (-21.6)\cos 60^\circ] \times 10^6 \text{ mm}^4\end{aligned}$$

$$\text{or } \bar{I}_{x'y'} = -38.3 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.81

Determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

SOLUTION

From Problem 9.73:

$$\bar{I}_{xy} = 337.5 \times 10^6 \text{ mm}^4$$

Now

$$\bar{I}_x = (I_x)_1 + (I_x)_2$$

where

$$(I_x)_1 = (I_x)_2 = \frac{\pi}{8}(150 \text{ mm})^4 \\ = (63.28125 \times 10^6)\pi \text{ mm}^4$$

Then

$$\bar{I}_x = 2I_{xj} = (126.5625 \times 10^6)\pi \text{ mm}^4$$

Also

$$\bar{I}_y = (I_y)_1 + (I_y)_2$$

where

$$(I_y)_1 = (I_y)_2 = \frac{\pi}{8}(150 \text{ mm})^4 + \left[\frac{\pi}{2}(150 \text{ mm})^2 \right] (75 \text{ mm})^2 = (126.5625 \times 10^6)\pi \text{ mm}^4$$

Then

$$\bar{I}_y = 2(126.5625 \times 10^6)\pi \text{ mm}^4 = (253.125 \times 10^6)\pi \text{ mm}^4$$

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{\pi}{2}(126.5625 \times 10^6 + 253.125 \times 10^6) = 189.84375\pi \times 10^6 \text{ mm}^4$$

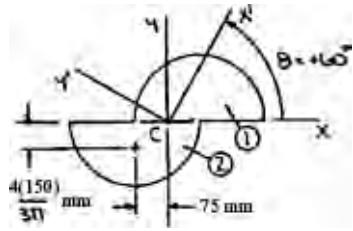
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{\pi}{2}(126.5625 \times 10^6 - 253.125 \times 10^6) - 63.28125 \times 10^6\pi \text{ mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):

$$\begin{aligned} \bar{I}_{x'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y) \cos 2\theta - \bar{I}_{xy} \sin 2\theta \\ &= [189.84375 + (-63.28125) \cos 120^\circ]\pi - 337.5(\sin 120^\circ) \times 10^6 \text{ mm}^4 \\ &= 403.53 \times 10^6 \text{ mm}^4 \end{aligned}$$

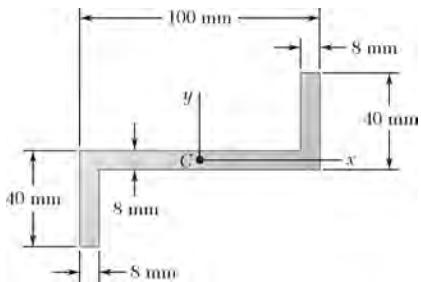
$$\text{or } \bar{I}_{x'} = 404 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.81 (Continued)

$$\begin{aligned}
 \text{Eq. (9.19): } \bar{I}_{y'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy} \sin 2\theta \\
 &= [\{189.84375 - (-63.28125)\cos 120^\circ\}\pi + 337.5 \sin 120^\circ] \times 10^6 \text{ mm}^4 = 789.29 \times 10^6 \text{ mm}^4 \\
 &\quad \text{or } \bar{I}_{y'} = 789 \times 10^6 \text{ mm}^4 \blacktriangleleft
 \end{aligned}$$

$$\begin{aligned}
 \text{Eq. (9.20): } \bar{I}_{x'y'} &= \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy} \cos 2\theta \\
 &= [(-63.28125\pi)\sin 120^\circ + 337.5 \cos 120^\circ] \times 10^6 \text{ mm}^4 \\
 &= -340.919 \times 10^6 \text{ mm}^4 \\
 &\quad \text{or } \bar{I}_{x'y'} = -341 \times 10^6 \text{ mm}^4 \blacktriangleleft
 \end{aligned}$$



PROBLEM 9.82

Determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

SOLUTION

From Problem 9.75:

$$\bar{I}_{xy} = 471,040 \text{ mm}^4$$

Now

$$\bar{I}_x = (\bar{I}_x)_1 + (I_x)_2 + (I_x)_3$$

where

$$\begin{aligned} (\bar{I}_x)_1 &= \frac{1}{12}(100 \text{ mm})(8 \text{ mm})^3 \\ &= 4266.67 \text{ mm}^4 \end{aligned}$$

$$\begin{aligned} (I_x)_2 &= (I_x)_3 = \frac{1}{12}(8 \text{ mm})(32 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](20 \text{ mm})^2 \\ &= 124,245.33 \end{aligned}$$

Then

$$\bar{I}_x = [4266.67 + 2(124,245.33)] \text{ mm}^4 = 252,757 \text{ mm}^4$$

Also

$$\bar{I}_y = (\bar{I}_y)_1 + (I_y)_2 + (I_y)_3$$

where

$$(\bar{I}_y)_1 = \frac{1}{12}(8 \text{ mm})(100 \text{ mm})^3 = 666,666.7 \text{ mm}^4$$

$$\begin{aligned} (\bar{I}_y)_2 &= (I_y)_3 = \frac{1}{12}(32 \text{ mm})(8 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](46 \text{ mm})^2 \\ &= 543,061.3 \text{ mm}^4 \end{aligned}$$

Then

$$\bar{I}_y = [666,666.7 + 2(543,061.3)] \text{ mm}^4 = 1,752,789 \text{ mm}^4$$

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(252,757 + 1,752,789) \text{ mm}^4 = 1,002,773 \text{ mm}^4$$

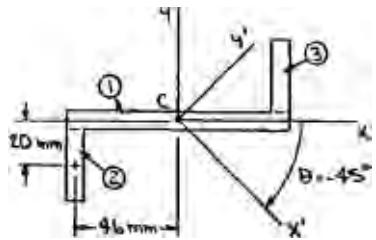
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(252,757 - 1,752,789) \text{ mm}^4 = -750,016 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):

$$\begin{aligned} \bar{I}_{x'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy} \sin 2\theta \\ &= [1,002,773 + (-750,016)\cos(-90^\circ) - 471,040 \sin(-90^\circ)] \end{aligned}$$

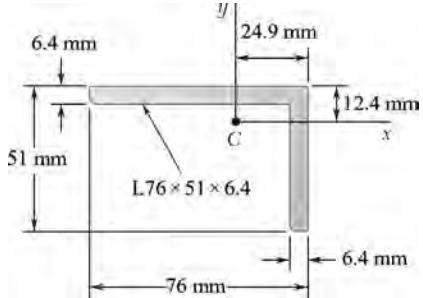
$$\text{or } \bar{I}_{x'} = 1.474 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.82 (Continued)

$$\begin{aligned} \text{Eq. (9.19): } \bar{I}_{y'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy} \sin 2\theta \\ &= [1,002,773 - (-750,016)\cos(-90^\circ) + 471,040 \sin(-90^\circ)] \\ &\quad \text{or} \quad \bar{I}_{y'} = 0.532 \times 10^6 \text{ mm}^4 \blacksquare \end{aligned}$$

$$\begin{aligned} \text{Eq. (9.20): } \bar{I}_{x'y'} &= \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy} \cos 2\theta \\ &= [(-750,016)\sin(-90^\circ) + 471,040 \cos(-90^\circ)] \\ &\quad \text{or} \quad \bar{I}_{x'y'} = 0.750 \times 10^6 \text{ mm}^4 \blacksquare \end{aligned}$$



PROBLEM 9.83

Determine the moments of inertia and the product of inertia of the L76×51×6.4 mm angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION

From Figure 9.13:

$$\bar{I}_x = 0.162 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.454 \times 10^6 \text{ mm}^4$$

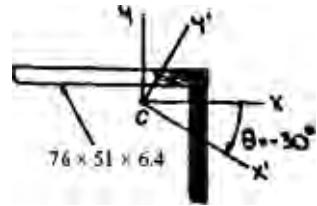
From Problem 9.74:

$$\bar{I}_{xy} = -0.159584 \times 10^6 \text{ mm}^4$$

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.162 + 0.454) \times 10^6 \text{ mm}^4 = 0.308 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(0.162 - 0.454) \times 10^6 \text{ mm}^4 = -0.146 \times 10^6 \text{ mm}^4$$



Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):

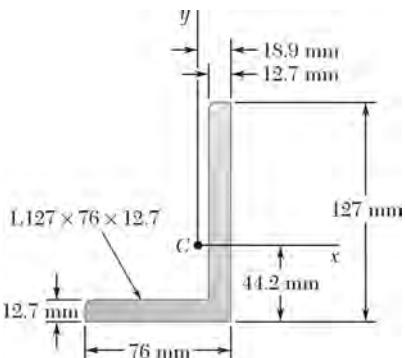
$$\begin{aligned}\bar{I}_{x'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y) \cos 2\theta - \bar{I}_{xy} \sin 2\theta \\ &= [0.308 + (-0.146) \cos(-60^\circ) - (-0.159584) \sin(-60^\circ)] \times 10^6 \text{ mm}^4 \\ &\quad \text{or } \bar{I}_{x'} = 96.8 \times 10^3 \text{ mm}^4\end{aligned}$$

Eq. (9.19):

$$\begin{aligned}\bar{I}_{y'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y) \cos 2\theta + \bar{I}_{xy} \sin 2\theta \\ &= [0.308 - (-0.146) \cos(-60^\circ) + (-0.159584) \sin(-60^\circ)] \times 10^6 \text{ mm}^4 \\ &\quad \text{or } \bar{I}_{y'} = 519 \times 10^3 \text{ mm}^4\end{aligned}$$

Eq. (9.20):

$$\begin{aligned}\bar{I}_{x'y'} &= \frac{1}{2}(\bar{I}_x - \bar{I}_y) \sin 2\theta + \bar{I}_{xy} \cos 2\theta \\ &= [(-0.146) \sin(-60^\circ) + (-0.159584) \cos(-60^\circ)] \times 10^6 \text{ mm}^4 \\ &\quad \text{or } I_{x'y'} = 46.6 \times 10^3 \text{ mm}^4\end{aligned}$$



PROBLEM 9.84

Determine the moments of inertia and the product of inertia of the L127 × 76 × 12.7 mm angle cross section of Problem 9.78 with respect to new centroidal axes obtained by rotating the x and y axes 45° counterclockwise.

SOLUTION

From Figure 9.13:

$$\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$$

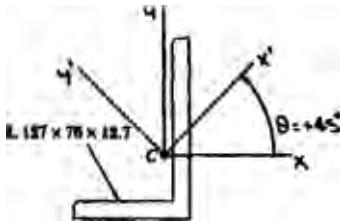
From Problem 9.78:

$$\bar{I}_{xy} = 1.165061 \times 10^6 \text{ mm}^4$$

Now

$$\begin{aligned} \frac{1}{2}(\bar{I}_x + \bar{I}_y) &= \frac{1}{2}(3.93 + 1.06) \times 10^6 \text{ mm}^4 \\ &= 2.495 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(3.93 - 1.06) \times 10^6 \text{ mm}^4 = 1.435 \times 10^6 \text{ mm}^4$$

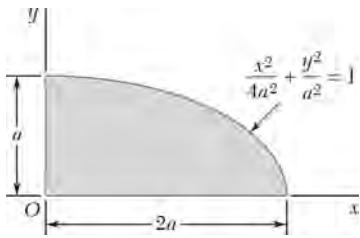


Using Eqs. (9.18), (9.19), and (9.20):

$$\begin{aligned} \text{Eq. (9.18): } \bar{I}_{x'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta \\ &= [2.495 + 1.435\cos 90^\circ - 1.165061\sin 90^\circ] \times 10^6 \text{ mm}^4 \\ &\quad \text{or } \bar{I}_{x'} = 1.330 \times 10^6 \text{ mm}^4 \blacksquare \end{aligned}$$

$$\begin{aligned} \text{Eq. (9.19): } \bar{I}_{y'} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta \\ &= [2.495 - 1.435\cos 90^\circ + 1.165061\sin 90^\circ] \times 10^6 \text{ mm}^4 \\ &\quad \text{or } \bar{I}_{y'} = 3.66 \times 10^6 \text{ mm}^4 \blacksquare \end{aligned}$$

$$\begin{aligned} \text{Eq. (9.20): } \bar{I}_{x'y'} &= \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta \\ &= [(1.435\sin 90^\circ + 1.165061\cos 90^\circ)] \times 10^6 \text{ mm}^4 \\ &\quad \text{or } \bar{I}_{x'y'} = 1.435 \times 10^6 \text{ mm}^4 \blacksquare \end{aligned}$$



PROBLEM 9.85

For the quarter ellipse of Problem 9.67, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79:

$$I_x = \frac{\pi}{8}a^4 \quad I_y = \frac{\pi}{2}a^4$$

Problem 9.67:

$$I_{xy} = \frac{1}{2}a^4$$

Now Eq. (9.25):

$$\begin{aligned} \tan 2\theta_m &= -\frac{2I_{xy}}{I_x - I_y} = -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4} \\ &= \frac{8}{3\pi} = 0.84883 \end{aligned}$$

Then

$$2\theta_m = 40.326^\circ \text{ and } 220.326^\circ$$

$$\text{or } \theta_m = 20.2^\circ \text{ and } 110.2^\circ \blacktriangleleft$$

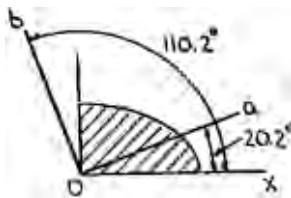
Also Eq. (9.27):

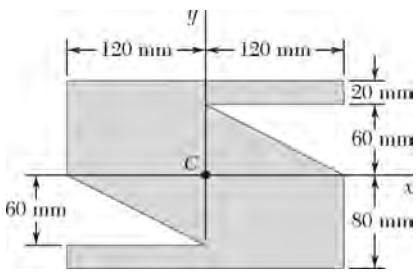
$$\begin{aligned} I_{\max, \min} &= \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \\ &= \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) \\ &\quad \pm \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2} \\ &= (0.981,748 \pm 0.772,644)a^4 \end{aligned}$$

$$\text{or } I_{\max} = 1.754a^4 \blacktriangleleft$$

$$\text{and } I_{\min} = 0.209a^4 \blacktriangleleft$$

By inspection, the a axis corresponds to I_{\min} and the b axis corresponds to I_{\max} .





PROBLEM 9.86

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.72.

SOLUTION

From Problem 9.80:

$$\bar{I}_x = 68.96 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 132.48 \times 10^6 \text{ mm}^4$$

Problem 9.72:

$$\bar{I}_{xy} = -21.6 \times 10^6 \text{ mm}^4$$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-21.6 \times 10^6)}{(68.96 - 132.48) \times 10^6}$$

$$= -0.68010$$

Then

$$2\theta_m = -34.220^\circ \text{ and } 145.780^\circ$$

$$\text{or } \theta_m = -17.11^\circ \text{ and } 72.9^\circ \blacktriangleleft$$

Also Eq. (9.27):

$$\bar{I}_{\max, \min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + I_{xy}^2}$$

Then

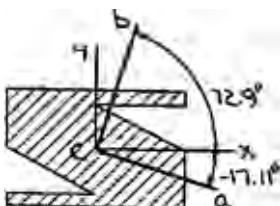
$$\bar{I}_{\max, \min} = \left[\frac{68.96 + 132.48}{2} \pm \sqrt{\left(\frac{68.96 - 132.48}{2}\right)^2 + (-21.6)^2} \right] \times 10^6$$

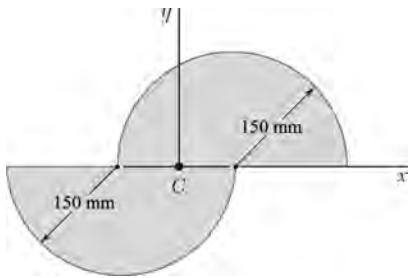
$$= (100.72 \pm 38.409) \times 10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_{\max} = 139.1 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\text{and } \bar{I}_{\min} = 62.3 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

By inspection, the *a* axis corresponds to \bar{I}_{\min} and the *b* axis corresponds to \bar{I}_{\max} .





PROBLEM 9.87

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.73.

SOLUTION

From Problem 9.81:

$$\bar{I}_x = (126.5625 \times 10^6) \pi \text{ mm}^4 \quad \bar{I}_y = (253.125 \times 10^6) \pi \text{ mm}^4$$

Problem 9.73:

$$\bar{I}_{xy} = 337.5 \times 10^6$$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(337.5 \times 10^6)}{(126.5625 - 253.125) \times 10^6 \pi} \\ = 1.69765$$

Then

$$2\theta_m = 59.500^\circ \text{ and } 239.500^\circ$$

$$\text{or } \theta_m = 29.7^\circ \text{ and } 119.7^\circ \blacktriangleleft$$

Also Eq. (9.27):

$$\bar{I}_{\max, \min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

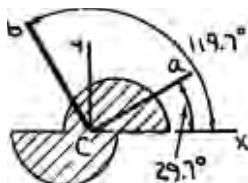
Then

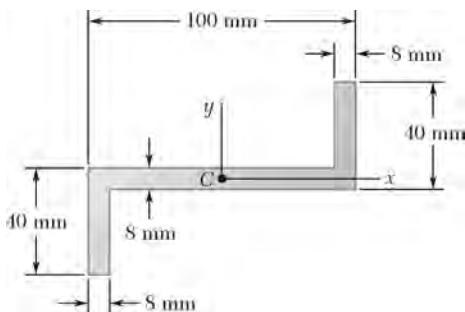
$$\bar{I}_{\max, \min} = \frac{(126.5625 + 253.125) \times 10^6 \pi}{2} \pm \sqrt{\left\{ \frac{(126.5625 - 253.125) \times 10^6 \pi}{2} \right\} + (337.5 \times 10^6)^2} \\ = (596.412 \times 10^6 \pm 391.700 \times 10^6) \text{ mm}^4$$

$$\text{or } \bar{I}_{\max} = 988 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\text{and } \bar{I}_{\min} = 205 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

By inspection, the *a* axis corresponds to \bar{I}_{\min} and the *b* axis corresponds to \bar{I}_{\max} .





PROBLEM 9.88

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.75.

SOLUTION

From Problem 9.82:

$$\bar{I}_x = 252,757 \text{ mm}^4$$

$$\bar{I}_y = 1,752,789 \text{ mm}^4$$

Problem 9.75:

$$\bar{I}_{xy} = 471,040 \text{ mm}^4$$

Now Eq. (9.25):

$$\begin{aligned}\tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(471,040)}{252,757 - 1,752,789} \\ &= 0.62804\end{aligned}$$

Then

$$2\theta_m = 32.130^\circ \quad \text{and} \quad 212.130^\circ$$

$$\text{or} \quad \theta_m = 16.07^\circ \text{ and } 106.1^\circ \quad \blacktriangleleft$$

Also Eq. (9.27):

$$\bar{I}_{\max, \min} = \frac{\bar{I}_x - \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

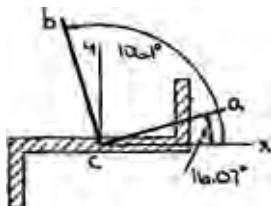
Then

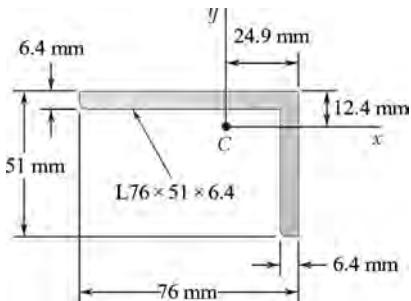
$$\begin{aligned}\bar{I}_{\max, \min} &= \frac{252,757 + 1,752,789}{2} \pm \sqrt{\left(\frac{252,757 - 1,752,789}{2}\right)^2 + 471,040^2} \\ &= (1,002,773 \pm 885,665) \text{ mm}^4\end{aligned}$$

$$\text{or} \quad \bar{I}_{\max} = 1.888 \times 10^6 \text{ mm}^4 \quad \blacktriangleleft$$

$$\text{and} \quad \bar{I}_{\min} = 0.1171 \times 10^6 \text{ mm}^4 \quad \blacktriangleleft$$

By inspection, the *a* axis corresponds to \bar{I}_{\min} and the *b* axis corresponds to \bar{I}_{\max} .





PROBLEM 9.89

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The L $76 \times 51 \times 6.4$ mm angle cross section of Problem 9.74.

SOLUTION

From Problem 9.83:

$$\bar{I}_x = 0.162 \times 10^6 \text{ mm}^4 \quad \bar{I}_y = 0.454 \times 10^6 \text{ mm}^4$$

Problem 9.74:

$$\bar{I}_{xy} = -0.159584 \times 10^6 \text{ mm}^4$$

Now Eq. (9.25):

$$\begin{aligned}\tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-0.159584 \times 10^6)}{(0.162 - 0.454) \times 10^6} \\ &= -1.09304\end{aligned}$$

Then

$$2\theta_m = -47.545^\circ \text{ and } 132.455^\circ$$

$$\text{or } \theta_m = -23.8^\circ \text{ and } 66.2^\circ \blacktriangleleft$$

Also Eq. (9.27):

$$\bar{I}_{\max, \min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

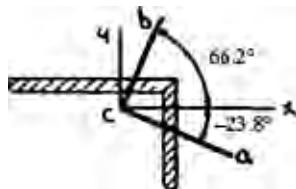
Then

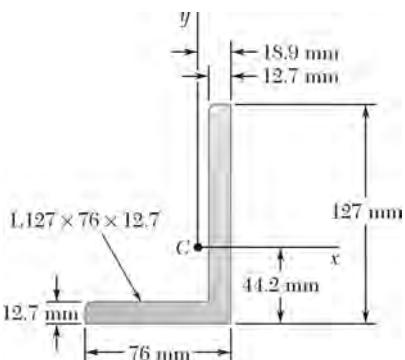
$$\begin{aligned}\bar{I}_{\max, \min} &= \frac{(0.162 + 0.454) \times 10^6}{2} \pm \sqrt{\left(\frac{(0.162 - 0.454) \times 10^6}{2}\right)^2 + (-0.159584 \times 10^6)^2} \\ &= (0.308 \times 10^6 \pm 0.21629) \times 10^6 \text{ mm}^4\end{aligned}$$

$$\text{or } \bar{I}_{\max} = 0.524 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\text{and } \bar{I}_{\min} = 0.0917 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

By inspection, the *a* axis corresponds to \bar{I}_{\min} and the *b* axis corresponds to \bar{I}_{\max} .





PROBLEM 9.90

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The L127 × 76 × 12.7-mm angle cross section of Problem 9.78.

SOLUTION

From Problem 9.84: $\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4$

$$\bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$$

Problem 9.78: $\bar{I}_{xy} = 1.165061 \times 10^6 \text{ mm}^4$

$$\begin{aligned} \text{Now Eq. (9.25): } \tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(1.165061 \times 10^6)}{(3.93 - 1.06) \times 10^6} \\ &= -0.81189 \end{aligned}$$

Then

$$2\theta_m = -39.073^\circ \text{ and } 140.927^\circ$$

$$\text{or } \theta_m = -19.54^\circ \text{ and } 70.5^\circ \blacktriangleleft$$

Also Eq. (9.27):

$$\bar{I}_{\max, \min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

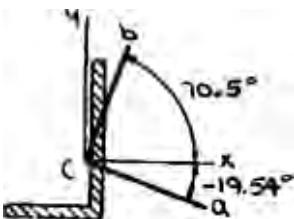
Then

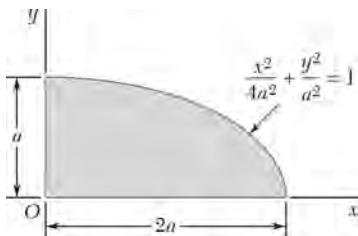
$$\begin{aligned} \bar{I}_{\max, \min} &= \left[\frac{3.93 + 1.06}{2} \pm \sqrt{\left(\frac{3.93 - 1.06}{2}\right)^2 + 1.165061^2} \right] \times 10^6 \text{ mm}^4 \\ &= (2.495 \pm 1.84840) \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\text{or } \bar{I}_{\max} = 4.34 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\text{and } \bar{I}_{\min} = 0.647 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

By inspection, the a axis corresponds to \bar{I}_{\max} and the b axis corresponds to \bar{I}_{\min} .





PROBLEM 9.91

Using Mohr's circle, determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (a) through 45° counterclockwise, (b) through 30° clockwise.

SOLUTION

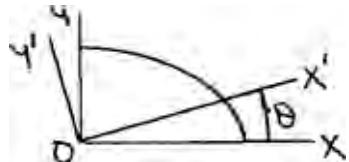
From Problem 9.79:

$$I_x = \frac{\pi}{8} a^4$$

$$I_y = \frac{\pi}{2} a^4$$

Problem 9.67:

$$I_{xy} = \frac{1}{2} a^4$$



The Mohr's circle is defined by the diameter XY , where

$$X\left(\frac{\pi}{8} a^4, \frac{1}{2} a^4\right) \quad \text{and} \quad Y\left(\frac{\pi}{2} a^4, -\frac{1}{2} a^4\right)$$

Now

$$I_{ave} = \frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8} a^4 + \frac{\pi}{2} a^4\right) = \frac{5}{16}\pi a^4 = 0.98175 a^4$$

and

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} = \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8} a^4 - \frac{\pi}{2} a^4\right)\right]^2 + \left(\frac{1}{2} a^4\right)^2}$$

$$= 0.77264 a^4$$

The Mohr's circle is then drawn as shown.

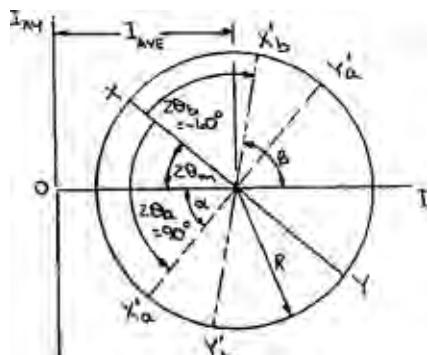
$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

$$= -\frac{2\left(\frac{1}{2} a^4\right)}{\frac{\pi}{8} a^4 - \frac{\pi}{2} a^4}$$

$$= 0.84883$$

or

$$2\theta_m = 40.326^\circ$$



PROBLEM 9.91 (Continued)

Then

$$\begin{aligned}\alpha &= 90^\circ - 40.326^\circ \\ &= 49.674^\circ\end{aligned}$$

$$\begin{aligned}\beta &= 180^\circ - (40.326^\circ + 60^\circ) \\ &= 79.674^\circ\end{aligned}$$

$$(a) \quad \theta = +45^\circ: \quad I_{x'} = I_{\text{ave}} - R \cos \alpha = 0.98175a^4 - 0.77264a^4 \cos 49.674^\circ$$

$$\text{or} \quad I_{x'} = 0.482a^4 \quad \blacktriangleleft$$

$$I_{y'} = I_{\text{ave}} + R \cos \alpha = 0.98175a^4 + 0.77264a^4 \cos 49.674^\circ$$

$$\text{or} \quad I_{y'} = 1.482a^4 \quad \blacktriangleleft$$

$$I_{x'y'} = -R \sin \alpha = -0.77264a^4 \sin 49.674^\circ$$

$$\text{or} \quad I_{x'y'} = -0.589a^4 \quad \blacktriangleleft$$

$$(b) \quad \theta = -30^\circ: \quad I_{x'} = I_{\text{ave}} + R \cos \beta = 0.98175a^4 + 0.77264a^4 \cos 79.674^\circ$$

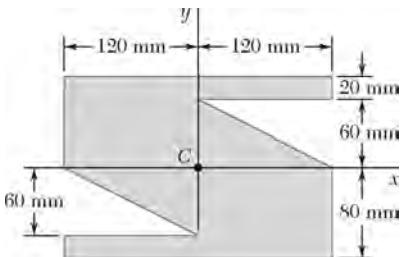
$$\text{or} \quad I_{x'} = 1.120a^4 \quad \blacktriangleleft$$

$$I_{y'} = I_{\text{ave}} - R \cos \beta = 0.98175a^4 - 0.77264a^4 \cos 79.674^\circ$$

$$\text{or} \quad I_{y'} = 0.843a^4 \quad \blacktriangleleft$$

$$I_{x'y'} = R \sin \beta = 0.77264a^4 \sin 79.674^\circ$$

$$\text{or} \quad I_{x'y'} = 0.760a^4 \quad \blacktriangleleft$$



PROBLEM 9.92

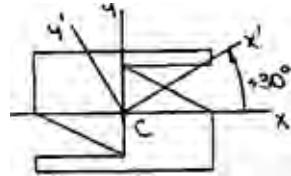
Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the x and y axes 30° counterclockwise.

SOLUTION

From Problem 9.80: $\bar{I}_x = 68.96 \times 10^6 \text{ mm}^4$

$$\bar{I}_y = 132.48 \times 10^6 \text{ mm}^4$$

Problem 9.72: $\bar{I}_{xy} = -21.6 \times 10^6 \text{ mm}^4$



The Mohr's circle is defined by the diameter XY , where $X(68.96 \times 10^6, -21.6 \times 10^6)$ and $Y(132.48 \times 10^6, 21.6 \times 10^6)$.

Now

$$I_{\text{ave}} = \frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(68.96 + 132.48) \times 10^6 = 100.72 \times 10^6 \text{ mm}^4$$

and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y) \right]^2 + I_{xy}^2} = \left\{ \sqrt{\left[\frac{1}{2}(68.96 - 132.48) \right]^2 + (-21.6)^2} \right\} \times 10^6 \text{ mm}^4 \\ = 38.409 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.

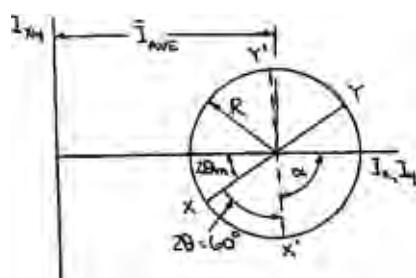
$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ = -\frac{2(-21.6 \times 10^6)}{(68.96 - 132.48) \times 10^6} \\ = -0.68010$$

or

$$2\theta_m = -34.220^\circ$$

Then

$$\alpha = 180^\circ - (34.220^\circ + 60^\circ) \\ = 85.780^\circ$$



PROBLEM 9.92 (Continued)

Then

$$\bar{I}_{x'} = \bar{I}_{\text{ave}} + R \cos \alpha = (100.72 + 38.409 \cos 85.780^\circ) \times 10^6$$

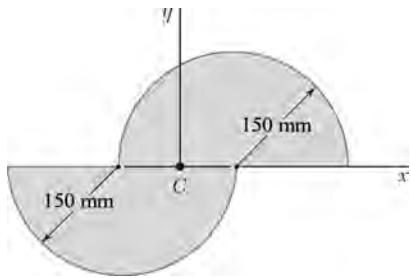
or $\bar{I}_{x'} = 103.5 \times 10^6 \text{ mm}^4 \blacktriangleleft$

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} - R \cos \alpha = (100.72 - 38.409 \cos 85.780^\circ) \times 10^6$$

or $\bar{I}_{y'} = 97.9 \times 10^6 \text{ mm}^4 \blacktriangleleft$

$$\bar{I}_{x'y'} = -R \sin \alpha = -(38.409 \times 10^6) \sin 85.780^\circ$$

or $\bar{I}_{x'y'} = -38.3 \times 10^6 \text{ mm}^4 \blacktriangleleft$



PROBLEM 9.93

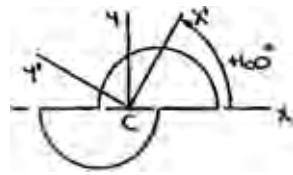
Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

SOLUTION

From Problem 9.81: $\bar{I}_x = (126.5625 \times 10^6) \pi \text{ mm}^4$

$$\bar{I}_y = (253.125 \times 10^6) \pi \text{ mm}^4$$

$$\text{Problem 9.73: } \bar{I}_{xy} = 337.5 \times 10^6 \text{ mm}^4$$



The Mohr's circle is defined by the diameter XY , where $X(126.5625 \times 10^6 \pi, 337.5 \times 10^6)$ and $Y(253.125 \times 10^6 \pi, -337.5 \times 10^6)$.

Now

$$\begin{aligned}\bar{I}_{\text{ave}} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) = (189.84375 \times 10^6) \pi \text{ mm}^4 \\ &= 596.412 \times 10^6 \text{ mm}^4\end{aligned}$$

and

$$\begin{aligned}R &= \sqrt{\frac{1}{2}(\bar{I}_x - \bar{I}_y)^2 + \bar{I}_{xy}^2} \\ &= 391.700 \times 10^6 \text{ mm}^4\end{aligned}$$

(from solution of Problem 9.87)

The Mohr's circle is then drawn as shown.

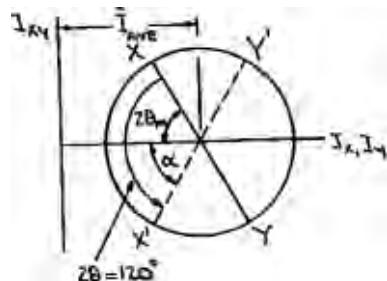
$$\begin{aligned}\tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{-2(337.5 \times 10^6)}{(126.5625 - 253.125) \times 10^6 \times \pi} \\ &= 1.69765\end{aligned}$$

or

$$2\theta_m = 59.500^\circ$$

Then

$$\begin{aligned}\alpha &= 120^\circ - 59.500^\circ \\ &= 60.500^\circ\end{aligned}$$



PROBLEM 9.93 (Continued)

Then

$$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R \cos \alpha = 596.412 \times 10^6 - 391.700 \times 10^6 \cos 60.500^\circ = 403.53 \times 10^6$$

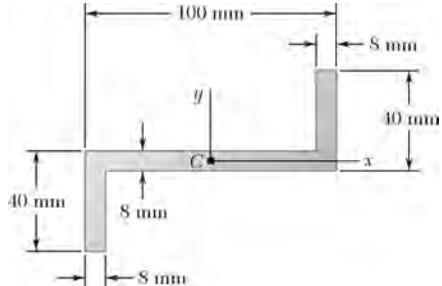
or $\bar{I}_{x'} = 404 \times 10^6 \text{ mm}^4 \blacktriangleleft$

$$\bar{I}_y = \bar{I}_{\text{ave}} + R \cos \alpha = 596.412 \times 10^6 + 391.7 \times 10^6 \cos(60.5) = 789.294 \times 10^6 \text{ mm}^4$$

or $\bar{I}_{y'} = 789 \times 10^6 \text{ mm}^4 \blacktriangleleft$

$$\bar{I}_{x'y'} = -R \sin \alpha = -391.700 \times 10^6 \sin(60.5^\circ)$$

or $\bar{I}_{x'y'} = -341 \times 10^6 \text{ mm}^4 \blacktriangleleft$



PROBLEM 9.94

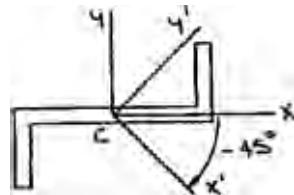
Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

SOLUTION

From Problem 9.82: $\bar{I}_x = 252,757 \text{ mm}^4$

$$\bar{I}_y = 1,752,789 \text{ mm}^4$$

Problem 9.75: $\bar{I}_{xy} = 471,040 \text{ mm}^4$



The Mohr's circle is defined by the diameter XY , where $X(252,757; 471,040)$ and $Y(1,752,789; -471,040)$.

$$\begin{aligned} \text{Now } \bar{I}_{\text{ave}} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(252,757 + 1,752,789) \\ &= 1,002,773 \text{ mm}^4 \end{aligned}$$

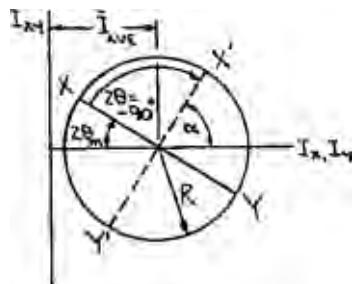
$$\begin{aligned} \text{and } R &= \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} \\ &= \sqrt{\left[\frac{1}{2}(252,757 - 1,752,789)^2\right] + 471,040^2} \\ &= 885,665 \text{ mm}^4 \end{aligned}$$

The Mohr's circle is then drawn as shown.

$$\begin{aligned} \tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(471,040)}{252,757 - 1,752,789} \\ &= 0.62804 \end{aligned}$$

$$\text{or } 2\theta_m = 32.130^\circ$$

$$\begin{aligned} \text{Then } \alpha &= 180^\circ - (32.130 + 90^\circ) \\ &= 57.870^\circ \end{aligned}$$



PROBLEM 9.94 (Continued)

Then

$$\bar{I}_{x'} = \bar{I}_{\text{ave}} + R \cos \alpha = 1,002,773 + 885,665 \cos 57.870^\circ$$

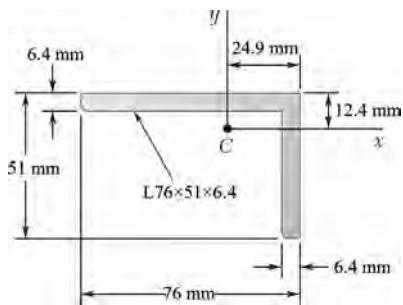
or $\bar{I}_{x'} = 1.474 \times 10^6 \text{ mm}^4 \blacktriangleleft$

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} - R \cos \alpha = 1,002,773 - 885,665 \cos 57.870^\circ$$

or $\bar{I}_{y'} = 0.532 \times 10^6 \text{ mm}^4 \blacktriangleleft$

$$\bar{I}_{x'y'} = R \sin \alpha = 885,665 \sin 57.870^\circ$$

or $\bar{I}_{x'y'} = 0.750 \times 10^6 \text{ mm}^4 \blacktriangleleft$



PROBLEM 9.95

Using Mohr's circle, determine the moments of inertia and the product of inertia of the L76×51×6.4 mm angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION

From Problem 9.83:

$$\bar{I}_x = 0.162 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.454 \times 10^6 \text{ mm}^4$$

Problem 9.74:

$$\bar{I}_{xy} = -0.159584 \times 10^6 \text{ mm}^4$$

The Mohr's circle is defined by the diameter XY, where X(0.162 – 10⁶, 0.159584 – 10⁶) and Y(0.454 – 10⁶, 0.159584 – 10⁶).

Now

$$\begin{aligned}\bar{I}_{\text{ave}} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) \\ &= 0.308 \times 10^6 \text{ mm}^4\end{aligned}$$

and

$$\begin{aligned}R &= \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} \\ &= 0.21629 \times 10^6 \text{ mm}^4\end{aligned}$$

(from Problem 9.89)

The Mohr's circle is then drawn as shown.

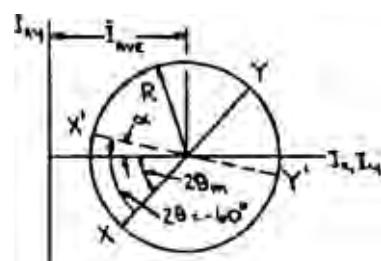
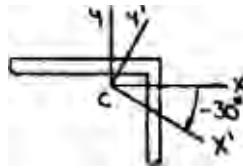
$$\begin{aligned}\tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(-0.159584 \times 10^6)}{(0.162 - 0.454) \times 10^6} \\ &= -1.09304\end{aligned}$$

or

$$2\theta_m = -47.545^\circ$$

Then

$$\alpha = 60^\circ - 47.545^\circ = 12.455^\circ$$



PROBLEM 9.95 (Continued)

Then

$$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R \cos \alpha = 0.308 \times 10^6 - 0.21629 \times 10^6 \cos 12.455^\circ = 0.0968 \times 10^6 \text{ mm}^4$$

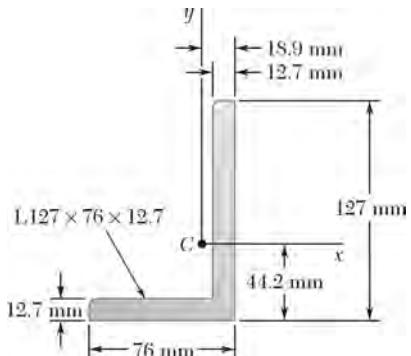
$$\text{or } \bar{I}_{x'} = 0.0968 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R \cos \alpha = 0.308 \times 10^6 + 0.21629 \times 10^6 \cos 12.455^\circ = 0.5192 \times 10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_y = 0.519 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{I}_{x'y'} = R \sin \alpha = 0.21629 \times 10^6 \sin(12.455^\circ)$$

$$\text{or } \bar{I}_{x'y'} = 0.0466 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.96

Using Mohr's circle, determine the moments of inertia and the product of inertia of the L127 × 76 × 12.7-mm angle cross section of Problem 9.78 with respect to new centroidal axes obtained by rotating the x and y axes 45° counterclockwise.

SOLUTION

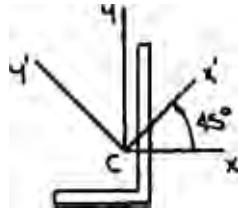
From Problem 9.84:

$$\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$$

Problem 9.78:

$$\bar{I}_{xy} = 1.165061 \times 10^6 \text{ mm}^4$$



The Mohr's circle is defined by the diameter XY , where $X(3.93 \times 10^6, 1.165061 \times 10^6)$, $Y(1.06 \times 10^6, -1.165061 \times 10^6)$.

Now

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.93 + 1.06) \times 10^6 = 2.495 \times 10^6 \text{ mm}^4$$

and

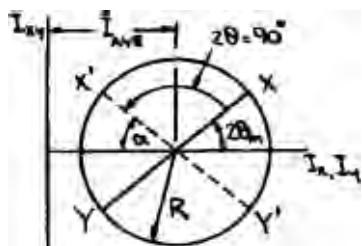
$$\begin{aligned} R &= \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} \\ &= \sqrt{\left[\frac{1}{2}(3.93 - 1.06)\right]^2 + 1.165061^2} \times 10^6 \text{ mm}^4 \\ &= 1.84840 \times 10^6 \text{ mm}^4 \end{aligned}$$

The Mohr's circle is then drawn as shown.

$$\begin{aligned} \tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(1.165061 \times 10^6)}{(3.93 - 1.06) \times 10^6} \\ &= -0.81189 \end{aligned}$$

or

$$2\theta_m = -39.073^\circ$$



PROBLEM 9.96 (Continued)

Then

$$\begin{aligned}\alpha &= 180^\circ - (39.073^\circ + 90^\circ) \\ &= 50.927^\circ\end{aligned}$$

Then

$$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R \cos \alpha = (2.495 - 1.84840 \cos 50.927^\circ) \times 10^6$$

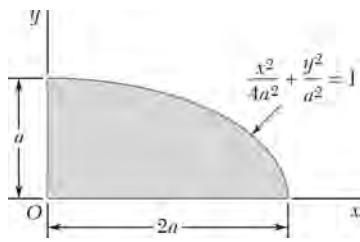
$$\text{or } \bar{I}_{x'} = 1.330 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R \cos \alpha = (2.495 + 1.84840 \cos 50.927^\circ) \times 10^6$$

$$\text{or } \bar{I}_{y'} = 3.66 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{I}_{x'y'} = R \sin \alpha = (1.84840 \times 10^6) \sin 50.927^\circ$$

$$\text{or } \bar{I}_{x'y'} = 1.435 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.97

For the quarter ellipse of Problem 9.67, use Mohr's circle to determine the orientation the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79:

$$I_x = \frac{\pi}{8}a^4 \quad I_y = \frac{\pi}{2}a^4$$

Problem 9.67:

$$I_{xy} = \frac{1}{2}a^4$$

The Mohr's circle is defined by the diameter XY, where

$$X\left(\frac{\pi}{8}a^4, \frac{1}{2}a^4\right) \text{ and } Y\left(\frac{\pi}{2}a^4, -\frac{1}{2}a^4\right)$$

Now

$$I_{ave} = \frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = 0.98175a^4$$

and

$$\begin{aligned} R &= \sqrt{\left[\frac{1}{2}(I_x - I_y)\right]^2 + I_{xy}^2} \\ &= \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2} \\ &= 0.77264a^4 \end{aligned}$$

The Mohr's circle is then drawn as shown.

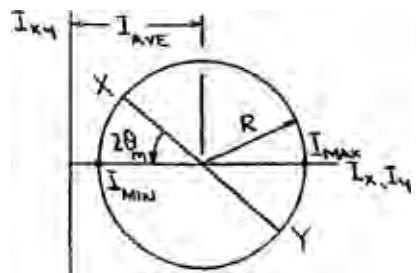
$$\begin{aligned} \tan 2\theta_m &= -\frac{2I_{xy}}{I_x - I_y} \\ &= -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4} \\ &= 0.84883 \end{aligned}$$

or

$$2\theta_m = 40.326^\circ$$

and

$$\theta_m = 20.2^\circ$$

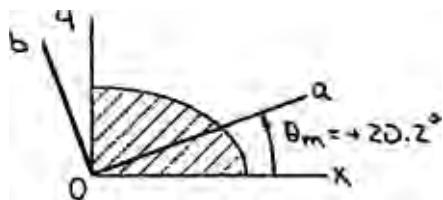


PROBLEM 9.97 (Continued)

The Principal axes are obtained by rotating the xy axes through

20.2° counterclockwise ◀

about O .



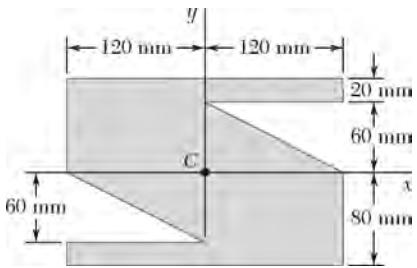
Now

$$I_{\max, \min} = I_{\text{ave}} \pm R = 0.98175a^4 \pm 0.77264a^4$$

$$\text{or } I_{\max} = 1.754a^4 \quad \blacktriangleleft$$

$$\text{and } I_{\min} = 0.209a^4 \quad \blacktriangleleft$$

From the Mohr's circle it is seen that the a axis corresponds to I_{\min} and the b axis corresponds to I_{\max} .



PROBLEM 9.98

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.72.

SOLUTION

From Problem 9.80:

$$\bar{I}_x = 68.96 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 132.48 \times 10^6 \text{ mm}^4$$

Problem 9.72:

$$I_{xy} = -21.6 \times 10^6 \text{ mm}^4$$

The Mohr's circle is defined by the diameter XY, where $X(68.96 \times 10^6, -21.6 \times 10^6)$ and $Y(132.48 \times 10^6, 21.6 \times 10^6)$

Now

$$\begin{aligned}\bar{I}_{\text{ave}} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) \\ &= \frac{1}{2}(68.96 + 132.48) \times 10^6 \\ &= 100.72 \times 10^6 \text{ mm}^4\end{aligned}$$

and

$$\begin{aligned}R &= \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + I_{xy}^2} \\ &= \sqrt{\left[\frac{1}{2}(68.96 - 132.48)\right]^2 + (-21.6)^2} \times 10^6 \\ &= 38.409 \times 10^6 \text{ mm}^4\end{aligned}$$

The Mohr's circle is then drawn as shown.

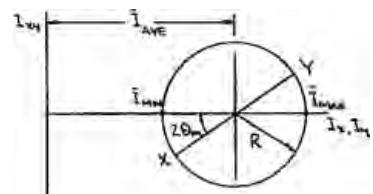
$$\begin{aligned}\tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(-21.6 \times 10^6)}{(68.96 - 132.48) \times 10^6} \\ &= -0.68010\end{aligned}$$

or

$$2\theta_m = -34.220^\circ$$

and

$$\theta_m = -17.11^\circ$$

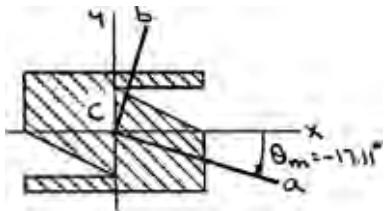


PROBLEM 9.98 (Continued)

The principal axes are obtained by rotating the xy axes through

17.11° clockwise ◀

about C .



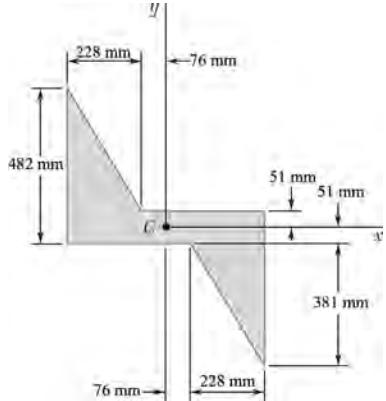
Now

$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (100.72 + 38.409) \times 10^6$$

$$\text{or } \bar{I}_{\max} = 139.1 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\text{and } \bar{I}_{\min} = 62.3 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

From the Mohr's circle it is seen that the a axis corresponds to \bar{I}_{\min} and the b axis corresponds to \bar{I}_{\max} .



PROBLEM 9.99

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.76.

SOLUTION

From Problem 9.76:

$$\bar{I}_{xy} = -3.7266 \times 10^9 \text{ mm}^4$$

Now

$$\bar{I}_x = (\bar{I}_x)_1 + (\bar{I}_x)_2 + (\bar{I}_x)_3$$

where

$$(\bar{I}_x)_1 = \frac{1}{12}(608 \text{ mm})(102 \text{ mm})^3 = 53.767872 \times 10^6 \text{ mm}^4$$

$$(\bar{I}_x)_2 = \frac{1}{36}(228 \text{ mm})(380 \text{ mm})^3 + \left[\frac{1}{2}(228 \text{ mm})(380 \text{ mm}) \right] \left(\frac{533}{3} \text{ mm} \right)^2 \\ = 1.71494 \times 10^9 \text{ mm}^4$$

$$(\bar{I}_x)_3 = \frac{1}{36}(228 \text{ mm})(381 \text{ mm})^3 + \left[\frac{1}{2}(228 \text{ mm})(381 \text{ mm}) \right] (178 \text{ mm})^2 \\ = 1.72644 \times 10^9 \text{ mm}^4$$

Then

$$\bar{I}_x = [0.053768 + 1.71494 + 1.72644] \times 10^9 \text{ mm}^4 \\ = 3.495148 \times 10^9 \text{ mm}^4$$

Also

$$\bar{I}_y = (\bar{I}_y)_1 + (\bar{I}_y)_2 + (\bar{I}_y)_3$$

where

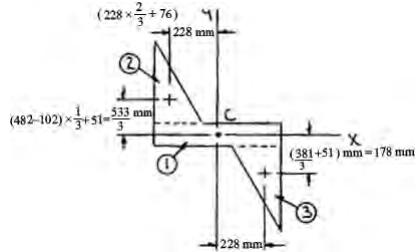
$$(\bar{I}_y)_1 = \frac{1}{12}(102 \text{ mm})(608 \text{ mm})^3 = 1.91042 \times 10^9 \text{ mm}^4$$

$$(\bar{I}_y)_2 = \frac{1}{36}(380 \text{ mm})(228 \text{ mm})^3 + \left[\frac{1}{2}(228 \text{ mm})(380 \text{ mm}) \right] (228 \text{ mm})^2 = 2.37706 \times 10^9 \text{ mm}^4$$

$$(\bar{I}_y)_3 = \frac{1}{36}(381 \text{ mm})(228 \text{ mm})^3 + \left[\frac{1}{2}(228 \text{ mm})(381 \text{ mm}) \right] (228 \text{ mm})^2 = 2.38331 \times 10^9 \text{ mm}^4$$

Then

$$\bar{I}_y = (1.91042 + 2.37706 + 2.38331) \times 10^9 \text{ mm}^4 = 6.67079 \times 10^9 \text{ mm}^4$$



PROBLEM 9.99 (Continued)

The Mohr's circle is defined by the diameter XY , where $X(3.495148 \times 10^9, -3.7266 \times 10^9)$ and $Y(6.67079 \times 10^9, 3.7266 \times 10^9)$

Now

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.495148 + 6.67079) \times 10^9 = 5.082969 \times 10^9$$

and

$$\begin{aligned} R &= \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} \\ &= \sqrt{\left[\frac{1}{2}(3.495148 - 6.67079) \times 10^9\right]^2 + (-3.7266 \times 10^9)^2} \\ &= 4.05077 \times 10^9 \text{ mm}^4 \end{aligned}$$

The Mohr's circle is then drawn as shown.

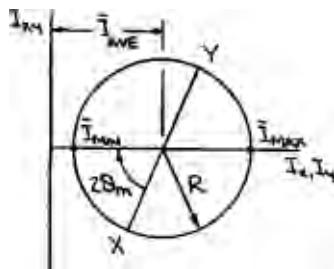
$$\begin{aligned} \tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(-3.7266 \times 10^9)}{(3.495148 - 6.67079) \times 10^9} \\ &= -2.34699 \end{aligned}$$

or

$$2\theta_m = -66.922^\circ$$

and

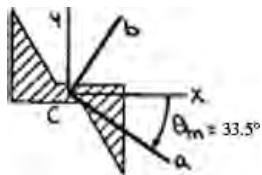
$$\theta_m = -33.5^\circ$$



The principal axes are obtained by rotating the xy axes through

33.5° clockwise ◀

about C .



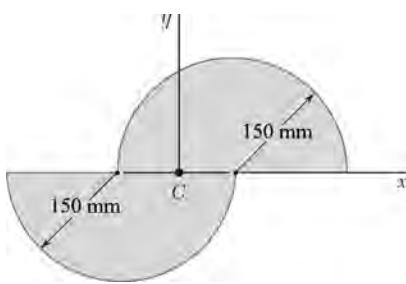
Now

$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (5.082969 \pm 4.05077) \times 10^9 \text{ mm}^4$$

or $\bar{I}_{\max} = 9.13 \times 10^9 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 1.032 \times 10^9 \text{ mm}^4$ ◀

From the Mohr's circle it is seen that the a axis corresponds to \bar{I}_{\min} and the b axis corresponds to \bar{I}_{\max} .



PROBLEM 9.100

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.73.

SOLUTION

From Problem 9.81:

$$\bar{I}_x = (126.5625 \times 10^6) \pi \text{ mm}^4 \quad \bar{I}_y = (253.125 \times 10^6) \pi \text{ mm}^4$$

Problem 9.73:

$$\bar{I}_{xy} = 337.5 \times 10^6 \text{ mm}^4$$

The Mohr's circle is defined by the diameter XY, where X(126.5625 $\times 10^6 \pi$, 337.5 $\times 10^6$) and Y(253.125 $\times 10^6 \pi$, -337.5 $\times 10^6$).

Now

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = 596.412 \times 10^6 \text{ mm}^4$$

and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)^2 \right] + I_{xy}^2}$$

$$= 391.700 \times 10^6 \text{ mm}^4 \quad (\text{from Solution of Problem 9.87})$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(337.5 \times 10^6)}{(126.5625 - 253.125) \times 10^6 \pi}$$

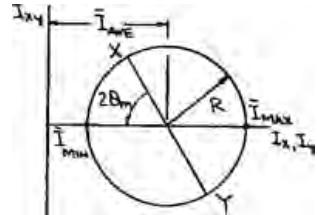
$$= 1.69765$$

or

$$2\theta_m = 59.4998^\circ$$

and

$$\theta_m = 29.7^\circ$$



The principal axes are obtained by rotating the xy axes through

29.7° counterclockwise

about C.

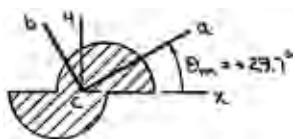
PROBLEM 9.100 (Continued)

Now

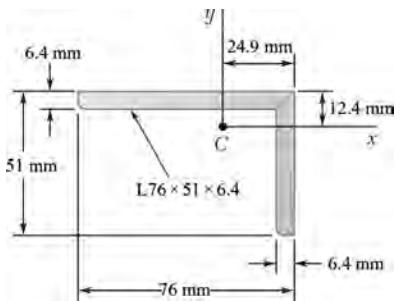
$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = 596.412 \times 10^6 \pm 391.700 \times 10^6$$

$$\text{or } I_{\max} = 988 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\text{and } \bar{I}_{\min} = 205 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



From the Mohr's circle it is seen that the a axis corresponds to I_{\min} and the b axis corresponds to \bar{I}_{\max} .



PROBLEM 9.101

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.74.

SOLUTION

From Problem 9.83:

$$\bar{I}_x = 0.162 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.454 \times 10^6 \text{ mm}^4$$

Problem 9.74:

$$\bar{I}_{xy} = -0.159584 \times 10^6 \text{ mm}^4$$

The Mohr's circle is defined by the diameter XY , where $X(0.162 \times 10^6, -0.159584 \times 10^6)$ and $Y(0.454 \times 10^6, 0.159584 \times 10^6)$.

Now

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = 0.308 \times 10^6 \text{ mm}^4$$

and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$

$$= 0.21629 \times 10^6 \text{ mm}^4$$

(from Solution of Problem 9.89)

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-0.159584 \times 10^6)}{(0.162 - 0.454) \times 10^6}$$

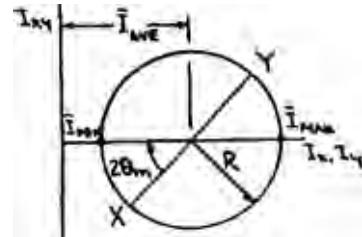
$$= -1.09304$$

Then

$$2\theta_m = -47.545$$

and

$$\theta_m = -23.8^\circ$$



The principal axes are obtained by rotating the xy axes through

23.8° clockwise \blacktriangleleft

about C .

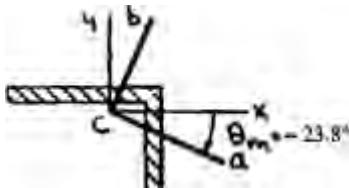
PROBLEM 9.101 (Continued)

Now

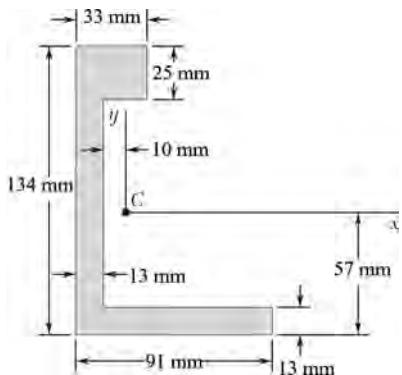
$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = 0.308 \times 10^6 \pm 0.21629 \times 10^6$$

or $\bar{I}_{\max} = 0.524 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 0.0917 \times 10^6 \text{ mm}^4$ ◀



From the Mohr's circle it is seen that the a axis corresponds to I_{\min} and the b axis corresponds to I_{\max} .



PROBLEM 9.102

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.77

(The moments of inertia \bar{I}_x and \bar{I}_y of the area of Problem 9.102 were determined in Problem 9.44).

SOLUTION

From Problem 9.44:

$$\bar{I}_x = 2.90051 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.721271 \times 10^6 \text{ mm}^4$$

Problem 9.77:

$$\bar{I}_{xy} = -0.680632 \times 10^6 \text{ mm}^4$$

The Mohr's circle is defined by the diameter XY, where X(2.90051 $\times 10^6$, -0.680632 $\times 10^6$) and Y(0.72127 $\times 10^6$, 0.680632 $\times 10^6$).

Now

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = 1.81089 \times 10^6 \text{ mm}^4$$

and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y) \right]^2 + \bar{I}_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}(2.90051 - 0.721271) \times 10^6 \right]^2 + (-0.680632 \times 10^6)^2}$$

$$= 1.28473 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

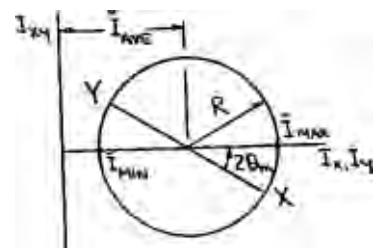
$$= -\frac{2(-0.680632 \times 10^6)}{(2.90051 - 0.721271) \times 10^6}$$

$$= 0.6246$$

or

$$2\theta_m = 31.99^\circ$$

$$\theta_m = 16.00^\circ$$



The principal axes are obtained by rotating the xy axes through

16.00° counterclockwise \blacktriangleleft
about C .

PROBLEM 9.102 (Continued)

Now

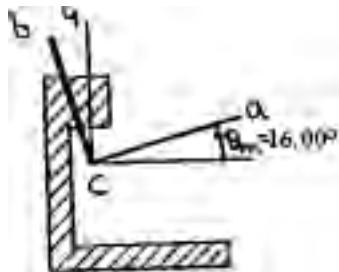
$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (1.81089 \pm 1.28473) \times 10^6$$

or

and

$$\bar{I}_{\max} = 3.10 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{I}_{\min} = 0.526 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



From the Mohr's circle it is seen that the a axis corresponds to I_{\max} and the b axis corresponds to I_{\min} .

PROBLEM 9.103

The moments and product of inertia of an L102 × 102 × 12.7-mm angle cross section with respect to two rectangular axes x and y through C are, respectively, $\bar{I}_x = 2.30 \times 10^6 \text{ mm}^4$, $\bar{I}_y = 2.30 \times 10^6 \text{ mm}^4$, and $\bar{I}_{xy} < 0$, with the minimum value of the moment of inertia of the area with respect to any axis through C being $\bar{I}_{\min} = 1.15 \times 10^6 \text{ mm}^4$. Using Mohr's circle, determine (a) the product of inertia I_{xy} of the area, (b) the orientation of the principal axes, (c) the value of \bar{I}_{\max} .

SOLUTION

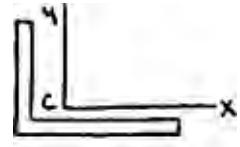
(Note: A review of a table of rolled-steel shapes reveals that the given values of \bar{I}_x and \bar{I}_y are obtained when the 104-mm leg of the angle is parallel to the x axis. Further, for $\bar{I}_{xy} < 0$, the angle must be oriented as shown.)

Now

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(2.30 \times 10^6) \times 2 = 2.30 \times 10^6 \text{ mm}^4$$

and

$$\begin{aligned}\bar{I}_{\min} &= \bar{I}_{\text{ave}} - R \quad \text{or} \quad R = 2.30 \times 10^6 - 1.15 \times 10^6 \\ &= 1.15 \times 10^6 \text{ mm}^4\end{aligned}$$



Using \bar{I}_{ave} and R , the Mohr's circle is then drawn as shown; note that for the diameter XY , $X(2.30 \times 10^6, \bar{I}_{xy})$ and $Y(2.30 \times 10^6, |\bar{I}_{xy}|)$.

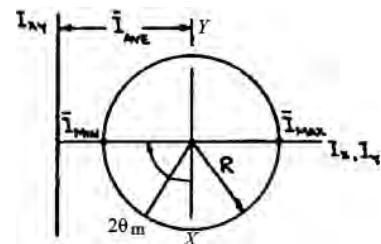
(a) We have

$$R^2 = \left[\frac{1}{2}(\bar{I}_x - \bar{I}_y) \right]^2 + \bar{I}_{xy}^2 ; \text{ Since } \bar{I}_x = \bar{I}_y$$

$$R = \bar{I}_{xy}$$

or

$$\bar{I}_{xy} = 1.15 \times 10^6 \text{ mm}^4$$



Solving for \bar{I}_{xy} and taking the negative root (since $\bar{I}_{xy} < 0$) yields

$$\bar{I}_{xy} = -1.15 \times 10^6 \text{ mm}^4$$

$$\bar{I}_{xy} = -1.150 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

(b) We have from formula or otherwise clearly

$$2\theta_m = -90^\circ$$

or

$$\theta_m = -45^\circ$$

The principal axes are obtained by rotating the xy axes through

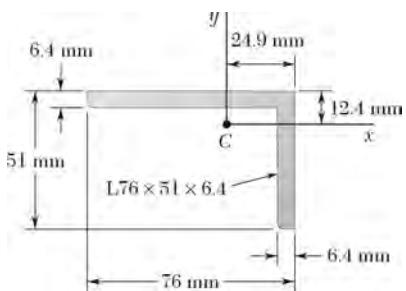
45° clockwise \blacktriangleleft

about C .

(c) We have

$$\begin{aligned}\bar{I}_{\max} &= \bar{I}_{\text{ave}} + R = 2.30 \times 10^6 + 1.15 \times 10^6 \\ &= 3.45 \times 10^6 \text{ mm}^4\end{aligned}$$

$$\text{or } \bar{I}_{\max} = 3.45 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.104

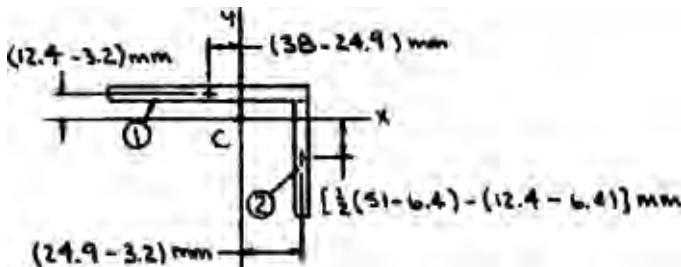
Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Figure 9.13.)

SOLUTION

From Figure 9.13B:

$$\bar{I}_x = 0.162 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.454 \times 10^6 \text{ mm}^4$$



We have

$$\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$$

For each rectangle

$$I_{xy} = \bar{I}_{x'y'} + \bar{x} \bar{y} A$$

and

$$\bar{I}_{x'y'} = 0 \quad (\text{symmetry})$$

$$I_{xy} = \sum \bar{x} \bar{y} A$$

	$A, \text{ mm}^2$	$\bar{x}, \text{ mm}$	$\bar{y}, \text{ mm}$	$\bar{x} \bar{y} A, \text{ mm}^4$
1	$76 \times 6.4 = 486.4$	-13.1	9.2	-58620.93
2	$6.4 \times (51 - 6.4) = 285.44$	21.7	-16.3	-100962.98
Σ				-159583.91

$$\bar{I}_{xy} = -159584 \text{ mm}^4$$

The Mohr's circle is defined by the diameter XY where $X(0.162 \times 10^6, -0.159584 \times 10^6)$ and $Y(0.454 \times 10^6, 0.159584 \times 10^6)$

Now

$$\begin{aligned} \bar{I}_{\text{ave}} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.162 + 0.454) \times 10^6 \\ &= 0.3080 \times 10^6 \text{ mm}^4 \end{aligned}$$

PROBLEM 9.104 (Continued)

and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y) \right]^2 + I_{xy}^2} = \left\{ \sqrt{\left[\frac{1}{2}(0.162 - 0.454) \right]^2 + (-0.159584)^2} \right\} \times 10^6 \\ = 0.21629 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.

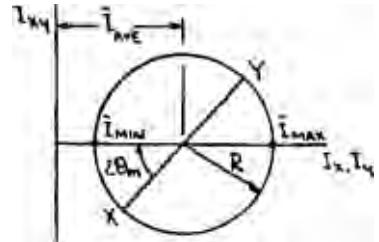
$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ = -\frac{2(-0.159584 \times 10^6)}{(0.162 - 0.454) \times 10^6} \\ = -1.09304$$

or

$$2\theta_m = -47.545$$

and

$$\theta_m = -23.8^\circ$$



The principal axes are obtained by rotating the xy axes through

23.8° clockwise ◀

About C .

Now

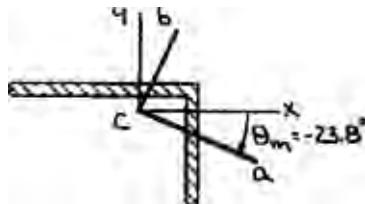
$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (0.3080 \pm 0.21629) \times 10^6$$

or

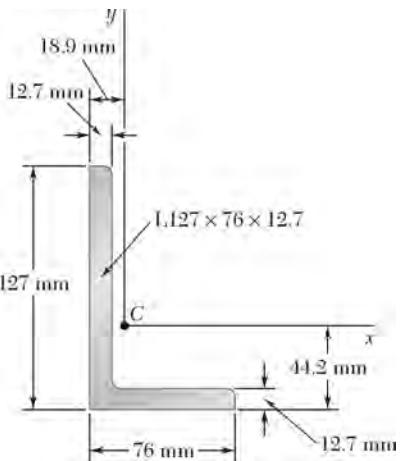
$$\bar{I}_{\max} = 0.524 \times 10^6 \text{ mm}^4$$

and

$$\bar{I}_{\min} = 0.0917 \times 10^6 \text{ mm}^4$$



From the Mohr's circle it is seen that the a axis corresponds to \bar{I}_{\min} and the b axis corresponds to \bar{I}_{\max} .



PROBLEM 9.105

Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Figure 9.13.)

SOLUTION

From Figure 9.13B:

$$\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$$

Problem 9.7B:

$$\bar{I}_{xy} = -1.165061 \times 10^6 \text{ mm}^4$$

(Note that the figure of Problem 9.105 is obtained by replacing x with $-x$ in the figure of Problem 9.78; thus the change in sign of \bar{I}_{xy} .)

The Mohr's circle is defined by the diameter XY , where $X(3.93 \times 10^6, -1.165061 \times 10^6)$ $Y(1.06 \times 10^6, 1.165061 \times 10^6)$.

Now

$$\begin{aligned} I_{\text{ave}} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) \\ &= \frac{1}{2}(3.93 + 1.06) \times 10^6 \\ &= 2.495 \times 10^6 \text{ mm}^4 \end{aligned}$$

and

$$\begin{aligned} R &= \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} \\ &= \left\{ \sqrt{\left[\frac{1}{2}(3.93 - 1.06)\right]^2 + (-1.165061)^2} \right\} \times 10^6 \\ &= 1.84840 \times 10^6 \text{ mm}^4 \end{aligned}$$

PROBLEM 9.105 (Continued)

The Mohr's circle is then drawn as shown.

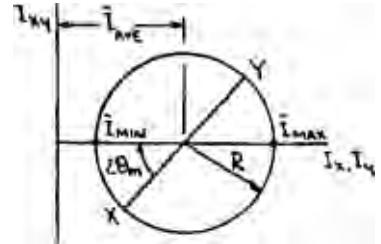
$$\begin{aligned}\tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(-1.165061 \times 10^6)}{(3.93 - 1.06) \times 10^6} \\ &= 0.81189\end{aligned}$$

or

$$2\theta_m = 39.073^\circ$$

and

$$\theta_m = 19.54^\circ$$



The principal axes are obtained by rotating the xy axes through

19.54° counterclockwise \blacktriangleleft

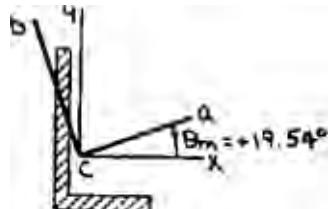
about C .

Now

$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (2.495 \pm 1.84840) \times 10^6$$

$$\text{or} \quad \bar{I}_{\max} = 4.34 \times 10^6 \text{ mm}^4 \quad \blacktriangleleft$$

$$\text{and} \quad \bar{I}_{\min} = 0.647 \times 10^6 \text{ mm}^4 \quad \blacktriangleleft$$



From the Mohr's circle it is seen that the a axis corresponds to \bar{I}_{\max} and the b axis corresponds to \bar{I}_{\min} .

PROBLEM 9.106*

For a given area the moments of inertia with respect to two rectangular centroidal x and y axes are $\bar{I}_x = 12 \times 10^6 \text{ mm}^4$ and $\bar{I}_y = 3 \times 10^6 \text{ mm}^4$, respectively. Knowing that after rotating the x and y axes about the centroid 30° counterclockwise, the moment of inertia relative to the rotated x axis is $14.5 \times 10^6 \text{ mm}^4$, use Mohr's circle to determine (a) the orientation of the principal axes, (b) the principal centroidal moments of inertia.

SOLUTION

We have

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(12 + 3) \times 10^6 = 7.5 \times 10^6 \text{ mm}^4$$

Now observe that $\bar{I}_x > \bar{I}_{\text{ave}}$, $\bar{I}_{x'} > \bar{I}_x$, and $2\theta = +60^\circ$. This is possible only if $\bar{I}_{xy} < 0$. Therefore, assume $\bar{I}_{xy} < 0$ and (for convenience) $\bar{I}_{x'y'} > 0$. Mohr's circle is then drawn as shown.

We have

$$2\theta_m + \alpha = 60^\circ$$

Now using ΔABD :

$$R = \frac{\bar{I}_x - \bar{I}_{\text{ave}}}{\cos 2\theta_m} = \frac{12 \times 10^6 - 7.5 \times 10^6}{\cos 2\theta_m}$$

$$= \frac{4.5 \times 10^6}{\cos 2\theta_m} \text{ (mm}^4\text{)}$$

Using ΔAEF :

$$R = \frac{\bar{I}_{x'} - \bar{I}_{\text{ave}}}{\cos \alpha} = \frac{14.5 \times 10^6 - 7.5 \times 10^6}{\cos \alpha}$$

$$= \frac{7 \times 10^6}{\cos \alpha} \text{ (mm}^4\text{)}$$

Then

$$\frac{4.5 \times 10^6}{\cos 2\theta_m} = \frac{7 \times 10^6}{\cos \alpha} \quad \alpha = 60^\circ - 2\theta_m$$

or

$$9 \cos(60^\circ - 2\theta_m) = 14 \cos 2\theta_m$$

Expanding:

$$9(\cos 60^\circ \cos 2\theta_m + \sin 60^\circ \sin 2\theta_m) = 14 \cos 2\theta_m$$

or

$$\tan 2\theta_m = \frac{14 - 9 \cos 60^\circ}{9 \sin 60^\circ} = 1.21885$$

or

$$2\theta_m = 50.633^\circ \quad \text{and} \quad \theta_m = 25.3^\circ$$

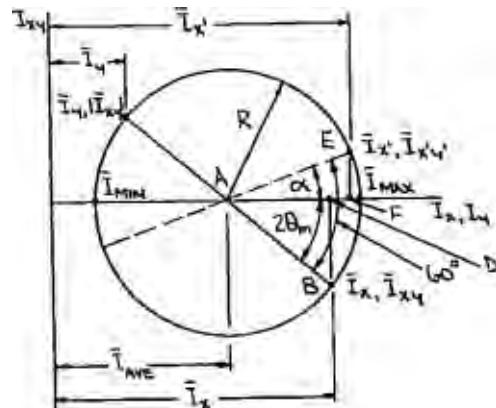
(Note: $2\theta_m < 60^\circ$ implies assumption $\bar{I}_{x'y'} > 0$ is correct.)

Finally,

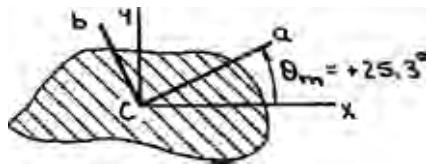
$$R = \frac{4.5 \times 10^6}{\cos 50.633^\circ} = 7.0946 \times 10^6 \text{ mm}^4$$

- (a) From the Mohr's circle it is seen that the principal axes are obtained by rotating the given centroidal x and y axes through θ_m about the centroid C or

25.3° counterclockwise 



PROBLEM 9.106* (Continued)



(b) We have

$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (7.5 \pm 7.0946) \times 10^6$$

or $\bar{I}_{\max} = 14.59 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 0.405 \times 10^6 \text{ mm}^4$ ◀

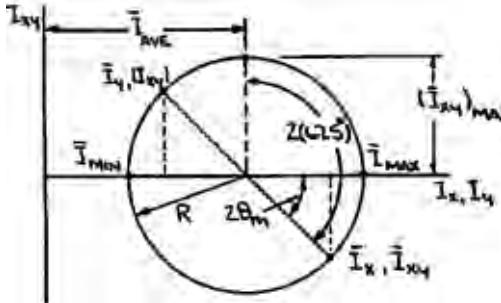
From the Mohr's circle it is seen that the *a* axis corresponds to \bar{I}_{\max} and the *b* axis corresponds to \bar{I}_{\min} .

PROBLEM 9.107

It is known that for a given area $\bar{I}_y = 48 \times 10^6 \text{ mm}^4$ and $\bar{I}_{xy} = -20 \times 10^6 \text{ mm}^4$, where the x and y axes are rectangular centroidal axes. If the axis corresponding to the maximum product of inertia is obtained by rotating the x axis 67.5° counterclockwise about C , use Mohr's circle to determine (a) the moment of inertia \bar{I}_x of the area, (b) the principal centroidal moments of inertia.

SOLUTION

First assume $\bar{I}_x > \bar{I}_y$ and then draw the Mohr's circle as shown. (Note: Assuming $\bar{I}_x < \bar{I}_y$ is not consistent with the requirement that the axis corresponding to $(\bar{I}_{xy})_{\max}$ is obtained after rotating the x axis through 67.5° CCW.)



From the Mohr's circle we have

$$2\theta_m = 2(67.5^\circ) - 90^\circ = 45^\circ$$

(a) From the Mohr's circle we have

$$\bar{I}_x = \bar{I}_y + 2 \frac{|I_{xy}|}{\tan 2\theta_m} = 48 \times 10^6 + 2 \frac{20 \times 10^6}{\tan 45^\circ} \quad \text{or} \quad \bar{I}_x = 88.0 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

(b) We have

$$\begin{aligned} \bar{I}_{\text{ave}} &= \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(88.0 + 48) \times 10^6 \\ &= 68.0 \times 10^6 \text{ mm}^4 \end{aligned}$$

and

$$R = \frac{|I_{xy}|}{\sin 2\theta_m} = \frac{20 \times 10^6}{\sin 45^\circ} = 28.284 \times 10^6 \text{ mm}^4$$

Now

$$\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (68.0 \pm 28.284) \times 10^6$$

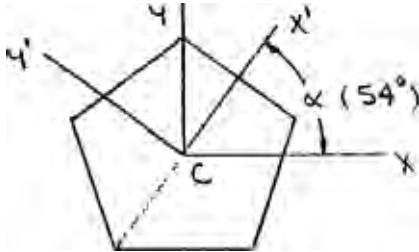
$$\begin{aligned} \text{or} \quad \bar{I}_{\max} &= 96.3 \times 10^6 \text{ mm}^4 \blacktriangleleft \\ \text{and} \quad \bar{I}_{\min} &= 39.7 \times 10^6 \text{ mm}^4 \blacktriangleleft \end{aligned}$$

PROBLEM 9.108

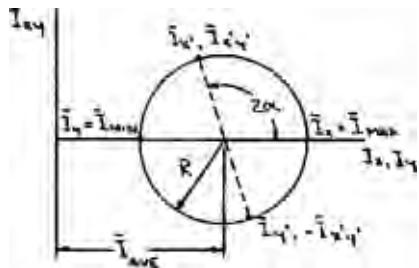
Using Mohr's circle, show that for any regular polygon (such as a pentagon) (a) the moment of inertia with respect to every axis through the centroid is the same, (b) the product of inertia with respect to every pair of rectangular axes through the centroid is zero.

SOLUTION

Consider the regular pentagon shown, with centroidal axes x and y .



Because the y axis is an axis of symmetry, it follows that $\bar{I}_{xy} = 0$. Since $\bar{I}_{xy} = 0$, the x and y axes must be principal axes. Assuming $\bar{I}_x = \bar{I}_{\max}$ and $\bar{I}_y = \bar{I}_{\min}$, the Mohr's circle is then drawn as shown.



Now rotate the coordinate axes through an angle α as shown; the resulting moments of inertia, $\bar{I}_{x'}$ and $\bar{I}_{y'}$, and product of inertia, $\bar{I}_{x'y'}$, are indicated on the Mohr's circle. However, the x' axis is an axis of symmetry, which implies $\bar{I}_{x'y'} = 0$. For this to be possible on the Mohr's circle, the radius R must be equal to zero (thus, the circle degenerates into a point). With $R = 0$, it immediately follows that

$$(a) \quad \bar{I}_x = \bar{I}_y = \bar{I}_{x'} = \bar{I}_{y'} = \bar{I}_{\text{ave}} \quad (\text{for all moments of inertia with respect to an axis through } C) \blacktriangleleft$$

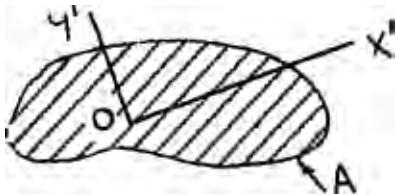
$$(b) \quad \bar{I}_{xy} = \bar{I}_{x'y'} = 0 \quad (\text{for all products of inertia with respect to all pairs of rectangular axes with origin at } C) \blacktriangleleft$$

PROBLEM 9.109

Using Mohr's circle, prove that the expression $I_{x'}I_{y'} - I_{x'y'}^2$ is independent of the orientation of the x' and y' axes, where I_x , I_y , and $I_{x'y'}$ represent the moments and product of inertia, respectively, of a given area with respect to a pair of rectangular axes x' and y' through a given Point O . Also show that the given expression is equal to the square of the length of the tangent drawn from the origin of the coordinate system to Mohr's circle.

SOLUTION

First observe that for a given area A and origin O of a rectangular coordinate system, the values of I_{ave} and R are the same for all orientations of the coordinate axes. Shown below is a Mohr's circle, with the moments of inertia, $I_{x'}$ and $I_{y'}$, and the product of inertia, $I_{x'y'}$, having been computed for an arbitrary orientation of the $x'y'$ axes.



From the Mohr's circle

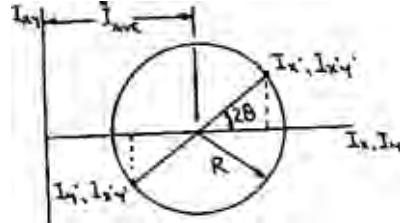
$$I_{x'} = I_{ave} + R \cos 2\theta$$

$$I_{y'} = I_{ave} - R \cos 2\theta$$

$$I_{x'y'} = R \sin 2\theta$$

Then, forming the expression

$$I_{x'}I_{y'} - I_{x'y'}^2$$



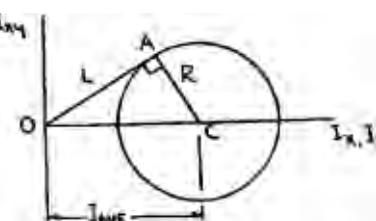
$$\begin{aligned} I_{x'}I_{y'} - I_{x'y'}^2 &= (I_{ave} + R \cos 2\theta)(I_{ave} - R \cos 2\theta) - (R \sin 2\theta)^2 \\ &= (I_{ave}^2 - R^2 \cos^2 2\theta) - (R^2 \sin^2 2\theta) \\ &= I_{ave}^2 - R^2 \quad \text{which is a constant} \end{aligned}$$

$I_{x'}I_{y'} - I_{x'y'}^2$ is independent of the orientation of the coordinate axes Q.E.D. ◀

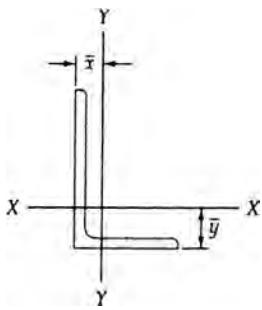
Shown is a Mohr's circle, with line \overline{OA} , of length L , the required tangent.

Noting that $\angle OAC$ is a right angle, it follows that

$$L^2 = I_{ave}^2 - R^2$$



or $L^2 = I_{x'}I_{y'} - I_{x'y'}^2$ Q.E.D. ◀



PROBLEM 9.110

Using the invariance property established in the preceding problem, express the product of inertia I_{xy} of an area A with respect to a pair of rectangular axes through O in terms of the moments of inertia I_x and I_y of A and the principal moments of inertia I_{\min} and I_{\max} of A about O . Use the formula obtained to calculate the product of inertia I_{xy} of the L76 \times 51 \times 6.4 mm angle cross section shown in Figure 9.13A, knowing that its maximum moment of inertia is $0.6 \times 10^6 \text{ mm}^4$.

SOLUTION

Consider the following two sets of moments and products of inertia, which correspond to two different orientations of the coordinate axes whose origin is at Point O .

Case 1:

$$I_{x'} = I_x, \quad I_{y'} = I_y, \quad I_{x'y'} = I_{xy}$$

Case 2:

$$I_{x'} = I_{\max}, \quad I_{y'} = I_{\min}, \quad I_{x'y'} = 0$$

The invariance property then requires

$$I_x I_y - I_{xy}^2 = I_{\max} I_{\min} \quad \text{or} \quad I_{xy} = \pm \sqrt{I_x I_y - I_{\max} I_{\min}} \blacktriangleleft$$

From Figure 9.13A:

$$\bar{I}_x = 0.454 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.162 \times 10^6 \text{ mm}^4$$

Using Eq. (9.21):

$$\bar{I}_x + \bar{I}_y = \bar{I}_{\max} + \bar{I}_{\min}$$

Substituting

$$(0.454 + 0.162) \times 10^6 = 0.6 \times 10^6 + \bar{I}_{\min}$$

or

$$\bar{I}_{\min} = 0.016 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_{xy} = \pm \sqrt{(0.454 \times 10^6)(0.162 \times 10^6) - (0.6 \times 10^6)(0.016 \times 10^6)}$$

$$= \pm 0.25288 \times 10^6 \text{ mm}^4$$

The two roots correspond to the following two orientations of the cross section.

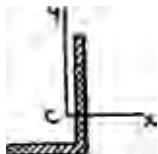
For

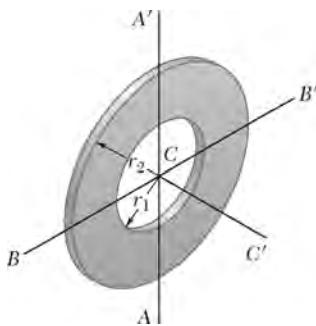
$$\bar{I}_{xy} = -0.253 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



and for

$$\bar{I}_{xy} = 0.253 \times 10^6 \text{ mm}^4 \blacktriangleleft$$





PROBLEM 9.111

Determine the mass moment of inertia of a ring of mass m , cut from a thin uniform plate, with respect to (a) the axis AA' , (b) the centroidal axis CC' that is perpendicular to the plane of the ring.

SOLUTION

$$\text{Mass} = m = \rho V = \rho t A$$

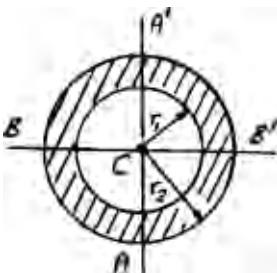
$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

We first determine

$$A = \pi r_2^2 - \pi r_1^2 = \pi(r_2^2 - r_1^2)$$

$$I_{AA', \text{area}} = \frac{\pi}{4} r_2^4 - \frac{\pi}{4} r_1^4 = \frac{\pi}{4}(r_2^4 - r_1^4)$$

$$(a) \quad I_{AA', \text{mass}} = \frac{m}{A} I_{AA', \text{area}} = \frac{m}{\pi(r_2^2 - r_1^2)} \frac{\pi}{4}(r_2^4 - r_1^4) = \frac{1}{4} m(r_2^2 + r_1^2)$$



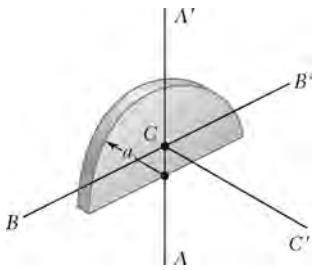
$$I_{AA'} = \frac{1}{4} m(r_1^2 + r_2^2) \blacktriangleleft$$

$$(b) \quad \text{By symmetry:} \quad I_{BB'} = I_{AA'}$$

$$\text{Eq. (9.38):} \quad I_{CC'} = I_{AA'} + I_{BB'} = 2I_{AA'} \quad I_{CC'} = \frac{1}{2} m(r_1^2 + r_2^2) \blacktriangleleft$$

PROBLEM 9.112

A thin semicircular plate has a radius a and a mass m . Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis B' , (b) the centroidal axis CC' that is perpendicular to the plate.



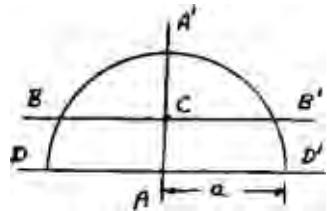
SOLUTION

$$\text{mass} = m = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

Area:

$$A = \frac{1}{2} \pi a^2$$



$$I_{AA', \text{area}} = I_{DD', \text{area}} = \frac{1}{2} \left(\frac{\pi}{4} a^4 \right) = \frac{1}{8} \pi a^4$$

$$I_{AA', \text{mass}} = I_{DD', \text{mass}} = \frac{m}{A} I_{AA', \text{area}} = \frac{m}{\frac{1}{2} \pi a^2} \left(\frac{1}{8} \pi a^4 \right) = \frac{1}{4} ma^2$$

(a)

$$I_{BB'} = I_{DD'} - m(AC)^2 = \frac{1}{4} ma^2 - m \left(\frac{4a}{3\pi} \right)^2$$

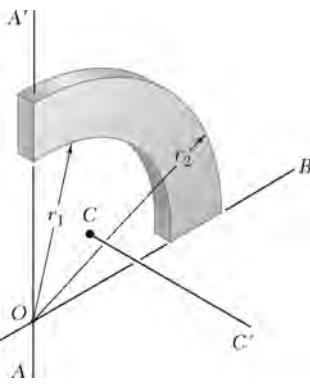
$$= (0.25 - 0.1801)ma^2$$

$$I_{BB'} = 0.0699ma^2 \blacktriangleleft$$

(b) Eq. (9.38):

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{4} ma^2 + 0.0699ma^2$$

$$I_{CC'} = 0.320ma^2 \blacktriangleleft$$



PROBLEM 9.113

The quarter ring shown has a mass m and was cut from a thin, uniform plate. Knowing that $r_1 = \frac{3}{4}r_2$, determine the mass moment of inertia of the quarter ring with respect to (a) the axis AA' , (b) the centroidal axis CC' that is perpendicular to the plane of the quarter ring.

SOLUTION

First note

$$\begin{aligned}\text{mass} &= m = \rho V = \rho t A \\ &= \rho t \frac{\pi}{4} (r_2^2 - r_1^2)\end{aligned}$$

Also

$$\begin{aligned}I_{\text{mass}} &= \rho t I_{\text{area}} \\ &= \frac{m}{\frac{\pi}{4} (r_2^2 - r_1^2)} I_{\text{area}}\end{aligned}$$

(a) Using Figure 9.12,

$$I_{AA', \text{area}} = \frac{\pi}{16} (r_2^4 - r_1^4)$$

Then

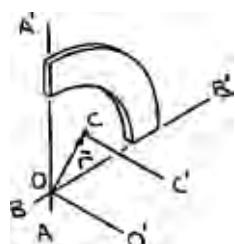
$$\begin{aligned}I_{AA', \text{mass}} &= \frac{m}{\frac{\pi}{4} (r_2^2 - r_1^2)} \times \frac{\pi}{16} (r_2^4 - r_1^4) \\ &= \frac{m}{4} (r_2^2 + r_1^2) \\ &= \frac{m}{4} \left[r_2^2 + \left(\frac{3}{4} r_2 \right)^2 \right] \quad \text{or } I_{AA'} = \frac{25}{64} m r_2^2 \blacktriangleleft\end{aligned}$$

(b) Symmetry implies

$$I_{BB', \text{mass}} = I_{AA', \text{mass}}$$

Then

$$\begin{aligned}I_{DD'} &= I_{AA'} + I_{BB'} \\ &= 2 \left(\frac{25}{64} m r_2^2 \right) \\ &= \frac{25}{32} m r_2^2\end{aligned}$$



PROBLEM 9.113 (Continued)

Now locate centroid C .

$$\bar{X} \Sigma A = \Sigma \bar{x} A$$

or $\bar{X} \left(\frac{\pi}{4} r_2^2 - \frac{\pi}{4} r_1^2 \right) = \frac{4r_2}{3\pi} \left(\frac{\pi}{4} r_2^2 \right) - \frac{4r_1}{3\pi} \left(\frac{\pi}{4} r_1^2 \right)$

or $\bar{X} = \frac{4}{3\pi} \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$

Now

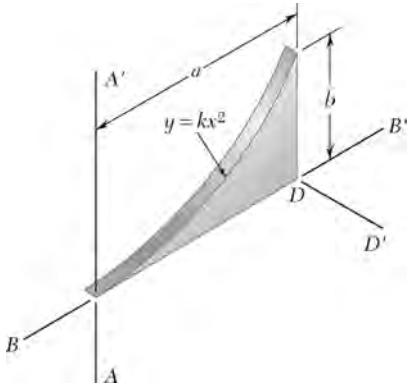
$$\bar{r} = \bar{X}\sqrt{2}$$

$$\begin{aligned} &= \frac{4\sqrt{2}}{3\pi} \frac{r_2^3 - \left(\frac{3}{4}r_2\right)^3}{r_2^2 - \left(\frac{3}{4}r_2\right)^2} \\ &= \frac{37\sqrt{2}}{21\pi} r_2 \end{aligned}$$

Finally,

$$I_{DD'} = I_{CC'} + m\bar{r}^2$$

or $\frac{25}{32}mr_2^2 = I_{CC'} + m\left(\frac{37\sqrt{2}}{21\pi}r_2\right)^2$ or $I_{CC'} = 0.1522mr_2^2$ ◀



PROBLEM 9.114

The parabolic spandrel shown was cut from a thin, uniform plate. Denoting the mass of the spandrel by m , determine its mass moment of inertia with respect to (a) the axis BB' , (b) the axis DD' that is perpendicular to the spandrel. (Hint: See Sample Problem 9.3.)

SOLUTION

First note

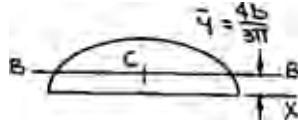
$$\begin{aligned} \text{mass} &= m = \rho V = \rho t A \\ &= \rho t \left(\frac{\pi}{2} ab \right) \end{aligned}$$

Also

$$\begin{aligned} I_{\text{mass}} &= \rho t I_{\text{area}} \\ &= \frac{2m}{\pi ab} I_{\text{area}} \end{aligned}$$

(a) We have

$$\begin{aligned} I_{x, \text{area}} &= I_{BB', \text{area}} + A \bar{y}^2 \\ I_{BB', \text{area}} &= \frac{\pi}{8} ab^3 - \left(\frac{\pi}{2} ab \right) \left(\frac{4b}{3\pi} \right)^2 \end{aligned}$$



(a) From Sample Problem 9.3:

$$I_{BB', \text{area}} = \frac{1}{21} ab^3$$

Then

$$I_{BB', \text{mass}} = \frac{3m}{ab} \times \frac{1}{21} ab^3$$

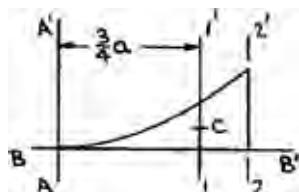
$$\text{or } I_{BB'} = \frac{1}{7} mb^2 \blacktriangleleft$$

(b) From Sample Problem 9.3:

$$I_{AA', \text{area}} = \frac{1}{5} a^3 b$$

Now

$$I_{AA', \text{area}} = I_{11', \text{area}} + A \left(\frac{3}{4} a \right)^2$$



PROBLEM 9.114 (Continued)

and

$$I_{22', \text{area}} = I_{11', \text{area}} + A \left(\frac{1}{4} a \right)^2$$

Then

$$\begin{aligned} I_{22', \text{area}} &= I_{AA', \text{area}} + A \left[\left(\frac{1}{4} a \right)^2 - \left(\frac{3}{4} a \right)^2 \right] \\ &= \frac{1}{5} a^3 b + \frac{1}{3} ab \left(\frac{1}{16} a^2 - \frac{9}{16} a^2 \right) \\ &= \frac{1}{30} a^3 b \end{aligned}$$

and

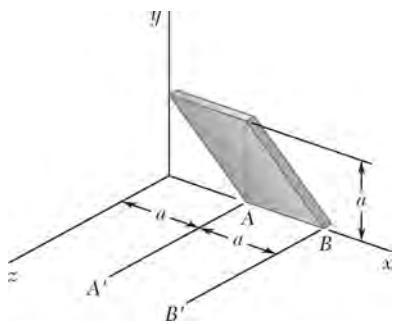
$$\begin{aligned} I_{22', \text{mass}} &= \frac{3m}{ab} \times \frac{1}{30} a^3 b \\ &= \frac{1}{10} m a^2 \end{aligned}$$

Finally,

$$I_{DD', \text{mass}} = I_{BB', \text{mass}} + I_{22', \text{mass}}$$

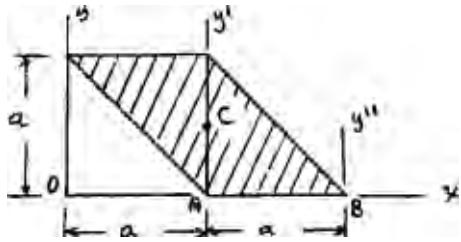
$$= \frac{1}{7} mb^2 + \frac{1}{10} ma^2 \quad \text{or} \quad I_{DD'} = \frac{1}{70} m(7a^2 + 10b^2) \blacktriangleleft$$

PROBLEM 9.115



A thin plate of mass m was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (a) the x axis, (b) the axis B' , which is perpendicular to the plate.

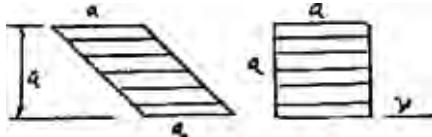
SOLUTION



$$\text{mass} = m = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

- (a) Consider parallelogram as made of horizontal strips and slide strips to form a square since distance from each strip to x axis is unchanged.



$$I_{x, \text{area}} = \frac{1}{3} a^4$$

$$I_{x, \text{mass}} = \frac{m}{A} I_{x, \text{area}} = \frac{m}{a^2} \left(\frac{1}{3} a^4 \right) \quad I_x = \frac{1}{3} ma^2 \blacktriangleleft$$

- (b) For centroidal axis y' :

$$\bar{I}_{y', \text{area}} = 2 \left[\frac{1}{12} a^4 \right] = \frac{1}{6} a^4$$

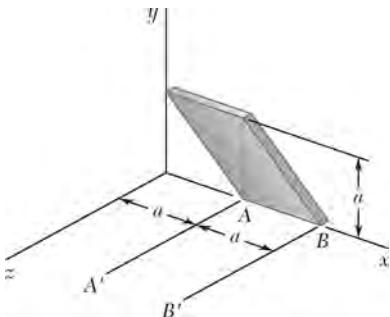
$$\bar{I}_{y', \text{mass}} = \frac{m}{A} I_{y', \text{area}} = \frac{m}{a^2} \left(\frac{1}{6} a^4 \right) = \frac{1}{6} ma^2$$

$$I_{y''} = \bar{I}_{y'} + ma^2 = \frac{1}{6} ma^2 + ma^2 = \frac{7}{6} ma^2$$

For axis $BB' \perp$ to plate, Eq. (9.38):

$$I_{BB'} = I_x + I_{y''} = \frac{1}{3} ma^2 + \frac{7}{6} ma^2$$

$$I_{BB'} = \frac{3}{2} ma^2 \blacktriangleleft$$



PROBLEM 9.116

A thin plate of mass m was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (a) the y axis, (b) the axis AA' , which is perpendicular to the plate.

SOLUTION

See sketch of the solution of Problem 9.115.

(a) From Part b of solution of Problem 9.115:

$$\bar{I}_{y'} = \frac{1}{6}ma^2$$

$$I_y = \bar{I}_{y'} + ma^2 = \frac{1}{6}ma^2 + ma^2$$

$$I_y = \frac{7}{6}ma^2 \blacktriangleleft$$

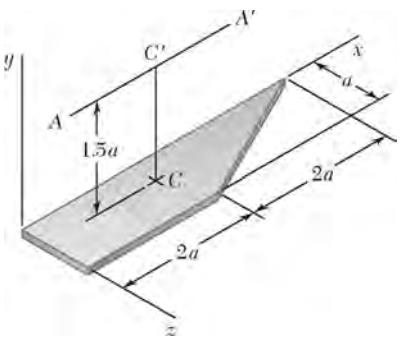
(b) From solution of Problem 9.115:

$$\bar{I}_{y'} = \frac{1}{6}ma^2 \quad \text{and} \quad I_x = \frac{1}{3}ma^2$$

$$\text{Eq. (9.38):} \quad I_{AA'} = \bar{I}_{y'} + I_x$$

$$= \frac{1}{6}ma^2 + \frac{1}{3}ma^2$$

$$I_{AA'} = \frac{1}{2}ma^2 \blacktriangleleft$$



PROBLEM 9.117

A thin plate of mass m has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (a) the x axis, (b) the y axis.

SOLUTION

First note

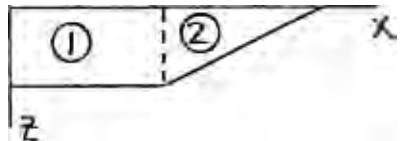
$$\begin{aligned} \text{mass} &= m = \rho V = \rho t A \\ &= \rho t \left[(2a)(a) + \frac{1}{2}(2a)(a) \right] = 3\rho t a^2 \end{aligned}$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{3a^2} I_{\text{area}}$$

(a) Now

$$\begin{aligned} I_{x, \text{area}} &= (I_x)_{1, \text{area}} + (I_x)_{2, \text{area}} \\ &= \frac{1}{3}(2a)(a)^3 + \frac{1}{12}(2a)(a)^3 \\ &= \frac{5}{6}a^4 \end{aligned}$$



Then

$$I_{x, \text{mass}} = \frac{m}{3a^2} \times \frac{5}{6}a^4 \quad \text{or} \quad I_{x, \text{mass}} = \frac{5}{18}ma^2 \blacktriangleleft$$

(b) We have

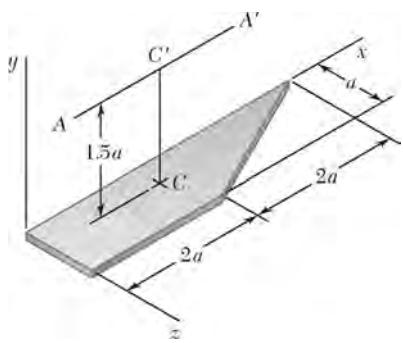
$$\begin{aligned} I_{z, \text{area}} &= (I_z)_{1, \text{area}} + (I_z)_{2, \text{area}} \\ &= \left[\frac{1}{3}(a)(2a)^3 \right] + \left[\frac{1}{36}(a)(2a)^3 + \frac{1}{2}(2a)(a) \left(2a + \frac{1}{3} \times 2a \right)^2 \right] = 10a^4 \end{aligned}$$

Then

$$I_{x, \text{mass}} = \frac{m}{3a^2} \times 10a^4 = \frac{10}{3}ma^2$$

Finally,

$$\begin{aligned} I_{y, \text{mass}} &= I_{x, \text{mass}} + I_{z, \text{mass}} \\ &= \frac{5}{18}ma^2 + \frac{10}{3}ma^2 \\ &= \frac{65}{18}ma^2 \quad \text{or} \quad I_{y, \text{mass}} = 3.61ma^2 \blacktriangleleft \end{aligned}$$



PROBLEM 9.118

A thin plate of mass m has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis CC' that is perpendicular to the plate, (b) the axis AA' that is parallel to the x axis and is located at a distance $1.5a$ from the plate.

SOLUTION

First locate the centroid C .

$$\bar{X} \Sigma A = \Sigma \bar{x} A: \quad \bar{X}(2a^2 + a^2) = a(2a^2) + \left(2a + \frac{1}{3} \times 2a\right)(a^2)$$

or

$$\bar{X} = \frac{14}{9}a$$

$$\bar{Z} \Sigma A = \Sigma \bar{z} A: \quad \bar{Z}(2a^2 + a^2) = \left(\frac{1}{2}a\right)(2a^2) + \left(\frac{1}{3}a\right)(a^2)$$

or

$$\bar{Z} = \frac{4}{9}a$$

(a) We have

$$I_{y, \text{mass}} = \bar{I}_{CC', \text{mass}} + m(\bar{X}^2 + \bar{Z}^2)$$

From the solution to Problem 9.117:

$$I_{y, \text{mass}} = \frac{65}{18}ma^2$$

Then

$$\bar{I}_{cc', \text{mass}} = \frac{65}{18}ma^2 - m\left[\left(\frac{14}{9}a\right)^2 + \left(\frac{4}{9}a\right)^2\right]$$

$$\text{or} \quad \bar{I}_{cc'} = 0.994ma^2 \blacktriangleleft$$

(b) We have

$$I_{x, \text{mass}} = \bar{I}_{BB', \text{mass}} + m(\bar{Z})^2$$

and

$$I_{AA', \text{mass}} = \bar{I}_{BB', \text{mass}} + m(1.5a)^2$$

Then

$$I_{AA', \text{mass}} = I_{x, \text{mass}} + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$$

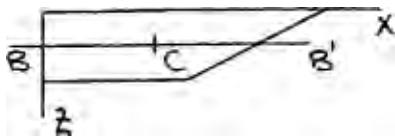
PROBLEM 9.118 (Continued)

From the solution to Problem 9.117:

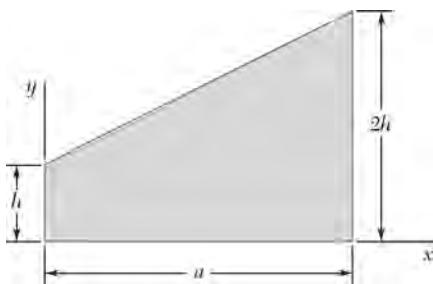
$$I_{x, \text{mass}} = \frac{5}{18}ma^2$$

Then

$$I_{AA', \text{mass}} = \frac{5}{18}ma^2 + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$$



$$\text{or } I_{AA'} = 2.33ma^2 \blacktriangleleft$$



PROBLEM 9.119

The area shown is revolved about the x axis to form a homogeneous solid of revolution of mass m . Using direct integration, express the mass moment of inertia of the solid with respect to the x axis in terms of m and h .

SOLUTION

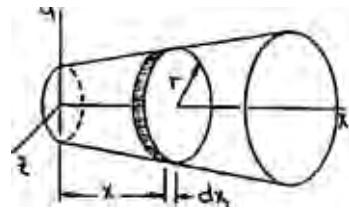
We have

$$y = \frac{2h - h}{a}x + h$$

so that

$$r = \frac{h}{a}(x + a)$$

For the element shown:



$$\begin{aligned} dm &= \rho\pi r^2 dx \quad dI_x = \frac{1}{2}r^2 dm \\ &= \rho\pi \left[\frac{h}{a}(x + a) \right]^2 dx \end{aligned}$$

Then

$$\begin{aligned} m &= \int dm = \int_0^a \rho\pi \frac{h^2}{a^2} (x + a)^2 dx = \frac{1}{3} \rho\pi \frac{h^2}{a^2} [(x + a)^3]_0^a \\ &= \frac{1}{3} \rho\pi \frac{h^2}{a^2} (8a^3 - a^3) = \frac{7}{3} \rho\pi ah^2 \end{aligned}$$

Now

$$\begin{aligned} I_x &= \int dI_x = \int \frac{1}{2}r^2 (\rho\pi r^2 dx) = \frac{1}{2}\rho\pi \int_0^a \left[\frac{h}{a}(x + a) \right]^4 dx \\ &= \frac{1}{2}\rho\pi \times \frac{1}{5} \frac{h^4}{a^4} [(x + a)^5]_0^a = \frac{1}{10}\rho\pi \frac{h^4}{a^4} (32a^5 - a^5) \\ &= \frac{31}{10}\rho\pi ah^4 \end{aligned}$$

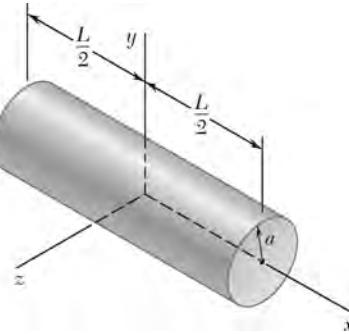
From above:

$$\rho\pi ah^2 = \frac{3}{7}m$$

Then

$$I_x = \frac{31}{10} \left(\frac{3}{7}m \right) h^2 = \frac{93}{70}mh^2$$

$$\text{or } I_x = 1.329 mh^2 \blacktriangleleft$$

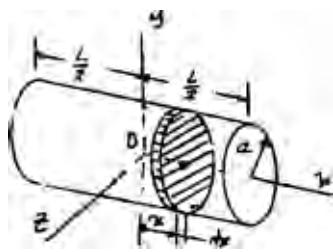


PROBLEM 9.120

Determine by direct integration the mass moment of inertia with respect to the y axis of the right circular cylinder shown, assuming that it has a uniform density and a mass m .

SOLUTION

For element shown:



$$dm = \rho dV = \rho \pi a^2 dx$$

$$\begin{aligned} dI_y &= d\bar{I}_y + x^2 dm \\ &= \frac{1}{4} a^2 dm + x^2 dm \\ &= \left(\frac{1}{4} a^2 + x^2 \right) \rho \pi a^2 dx \end{aligned}$$

$$\begin{aligned} I_y &= \int dI_y = \int_{-L/2}^{+L/2} \rho \pi a^2 \left(\frac{1}{4} a^2 + x^2 \right) dx \\ &= \rho \pi a^2 \left[\frac{1}{4} a^2 x + \frac{x^3}{3} \right]_{-L/2}^{+L/2} = 2 \rho \pi a^2 \left[\frac{1}{4} a^2 \frac{L}{2} + \frac{1}{3} \left(\frac{L}{2} \right)^3 \right] \end{aligned}$$

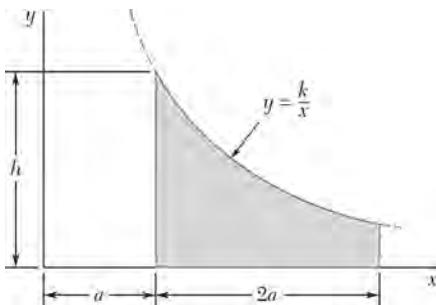
$$I_y = \frac{1}{12} \rho \pi a^2 L (3a^2 + L^2) \quad \blacktriangleleft$$

But for whole cylinder:

$$m = \rho V = \rho \pi a^2 L$$

Thus,

$$I_y = \frac{1}{12} m (3a^2 + L^2) \quad \blacktriangleleft$$



PROBLEM 9.121

The area shown is revolved about the x axis to form a homogeneous solid of revolution of mass m . Determine by direct integration the mass moment of inertia of the solid with respect to (a) the x axis, (b) the y axis. Express your answers in terms of m and the dimensions of the solid.

SOLUTION

We have at

$$(a, h): \quad h = \frac{k}{a}$$

or

$$k = ah$$

For the element shown:

$$r = 4$$

$$\begin{aligned} dm &= \rho \pi r^2 dx \\ &= \rho \pi \left(\frac{ah}{x} \right)^2 dx \end{aligned}$$

Then

$$\begin{aligned} m &= \int dm = \int_a^{3a} \rho \pi \left(\frac{ah}{x} \right)^2 dx \\ &= \rho \pi a^2 h^2 \left[-\frac{1}{x} \right]_a^{3a} \\ &= \rho \pi a^2 h^2 \left[-\frac{1}{3a} - \left(-\frac{1}{a} \right) \right] = \frac{2}{3} \rho \pi a h^2 \end{aligned}$$

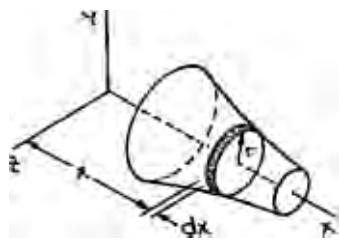
(a) For the element:

$$dI_x = \frac{1}{2} r^2 dm = \frac{1}{2} \rho \pi r^4 dx$$

Then

$$\begin{aligned} I_x &= \int dI_x = \int_a^{3a} \frac{1}{2} \rho \pi \left(\frac{ah}{x} \right)^4 dx = \frac{1}{2} \rho \pi a^4 h^4 \left[-\frac{1}{3} \frac{1}{x^3} \right]_a^{3a} \\ &= -\frac{1}{6} \rho \pi a^4 h^4 \left[\left(\frac{1}{3a} \right)^3 - \left(\frac{1}{a} \right)^3 \right] = \frac{1}{6} \times \frac{26}{27} \rho \pi a h^4 \\ &= \frac{1}{6} \times \frac{2}{3} \rho \pi a h^2 \times \frac{13}{9} h^2 = \frac{13}{54} m h^2 \end{aligned}$$

$$\text{or } I_x = 0.241 m h^2 \blacktriangleleft$$



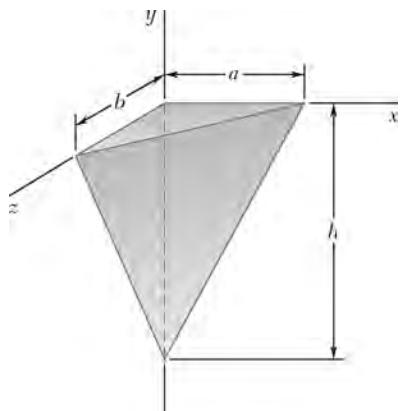
PROBLEM 9.121 (Continued)

(b) For the element:

$$\begin{aligned} dI_y &= dI_y + x^2 dm \\ &= \frac{1}{4} r^2 dm + x^2 dm \end{aligned}$$

Then

$$\begin{aligned} I_y &= \int dI_y = \int_a^{3a} \left[\frac{1}{4} \left(\frac{ah}{x} \right)^2 + x^2 \right] \rho \pi \left(\frac{ah}{x} \right)^2 dx \\ &= \rho \pi a^2 h^2 \int_a^{3a} \left(\frac{1}{4} \frac{a^2 h^2}{x^4} + 1 \right) dx = \rho \pi a^2 h^2 \left[-\frac{1}{12} \frac{a^2 h^2}{x^3} + x \right]_a^{3a} \\ &= \left(\frac{3}{2} m \right) a \left\{ \left[-\frac{1}{12} \frac{a^2 h^2}{(3a)^3} + 3a \right] - \left[-\frac{1}{12} \frac{a^2 h^2}{(a)^3} + a \right] \right\} \\ &= \frac{3}{2} ma \left(\frac{1}{12} \times \frac{26}{27} \frac{h^2}{a} + 2a \right) = m \left(\frac{13}{108} h^2 + 3a^2 \right) \\ \text{or } I_y &= m(3a^2 + 0.1204h^2) \blacktriangleleft \end{aligned}$$



PROBLEM 9.122

Determine by direct integration the mass moment of inertia with respect to the x axis of the tetrahedron shown, assuming that it has a uniform density and a mass m .

SOLUTION

We have

$$x = \frac{a}{h}y + a = a\left(1 + \frac{y}{h}\right)$$

and

$$z = \frac{b}{h}y + b = b\left(1 + \frac{y}{h}\right)$$

For the element shown:

$$dm = \rho \left(\frac{1}{2}xz dy \right) = \frac{1}{2} \rho ab \left(1 + \frac{y}{h}\right)^2 dy$$

Then

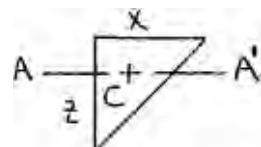
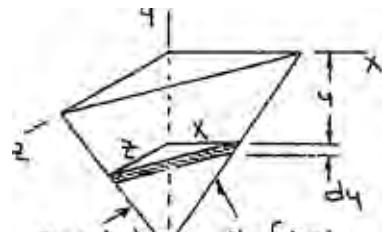
$$\begin{aligned} m &= \int dm = \int_{-h}^0 \frac{1}{2} \rho ab \left(1 + \frac{y}{h}\right)^2 dy \\ &= \frac{1}{2} \rho ab \times \frac{h}{3} \left[\left(1 + \frac{y}{h}\right)^3 \right]_{-h}^0 \\ &= \frac{1}{6} \rho abh \left[(1)^3 - (-1)^3 \right] \\ &= \frac{1}{6} \rho abh \end{aligned}$$

Now, for the element:

$$I_{AA', \text{area}} = \frac{1}{36} xz^3 = \frac{1}{36} ab^3 \left(1 + \frac{y}{h}\right)^4$$

Then

$$dI_{AA', \text{mass}} = \rho t I_{AA', \text{area}} = \rho(dy) \left[\frac{1}{3} ab^3 \left(1 + \frac{y}{h}\right)^4 \right]$$



PROBLEM 9.122 (Continued)

Now

$$\begin{aligned}
 dI_x &= dI_{AA', \text{mass}} + \left[y^2 + \left(\frac{1}{3} z \right)^2 \right] dm \\
 &= \frac{1}{36} \rho ab^3 \left(1 + \frac{y}{h} \right)^4 dy \\
 &\quad + \left\{ y^2 + \left[\frac{1}{3} b \left(1 + \frac{y}{h} \right) \right]^2 \right\} \left[\frac{1}{2} \rho ab \left(1 + \frac{y}{h} \right)^2 dy \right] \\
 &= \frac{1}{12} \rho ab^3 \left(1 + \frac{y}{h} \right)^4 dy + \frac{1}{2} \rho ab \left(y^2 + 2 \frac{y^3}{h} + \frac{y^4}{h^2} \right) dy
 \end{aligned}$$

Now

$$m = \frac{1}{6} \rho abh$$

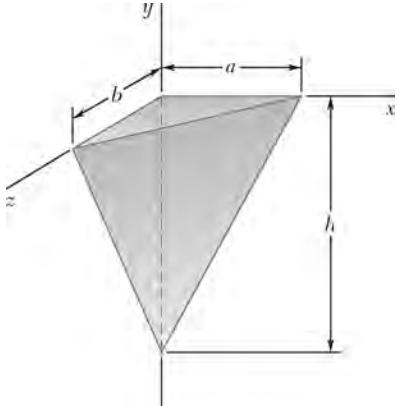
Then

$$dI_x = \left[\frac{1}{2} m \frac{b^2}{h} \left(1 + \frac{y}{h} \right)^4 + \frac{3m}{h} \left(y^2 + 2 \frac{y^3}{h} + \frac{y^4}{h^2} \right) \right] dy$$

and

$$\begin{aligned}
 I_x &= \int dI_x = \int_{-h}^0 \frac{m}{2h} \left[b^2 \left(1 + \frac{y}{h} \right)^4 + 6 \left(y^2 + 2 \frac{y^3}{h} + \frac{y^4}{h^2} \right) \right] dy \\
 &= \frac{m}{2h} \left[b^2 \times \frac{h}{5} \left(1 + \frac{y}{h} \right)^5 + 6 \left(\frac{1}{3} y^3 + \frac{1}{2} \frac{y^4}{h} + \frac{y^5}{5h^2} \right) \right]_{-h}^0 \\
 &= \frac{m}{2h} \left\{ \frac{1}{5} b^2 h (1)^5 - 6 \left[\frac{1}{3} (-h)^3 + \frac{1}{2h} (-h)^4 + \frac{1}{5h^2} (-h)^5 \right] \right\}
 \end{aligned}$$

$$\text{or } I_x = \frac{1}{10} m(b^2 + h^2) \blacktriangleleft$$



PROBLEM 9.123

Determine by direct integration the mass moment of inertia with respect to the y axis of the tetrahedron shown, assuming that it has a uniform density and a mass m .

SOLUTION

We have

$$x = \frac{a}{h}y + a = a\left(1 + \frac{y}{h}\right)$$

and

$$z = \frac{b}{h}y + b = b\left(1 + \frac{y}{h}\right)$$

For the element shown:

$$dm = \rho \left(\frac{1}{2}xz dy \right) = \frac{1}{2}\rho ab \left(1 + \frac{y}{h} \right)^2 dy$$

Then

$$\begin{aligned} m &= \int dm = \int_{-h}^0 \frac{1}{2}\rho ab \left(1 + \frac{y}{h} \right)^2 dy \\ &= \frac{1}{2}\rho ab \times \frac{h}{3} \left[\left(1 + \frac{y}{h} \right)^3 \right]_{-h}^0 \\ &= \frac{1}{6}\rho abh \left[(1)^3 - (-1)^3 \right] \\ &= \frac{1}{6}\rho abh \end{aligned}$$

Also

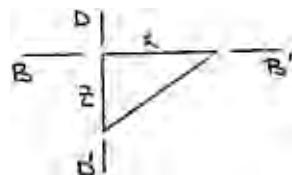
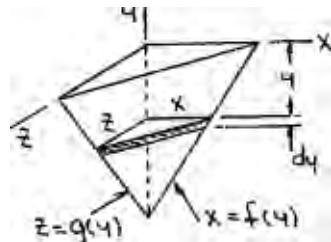
$$I_{BB', \text{area}} = \frac{1}{12}x_z^3 \quad I_{DD', \text{area}} = \frac{1}{12}z_x^3$$

Then, using

$$I_{\text{mass}} = \rho t I_{\text{area}}$$

we have

$$dI_{BB', \text{mass}} = \rho(dy) \left(\frac{1}{12}x_z^3 \right) \quad dI_{DD', \text{mass}} = \rho(dy) \left(\frac{1}{12}z_x^3 \right)$$



PROBLEM 9.123 (Continued)

Now

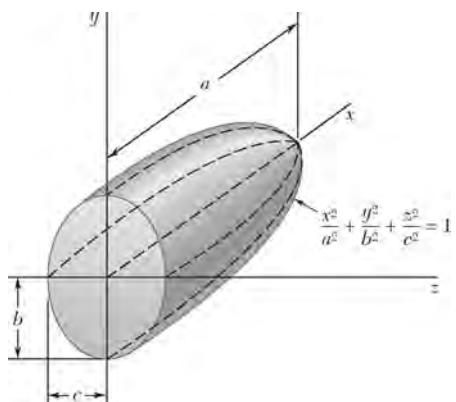
$$\begin{aligned}
 dI_y &= dI_{BB', \text{ mass}} + dI_{DD', \text{ mass}} \\
 &= \frac{1}{12} \rho xz(x^2 + z^2) dy \\
 &= \frac{1}{12} \rho ab \left(1 + \frac{y}{h}\right)^2 \left[(a^2 + b^2) \left(1 + \frac{y}{h}\right)^2 \right] dy
 \end{aligned}$$

We have

$$m = \frac{1}{6} \rho abh \Rightarrow dI_y = \frac{m}{2h} (a^2 + b^2) \left(1 + \frac{y}{h}\right)^4 dy$$

Then

$$\begin{aligned}
 I_y &= \int dI_y = \int_{-h}^0 \frac{m}{2h} (a^2 + b^2) \left(1 + \frac{y}{h}\right)^4 dy \\
 &= \frac{m}{2h} (a^2 + b^2) \times \frac{h}{5} \left[\left(1 + \frac{y}{h}\right)^5 \right]_{-h}^0 \\
 &= \frac{m}{10} (a^2 + b^2) \left[(1)^5 - (-1)^5 \right] \\
 \text{or } I_y &= \frac{1}{10} m(a^2 + b^2) \blacktriangleleft
 \end{aligned}$$



PROBLEM 9.124*

Determine by direct integration the mass moment of inertia with respect to the z axis of the semiellipsoid shown, assuming that it has a uniform density and a mass m .

SOLUTION

First note that when

$$z = 0: \quad y = b \left(1 - \frac{x^2}{a^2} \right)^{1/2}$$

$$y = 0: \quad z = c \left(1 - \frac{x^2}{a^2} \right)^{1/2}$$

For the element shown: $dm = \rho(\pi yzdx) = \pi\rho bc \left(1 - \frac{x^2}{a^2} \right) dx$

Then $m = \int dm = \int_0^a \pi\rho bc \left(1 - \frac{x^2}{a^2} \right) dx$

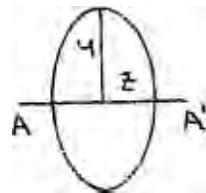
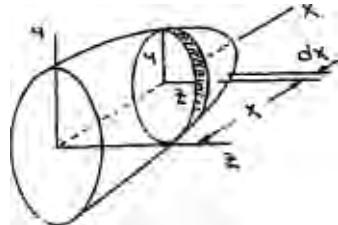
$$= \pi\rho bc \left[x - \frac{1}{3a^2} x^3 \right]_0^a = \frac{2}{3} \pi\rho abc$$

For the element: $I_{AA', \text{area}} = \frac{\pi}{4} zy^3$

Then $dI_{AA', \text{mass}} = \rho t I_{AA', \text{area}} = \rho(dx) \left(\frac{\pi}{4} zy^3 \right)$

Now $dI_z = dI_{AA', \text{mass}} + x^2 dm$

$$\begin{aligned} &= \frac{\pi}{4} \rho b^3 c \left(1 - \frac{x^2}{a^2} \right)^2 dx + x^2 \left[\pi\rho bc \left(1 - \frac{x^2}{a^2} \right) dx \right] \\ &= \frac{3m}{2a} \left[\frac{b^2}{4} \left(1 - 2\frac{x^2}{a^2} + \frac{x^4}{a^4} \right) + \left(x^2 - \frac{x^4}{a^2} \right) \right] dx \end{aligned}$$

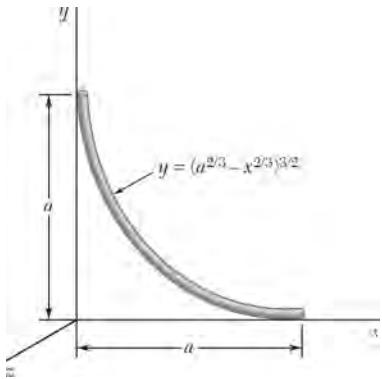


PROBLEM 9.124* (Continued)

Finally,

$$\begin{aligned} I_z &= \int dI_z \\ &= \frac{3m}{2a} \int_0^a \left[\frac{b^2}{4} \left(1 - 2 \frac{x^2}{a^2} + \frac{x^4}{a^4} \right) + \left(x^2 - \frac{x^4}{a^2} \right) \right] dx \\ &= \frac{3m}{2a} \left[\frac{b^2}{4} \left(x - \frac{2}{3} \frac{x^3}{a^2} + \frac{1}{5} \frac{x^5}{a^4} \right) + \left(\frac{1}{3} x^3 - \frac{1}{5} \frac{x^5}{a^2} \right) \right]_0^a \\ &= \frac{3}{2} m \left[\frac{b^2}{4} \left(1 - \frac{2}{3} + \frac{1}{5} \right) + a^2 \left(\frac{1}{3} - \frac{1}{5} \right) \right] \\ \text{or } I_z &= \frac{1}{5} m (a^2 + b^2) \blacktriangleleft \end{aligned}$$

PROBLEM 9.125*



A thin steel wire is bent into the shape shown. Denoting the mass per unit length of the wire by m' , determine by direct integration the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION

First note

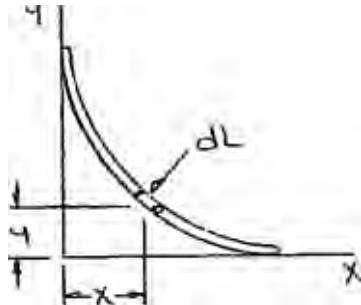
$$\frac{dy}{dx} = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$$

Then

$$1 + \left(\frac{dy}{dx} \right)^2 = 1 + x^{-2/3}(a^{2/3} - x^{2/3}) \\ = \left(\frac{a}{x} \right)^{2/3}$$

For the element shown:

$$dm = m'dL = m' \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ = m' \left(\frac{a}{x} \right)^{1/3} dx$$



Then

$$m = \int dm = \int_0^a m' \frac{a^{1/3}}{x^{1/3}} dx = \frac{3}{2} m' a^{1/3} \left[x^{2/3} \right]_0^a = \frac{3}{2} m' a$$

Now

$$I_x = \int y^2 dm = \int_0^a (a^{2/3} - x^{2/3})^3 \left(m' \frac{a^{1/3}}{x^{1/3}} dx \right) \\ = m' a^{1/3} \int_0^a \left(\frac{a^2}{x^{1/3}} - 3a^{4/3} x^{1/3} + 3a^{2/3} x - x^{5/3} \right) dy \\ = m' a^{1/3} \left[\frac{3}{2} a^2 x^{2/3} - \frac{9}{4} a^{4/3} x^{4/3} + \frac{3}{2} a^{2/3} x^2 - \frac{3}{8} x^{8/3} \right]_0^a \\ = m' a^3 \left(\frac{3}{2} - \frac{9}{4} + \frac{3}{2} - \frac{3}{8} \right) = \frac{3}{8} m' a^3$$

$$\text{or } I_x = \frac{1}{4} m a^2 \blacktriangleleft$$

Symmetry implies

$$I_y = \frac{1}{4} m a^2 \blacktriangleleft$$

PROBLEM 9.125* (Continued)

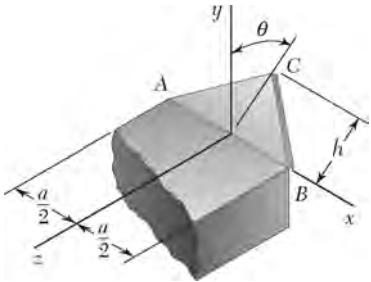
Alternative solution:

$$\begin{aligned}I_y &= \int x^2 dm = \int_0^a x^2 \left(m' \frac{a^{1/3}}{x^{1/3}} dx \right) = m' a^{1/3} \int_0^a x^{5/3} dx \\&= m' a^{1/3} \times \frac{3}{8} \left[x^{8/3} \right]_0^a = \frac{3}{8} m' a^3 \\&= \frac{1}{4} ma^2\end{aligned}$$

Also

$$I_z = \int (x^2 + y^2) dm = I_y + I_x$$

$$\text{or } I_z = \frac{1}{2} ma^2 \blacktriangleleft$$



PROBLEM 9.126

A thin triangular plate of mass m is welded along its base AB to a block as shown. Knowing that the plate forms an angle θ with the y axis, determine by direct integration the mass moment of inertia of the plate with respect to (a) the x axis, (b) the y axis, (c) the z axis.

SOLUTION

For line BC :

$$\begin{aligned}\zeta &= -\frac{h}{a}x + h \\ &\quad \frac{2}{2} \\ &= \frac{h}{a}(a - 2x)\end{aligned}$$

Also

$$\begin{aligned}m &= \rho V = \rho t \left(\frac{1}{2}ah \right) \\ &= \frac{1}{2}\rho t ah\end{aligned}$$

(a) We have

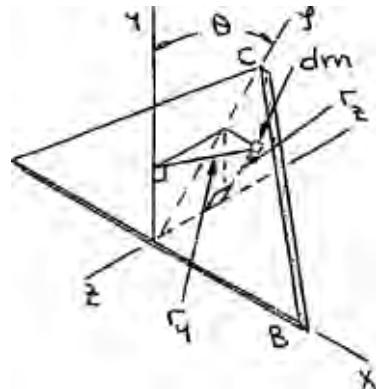
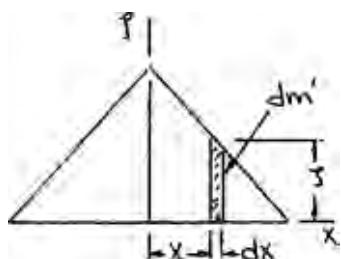
$$\begin{aligned}dI_x &= \frac{1}{12}\zeta^2 dm' + \left(\frac{\zeta}{2}\right)^2 dm' \\ &= \frac{1}{3}\zeta^2 dm'\end{aligned}$$

where

$$dm' = \rho t \zeta dx$$

Then

$$\begin{aligned}I_x &= \int dI_x = 2 \int_0^{a/2} \frac{1}{3}\zeta^2 (\rho t \zeta dx) \\ &= \frac{2}{3}\rho t \int_0^{a/2} \left[\frac{h}{a}(a - 2x) \right]^3 dx \\ &= \frac{2}{3}\rho t \frac{h^3}{a^3} \times \frac{1}{4} \left(-\frac{1}{2} \right) \left[(a - 2x)^4 \right]_0^{a/2} \\ &= -\frac{1}{12}\rho t \frac{h^3}{a^3} [(a - a)^4 - (a)^4] \\ &= \frac{1}{12}\rho t ah^3\end{aligned}$$



$$\text{or } I_x = \frac{1}{6}mh^2 \blacktriangleleft$$

PROBLEM 9.126 (Continued)

Now

$$I_\zeta = \int x^2 dm$$

and

$$I_\zeta = \int x^2 dm' = 2 \int_0^{a/2} x^2 (\rho t \zeta dx)$$

$$\begin{aligned} &= 2\rho t \int_0^{a/2} x^2 \left[\frac{h}{a}(a - 2x) \right] dx \\ &= 2\rho t \frac{h}{a} \left[\frac{a}{3}x^3 - \frac{1}{4}x^4 \right]_0^{a/2} \\ &= 2\rho t \frac{h}{a} \left[\frac{a}{3}\left(\frac{a}{2}\right)^3 - \frac{1}{4}\left(\frac{a}{2}\right)^4 \right] \\ &= \frac{1}{48} \rho t a^3 h = \frac{1}{24} m a^2 \end{aligned}$$

(b) We have

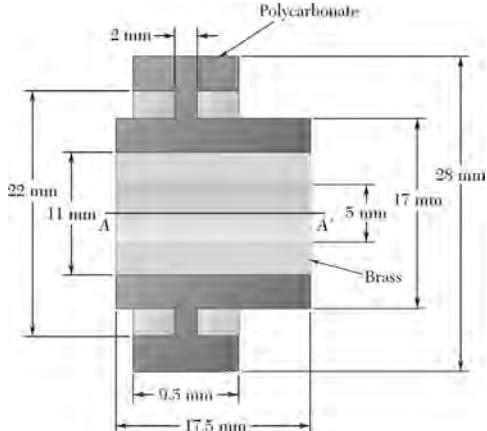
$$\begin{aligned} I_y &= \int r_y^2 dm = \int [x^2 + (\zeta \sin \theta)^2] dm \\ &= \int x^2 dm + \sin^2 \theta \int \zeta^2 dm \end{aligned}$$

Now

$$\begin{aligned} I_x &= \int \zeta^2 dm \Rightarrow I_y = I_\zeta + I_x \sin^2 \theta \\ &= \frac{1}{24} m a^2 + \frac{1}{6} m h^2 \sin^2 \theta \\ \text{or } I_y &= \frac{m}{24} (a^2 + 4h^2 \sin^2 \theta) \blacktriangleleft \end{aligned}$$

(c) We have

$$\begin{aligned} I_z &= \int r_z^2 dm = \int (x^2 + y^2) dm \\ &= \int [x^2 + (\zeta \cos \theta)^2] dm \\ &= \int x^2 dm + \cos^2 \theta \int \zeta^2 dm \\ &= I_\zeta + I_x \cos^2 \theta \\ &= \frac{1}{24} m a^2 + \frac{1}{6} m h^2 \cos^2 \theta \\ \text{or } I_z &= \frac{m}{24} (a^2 + 4h^2 \cos^2 \theta) \blacktriangleleft \end{aligned}$$



PROBLEM 9.127

Shown is the cross section of a molded flat-belt pulley. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA' . (The density of brass is 8650 kg/m^3 and the density of the fiber-reinforced polycarbonate used is 1250 kg/m^3 .)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4} (d_2^2 - d_1^2)$$

and, using Figure 9.28, that

$$\begin{aligned} I_{AA'} &= \frac{1}{2} m_2 \left(\frac{d_2}{2} \right)^2 - \frac{1}{2} m_1 \left(\frac{d_1}{2} \right)^2 \\ &= \frac{1}{8} \left[\left(\rho t \times \frac{\pi}{4} d_2^2 \right) d_2^2 - \left(\rho t \times \frac{\pi}{4} d_1^2 \right) d_1^2 \right] \\ &= \frac{1}{8} \left(\frac{\pi}{4} \rho t \right) (d_2^4 - d_1^4) \\ &= \frac{1}{8} \left(\frac{\pi}{4} \rho t \right) (d_2^2 - d_1^2)(d_2^2 + d_1^2) \\ &= \frac{1}{8} m (d_1^2 + d_2^2) \end{aligned}$$

Now treat the pulley as four concentric rings and, working from the brass outward, we have

$$\begin{aligned} m &= \frac{\pi}{4} \left\{ 8650 \text{ kg/m}^3 \times (0.0175 \text{ m}) \times (0.011^2 - 0.005^2) \text{ m}^2 \right. \\ &\quad + 1250 \text{ kg/m}^3 \left[(0.0175 \text{ m}) \times (0.017^2 - 0.011^2) \text{ m}^2 \right. \\ &\quad + (0.002 \text{ m}) \times (0.022^2 - 0.017^2) \text{ m}^2 \\ &\quad \left. \left. + (0.0095 \text{ m}) \times (0.028^2 - 0.022^2) \text{ m}^2 \right] \right\} \\ &= (11.4134 + 2.8863 + 0.38288 + 2.7980) \times 10^{-3} \text{ kg} \\ &= 17.4806 \times 10^{-3} \text{ kg} \end{aligned}$$

PROBLEM 9.127 (Continued)

and

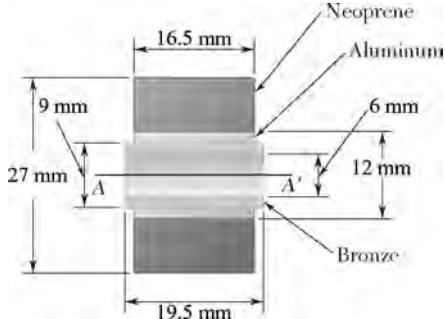
$$\begin{aligned}
 I_{AA'} &= \frac{1}{8} \left[(11.4134)(0.005^2 + 0.011^2) + (2.8863)(0.011^2 + 0.017^2) \right. \\
 &\quad \left. + (0.38288)(0.017^2 + 0.022^2) \right. \\
 &\quad \left. + (2.7980)(0.022^2 + 0.028^2) \right] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (208.29 + 147.92 + 37.00 + 443.48) \times 10^{-9} \text{ kg} \cdot \text{m}^2 \\
 &= 836.69 \times 10^{-9} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_{AA'} = 837 \times 10^{-9} \text{ kg} \cdot \text{m}^2$ ◀

Now

$$k_{AA'}^2 = \frac{I_{AA'}}{m} = \frac{836.69 \times 10^{-9} \text{ kg} \cdot \text{m}^2}{17.4806 \times 10^{-3} \text{ kg}} = 47.864 \times 10^{-6} \text{ m}^2$$

or $k_{AA'} = 6.92 \text{ mm}$ ◀



PROBLEM 9.128

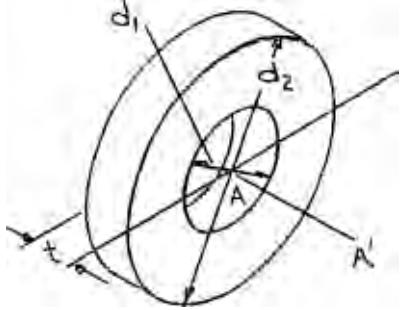
Shown is the cross section of an idler roller. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA' . (The specific weight of bronze is 8580 kg/m^3 ; of aluminum, 2770 kg/m^3 ; and of neoprene, 1250 kg/m^3 .)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4} (d_2^2 - d_1^2) = \frac{\pi}{4} \rho t (d_2^2 - d_1^2)$$

and, using Figure 9.28, that



$$\begin{aligned} I_{AA'} &= \frac{1}{2} m_2 \left(\frac{d_2}{2} \right)^2 - \frac{1}{2} m_1 \left(\frac{d_1}{2} \right)^2 \\ &= \frac{1}{8} \left[\left(\rho t \times \frac{\pi}{4} d_2^2 \right) d_2^2 - \left(\rho t \times \frac{\pi}{4} d_1^2 \right) d_1^2 \right] \\ &= \frac{1}{8} \left(\frac{\pi}{4} \rho t \right) (d_2^4 - d_1^4) \\ &= \frac{1}{8} \left(\frac{\pi}{4} \rho t \right) (d_2^2 - d_1^2)(d_2^2 + d_1^2) \\ &= \frac{1}{8} m (d_1^2 + d_2^2) \end{aligned}$$

Now treat the roller as three concentric rings and, working from the bronze outward, have

$$\begin{aligned} \text{Have } m &= \frac{\pi}{4} \times \left\{ (8580 \text{ kg/m}^3) (0.0195 \text{ m}) [(0.009 \text{ m})^2 - (0.006 \text{ m})^2] \right. \\ &\quad + (2770 \text{ kg/m}^3) (0.0165 \text{ m}) [(0.012 \text{ m})^2 - (0.009 \text{ m})^2] \\ &\quad \left. + (1250 \text{ kg/m}^3) (0.0165 \text{ m}) [(0.027 \text{ m})^2 - (0.012 \text{ m})^2] \right\} \\ &= \frac{\pi}{4} [7.52895 + 2.87942 + 12.06563] \times 10^{-3} \text{ kg} \\ &= 5.9132 \times 10^{-3} \text{ kg} + 2.26149 \times 10^{-3} \text{ kg} + 9.47632 \times 10^{-3} \text{ kg} \\ &= 17.6510 \times 10^{-3} \text{ kg} \end{aligned}$$

PROBLEM 9.128 (Continued)

And

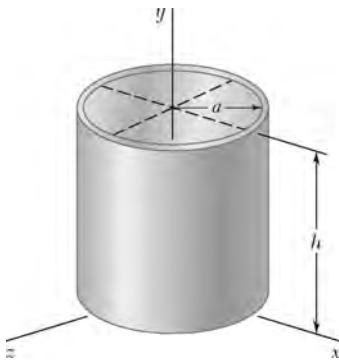
$$\begin{aligned}
 I_{AA'} &= \frac{1}{8} \{ (5.9132 \times 10^{-3} \text{ kg}) [(0.006)^2 + (0.009)^2] \text{ m}^2 \\
 &\quad + (2.26149 \times 10^{-3} \text{ kg}) [(0.009)^2 + (0.012)^2] \text{ m}^2 \\
 &\quad + (9.47632 \times 10^{-3} \text{ kg}) [(0.012)^2 + (0.027)^2] \text{ m}^2 \} \\
 &= \frac{1}{8} (691.844 + 508.835 + 8272.827) 10^{-9} \text{ kg} \cdot \text{m}^2 \\
 &= 1.18419 \times 10^{-6} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_{AA'} = 1.184 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$

Now

$$\begin{aligned}
 k_{AA'}^2 &= \frac{I_{AA'}}{m} = \frac{1.18419 \times 10^{-6} \text{ kg} \cdot \text{m}^2}{17.6510 \times 10^{-3} \text{ kg}} \\
 &= 67.08902 \times 10^{-6} \text{ m}^2 \\
 k_{AA'} &= 8.19079 \times 10^{-3} \text{ m}
 \end{aligned}$$

or $k_{AA'} = 8.19 \text{ mm} \blacktriangleleft$

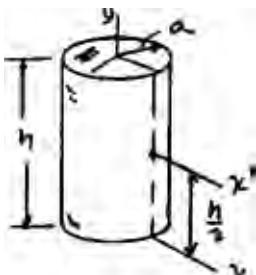


PROBLEM 9.129

Knowing that the thin cylindrical shell shown is of mass m , thickness t , and height h , determine the mass moment of inertia of the shell with respect to the x axis. (Hint: Consider the shell as formed by removing a cylinder of radius a and height h from a cylinder of radius $a + t$ and height h ; then neglect terms containing t^2 and t^3 and keep those terms containing t .)

SOLUTION

From Figure 9.28 for a solid cylinder (change orientation of axes):



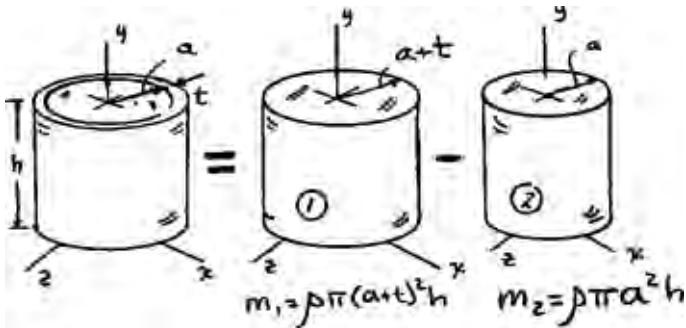
$$\bar{I}_{x'} = \frac{1}{12}m(3a^2 + h^2)$$

$$I_x = \bar{I}_{x'} + m\left(\frac{h}{2}\right)^2$$

$$= \frac{1}{12}m(3a^2 + h^2) + m\frac{h^2}{4}$$

$$= m\left(\frac{3}{12}a^2 + \frac{h^2}{3}\right)$$

$$I_x = \frac{1}{12}m(3a^2 + 4h^2) \quad \square$$



$$\text{Shell} = \text{Cylinder } ① - \text{Cylinder } ②$$

$$\begin{aligned} I_x &= \frac{1}{12}m_1[3(a+t)^2 + 4h^2] - \frac{1}{12}m_2(3a^2 + 4h^2) \\ &= \frac{1}{12}[\rho\pi(a+t)^2h][3(a+t)^2 + 4h^2] - \frac{1}{12}(\rho\pi a^2h)(3a^2 + 4h^2) \end{aligned}$$

PROBLEM 9.129 (Continued)

Expand $(a+t)$ factors and neglecting terms in t^2 , t^3 , and t^4 :

$$\begin{aligned} I_x &= \frac{\rho\pi h}{12} \left[(a^2 + 2at + t^2)(3a^2 + 6at + 3t^2 + 4h^2) \right] - 3a^4 - 4a^2h^2 \\ &= \frac{\rho\pi h}{12} (3a^4 + 6a^3t + 3a^2t^2 + 4a^2h^2 + 6a^3t + 8ath^2 - 3a^4 - 4a^2h^2) \\ &= \frac{\rho\pi h}{12} (12a^3t + 8ath^2) \\ &= \frac{\rho\pi hat}{12} (12a^2 + 8h^2) \end{aligned}$$

Mass of shell:

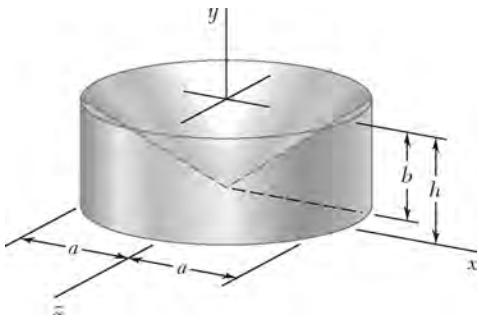
$$m = \rho(2\pi a h t) = 2\rho\pi h a t$$

$$\begin{aligned} I_x &= \frac{2\rho\pi h a t}{24} (12a^2 + 8h^2) \\ &= \frac{4m}{24} (3a^2 + 2h^2) \end{aligned}$$

$$\text{or } I_x = \frac{m}{6} (3a^2 + 2h^2) \blacktriangleleft$$

Checks:

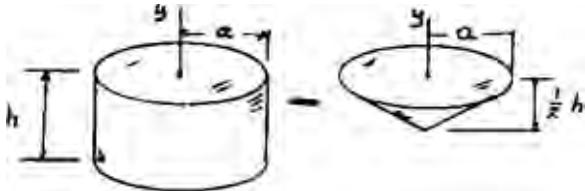
$$\left. \begin{array}{l} a = 0; \quad I_x = \frac{1}{3} m h^2 - \text{slender rod} \\ h = 0; \quad I_x = \frac{1}{2} m a^2 - \text{thin ring} \end{array} \right\} \text{OK}$$



PROBLEM 9.130

The machine part shown is formed by machining a conical surface into a circular cylinder. For $b = \frac{1}{2}h$, determine the mass moment of inertia and the radius of gyration of the machine part with respect to the y axis.

SOLUTION



Mass:

$$m_{\text{cyl}} = \rho \pi a^2 h \quad m_{\text{cone}} = \frac{1}{3} \rho \pi a^2 \frac{h}{2} = \frac{1}{6} \rho \pi a^2 h$$

$$\begin{aligned} I_y: \quad I_{\text{cyl}} &= \frac{1}{2} m_{\text{cyl}} a^2 & I_{\text{cone}} &= \frac{3}{10} m_{\text{cone}} a^2 \\ &= \frac{1}{2} \rho \pi a^4 h & &= \frac{1}{20} \rho \pi a^4 h \end{aligned}$$

For entire machine part:

$$m = m_{\text{cyl}} - m_{\text{cone}} = \rho \pi a^2 h - \frac{1}{6} \rho \pi a^2 h = \frac{5}{6} \rho \pi a^2 h$$

$$I_y = I_{\text{cyl}} - I_{\text{cone}} = \frac{1}{2} \rho \pi a^4 h - \frac{1}{20} \rho \pi a^4 h = \frac{9}{20} \rho \pi a^4 h$$

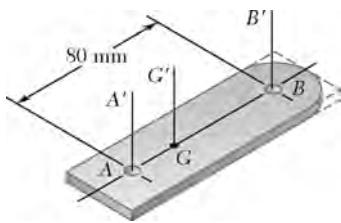
or

$$I_y = \left(\frac{5}{6} \rho \pi a^2 h \right) \left(\frac{6}{5} \right) \left(\frac{9}{20} \right) a^2 \quad I_y = \frac{27}{50} m a^2 \blacktriangleleft$$

Then

$$k_y^2 = \frac{I}{m} = \frac{27}{50} a^2$$

$$k_y = 0.735 a \blacktriangleleft$$



PROBLEM 9.131

After a period of use, one of the blades of a shredder has been worn to the shape shown and is of mass 0.18 kg. Knowing that the mass moments of inertia of the blade with respect to the AA' and $B'B$ axes are $0.320 \text{ g} \cdot \text{m}^2$ and $0.680 \text{ g} \cdot \text{m}^2$, respectively, determine (a) the location of the centroidal axis GG' , (b) the radius of gyration with respect to axis GG' .

SOLUTION

(a) We have

$$d_B = (0.08 - d_A) \text{ m}$$

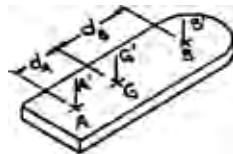
and, using the parallel axis

$$I_{AA'} = \bar{I}_{GG'} + md_A^2$$

$$I_{BB'} = \bar{I}_{GG'} + md_B^2$$

Then

$$\begin{aligned} I_{BB'} - I_{AA'} &= m(d_B^2 - d_A^2) \\ &= m[(0.08 - d_A)^2 - d_A^2] \\ &= m(0.0064 - 0.16d_A) \end{aligned}$$



Substituting: $(0.68 - 0.32) \times 10^{-3} \text{ kg} \cdot \text{m}^2 = 0.18 \text{ kg}(0.0064 - 0.16d_A) \text{ m}^2$

$$\text{or } d_A = 27.5 \text{ mm} \blacktriangleleft$$

(b) We have

$$I_{AA'} = \bar{I}_{GG'} + md_A^2$$

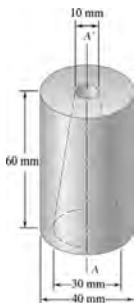
or

$$\begin{aligned} I_{GG'} &= 0.32 \times 10^{-3} \text{ kg} \cdot \text{m}^2 - 0.18 \text{ kg} \cdot (0.0275 \text{ m})^2 \\ &= 0.183875 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

Then

$$k_{GG'}^2 = \frac{I_{GG'}}{m} = \frac{0.183875 \times 10^{-3} \text{ kg} \cdot \text{m}^2}{0.18 \text{ kg}} = 1.02153 \times 10^{-3} \text{ m}^2$$

$$\text{or } k_{GG'} = 32.0 \text{ mm} \blacktriangleleft$$

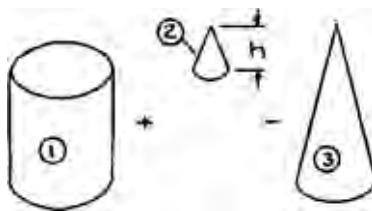


PROBLEM 9.132

Determine the mass moment of inertia of the 400-g machine component shown with respect to the axis AA'.

SOLUTION

First note that the given shape can be formed adding a small cone to a cylinder and then removing a larger cone as indicated.



Now

$$\frac{h}{10} = \frac{h+60}{30} \quad \text{or} \quad h = 30 \text{ mm}$$

The weight of the body is given by

$$\begin{aligned} m &= (m_1 + m_2 - m_3) = \rho(V_1 + V_2 - V_3) \\ \text{or} \quad 0.4 \text{ kg} &= \rho \left[\pi(0.02)^2(0.06) + \frac{\pi}{3}(0.005)^2(0.03) - \frac{\pi}{3}(0.015)^2(0.09) \right] \text{m}^3 \\ &= \rho(7.54 + 0.079 - 2.12) \times 10^{-5} \text{ m}^3 \end{aligned}$$

or

$$\rho = 7274 \text{ kg/m}^3$$

Then

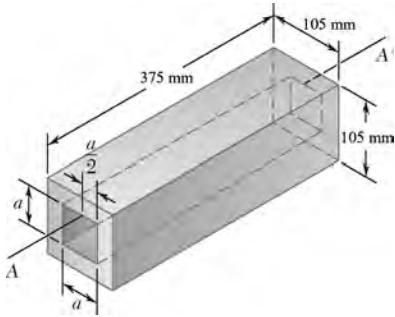
$$m_1 = (7274)(7.54 \times 10^{-5}) = 0.5485 \text{ kg}$$

$$m_2 = (7274)(0.079 \times 10^{-5}) = 0.00575 \text{ kg}$$

$$m_3 = (7274)(2.12 \times 10^{-5}) = 0.1542 \text{ kg}$$

Finally, using Figure 9.28 have,

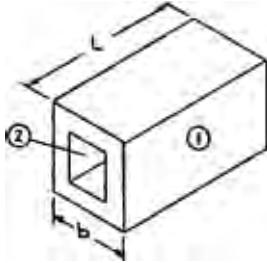
$$\begin{aligned} I_{AA'} &= (I_{AA'})_1 + (I_{AA'})_2 - (I_{AA'})_3 \\ &= \frac{1}{2}m_1a_1^2 + \frac{3}{10}m_2a_2^2 - \frac{3}{10}m_3a_3^2 \\ &= \left[\frac{1}{2}(0.5485)(0.02)^2 + \frac{3}{10}(0.00575)(0.005)^2 - \frac{3}{10}(0.1542)(0.015)^2 \right] \\ &= (10.97 + 0.00431 - 1.042) \times 10^{-5} \text{ kg} \cdot \text{m}^2 \\ \text{or} \quad I_{AA'} &= 9.93 \times 10^{-5} \text{ kg} \cdot \text{m}^2 \blacksquare \end{aligned}$$



PROBLEM 9.133

A square hole is centered in and extends through the aluminum machine component shown. Determine (a) the value of a for which the mass moment of inertia of the component with respect to the axis AA' , which bisects the top surface of the hole, is maximum, (b) the corresponding values of the mass moment of inertia and the radius of gyration with respect to the axis AA' . (The density of aluminum is 2770 kg/m^3 .)

SOLUTION



$$\text{First note } m_1 = \rho V_1 = \rho b^2 L$$

$$\text{And } m_2 = \rho V_2 = \rho a^2 L$$

(a) Using Figure 9.28 and the parallel-axis theorem, have

$$\begin{aligned} I_{AA'} &= (I_{AA'})_1 - (I_{AA'})_2 \\ &= \left[\frac{1}{12} m_1 (b^2 + b^2) + m_1 \left(\frac{a}{2} \right)^2 \right] \\ &\quad - \left[\frac{1}{12} m_2 (a^2 + a^2) + m_2 \left(\frac{a}{2} \right)^2 \right] \\ &= (\rho b^2 L) \left(\frac{1}{6} b^2 + \frac{1}{4} a^2 \right) - (\rho a^2 L) \left(\frac{5}{12} a^2 \right) \\ &= \frac{\rho L}{12} (2b^4 + 3b^2 a^2 - 5a^4) \end{aligned}$$

$$\text{Then } \frac{dI_{AA'}}{da} = \frac{\rho L}{12} (6b^2 a - 20a^3) = 0$$

$$\text{or } a = 0 \quad \text{and} \quad a = b \sqrt{\frac{3}{10}}$$

$$\text{Also } \frac{d^2 I_{AA'}}{da^2} = \frac{\rho L}{12} (6b^2 - 60a^2) = \frac{1}{2} \rho L (b^2 - 10a^2)$$

Now, for

$$a = 0, \quad \frac{d^2 I_{AA'}}{da^2} > 0 \quad \text{and for } a = b \sqrt{\frac{3}{10}}, \quad \frac{d^2 I_{AA'}}{da^2} < 0$$

$$\therefore (I_{AA'})_{\max} \text{ occurs when } a = b \sqrt{\frac{3}{10}}$$

$$a = 105 \sqrt{\frac{3}{10}} = 57.511 \text{ mm}$$

or

$$a = 57.5 \text{ mm} \blacktriangleleft$$

PROBLEM 9.133 (Continued)

(b) From part (a)

$$\begin{aligned}
 (I_{AA'})_{\text{mass}} &= \frac{\rho L}{12} \left[2b^4 + 3b^2 \left(b \sqrt{\frac{3}{10}} \right)^2 - 5 \left(b \sqrt{\frac{3}{10}} \right)^4 \right] \\
 &= \frac{49}{240} \rho L b^4 = \frac{49}{240} (2770 \text{ kg/m}^3)(0.375 \text{ m})(0.105 \text{ m})^4 \\
 &= 25.778 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $(I_{AA'})_{\text{mass}} = 25.8 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

and

$$k_{AA'}^2 = \frac{(I_{AA'})_{\text{mass}}}{m}$$

where

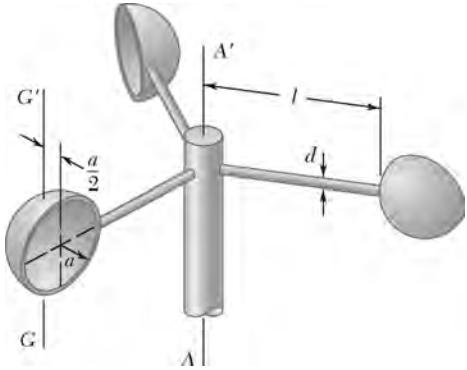
$$m = m_1 - m_2 = \rho L(b^2 - a^2) = \rho L \left[b^2 - \left(b \sqrt{\frac{3}{10}} \right)^2 \right] = \frac{7}{10} \rho L b^2$$

Then

$$k_{AA'}^2 = \frac{\frac{49}{240} \rho L b^4}{\frac{7}{10} \rho L b^2} = \frac{7}{24} b^2 = \frac{7}{24} (105 \text{ mm})^2 = 3215.625 \text{ mm}^2$$

$k_{AA'} = 56.706 \text{ mm}$

or $k_{AA'} = 56.7 \text{ mm}$ ◀



PROBLEM 9.134

The cups and the arms of an anemometer are fabricated from a material of density ρ . Knowing that the mass moment of inertia of a thin, hemispherical shell of mass m and thickness t with respect to its centroidal axis GG' is $5ma^2/12$, determine (a) the mass moment of inertia of the anemometer with respect to the axis AA' , (b) the ratio of a to l for which the centroidal moment of inertia of the cups is equal to 1 percent of the moment of inertia of the cups with respect to the axis AA' .

SOLUTION

(a) First note

$$m_{\text{arm}} = \rho V_{\text{arm}} = \rho \times \frac{\pi}{4} d^2 l$$

and

$$\begin{aligned} dm_{\text{cup}} &= \rho dV_{\text{cup}} \\ &= \rho [(2\pi a \cos \theta)(t)(ad\theta)] \end{aligned}$$

Then

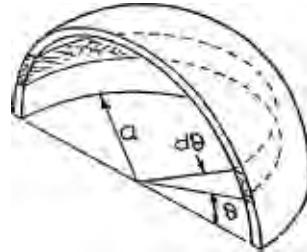
$$\begin{aligned} m_{\text{cup}} &= \int dm_{\text{cup}} = \int_0^{\pi/2} 2\pi\rho a^2 t \cos \theta d\theta \\ &= 2\pi\rho a^2 t [\sin \theta]_0^{\pi/2} \\ &= 2\pi\rho a^2 t \end{aligned}$$

Now

$$(I_{AA'})_{\text{anem}} = (I_{AA'})_{\text{cups}} + (I_{AA'})_{\text{arms}}$$

Using the parallel-axis theorem and assuming the arms are slender rods, we have

$$\begin{aligned} (I_{AA'})_{\text{anem}} &= 3[(I_{GG'})_{\text{cup}} + m_{\text{cup}} d_{AG}^2] + 3[\bar{I}_{\text{arm}} + m_{\text{arm}} d_{AG_{\text{arm}}}^2] \\ &= 3\left\{\frac{5}{12}m_{\text{cup}}a^2 + m_{\text{cup}}\left[(l+a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} + 3\left[\frac{1}{2}m_{\text{arm}}l^2 + m_{\text{arm}}\left(\frac{l}{2}\right)^2\right] \\ &= 3m_{\text{cup}}\left(\frac{5}{3}a^2 + 2la + l^2\right) + m_{\text{arm}}l^2 \\ &= 3(2\pi\rho a^2 t)\left(\frac{5}{3}a^2 + 2la + l^2\right) + \left(\frac{\pi}{4}\rho d^2 l\right)(l^2) \end{aligned}$$



$$\text{or } (I_{AA'})_{\text{anem}} = \pi\rho l^2 \left[6a^2 t \left(\frac{5}{3} \frac{a^2}{l^2} + 2 \frac{a}{l} + 1 \right) + \frac{d^2 l}{4} \right] \blacktriangleleft$$

PROBLEM 9.134 (Continued)

(b) We have

$$\frac{(I_{GG'})_{\text{cup}}}{(I_{AA'})_{\text{cup}}} = 0.01$$

or

$$\frac{5}{12}m_{\text{cup}}a^2 = 0.01m_{\text{cup}}\left(\frac{5}{3}a^2 + 2la + l^2\right) \quad (\text{from Part } a)$$

Now let $\zeta = \frac{a}{l}$.

Then

$$5\zeta^2 = 0.12\left(\frac{5}{3}\zeta^2 + 2\zeta + 1\right)$$

or

$$40\zeta^2 - 2\zeta - 1 = 0$$

Then

$$\zeta = \frac{2 \pm \sqrt{(-2)^2 - 4(40)(-1)}}{2(40)}$$

or

$$\zeta = 0.1851 \quad \text{and} \quad \zeta = -0.1351$$

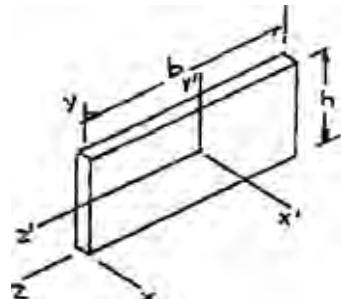
$$\frac{a}{l} = 0.1851 \blacktriangleleft$$

To the instructor:

The following formulas for the mass moment of inertia of thin plates and a half cylindrical shell are derived at this time for use in the solutions of Problems 9.135 through 9.140.

Thin rectangular plate

$$\begin{aligned}(I_x)_m &= (\bar{I}_{x'})_m + md^2 \\&= \frac{1}{12}m(b^2 + h^2) + m\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right] \\&= \frac{1}{3}m(b^2 + h^2)\end{aligned}$$



$$\begin{aligned}(I_y)_m &= (\bar{I}_{y'})_m + md^2 \\&= \frac{1}{12}mb^2 + m\left(\frac{b}{2}\right)^2 = \frac{1}{3}mb^2\end{aligned}$$

$$\begin{aligned}I_z &= (\bar{I}_{z'})_m + md^2 \\&= \frac{1}{12}mh^2 + m\left(\frac{h}{2}\right)^2 = \frac{1}{3}mh^2\end{aligned}$$

PROBLEM 9.134 (Continued)

Thin triangular plate

We have

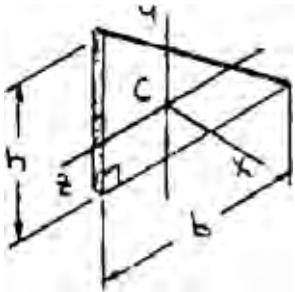
$$m = \rho V = \rho \left(\frac{1}{2} bht \right)$$

and

$$\bar{I}_{z, \text{area}} = \frac{1}{36} bh^3$$

Then

$$\begin{aligned}\bar{I}_{z, \text{mass}} &= \rho t I_{z, \text{area}} \\ &= \rho t \times \frac{1}{36} bh^3 \\ &= \frac{1}{18} mh^2\end{aligned}$$



Similarly,

$$\bar{I}_{y, \text{mass}} = \frac{1}{18} mb^2$$

Now

$$\bar{I}_{x, \text{mass}} = \bar{I}_{y, \text{mass}} + \bar{I}_{z, \text{mass}} = \frac{1}{18} m(b^2 + h^2)$$

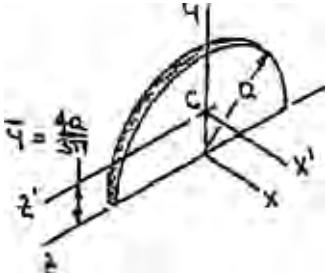
Thin semicircular plate

We have

$$m = \rho V = \rho \left(\frac{\pi}{2} a^2 t \right)$$

and

$$\bar{I}_{y, \text{area}} = I_{z, \text{area}} = \frac{\pi}{8} a^4$$



Then

$$\begin{aligned}\bar{I}_{y, \text{mass}} &= I_{z, \text{mass}} = \rho t \bar{I}_{y, \text{area}} \\ &= \rho t \times \frac{\pi}{8} a^4 \\ &= \frac{1}{4} ma^2\end{aligned}$$

Now

$$\bar{I}_{x, \text{mass}} = \bar{I}_{y, \text{mass}} + I_{z, \text{mass}} = \frac{1}{2} ma^2$$

Also

$$I_{x, \text{mass}} = \bar{I}_{x', \text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{x', \text{mass}} = m \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) a^2$$

and

$$I_{z, \text{mass}} = \bar{I}_{z', \text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{z', \text{mass}} = m \left(\frac{1}{4} - \frac{16}{9\pi^2} \right) a^2$$

PROBLEM 9.134 (Continued)

$$\bar{y} = \bar{z} = \frac{4a}{3\pi}$$

Thin Quarter-Circular Plate

We have

$$m = \rho V = \rho \left(\frac{\pi}{4} a^2 t \right)$$

and

$$I_{y,\text{area}} = I_{z,\text{area}} = \frac{\pi}{16} a^4$$

Then

$$\begin{aligned} I_{y,\text{mass}} &= I_{z,\text{mass}} = \rho t I_{y,\text{area}} \\ &= \rho t \times \frac{\pi}{16} a^4 \\ &= \frac{1}{4} m a^2 \end{aligned}$$

Now

$$I_{x,\text{mass}} = I_{y,\text{mass}} + I_{z,\text{mass}} = \frac{1}{2} m a^2$$

Also

$$I_{x,\text{mass}} = I_{x',\text{mass}} + m(\bar{y}^2 + \bar{z}^2)$$

or

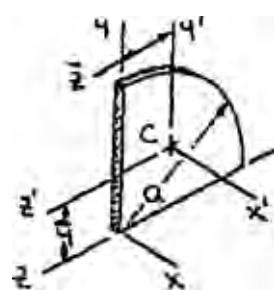
$$\bar{I}_{x',\text{mass}} = m \left(\frac{1}{2} - \frac{32}{9\pi^2} \right) a^2$$

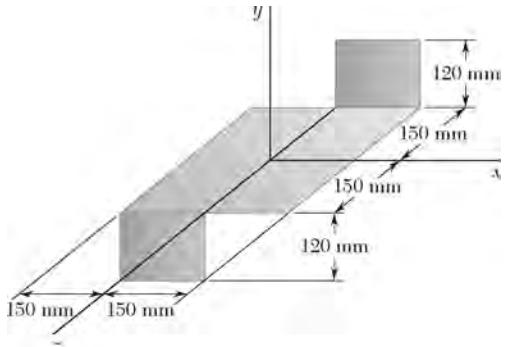
and

$$I_{y,\text{mass}} = I_{y',\text{mass}} + m \bar{z}^2$$

or

$$\bar{I}_{y',\text{mass}} = m \left(\frac{1}{4} - \frac{16}{9\pi^2} \right) a^2$$





PROBLEM 9.135

A 2 mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3 , determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION

First compute the mass of each component. We have

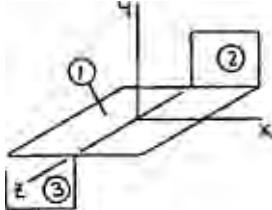
$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} t A$$

Then

$$\begin{aligned} m_1 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})(0.3 \text{ m})^2 \\ &= 1.413 \text{ kg} \\ m_2 = m_3 &= (7850 \text{ kg/m}^3)(0.002 \text{ m}) \times (0.15 \times 0.12) \text{ m}^2 \\ &= 0.2826 \text{ kg} \end{aligned}$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$\begin{aligned} I_x &= (I_x)_1 + 2(I_x)_2 \\ &= \left[\frac{1}{12}(1.413 \text{ kg})(0.3 \text{ m})^2 \right] \\ &\quad + 2 \left[\frac{1}{12}(0.2826 \text{ kg})(0.12 \text{ m})^2 + (0.2826 \text{ kg})(0.15^2 + 0.06^2) \text{ m}^2 \right] \\ &= [(0.0105975) + 2(0.0003391 + 0.0073759)] \text{ kg} \cdot \text{m}^2 \\ &= [(0.0105975) + 2(0.0077150)] \text{ kg} \cdot \text{m}^2 \\ &\quad \text{or } I_x = 26.0 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft \end{aligned}$$

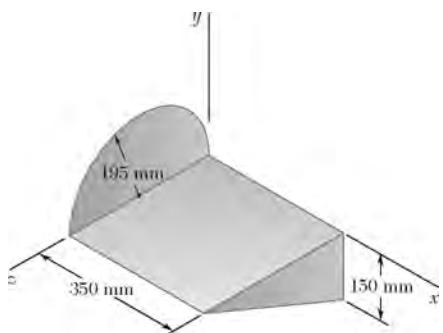


$$\begin{aligned} I_y &= (I_y)_1 + 2(I_y)_2 \\ &= \left[\frac{1}{12}(1.413 \text{ kg})(0.3^2 + 0.3^2) \text{ m}^2 \right] \\ &\quad + 2 \left[\frac{1}{12}(0.2826 \text{ kg})(0.15 \text{ m})^2 + (0.2826 \text{ kg})(0.075^2 + 0.15^2) \text{ m}^2 \right] \\ &= [(0.0211950) + 2(0.0005299 + 0.0079481)] \text{ kg} \cdot \text{m}^2 \\ &= [(0.0211950) + 2(0.0084780)] \text{ kg} \cdot \text{m}^2 \\ &\quad \text{or } I_y = 38.2 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft \end{aligned}$$

PROBLEM 9.135 (Continued)

$$\begin{aligned}I_z &= (I_z)_1 + 2(I_z)_2 \\&= \left[\frac{1}{12}(1.413 \text{ kg})(0.3 \text{ m})^2 \right] \\&\quad + 2 \left[\frac{1}{12}(0.2826 \text{ kg})(0.15^2 + 0.12^2) \text{ m}^2 + (0.2826 \text{ kg})(0.075^2 + 0.06^2) \text{ m}^2 \right] \\&= [(0.0105975) + 2(0.0008690 + 0.0026070)] \text{ kg} \cdot \text{m}^2 \\&= [(0.0105975) + 2(0.0034760)] \text{ kg} \cdot \text{m}^2\end{aligned}$$

or $I_z = 17.55 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀



PROBLEM 9.136

A 2 mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3 , determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION

First compute the mass of each component. We have

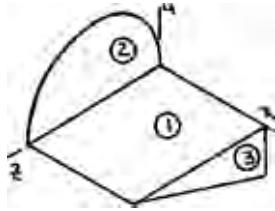
$$m = \rho_{ST}V = \rho_{ST}tA$$

Then

$$\begin{aligned} m_1 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})(0.35 \times 0.39) \text{ m}^2 \\ &= 2.14305 \text{ kg} \end{aligned}$$

$$m_2 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{2} \times 0.195^2\right) \text{ m}^2 = 0.93775 \text{ kg}$$

$$m_3 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{1}{2} \times 0.39 \times 0.15\right) \text{ m}^2 = 0.45923 \text{ kg}$$



Using Figure 9.28 for component 1 and the equations derived above for components 2 and 3, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 \\ &= \left[\frac{1}{12}(2.14305 \text{ kg})(0.39 \text{ m})^2 + (2.14305 \text{ kg})\left(\frac{0.39}{2} \text{ m}\right)^2 \right] \\ &\quad + \left\{ \left(\frac{1}{2} - \frac{16}{9\pi^2} \right)(0.93775 \text{ kg})(0.195 \text{ m})^2 + (0.93775 \text{ kg})\left[\left(\frac{4 \times 0.195}{3\pi}\right)^2 + (0.195)^2\right] \text{ m}^2 \right\} \\ &\quad + \left\{ \frac{1}{18}(0.45923 \text{ kg})[(0.39)^2 + (0.15)^2] \text{ m}^2 + (0.45923 \text{ kg})\left[\left(\frac{0.39}{3}\right)^2 + \left(\frac{0.15}{3}\right)^2\right] \text{ m}^2 \right\} \\ &= [(0.027163 + 0.081489) + (0.011406 + 0.042081) + (0.004455 + 0.008909)] \text{ kg} \cdot \text{m}^2 \\ &= (0.108652 + 0.053487 + 0.013364) \text{ kg} \cdot \text{m}^2 \\ &= 0.175503 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_x = 175.5 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

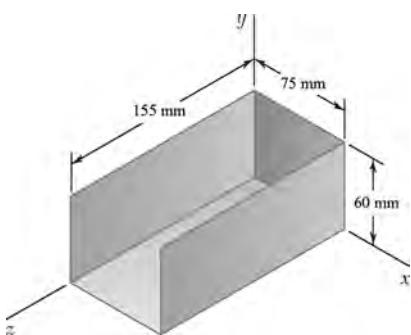
PROBLEM 9.136 (Continued)

$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
 &= \left\{ \frac{1}{12} (2.14305 \text{ kg})[(0.35)^2 + (0.39)^2] \text{ m}^2 + (2.14305 \text{ kg}) \left[\left(\frac{0.35}{2} \right)^2 + \left(\frac{0.39}{2} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad + \left[\frac{1}{4} (0.93775 \text{ kg})(0.195 \text{ m})^2 + (0.93775 \text{ kg})(0.195 \text{ m})^2 \right] \\
 &\quad + \left\{ \frac{1}{18} (0.45923 \text{ kg})(0.39 \text{ m})^2 + (0.45923 \text{ kg}) \left[(0.35)^2 + \left(\frac{0.39}{3} \right)^2 \right] \text{ m}^2 \right\} \\
 &= [(0.049040 + 0.147120) + (0.008914 + 0.035658) \\
 &\quad + (0.003880 + 0.064017)] \text{ kg} \cdot \text{m}^2 \\
 &= (0.196160 + 0.044572 + 0.067897) \text{ kg} \cdot \text{m}^2 \\
 &= 0.308629 \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 309 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 \\
 &= \left[\frac{1}{12} (2.14305 \text{ kg})(0.35 \text{ m})^2 + (2.14305 \text{ kg}) \left(\frac{0.35}{2} \text{ m} \right)^2 \right] \\
 &\quad + \left[\frac{1}{4} (0.93775 \text{ kg})(0.195 \text{ m})^2 \right] \\
 &\quad + \left\{ \frac{1}{18} (0.45923 \text{ kg})(0.15 \text{ m})^2 + (0.45923 \text{ kg}) \left[(0.35)^2 + \left(\frac{0.15}{3} \right)^2 \right] \text{ m}^2 \right\} \\
 &= [(0.021877 + 0.065631) + 0.008914] + (0.000574 + 0.057404) \text{ kg} \cdot \text{m}^2 \\
 &= (0.087508 + 0.008914 + 0.057978) \text{ kg} \cdot \text{m}^2 \\
 &= 0.154400 \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

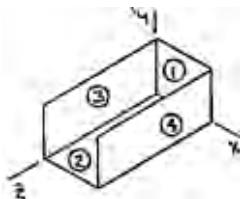
or $I_z = 154.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀



PROBLEM 9.137

The cover for an electronic device is formed from sheet aluminum that is 1.25 mm thick. Determine the mass moment of inertia of the cover with respect to each of the coordinate axes. (The density of aluminum is 2770 kg/m^3 .)

SOLUTION



Have

$$m = \rho V$$

Now

$$\begin{aligned} m_1 &= (2770 \text{ kg/m}^3)(0.00125 \text{ m})(0.060 \text{ m})(0.075 \text{ m}) \\ &= 0.015581 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= (2770 \text{ kg/m}^3)(0.00125 \text{ m})(0.075 \text{ m})(0.155 \text{ m}) \\ &= 0.040252 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 &= (2770 \text{ kg/m}^3)(0.00125 \text{ m})(0.060 \text{ m})(0.155 \text{ m}) \\ &= 0.032201 \text{ kg} \end{aligned}$$

Using Figure 9.28 and the parallel-axis theorem, have

$$\begin{aligned} (I_x) &= (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 \quad \text{and} \quad (I_x)_3 = (I_x)_4 \\ &= (0.015581 \text{ kg}) \left[(0.06)^2 \left(\frac{1}{12} + \frac{1}{4} \right) \right] \text{m}^2 + (0.040252 \text{ kg}) \left[(0.155)^2 \left(\frac{1}{12} + \frac{1}{4} \right) \right] \text{m}^2 \\ &\quad + 2(0.032201 \text{ kg}) \left\{ \left[(0.06)^2 + (0.155)^2 \right] \left(\frac{1}{12} + \frac{1}{4} \right) \right\} \text{m}^2 \\ &= (1.86972 \times 10^{-5} + 3.2235 \times 10^{-4} + 5.9304 \times 10^{-4}) \text{kg} \cdot \text{m}^2 \\ &= 934.087 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_x = 934 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.137 (Continued)

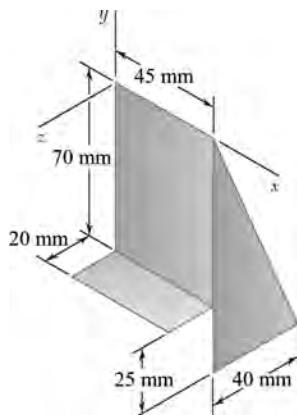
$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 \\
 &= (0.015581\text{kg}) \left[\frac{1}{3}(0.075)^2 \right] \text{m}^2 + (0.040252\text{kg}) \left\{ \left[(0.075)^2 + (0.155)^2 \right] \left(\frac{1}{3} \right) \right\} \text{m}^2 \\
 &\quad + (0.032201\text{kg}) \left[(0.155)^2 \left(\frac{1}{3} \right) \right] \text{m}^2 + (0.032201\text{kg}) \left[\frac{1}{2}(0.155)^2 + (0.075)^2 + \frac{1}{4}(0.155)^2 \right] \text{m}^2 \\
 &= (2.9214 \times 10^{-5} + 3.9782 \times 10^{-4} + 2.5788 \times 10^{-4} + 4.3901 \times 10^{-4}) \text{kg} \cdot \text{m}^2 = 1123.924 \times 10^{-6} \text{kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 1124 \times 10^{-6} \text{ kg} \cdot \text{m}^2$ ◀

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4 \\
 &= (0.015581\text{kg}) \left\{ \left[(0.075)^2 + (0.06)^2 \right] \left(\frac{1}{3} \right) \right\} \text{m}^2 + (0.040252\text{kg}) \left[(0.075)^2 \left(\frac{1}{3} \right) \right] \text{m}^2 \\
 &\quad + (0.032201\text{kg}) \left[(0.06)^2 \left(\frac{1}{3} \right) \right] \text{m}^2 + (0.032201\text{kg}) \left[\left(\frac{1}{3} \right)(0.06)^2 + (0.075)^2 \right] \text{m}^2 \\
 &= (4.7912 \times 10^{-5} + 7.5473 \times 10^{-5} + 3.8641 \times 10^{-5} + 2.1977 \times 10^{-4}) \text{kg} \cdot \text{m}^2 \\
 &= 381.796 \times 10^{-6} \text{kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_z = 382 \times 10^{-6} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.138



A framing anchor is formed of 2-mm-thick galvanized steel. Determine the mass moment of inertia of the anchor with respect to each of the coordinate axes. (The density of galvanized steel is 7530 kg/m^3 .)

SOLUTION

First compute the mass of each component.

Have

$$m = \rho V = \rho At$$

Then

$$\begin{aligned} m_1 &= (7530 \text{ kg/m}^3)(0.002 \text{ m})(0.045 \text{ m})(0.070 \text{ m}) \\ &= 0.047439 \text{ kg} \end{aligned}$$

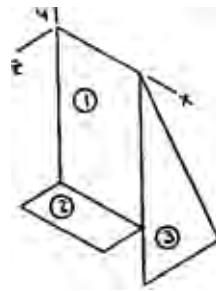
$$\begin{aligned} m_2 &= (7530 \text{ kg/m}^3)(0.002 \text{ m})(0.045 \text{ m})(0.045 \text{ m}) \\ &= 0.013554 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 &= (7530 \text{ kg/m}^3)(0.002 \text{ m}) \times \frac{1}{2}(0.04 \text{ m})(0.095 \text{ m}) \\ &= 0.028614 \text{ kg} \end{aligned}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 \\ &= (0.047439 \text{ kg}) \left[\frac{1}{3}(0.07)^2 \right] \text{m}^2 + (0.013554 \text{ kg}) \left\{ \frac{1}{12}(0.020)^2 + [(0.07)^2 + (0.01)^2] \right\} \text{m}^2 \\ &\quad + (0.028614 \text{ kg}) \left\{ \frac{1}{18}[(0.095)^2 + (0.04)^2] + \frac{1}{9}[(2 \times 0.095)^2 + (0.040)^2] \right\} \text{m}^2 \\ &= [(0.077484 + 0.068222 + 0.136751) \times 10^{-3}] \text{kg} \cdot \text{m}^2 = 0282457 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_x = 0.2825 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



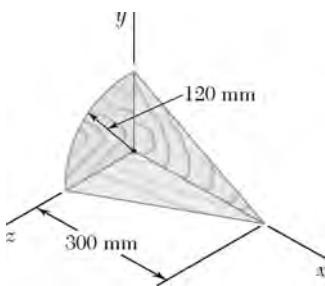
PROBLEM 9.138 (Continued)

$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
 &= (0.047439 \text{ kg}) \left[\frac{1}{3} (0.045)^2 \right] \text{m}^2 + (0.013554 \text{ kg}) \left\{ [(0.045)^2 + (0.02)^2] \left(\frac{1}{3} \right) \right\} \text{m}^2 \\
 &\quad + (0.028614 \text{ kg}) \left[\frac{1}{18} (0.04)^2 (0.045)^2 + \frac{1}{9} (0.04)^2 \right] \text{m}^2 \\
 &= [(0.03202 + 0.010956 + 0.065574) \times 10^{-3}] \text{kg} \cdot \text{m}^2 \\
 &= 0.10855 \times 10^3 \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 0.1086 \times 10^3 \text{ kg} \cdot \text{m}^2$ ◀

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 \\
 &= (0.047439 \text{ kg}) \left\{ [(0.045)^2 + (0.070)^2] \left(\frac{1}{3} \right) \right\} \text{m}^2 + (0.013554 \text{ kg}) [(0.045)^2 \left(\frac{1}{3} \right) \\
 &\quad + (0.070)^2] \text{m}^2 \\
 &\quad + (0.028614 \text{ kg}) \left[\frac{1}{18} (0.095)^2 + (0.045)^2 + \left(\frac{2}{3} 0.095 \right)^2 \right] \text{m}^2 \\
 &= [(0.0109505 + 0.075564 + 0.187064) \times 10^{-3}] \text{kg} \cdot \text{m}^3 \\
 &= 0.37213 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_z = 0.372 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀



PROBLEM 9.139

A subassembly for a model airplane is fabricated from three pieces of 1.5 mm plywood. Neglecting the mass of the adhesive used to assemble the three pieces, determine the mass moment of inertia of the subassembly with respect to each of the coordinate axes. (The density of the plywood is 780 kg/m^3 .)

SOLUTION

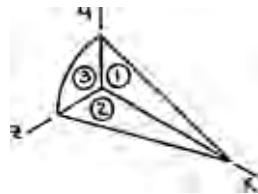
First compute the mass of each component. We have

$$m = \rho V = \rho tA$$

Then

$$\begin{aligned} m_1 = m_2 &= (780 \text{ kg/m}^3)(0.0015 \text{ m})\left(\frac{1}{2} \times 0.3 \times 0.12\right) \text{ m}^2 \\ &= 21.0600 \times 10^{-3} \text{ kg} \end{aligned}$$

$$m_3 = (780 \text{ kg/m}^3)(0.0015 \text{ m})\left(\frac{\pi}{4} \times 0.12^2\right) \text{ m}^2 = 13.2324 \times 10^{-3} \text{ kg}$$



Using the equations derived above and the parallel-axis theorem, we have

$$\begin{aligned} (I_x)_1 &= (I_x)_2 \\ I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 \\ &= 2 \left[\frac{1}{18} (21.0600 \times 10^{-3} \text{ kg})(0.12 \text{ m})^2 + (21.0600 \times 10^{-3} \text{ kg}) \left(\frac{0.12}{3} \text{ m} \right)^2 \right] \\ &\quad + \left[\frac{1}{2} (13.2324 \times 10^{-3} \text{ kg})(0.12 \text{ m})^2 \right] \\ &= [2(16.8480 + 33.6960) + (95.2733)] \times 10^{-6} \text{ kg} \cdot \text{m}^2 \\ &= [2(50.5440) + (95.2733)] \times 10^{-6} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

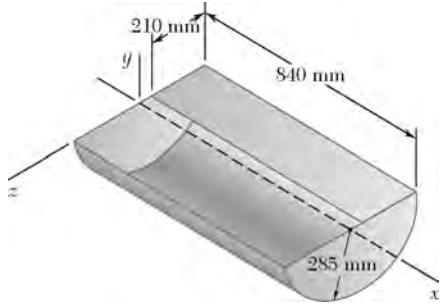
$$\text{or } I_x = 196.4 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.139 (Continued)

$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
 &= \left[\frac{1}{18} (21.0600 \times 10^{-3} \text{ kg})(0.3 \text{ m})^2 + (21.0600 \text{ kg}) \left(\frac{0.3}{3} \text{ m} \right)^2 \right] \\
 &\quad + \left\{ \frac{1}{18} (21.0600 \times 10^{-3} \text{ kg})[(0.3)^2 + (0.12)^2] \text{ m}^2 \right. \\
 &\quad \left. + (21.0600 \times 10^{-3} \text{ kg}) \left[\left(\frac{0.3}{3} \right)^2 + \left(\frac{0.12}{3} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad + \left[\frac{1}{4} (13.2324 \times 10^{-3} \text{ kg})(0.12 \text{ m})^2 \right] \\
 &= [(105.300 + 210.600) + (122.148 + 244.296) \\
 &\quad + (47.637)] \times 10^{-6} \text{ kg} \cdot \text{m}^2 \\
 &= (315.900 + 366.444 + 47.637) \times 10^{-6} \text{ kg} \cdot \text{m}^2 \\
 &\quad \text{or } I_y = 730 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$

Symmetry implies $I_y = I_z$

$$I_z = 730 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



PROBLEM 9.140*

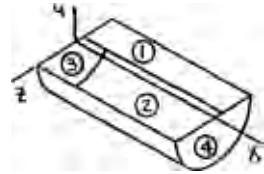
A farmer constructs a trough by welding a rectangular piece of 2 mm-thick sheet steel to half of a steel drum. Knowing that the density of steel is 7850 kg/m^3 and that the thickness of the walls of the drum is 1.8 mm, determine the mass moment of inertia of the trough with respect to each of the coordinate axes. Neglect the mass of the welds.

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{ST}V = \rho_{ST}tA$$

Then



$$\begin{aligned} m_1 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})(0.84 \times 0.21) \text{ m}^2 \\ &= 2.76948 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= (7850 \text{ kg/m}^3)(0.0018 \text{ m})(\pi \times 0.285 \times 0.84) \text{ m}^2 \\ &= 10.62713 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 = m_4 &= (7850 \text{ kg/m}^3)(0.0018 \text{ m})\left(\frac{\pi}{2} \times 0.285^2\right) \text{ m}^2 \\ &= 1.80282 \text{ kg} \end{aligned}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 through 4, we have

$$\begin{aligned} (I_x)_3 &= (I_x)_4 \\ I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 \\ &= \left[\frac{1}{12}(2.76948 \text{ kg})(0.21 \text{ m})^2 + (2.76948 \text{ kg})\left(0.285 - \frac{0.21}{2}\right)^2 \text{ m}^2 \right] \\ &\quad + [(10.62713 \text{ kg})(0.285 \text{ m})^2] + 2\left[\frac{1}{2}(1.80282 \text{ kg})(0.285 \text{ m})^2\right] \\ &= [(0.01018 + 0.08973) + (0.86319) + 2(0.07322)] \text{ kg} \cdot \text{m}^2 \\ &= [(0.09991 + 0.86319 + 2(0.07322))] \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_x = 1.110 \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

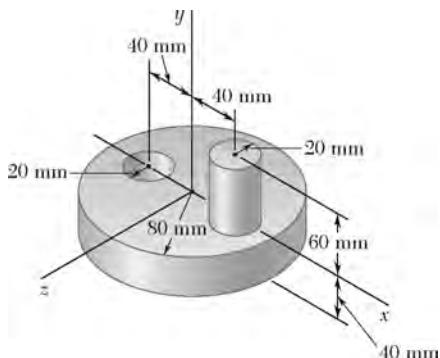
PROBLEM 9.140* (Continued)

$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 \\
 &= \left\{ \frac{1}{12} (2.76948 \text{ kg}) [(0.84)^2 + (0.21)^2] \text{ m}^2 \right. \\
 &\quad + (2.76948 \text{ kg}) \left[\left(\frac{0.84}{2} \right)^2 + \left(0.285 - \frac{0.21}{2} \right)^2 \right] \text{ m}^2 \Bigg\} \\
 &\quad + \left\{ \frac{1}{12} (10.62713 \text{ kg}) [(0.84)^2 + 6(0.285)^2] \text{ m}^2 + (10.62713 \text{ kg}) \left(\frac{0.84}{2} \text{ m} \right)^2 \right\} \\
 &\quad + \left[\frac{1}{4} (1.80282 \text{ kg}) (0.285 \text{ m})^2 \right] \\
 &\quad + \left[\frac{1}{4} (1.80282 \text{ kg}) (0.285 \text{ m})^2 + (1.80282 \text{ kg}) (0.84 \text{ m})^2 \right] \\
 &= [(0.17302 + 0.57827) + (1.05647 + 1.87463) \\
 &\quad + (0.03661) + (0.03661 + 1.27207)] \text{ kg} \cdot \text{m}^2 \\
 &= (0.75129 + 2.93110 + 0.03661 + 1.30868) \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 5.03 \text{ kg} \cdot \text{m}^2$ ◀

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4 \\
 &= \left[\frac{1}{12} (2.76948 \text{ kg}) (0.84 \text{ m})^2 + (2.76948 \text{ kg}) \left(\frac{0.84}{2} \text{ m} \right)^2 \right] \\
 &\quad + \left\{ \frac{1}{12} (10.62713 \text{ kg}) [(0.84)^2 + 6(0.285)^2] \text{ m}^2 + (10.62713 \text{ kg}) \left(\frac{0.84}{2} \text{ m} \right)^2 \right\} \\
 &\quad + \left[\frac{1}{4} (1.80282 \text{ kg}) (0.285 \text{ m})^2 \right] \\
 &\quad + \left[\frac{1}{4} (1.80282 \text{ kg}) (0.285 \text{ m})^2 + (1.80282 \text{ kg}) (0.84 \text{ m})^2 \right] \\
 &= [(0.16285 + 0.48854) + (1.05647 + 1.87463) \\
 &\quad + (0.03661) + (0.03661 + 1.27207)] \text{ kg} \cdot \text{m}^2 \\
 &= (0.65139 + 2.93110 + 0.03661 + 1.30868) \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_z = 4.93 \text{ kg} \cdot \text{m}^2$ ◀



PROBLEM 9.141

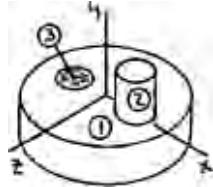
The machine element shown is fabricated from steel. Determine the mass moment of inertia of the assembly with respect to (a) the x axis, (b) the y axis, (c) the z axis. (The density of steel is 7850 kg/m^3 .)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then



$$m_1 = (7850 \text{ kg/m}^3)[(\pi(0.08 \text{ m})^2)(0.04 \text{ m})] \\ = 6.31334 \text{ kg}$$

$$m_2 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.06 \text{ m})] = 0.59188 \text{ kg}$$

$$m_3 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.04 \text{ m})] = 0.39458 \text{ kg}$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$(a) \quad I_x = (I_x)_1 + (I_x)_2 - (I_x)_3 \\ = \left\{ \frac{1}{12}(6.31334 \text{ kg})[3(0.08)^2 + (0.04)^2] \text{ m}^2 + (6.31334 \text{ kg})(0.02 \text{ m})^2 \right\} \\ + \left\{ \frac{1}{12}(0.59188 \text{ kg})[3(0.02)^2 + (0.06)^2] \text{ m}^2 + (0.59188 \text{ kg})(0.03 \text{ m})^2 \right\} \\ - \left\{ \frac{1}{12}(0.39458 \text{ kg})[3(0.02)^2 + (0.04)^2] \text{ m}^2 + (0.39458 \text{ kg})(0.02 \text{ m})^2 \right\} \\ = [(10.94312 + 2.52534) + (0.23675 + 0.53269) \\ - (0.09207 + 0.15783)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ = (13.46846 + 0.76944 - 0.24990) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ = 13.98800 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_x = 13.99 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.141 (Continued)

(b)

$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 - (I_y)_3 \\
 &= \left[\frac{1}{2} (6.31334 \text{ kg}) (0.08 \text{ m})^2 \right] \\
 &\quad + \left[\frac{1}{2} (0.59188 \text{ kg}) (0.02 \text{ m})^2 + (0.59188 \text{ kg}) (0.04 \text{ m})^2 \right] \\
 &\quad - \left[\frac{1}{2} (0.39458 \text{ kg}) (0.02 \text{ m})^2 + (0.39458 \text{ kg}) (0.04 \text{ m})^2 \right] \\
 &= [(20.20269) + (0.11838 + 0.94701) \\
 &\quad - (0.07892 + 0.63133)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (20.20269 + 1.06539 - 0.71025) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 20.55783 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 20.6 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

(c)

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 - (I_z)_3 \\
 &= \left\{ \frac{1}{12} (6.31334 \text{ kg}) [3(0.08)^2 + (0.04)^2] \text{ m}^2 + (6.31334 \text{ kg}) (0.02 \text{ m})^2 \right\} \\
 &\quad + \left\{ \frac{1}{12} (0.59188 \text{ kg}) [3(0.02)^2 + (0.06)^2] \text{ m}^2 + (0.59188 \text{ kg}) [(0.04)^2 + (0.03)^2] \text{ m}^2 \right\} \\
 &\quad - \left\{ \frac{1}{12} (0.39458 \text{ kg}) [3(0.02)^2 + (0.04)^2] \text{ m}^2 + (0.03958 \text{ kg}) [(0.04)^2 + (0.02)^2] \text{ m}^2 \right\} \\
 &= [(10.94312 + 2.52534) + (0.23675 + 1.47970) \\
 &\quad - (0.09207 + 0.78916)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (13.46846 + 1.71645 - 0.88123) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_z = 14.30 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

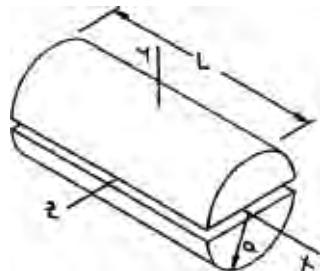
To the Instructor:

The following formulas for the mass of inertia of a semicylinder are derived at this time for use in the solutions of Problems 9.142 through 9.145.

From Figure 9.28:

Cylinder $(I_x)_{\text{cyl}} = \frac{1}{2} m_{\text{cyl}} a^2$

$$(I_y)_{\text{cyl}} = (I_z)_{\text{cyl}} = \frac{1}{12} m_{\text{cyl}} (3a^2 + L^2)$$



PROBLEM 9.141 (Continued)

Symmetry and the definition of the mass moment of inertia ($I = \int r^2 dm$) imply

$$(I)_{\text{semicylinder}} = \frac{1}{2}(I)_{\text{cylinder}}$$

$$(I_x)_{\text{sc}} = \frac{1}{2} \left(\frac{1}{2} m_{\text{cyl}} a^2 \right)$$

and

$$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{2} \left[\frac{1}{12} m_{\text{cyl}} (3a^2 + L^2) \right]$$

However,

$$m_{\text{sc}} = \frac{1}{2} m_{\text{cyl}}$$

Thus,

$$(I_x)_{\text{sc}} = \frac{1}{2} m_{\text{sc}} a^2$$

and

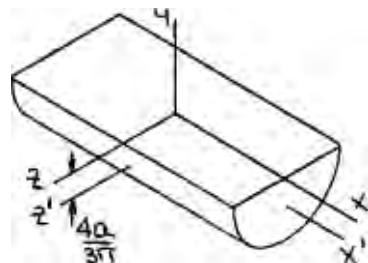
$$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{12} m_{\text{sc}} (3a^2 + L^2)$$

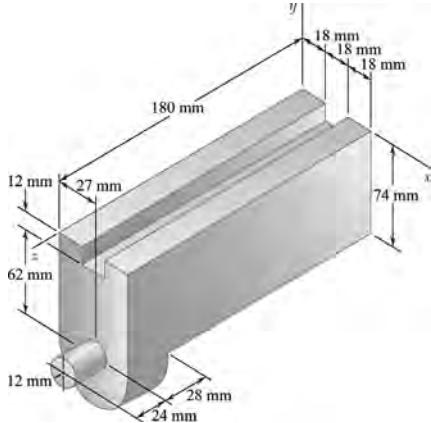
Also, using the parallel axis theorem find

$$\bar{I}_{x'} = m_{\text{sc}} \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) a^2$$

$$\bar{I}_{z'} = m_{\text{sc}} \left[\left(\frac{1}{4} - \frac{16}{9\pi^2} \right) a^2 + \frac{1}{12} L^2 \right]$$

where x' and z' are centroidal axes.





PROBLEM 9.142

Determine the mass moment of inertia of the steel machine element shown with respect to the y axis. (The density of steel is 7850 kg/m^3 .)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{ST} V$$

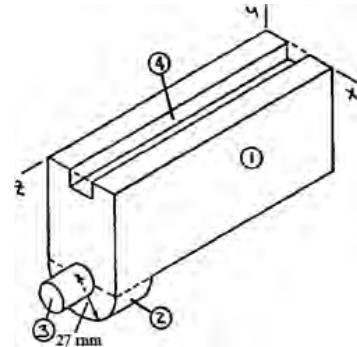
Then

$$\begin{aligned} m_1 &= 7850 \text{ kg/m}^3 \times (0.18 \times 0.074 \times 0.054) \text{ m}^3 \\ &= 5.6463 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= 7850 \text{ kg/m}^3 \left[\frac{\pi}{2} \times (0.027)^2 \times (0.028) \right] \text{ m}^3 \\ &= 0.25170 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 &= 7850 \text{ kg/m}^3 \times [\pi \times (0.012)^2 \times (0.024)] \text{ m}^3 \\ &= 0.085230 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_4 &= 7850 \text{ kg/m}^3 \times (0.018 \times 0.18 \times 0.012) \text{ m}^3 \\ &= 0.30521 \text{ kg} \end{aligned}$$



The mass moments of inertia are now computed using Figure 9.28 (components 1, 3, and 4) and the equations derived above (component 2).

PROBLEM 9.142 (Continued)

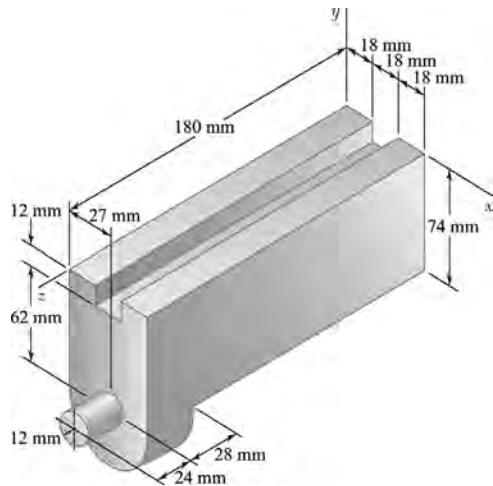
Find: I_y

We have

$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 - (I_y)_4 \\
 &= \left\{ \frac{1}{12} (5.6463 \text{ kg}) \left[(0.054)^2 + (0.18)^2 \right] \text{m}^2 \right. \\
 &\quad + (5.6463 \text{ kg}) \left[\left(\frac{0.054}{2} \right)^2 + \left(\frac{0.18}{2} \right)^2 \right] \text{m}^2 \Big\} \\
 &\quad + \left\{ \frac{1}{12} (0.25170 \text{ kg}) [3(0.027)^2 + (0.028)^2] \text{m}^2 \right. \\
 &\quad + (0.25170 \text{ kg}) \left[(0.027)^2 + \left(0.18 - \frac{0.028}{2} \right)^2 \right] \text{m}^2 \Big\} \\
 &\quad + \left\{ \frac{1}{12} (0.08523 \text{ kg}) [3(0.012)^2 + (0.024)^2] \text{m}^2 \right. \\
 &\quad + (0.08523 \text{ kg}) \left[(0.027)^2 + \left(0.18 + \frac{0.024}{2} \right)^2 \right] \text{m}^2 \Big\} \\
 &\quad - \left\{ \frac{1}{12} (0.30521 \text{ kg}) [(0.018)^2 + (0.18)^2] \text{m}^2 \right. \\
 &\quad + (0.30521 \text{ kg}) \left[(0.027)^2 + \left(\frac{0.180}{2} \right)^2 \right] \text{m}^2 \Big\} \\
 &= [(16617.1 + 49851.2) + (62.317 + 7119.3) \\
 &\quad + (7.1593 + 3204.1) - (832.31 + 2694.7)] \times 10^{-6} \text{ kg} \cdot \text{m}^2 \\
 &= 73.334 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 73.3 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$

PROBLEM 9.143



Determine the mass moment of inertia of the steel machine element shown with respect to the z axis. (The density of steel is 7850 kg/m^3 .)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

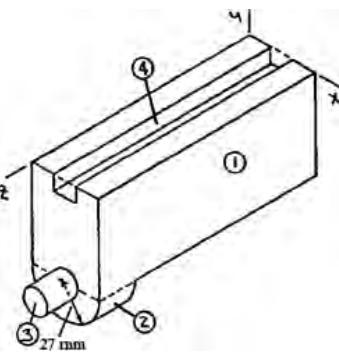
Then

$$\begin{aligned} m_1 &= 7850 \text{ kg/m}^3 \times (0.18 \times 0.074 \times 0.054) \text{ m}^3 \\ &= 5.6463 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= 7850 \text{ kg/m}^3 \left[\frac{\pi}{2} \times (0.027)^2 \times (0.028) \right] \text{ m}^3 \\ &= 0.25170 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 &= 7850 \text{ kg/m}^3 \times [\pi \times (0.012)^2 \times (0.024)] \text{ m}^3 \\ &= 0.085230 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_4 &= 7850 \text{ kg/m}^3 \times (0.018 \times 0.18 \times 0.012) \text{ m}^3 \\ &= 0.30521 \text{ kg} \end{aligned}$$



The mass moments of inertia are now computed using Figure 9.28 (components 1, 3, and 4) and the equations derived above (component 2).

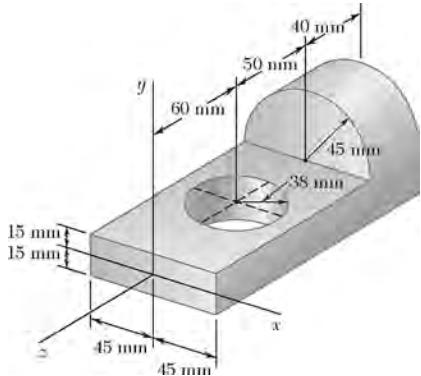
PROBLEM 9.143 (Continued)

Find: I_z

We have

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 - (I_z)_4 \\
 &= \left\{ \frac{1}{12} (5.6463 \text{ kg}) \left[(0.054)^2 + (0.074)^2 \right] \text{ m}^2 \right. \\
 &\quad + (5.6463 \text{ kg}) \left[\left(\frac{0.054}{2} \right)^2 + \left(\frac{0.074}{2} \right)^2 \right] \text{ m}^2 \Big\} \\
 &\quad + \left\{ (0.2517 \text{ kg}) \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) (0.027)^2 \text{ m}^2 \right. \\
 &\quad + (0.2517 \text{ kg}) \left[(0.027)^2 + \left(0.074 + \frac{4 \times 0.027}{3\pi} \right)^2 \right] \text{ m}^2 \Big\} \\
 &\quad + \left\{ \frac{1}{2} (0.08523 \text{ kg}) (0.012 \text{ m})^2 \right. \\
 &\quad + (0.08523 \text{ kg}) [(0.027)^2 + (0.074)^2] \text{ m}^2 \Big\} \\
 &\quad - \left\{ \frac{1}{2} (0.30521 \text{ kg}) [(0.018)^2 + (0.012)^2] \text{ m}^2 \right. \\
 &\quad + (0.30521 \text{ kg}) \left[(0.027)^2 + \left(\frac{0.012}{2} \right)^2 \right] \text{ m}^2 \Big\} \\
 I_z &= [(3948.6 + 11845.9) + (58.693 + 2021.7) \\
 &\quad + (6.1366 + 528.85) - (11.903 + 233.49)] \times 10^{-6} \text{ kg} \cdot \text{m}^2 \\
 &= 18.1645 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_z = 18.16 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀



PROBLEM 9.144

Determine the mass moment of inertia and the radius of gyration of the steel machine element shown with respect to the x axis. (The density of steel is 7850 kg/m^3 .)

SOLUTION

First compute the mass of each component. We have

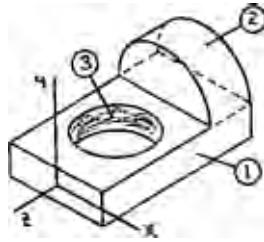
$$m = \rho_{ST}V$$

Then

$$\begin{aligned} m_1 &= (7850 \text{ kg/m}^3)(0.09 \times 0.03 \times 0.15) \text{ m}^3 \\ &= 3.17925 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= (7850 \text{ kg/m}^3) \left[\frac{\pi}{2} (0.045)^2 \times 0.04 \right] \text{ m}^3 \\ &= 0.998791 \text{ kg} \end{aligned}$$

$$m_3 = (7850 \text{ kg/m}^3)[\pi(0.038)^2 \times 0.03] \text{ m}^3 = 1.06834 \text{ kg}$$



Using Figure 9.28 for components 1 and 3 and the equation derived above (before the solution to Problem 9.142) for a semicylinder, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 - (I_x)_3 \\ &= \left[\frac{1}{12}(3.17925 \text{ kg})(0.03^2 + 0.15^2) \text{ m}^2 + (3.17925 \text{ kg})(0.075 \text{ m})^2 \right] \\ &\quad + \left\{ (0.998791 \text{ kg}) \left[\left(\frac{1}{4} - \frac{16}{9\pi^2} \right) (0.045 \text{ m})^2 + \frac{1}{12}(0.04 \text{ m})^2 \right] \right. \\ &\quad \left. + (0.998791 \text{ kg}) \left[(0.13)^2 + \left(\frac{4 \times 0.045}{3\pi} + 0.015 \right)^2 \right] \text{ m}^2 \right\} \\ &\quad - \left\{ \frac{1}{12}(1.06834 \text{ kg})[3(0.038 \text{ m})^2 + (0.03 \text{ m})^2] + (1.06834 \text{ kg})(0.06 \text{ m})^2 \right\} \end{aligned}$$

PROBLEM 9.144 (Continued)

$$\begin{aligned} &= [(0.0061995 + 0.0178833) + (0.0002745 + 0.0180409) \\ &\quad - (0.0004658 + 0.0038460)] \text{ kg} \cdot \text{m}^2 \\ &= (0.0240828 + 0.0183154 - 0.0043118) \text{ kg} \cdot \text{m}^2 \\ &= 0.0380864 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

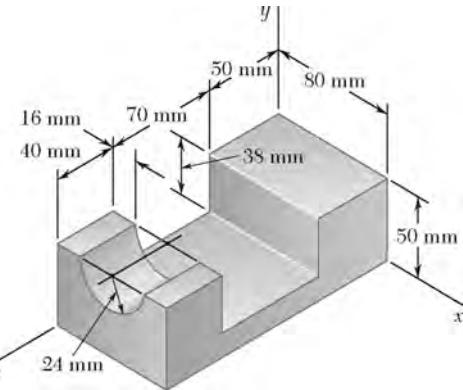
$$\text{or } I_x = 38.1 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

Now $m = m_1 + m_2 - m_3 = (3.17925 + 0.998791 - 1.06834) \text{ kg}$
 $= 3.10970 \text{ kg}$

and $k_x^2 = \frac{I_x}{m} = \frac{0.0380864 \text{ kg} \cdot \text{m}^2}{3.10970 \text{ kg}}$

$$\text{or } kx = 110.7 \text{ mm} \blacktriangleleft$$

PROBLEM 9.145



Determine the mass moment of inertia of the steel fixture shown with respect to (a) the x axis, (b) the y axis, (c) the z axis. (The density of steel is 7850 kg/m^3 .)

SOLUTION

First compute the mass of each component. we have

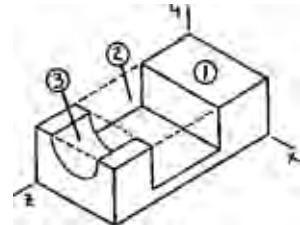
$$m = \rho_{\text{ST}} V$$

Then

$$\begin{aligned} m_1 &= 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.160) \text{ m}^3 \\ &= 5.02400 \text{ kg} \end{aligned}$$

$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$

$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04 \right) \text{ m}^3 = 0.28410 \text{ kg}$$



Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

$$(a) \quad I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$$

$$\begin{aligned} &= \left\{ \frac{1}{12} (5.02400 \text{ kg}) [(0.05)^2 + (0.16)^2] \text{ m}^2 + (5.02400 \text{ kg}) \left[\left(\frac{0.05}{2} \right)^2 + \left(\frac{0.16}{2} \right)^2 \right] \text{ m}^2 \right\} \\ &\quad - \left\{ \frac{1}{12} (1.67048 \text{ kg}) [(0.038)^2 + (0.07)^2] \text{ m}^2 + (1.67048 \text{ kg}) \left[\left(0.05 - \frac{0.038}{2} \right)^2 + \left(0.05 + \frac{0.07}{2} \right)^2 \right] \text{ m}^2 \right\} \end{aligned}$$

$$\begin{aligned} &\quad - \left\{ (0.28410 \text{ kg}) \left[\left(\frac{1}{4} - \frac{16}{9\pi^2} \right) (0.024)^2 + \frac{1}{12} (0.04)^2 \right] \text{ m}^2 \right. \\ &\quad \left. + (0.28410 \text{ kg}) \left[\left(0.05 - \frac{4 \times 0.024}{3\pi} \right)^2 + \left(0.16 - \frac{0.04}{2} \right)^2 \right] \text{ m}^2 \right\} \end{aligned}$$

$$= [(11.7645 + 35.2936) - (0.8831 + 13.6745) - (0.0493 + 6.0187)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (47.0581 - 14.5576 - 6.0680) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= 26.4325 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_x = 26.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.145 (Continued)

$$\begin{aligned}
 (b) \quad I_y &= (I_y)_1 - (I_y)_2 - (I_y)_3 \\
 &= \left\{ \frac{1}{12} (5.02400 \text{ kg}) [(0.08)^2 + (0.16)^2] \text{ m}^2 + (5.02400 \text{ kg}) \left[\left(\frac{0.08}{2} \right)^2 + \left(\frac{0.16}{2} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad - \left\{ \frac{1}{12} (1.67048 \text{ kg}) [(0.08)^2 + (0.07)^2] \text{ m}^2 + (1.67048 \text{ kg}) \left[\left(\frac{0.08}{2} \right)^2 + \left(0.05 + \frac{0.07}{2} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad - \left\{ \frac{1}{12} (0.28410 \text{ kg}) [3(0.024)^2 + (0.04)^2] \text{ m}^2 + (0.28410 \text{ kg}) \left[\left(\frac{0.08}{2} \right)^2 + \left(0.16 - \frac{0.04}{2} \right)^2 \right] \text{ m}^2 \right\} \\
 &= [(13.3973 + 40.1920) - (1.5730 + 14.7420) - (0.0788 + 6.0229)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (53.5893 - 16.3150 - 6.1017) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 31.1726 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \quad \text{or } I_y = 31.2 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad I_z &= (I_z)_1 - (I_z)_2 - (I_z)_3 \\
 &= \left\{ \frac{1}{12} (5.02400 \text{ kg}) [(0.08)^2 + (0.05)^2] \text{ m}^2 + (5.02400 \text{ kg}) \left[\left(\frac{0.08}{2} \right)^2 + \left(\frac{0.05}{2} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad - \left\{ \frac{1}{12} (1.67048 \text{ kg}) [(0.08)^2 + (0.038)^2] \text{ m}^2 + (1.67048 \text{ kg}) \left[\left(\frac{0.08}{2} \right)^2 + \left(0.05 - \frac{0.038}{2} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad - \left\{ (0.28410 \text{ kg}) \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) (0.024 \text{ m})^2 + (0.28410 \text{ kg}) \left[\left(\frac{0.08}{2} \right)^2 + \left(0.05 - \frac{4 \times 0.024}{3\pi} \right)^2 \right] \text{ m}^2 \right\} \\
 &= [(3.7261 + 11.1784) - (1.0919 + 4.2781) - (0.0523 + 0.9049)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (14.9045 - 5.3700 - 0.9572) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 8.5773 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \quad \text{or } I_z = 8.58 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$

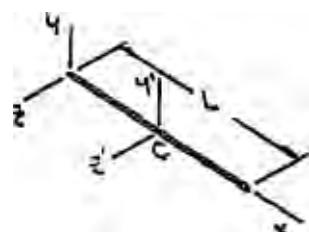
To the instructor:

The following formulas for the mass moment of inertia of wires are derived or summarized at this time for use in the solutions of Problems 9.146 through 9.148.

Slender Rod

$$I_x = 0 \quad \bar{I}_{y'} = \bar{I}_{z'} = \frac{1}{12} mL^2 \text{ (Figure 9.28)}$$

$$I_y = I_z = \frac{1}{3} mL^2 \text{ (Sample Problem 9.9)}$$



PROBLEM 9.145 (Continued)

Circle

We have

$$\bar{I}_y = \int r^2 dm = ma^2$$

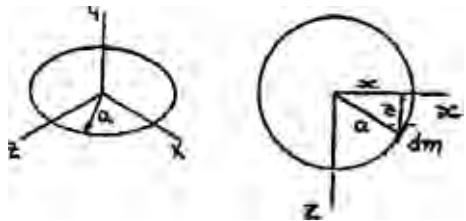
Now

$$\bar{I}_y = \bar{I}_x + \bar{I}_z$$

And symmetry implies

$$\bar{I}_x = \bar{I}_z$$

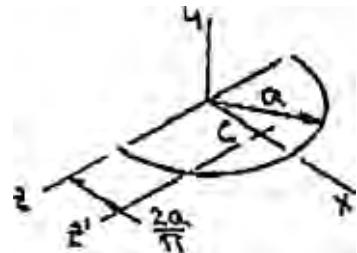
$$\bar{I}_x = \bar{I}_z = \frac{1}{2}ma^2$$



Semicircle

Following the above arguments for a circle, we have

$$\bar{I}_x = I_z = \frac{1}{2}ma^2 \quad I_y = ma^2$$

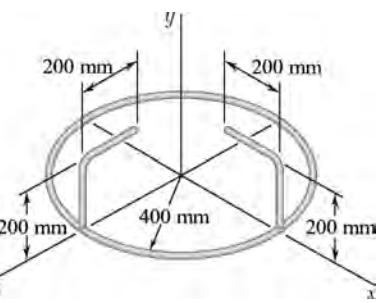


Using the parallel-axis theorem

$$I_z = \bar{I}_{z'} + m\bar{x}^2 \quad x = \frac{2a}{\pi}$$

or

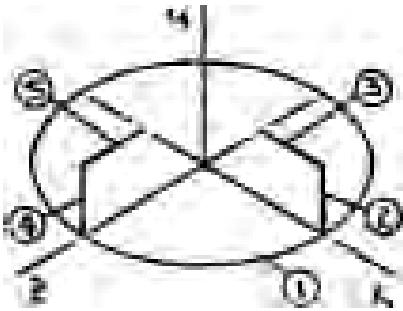
$$I_{z'} = m\left(\frac{1}{2} - \frac{4}{\pi^2}\right)a^2$$



PROBLEM 9.146

Aluminum wire with a weight per unit length of 0.05 kg/m is used to form the circle and the straight members of the figure shown. Determine the mass moment of inertia of the assembly with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component.

$$\text{Have } m = \rho L$$

$$\begin{aligned} \text{Then } m_1 &= (0.05 \text{ kg/m})[2\pi(0.4 \text{ m})] \\ &= 0.125664 \text{ kg} \\ m_2 &= m_3 = m_4 = m_5 \\ &= (0.05 \text{ kg/m})(0.2 \text{ m}) = 0.01 \text{ kg} \end{aligned}$$

Using the equations given above and the parallel-axis theorem, have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 + (I_x)_5 \\ &= (0.125664 \text{ kg}) \left[\left(\frac{1}{2}\right)(0.4 \text{ m})^2 \right] + (0.01 \text{ kg}) \left[\left(\frac{1}{3}\right)(0.2 \text{ m})^2 \right] \\ &\quad + (0.01 \text{ kg}) \left[0 + (0.2 \text{ m})^2 \right] \\ &\quad + (0.01 \text{ kg}) \left[\left(\frac{1}{12}\right)(0.2 \text{ m})^2 + (0.1 \text{ m})^2 + (0.4 \text{ m})^2 \right] \\ &\quad + (0.01 \text{ kg}) \left[\left(\frac{1}{12}\right)(0.2 \text{ m})^2 + (0.2 \text{ m})^2 + (0.4 \text{ m} - 0.1 \text{ m})^2 \right] \\ &= [(10.053 + 0.13333 + 0.4 + 1.7333 + 1.3333) \times 10^{-3}] \text{ kg} \cdot \text{m}^2 \\ &= 13.6529 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_x = 13.65 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.146 (Continued)

Have

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5$$

where

$$(I_y)_2 = (I_y)_4 \quad \text{and} \quad (I_y)_3 = (I_y)_5$$

Then

$$I_y = (0.125664 \text{ kg})[(0.4 \text{ m})^2] + 2(0.01 \text{ kg})[0 + (0.4 \text{ m})^2]$$

$$+ 2(0.01 \text{ kg})\left[\frac{1}{12}(0.2 \text{ m})^2 + (0.3 \text{ m})^2\right]$$

$$= [20.106 + 2(1.6) + 2(0.9333)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

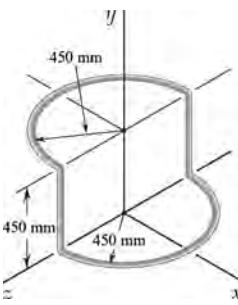
$$= 25.1726 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_y = 25.2 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

By symmetry

$$I_z = I_x$$

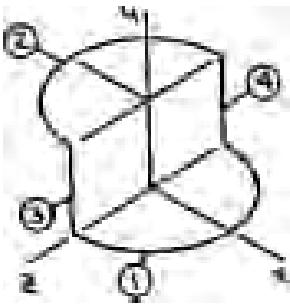
$$\text{or } I_z = 13.65 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



PROBLEM 9.147

The figure shown is formed of 3-mm steel wire. Knowing that the density of the steel is 7850 kg/m^3 determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION



Have

$$m = \rho V = \rho AL$$

Then

$$m_1 = m_2 = (7850 \text{ kg/m}^3) [\pi(0.0015 \text{ m})^2] \times (\pi \times 0.45 \text{ m})$$

$$m_2 = m_1 = 0.078445 \text{ kg}$$

$$\begin{aligned} m_3 = m_4 &= (7850 \text{ kg/m}^3) [\pi(0.0015 \text{ m})^2] \times (0.45 \text{ m}) \\ &= 0.024970 \text{ kg} \end{aligned}$$

Using the equations given above and the parallel-axis theorem, have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4$$

where

$$(I_x)_3 = (I_x)_4$$

Then

$$\begin{aligned} I_x &= (0.078445 \text{ kg}) \left[\frac{1}{2} (0.45 \text{ m})^2 \right] + (0.078445 \text{ kg}) \left[\frac{1}{2} (0.45 \text{ m})^2 + (0.45 \text{ m})^2 \right] \\ &\quad + 2(0.02497) \left[\frac{1}{12} (0.45 \text{ m})^2 + (0.225 \text{ m})^2 + (0.45 \text{ m})^2 \right] \\ &= [794256 + 23.8277 + 2(6.7419)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ &= 45.254 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_x = 45.3 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

Have

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4$$

where

$$(I_y)_1 = (I_y)_2 \quad \text{and} \quad (I_y)_3 = (I_y)_4$$

PROBLEM 9.147 (Continued)

Then

$$\begin{aligned}
 I_y &= 2(0.078445 \text{ kg})[(0.45 \text{ m})^2] + 2(0.02497 \text{ kg})[0 + (0.45 \text{ m})^2] \\
 &= 2(15.8851 \times 10^{-3} \text{ kg} \cdot \text{m}^2) + 2(5.0564 \times 10^{-3} \text{ kg} \cdot \text{m}^2) \\
 &= 41.883 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 41.9 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

Have

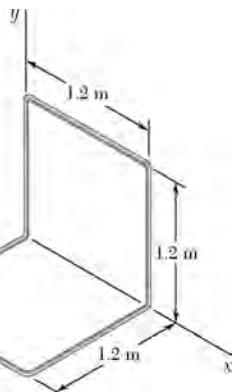
$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4$$

where $(I_z)_3 = (I_z)_4$

Then

$$\begin{aligned}
 I_z &= (0.078445 \text{ kg}) \left[\frac{1}{2}(0.45 \text{ m})^2 \right] \\
 &\quad + (0.078445 \text{ kg}) \left[\left(\frac{1}{2} - \frac{4}{\pi^2} \right)(0.45 \text{ m})^2 + \left(\frac{2 \times 0.45 \text{ m}}{\pi} \right)^2 + (0.45 \text{ m})^2 \right] \\
 &\quad + 2(0.02497 \text{ kg}) + \left[\frac{1}{3}(0.45 \text{ m})^2 \right] \\
 &= [7.94256 + 23.8277 + 2(1.6855)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 35.141 \times 10^{-3} \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_z = 35.1 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀



PROBLEM 9.148

A homogeneous wire with a mass per unit length of 0.056 kg/m is used to form the figure shown. Determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION

First compute the mass m of each component. We have

$$\begin{aligned}m &= (m/L)L \\&= 0.056 \text{ kg/m} \times 1.2 \text{ m} \\&= 0.0672 \text{ kg}\end{aligned}$$

Using the equations given above and the parallel-axis theorem, we have

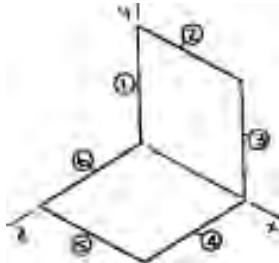
$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 + (I_x)_5 + (I_x)_6$$

Now

$$(I_x)_1 = (I_x)_3 = (I_x)_4 = (I_x)_6 \quad \text{and} \quad (I_x)_2 = (I_x)_5$$

Then

$$\begin{aligned}I_x &= 4\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + 2[0 + (0.0672 \text{ kg})(1.2 \text{ m})^2] \\&= [4(0.03226) + 2(0.09677)] \text{ kg} \cdot \text{m}^2 \\&= 0.32258 \text{ kg} \cdot \text{m}^2 \quad \text{or} \quad I_x = 0.323 \text{ kg} \cdot \text{m}^2 \blacktriangleleft\end{aligned}$$



$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5 + (I_y)_6$$

Now

$$(I_y)_1 = 0, \quad (I_y)_2 = (I_y)_6, \quad \text{and} \quad (I_y)_4 = (I_y)_5$$

Then

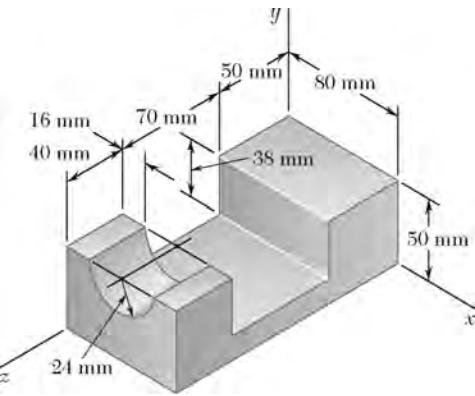
$$\begin{aligned}I_y &= 2\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + [0 + (0.0672 \text{ kg})(1.2 \text{ m})^2] \\&\quad + 2\left[\frac{1}{12}(0.0672 \text{ kg})(1.2 \text{ m})^2 + (0.0672 \text{ kg})(1.2^2 + 0.6^2) \text{ m}^2\right] \\&= [2(0.03226) + (0.09677) + 2(0.00806 + 0.12096)] \text{ kg} \cdot \text{m}^2 \\&= [2(0.03226) + (0.09677) + 2(0.12902)] \text{ kg} \cdot \text{m}^2 \\&= 0.41933 \text{ kg} \cdot \text{m}^2 \quad \text{or} \quad I_y = 0.419 \text{ kg} \cdot \text{m}^2 \blacktriangleleft\end{aligned}$$

Symmetry implies

$$I_y = I_z$$

$$I_z = 0.419 \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.149



Determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the steel fixture shown. (The density of steel is 7850 kg/m^3 .)

SOLUTION

First compute the mass of each component. We have

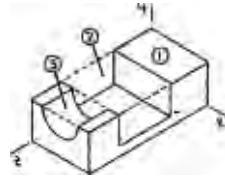
$$m = \rho V$$

Then

$$m_1 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.16) \text{ m}^3 = 5.02400 \text{ kg}$$

$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$

$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04 \right) \text{ m}^3 = 0.28410 \text{ kg}$$



Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{y}_2 = \left(0.05 - \frac{0.038}{2} \right) \text{ m} = 0.031 \text{ m}$$

$$\bar{y}_3 = \left(0.05 - \frac{4 \times 0.024}{3\pi} \right) \text{ m} = 0.039814 \text{ m}$$

and then

	m , kg	\bar{x} , m	\bar{y} , m	\bar{z} , m	$m\bar{x}\bar{y}$, $\text{kg} \cdot \text{m}^2$	$m\bar{y}\bar{z}$, $\text{kg} \cdot \text{m}^2$	$m\bar{z}\bar{x}$, $\text{kg} \cdot \text{m}^2$
1	5.02400	0.04	0.025	0.08	5.0240×10^{-3}	10.0480×10^{-3}	16.0768×10^{-3}
2	1.67048	0.04	0.031	0.085	2.0714×10^{-3}	4.4017×10^{-3}	5.6796×10^{-3}
3	0.28410	0.04	0.039814	0.14	0.4524×10^{-3}	1.5836×10^{-3}	1.5910×10^{-3}

Finally,

$$I_{xy} = (I_{xy})_1 - (I_{xy})_2 - (I_{xy})_3 = [(0 + 5.0240) - (0 + 2.0714) - (0 + 0.4524)] \times 10^{-3}$$

$$= 2.5002 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_{xy} = 2.50 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.149 (Continued)

$$I_{yz} = (I_{yz})_1 - (I_{yz})_2 - (I_{yz})_3 = [(0 + 10.0480) - (0 + 4.4017) - (0 + 1.5836)] \times 10^{-3}$$

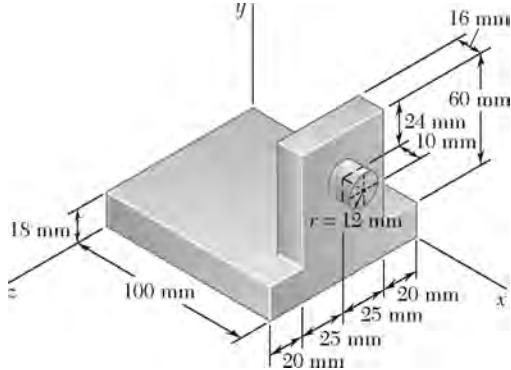
$$= 4.0627 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_{yz} = 4.06 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{zx} = (I_{zx})_1 - (I_{zx})_2 - (I_{zx})_3 = [(0 + 16.0768) - (0 + 5.6796) - (0 + 1.5910)] \times 10^{-3}$$

$$= 8.8062 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_{zx} = 8.81 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



PROBLEM 9.150

Determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the steel machine element shown. (The density of steel is 7850 kg/m^3 .)

SOLUTION

First compute the mass of each component. We have

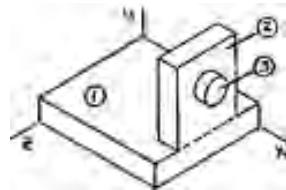
$$m = \rho_{ST}V$$

Then

$$m_1 = 7850 \text{ kg/m}^3 \times (0.1 \times 0.018 \times 0.09) \text{ m}^3 = 1.27170 \text{ kg}$$

$$m_2 = 7850 \text{ kg/m}^3 \times (0.016 \times 0.06 \times 0.05) \text{ m}^3 = 0.37680 \text{ kg}$$

$$m_3 = 7850 \text{ kg/m}^3 \times (\pi \times 0.012^2 \times 0.01) \text{ m}^3 = 0.03551 \text{ kg}$$



Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

	$m, \text{ kg}$	$\bar{x}, \text{ m}$	$\bar{y}, \text{ m}$	$\bar{z}, \text{ m}$	$m\bar{x}\bar{y}, \text{ kg} \cdot \text{m}^2$	$m\bar{y}\bar{z}, \text{ kg} \cdot \text{m}^2$	$m\bar{z}\bar{x}, \text{ kg} \cdot \text{m}^2$
1	1.27170	0.05	0.009	0.045	0.57227×10^{-3}	0.51504×10^{-3}	2.86133×10^{-3}
2	0.37680	0.092	0.048	0.045	1.66395×10^{-3}	0.81389×10^{-3}	1.55995×10^{-3}
3	0.03551	0.105	0.054	0.045	0.20134×10^{-3}	0.08629×10^{-3}	0.16778×10^{-3}
Σ					2.43756×10^{-3}	1.41522×10^{-3}	4.58906×10^{-3}

Then

$$I_{xy} = \sum(\bar{I}_{x'y'} + m\bar{x}\bar{y})$$

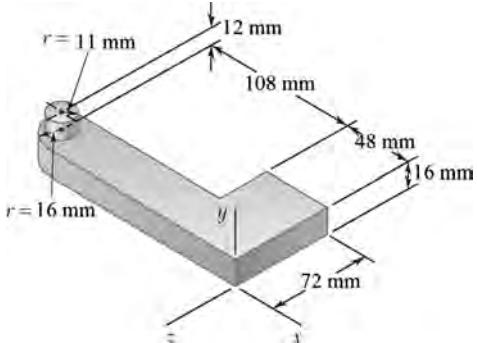
$$\text{or } I_{xy} = 2.44 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{yz} = \sum(\bar{I}_{y'z'} + m\bar{y}\bar{z})$$

$$\text{or } I_{yz} = 1.415 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{zx} = \sum(\bar{I}_{z'x'} + m\bar{z}\bar{x})$$

$$\text{or } I_{zx} = 4.59 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



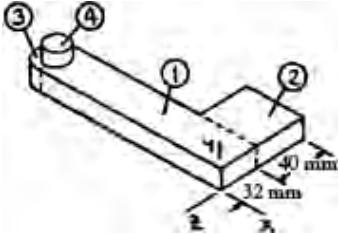
PROBLEM 9.151

Determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 2770 kg/m^3)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{AL} V$$



Then

$$\begin{aligned} m_1 &= 2770 \frac{\text{kg}}{\text{m}^3} \times (0.156 \times 0.016 \times 0.032) \text{ m}^3 \\ &= 221.245 \times 10^{-3} \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= 2770 \frac{\text{kg}}{\text{m}^3} \times (0.048 \times 0.016 \times 0.04) \text{ m}^3 \\ &= 85.0944 \times 10^{-3} \text{ kg} \end{aligned}$$

$$m_3 = 2770 \frac{\text{kg}}{\text{m}^3} \times \left[\frac{\pi}{2} \times (0.016)^2 \times 0.016 \right] \text{ m}^3 = 17.8221 \times 10^{-3} \text{ kg}$$

$$m_4 = 2770 \frac{\text{kg}}{\text{m}^3} \times [\pi \times (0.011)^2 \times 0.012] \text{ m}^3 = 12.6356 \times 10^{-3} \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\begin{aligned} \bar{x}_3 &= -\left(156 + \frac{4 \times 16}{3\pi} \right) \text{ mm} = -162.791 \text{ mm} \\ &= -0.162791 \text{ m} \end{aligned}$$

PROBLEM 9.151 (Continued)

and then

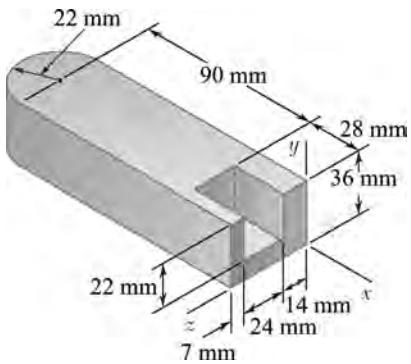
	$m, \text{ kg}$	$\bar{x}, \text{ m}$	$\bar{y}, \text{ m}$	$\bar{z}, \text{ m}$	$m\bar{x}\bar{y}, \text{ kg} \cdot \text{m}^2$	$m\bar{y}\bar{z}, \text{ kg} \cdot \text{m}^2$	$m\bar{z}\bar{x}, \text{ kg} \cdot \text{m}^2$
1	221.245×10^{-3}	-0.078	0.008	-0.016	-138.057×10^{-6}	-28.3194×10^{-6}	276.114×10^{-6}
2	85.0944×10^{-3}	-0.024	0.008	-0.052	-16.3381×10^{-6}	-35.3993×10^{-6}	106.198×10^{-6}
3	17.8221×10^{-3}	-0.1628	0.008	-0.016	-23.2115×10^{-6}	-2.28123×10^{-6}	46.423×10^{-6}
4	12.6356×10^{-3}	-0.156	0.022	-0.016	-43.3654×10^{-6}	-4.44773×10^{-6}	31.5385×10^{-6}
Σ					-220.972×10^{-6}	-70.448×10^{-6}	460.274×10^{-6}

Then

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\bar{y}) \quad \begin{matrix} 0 \\ \nearrow \end{matrix} \quad \text{or} \quad I_{xy} = -221 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \quad \blacktriangleleft$$

$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\bar{y}\bar{z}) \quad \begin{matrix} 0 \\ \nearrow \end{matrix} \quad \text{or} \quad I_{yz} = -70.4 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \quad \blacktriangleleft$$

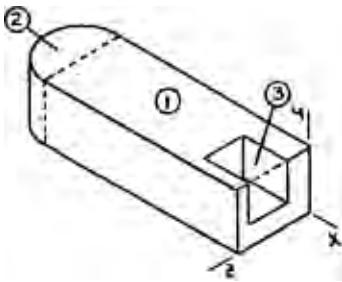
$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\bar{x}) \quad \begin{matrix} 0 \\ \nearrow \end{matrix} \quad \text{or} \quad I_{zx} = 460 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \quad \blacktriangleleft$$



PROBLEM 9.152

Determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the cast aluminum machine component shown. (The density of aluminum is 2770 kg/m^3 .)

SOLUTION



Have

$$m = \rho V$$

Then

$$m_1 = (2770 \text{ kg/m}^3)(0.118 \times 0.036 \times 0.044) \text{ m}^3 = 0.51775 \text{ kg}$$

$$m_2 = (2770 \text{ kg/m}^3) \left[\frac{\pi}{2} (0.022)^2 \times 0.036 \right] \text{ m}^3 = 0.075814 \text{ kg}$$

$$m_3 = (2770 \text{ kg/m}^3)(0.028 \times 0.022 \times 0.024) \text{ m}^3 = 0.04095 \text{ kg}$$

Now observe that $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$ are zero because of symmetry

Now

$$\bar{x}_2 = -\left(0.118 + \frac{4 \times 0.023}{3\pi}\right) \text{ m} = -0.12734 \text{ m}$$

$$\bar{y}_3 = \left(0.036 - \frac{0.062}{2}\right) \text{ m} = 0.025 \text{ m}$$

	$m, \text{ kg}$	$\bar{x}, \text{ m}$	$\bar{y}, \text{ m}$	$\bar{z}, \text{ m}$	$m\bar{x}\bar{y}, \text{ kg} \cdot \text{m}^2$	$m\bar{y}\bar{z}, \text{ kg} \cdot \text{m}^2$	$m\bar{z}\bar{x}, \text{ kg} \cdot \text{m}^2$
1	0.51775	-0.059	0.018	0.022	-0.54985×10^{-3}	0.20503×10^{-3}	-0.67204×10^{-3}
2	0.075814	-0.12734	0.018	0.022	-0.17377×10^{-3}	0.03002×10^{-3}	-0.21239×10^{-3}
3	0.04095 kg	-0.041	0.025	0.026	-0.014198×10^{-3}	0.02662×10^{-3}	-0.04365×10^{-3}

PROBLEM 9.152 (Continued)

And

$$I_{xy} = \Sigma(V'_{x'y'} + m\bar{x}\bar{y})$$

$$I_{yz} = \Sigma(V'_{y'z'} + m\bar{y}\bar{z})$$

$$I_{zx} = \Sigma(V'_{z'x'} + m\bar{z}\bar{x})$$

Finally,

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 - (I_{xy})_3 = -0.70942 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

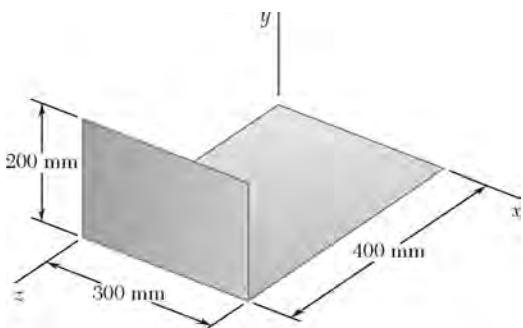
$$\text{or } I_{xy} = -0.709 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{yz} = (I_{yz})_1 + (I_{yz})_2 - (I_{yz})_3 = 0.20843 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_{yz} = 0.208 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{zx} = (I_{zx})_1 + (I_{zx})_2 - (I_{zx})_3 = -0.84078 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

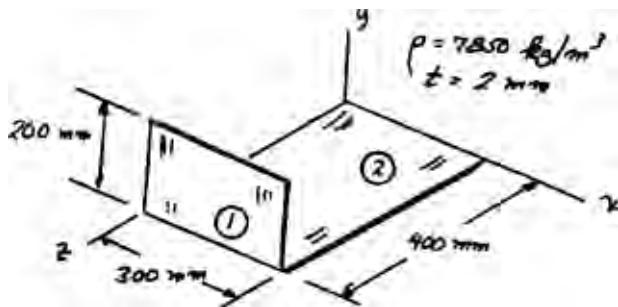
$$\text{or } I_{zx} = -0.841 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



PROBLEM 9.153

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3 , determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the component.

SOLUTION



$$m_1 = \rho V = 7850 \text{ kg/m}^3 (0.2 \text{ m})(0.3 \text{ m})(0.002 \text{ m}) = 0.942 \text{ kg}$$

$$m_2 = \rho V = 7850 \text{ kg/m}^3 (0.4 \text{ m})(0.3 \text{ m})(0.002 \text{ m}) = 1.884 \text{ kg}$$

For each panel the centroidal product of inertia is zero with respect to each pair of coordinate axes.

	$m, \text{ kg}$	$\bar{x}, \text{ m}$	$\bar{y}, \text{ m}$	$\bar{z}, \text{ m}$	$m\bar{x}\bar{y}$	$m\bar{y}\bar{z}$	$m\bar{z}\bar{x}$
①	0.942	0.15	0.1	0.4	$+14.13 \times 10^{-3}$	$+37.68 \times 10^{-3}$	$+56.52 \times 10^{-3}$
②	1.884	0.15	0	0.2	0	0	$+56.52 \times 10^{-3}$
Σ					$+14.13 \times 10^{-3}$	$+37.68 \times 10^{-3}$	$+113.02 \times 10^{-3}$

$$I_{xy} = +14.13 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

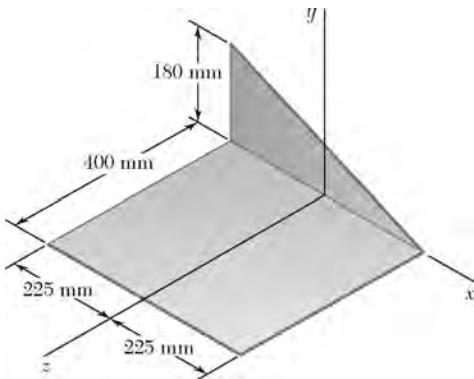
$$I_{xy} = +14.13 \text{ g} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{yz} = +37.68 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_{yz} = +37.7 \text{ g} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{zx} = +113.02 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

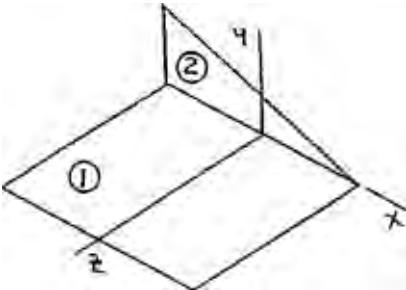
$$I_{zx} = +113.0 \text{ g} \cdot \text{m}^2 \blacktriangleleft$$



PROBLEM 9.154

A section of sheet steel 2-mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3 , determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the component.

SOLUTION



First compute the mass of each component.

We have

$$m = \rho_{ST}V = \rho_{ST}tA$$

Then

$$\begin{aligned} m_1 &= (7850 \text{ kg/m}^3)[(0.002)(0.4)(0.45)]\text{m}^3 \\ &= 2.8260 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= 7850 \text{ kg/m}^3 \left[(0.002) \left(\frac{1}{2} \times 0.45 \times 0.18 \right) \right] \text{m}^3 \\ &= 0.63585 \text{ kg} \end{aligned}$$

Now observe that

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = (\bar{I}_{z'x'})_1 = 0$$

$$(\bar{I}_{y'z'})_2 = (\bar{I}_{z'x'})_2 = 0$$

From Sample Problem 9.6:

$$(\bar{I}_{x'y'})_{2,\text{area}} = -\frac{1}{72} b_2^2 h_2^2$$

Then

$$(\bar{I}_{x'y'})_2 = \rho_{ST}t(I_{x'y'})_{2,\text{area}} = \rho_{ST}t \left(-\frac{1}{72} b_2^2 h_2^2 \right) = -\frac{1}{36} m_2 b_2 h_2$$

Also

$$\bar{x}_1 = \bar{y}_1 = \bar{z}_2 = 0 \quad \bar{x}_2 = \left(-0.225 + \frac{0.45}{3} \right) \text{m} = -0.075 \text{ m}$$

PROBLEM 9.154 (Continued)

Finally,

$$\begin{aligned} I_{xy} &= \Sigma(\bar{I}_{xy} + m\bar{x}\bar{y}) = (0+0) + \left[-\frac{1}{36}(0.63585 \text{ kg})(0.45 \text{ m})(0.18 \text{ m}) \right. \\ &\quad \left. + (0.63585 \text{ kg})(-0.075 \text{ m})\left(\frac{0.18 \text{ m}}{3}\right) \right] \\ &= (-1.43066 \times 10^{-3} - 2.8613 \times 10^{-3}) \text{ kg} \cdot \text{m}^2 \end{aligned}$$

or $I_{xy} = -4.29 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$

and

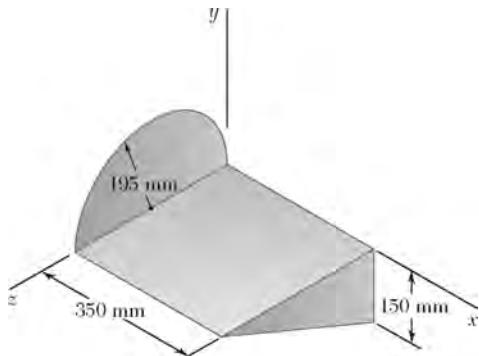
$$I_{yz} = \Sigma(\bar{I}_{yz'} + m\bar{y}\bar{z}) = (0+0) + (0+0) = 0$$

or $I_{yz} = 0 \blacktriangleleft$

$$I_{zx} = \Sigma(\bar{I}_{zx'} + m\bar{z}\bar{x}) = (0+0) + (0+0) = 0$$

or $I_{zx} = 0 \blacktriangleleft$

PROBLEM 9.155



A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3 , determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the component.

SOLUTION

First compute the mass of each component. We have

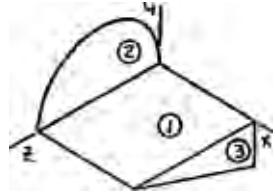
$$m = \rho_{ST}V = \rho_{ST}tA$$

Then

$$\begin{aligned} m_1 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})(0.35 \times 0.39) \text{ m}^2 \\ &= 2.14305 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{2} \times 0.195^2\right) \text{ m}^2 \\ &= 0.93775 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{1}{2} \times 0.39 \times 0.15\right) \text{ m}^2 \\ &= 0.45923 \text{ kg} \end{aligned}$$



Now observe that because of symmetry the centroidal products of inertia of components 1 and 2 are zero and

$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0$$

Also

$$(\bar{I}_{y'z'})_{3,\text{mass}} = \rho_{ST}t(\bar{I}_{y'z'})_{3,\text{area}}$$

Using the results of Sample Problem 9.6 and noting that the orientation of the axes corresponds to a 90° rotation, we have

$$(\bar{I}_{y'z'})_{3,\text{area}} = \frac{1}{72}b_3^2h_3^2$$

Then

$$(I_{y'z'})_3 = \rho_{ST}t\left(\frac{1}{72}b_3^2h_3^2\right) = \frac{1}{36}m_3b_3h_3$$

Also

$$\bar{y}_1 = \bar{x}_2 = 0$$

$$\bar{y}_2 = \frac{4 \times 0.195}{3\pi} \text{ m} = 0.082761 \text{ m}$$

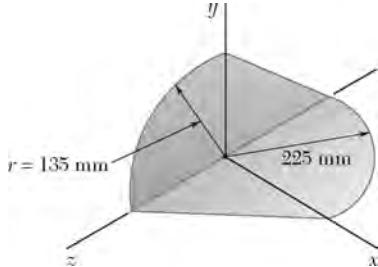
PROBLEM 9.155 (Continued)

Finally,

$$\begin{aligned}
 I_{xy} &= \Sigma(\bar{I}_{x'y'} + m\bar{x}\bar{y}) \\
 &= (0+0) + (0+0) + \left[0 + (0.45923 \text{ kg})(0.35 \text{ m}) \left(-\frac{0.15}{3} \text{ m} \right) \right] \\
 &= -8.0365 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \quad \text{or } I_{xy} = -8.04 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$

$$\begin{aligned}
 I_{yz} &= \Sigma(\bar{I}_{y'z'} + m\bar{y}\bar{z}) \\
 &= (0+0) + [0 + (0.93775 \text{ kg})(0.082761 \text{ m})(0.195 \text{ m})] \\
 &\quad + \left[\frac{1}{36}(0.45923 \text{ kg})(0.39 \text{ m})(0.15 \text{ m}) + (0.45923 \text{ kg}) \left(-\frac{0.15}{3} \text{ m} \right) \left(\frac{0.39}{3} \text{ m} \right) \right] \\
 &= [(15.1338) + (0.7462 - 2.9850)] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (15.1338 - 2.2388) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 12.8950 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \quad \text{or } I_{yz} = 12.90 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$

$$\begin{aligned}
 I_{zx} &= \Sigma(\bar{I}_{z'x'} + m\bar{z}\bar{x}) \\
 &= [0 + (2.14305 \text{ kg})(0.175 \text{ m})(0.195 \text{ m})] + (0+0) \\
 &\quad + \left[0 + (0.45923 \text{ kg}) \left(\frac{0.39}{3} \text{ m} \right) (0.35 \text{ m}) \right] \\
 &= (73.1316 + 20.8950) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 94.0266 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \quad \text{or } I_{zx} = 94.0 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$



PROBLEM 9.156

A section of sheet steel 2mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850kg/m^3 , determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the component.

SOLUTION

First compute the mass of each component. We have

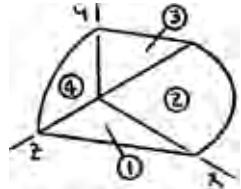
$$m = \rho_{ST}V = \rho_{ST}tA$$

Then

$$m_1 = m_3 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{1}{2} \times 0.225 \times 0.135\right) \text{ m}^2 = 0.23844 \text{ kg}$$

$$m_2 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{4} \times 0.225^2\right) \text{ m}^2 = 0.62424 \text{ kg}$$

$$m_4 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{4} \times 0.135^2\right) \text{ m}^2 = 0.22473 \text{ kg}$$



Now observe that the following centroidal products of inertia are zero because of symmetry.

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = 0 \quad (\bar{I}_{x'y'})_2 = (\bar{I}_{y'z'})_2 = 0$$

$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0 \quad (\bar{I}_{x'y'})_4 = (\bar{I}_{z'x'})_4 = 0$$

Also

$$\bar{y}_1 = \bar{y}_2 = 0 \quad \bar{x}_3 = \bar{x}_4 = 0$$

Now

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\bar{y})$$

so that

$$I_{xy} = 0 \blacktriangleleft$$

Using the results of Sample Problem 9.6, we have

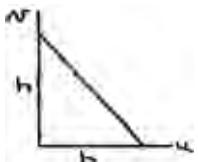
$$I_{uv, \text{area}} = \frac{1}{24}b^2h^2$$

Now

$$\begin{aligned} I_{uv, \text{mass}} &= \rho_{ST}t I_{uv, \text{area}} \\ &= \rho_{ST}t \left(\frac{1}{24}b^2h^2 \right) \\ &= \frac{1}{12}mbh \end{aligned}$$

Thus,

$$(I_{zx})_1 = \frac{1}{12}m_1b_1h_1$$



PROBLEM 9.156 (Continued)

while

$$(I_{yz})_3 = -\frac{1}{12}m_3b_3h_3$$

because of a 90° rotation of the coordinate axes.

To determine I_{uv} for a quarter circle, we have

$$dI_{uv} = d\bar{V}_{u'v'} + u_{EL}v_{EL}dm$$

where

$$\bar{u}_{EL} = u \quad \bar{v}_{EL} = \frac{1}{2}v = \frac{1}{2}\sqrt{a^2 - u^2}$$

$$dm = \rho_{ST}t \, dA = \rho_{ST}tv \, du = \rho_{ST}t\sqrt{a^2 - u^2} \, du$$

Then

$$\begin{aligned} I_{uv} &= \int dI_{uv} = \int_0^a (u) \left(\frac{1}{2}\sqrt{a^2 - u^2} \right) (\rho_{ST}t\sqrt{a^2 - u^2} \, du) \\ &= \frac{1}{2}\rho_{ST}t \int_0^a u(a^2 - u^2) \, du \\ &= \frac{1}{2}\rho_{ST}t \left[\frac{1}{2}a^2u^2 - \frac{1}{4}u^4 \right]_0^a = \frac{1}{8}\rho_{ST}t a^4 = \frac{1}{2\pi}ma^4 \end{aligned}$$

Thus

$$(I_{zx})_2 = -\frac{1}{2\pi}m_2a_2^2$$

because of a 90° rotation of the coordinate axes. Also

$$(I_{yz})_4 = \frac{1}{2\pi}m_4a_4^2$$

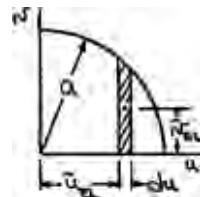
Finally,

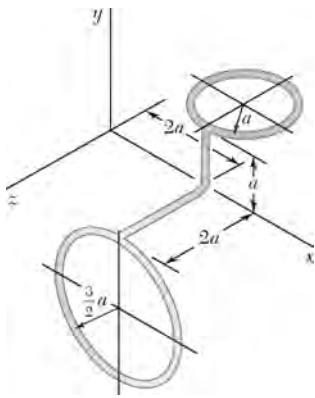
$$\begin{aligned} I_{yz} &= \Sigma(I_{yz}) = [(\bar{I}_{y'z'}) + m_1\bar{y}_1\bar{z}_1] + [(\bar{I}_{y'z'}) + m_2\bar{y}_2\bar{z}_2] + (I_{yz})_3 + (I_{yz})_4 \\ &= \left[-\frac{1}{12}(0.23844 \text{ kg})(0.225 \text{ m})(0.135 \text{ m}) \right] + \left[\frac{1}{2\pi}(0.22473 \text{ kg})(0.135 \text{ m})^2 \right] \\ &= (-0.60355 + 0.65185) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_{yz} = 48.3 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$\begin{aligned} I_{zx} &= \Sigma(I_{zx}) = (I_{zx})_1 + (I_{zx})_2 + [(\bar{I}_{z'x'})_3 + m_3\bar{z}_3\bar{x}_3] + [(\bar{I}_{z'x'})_4 + m_4\bar{z}_4\bar{x}_4] \\ &= \left[\frac{1}{12}(0.23844 \text{ kg})(0.225 \text{ m})(0.135 \text{ m}) \right] + \left[-\frac{1}{2\pi}(0.62424 \text{ kg})(0.225 \text{ m})^2 \right] \\ &= (0.60355 - 5.02964) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_{zx} = -4.43 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$





PROBLEM 9.157

Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g} w L$$

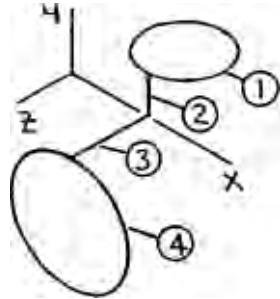
Then

$$m_1 = \frac{W}{g} (2\pi \times a) = 2\pi \frac{w}{g} a$$

$$m_2 = \frac{w}{g} (a) = \frac{w}{g} a$$

$$m_3 = \frac{w}{g} (2a) = 2\frac{w}{g} a$$

$$m_4 = \frac{w}{g} \left(2\pi \times \frac{3}{2} a \right) = 3\pi \frac{w}{g} a$$



Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m	\bar{x}	\bar{y}	\bar{z}	$m\bar{x}\bar{y}$	$m\bar{y}\bar{z}$	$m\bar{z}\bar{x}$
1	$2\pi \frac{w}{g} a$	$2a$	a	$-a$	$4\pi \frac{w}{g} a^3$	$-2\pi \frac{w}{g} a^3$	$-4\pi \frac{w}{g} a^3$
2	$\frac{w}{g} a$	$2a$	$\frac{1}{2}a$	0	$\frac{w}{g} a^3$	0	0
3	$2\frac{w}{g} a$	$2a$	0	a	0	0	$4\frac{w}{g} a^3$
4	$3\pi \frac{w}{g} a$	$2a$	$-\frac{3}{2}a$	$2a$	$-9\pi \frac{w}{g} a^3$	$-9\pi \frac{w}{g} a^3$	$12\pi \frac{w}{g} a^3$
Σ					$\frac{w}{g}(1-5\pi)a^3$	$-11\pi \frac{w}{g} a^3$	$4\frac{w}{g}(1+2\pi)a^3$

PROBLEM 9.157 (Continued)

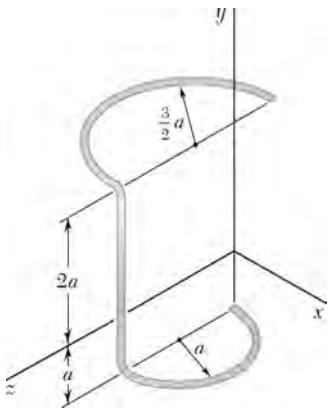
Then

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\bar{y}) \quad \text{or} \quad I_{xy} = \frac{w}{g}a^3(1 - 5\pi) \blacktriangleleft$$

$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\bar{y}\bar{z}) \quad \text{or} \quad I_{yz} = -11\pi \frac{w}{g}a^3 \blacktriangleleft$$

$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\bar{x}) \quad \text{or} \quad I_{zx} = 4\frac{w}{g}a^3(1 + 2\pi) \blacktriangleleft$$

PROBLEM 9.158



Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have

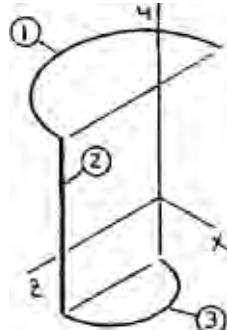
$$m = \frac{W}{g} = \frac{1}{g} wL$$

Then

$$m_1 = \frac{W}{g} \left(\pi \times \frac{3}{2} a \right) = \frac{3}{2} \pi \frac{w}{g} a$$

$$m_2 = \frac{w}{g} (3a) = 3 \frac{w}{g} a$$

$$m_3 = \frac{w}{g} (\pi \times a) = \pi \frac{w}{g} a$$



Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m	\bar{x}	\bar{y}	\bar{z}	$m\bar{x}\bar{y}$	$m\bar{y}\bar{z}$	$m\bar{z}\bar{x}$
1	$\frac{3}{2} \pi \frac{w}{g} a$	$-\frac{2}{\pi} \left(\frac{3}{2} a \right)$	$2a$	$\frac{1}{2} a$	$-9 \frac{w}{g} a^3$	$\frac{3}{2} \pi \frac{w}{g} a^3$	$-\frac{9}{4} \frac{w}{g} a^3$
2	$3 \frac{w}{g} a$	0	$\frac{1}{2} a$	$2a$	0	$3 \frac{w}{g} a^3$	0
3	$\pi \frac{w}{g} a$	$\frac{2}{\pi} (a)$	$-a$	a	$-2 \frac{w}{g} a^3$	$-\pi \frac{w}{g} a^3$	$2 \frac{w}{g} a^3$
Σ					$-11 \frac{w}{g} a^3$	$\frac{w}{g} \left(\frac{\pi}{2} + 3 \right) a^3$	$-\frac{1}{4} \frac{w}{g} a^3$

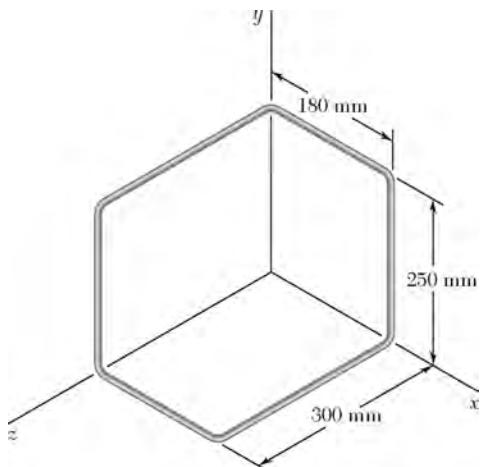
PROBLEM 9.158 (Continued)

Then

$$I_{xy} = \Sigma(\bar{J}_{x'y'} + m\bar{x}\bar{y}) \quad \text{or} \quad I_{xy} = -11\frac{w}{g}a^3 \blacktriangleleft$$

$$I_{yz} = \Sigma(\bar{J}_{y'z'} + m\bar{y}\bar{z}) \quad \text{or} \quad I_{yz} = \frac{1}{2}\frac{w}{g}a^3(\pi + b) \blacktriangleleft$$

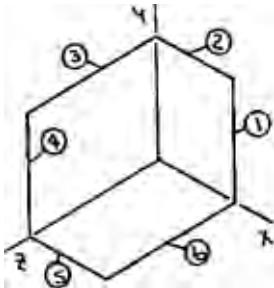
$$I_{zx} = \Sigma(\bar{J}_{z'x'} + m\bar{z}\bar{x}) \quad \text{or} \quad I_{zx} = -\frac{1}{4}\frac{w}{g}a^3 \blacktriangleleft$$



PROBLEM 9.159

The figure shown is formed of 1.5-mm-diameter aluminum wire. Knowing that the density of aluminum is 2800 kg/m^3 , determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{AL}V = \rho_{AL}AL$$

Then

$$\begin{aligned} m_1 = m_4 &= (2800 \text{ kg/m}^3) \left[\frac{\pi}{4} (0.0015 \text{ m})^2 \right] (0.25 \text{ m}) \\ &= 1.23700 \times 10^{-3} \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 = m_5 &= (2800 \text{ kg/m}^3) \left[\frac{\pi}{4} (0.0015 \text{ m})^2 \right] (0.18 \text{ m}) \\ &= 0.89064 \times 10^{-3} \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 = m_6 &= (2800 \text{ kg/m}^3) \left[\frac{\pi}{4} (0.0015 \text{ m})^2 \right] (0.3 \text{ m}) \\ &= 1.48440 \times 10^{-3} \text{ kg} \end{aligned}$$

PROBLEM 9.159 (Continued)

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	$m, \text{ kg}$	$\bar{x}, \text{ m}$	$\bar{y}, \text{ m}$	$\bar{z}, \text{ m}$	$m\bar{x}\bar{y}, \text{ kg} \cdot \text{m}^2$	$m\bar{y}\bar{z}, \text{ kg} \cdot \text{m}^2$	$m\bar{z}\bar{x}, \text{ kg} \cdot \text{m}^2$
1	1.23700×10^{-3}	0.18	0.125	0	27.8325×10^{-6}	0	0
2	0.89064×10^{-3}	0.09	0.25	0	20.0394×10^{-6}	0	0
3	1.48440×10^{-3}	0	0.25	0.15	0	55.6650×10^{-6}	0
4	1.23700×10^{-3}	0	0.125	0.3	0	46.3875×10^{-6}	0
5	0.89064×10^{-3}	0.09	0	0.3	0	0	24.0473×10^{-6}
6	1.48440×10^{-3}	0.18	0	0.15	0	0	40.0788×10^{-6}
Σ					47.8719×10^{-6}	102.0525×10^{-6}	64.1261×10^{-6}

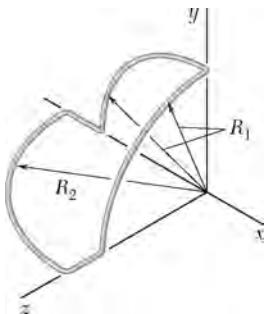
Then

$$I_{xy} = \Sigma(\overset{0}{\cancel{\bar{I}}}_{x'y'} + m\bar{x}\bar{y}) \quad \text{or} \quad I_{xy} = 47.9 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{yz} = \Sigma(\overset{0}{\cancel{\bar{I}}}_{y'z'} + m\bar{y}\bar{z}) \quad \text{or} \quad I_{yz} = 102.1 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{zx} = \Sigma(\overset{0}{\cancel{\bar{I}}}_{z'x'} + m\bar{z}\bar{x}) \quad \text{or} \quad I_{zx} = 64.1 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.160



Thin aluminum wire of uniform diameter is used to form the figure shown. Denoting by m' the mass per unit length of the wire, determine the mass products of inertia I_{xy} , I_{yz} , and I_{zx} of the wire figure.

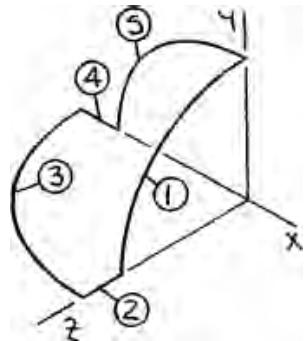
SOLUTION

First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = m'L$$

Then

$$\begin{aligned} m_1 &= m_5 = m'\left(\frac{\pi}{2}R_1\right) = \frac{\pi}{2}m'R_1 \\ m_2 &= m_4 = m'(R_2 - R_1) \\ m_3 &= m'\left(\frac{\pi}{2}R_2\right) = \frac{\pi}{2}m'R_2 \end{aligned}$$



Now observe that because of symmetry the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of components 2 and 4 are zero and

$$\begin{aligned} (\bar{I}_{x'y'})_1 &= (\bar{I}_{z'x'})_1 = 0 & (\bar{I}_{x'y'})_3 &= (\bar{I}_{y'z'})_3 = 0 \\ (\bar{I}_{y'z'})_5 &= (\bar{I}_{z'x'})_5 = 0 \end{aligned}$$

Also

$$\bar{x}_1 = \bar{x}_2 = 0 \quad \bar{y}_2 = \bar{y}_3 = \bar{y}_4 = 0 \quad \bar{z}_4 = \bar{z}_5 = 0$$

Using the parallel-axis theorem [Equations (9.47)], it follows that $I_{xy} = I_{yz} = I_{zx}$ for components 2 and 4.

To determine I_{uv} for one quarter of a circular arc, we have $dI_{uv} = uvdm$

where

$$u = a \cos \theta \quad v = a \sin \theta$$

and

$$dm = \rho dV = \rho[A(ad\theta)]$$

where A is the cross-sectional area of the wire. Now

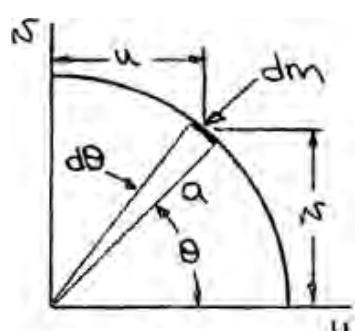
$$m = m'\left(\frac{\pi}{2}a\right) = \rho A\left(\frac{\pi}{2}a\right)$$

so that

$$dm = m'ad\theta$$

and

$$\begin{aligned} dI_{uv} &= (a \cos \theta)(a \sin \theta)(m'ad\theta) \\ &= m'a^3 \sin \theta \cos \theta d\theta \end{aligned}$$



PROBLEM 9.160 (Continued)

Then

$$I_{uv} = \int dI_{uv} = \int_0^{\pi/2} m'a^3 \sin \theta \cos \theta d\theta \\ = m'a^3 \left[\frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} = \frac{1}{2} m'a^3$$

Thus,

$$(I_{yz})_1 = \frac{1}{2} m'R_l^3$$

and

$$(I_{zx})_3 = -\frac{1}{2} m'R_2^3 \quad (I_{xy})_5 = -\frac{1}{2} m'R_l^3$$

because of 90° rotations of the coordinate axes. Finally,

$$I_{xy} = \Sigma(I_{xy}) = [(\bar{I}_{x'y'})_1 + m_1 \bar{x}_1 \bar{y}_1] + [(\bar{I}_{x'y'})_3 + m_3 \bar{x}_3 \bar{y}_3] + (I_{xy})_5 \\ \text{or} \quad I_{xy} = -\frac{1}{2} m'R_l^3 \blacktriangleleft$$

$$I_{yz} = \Sigma(I_{yz}) = (I_{yz})_1 + [(\bar{I}_{y'z'})_3 + m_3 \bar{y}_3 \bar{z}_3] + [(\bar{I}_{y'z'})_5 + m_5 \bar{y}_5 \bar{z}_5] \\ \text{or} \quad I_{yz} = \frac{1}{2} m'R_l^3 \blacktriangleleft$$

$$I_{zx} = \Sigma(I_{zx}) = [(\bar{I}_{z'x'})_1 + m_1 \bar{z}_1 \bar{x}_1] + (I_{zx})_3 + [(\bar{I}_{z'x'})_5 + m_5 \bar{z}_5 \bar{x}_5] \\ \text{or} \quad I_{zx} = -\frac{1}{2} m'R_2^3 \blacktriangleleft$$

PROBLEM 9.161

Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

SOLUTION

We have

$$I_{xy} = \int xydm \quad I_{yz} = \int yzdm \quad I_{zx} = \int zx dm \quad (9.45)$$

and

$$x = x' + \bar{x} \quad y = y' + \bar{y} \quad z = z' + \bar{z} \quad (9.31)$$

Consider

$$I_{xy} = \int xydm$$

Substituting for x and for y

$$\begin{aligned} I_{xy} &= \int (x' + \bar{x})(y' + \bar{y}) dm \\ &= \int x'y'dm + \bar{y} \int x'dm + \bar{x} \int y'dm + \bar{x}\bar{y} \int dm \end{aligned}$$

By definition

$$\bar{I}_{x'y'} = \int x'y'dm$$

and

$$\int x'dm = m\bar{x}'$$

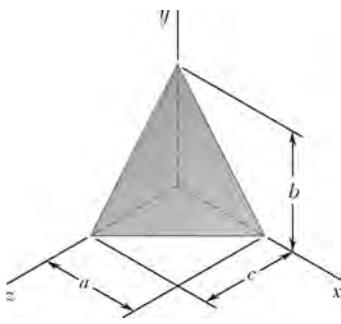
$$\int y'dm = m\bar{y}'$$

However, the origin of the primed coordinate system coincides with the mass center G , so that

$$\bar{x}' = \bar{y}' = 0$$

$$I_{xy} = \bar{I}_{x'y'} + m\bar{x}\bar{y} \quad \text{Q.E.D.} \blacktriangleleft$$

The expressions for I_{yz} and I_{zx} are obtained in a similar manner.



PROBLEM 9.162

For the homogeneous tetrahedron of mass m shown, (a) determine by direct integration the mass product of inertia I_{zx} , (b) deduce I_{yz} and I_{xy} from the result obtained in Part a.

SOLUTION

- (a) First divide the tetrahedron into a series of thin vertical slices of thickness dz as shown.

Now

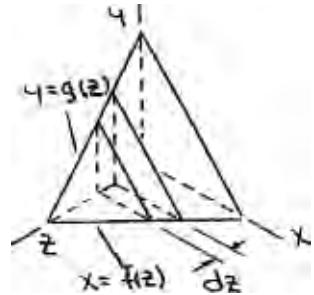
$$x = -\frac{a}{c}z + a = a\left(1 - \frac{z}{c}\right)$$

and

$$y = -\frac{b}{c}z + b = b\left(1 - \frac{z}{c}\right)$$

The mass dm of the slab is

$$\begin{aligned} dm &= \rho dV = \rho \left(\frac{1}{2}xydz\right) = \frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz \\ \text{Then } m &= \int dm = \int_0^c \frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz \\ &= \frac{1}{2}\rho ab \left[\left(-\frac{c}{3}\right)\left(1 - \frac{z}{c}\right)^3 \right]_0^c = \frac{1}{6}\rho abc \end{aligned}$$



Now

$$dI_{zx} = d\bar{I}_{z'x'} + \bar{z}_{\text{EL}} \bar{x}_{\text{EL}} dm$$

where

$$d\bar{I}_{z'x'} = 0 \quad (\text{symmetry})$$

and

$$\bar{z}_{\text{EL}} = z \quad \bar{x}_{\text{EL}} = \frac{1}{3}x = \frac{1}{3}a\left(1 - \frac{z}{c}\right)$$

Then

$$\begin{aligned} I_{zx} &= \int dI_{zx} = \int_0^c z \left[\frac{1}{3}a\left(1 - \frac{z}{c}\right) \right] \left[\frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz \right] \\ &= \frac{1}{6}\rho a^2 b \int_0^c \left(z - 3\frac{z^2}{c} + 3\frac{z^3}{c^2} - \frac{z^4}{c^3} \right) dz \\ &= \frac{m}{c}a \left[\frac{1}{2}z^2 - \frac{z^3}{c} + \frac{3}{4}\frac{z^4}{c^2} - \frac{1}{5}\frac{z^5}{c^3} \right]_0^c \\ &\quad \text{or } I_{zx} = \frac{1}{20}mac \blacktriangleleft \end{aligned}$$

PROBLEM 9.162 (Continued)

- (b) Because of the symmetry of the body, I_{xy} and I_{yz} can be deduced by considering the circular permutation of (x, y, z) and (a, b, c) .

Thus,

$$I_{xy} = \frac{1}{20} mab \blacktriangleleft$$

$$I_{yz} = \frac{1}{20} mbc \blacktriangleleft$$

Alternative solution for Part a:

First divide the tetrahedron into a series of thin horizontal slices of thickness dy as shown.

Now

$$x = -\frac{a}{b}y + a = a\left(1 - \frac{y}{b}\right)$$

and

$$z = -\frac{c}{b}y + c = c\left(1 - \frac{y}{b}\right)$$

The mass dm of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xzdy\right) = \frac{1}{2}\rho ac\left(1 - \frac{y}{b}\right)^2 dy$$

Now

$$dI_{zx} = \rho t dI_{zx, \text{area}}$$

where

$$t = dy$$

and $dI_{zx, \text{area}} = \frac{1}{24}x^2z^2$ from the results of Sample Problem 9.6.

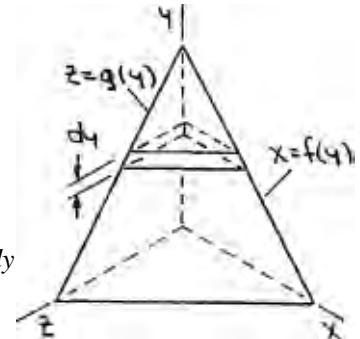
Then

$$\begin{aligned} dI_{zx} &= \rho(dy) \left\{ \frac{1}{24} \left[a\left(1 - \frac{y}{b}\right)^2 \right] \left[c\left(1 - \frac{y}{b}\right)^2 \right] \right\} \\ &= \frac{1}{24} \rho a^2 c^2 \left(1 - \frac{y}{b}\right)^4 dy = \frac{1}{4} \frac{m}{b} ac \left(1 - \frac{y}{b}\right)^4 dy \end{aligned}$$

Finally,

$$\begin{aligned} I_{zx} &= \int dI_{zx} = \int_0^b \frac{1}{4} \frac{m}{b} ac \left(1 - \frac{y}{b}\right)^4 dy \\ &= \frac{1}{4} \frac{m}{b} ac \left[\left(-\frac{b}{5}\right) \left(1 - \frac{y}{b}\right)^5 \right]_0^b \end{aligned}$$

$$\text{or } I_{zx} = \frac{1}{20} mac \blacktriangleleft$$



PROBLEM 9.162 (Continued)

Alternative solution for Part *a*:

The equation of the included face of the tetrahedron is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

so that

$$y = b \left(1 - \frac{x}{a} - \frac{z}{c} \right)$$

For an infinitesimal element of sides dx , dy , and dz :

$$dm = \rho dV = \rho dy dx dz$$

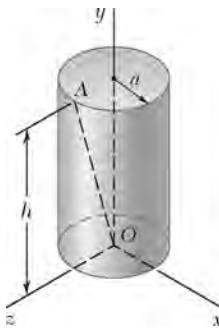
From Part *a*

$$x = a \left(1 - \frac{z}{c} \right)$$

Now

$$\begin{aligned} I_{zx} &= \int z x dm = \int_0^c \int_0^{a(1-z/c)} \int_0^{b(1-x/a-z/c)} zx (\rho dy dx dz) \\ &= \rho \int_0^c \int_0^{a(1-z/c)} zx \left[b \left(1 - \frac{x}{a} - \frac{z}{c} \right) \right] dx dz \\ &= \rho b \int_0^c z \left[\frac{1}{2} x^2 - \frac{1}{3} \frac{x^3}{a} - \frac{1}{2} \frac{z}{c} x^2 \right]_0^{a(1-z/c)} dz \\ &= \rho b \int_0^c z \left[\frac{1}{2} a^2 \left(1 - \frac{z}{c} \right)^2 - \frac{1}{3a} a^3 \left(1 - \frac{z}{c} \right)^3 - \frac{1}{2} \frac{z}{c} a^2 \left(1 - \frac{z}{c} \right)^2 \right] dz \\ &= \rho b \int_0^c \frac{1}{6} a^2 z \left(1 - \frac{z}{c} \right)^3 dz \\ &= \frac{1}{6} \rho a^2 b \int_0^c \left(z - 3 \frac{z^2}{c} + 3 \frac{z^3}{c^2} - \frac{z^4}{c^3} \right) dz \\ &= \frac{m}{c} a \left[\frac{1}{2} z^2 - \frac{z^3}{c} + \frac{3}{4} \frac{z^4}{c^2} - \frac{1}{5} \frac{z^5}{c^3} \right]_0^c \end{aligned}$$

or $I_{zx} = \frac{1}{20} mac \blacktriangleleft$



PROBLEM 9.163

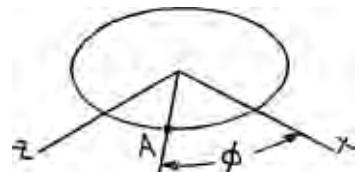
The homogeneous circular cylinder shown has a mass m . Determine the mass moment of inertia of the cylinder with respect to the line joining the origin O and Point A that is located on the perimeter of the top surface of the cylinder.

SOLUTION

From Figure 9.28:

$$I_y = \frac{1}{2}ma^2$$

and using the parallel-axis theorem



$$I_x = I_z = \frac{1}{12}m(3a^2 + h^2) + m\left(\frac{h}{2}\right)^2 = \frac{1}{12}m(3a^2 + 4h^2)$$

Symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

For convenience, let Point A lie in the yz plane. Then

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(h\mathbf{j} + a\mathbf{k})$$

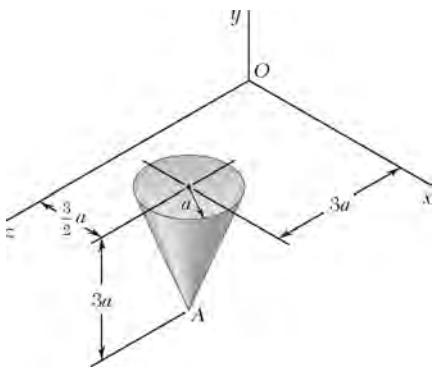
With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$\begin{aligned} I_{OA} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 \\ &= \frac{1}{2}ma^2 \left(\frac{h}{\sqrt{h^2 + a^2}} \right)^2 + \frac{1}{12}m(3a^2 + 4h^2) \left(\frac{a}{\sqrt{h^2 + a^2}} \right)^2 \\ \text{or } I_{OA} &= \frac{1}{12}ma^2 \frac{10h^2 + 3a^2}{h^2 + a^2} \end{aligned}$$

Note: For Point A located at an arbitrary point on the perimeter of the top surface, λ_{OA} is given by

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(a \cos \phi \mathbf{i} + h\mathbf{j} + a \sin \phi \mathbf{k})$$

which results in the same expression for I_{OA} .



PROBLEM 9.164

The homogeneous circular cone shown has a mass m . Determine the mass moment of inertia of the cone with respect to the line joining the origin O and Point A .

SOLUTION

First note that

$$d_{OA} = \sqrt{\left(\frac{3}{2}a\right)^2 + (-3a)^2 + (3a)^2} = \frac{9}{2}a$$

Then

$$\lambda_{OA} = \frac{1}{\frac{9}{2}a} \left(\frac{3}{2}a\mathbf{i} - 3a\mathbf{j} + 3a\mathbf{k} \right) = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

For a rectangular coordinate system with origin at Point A and axes aligned with the given x , y , z axes, we have (using Figure 9.28)

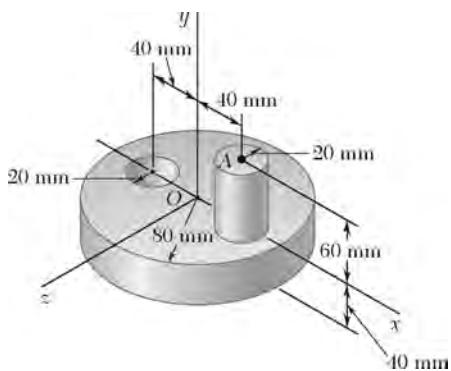
$$\begin{aligned} I_x = I_z &= \frac{3}{5}m \left[\frac{1}{4}a^2 + (3a)^2 \right] & I_y &= \frac{3}{10}ma^2 \\ &= \frac{111}{20}ma^2 \end{aligned}$$

Also, symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

With the mass products of inertia equal to zero, Equation (9.46) reduces to

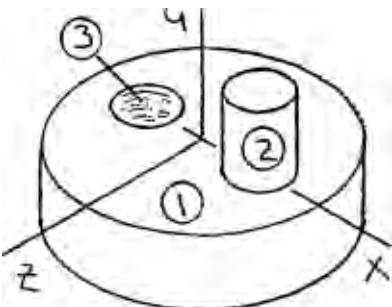
$$\begin{aligned} I_{OA} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 \\ &= \frac{111}{20}ma^2 \left(\frac{1}{3} \right)^2 + \frac{3}{10}ma^2 \left(-\frac{2}{3} \right)^2 + \frac{111}{20}ma^2 \left(\frac{2}{3} \right)^2 \\ &= \frac{193}{60}ma^2 & \text{or } I_{OA} &= 3.22ma^2 \blacktriangleleft \end{aligned}$$



PROBLEM 9.165

Shown is the machine element of Problem 9.141. Determine its mass moment of inertia with respect to the line joining the origin O and Point A.

SOLUTION



First compute the mass of each component.

We have

$$m = \rho_{ST}V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)[\pi(0.08 \text{ m})^2(0.04 \text{ m})] = 6.31334 \text{ kg}$$

$$m_2 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.06 \text{ m})] = 0.59188 \text{ kg}$$

$$m_3 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.04 \text{ m})] = 0.39458 \text{ kg}$$

Symmetry implies

$$I_{yz} = I_{zx} = 0 \quad (I_{xy})_1 = 0$$

and

$$(\bar{I}_{x'y'})_2 = (\bar{I}_{x'y'})_3 = 0$$

Now

$$I_{xy} = \sum(\bar{I}_{x'y'} + m\bar{x}\bar{y}) = m_2\bar{x}_2\bar{y}_2 - m_3\bar{x}_3\bar{y}_3$$

$$= [0.59188 \text{ kg} (0.04 \text{ m})(0.03 \text{ m})] - [0.39458 \text{ kg} (-0.04 \text{ m})(-0.02 \text{ m})]$$

$$= (0.71026 - 0.31566) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= 0.39460 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

PROBLEM 9.165 (Continued)

From the solution to Problem 9.141, we have

$$I_x = 13.98800 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = 20.55783 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

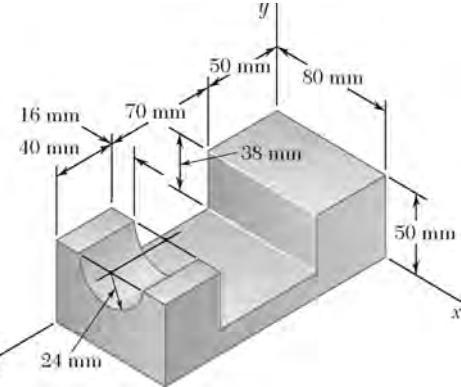
By observation

$$\lambda_{OA} = \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$$

Substituting into Eq. (9.46)

$$\begin{aligned} I_{OA} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x \\ &= \left[(13.98800) \left(\frac{2}{\sqrt{13}} \right)^2 + (20.55783) \left(\frac{3}{\sqrt{13}} \right)^2 \right. \\ &\quad \left. - 2(0.39460) \left(\frac{2}{\sqrt{13}} \right) \left(\frac{2}{\sqrt{13}} \right) \right] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ &= (4.30400 + 14.23234 - 0.36425) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_{OA} = 18.17 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



PROBLEM 9.166

Determine the mass moment of inertia of the steel fixture of Problems 9.145 and 9.149 with respect to the axis through the origin that forms equal angles with the x , y , and z axes.

SOLUTION

From the solutions to Problems 9.145 and 9.149, we have

Problem 9.145:

$$I_x = 26.4325 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = 31.1726 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = 8.5773 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Problem 9.149:

$$I_{xy} = 2.5002 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_{yz} = 4.0627 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_{zx} = 8.8062 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

From the problem statement it follows that

$$\lambda_x = \lambda_y = \lambda_z$$

Now

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \Rightarrow 3\lambda_x^2 = 1$$

or

$$\lambda_x = \lambda_y = \lambda_z = \frac{1}{\sqrt{3}}$$

Substituting into Eq. (9.46)

$$I_{OL} = I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x$$

Noting that

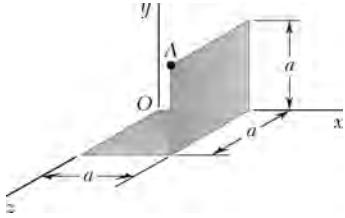
$$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x \lambda_y = \lambda_y \lambda_z = \lambda_z \lambda_x = \frac{1}{3}$$

We have

$$I_{OL} = \frac{1}{3} [26.4325 + 31.1726 + 8.5773]$$

$$-2(2.5002 + 4.0627 + 8.8062)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or } I_{OL} = 11.81 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$



PROBLEM 9.167

The thin bent plate shown is of uniform density and weight W . Determine its mass moment of inertia with respect to the line joining the origin O and Point A .

SOLUTION

First note that

$$m_1 = m_2 = \frac{1}{2} \frac{W}{g}$$

and that

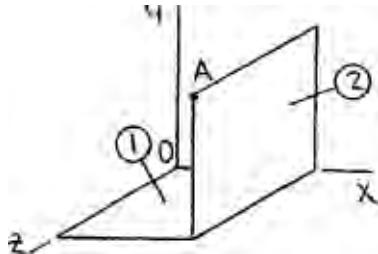
$$\lambda_{OA} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 \\ &= \left[\frac{1}{12} \left(\frac{1}{2} \frac{W}{g} \right) a^2 + \frac{1}{2} \frac{W}{g} \left(\frac{a}{2} \right)^2 \right] \\ &\quad + \left\{ \frac{1}{12} \left(\frac{1}{2} \frac{W}{g} \right) (a^2 + a^2) + \frac{1}{2} \frac{W}{g} \left[\left(\frac{a}{2} \right)^2 + \left(\frac{a}{2} \right)^2 \right] \right\} \\ &= \frac{1}{2} \frac{W}{g} \left[\left(\frac{1}{12} + \frac{1}{4} \right) a^2 + \left(\frac{1}{6} + \frac{1}{2} \right) a^2 \right] = \frac{1}{2} \frac{W}{g} a^2 \end{aligned}$$

$$\begin{aligned} I_y &= (I_y)_1 + (I_y)_2 \\ &= \left\{ \frac{1}{12} \left(\frac{1}{2} \frac{W}{g} \right) (a^2 + a^2) + \frac{1}{2} \frac{W}{g} \left[\left(\frac{a}{2} \right)^2 + \left(\frac{a}{2} \right)^2 \right] \right\} \\ &\quad + \left\{ \frac{1}{12} \left(\frac{1}{2} \frac{W}{g} \right) a^2 + \frac{1}{2} \frac{W}{g} \left[(a)^2 + \left(\frac{a}{2} \right)^2 \right] \right\} \\ &= \frac{1}{2} \frac{W}{g} \left[\left(\frac{1}{6} + \frac{1}{2} \right) a^2 + \left(\frac{1}{12} + \frac{5}{4} \right) a^2 \right] = \frac{W}{g} a^2 \end{aligned}$$

$$\begin{aligned} I_z &= (I_z)_1 + (I_z)_2 \\ &= \left[\frac{1}{12} \left(\frac{1}{2} \frac{W}{g} \right) a^2 + \frac{1}{2} \frac{W}{g} \left(\frac{a}{2} \right)^2 \right] \\ &\quad + \left\{ \frac{1}{12} \left(\frac{1}{2} \frac{W}{g} \right) a^2 + \frac{1}{2} \frac{W}{g} \left[(a)^2 + \left(\frac{a}{2} \right)^2 \right] \right\} \\ &= \frac{1}{2} \frac{W}{g} \left[\left(\frac{1}{12} + \frac{1}{4} \right) a^2 + \left(\frac{1}{12} + \frac{5}{4} \right) a^2 \right] = \frac{5}{6} \frac{W}{g} a^2 \end{aligned}$$



PROBLEM 9.167 (Continued)

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of both components are zero because of symmetry. Also, $\bar{y}_1 = 0$

Then

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\bar{y}) \xrightarrow{0} = m_2\bar{x}_2\bar{y}_2 = \frac{1}{2}\frac{W}{g}(a)\left(\frac{a}{2}\right) = \frac{1}{4}\frac{W}{g}a^2$$

$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\bar{y}\bar{z}) \xrightarrow{0} = m_2\bar{y}_2\bar{z}_2 = \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{1}{8}\frac{W}{g}a^2$$

$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\bar{x}) \xrightarrow{0} = m_1\bar{z}_1\bar{x}_1 + m_2\bar{z}_2\bar{x}_2$$

$$= \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)(a) = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Equation (9.46)

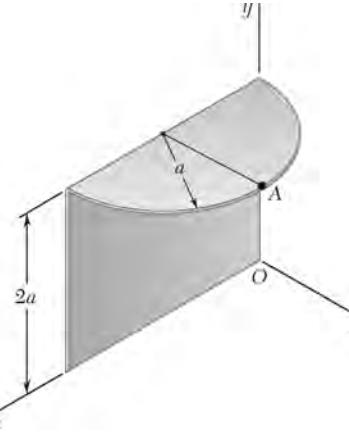
$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that

$$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \frac{1}{3}$$

We have

$$\begin{aligned} I_{OA} &= \frac{1}{3}\left[\frac{1}{2}\frac{W}{g}a^2 + \frac{W}{g}a^2 + \frac{5}{6}\frac{W}{g}a^2 - 2\left(\frac{1}{4}\frac{W}{g}a^2 + \frac{1}{8}\frac{W}{g}a^2 + \frac{3}{8}\frac{W}{g}a^2\right)\right] \\ &= \frac{1}{3}\left[\frac{14}{6} - 2\left(\frac{3}{4}\right)\right]\frac{W}{g}a^2 \quad \text{or } I_{OA} = \frac{5}{18}\frac{W}{g}a^2 \blacktriangleleft \end{aligned}$$



PROBLEM 9.168

A piece of sheet steel of thickness t and density ρ is cut and bent into the machine component shown. Determine the mass moment of inertia of the component with respect to the line joining the origin O and Point A .

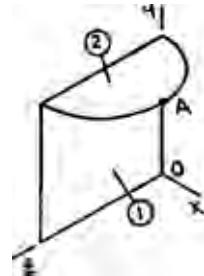
SOLUTION

Let γ be the specific weight = ρg

First note that

$$\lambda_{OA} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

Next compute the mass of each component. We have



Then

$$m_1 = \frac{\gamma}{g} t (2a \times 2a) = 4 \frac{\gamma t}{g} a^2$$

$$m_2 = \frac{\gamma}{g} t \left(\frac{\pi}{2} \times a^2 \right) = \frac{\pi \gamma t}{2 g} a^2$$

Using Figure 9.28 for component 1 and the equations derived above (following the solution to Problem 9.134) for a semicircular plate for component 2, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 \\ &= \left\{ \frac{1}{12} \left(4 \frac{\gamma t}{g} a^2 \right) [(2a)^2 + (2a)^2] + 4 \frac{\gamma t}{g} a^2 (a^2 + a^2) \right\} \\ &\quad + \left\{ \frac{1}{4} \left(\frac{\pi \gamma t}{2 g} a^2 \right) a^2 + \frac{\pi \gamma t}{2 g} a^2 [(2a)^2 + (a)^2] \right\} \\ &= 4 \frac{\gamma t}{g} a^2 \left(\frac{2}{3} + 2 \right) a^2 + \frac{\pi \gamma t}{2 g} a^2 \left(\frac{1}{4} + 5 \right) a^2 \\ &= 18.91335 \frac{\gamma t}{g} a^4 \end{aligned}$$

PROBLEM 9.168 (Continued)

$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 \\
 &= \left[\frac{1}{12} \left(4 \frac{\gamma t}{g} a^2 \right) (2a)^2 + 4 \frac{\gamma t}{g} a^2 (a^2) \right] \\
 &\quad + \left\{ \frac{\pi}{2} \frac{\gamma t}{g} a^2 \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) a^2 + \frac{\pi}{2} \frac{\gamma t}{g} a^2 \left[\left(\frac{4a}{3\pi} \right)^2 + (a)^2 \right] \right\} \\
 &= 4 \frac{\gamma t}{g} a^2 \left(\frac{1}{3} + 1 \right) a^2 + \frac{\pi}{2} \frac{\gamma t}{g} a^2 \left(\frac{1}{2} - \frac{16}{9\pi^2} + \frac{16}{9\pi^2} + 1 \right) a^2 \\
 &= 7.68953 \frac{\gamma t}{g} a^4
 \end{aligned}$$

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 \\
 &= \left[\frac{1}{12} \left(4 \frac{\gamma t}{g} a^2 \right) (2a)^2 + 4 \frac{\gamma t}{g} a^2 (a^2) \right] \\
 &\quad + \left\{ \frac{\pi}{2} \frac{\gamma t}{g} a^2 \left(\frac{1}{4} - \frac{16}{9\pi^2} \right) a^2 + \frac{\pi}{2} \frac{\gamma t}{g} a^2 \left[\left(\frac{4a}{3\pi} \right)^2 + (2a)^2 \right] \right\} \\
 &= 4 \frac{\gamma t}{g} a^2 \left(\frac{1}{3} + 1 \right) a^2 + \frac{\pi}{2} \frac{\gamma t}{g} a^2 \left(\frac{1}{4} - \frac{16}{9\pi^2} + \frac{16}{9\pi^2} + 4 \right) a^2 \\
 &= 12.00922 \frac{\gamma t}{g} a^4
 \end{aligned}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of both components are zero because of symmetry. Also $\bar{x}_1 = 0$.

Then

$$\begin{aligned}
 I_{xy} &= \Sigma(\bar{I}_{x'y'} + m\bar{x}\bar{y}) \stackrel{0}{\nearrow} = m_2 \bar{x}_2 \bar{y}_2 = \frac{\pi}{2} \frac{\gamma t}{g} a^2 \left(\frac{4a}{3\pi} \right) (2a) \\
 &= 1.33333 \frac{\gamma t}{g} a^4
 \end{aligned}$$

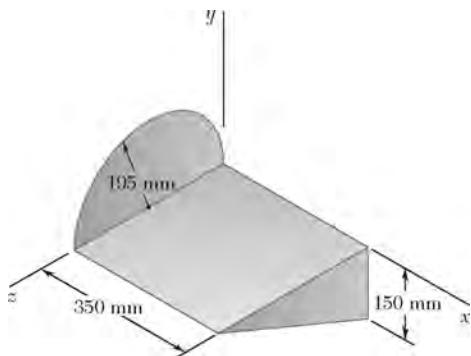
$$\begin{aligned}
 I_{yz} &= \Sigma(\bar{I}_{y'z'} + m\bar{y}\bar{z}) \stackrel{0}{\nearrow} = m_1 \bar{y}_1 \bar{z}_1 + m_2 \bar{y}_2 \bar{z}_2 \\
 &= 4 \frac{\gamma t}{g} a^2 (a)(a) + \frac{\pi}{2} \frac{\gamma t}{g} a^2 (2a)(a) \\
 &= 7.14159 \frac{\gamma t}{g} a^4
 \end{aligned}$$

$$\begin{aligned}
 I_{zx} &= \Sigma(\bar{I}_{z'x'} + m\bar{z}\bar{x}) \stackrel{0}{\nearrow} = m_2 \bar{z}_2 \bar{x}_2 = \frac{\pi}{2} \frac{\gamma t}{g} a^2 (a) \left(\frac{4a}{3\pi} \right) \\
 &= 0.66667 \frac{\gamma t}{g} a^4
 \end{aligned}$$

PROBLEM 9.168 (Continued)

Substituting into Eq. (9.46)

$$\begin{aligned}
 I_{OA} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x \\
 &= 18.91335 \frac{\gamma t}{g} a^4 \left(\frac{1}{\sqrt{6}} \right)^2 + 7.68953 \frac{\gamma t}{g} a^4 \left(\frac{2}{\sqrt{6}} \right)^2 \\
 &\quad + 12.00922 \frac{\gamma t}{g} a^4 \left(\frac{1}{\sqrt{6}} \right)^2 - 2 \left(1.33333 \frac{\gamma t}{g} a^4 \right) \left(\frac{1}{\sqrt{6}} \right) \left(\frac{2}{\sqrt{6}} \right) \\
 &\quad - 2 \left(7.14159 \frac{\gamma t}{g} a^4 \right) \left(\frac{2}{\sqrt{6}} \right) \left(\frac{2}{\sqrt{6}} \right) - 2 \left(0.66667 \frac{\gamma t}{g} a^4 \right) \left(\frac{1}{\sqrt{6}} \right) \left(\frac{1}{\sqrt{6}} \right) \\
 &= (3.15223 + 5.12635 + 2.00154 - 0.88889 - 4.76106 - 0.22222) \frac{\gamma t}{g} a^4 \\
 &\quad \text{or} \qquad I_{OA} = 4.41 \rho t a^4 \blacktriangleleft
 \end{aligned}$$



PROBLEM 9.169

Determine the mass moment of inertia of the machine component of Problems 9.136 and 9.155 with respect to the axis through the origin characterized by the unit vector $\lambda = (-4\mathbf{i} + 8\mathbf{j} + \mathbf{k}) / 9$.

SOLUTION

From the solutions to Problems 9.136 and 9.155, we have

Problem 9.136:

$$I_x = 175.503 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = 308.629 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = 154.400 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Problem 9.155:

$$I_{xy} = -8.0365 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

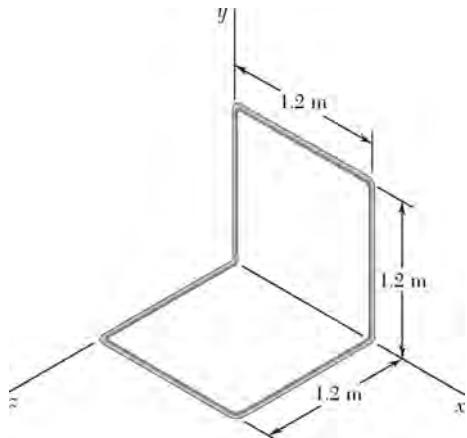
$$I_{yz} = 12.8950 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_{zx} = 94.0266 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Substituting into Eq. (9.46)

$$\begin{aligned} I_{OL} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x \\ &= \left[175.503 \left(-\frac{4}{9} \right)^2 + 308.629 \left(\frac{8}{9} \right)^2 + 154.400 \left(\frac{1}{9} \right)^2 \right. \\ &\quad \left. - 2(-8.0365) \left(-\frac{4}{9} \right) \left(\frac{8}{9} \right) - 2(12.8950) \left(\frac{8}{9} \right) \left(\frac{1}{9} \right) \right. \\ &\quad \left. - 2(94.0266) \left(\frac{1}{9} \right) \left(-\frac{4}{9} \right) \right] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ &= (34.6673 + 243.855 + 1.906 - 6.350 \\ &\quad - 2.547 + 9.287) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ &\quad \text{or } I_{OL} = 281 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft \end{aligned}$$

PROBLEM 9.170



For the wire figure of Problem 9.148, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\lambda = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) / 7$.

SOLUTION

First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = 0.056 \text{ kg/m} \times 1.2 \text{ m} \\ = 0.0672 \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, for each component are zero because of symmetry.

Also

$$\bar{x}_1 = \bar{x}_6 = 0 \quad \bar{y}_4 = \bar{y}_5 = \bar{y}_6 = 0 \quad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

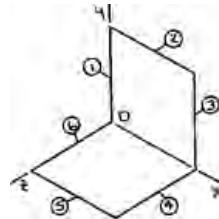
Then

$$\begin{aligned} I_{xy} &= \Sigma(\bar{I}_{x'y'} + m\bar{x}\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3 \\ &= (0.0672 \text{ kg})(0.6 \text{ m})(1.2 \text{ m}) + (0.0672 \text{ kg})(1.2 \text{ m})(0.6 \text{ m}) \\ &= 0.096768 \text{ kg} \cdot \text{m}^2 \\ I_{yz} &= \Sigma(\bar{I}_{y'z'} + m\bar{y}\bar{z}) = 0 \\ I_{zx} &= \Sigma(\bar{I}_{z'x'} + m\bar{z}\bar{x}) = m_4\bar{z}_4\bar{x}_4 + m_5\bar{z}_5\bar{x}_5 \\ &= (0.0672 \text{ kg})(0.6 \text{ m})(1.2 \text{ m}) + (0.0672 \text{ kg})(1.2 \text{ m})(0.6 \text{ m}) \\ &= 0.096768 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

From the solution to Problem 9.148, we have

$$I_x = 0.32258 \text{ kg} \cdot \text{m}^2$$

$$I_y = I_z = 0.41933 \text{ kg} \cdot \text{m}^2$$



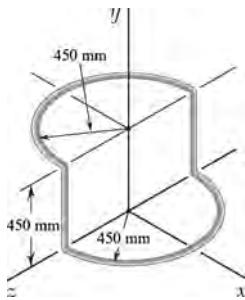
PROBLEM 9.170 (Continued)

Substituting into Eq. (9.46)

$$\begin{aligned} I_{OL} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz}^0 \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x \\ &= \left[0.32258 \left(-\frac{3}{7} \right)^2 + 0.41933 \left(-\frac{6}{7} \right)^2 + 0.41933 \left(\frac{2}{7} \right)^2 \right. \\ &\quad \left. - 2(0.096768) \left(-\frac{3}{7} \right) \left(-\frac{6}{7} \right) - 2(0.096768) \left(\frac{2}{7} \right) \left(-\frac{3}{7} \right) \right] \text{kg} \cdot \text{m}^2 \end{aligned}$$

$$I_{OL} = (0.059249 + 0.30808 + 0.034231 - 0.071095 + 0.023698) \text{kg} \cdot \text{m}^2$$

$$\text{or } I_{OL} = 0.354 \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

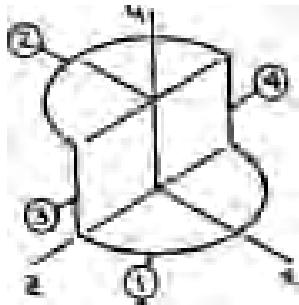


PROBLEM 9.171

For the wire figure of the problem indicated, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\lambda = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION

Have



$$m = \rho V = \rho A L$$

Then

$$m_1 = m_2 = (7850 \text{ kg/m}^3) [\pi(0.0015 \text{ m})^2] \times (\pi \times 0.45 \text{ m})$$

$$m_2 = m_1 = 0.078445 \text{ kg}$$

$$m_3 = m_4 = (7850 \text{ kg/m}^3) [\pi(0.0015 \text{ m})^2] \times (0.45 \text{ m})$$

$$= 0.024970 \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{z'x'} = 0$ for each component.

$$\text{Also } \bar{x}_3 = \bar{x}_4 = 0, \quad \bar{y}_1 = 0, \quad \bar{z}_1 = \bar{z}_2 = 0$$

$$\begin{aligned} \text{Then } I_{xy} &= \Sigma(\bar{I}_{x'y'}^0 + m\bar{x}\bar{y}) = m_2\bar{x}_2\bar{y}_2 \\ &= (0.078445 \text{ kg}) \left(-\frac{2 \times 0.45 \text{ m}}{\pi} \right) (0.45 \text{ m}) \\ &= -10.1128 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$I_{yz} = \Sigma(\bar{I}_{y'z'}^0 + m\bar{y}\bar{z}) = m_3\bar{y}_3\bar{z}_3 + m_4\bar{y}_4\bar{z}_4$$

$$\text{where } m_3 = m_4, \quad \bar{y}_3 = \bar{y}_4, \quad \bar{z}_4 = -\bar{z}_3, \quad \text{so that } I_{yz} = 0$$

$$I_{zx} = \Sigma(I_{z'x'} + m\bar{z}\bar{x}) = m_1\bar{z}_1\bar{x}_1 + m_2\bar{z}_2\bar{x}_2 = 0$$

PROBLEM 9.171 (Continued)

From the solution to Problem 9.147

$$I_x = 45.254 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = 41.883 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

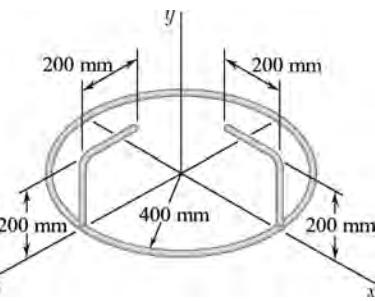
$$I_z = 35.141 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Now

$$\lambda_{OL} = \frac{1}{7}(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$$

Have

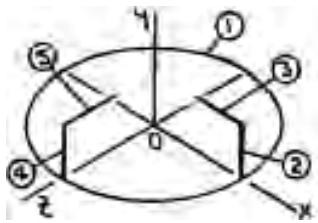
$$\begin{aligned}
 I_{OL} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \overset{0}{\lambda}_y \lambda_z - 2I_{zx} \overset{0}{\lambda}_z \lambda_x \quad [\text{Eq. (9.46)}] \\
 &= \left[45.254 \left(-\frac{3}{7} \right)^2 + 41.883 \left(-\frac{6}{7} \right)^2 + 35.141 \left(\frac{2}{7} \right)^2 \right. \\
 &\quad \left. - 2(-10.1128) \left(-\frac{3}{7} \right) \left(-\frac{6}{7} \right) \right] \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (8.3120 + 30.7712 + 2.8687 + 7.4298) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 49.3817 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 \text{or } I_{OL} &= 49.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$



PROBLEM 9.172

For the wire figure of Problem indicated, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\lambda = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component.

Have

$$m = \rho L$$

Then

$$\begin{aligned} m_1 &= (0.05 \text{ kg/m})[2\pi(0.4 \text{ m})] \\ &= 0.125664 \text{ kg} \\ m_2 &= m_3 = m_4 = m_5 \\ &= (0.05 \text{ kg/m})(0.2 \text{ m}) = 0.01 \text{ kg} \end{aligned}$$

Now observe that $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{z'x'} = 0$ for each component.

Also,

$$\bar{x}_1 = \bar{x}_4 = \bar{x}_5 = 0, \quad \bar{y}_1 = 0, \quad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

Then

$$\begin{aligned} I_{xy} &= \Sigma(\bar{J}_{x'y'} + m\bar{x}\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3 \\ &= (0.01 \text{ kg})[(0.40 \text{ m})(0.1 \text{ m}) + (0.3 \text{ m})(0.2 \text{ m})] \\ &= 1.000 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

By symmetry

$$I_{yz} = I_{xy} \quad I_{yz} = 1.000 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Now

$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\bar{x}) = 0$$

From the solution to Problem 9.146

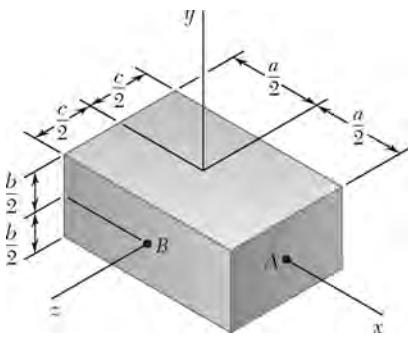
$$\begin{aligned} I_x &= I_z = 13.6529 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ I_y &= 25.1726 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

PROBLEM 9.172 (Continued)

Now $\lambda_{OL} = \frac{1}{7}(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

Have

$$\begin{aligned}
 I_{OL} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \overset{0}{\lambda_z} \lambda_x \\
 &= \left\{ (13.6529) \left[\left(-\frac{3}{7} \right)^2 + \left(\frac{2}{7} \right)^2 \right] \right\} \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &\quad + \left\{ (25.1726) \left(\frac{-6}{7} \right)^2 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \right. \\
 &\quad \left. - 2(1.000) \left[\left(-\frac{3}{7} \right) \left(-\frac{6}{7} \right) + \left(-\frac{6}{7} \right) \left(\frac{2}{7} \right) \right] \right\} \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= (3.6222 + 18.4942 - 0.2449) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 &= 21.8715 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
 \text{or } I_{OL} &= 21.9 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft
 \end{aligned}$$



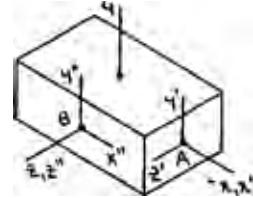
PROBLEM 9.173

For the rectangular prism shown, determine the values of the ratios b/a and c/a so that the ellipsoid of inertia of the prism is a sphere when computed (a) at Point A, (b) at Point B.

SOLUTION

- (a) Using Figure 9.28 and the parallel-axis theorem, we have
at Point A

$$\begin{aligned} I_{x'} &= \frac{1}{12}m(b^2 + c^2) \\ I_{y'} &= \frac{1}{12}m(a^2 + c^2) + m\left(\frac{a}{2}\right)^2 \\ &= \frac{1}{12}m(4a^2 + c^2) \\ I_{z'} &= \frac{1}{12}m(a^2 + b^2) + m\left(\frac{a}{2}\right)^2 = \frac{1}{12}m(4a^2 + b^2) \end{aligned}$$



Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_{x'}x^2 + I_{y'}y^2 + I_{z'}z^2 = 1: \quad \frac{1}{12}m(b^2 + c^2)x^2 + \frac{1}{12}m(4a^2 + c^2)y^2 + \frac{1}{12}m(4a^2 + b^2)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\begin{aligned} \frac{1}{12}m(b^2 + c^2) &= \frac{1}{12}m(4a^2 + c^2) \\ &= \frac{1}{12}m(4a^2 + b^2) \end{aligned}$$

Then

$$b^2 + c^2 = 4a^2 + c^2 \quad \text{or } \frac{b}{a} = 2 \blacktriangleleft$$

and

$$b^2 + c^2 = 4a^2 + b^2 \quad \text{or } \frac{c}{a} = 2 \blacktriangleleft$$

PROBLEM 9.173 (Continued)

- (b) Using Figure 9.28 and the parallel-axis theorem, we have at Point B

$$I_{x''} = \frac{1}{12}m(b^2 + c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(b^2 + 4c^2)$$

$$I_{y''} = \frac{1}{12}m(a^2 + c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(a^2 + 4c^2)$$

$$I_{z''} = \frac{1}{12}m(a^2 + b^2)$$

Now observe that symmetry implies

$$I_{x''y''} = I_{y''z''} = I_{z''x''} = 0$$

From Part *a* it then immediately follows that

$$\frac{1}{12}m(b^2 + 4c^2) = \frac{1}{12}m(a^2 + 4c^2) = \frac{1}{12}m(a^2 + b^2)$$

Then

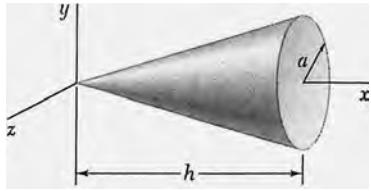
$$b^2 + 4c^2 = a^2 + 4c^2$$

or $\frac{b}{a} = 1$ ◀

and

$$b^2 + 4c^2 = a^2 + b^2$$

or $\frac{c}{a} = \frac{1}{2}$ ◀



PROBLEM 9.174

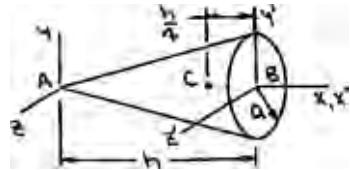
For the right circular cone of Sample Problem 9.11, determine the value of the ratio a/h for which the ellipsoid of inertia of the cone is a sphere when computed (a) at the apex of the cone, (b) at the center of the base of the cone.

SOLUTION

- (a) From Sample Problem 9.11, we have at the apex A

$$I_x = \frac{3}{10}ma^2$$

$$I_y = I_z = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$$



Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{3}{10}ma^2 x^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)y^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore,

$$\frac{3}{10}ma^2 = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right) \quad \text{or} \quad \frac{a}{h} = 2 \blacktriangleleft$$

- (b) From Sample Problem 9.11, we have

$$I_{x'} = \frac{3}{10}ma^2$$

and at the centroid C $I_{y''} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right)$

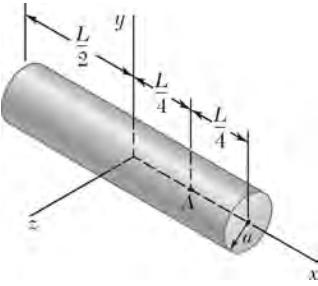
Then $I_{y'} = I_{z'} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right) + m\left(\frac{h}{4}\right)^2 = \frac{1}{20}m(3a^2 + 2h^2)$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part a it then immediately follows that

$$\frac{3}{10}ma^2 = \frac{1}{20}m(3a^2 + 2h^2) \quad \text{or} \quad \frac{a}{h} = \sqrt{\frac{2}{3}} \blacktriangleleft$$



PROBLEM 9.175

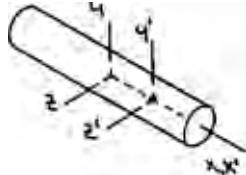
For the homogeneous circular cylinder shown, of radius a and length L , determine the value of the ratio a/L for which the ellipsoid of inertia of the cylinder is a sphere when computed (a) at the centroid of the cylinder, (b) at Point A.

SOLUTION

(a) From Figure 9.28:

$$\bar{I}_x = \frac{1}{2}ma^2$$

$$\bar{I}_y = \bar{I}_z = \frac{1}{12}m(3a^2 + L^2)$$



Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{1}{2}ma^2 x^2 + \frac{1}{12}m(3a^2 + L^2)y^2 + \frac{1}{12}m(3a^2 + L^2)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\frac{1}{2}ma^2 = \frac{1}{12}m(3a^2 + L^2) \quad \text{or} \quad \frac{a}{L} = \frac{1}{\sqrt{3}}$$

(b) Using Figure 9.28 and the parallel-axis theorem, we have

$$I_{x'} = \frac{1}{2}ma^2$$

$$I_{y'} = I_{z'} = \frac{1}{12}m(3a^2 + L^2) + m\left(\frac{L}{4}\right)^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part a it then immediately follows that

$$\frac{1}{2}ma^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right) \quad \text{or} \quad \frac{a}{L} = \sqrt{\frac{7}{12}}$$

PROBLEM 9.176

Given an arbitrary body and three rectangular axes x , y , and z , prove that the mass moment of inertia of the body with respect to any one of the three axes cannot be larger than the sum of the mass moments of inertia of the body with respect to the other two axes. That is, prove that the inequality $I_x \leq I_y + I_z$ and the two similar inequalities are satisfied. Further, prove that $I_y \geq \frac{1}{2}I_x$ if the body is a homogeneous solid of revolution, where x is the axis of revolution and y is a transverse axis.

SOLUTION

(i) To prove $I_y + I_z \geq I_x$

By definition $I_y = \int (z^2 + x^2) dm$

$$I_z = \int (x^2 + y^2) dm$$

Then
$$\begin{aligned} I_y + I_z &= \int (z^2 + x^2) dm + \int (x^2 + y^2) dm \\ &= \int (y^2 + z^2) dm + 2 \int x^2 dm \end{aligned}$$

Now $\int (y^2 + z^2) dm = I_x$ and $\int x^2 dm \geq 0$

$$I_y + I_z \geq I_x \quad \text{Q.E.D.}$$

The proofs of the other two inequalities follow similar steps.

(ii) If the x axis is the axis of revolution, then

$$I_y = I_z$$

and from Part (i) $I_y + I_z \geq I_x$

or $2I_y \geq I_x$

or $I_y \geq \frac{1}{2}I_x \quad \text{Q.E.D.}$

PROBLEM 9.177

Consider a cube of mass m and side a . (a) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the moment of inertia of the cube with respect to one of its diagonals. (b) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

SOLUTION

- (a) At the center of the cube have (using Figure 9.28)

$$I_x = I_y = I_z = \frac{1}{12}m(a^2 + a^2) = \frac{1}{6}ma^2$$

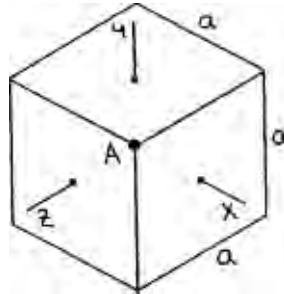
Now observe that symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

Using Equation (9.48), the equation of the ellipsoid of inertia is

$$\left(\frac{1}{6}ma^2\right)x^2 + \left(\frac{1}{6}ma^2\right)y^2 + \left(\frac{1}{6}ma^2\right)z^2 = 1$$

$$\text{or } x^2 + y^2 + z^2 = \frac{6}{ma^2} (= R^2) \blacktriangleleft$$



which is the equation of a sphere.

Since the ellipsoid of inertia is a sphere, the moment of inertia with respect to any axis OL through the center O of the cube must always

$$\text{be the same } \left(R = \frac{1}{\sqrt{I_{OL}}}\right). \quad I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

- (b) The above sketch of the cube is the view seen if the line of sight is along the diagonal that passes through corner A . For a rectangular coordinate system at A and with one of the coordinate axes aligned with the diagonal, an ellipsoid of inertia at A could be constructed. If the cube is then rotated 120° about the diagonal, the mass distribution will remain unchanged. Thus, the ellipsoid will also remain unchanged after it is rotated. As noted at the end of Section 9.17, this is possible only if the ellipsoid is an ellipsoid of revolution, where the diagonal is both the axis of revolution and a principal axis.

It then follows that

$$I_{x'} = I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

In addition, for an ellipsoid of revolution, the two transverse principal moments of inertia are equal and any axis perpendicular to the axis of revolution is a principal axis. Then, applying the parallel-axis theorem between the center of the cube and corner A for any perpendicular axis

$$I_{y'} = I_{z'} = \frac{1}{6}ma^2 + m\left(\frac{\sqrt{3}}{2}a\right)^2$$

$$\text{or } I_{y'} = I_{z'} = \frac{11}{12}ma^2 \blacktriangleleft$$

(Note: Part b can also be solved using the method of Section 9.18.)

PROBLEM 9.177 (Continued)

First note that at corner A

$$I_x = I_y = I_z = \frac{2}{3}ma^2$$

$$I_{xy} = I_{yz} = I_{zx} = \frac{1}{4}ma^2$$

Substituting into Equation (9.56) yields

$$k^3 - 2ma^2k^2 + \frac{55}{48}m^2a^6k - \frac{121}{864}m^3a^9 = 0$$

For which the roots are

$$k_1 = \frac{1}{6}ma^2$$

$$k_2 = k_3 = \frac{11}{12}ma^2$$

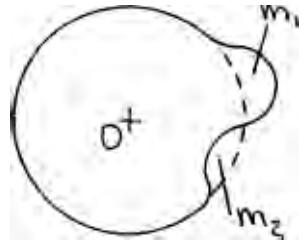
PROBLEM 9.178

Given a homogeneous body of mass m and of arbitrary shape and three rectangular axes x , y , and z with origin at O , prove that the sum $I_x + I_y + I_z$ of the mass moments of inertia of the body cannot be smaller than the similar sum computed for a sphere of the same mass and the same material centered at O . Further, using the result of Problem 9.176, prove that if the body is a solid of revolution, where x is the axis of revolution, its mass moment of inertia I_y about a transverse axis y cannot be smaller than $3ma^2/10$, where a is the radius of the sphere of the same mass and the same material.

SOLUTION

- (i) Using Equation (9.30), we have

$$\begin{aligned} I_x + I_y + I_z &= \int (y^2 + z^2) dm + \int (z^2 + x^2) dm + \int (x^2 + y^2) dm \\ &= 2 \int (x^2 + y^2 + z^2) dm \\ &= 2 \int r^2 dm \end{aligned}$$



where r is the distance from the origin O to the element of mass dm . Now assume that the given body can be formed by adding and subtracting appropriate volumes V_1 and V_2 from a sphere of mass m and radius a which is centered at O ; it then follows that

$$m_1 = m_2 \quad (m_{\text{body}} = m_{\text{sphere}} = m)$$

Then

$$\begin{aligned} (I_x + I_y + I_z)_{\text{body}} &= (I_x + I_y + I_z)_{\text{sphere}} + (I_x + I_y + I_z)_{V_1} \\ &\quad - (I_x + I_y + I_z)_{V_2} \end{aligned}$$

or
$$(I_x + I_y + I_z)_{\text{body}} = (I_x + I_y + I_z)_{\text{sphere}} + 2 \int_{m_1} r^2 dm - 2 \int_{m_2} r^2 dm$$

Now, $m_1 = m_2$ and $r_1 \geq r_2$ for all elements of mass dm in volumes 1 and 2.

$$\int_{m_1} r^2 dm - \int_{m_2} r^2 dm \geq 0$$

so that

$$(I_x + I_y + I_z)_{\text{body}} \geq (I_x + I_y + I_z)_{\text{sphere}} \quad \text{Q.E.D.}$$

PROBLEM 9.178 (Continued)

(ii) First note from Figure 9.28 that for a sphere

$$I_x = I_y = I_z = \frac{2}{5}ma^2$$

Thus $(I_x + I_y + I_z)_{\text{sphere}} = \frac{6}{5}ma^2$

For a solid of revolution, where the x axis is the axis of revolution, we have

$$I_y = I_z$$

Then, using the results of Part (i)

$$(I_x + 2I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

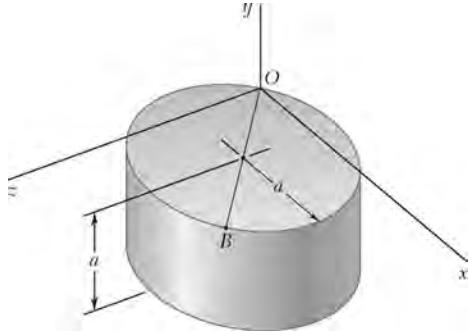
From Problem 9.178 we have $I_y \geq \frac{1}{2}I_x$

or $(2I_y - I_x)_{\text{body}} \geq 0$

Adding the last two inequalities yields

$$(4I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

or $(I_y)_{\text{body}} \geq \frac{3}{10}ma^2 \quad \text{Q.E.D.}$



PROBLEM 9.179*

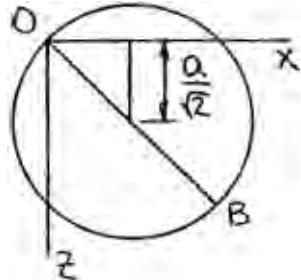
The homogeneous circular cylinder shown has a mass m , and the diameter OB of its top surface forms 45° angles with the x and z axes. (a) Determine the principal mass moments of inertia of the cylinder at the origin O . (b) Compute the angles that the principal axes of inertia at O form with the coordinate axes. (c) Sketch the cylinder, and show the orientation of the principal axes of inertia relative to the x , y , and z axes.

SOLUTION

- (a) First compute the moments of inertia using Figure 9.28 and the parallel-axis theorem.

$$I_x = I_z = \frac{1}{12}m(3a^2 + a^2) + m\left[\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2\right] = \frac{13}{12}ma^2$$

$$I_y = \frac{1}{2}ma^2 + m(a)^2 = \frac{3}{2}ma^2$$



Next observe that the centroidal products of inertia are zero because of symmetry. Then

$$I_{xy} = \bar{J}_{x'y'}^0 + m\bar{x}\bar{y} = m\left(\frac{a}{\sqrt{2}}\right)\left(-\frac{a}{2}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{yz} = \bar{J}_{y'z'}^0 + m\bar{y}\bar{z} = m\left(-\frac{a}{\sqrt{2}}\right)\left(\frac{a}{2}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{zx} = \bar{J}_{z'x'}^0 + m\bar{z}\bar{x} = m\left(\frac{a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right) = \frac{1}{2}ma^2$$

Substituting into Equation (9.56)

$$K^3 - \left(\frac{13}{12} + \frac{3}{2} + \frac{13}{12}\right)ma^2 K^2 + \left[\left(\frac{13}{12} \times \frac{3}{2}\right) + \left(\frac{3}{2} \times \frac{13}{12}\right) + \left(\frac{13}{12} \times \frac{13}{12}\right) - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2\right] (ma^2)^2 K - \left[\left(\frac{13}{12} \times \frac{3}{2} \times \frac{13}{12}\right) - \left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2 - \left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - 2\left(-\frac{1}{2\sqrt{2}}\right)\left(-\frac{1}{2\sqrt{2}}\right)\left(\frac{1}{2}\right)\right] (ma^2)^3 = 0$$

PROBLEM 9.179* (Continued)

Simplifying and letting $K = ma^2\zeta$ yields

$$\zeta^3 - \frac{11}{3}\zeta^2 + \frac{565}{144}\zeta - \frac{95}{96} = 0$$

Solving yields

$$\zeta_1 = 0.363383 \quad \zeta_2 = \frac{19}{12} \quad \zeta_3 = 1.71995$$

The principal moments of inertia are then

$$K_1 = 0.363ma^2 \blacktriangleleft$$

$$K_2 = 1.583ma^2 \blacktriangleleft$$

$$K_3 = 1.720ma^2 \blacktriangleleft$$

- (b) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, we use two of the equations of Equations (9.54) and (9.57).

Thus,

$$(I_x - K)\lambda_x - I_{xy}\lambda_y - I_{zx}\lambda_z = 0 \quad (9.54a)$$

$$-I_{zx}\lambda_x - I_{yz}\lambda_y + (I_z - K)\lambda_z = 0 \quad (9.54c)$$

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \quad (9.57)$$

(Note: Since $I_{xy} = I_{yz}$, Equations (9.54a) and (9.54c) were chosen to simplify the “elimination” of λ_y during the solution process.)

Substituting for the moments and products of inertia in Equations (9.54a) and (9.54c)

$$\begin{aligned} & \left(\frac{13}{12}ma^2 - K\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y - \left(\frac{1}{2}ma^2\right)\lambda_z = 0 \\ & -\left(\frac{1}{2}ma^2\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y + \left(\frac{13}{12}ma^2 - K\right)\lambda_z = 0 \end{aligned}$$

or

$$\left(\frac{13}{12} - \zeta\right)\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y - \frac{1}{2}\lambda_z = 0 \quad (i)$$

and

$$-\frac{1}{2}\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y + \left(\frac{13}{12} - \zeta\right)\lambda_z = 0 \quad (ii)$$

Observe that these equations will be identical, so that one will need to be replaced, if

$$\frac{13}{12} - \zeta = -\frac{1}{2} \quad \text{or} \quad \zeta = \frac{19}{12}$$

PROBLEM 9.179* (Continued)

Thus, a third independent equation will be needed when the direction cosines associated with K_2 are determined. Then for K_1 and K_3

$$\text{Eq. (i) through Eq. (ii): } \left[\frac{13}{12} - \zeta - \left(-\frac{1}{2} \right) \right] \lambda_x + \left[-\frac{1}{2} - \left(\frac{13}{12} - \zeta \right) \right] \lambda_z = 0$$

or

$$\lambda_z = \lambda_x$$

$$\text{Substituting into Eq. (i): } \left(\frac{13}{12} - \zeta \right) \lambda_x + \frac{1}{2\sqrt{2}} \lambda_y - \frac{1}{2} \lambda_x = 0$$

or

$$\lambda_y = 2\sqrt{2} \left(\zeta - \frac{7}{12} \right) \lambda_x$$

Substituting into Equation (9.57):

$$\lambda_x^2 + \left[2\sqrt{2} \left(\zeta - \frac{7}{12} \right) \lambda_x \right]^2 + (\lambda_x)^2 = 1$$

or

$$\left[2 + 8 \left(\zeta - \frac{7}{12} \right)^2 \right] \lambda_x^2 = 1 \quad (\text{iii})$$

K₁: Substituting the value of ζ_1 into Eq. (iii):

$$\left[2 + 8 \left(0.363383 - \frac{7}{12} \right)^2 \right] (\lambda_x)_1^2 = 1$$

or

$$(\lambda_x)_1 = (\lambda_z)_1 = 0.647249$$

and then

$$(\lambda_y)_1 = 2\sqrt{2} \left(0.363383 - \frac{7}{12} \right) (0.647249)$$

$$= -0.402662$$

$$(\theta_x)_1 = (\theta_z)_1 = 49.7^\circ \quad (\theta_y)_1 = 113.7^\circ \quad \blacktriangleleft$$

K₃: Substituting the value of ζ_3 into Eq. (iii):

$$\left[2 + 8 \left(1.71995 - \frac{7}{12} \right)^2 \right] (\lambda_x)_3^2 = 1$$

or

$$(\lambda_x)_3 = (\lambda_z)_3 = 0.284726$$

and then

$$(\lambda_y)_3 = 2\sqrt{2} \left(1.71995 - \frac{7}{12} \right) (0.284726)$$

$$= 0.915348$$

$$(\theta_x)_3 = (\theta_z)_3 = 73.5^\circ \quad (\theta_y)_3 = 23.7^\circ \quad \blacktriangleleft$$

PROBLEM 9.179* (Continued)

K₂: For this case, the set of equations to be solved consists of Equations (9.54a), (9.54b), and (9.57). Now

$$-I_{xy}\lambda_x + (I_y - K)\lambda_y - I_{yz}\lambda_z = 0 \quad (9.54b)$$

Substituting for the moments and products of inertia.

$$-\left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_x + \left(\frac{3}{2}ma^2 - K\right)\lambda_y - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_z = 0$$

or $\frac{1}{2\sqrt{2}}\lambda_x + \left(\frac{3}{2} - \xi\right)\lambda_y + \frac{1}{2\sqrt{2}}\lambda_z = 0 \quad (\text{iv})$

Substituting the value of ξ_2 into Eqs. (i) and (iv):

$$\left(\frac{13}{12} - \frac{19}{12}\right)(\lambda_x)_2 + \frac{1}{2\sqrt{2}}(\lambda_y)_2 - \frac{1}{2}(\lambda_z)_2 = 0$$

$$\frac{1}{2\sqrt{2}}(\lambda_x)_2 + \left(\frac{3}{2} - \frac{19}{12}\right)(\lambda_y)_2 + \frac{1}{2\sqrt{2}}(\lambda_z)_2 = 0$$

or $-(\lambda_x)_2 + \frac{1}{\sqrt{2}}(\lambda_y)_2 - (\lambda_z)_2 = 0$

and $(\lambda_x)_2 - \frac{\sqrt{2}}{6}(\lambda_y)_2 + (\lambda_z)_2 = 0$

Adding yields $(\lambda_y)_2 = 0$

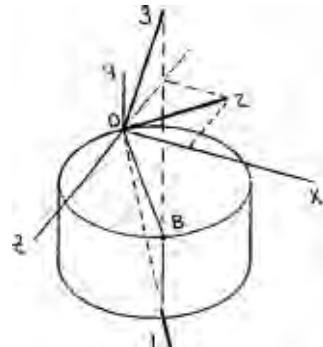
and then $(\lambda_y)_2 = -(\lambda_x)_2$

Substituting into Equation (9.57)

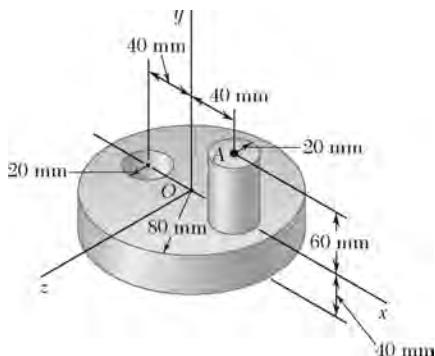
$$(\lambda_x)_2^2 + (\lambda_y)_2^2 + (-\lambda_x)_2^2 = 1$$

or $(\lambda_x)_2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad (\lambda_z)_2 = -\frac{1}{\sqrt{2}}$

$$(\theta_x)_2 = 45.0^\circ \quad (\theta_y)_2 = 90.0^\circ \quad (\theta_z)_2 = 135.0^\circ \quad \blacktriangleleft$$



- (c) Principal axes 1 and 3 lie in the vertical plane of symmetry passing through Points *O* and *B*. Principal axis 2 lies in the *xz* plane.



PROBLEM 9.180

For the component described in Problem 9.165, determine (a) the principal mass moments of inertia at the origin, (b) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x , y , and z axes.

SOLUTION

(a) From the solutions to Problems 9.141 and 9.165 we have

Problem 9.141:

$$I_x = 13.98800 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = 20.55783 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Problem 9.165:

$$I_{yz} = I_{zx} = 0$$

$$I_{xy} = 0.39460 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Eq. (9.55) then becomes

$$\begin{vmatrix} I_x - K & -I_{xy} & 0 \\ -I_{xy} & I_y - K & 0 \\ 0 & 0 & I_z - K \end{vmatrix} = 0 \quad \text{or} \quad (I_x - K)(I_y - K)(I_z - K) - (I_z - K)I_{xy}^2 = 0$$

Thus

$$I_z - K = 0 \quad I_x I_y - (I_x + I_y)K + K^2 - I_{xy}^2 = 0$$

Substituting:

$$K_1 = 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or} \quad K_1 = 14.30 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$\text{and} \quad (13.98800 \times 10^{-3})(20.55783 \times 10^{-3}) - (13.98800 + 20.55783)(10^{-3})K + K^2 - (0.39460 \times 10^{-3})^2 = 0$$

or

$$K^2 - (34.54583 \times 10^{-3})K + 287.4072 \times 10^{-6} = 0$$

Solving yields

$$K_2 = 13.96438 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or} \quad K_2 = 13.96 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

and

$$K_3 = 20.58145 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$\text{or} \quad K_3 = 20.6 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

PROBLEM 9.180 (Continued)

- (b) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eqs. (9.54) and Eq. (9.57). Then

K_1 : Begin with Eqs. (9.54a) and (9.54b) with $I_{yz} = I_{zx} = 0$.

$$\begin{aligned}(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 &= 0 \\ -I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 &= 0\end{aligned}$$

Substituting:

$$\begin{aligned}[(13.98800 - 14.30368) \times 10^{-3}](\lambda_x)_1 - (0.39460 \times 10^{-3})(\lambda_y)_1 &= 0 \\ -(0.39460 \times 10^{-3})(\lambda_x)_1 + [(20.55783 - 14.30368) \times 10^{-3}](\lambda_y)_1 &= 0\end{aligned}$$

Adding yields

$$(\lambda_x)_1 = (\lambda_y)_1 = 0$$

Then using Eq. (9.57) $(\lambda_x)_1^2 + (\lambda_y)_1^2 + (\lambda_z)_1^2 = 1$

or

$$(\lambda_z)_1 = 1$$

$$(\theta_x)_1 = 90.0^\circ \quad (\theta_y)_1 = 90.0^\circ \quad (\theta_z)_1 = 0^\circ \quad \blacktriangleleft$$

K_2 : Begin with Eqs. (9.54b) and (9.54c) with $I_{yz} = I_{zx} = 0$.

$$\begin{aligned}-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 &= 0 \\ (I_z - K_2)(\lambda_z)_2 &= 0\end{aligned} \quad (i)$$

Now

$$I_z \neq K_2 \Rightarrow (\lambda_z)_2 = 0$$

Substituting into Eq. (i):

$$-(0.39460 \times 10^{-3})(\lambda_x)_2 + [(20.55783 - 13.96438) \times 10^{-3}](\lambda_y)_2 = 0$$

or

$$(\lambda_y)_2 = 0.059847(\lambda_x)_2$$

Using Eq. (9.57):

$$(\lambda_x)_2^2 + [0.05984](\lambda_x)_2^2 + (\lambda_z)_2^2 = 1$$

or

$$(\lambda_x)_2 = 0.998214$$

and

$$(\lambda_y)_2 = 0.059740$$

$$(\theta_x)_2 = 3.4^\circ \quad (\theta_y)_2 = 86.6^\circ \quad (\theta_z)_2 = 90.0^\circ \quad \blacktriangleleft$$

K_3 : Begin with Eqs. (9.54b) and (9.54c) with $I_{yz} = I_{zx} = 0$.

$$\begin{aligned}-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 &= 0 \\ (I_z - K_3)(\lambda_z)_3 &= 0\end{aligned} \quad (ii)$$

PROBLEM 9.180 (Continued)

Now

$$I_z \neq K_3 \Rightarrow (\lambda_z)_3 = 0$$

Substituting into Eq. (ii):

$$-(0.39460 \times 10^{-3})(\lambda_x)_3 + [(20.55783 - 20.58145) \times 10^{-3}](\lambda_y)_3 = 0$$

or

$$(\lambda_y)_3 = -16.70618(\lambda_x)_3$$

Using Eq. (9.57):

$$(\lambda_x)_3^2 + [-16.70618(\lambda_x)_3]^2 + (\lambda_z)_3^2 = 1$$

or

$$(\lambda_x)_3 = -0.059751 \quad (\text{axes right-handed set} \Rightarrow " -")$$

and

$$(\lambda_y)_3 = 0.998211$$

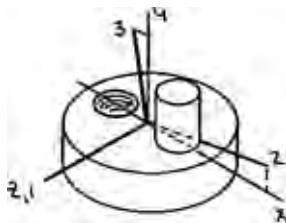
$$(\theta_x)_3 = 93.4^\circ \quad (\theta_y)_3 = 3.43^\circ \quad (\theta_z)_3 = 90.0^\circ \blacktriangleleft$$

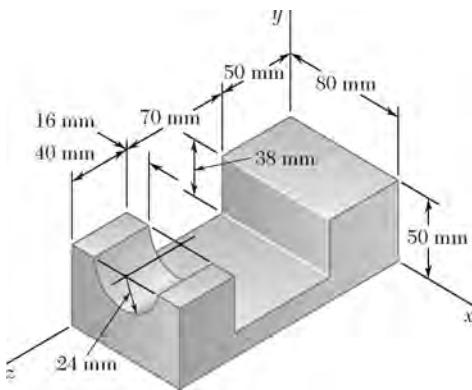
Note: Since the principal axes are orthogonal and $(\theta_z)_2 = (\theta_z)_3 = 90^\circ$, it follows that

$$|(\lambda_x)_2| = |(\lambda_y)_3| \quad |(\lambda_y)_2| = |(\lambda_z)_3|$$

The differences in the above values are due to round-off errors.

- (c) Principal axis 1 coincides with the z axis, while principal axes 2 and 3 lie in the xy plane.





PROBLEM 9.181*

For the component described in Problems 9.145 and 9.149, determine (a) the principal mass moments of inertia at the origin, (b) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x , y , and z axes.

SOLUTION

(a) From the solutions to Problems 9.145 and 9.149, we have

$$\begin{array}{ll} \text{Problem 9.145: } I_x = 26.4325 \times 10^{-3} \text{ kg} \cdot \text{m}^2 & \text{Problem 9.149: } I_{xy} = 2.5002 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ I_y = 31.1726 \times 10^{-3} \text{ kg} \cdot \text{m}^2 & I_{yz} = 4.0627 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ I_z = 8.5773 \times 10^{-3} \text{ kg} \cdot \text{m}^2 & I_{zx} = 8.8062 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{array}$$

Substituting into Eq. (9.56):

$$\begin{aligned} K^3 - & [(26.4325 + 31.1726 + 8.5773)(10^{-3})]K^2 \\ & + [(26.4325)(31.1726) + (31.1726)(8.5773) + (8.5773)(26.4325) \\ & - (2.5002)^2 - (4.0627)^2 - (8.8062)^2](10^{-6})K \\ & - [(26.4325)(31.1726)(8.5773) - (26.4325)(4.0627)^2 \\ & - (31.1726)(8.8062)^2 - (8.5773)(2.5002)^2 \\ & - 2(2.5002)(4.0627)(8.8062)](10^{-9}) = 0 \end{aligned}$$

$$\text{or } K^3 - (66.1824 \times 10^{-3})K^2 + (1217.76 \times 10^{-6})K - (3981.23 \times 10^{-9}) = 0$$

Solving yields

$$\begin{array}{ll} K_1 = 4.1443 \times 10^{-3} \text{ kg} \cdot \text{m}^2 & \text{or } K_1 = 4.14 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft \\ K_2 = 29.7840 \times 10^{-3} \text{ kg} \cdot \text{m}^2 & \text{or } K_2 = 29.8 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft \\ K_3 = 32.2541 \times 10^{-3} \text{ kg} \cdot \text{m}^2 & \text{or } K_3 = 32.3 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft \end{array}$$

PROBLEM 9.181* (Continued)

- (b) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eqs. (9.54) and Eq. (9.57). Then

K₁: Begin with Eqs. (9.54a) and (9.54b).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$

$$I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting:

$$[(26.4325 - 4.1443)(10^{-3})](\lambda_x)_1 - (2.5002 \times 10^{-3})(\lambda_y)_1 - (8.8062 \times 10^{-3})(\lambda_z)_1 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_1 + [(31.1726 - 4.1443)(10^{-3})](\lambda_y)_1 - (4.0627 \times 10^{-3})(\lambda_z)_1 = 0$$

Simplifying

$$8.9146(\lambda_x)_1 - (\lambda_y)_1 - 3.5222(\lambda_z)_1 = 0$$

$$-0.0925(\lambda_x)_1 + (\lambda_y)_1 - 0.1503(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 2.4022(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [8.9146 - 3.5222(2.4022)](\lambda_x)_1$$

$$= 0.45357(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.45357(\lambda_x)_1]^2 + [2.4022(\lambda_x)_1]^2 = 1$$

or

$$(\lambda_x)_1 = 0.37861$$

and

$$(\lambda_y)_1 = 0.17173$$

$$(\lambda_z)_1 = 0.90950$$

$$(\theta_x)_1 = 67.8^\circ \quad (\theta_y)_1 = 80.1^\circ \quad (\theta_z)_1 = 24.6^\circ \quad \blacktriangleleft$$

K₂: Begin with Eqs. (9.54a) and (9.54b).

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$

$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting:

$$[(26.4325 - 29.7840)(10^{-3})](\lambda_x)_2 - (2.5002 \times 10^{-3})(\lambda_y)_2 - (8.8062 \times 10^{-3})(\lambda_z)_2 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_2 + [(31.1726 - 29.7840)(10^{-3})](\lambda_y)_2 - (4.0627 \times 10^{-3})(\lambda_z)_2 = 0$$

PROBLEM 9.181* (Continued)

Simplifying

$$\begin{aligned}-1.3405(\lambda_x)_2 - (\lambda_y)_2 - 3.5222(\lambda_z)_2 &= 0 \\ -1.8005(\lambda_x)_2 + (\lambda_y)_2 - 2.9258(\lambda_z)_2 &= 0\end{aligned}$$

Adding and solving for $(\lambda_z)_2$:

$$(\lambda_z)_2 = -0.48713(\lambda_x)_2$$

and then

$$\begin{aligned}(\lambda_y)_2 &= [-1.3405 - 3.5222(-0.48713)](\lambda_x)_2 \\ &= 0.37527(\lambda_x)_2\end{aligned}$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [0.37527(\lambda_x)_2]^2 + [-0.48713(\lambda_x)_2]^2 = 1$$

or

$$(\lambda_x)_2 = 0.85184$$

and

$$(\lambda_y)_2 = 0.31967 \quad (\lambda_z)_2 = -0.41496$$

$$(\theta_x)_1 = 31.6^\circ \quad (\theta_y)_2 = 71.4^\circ \quad (\theta_z)_2 = 114.5^\circ \quad \blacktriangleleft$$

K₃: Begin with Eqs. (9.54a) and (9.54b).

$$\begin{aligned}(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 &= 0 \\ -I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_z)_3 &= 0\end{aligned}$$

Substituting:

$$\begin{aligned}[(26.4325 - 32.2541)(10^{-3})](\lambda_x)_3 - (2.5002 \times 10^{-3})(\lambda_y)_3 - (8.8062 \times 10^{-3})(\lambda_z)_3 &= 0 \\ -(2.5002 \times 10^{-3})(\lambda_x)_3 + [(31.1726 - 32.2541)(10^{-3})](\lambda_y)_3 - (4.0627 \times 10^{-3})(\lambda_z)_3 &= 0\end{aligned}$$

Simplifying

$$\begin{aligned}-2.3285(\lambda_x)_3 - (\lambda_y)_3 - 3.5222(\lambda_z)_3 &= 0 \\ 2.3118(\lambda_x)_3 + (\lambda_y)_3 + 3.7565(\lambda_z)_3 &= 0\end{aligned}$$

Adding and solving for $(\lambda_z)_3$:

$$(\lambda_z)_3 = 0.071276(\lambda_x)_3$$

and then

$$\begin{aligned}(\lambda_y)_3 &= [-2.3285 - 3.5222(0.071276)](\lambda_x)_3 \\ &= -2.5795(\lambda_x)_3\end{aligned}$$

PROBLEM 9.181* (Continued)

Now substitute into Eq. (9.57):

$$(\lambda_x)_3^2 + [-2.5795(\lambda_x)_3]^2 + [0.071276(\lambda_x)_3]^2 = 1 \quad (\text{i})$$

or

$$(\lambda_x)_3 = 0.36134$$

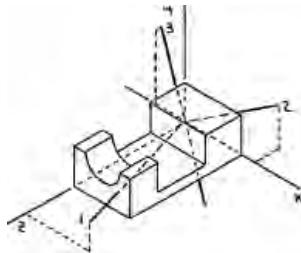
and

$$(\lambda_y)_3 = 0.93208$$

$$(\lambda_z)_3 = 0.025755$$

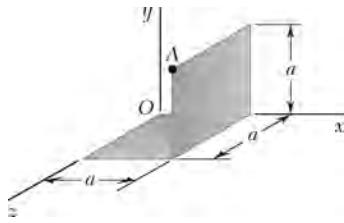
$$(\theta_x)_3 = 68.8^\circ \quad (\theta_y)_3 = 158.8^\circ \quad (\theta_z)_3 = 88.5^\circ \quad \blacktriangleleft$$

- (c) Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.36134$.



Then

$$(\theta_x)_3 = 111.2^\circ \quad (\theta_y)_3 = 21.2^\circ \quad (\theta_z)_3 = 91.5^\circ \quad \blacktriangleleft$$



PROBLEM 9.182*

For the component described in Problem 9.167, determine (a) the principal mass moments of inertia at the origin, (b) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x , y , and z axes.

SOLUTION

(a) From the solution of Problem 9.167, we have

$$\begin{aligned} I_x &= \frac{1}{2} \frac{W}{g} a^2 & I_{xy} &= \frac{1}{4} \frac{W}{g} a^2 \\ I_y &= \frac{W}{g} a^2 & I_{yz} &= \frac{1}{8} \frac{W}{g} a^2 \\ I_z &= \frac{5}{6} \frac{W}{g} a^2 & I_{zx} &= \frac{3}{8} \frac{W}{g} a^2 \end{aligned}$$

Substituting into Eq. (9.56):

$$\begin{aligned} K^3 - &\left[\left(\frac{1}{2} + 1 + \frac{5}{6} \right) \left(\frac{W}{g} a^2 \right) \right] K^2 \\ &+ \left[\left(\frac{1}{2} \right)(1) + (1) \left(\frac{5}{6} \right) + \left(\frac{5}{6} \right) \left(\frac{1}{2} \right) - \left(\frac{1}{4} \right)^2 - \left(\frac{1}{8} \right)^2 - \left(\frac{3}{8} \right)^2 \right] \left(\frac{W}{g} a^2 \right)^2 K \\ &- \left[\left(\frac{1}{2} \right)(1) \left(\frac{5}{6} \right) - \left(\frac{1}{2} \right) \left(\frac{1}{8} \right)^2 - (1) \left(\frac{3}{8} \right)^2 - \left(\frac{5}{6} \right) \left(\frac{1}{4} \right)^2 - 2 \left(\frac{1}{4} \right) \left(\frac{1}{8} \right) \left(\frac{3}{8} \right) \right] \left(\frac{W}{g} a^2 \right)^3 = 0 \end{aligned}$$

Simplifying and letting $K = \frac{W}{g} a^2 K$ yields

$$K^3 - 2.33333K^2 + 1.53125K - 0.192708 = 0$$

Solving yields

$$K_1 = 0.163917 \quad K_2 = 1.05402 \quad K_3 = 1.11539$$

The principal moments of inertia are then

$$K_1 = 0.1639 \frac{W}{g} a^2 \blacktriangleleft$$

$$K_2 = 1.054 \frac{W}{g} a^2 \blacktriangleleft$$

$$K_3 = 1.115 \frac{W}{g} a^2 \blacktriangleleft$$

PROBLEM 9.182* (Continued)

- (b) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eqs. (9.54) and Eq. (9.57). Then

K₁: Begin with Eqs. (9.54a) and (9.54b).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$

$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[\left(\frac{1}{2} - 0.163917 \right) \left(\frac{W}{g} a^2 \right) \right] (\lambda_x)_1 - \left(\frac{1}{4} \frac{W}{g} a^2 \right) (\lambda_y)_1 - \left(\frac{3}{8} \frac{W}{g} a^2 \right) (\lambda_z)_1 = 0$$

$$- \left(\frac{1}{4} \frac{W}{g} a^2 \right) (\lambda_x)_1 + \left[(1 - 0.163917) \left(\frac{W}{g} a^2 \right) \right] (\lambda_y)_1 - \left(\frac{1}{8} \frac{W}{g} a^2 \right) (\lambda_z)_1 = 0$$

Simplifying

$$1.34433(\lambda_x)_1 - (\lambda_y)_1 - 1.5(\lambda_z)_1 = 0$$

$$-0.299013(\lambda_x)_1 + (\lambda_y)_1 - 0.149507(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 0.633715(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [1.34433 - 1.5(0.633715)](\lambda_x)_1$$

$$= 0.393758(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.393758(\lambda_x)_1]^2 + [0.633715(\lambda_x)_1]^2 = 1$$

or

$$(\lambda_x)_1 = 0.801504$$

and

$$(\lambda_y)_1 = 0.315599 \quad (\lambda_z)_1 = 0.507925$$

$$(\theta_x)_1 = 36.7^\circ \quad (\theta_y)_1 = 71.6^\circ \quad (\theta_z)_1 = 59.5^\circ \blacktriangleleft$$

K₂: Begin with Eqs. (9.54a) and (9.54b):

$$(I_x - k_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$

$$-I_{xy}(\lambda_x)_2 + (I_y - k_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

PROBLEM 9.182* (Continued)

Substituting

$$\begin{aligned} & \left[\left(\frac{1}{2} - 1.05402 \right) \left(\frac{W}{g} a^2 \right) \right] (\lambda_x)_2 - \left(\frac{1}{4} \frac{W}{g} a^2 \right) (\lambda_y)_2 - \left(\frac{3}{8} \frac{W}{g} a^2 \right) (\lambda_z)_2 = 0 \\ & - \left(\frac{1}{4} \frac{W}{g} a^2 \right) (\lambda_x)_2 + \left[(1 - 1.05402) \left(\frac{W}{g} a^2 \right) \right] (\lambda_y)_2 - \left(\frac{1}{8} \frac{W}{g} a^2 \right) (\lambda_z)_2 = 0 \end{aligned}$$

Simplifying

$$-2.21608(\lambda_x)_2 - (\lambda_y)_2 - 1.5(\lambda_z)_2 = 0$$

$$4.62792(\lambda_x)_2 + (\lambda_y)_2 + 2.31396(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2 = -2.96309(\lambda_x)_2$$

and then

$$\begin{aligned} (\lambda_y)_2 &= [-2.21608 - 1.5(-2.96309)](\lambda_x)_2 \\ &= 2.22856(\lambda_x)_2 \end{aligned}$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [2.22856(\lambda_x)_2]^2 + [-2.96309(\lambda_x)_2]^2 = 1$$

or

$$(\lambda_x)_2 = 0.260410$$

and

$$(\lambda_y)_2 = 0.580339 \quad (\lambda_z)_2 = -0.771618$$

$$(\theta_x)_2 = 74.9^\circ \quad (\theta_y)_2 = 54.5^\circ \quad (\theta_z)_2 = 140.5^\circ \quad \blacktriangleleft$$

K₃: Begin with Eqs. (9.54a) and (9.54b):

$$\begin{aligned} (I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 &= 0 \\ -I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_z)_3 &= 0 \end{aligned}$$

Substituting

$$\begin{aligned} & \left[\left(\frac{1}{2} - 1.11539 \right) \left(\frac{W}{g} a^2 \right) \right] (\lambda_x)_3 - \left(\frac{1}{4} \frac{W}{g} a^2 \right) (\lambda_y)_3 - \left(\frac{3}{8} \frac{W}{g} a^2 \right) (\lambda_z)_3 = 0 \\ & - \left(\frac{1}{4} \frac{W}{g} a^2 \right) (\lambda_x)_3 + \left[(1 - 1.11539) \left(\frac{W}{g} a^2 \right) \right] (\lambda_y)_3 - \left(\frac{1}{8} \frac{W}{g} a^2 \right) (\lambda_z)_3 = 0 \end{aligned}$$

Simplifying

$$-2.46156(\lambda_x)_3 - (\lambda_y)_3 - 1.5(\lambda_z)_3 = 0$$

$$2.16657(\lambda_x)_3 + (\lambda_y)_3 + 1.08328(\lambda_z)_3 = 0$$

PROBLEM 9.182* (Continued)

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = -0.707885(\lambda_x)_3$$

and then

$$\begin{aligned} (\lambda_y)_3 &= [-2.46156 - 1.5(-0.707885)](\lambda_x)_3 \\ &= -1.39973(\lambda_x)_3 \end{aligned}$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_3^2 + [-1.39973(\lambda_x)_3]^2 + [-0.707885(\lambda_x)_3]^2 = 1 \quad (i)$$

or

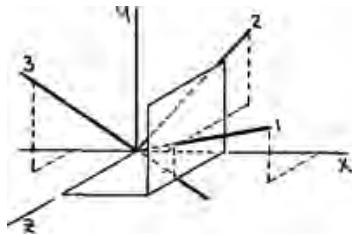
$$(\lambda_x)_3 = 0.537577$$

and

$$(\lambda_y)_3 = -0.752463 \quad (\lambda_z)_3 = -0.380543$$

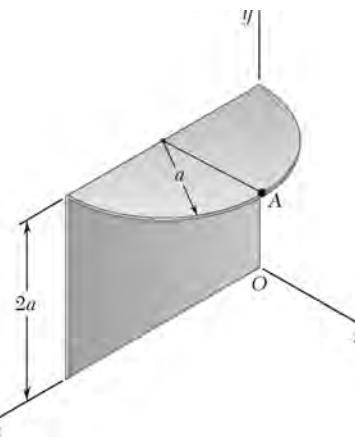
$$(\theta_x)_3 = 57.5^\circ \quad (\theta_y)_3 = 138.8^\circ \quad (\theta_z)_3 = 112.4^\circ \blacksquare$$

- (c) Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.537577$



Then

$$(\theta_x)_3 = 122.5^\circ \quad (\theta_y)_3 = 41.2^\circ \quad (\theta_z)_3 = 67.6^\circ \blacksquare$$



PROBLEM 9.183*

For the component described in Problem 9.168, determine (a) the principal mass moments of inertia at the origin, (b) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x , y , and z axes.

SOLUTION

(a) From the solution to Problem 9.168, we have

$$I_x = 18.91335 \frac{\gamma t}{g} a^4 \quad I_{xy} = 1.33333 \frac{\gamma t}{g} a^4$$

$$I_y = 7.68953 \frac{\gamma t}{g} a^4 \quad I_{yz} = 7.14159 \frac{\gamma t}{g} a^4$$

$$I_z = 12.00922 \frac{\gamma t}{g} a^4 \quad I_{zx} = 0.66667 \frac{\gamma t}{g} a^4$$

Substituting into Eq. (9.56):

$$\begin{aligned} K^3 - & \left[(18.91335 + 7.68953 + 12.00922) \left(\frac{\gamma t}{g} a^4 \right) \right] K^2 \\ & + [(18.91335)(7.68953) + (7.68953)(12.00922) + (12.00922)(18.91335) \\ & - (1.33333)^2 - (7.14159)^2 + (0.66667)^2] \left(\frac{\gamma t}{g} a^4 \right)^2 K \\ & - [(18.91335)(7.68953)(12.00922) - (18.91335)(7.14159)^2 \\ & - (7.68953)(0.66667)^2 - (12.00922)(1.33333)^2 \\ & - 2(1.33333)(7.14159)(0.66667)] \left(\frac{\gamma t}{g} a^4 \right)^3 = 0 \end{aligned}$$

Simplifying and letting

$$K = \frac{\gamma t}{g} a^4 K \quad \text{yields}$$

$$K^3 - 38.61210 K^2 + 411.69009 K - 744.47027 = 0$$

PROBLEM 9.183* (Continued)

Solving yields

$$K_1 = 2.25890 \quad K_2 = 17.27274 \quad K_3 = 19.08046$$

The principal moments of inertia are then

$$K_1 = 2.26 \frac{\gamma t}{g} a^4 \blacktriangleleft$$

$$K_2 = 17.27 \frac{\gamma t}{g} a^4 \blacktriangleleft$$

$$K_3 = 19.08 \frac{\gamma t}{g} a^4 \blacktriangleleft$$

- (b) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eqs. (9.54) and Eq. (9.57). Then

K₁: Begin with Eqs. (9.54a) and (9.54b):

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$

$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[(18.91335 - 2.25890) \left(\frac{\gamma t}{g} a^4 \right) \right] (\lambda_x)_1 - \left(1.33333 \frac{\gamma t}{a} a^4 \right) (\lambda_y)_1 - \left(0.66667 \frac{\gamma t}{g} a^4 \right) (\lambda_z)_1 = 0$$

$$- \left(1.33333 \frac{\gamma t}{g} a^4 \right) (\lambda_x)_1 + \left[(7.68953 - 2.25890) \left(\frac{\gamma t}{g} a^4 \right) \right] (\lambda_y)_1 - \left(7.14159 \frac{\gamma t}{g} a^4 \right) (\lambda_z)_1 = 0$$

Simplifying

$$12.49087(\lambda_x)_1 - (\lambda_y)_1 - 0.5(\lambda_z)_1 = 0$$

$$-0.24552(\lambda_x)_1 + (\lambda_y)_1 - 1.31506(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$

$$(\lambda_z)_1 = 6.74653(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [12.49087 - (0.5)(6.74653)](\lambda_x)_1$$

$$= 9.11761(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [9.11761(\lambda_x)_1]^2 + [6.74653(\lambda_x)_1]^2 = 1$$

or

$$(\lambda_x)_1 = 0.087825$$

and

$$(\lambda_y)_1 = 0.80075 \quad (\lambda_z)_1 = 0.59251$$

$$(\theta_x)_1 = 85.0^\circ \quad (\theta_y)_1 = 36.8^\circ \quad (\theta_z)_1 = 53.7^\circ \blacktriangleleft$$

PROBLEM 9.183* (Continued)

K₂: Begin with Eqs. (9.54a) and (9.54b):

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$

$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting

$$[(18.91335 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_2 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$

$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 + \left[(7.68953 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_2 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$

Simplifying

$$1.23046(\lambda_x)_2 - (\lambda_y)_2 - 0.5(\lambda_z)_2 = 0$$

$$0.13913(\lambda_x)_2 + (\lambda_y)_2 + 0.74522(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2 = -5.58515(\lambda_x)_2$$

and then

$$(\lambda_y)_2 = [1.23046 - (0.5)(-5.58515)](\lambda_x)_2$$

$$= 4.02304(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [4.02304(\lambda_x)_2]^2 + [-5.58515(\lambda_x)_2]^2 = 1$$

or

$$(\lambda_x)_2 = 0.14377$$

and

$$(\lambda_y)_2 = 0.57839 \quad (\lambda_z)_2 = -0.80298$$

$$(\theta_x)_2 = 81.7^\circ \quad (\theta_y)_2 = 54.7^\circ \quad (\theta_z)_2 = 143.4^\circ \blacktriangleleft$$

K₃: Begin with Eqs. (9.54a) and (9.54b):

$$(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 = 0$$

$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_z)_3 = 0$$

Substituting

$$[(18.91335 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_3 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$

$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 + \left[(7.68953 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_3 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$

PROBLEM 9.183* (Continued)

Simplifying

$$\begin{aligned}-0.12533(\lambda_x)_3 - (\lambda_y)_3 - 0.5(\lambda_z)_3 &= 0 \\ 0.11705(\lambda_x)_3 + (\lambda_y)_3 + 0.62695(\lambda_z)_3 &= 0\end{aligned}$$

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = 0.06522(\lambda_x)_3$$

and then

$$\begin{aligned}(\lambda_y)_3 &= [-0.12533 - (0.5)(0.06522)](\lambda_x)_3 \\ &= -0.15794(\lambda_x)_3\end{aligned}$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_3^2 + [-0.15794(\lambda_x)_3]^2 + [0.06522(\lambda_x)_3]^2 = 1 \quad (i)$$

or

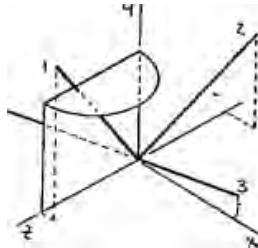
$$(\lambda_x)_3 = 0.98571$$

and

$$(\lambda_y)_3 = -0.15568 \quad (\lambda_z)_3 = 0.06429$$

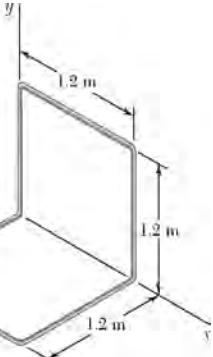
$$(\theta_x)_3 = 9.7^\circ \quad (\theta_y)_3 = 99.0^\circ \quad (\theta_z)_3 = 86.3^\circ \quad \blacktriangleleft$$

- (c) Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.98571$.



Then

$$(\theta_x)_3 = 170.3^\circ \quad (\theta_y)_3 = 81.0^\circ \quad (\theta_z)_3 = 93.7^\circ \quad \blacktriangleleft$$



PROBLEM 9.184*

For the component described in Problems 9.148 and 9.170, determine (a) the principal mass moments of inertia at the origin, (b) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x , y , and z axes.

SOLUTION

- (a) From the solutions to Problems 9.148 and 9.170. We have

$$\begin{aligned} I_x &= 0.32258 \text{ kg} \cdot \text{m}^2 & I_y = I_z &= 0.41933 \text{ kg} \cdot \text{m}^2 \\ I_{xy} = I_{zx} &= 0.096768 \text{ kg} \cdot \text{m}^2 & I_{yz} &= 0 \end{aligned}$$

Substituting into Eq. (9.56) and using

$$\begin{aligned} I_y &= I_z & I_{xy} &= I_{zx} & I_{yz} &= 0 \\ K^3 - [0.32258 + 2(0.41933)]K^2 & & & & & \\ + [2(0.32258)(0.41933) + (0.41933)^2 - 2(0.096768)^2]K & & & & & \\ - [(0.32258)(0.41933)^2 - 2(0.41933)(0.096768)^2] &= 0 & & & & \end{aligned}$$

Simplifying

$$K^3 - 1.16124K^2 + 0.42764K - 0.048869 = 0$$

Solving yields

$$\begin{aligned} K_1 &= 0.22583 \text{ kg} \cdot \text{m}^2 & \text{or } K_1 &= 0.226 \text{ kg} \cdot \text{m}^2 \blacktriangleleft \\ K_2 &= 0.41920 \text{ kg} \cdot \text{m}^2 & \text{or } K_2 &= 0.419 \text{ kg} \cdot \text{m}^2 \blacktriangleleft \\ K_3 &= 0.51621 \text{ kg} \cdot \text{m}^2 & \text{or } K_3 &= 0.516 \text{ kg} \cdot \text{m}^2 \blacktriangleleft \end{aligned}$$

- (b) To determine the direction cosines λ_x , λ_y , λ_z of each principal axis, use two of the equations of Eqs. (9.54) and (9.57). Then

K_1 : Begin with Eqs. (9.54b) and (9.54c):

$$\begin{aligned} -I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - \overset{0}{I_{yz}}(\lambda_z)_3 &= 0 \\ -I_{zx}(\lambda_x)_1 - \overset{0}{I_{yz}}(\lambda_y)_1 + (I_z - K_1)(\lambda_z)_3 &= 0 \end{aligned}$$

PROBLEM 9.184* (Continued)

Substituting

$$\begin{aligned} -(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_y)_1 &= 0 \\ -(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_z)_1 &= 0 \end{aligned}$$

Simplifying yields

$$(\lambda_y)_1 = (\lambda_z)_1 = 0.50009(\lambda_x)_1$$

Now substitute into Eq. (9.54):

$$(\lambda_x)^2 + 2[0.50009(\lambda_x)_1]^2 = 1$$

or

$$(\lambda_x)_1 = 0.81645$$

and

$$(\lambda_y)_1 = (\lambda_z)_1 = 0.40830$$

$$(\theta_x)_1 = 35.3^\circ \quad (\theta_y)_1 = (\theta_z)_1 = 65.9^\circ \blacksquare$$

K₂: Begin with Eqs. (9.54a) and (9.54b)

$$\begin{aligned} (I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 &= 0 \\ -I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 &= 0 \end{aligned}$$

Substituting

$$(0.32258 - 0.41920)(\lambda_x)_2 - (0.096768)(\lambda_y)_2 - (0.096768)(\lambda_z)_2 = 0 \quad (i)$$

$$-(0.96768)(\lambda_x)_2 + (0.41933 - 0.41920)(\lambda_y)_2 = 0 \quad (ii)$$

$$\text{Eq. (ii)} \Rightarrow (\lambda_x)_2 = 0$$

$$\text{and then Eq. (i)} \Rightarrow (\lambda_z)_2 = -(\lambda_y)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + (\lambda_y)_2^2 + [-(\lambda_y)_2]^2 = 1$$

or

$$(\lambda_y)_2 = \frac{1}{\sqrt{2}}$$

and

$$(\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$

$$(\theta_x)_2 = 90^\circ \quad (\theta_y)_2 = 45^\circ \quad (\theta_z)_2 = 135^\circ \blacksquare$$

PROBLEM 9.184* (Continued)

K₃: Begin with Eqs. (9.54b) and (9.54c):

$$\begin{aligned} I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 + I_{yz}(\lambda_z)_3 &= 0 \\ -I_{zx}(\lambda_x)_3 - I_{yz}(\lambda_y)_3 + (I_z - K_3)(\lambda_z)_3 &= 0 \end{aligned}$$

Substituting

$$\begin{aligned} -(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_y)_3 &= 0 \\ -(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_z)_3 &= 0 \end{aligned}$$

Simplifying yields

$$(\lambda_y)_3 = (\lambda_z)_3 = -(\lambda_x)_3$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_3^2 + 2[-(\lambda_x)_3]^2 = 1 \quad (i)$$

or

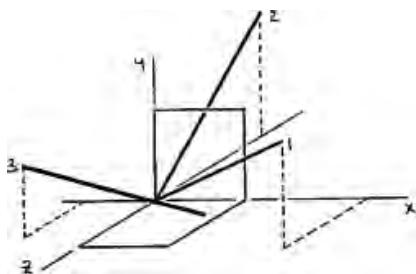
$$(\lambda_x)_3 = \frac{1}{\sqrt{3}}$$

and

$$(\lambda_y)_3 = (\lambda_z)_3 = -\frac{1}{\sqrt{3}}$$

$$(\theta_x)_3 = 54.7^\circ \quad (\theta_y)_3 = (\theta_z)_3 = 125.3^\circ \quad \blacktriangleleft$$

(c)



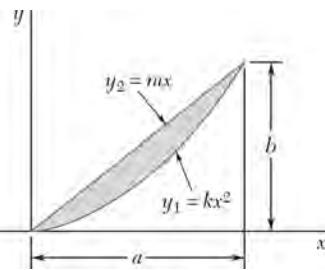
Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen;

That is,

$$(\lambda_x)_3 = -\frac{1}{\sqrt{3}}$$

Then

$$(\theta_x)_3 = 125.3^\circ \quad (\theta_y)_3 = (\theta_z)_3 = 54.7^\circ \quad \blacktriangleleft$$



PROBLEM 9.185

Determine by direct integration the moments of inertia of the shaded area with respect to the x and y axes.

SOLUTION

At

$$x_1 = a, \quad y_1 = y_2 = b$$

$$y_1: \quad b = ka^2 \quad \text{or} \quad k = \frac{b}{a^2}$$

$$y_2: \quad b = ma \quad \text{or} \quad m = \frac{b}{a}$$

Then

$$y_1 = \frac{b}{a^2} x^2$$

$$y_2 = \frac{b}{a} x$$

Now

$$\begin{aligned} dI_x &= \frac{1}{3} (y_2^3 - y_1^3) dx \\ &= \frac{1}{3} \left(\frac{b^3}{a^3} x^3 - \frac{b^3}{a^6} x^6 \right) dx \end{aligned}$$

Then

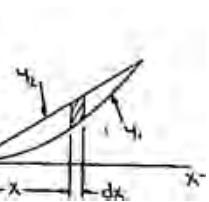
$$\begin{aligned} I_x &= \int dI_x = \int_0^a \frac{b^3}{3} \left(\frac{1}{a^3} x^3 - \frac{1}{a^6} x^6 \right) dx \\ &= \frac{b^3}{3} \left[\frac{1}{4a^3} x^4 - \frac{1}{7a^6} x^7 \right]_0^a \\ &\quad \text{or } I_x = \frac{1}{28} ab^3 \blacktriangleleft \end{aligned}$$

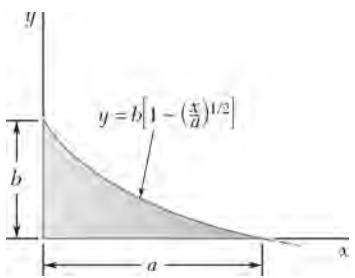
Also

$$dI_y = x^2 dA = x^2 [(y_2 - y_1) dx] = x^2 \left[\left(\frac{b}{a} x - \frac{b}{a^2} x^2 \right) dx \right]$$

Then

$$\begin{aligned} I_y &= \int dI_y = \int_0^a b \left(\frac{1}{a} x^3 - \frac{1}{a^2} x^4 \right) dx \\ &= b \left[\frac{1}{4a} x^4 - \frac{1}{5a^2} x^5 \right]_0^a \end{aligned}$$





PROBLEM 9.186

Determine the moments of inertia and the radii of gyration of the shaded area shown with respect to the x and y axes.

SOLUTION

We have

$$dA = ydx = b \left[1 - \left(\frac{x}{a} \right)^{1/2} \right] dx$$

Then

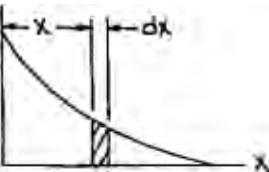
$$\begin{aligned} A &= \int dA = \int_0^a b \left[1 - \left(\frac{x}{a} \right)^{1/2} \right] dx \\ &= b \left[x - \frac{2}{3\sqrt{a}} x^{3/2} \right]_0^a = \frac{1}{3} ab \end{aligned}$$

Now

$$\begin{aligned} dI_x &= \frac{1}{3} y^3 dx = \frac{1}{3} \left\{ b \left[1 - \left(\frac{x}{a} \right)^{1/2} \right] \right\}^3 dx \\ &= \frac{b^3}{3} \left[1 - 3 \left(\frac{x}{a} \right)^{1/2} + 3 \left(\frac{x}{a} \right) - \left(\frac{x}{a} \right)^{3/2} \right] dx \end{aligned}$$

Then

$$\begin{aligned} I_x &= \int dI_x = \int_0^a \frac{b^3}{3} \left[1 - 3 \left(\frac{x}{a} \right)^{1/2} + 3 \left(\frac{x}{a} \right) - \left(\frac{x}{a} \right)^{3/2} \right] dx \\ &= \frac{b^3}{3} \left[x - \frac{2}{\sqrt{a}} x^{3/2} + \frac{3}{2a} x^2 + \frac{2}{5a^{3/2}} x^{5/2} \right]_0^a \end{aligned}$$



$$\text{or } I_x = \frac{1}{30} ab^3 \blacktriangleleft$$

and

$$k_x^2 = \frac{I_x}{A} = \frac{\frac{1}{30} ab^3}{\frac{1}{3} ab}$$

$$\text{or } k_x = \frac{b}{\sqrt{10}} \blacktriangleleft$$

Also

$$dI_y = x^2 dA = x^2 (ydx) = x^2 \left\{ b \left[1 - \left(\frac{x}{a} \right)^{1/2} \right] dx \right\}$$

PROBLEM 9.186 (Continued)

Then

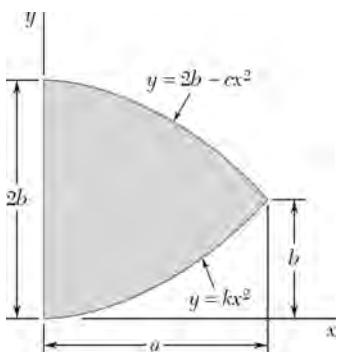
$$I_y = \int dI_y = \int_0^a b \left(x^2 - \frac{1}{\sqrt{a}} x^{5/2} \right) dx = b \left[\frac{1}{3} x^3 - \frac{2}{7\sqrt{a}} x^{7/2} \right]_0^a$$

or $I_y = \frac{1}{21} a^3 b \blacktriangleleft$

and

$$k_y^2 = \frac{I_y}{A} = \frac{\frac{1}{21} a^3 b}{\frac{1}{3} ab}$$

or $k_y = \frac{a}{\sqrt{7}} \blacktriangleleft$



PROBLEM 9.187

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION

At $x = a$, $y_1 = y_2 = b$:

$$y_1: b = ka^2 \quad \text{or} \quad k = \frac{b}{a^2}$$

$$y_2: b = 2b - ca^2 \quad \text{or} \quad c = \frac{b}{a^2}$$

Then

$$y_1 = \frac{b}{a^2} x^2$$

$$y_2 = b \left(2 - \frac{x^2}{a^2} \right)$$

Now

$$dA = (y_2 - y_1) dx$$

$$\begin{aligned} &= \left[b \left(2 - \frac{x^2}{a^2} \right) - \frac{b}{a^2} x^2 \right] dx \\ &= \frac{2b}{a^2} (a^2 - x^2) dx \end{aligned}$$

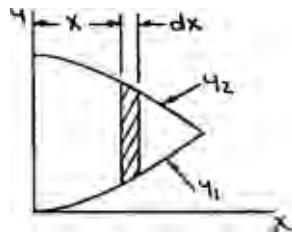
Then

$$A = \int dA = \int_0^a \frac{2b}{a^2} (a^2 - x^2) dx$$

$$\begin{aligned} &= \frac{2b}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\ &= \frac{4}{3} ab \end{aligned}$$

Now

$$dI_y = x^2 dA = x^2 \left[\frac{2b}{a^2} (a^2 - x^2) dx \right]$$



PROBLEM 9.187 (Continued)

Then

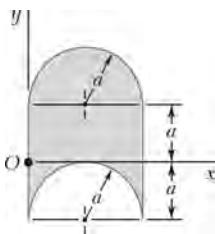
$$I_y = \int dI_y = \int_0^a \frac{2b}{a^2} x^2 (a^2 - x^2) dx \\ = \frac{2b}{a^2} \left[\frac{1}{3} a^2 x^3 - \frac{1}{5} x^5 \right]_0^a$$

or $I_y = \frac{4}{15} a^3 b \blacktriangleleft$

and

$$k_y^2 = \frac{I_y}{A} = \frac{\frac{4}{15} a^3 b}{\frac{4}{3} ab} = \frac{1}{5} a^2$$

or $k_y = \frac{a}{\sqrt{5}} \blacktriangleleft$



PROBLEM 9.188

Determine the moments of inertia of the shaded area shown with respect to the x and y axes.

SOLUTION

We have

$$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$

where

$$(I_x)_1 = \frac{1}{12}(2a)(2a)^3 = \frac{4}{3}a^4$$

Now

$$(I_{AA})_2 = (I_{BB})_3 = \frac{\pi}{8}a^4$$

and

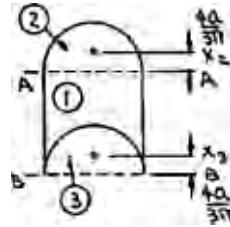
$$(I_{AA})_2 = (\bar{I}_{x_2})_2 + Ad^2$$

or

$$(\bar{I}_{x_2})_2 = (\bar{I}_{x_3})_3 = \frac{\pi}{8}a^4 - \left(\frac{\pi}{2}a^2\right)\left(\frac{4a}{3\pi}\right)^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a_1^4$$

Then

$$\begin{aligned} (I_x)_2 &= (\bar{I}_{x_2})_2 + Ad_2^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4 + \left(\frac{\pi}{2}a^2\right)\left(a + \frac{4a}{3\pi}\right)^2 \\ &= \left(\frac{4}{3} + \frac{5\pi}{8}\right)a^4 \end{aligned}$$



$$\begin{aligned} (I_x)_3 &= (\bar{I}_{x_3})_3 + Ad_3^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4 + \left(\frac{\pi}{2}a^2\right)\left(a - \frac{4a}{3\pi}\right)^2 \\ &= \left(-\frac{4}{3} + \frac{5\pi}{8}\right)a^4 \end{aligned}$$

Finally

$$I_x = \frac{4}{3}a^4 + \left[\left(\frac{4}{3} + \frac{5\pi}{8}\right)a^4\right] - \left[\left(-\frac{4}{3} + \frac{5\pi}{8}\right)a^4\right]$$

$$I_x = 4a^4 \blacktriangleleft$$

Also

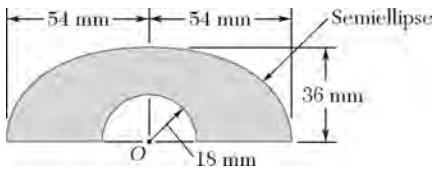
$$I_y = (I_y)_1 + (I_y)_2 - (I_y)_3$$

where

$$(I_y)_1 = \frac{1}{12}(2a)(2a)^3 + (2a)^2(a)^2 = \frac{16}{3}a^4$$

$$(I_y)_2 = (I_y)_3 \left[= \frac{\pi}{8}a^4 + \left(\frac{\pi}{2}a^2\right)(a)^2 \right]$$

$$I_y = \frac{16}{3}a^4 \blacktriangleleft$$

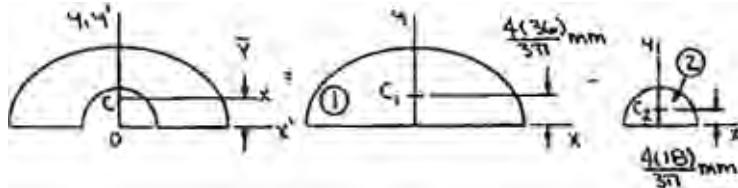


PROBLEM 9.189

Determine the polar moment of inertia of the area shown with respect to (a) Point O, (b) the centroid of the area.

SOLUTION

First locate centroid C of the area.



	A, mm^2	\bar{y}, mm	$\bar{y}A, \text{mm}^3$
1	$\frac{\pi}{2}(54)(36) = 3053.6$	$\frac{48}{\pi} = 15.2789$	46,656
2	$-\frac{\pi}{2}(18)^2 = -508.9$	$\frac{24}{\pi} = 7.6394$	-3888
Σ	2544.7		42,768

Then

$$\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(2544.7 \text{ mm}^2) = 42,768 \text{ mm}^3$$

$$\text{or} \quad \bar{Y} = 16.8067 \text{ mm} \quad \blacktriangleleft$$

$$(a) \quad J_O = (J_O)_1 - (J_O)_2$$

$$\begin{aligned}
 &= \frac{\pi}{8}(54 \text{ mm})(36 \text{ mm})[(54 \text{ mm})^2 + (36 \text{ mm})^2] - \frac{\pi}{4}(18 \text{ mm})^4 \\
 &= (3.2155 \times 10^6 - 0.0824 \times 10^6) \text{ mm}^4 \\
 &= 3.1331 \times 10^6 \text{ mm}^4
 \end{aligned}$$

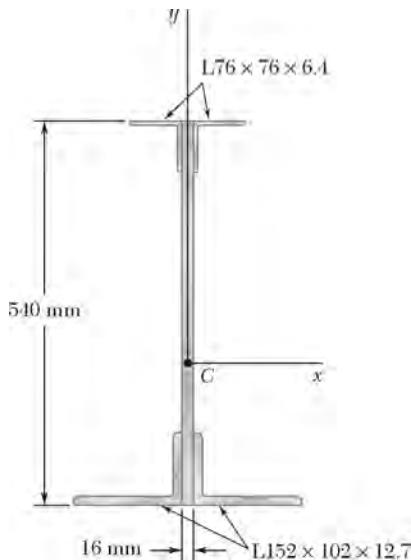
$$\text{or} \quad J_O = 3.13 \times 10^6 \text{ mm}^4 \quad \blacktriangleleft$$

$$(b) \quad J_O = \bar{J}_C + A(\bar{Y})^2$$

$$\text{or} \quad \bar{J}_C = 3.1331 \times 10^6 \text{ mm}^4 - (2544.7 \text{ mm}^2)(16.8067 \text{ mm})^2$$

$$\text{or} \quad \bar{J}_C = 2.41 \times 10^6 \text{ mm}^4 \quad \blacktriangleleft$$

PROBLEM 9.190



To form an unsymmetrical girder, two L76 × 76 × 6.4 mm angles and two L152 × 102 × 12.7-mm angles are welded to a 16-mm steel plate as shown. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

SOLUTION

L76 × 76 × 6.4:

$$A = 929 \text{ mm}^2$$

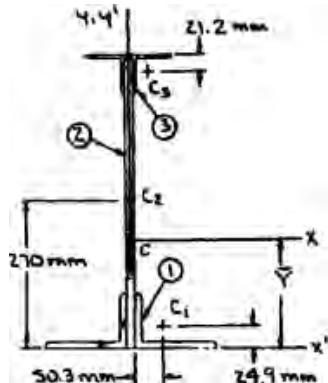
$$\bar{I}_x \times \bar{I}_y = 0.512 \times 10^6 \text{ mm}^4$$

L152 × 102 × 12.7:

$$A = 3060 \text{ mm}^2$$

$$\bar{I}_x = 2.59 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 7.20 \times 10^6 \text{ mm}^4$$



First locate centroid C of the section:

	A, mm^2	\bar{y}	$\bar{y}A, \text{mm}^3$
1	$2(3060) = 6120$	24.9	152,388
2	$(16)(540) = 8640$	270	2,332,800
3	$2(929) = 1858$	518.8	963,930
Σ	16,618		3,449,118

Then

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

or

$$\bar{Y}(16,618 \text{ mm}^2) = 3,449,118$$

or

$$\bar{Y} = 207.553 \text{ mm}$$

Now

$$\bar{I}_x = 2(I_x)_1 + (I_x)_2 + 2(I_x)_3$$

PROBLEM 9.190 (Continued)

where

$$(I_x)_1 = \bar{I}_x + Ad^2 = 2.59 \times 10^6 \text{ mm}^4 + (3060 \text{ mm}^2)[(207.553 - 24.9) \text{ mm}]^2 \\ = 104.678 \times 10^6 \text{ mm}^4$$

$$(I_x)_2 = \bar{I}_x + Ad^2 = \frac{1}{12}(16 \text{ mm})(540 \text{ mm})^3 + (16 \text{ mm})(540 \text{ mm})[(270 - 207.553) \text{ mm}]^2 \\ = 243.645 \times 10^6 \text{ mm}^4$$

$$(I_x)_3 = \bar{I}_x + Ad^2 = 0.512 \times 10^6 \text{ mm}^4 + (929 \text{ mm}^2)[(518.8 - 207.553) \text{ mm}]^2 \\ = 90.5086 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_x = [2(104.678) + 243.645 + 2(90.5086)] \times 10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_x = 634 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

Also

$$\bar{I}_y = 2(I_y)_1 + (\bar{I}_y)_2 + 2(I_y)_3$$

where

$$(I_y)_1 = \bar{I}_y + Ad^2 = 7.20 \times 10^6 \text{ mm}^4 + (3060 \text{ mm}^2)[(8 + 50.3) \text{ mm}]^2 \\ = 17.6006 \times 10^6 \text{ mm}^4$$

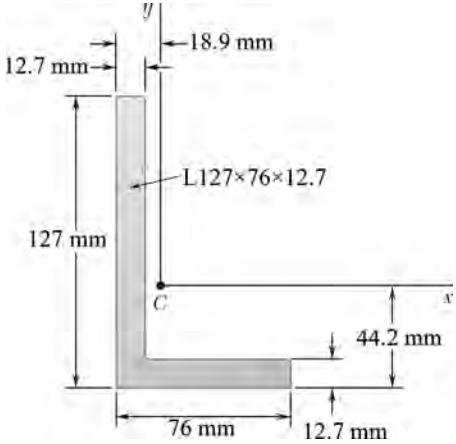
$$(\bar{I}_y)_2 = \frac{1}{12}(540 \text{ mm})(16 \text{ mm})^3 \\ = 0.1843 \times 10^6 \text{ mm}^4$$

$$(I_y)_3 = \bar{I}_y + Ad^2 = 0.512 \times 10^6 \text{ mm}^4 + (929 \text{ mm}^2)[(8 + 21.2) \text{ mm}]^2 \\ = 1.3041 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_y = [2(17.6006) + 0.1843 + 2(1.3041)] \times 10^6 \text{ mm}^4$$

$$\text{or } \bar{I}_y = 38.0 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.191

Using the parallel-axis theorem, determine the product of inertia of the L127 × 76 × 12.7 mm angle cross section shown with respect to the centroidal x and y axes.

SOLUTION

We have

$$\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$$

For each rectangle:

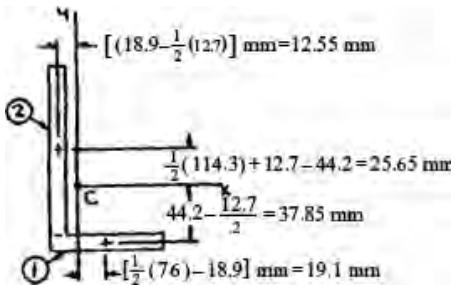
$$I_{xy} = \bar{I}_{x'y'} + \bar{x} \bar{y} A$$

and

$$\bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

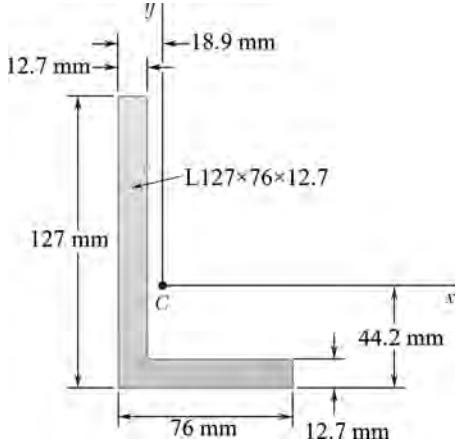
Thus

$$I_{xy} = \Sigma \bar{x} \bar{y} A$$



	$A, \text{ mm}^2$	$\bar{x}, \text{ mm}$	$\bar{y}, \text{ mm}$	$\bar{x} \bar{y} A, \text{ mm}^4$
1	$76 \times 12.7 = 965.2$	19.1	-37.85	-697.777×10^3
2	$114.3 \times 12.7 = 1451.61$	-12.55	25.65	-467.284×10^3
Σ				-1165.061×10^3

$$\bar{I}_{xy} = -1.165 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.192

For the L127 × 76 × 12.7 mm angle cross section shown, use Mohr's circle to determine (a) the moments of inertia and the product of inertia with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise, (b) the orientation of the principal axes through the centroid and the corresponding values of the moments of inertia.

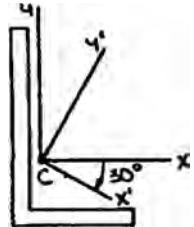
SOLUTION

From Figure 9.13a:

$$\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4 \quad \bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$$

From the solution to Problem 9.191

$$\bar{I}_{xy} = -1.16506 \times 10^6 \text{ mm}^4$$



The Mohr's circle is defined by the diameter XY , where

$$X(3.93 \times 10^6, -1.1651 \times 10^6), \quad Y(1.06 \times 10^6, 1.1651 \times 10^6)$$

Now

$$\bar{I}_{ave} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.93 + 1.06) \times 10^6 = 2.495 \times 10^6 \text{ mm}^4$$

$$\text{and} \quad R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y) \right]^2 + \bar{I}_{xy}^2} = \sqrt{\left[\frac{1}{2}(3.93 - 1.06) \times 10^6 \right]^2 + (-1.1651 \times 10^6)^2} \\ = 1.84843 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ = \frac{-2(-1.1651 \times 10^6)}{(3.93 - 1.06) \times 10^6} \\ = 0.81192$$

or

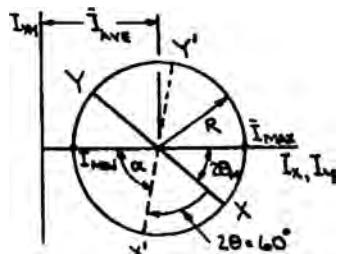
$$2\theta_m = 39.074^\circ$$

and

$$\theta_m = 19.537^\circ$$

Then

$$\alpha = 180^\circ - (39.074^\circ + 60^\circ) \\ = 80.926^\circ$$



PROBLEM 9.192 (Continued)

(a) We have

$$\begin{aligned}\bar{I}_{x'} &= \bar{I}_{\text{ave}} - R \cos \alpha = 2.495 \times 10^6 - 1.84843 \times 10^6 \cos 80.926^\circ \\ &= 2.2035 \times 10^6 \text{ mm}^4\end{aligned}$$

or $\bar{I}_{x'} = 2.20 \times 10^6 \text{ mm}^4$ ◀

$$\begin{aligned}\bar{I}_{y'} &= \bar{I}_{\text{ave}} + R \cos \alpha = 2.495 \times 10^6 + 1.84843 \times 10^6 \cos(80.926^\circ) \\ &= 2.7865 \times 10^6 \text{ mm}^4\end{aligned}$$

or $\bar{I}_{y'} = 2.79 \times 10^6 \text{ mm}^4$ ◀

$$\begin{aligned}\bar{I}_{x'y'} &= -R \sin \alpha = -1.84843 \times 10^6 \sin 80.926^\circ \\ &= -1.82530 \times 10^6 \text{ mm}^4\end{aligned}$$

or $\bar{I}_{x'y'} = -1.825 \times 10^6 \text{ mm}^4$ ◀

(b) First observe that the principal axes are obtained by rotating the xy axes through

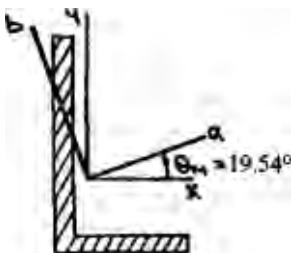
19.54° counterclockwise ◀

about C .

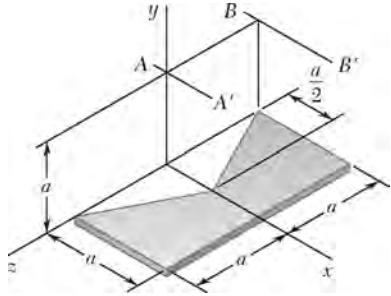
Now $\bar{I}_{\max, \min} = \bar{I}_{\text{ave}} \pm R = (2.495 \pm 1.84843) \times 10^6 \text{ mm}^4$

or $\bar{I}_{\max} = 4.34 \times 10^6 \text{ mm}^4$ ◀

$\bar{I}_{\min} = 0.647 \times 10^6 \text{ mm}^4$ ◀



From the Mohr's circle it is seen that the a axis corresponds to \bar{I}_{\max} and the b axis corresponds to \bar{I}_{\min} .



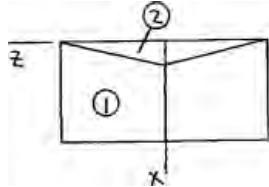
PROBLEM 9.193

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m , determine its mass moment of inertia with respect to (a) the x axis, (b) the y axis.

SOLUTION

First note

$$\begin{aligned}\text{mass} &= m = \rho V = \rho t A \\ &= \rho t \left[(2a)(a) - \frac{1}{2}(2a)\left(\frac{a}{2}\right)\right] \\ &= \frac{3}{2} \rho t a^2\end{aligned}$$



Also

$$\begin{aligned}I_{\text{mass}} &= \rho t I_{\text{area}} \\ &= \frac{2m}{3a^2} I_{\text{area}}\end{aligned}$$

(a) Now

$$\begin{aligned}\bar{I}_{x,\text{area}} &= (I_x)_{1,\text{area}} - 2(I_x)_{2,\text{area}} \\ &= \frac{1}{12}(a)(2a)^3 - 2\left[\frac{1}{12}\left(\frac{a}{2}\right)(a)^3\right] \\ &= \frac{7}{12}a^4\end{aligned}$$

Then

$$\bar{I}_{x,\text{mass}} = \frac{2m}{3a^2} \times \frac{7}{12}a^4$$

or $\bar{I}_x = \frac{7}{18}ma^2$ ◀

(b) We have

$$\begin{aligned}\bar{I}_{z,\text{area}} &= (I_z)_{1,\text{area}} - 2(I_z)_{2,\text{area}} \\ &= \frac{1}{3}(2a)(a)^3 - 2\left[\frac{1}{12}(a)\left(\frac{a}{2}\right)^3\right] \\ &= \frac{31}{48}a^4\end{aligned}$$

Then

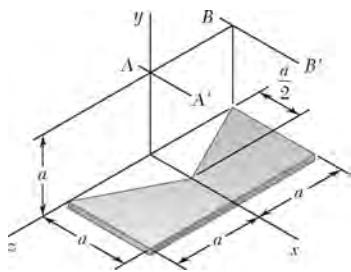
$$\begin{aligned}I_{z,\text{mass}} &= \frac{2m}{3a^2} \times \frac{31}{48}a^4 \\ &= \frac{31}{72}ma^2\end{aligned}$$

PROBLEM 9.193 (Continued)

Finally,

$$\begin{aligned} I_{y, \text{mass}} &= \bar{I}_{x, \text{mass}} + I_{z, \text{mass}} \\ &= \frac{7}{18}ma^2 + \frac{31}{72}ma^2 \\ &= \frac{59}{72}ma^2 \end{aligned}$$

or $I_y = 0.819ma^2$ ◀



PROBLEM 9.194

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m , determine its mass moment of inertia with respect to (a) the axis AA' , (b) the axis BB' , where the AA' and BB' axes are parallel to the x axis and lie in a plane parallel to and at a distance a above the xz plane.

SOLUTION

First note that the x axis is a centroidal axis so that

$$I = \bar{I}_{x, \text{mass}} + md^2$$

and that from the solution to Problem 9.115,

$$\bar{I}_{x, \text{mass}} = \frac{7}{18}ma^2$$

(a) We have

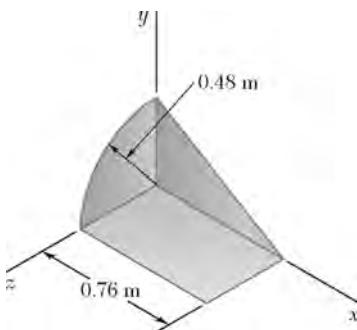
$$I_{AA', \text{mass}} = \frac{7}{18}ma^2 + m(a)^2$$

$$\text{or } I_{AA'} = 1.389ma^2 \blacktriangleleft$$

(b) We have

$$I_{BB', \text{mass}} = \frac{7}{18}ma^2 + m(a\sqrt{2})^2$$

$$\text{or } I_{BB'} = 2.39ma^2 \blacktriangleleft$$



PROBLEM 9.195

A 2 mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3 , determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION

First compute the mass of each component. We have

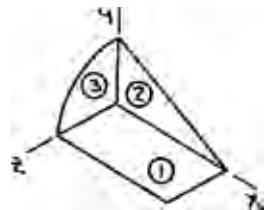
$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} t A$$

Then

$$\begin{aligned} m_1 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})(0.76 \times 0.48) \text{ m}^2 \\ &= 5.72736 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_2 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{1}{2} \times 0.76 \times 0.48\right) \text{ m}^2 \\ &= 2.86368 \text{ kg} \end{aligned}$$

$$\begin{aligned} m_3 &= (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{4} \times 0.48^2\right) \text{ m}^2 \\ &= 2.84101 \text{ kg} \end{aligned}$$



Using Figure 9.28 for component 1 and the equations derived above for components 2 and 3, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 \\ &= \left[\frac{1}{12}(5.72736 \text{ kg})(0.48 \text{ m})^2 + (5.72736 \text{ kg})\left(\frac{0.48}{2} \text{ m}\right)^2 \right] \\ &\quad + \left[\frac{1}{18}(2.86368 \text{ kg})(0.48 \text{ m})^2 + (2.86368 \text{ kg})\left(\frac{0.48}{3} \text{ m}\right)^2 \right] + \left[\frac{1}{2}(2.84101 \text{ kg})(0.48 \text{ m})^2 \right] \\ &= [(0.109965 + 0.329896) + (0.036655 + 0.073310) + (0.327284)] \text{ kg} \cdot \text{m}^2 \\ &= (0.439861 + 0.109965 + 0.327284) \text{ kg} \cdot \text{m}^2 \end{aligned}$$

or $I_x = 0.877 \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.195 (Continued)

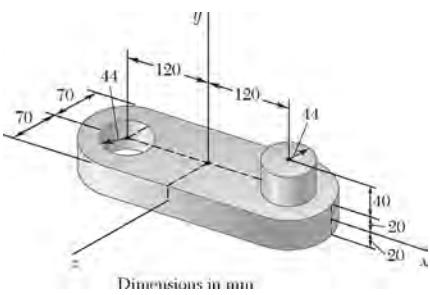
$$\begin{aligned}
 I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
 &= \left\{ \frac{1}{12} (5.72736 \text{ kg}) (0.76^2 + 0.48^2) \text{ m}^2 + (5.72736 \text{ kg}) \left[\left(\frac{0.76}{2} \right)^2 + \left(\frac{0.48}{2} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad + \left[\frac{1}{18} (2.86368 \text{ kg}) (0.76 \text{ m})^2 + (2.86368 \text{ kg}) \left(\frac{0.76}{3} \text{ m} \right)^2 \right] + \left[\frac{1}{4} (2.84101 \text{ kg}) (0.48 \text{ m})^2 \right] \\
 &= [(0.385642 + 1.156927) + (0.091892 + 0.183785) + (0.163642)] \text{ kg} \cdot \text{m}^2 \\
 &= (1.542590 + 0.275677 + 0.163642) \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

or $I_y = 1.982 \text{ kg} \cdot \text{m}^2$ ◀

$$\begin{aligned}
 I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 \\
 &= \left[\frac{1}{12} (5.72736 \text{ kg}) (0.76 \text{ m})^2 + (5.72736 \text{ kg}) \left(\frac{0.76}{2} \text{ m} \right)^2 \right] \\
 &\quad + \left\{ \frac{1}{18} (2.86368 \text{ kg}) (0.76^2 + 0.48^2) \text{ m}^2 + (2.86368 \text{ kg}) \left[\left(\frac{0.76}{3} \right)^2 + \left(\frac{0.48}{3} \right)^2 \right] \text{ m}^2 \right\} \\
 &\quad + \left[\frac{1}{4} (2.84101 \text{ kg}) (0.48 \text{ m})^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 I_z &= [(0.275677 + 0.827031) + (0.128548 + 0.257095) \\
 &\quad + (0.163642)] \text{ kg} \cdot \text{m}^2 \\
 &= (1.102708 + 0.385643 + 0.163642) \text{ kg} \cdot \text{m}^2
 \end{aligned}$$

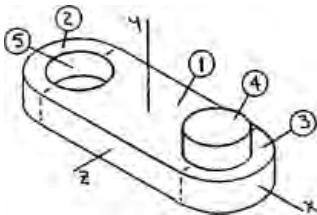
or $I_z = 1.652 \text{ kg} \cdot \text{m}^2$ ◀



PROBLEM 9.196

Determine the mass moments of inertia and the radii of gyration of the steel machine element shown with respect to the x and y axes. (The density of steel is 7850 kg/m^3 .)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{ST} V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)(0.24 \times 0.04 \times 0.14) \text{ m}^3 \\ = 10.5504 \text{ kg}$$

$$m_2 = m_3 = (7850 \text{ kg/m}^3) \left[\frac{\pi}{2} (0.07)^2 \times 0.04 \right] \text{ m}^3 = 2.41683 \text{ kg}$$

$$m_4 = m_5 = (7850 \text{ kg/m}^3) [\pi(0.044)^2 \times (0.04)] \text{ m}^3 = 1.90979 \text{ kg}$$

Using Figure 9.28 for components 1, 4, and 5 and the equations derived above (before the solution to Problem 9.144) for a semicylinder, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 - (I_x)_5 \quad \text{where } (I_x)_2 = (I_x)_3 \\ &= \left[\frac{1}{12} (10.5504 \text{ kg}) (0.04^2 + 0.14^2) \text{ m}^2 \right] + 2 \left\{ \frac{1}{12} (2.41683 \text{ kg}) \left[3(0.07 \text{ m})^2 + (0.04 \text{ m})^2 \right] \right\} \\ &\quad + \left\{ \frac{1}{12} (1.90979 \text{ kg}) \left[3(0.044 \text{ m})^2 + (0.04 \text{ m})^2 \right] + (1.90979 \text{ kg})(0.04 \text{ m})^2 \right\} \\ &\quad - \left\{ \frac{1}{12} (1.90979 \text{ kg}) \left[3(0.044 \text{ m})^2 + (0.04 \text{ m})^2 \right] \right\} \\ &= [(0.0186390) + 2(0.0032829) + (0.0011790 + 0.0030557) - (0.0011790)] \text{ kg} \cdot \text{m}^2 \\ &= 0.0282605 \text{ kg} \cdot \text{m}^2 \\ \text{or } I_x &= 28.3 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacksquare \end{aligned}$$

PROBLEM 9.196 (Continued)

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 - (I_y)_5$$

where

$$(I_y)_2 = (I_y)_3 \quad (I_y)_4 = |(I_y)_5|$$

Then

$$\begin{aligned} I_y &= \left[\frac{1}{12}(10.5504 \text{ kg})(0.24^2 + 0.14^2) \text{ m}^2 \right] \\ &\quad + 2 \left[(2.41683 \text{ kg}) \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) (0.07 \text{ m}^2) + (2.41683 \text{ kg}) \left(0.12 + \frac{4 \times 0.07}{3\pi} \right)^2 \text{ m}^2 \right] \\ &= [(0.0678742) + 2(0.0037881 + 0.0541678)] \text{ kg} \cdot \text{m}^2 \\ &= 0.1837860 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

$$\text{or } I_y = 183.8 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

Also

$$\begin{aligned} m &= m_1 + m_2 + m_3 + m_4 - m_5 \text{ where } m_2 = m_3, \quad m_4 = |m_5| \\ &= (10.5504 + 2 \times 2.41683) \text{ kg} = 15.38406 \text{ kg} \end{aligned}$$

Then

$$k_x^2 = \frac{I_x}{m} = \frac{0.0282605 \text{ kg} \cdot \text{m}^2}{15.38406 \text{ kg}}$$

$$\text{or } k_x = 42.9 \text{ mm} \blacktriangleleft$$

and

$$k_y^2 = \frac{I_y}{m} = \frac{0.1837860 \text{ kg} \cdot \text{m}^2}{15.38406 \text{ kg}}$$

$$\text{or } k_y = 109.3 \text{ mm} \blacktriangleleft$$