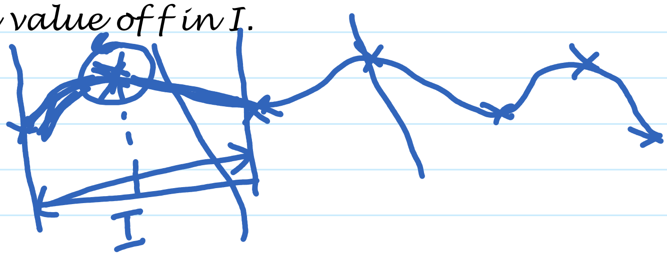
Maxima and Minima

Let f be a function defined on an interval I. Then

(A) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in I$. The number $f(c)$ is called the maximum value of f in I and the point c is called a point of maximum value of f in I.

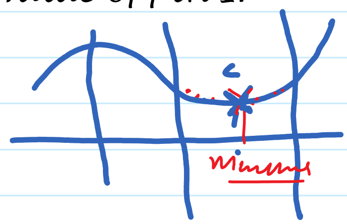
$f(x) \leq f(c)$



$f(c) \geq f(x)$ $\forall x \in I$

(B) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in I$. The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a point of minimum value of f in I.

$c \in I$ $\quad f(c) \leq f(x)$ $\quad f(x) \geq f(c)$ $\forall x \in I$



$f(c) \leq f(x)$ $\forall x \in I$

(C) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I. The number $f(c)$, in this case, is called an extreme value of f in I and the point c is called an extreme point.

Problem 1. *Find the maximum and minimum values, if any, of the function:*

$f(x) = x^2, x \in R$

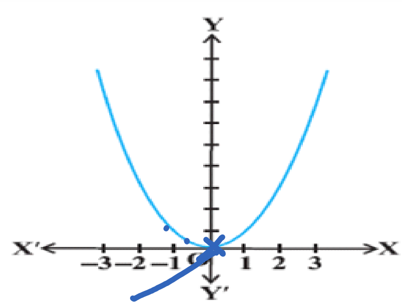
Solution.

$f(x) = x^2$

$f(0) = 0$

$f(x) \geq 0$ $\forall x$ in the nbd of 0

∴ From graph min value is $= 0$ attained at $(0,0)$



$(0,0)$

The given function has minima but no maxima

Problem 2. *Find the maximum and minimum values, if any, of the function:*

$f(x) = |x|, x \in R$

Solution. $f(x) = y = |x|$

Solution:

$f(x) = y = |x|$

$f(x) \geq 0 \quad \forall x$ in nbd of 0

i.e. $f(x) \geq f(0) \quad \forall x$ in the D_f

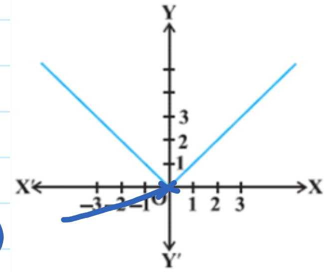
$\therefore f(0)$ is a minimum value

& Minimum Value $= 0$

& the point of minima is $(0,0)$

There is no maxima

$(0,0)$



Polling Quiz

The minimum value of function $f(x) = (2x-1)^2 + 3$ is:

(A) 0

(B) 4

(C) 3 ✓

(D) No minimum value exists.

$f(x) = (2x-1)^2 + 3$

$f'(x) = 2(2x-1) = 0$

$x = 1/2$

At $f''(x) = 4 > 0$

$(2x-1)^2$

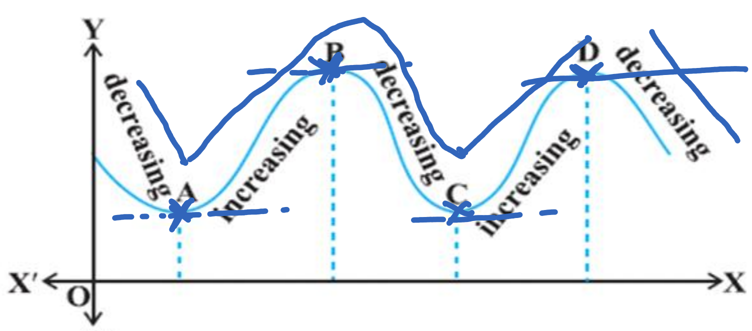
$0 + 3 = 3$

Maxima and Minima

Let us now examine the graph of a function as shown in Figure. Observe that at points A, B, C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called turning points(critical points) of the given function. Further, observe that at turning points, the graph has either a little hill or a little valley.

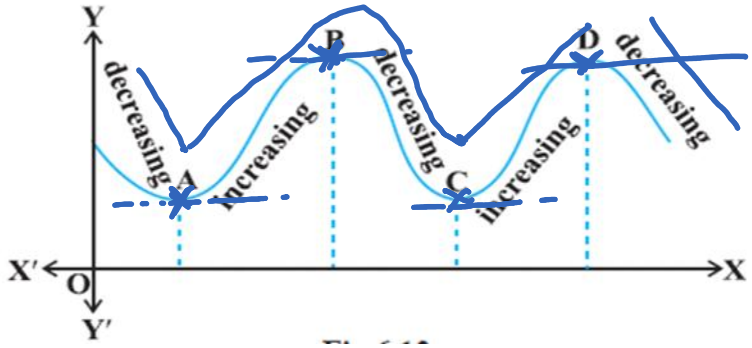
$f'(x) = 0$ all the points like $\boxed{A, B, C \& C}$

$\boxed{f'(x) = 0}$



$f'(x)=0$
all the points like $\boxed{A, B, C \,\&\, C}$

$\boxed{f'(x)=0}$



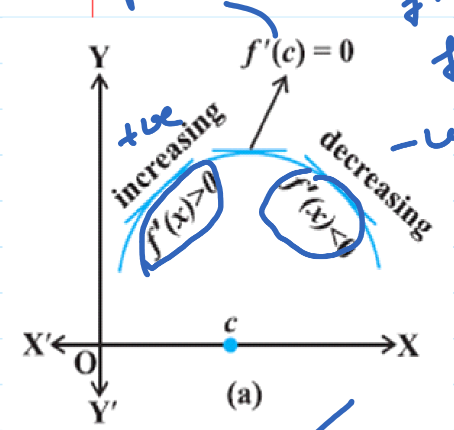
For this reason, the points A and C may be regarded as points of local minimum value (or relative minimum value) and points B and D may be regarded as points of local maximum value (or relative maximum value) for the function. The local maximum value and local minimum value of the function are referred to as local maxima and local minima, respectively, of the function.

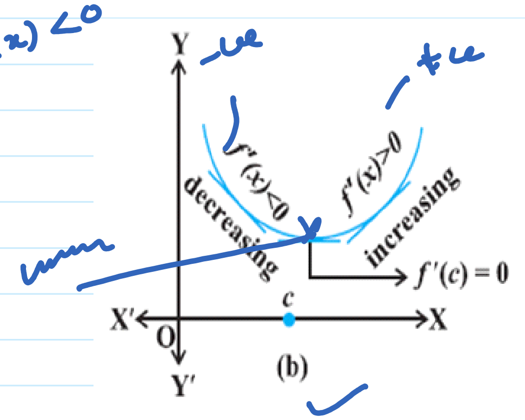
Critical Point

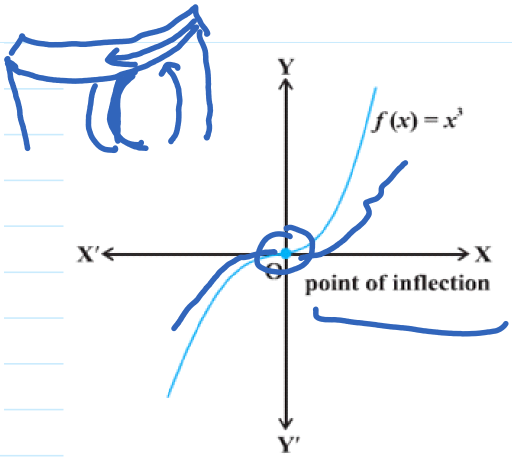
A point c in the domain of a function f at which $\boxed{f'(c) = 0}$ is called critical point of f. A critical point can be a point of maxima, minima or a point of inflexion.

Maxima

$f'(x) > 0$
$f'(x) < 0$







Second Derivative Test

Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$

$f'(x)=0$
$x = a, b, c$
$f''(x)$
$\boxed{f'(x)=0}$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, point c is a point of inflexion.

$\boxed{f'(x)=0}$

Problem 1. Find local maximum and local minimum values of the function f given by $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

Solution.

$$f(x) = 3x^4+4x^3-12x^2+12 \quad \text{--- (1)}$$

Step I Critical points

Diff (1) w.r.t x, we get

$$f'(x) = 12x^3+12x^2-24x$$

$\boxed{f'(c)=0}$

For critical points $f'(x)=0$

$$12x^3+12x^2-24x=0$$
$$12x(x^2+x-2)=0$$
$$x=0 \quad \Big| \quad x^2+x-2=0$$
$$x=1,-2$$

$\therefore \; x = 0, +1, -2$

At $x=0$

$\dfrac{d^2y}{dx^2} < 0$

> 0

$= 0$

At $x=0$

$$f''(x) = 36x^2+24x-24$$
$$=12[3x^2+2x-2]$$

$$f''(0) = 12[0+0-2] = -24 < 0$$

$\therefore$ $x=0$ is a point of maxima

& Maximum value $= f(0) = 12$

At $x=1$

$$f''(x) = 36x^2+24x-24$$
$$f''(1) = 36 > 0$$

$\therefore$ $x=1$ is a point of minima

Min value $= f(1) = 7$

At $x=-2$, $f''(-2) = 72 > 0$

$\therefore$ $x=-2$ is point of minima.

Min value $= f(-2) = \underline{-56}$

Problem 2. *Find local maximum and local minimum values of the function f given by*

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Sol

$f'(x) = 6x^2 - 12x + 6$

Put $f'(x) = 0$

$6x^2 - 12x + 6 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1, 1$

$f''(x) = 12x - 12$

At $x=1$ $f''(1) = 12 \cdot 1 - 12 = \underline{0}$

$x = 1$ is a point of inflexion.

Problem 3. *Find two positive numbers whose sum is 15 and the sum of whose squares is minimum*

Sol let the first number be x then second Number $= 15-x$

let $f(x) = x^2 + (15-x)^2$

$= x^2 + 225 + x^2 - 30x$

$f(x) = 2x^2 - 30x + 225$

$f'(x) = 4x - 30$

Put $f'(x) = 0$

$4x - 30 = 0$

$x = 15/2$

$f''(x) = 4$ $\therefore$ $x = 15/2$ is a point of minima

At $x = 15/2$, $f''(x) = 4 > 0$. Nr all $= 15/2$ & $15 - x$
$15 - 15/2 = 15/2$
$15/2$ & $15/2$ #

Problem 4. *Find the maximum profit that a company can make, if the profit function is given by* $p(x) = 41 - 24x - 18x^2$

Sol

$p'(x) = -24 - 36x$

Put $p'(x) = 0 \Rightarrow -36x - 24 = 0$

$x = -\frac{24}{36} = -2/3$

$x = -2/3$

$p''(x) = -36 < 0$

$\therefore x = -2/3$ is a point of Maxima

Max value $= p(-2/3) = 41 - 24 \times -\frac{2}{3} - 18\left(-\frac{2}{3}\right)^2$

$= 49$ Ans