31. $f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^3+y^3}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$

at (0, 0).

32. $f(x, y) = \begin{cases} \dots \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$

at (0, 0).

33. $f(x, y) = \begin{cases} \dfrac{x^2y}{1+x}, & x \neq -1 \\ y \quad , & (x, y) = (-1, \alpha) \end{cases}$

at $(-1, \alpha)$.

34. $f(x, y, z) = \begin{cases} \dfrac{xyz}{x^2+y^2+z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0 \quad , & (x, y, z) = (0, 0, 0) \end{cases}$

at (0, 0, 0).

35. $f(x, y, z) = \begin{cases} \dfrac{2xy}{x^2-3z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0 \quad , & (x, y, z) = (0, 0, 0) \end{cases}$

at (0, 0, 0).

2.3 Partial Derivatives

The derivative of a function of several variables with respect to one of the independent variables keeping all the other independent variables as constant is called the *partial derivative* of the function with respect to that variable.

Consider the function of two variables $z = f(x, y)$ defined in some domain D of the x-y plane. Let y be held constant, say $y = y_0$. Then, the function $f(x, y_0)$ depends on x alone and is defined in an interval about x, that is $f(x, y_0)$ is a function of one variable x. Let the points (x, y_0) and $(x + \Delta x, y_0)$ be in D, where Δx is an increment in the independent variable x. Then

$$\Delta_x z = f(x + \Delta x, y_0) - f(x, y_0) \tag{2.10}$$

is called the *partial increment* in z with respect to x and is a function of x and Δx.

Similarly, if x is held constant, say $x = x_0$, then the function $f(x_0, y)$ depends only on y and is defined in some interval about y, that is $f(x_0, y)$ is a function of one variable y. Let the points (x_0, y) and $(x_0, y + \Delta y)$ be in D, where Δy is an increment in the independent variable y. Then

$$\Delta_y z = f(x_0, y + \Delta y) - f(x_0, y) \tag{2.11}$$

is called the partial increment in z with respect to y and is a function of y and Δy.

When both x and y are given increments Δx and Δy respectively, then the increment Δz in z is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{2.12}$$

This increment is called the *total increment* in z and is a function of x, y, Δx and Δy.

In general, $\Delta z \neq \Delta_x z + \Delta_y z$. For example, consider the function $z = f(x, y) = xy$ and a point (x_0, y_0). We have

$$\Delta_x z = (x_0 + \Delta x) y_0 - x_0 y_0 = y_0 \Delta x$$

$$\Delta_y z = x_0(y_0 + \Delta y) - x_0 y_0 = x_0 \Delta y$$

$$\Delta z = (x_0 + \Delta x)(y_0 + \Delta y) - x_0 y_0 = x_0\Delta y + y_0 \Delta y + \Delta x \Delta y \neq \Delta_x z + \Delta_y z.$$

Now, consider the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}. \tag{2.13}$$

If this limit exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to x at the point (x_0, y_0) and is denoted by $z_x(x_0, y_0)$ or $f_x(x_0, y_0)$ or $(\partial f/\partial x)(x_0, y_0)$ or $(\partial z/\partial x)(x_0, y_0)$.

Similarly, if the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \tag{2.14}$$

exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to y at the point (x_0, y_0) and is denoted by $z_y(x_0, y_0)$ or $f_y(x_0, y_0)$ or $(\partial z/\partial y)(x_0, y_0)$ or $(\partial f/\partial y)(x_0, y_0)$.

Remark 4

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n variables defined in some domain D in $\mathbb{R}^n$. Let $P_0(x_1, x_2, \ldots, x_n)$ be a point in D. If the limit

$$\lim_{\Delta x_i \to 0} \frac{\Delta_{x_i} z}{\Delta x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots x_n)}{\Delta x_i}$$

exists, then it is called the partial derivative of f at the point P_0 and is denoted by $(\partial f/\partial x_i)(P_0)$.

Remark 5

The definition of continuity, $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$ can be written in alternate forms. Set $x = x_0 + \Delta x$, $y = y_0 + \Delta y$. Define $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Then, $\Delta x \to 0$, $\Delta y \to 0$ implies that $\Delta\rho \to 0$. We note that $|\Delta x| < \Delta\rho$ and $|\Delta y| < \Delta\rho$.

The above definition of continuity is equivalent to the following forms:

(i) $\lim_{\Delta x \to 0, \Delta y \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0.$

(ii) $\lim_{\Delta\rho \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0.$

(iii) $\lim_{\Delta\rho \to 0} \Delta z = 0.$

Example 2.7 Find the first order partial derivatives of the following functions

(i) $f(x, y) = x^2 + y^2 + x$, (ii) $f(x, y) = y\, e^{-x}$, (iii) $f(x, y) = \sin(2x + 3y)$

at the point (x, y) from the first principles.

Solution we have

(i) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + y^2 + (x + \Delta x)] - [x^2 + y^2 + x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x + 1)\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} [2x + 1 + \Delta x] = 2x + 1.$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x^2 + (y+\Delta y)^2 + x] - [x^2 + y^2 + x]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y\,\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y \to 0} [2y + \Delta y] = 2y.$$

(ii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-ye^{-x}(1 - e^{-\Delta x})}{\Delta x} = -ye^{-x} \lim_{\Delta x \to 0} \frac{1 - e^{-\Delta x}}{\Delta x} = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{(y+\Delta y)e^{-x} - ye^{-x}}{\Delta y} = e^{-x}.$$

(iii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin(2(x+\Delta x) + 3y) - \sin(2x + 3y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos(2x + 3y + \Delta x)\sin \Delta x}{\Delta x}$$

$$= 2\cos(2x + 3y).$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\sin(2x + 3(y+\Delta y)) - \sin(2x + 3y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\cos(2x + 3y + 3\Delta y/2)\sin(3\Delta y/2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} [3\cos(2x + 3y + 3\Delta y/2)] \frac{\sin(3\Delta y/2)}{(3\Delta y/2)} = 3\cos(2x + 3y).$$

Example 2.8 Show that the function

$$f(x, y) = \begin{cases} (x+y)\sin\left(\dfrac{1}{x+y}\right), & x + y \neq 0 \\ 0, & x + y = 0 \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left|(x+y)\sin\left(\frac{1}{x+y}\right)\right| \leq |x + y| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2} < \varepsilon.$$

If we choose $\delta < \varepsilon/2$, then

$$|f(x, y) - 0| < \varepsilon, \text{ whenever } 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, the given function is continuous at (0, 0).

Now, at (0, 0), the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x \sin(1/\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

does not exist. Therefore, the partial derivative f_x does not exist at (0, 0).

Similarly at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y \sin(1/\Delta y)}{\Delta y} = \lim_{\Delta y \to 0} \sin\left(\frac{1}{\Delta y}\right)$$

does not exist. Therefore, the partial derivative f_y does not exist at (0, 0).

Example 2.9 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left| \frac{x^2 + y^2}{|x| + |y|} \right| \leq \frac{[|x| + |y|]^2}{|x| + |y|} = |x| + |y| \leq 2\sqrt{x^2 + y^2} < \varepsilon.$$

Taking $\delta < \varepsilon / 2$, we find that

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0).$

Hence, the given function is continuous at (0, 0).

Now, at (0, 0) we have

$$\lim_{\Delta x \to 0} \frac{\Delta_x f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{|\Delta x|} = \begin{cases} 1, \text{ when } \Delta x > 0 \\ -1, \text{ when } \Delta x < 0. \end{cases}$$

Hence, the limit does not exist. Therefore, f_x does not exist at (0, 0).

Also at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y f}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{|\Delta y|} = \begin{cases} 1, \text{ when } \Delta y > 0 \\ -1, \text{ when } \Delta y < 0 \end{cases}$$

does not exist. Therefore, f_y does not exist at (0, 0).

Example 2.10 Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x and f_y exist at (0, 0).

Solution Choose the path $y = mx$. Since the limit

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{mx^2}{(1 + 2m^2)x^2} = \frac{m}{1 + 2m^2}$$

depends on m, the function is not continuous at (0, 0). We now have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0.$$

Therefore, the partial derivatives f_x and f_y exist at (0, 0).

Theorem 2.1 (Sufficient condition for continuity) A sufficient condition for a function $f(x, y)$ to be continuous at a point (x_0, y_0) is that one of its first order partial derivatives exists and is bounded in the neighborhood of (x_0, y_0) and that the other exists at (x_0, y_0).

Proof Let the partial derivative f_x exist and be bounded in the neighborhood of the point (x_0, y_0) and f_y exist at (x_0, y_0). Since f_y exists at (x_0, y_0), we have

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f_y(x_0, y_0).$$

Therefore, we can write

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0) = \Delta y f_y(x_0, y_0) + \varepsilon_1 \Delta y \tag{2.15}$$

where ε_1 depends on Δy and tends to zero as $\Delta y \to 0$. Since f_x exists in the neighborhood of (x_0, y_0), we can write using the Lagrange mean value theorem

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = \Delta x f_x(x_0 + \theta \Delta x, y_0 + \Delta y), 0 < \theta < 1. \tag{2.16}$$

Now, using Eqs. (2.15) and (2.16), we obtain

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] + [f(x_0, y_0 + \Delta y) - f(x_0, y_0)] \\ &= \Delta x f_x(x_0 + \theta \Delta x, y_0 + \Delta y) + \Delta y f_y(x_0, y_0) + \varepsilon_1 \Delta y \end{aligned} \tag{2.17}$$

Since f_x is bounded in the neighborhood of the point (x_0, y_0), we obtain from Eq. (2.17)

$$\lim_{\Delta x \to 0, \Delta y \to 0} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0).$$

Hence, the function $f(x, y)$ is continuous at the point (x_0, y_0).

Geometrical interpretation of partial derivatives

Let $z = f(x, y)$ represent a surface as shown in Fig. 2.3. Let the plane $x = x_0$ = constant intersect the surface $z = f(x, y)$ along the curve $z = f(x_0, y)$. Let $P(x_0, y, 0)$ be a particular point in the x-y plane and $R(x_0, y, z)$ be the corresponding point on the surface, where $z = f(x_0, y)$. Let $Q(x_0, y + \Delta y, 0)$ be a point in the x-y plane in the neighborhood of P and $S(x_0, y + \Delta y, z + \Delta_y z)$ be the corresponding point on the surface $z = f(x, y)$. From Fig. 2.3, we find that $\Delta y = PQ = RS'$ and the function z is increased by $SS' = (z + \Delta_y z) - z = \Delta_y z$. Now, let θ^* be the angle which the chord RS makes with the positive y-axis. Then, from $\Delta RSS'$, we have

$$\tan \theta^* = \frac{SS'}{RS'} = \frac{\Delta_y z}{\Delta y}.$$

Let $\Delta y \to 0$. Then, $\Delta_y z \to 0$. Hence,

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \frac{\partial z}{\partial y} = \tan \theta$$

where in the limit, θ is the angle made by the tangent to the curve $z = f(x_0, y)$ at the point $R(x_0, y, z)$ on the surface $z = f(x, y)$ with the positive y-axis.

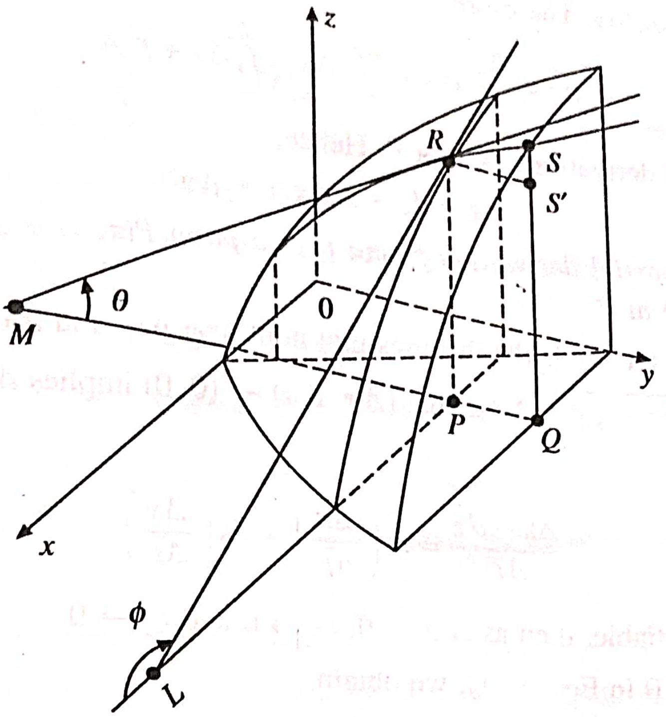

Fig. 2.3. Geometrical representation of partial derivatives.

Now, consider the intersection of the plane $y = y_0$ = constant with the surface $z = f(x, y)$. Following the similar procedure, we obtain $\partial z/\partial x = \tan \phi$, where ϕ is the angle made by the tangent to the curve $z = f(x, y_0)$ at the point (x, y_0, z) on the surface $z = f(x, y)$ with the positive x-axis.

It can be observed that this representation of partial derivatives is a direct extension of the one dimensional case.

2.3.1 Total Differential and Differentiability

Let a function of two variables $z = f(x, y)$ be defined in some domain D in the x-y plane. Let $P(x, y)$ be any point in D and $(x + \Delta x, y + \Delta y)$ be a point in the neighborhood of (x, y), in D. Then,

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

is called the *total increment* in z corresponding to the increments Δx in x and Δy in y.

The function $z = f(x, y)$ is said to be *differentiable* at the point (x, y), if at this point Δz can be written as

$$\Delta z = (a\, \Delta x + b\, \Delta y) + (\varepsilon_1 \Delta x + \varepsilon_2 \Delta y) \tag{2.18}$$

where a, b are independent of Δx, Δy and $\varepsilon_1 = \varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2 = \varepsilon_2(\Delta x, \Delta y)$ are infinitesimals and functions of Δx, Δy such that $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

The first part $a\, \Delta x + b\, \Delta y$ in Eq. (2.18) which is linear in Δx and Δy is called the *total differential* or simply the differential of z at the point (x, y) and is denoted by dz or df. That is

$$dz = a\, \Delta x + b \Delta y \quad \text{or} \quad dz = a\, dx + b\, dy.$$

Let $\Delta y = 0$ in Eq. (2.18). Then, $\Delta z = a\,\Delta x + \varepsilon_1 \Delta x$. Dividing by Δx and taking limits as $\Delta x \to 0$, we obtain $a = \partial z/\partial x$. Similarly, letting $\Delta x = 0$ in Eq. (2.18), dividing by Δy and taking limits as $\Delta y \to 0$, we obtain $b = \partial z/\partial y$. Therefore,

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = f_x \Delta x + f_y \Delta y. \tag{2.19}$$

assuming that the partial derivatives exist at P. Hence,

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y. \tag{2.20}$$

Therefore, *existence of partial derivatives f_x and f_y at a point $P(x, y)$ is a necessary condition for differentiability of $f(x, y)$ at P.*

The second part $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ is the infinitesimal nonlinear part and is of higher order relative to Δx, Δy or $\Delta \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Note that $(\Delta x, \Delta y) \to (0, 0)$ implies $\Delta \rho \to 0$. Eq. (2.20) can be written as

$$\frac{\Delta z - dz}{\Delta \rho} = \varepsilon_1\left(\frac{\Delta x}{\Delta \rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta \rho}\right) \tag{2.21}$$

Now, if $f(x, y)$ is differentiable, then as $\Delta \rho \to 0$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Taking the limit as $\Delta \rho \to 0$ in Eq. (2.21), we obtain

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = \lim_{\Delta\rho \to 0}\left[\varepsilon_1\left(\frac{\Delta x}{\Delta \rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta \rho}\right)\right] = 0 \tag{2.22}$$

since $|\Delta x/\Delta \rho| \le 1$ and $|\Delta y/\Delta \rho| \le 1$.

Therefore, to test differentiability at a point $P(x, y)$, we can use either of the following two approaches.

(i) Show that $\lim_{\Delta\rho \to 0} \dfrac{\Delta z - dz}{\Delta \rho} = 0$ (2.23)

(ii) Find the expressions for $\varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2(\Delta x, \Delta y)$ from Eq. (2.20) and then show that $\lim \varepsilon_1 \to 0$ and $\lim \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ or $\Delta \rho \to 0$.

Note that the function $f(x, y)$ may not be differentiable at a point $P(x, y)$, even if the partial derivatives f_x, f_y exist at P (see Example 2.12). However, if the first order partial derivatives are continuous at the point P, then the function is differentiable at P.

We present this result in the following theorem.

Theorem 2.2 (Sufficient condition for differentiability) If the function $z = f(x, y)$ has continuous first order partial derivatives at a point $P(x, y)$ in D, then $f(x, y)$ is differentiable at P.

Proof Let $P(x, y)$ be a fixed point in D. By the Lagrange mean value theorem, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x\, f_x(x + \theta_1 \Delta x, y),\ 0 < \theta_1 < 1$$

and

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y\, f_y(x + \Delta x, y + \theta_2 \Delta y),\ 0 < \theta_2 < 1.$$

Since f_x and f_y are continuous at (x, y), we can write

$$f_x(x + \theta_1 \Delta x, y) = f_x(x, y) + \varepsilon_1$$

and

$$f_y(x + \Delta x, y + \theta_2 \Delta y) = f_y(x, y) + \varepsilon_2$$

where ε_1, ε_2 are infinitesimals, are functions of Δx, Δy and tend to zero as $\Delta x \to 0$, $\Delta y \to 0$, that is, as $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$. Therefore, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x f_x(x, y) + \varepsilon_1 \Delta x \tag{2.24}$$

and

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y f_y(x, y) + \varepsilon_2 \Delta y \tag{2.25}$$

Now, the total increment is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$
$$= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)].$$

Using Eqs. (2.24) and (2.25), we obtain

$$\Delta z = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{2.26}$$

where the partial derivatives are evaluated at the point $P(x, y)$. Hence, $f(x, y)$ is differentiable at P.

Remark 6

(a) For a function of n variables $z = f(x_1, x_2, \ldots, x_n)$, we write the total differential as

$$dz = f_{x_1} dx_1 + f_{x_2} dx_2 + \ldots + f_{x_n} dx_n. \tag{2.27}$$

(b) Note that continuity of the first partial derivatives f_x and f_y at a point P is a sufficient condition for differentiability at P, that is, a function may be differentiable even if f_x and f_y are not continuous (Problem 5, Exercise 2.2).

(c) The conditions of Theorem 2.2 can be relaxed. It is sufficient that one of the first order partial derivatives is continuous at (x_0, y_0) and the other exists at (x_0, y_0).

Example 2.11 Find the total differential of the following functions

(i) $z = \tan^{-1}(x/y)$, $(x, y) \neq (0, 0)$, (ii) $u = \left(xz + \dfrac{x}{z}\right)^y$, $z \neq 0$.

Solution

(i)
$$f(x, y) = \tan^{-1}\left(\frac{x}{y}\right),\ f_x = \frac{1}{1 + (x/y)^2}\left(\frac{1}{y}\right) = \frac{y}{x^2 + y^2}$$

and

$$f_y = \frac{1}{1 + (x/y)^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}.$$

Therefore, we obtain the total differential as

$$dz = f_x\, dx + f_y\, dy = \frac{1}{x^2 + y^2}(y\, dx - x\, dy).$$

(ii)
$$f(x, y, z) = \left(xz + \frac{x}{z}\right)^y,\ f_x = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$$

$$f_y = \left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right),\ f_z = y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right).$$

Therefore, we obtain the total differential as

$$du = \left(xz + \frac{x}{z}\right)^{y-1}\left[y\left(z + \frac{1}{z}\right)dx + xy\left(1 - \frac{1}{z^2}\right)dz\right] + \left[\left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right)\right]dy.$$

Example 2.12 Show that the function

$$f(x,y)=\begin{cases}\dfrac{x^3+2y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 \quad , & (x,y)=(0,0)\end{cases}$$

(i) is continuous at (0, 0),

(ii) possesses partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$,

(iii) is not differentiable at (0, 0).

Solution

(i) Let $x = r\cos\theta$ and $y = r\sin\theta$. We have

$$|f(x,y)-f(0,0)|=\left|\frac{r^3(\cos^3\theta+2\sin^3\theta)}{r^2}\right|\le r[\,|\cos^3\theta|+2\,|\sin^3\theta|\,]$$

$$\le 3r = 3\sqrt{x^2+y^2}<\varepsilon.$$

Taking $\delta < \varepsilon/3$, we find that

$$|f(x,y)-0|<\varepsilon, \quad \text{whenever} \quad 0<\sqrt{x^2+y^2}<\delta.$$

Therefore, $\lim\limits_{(x,y)\to f(0,0)} f(x,y)=0=f(0,0).$

Hence, $f(x, y)$ is continuous at (0, 0).

(ii) $$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x-0}{\Delta x}=1$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{2\Delta y-0}{\Delta y}=2.$$

Therefore, the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist.

(iii) We have $dz = \Delta x + 2\Delta y$. Using Eq. (2.20), we get

$$\Delta z=\Delta x+2\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$$

Let $\Delta\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$. Now,

$$\Delta z=f(\Delta x,\Delta y)-f(0,0)=\frac{(\Delta x)^3+2(\Delta y)^3}{(\Delta x)^2+(\Delta y)^2}$$

Hence

$$\lim_{\Delta\rho\to 0}\frac{\Delta z-dz}{\Delta\rho}=\lim_{\Delta\rho\to 0}\frac{1}{\Delta\rho}\left[\frac{(\Delta x)^3+2(\Delta y)^3}{(\Delta x)^2+(\Delta y)^2}-(\Delta x+2\Delta y)\right]$$

$$=\lim_{\Delta\rho\to 0}-\left[\frac{\Delta x\Delta y(\Delta y+2\Delta x)}{\{(\Delta x)^2+(\Delta y)^2\}^{3/2}}\right]$$

Let $\Delta x = r\cos\theta$ and $\Delta y = r\sin\theta$. As $(\Delta x, \Delta y) \to (0, 0)$, $\Delta\rho = r \to 0$ for arbitrary θ. Therefore,

$$\lim_{\Delta\rho\to 0}\frac{\Delta z - dz}{\Delta\rho} = -\lim_{r\to 0}\,[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)]$$

$$= -[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)].$$

The limit depends on θ and does not tend to zero for arbitrary θ. Hence, the given function is not differentiable. Alternately, we can write

$$\frac{\Delta z - dz}{\Delta\rho} = -\frac{1}{\Delta\rho}\left[\frac{\Delta x(\Delta y)^2 + 2(\Delta x)^2\Delta y}{(\Delta x)^2 + (\Delta y)^2}\right] = \varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right)$$

where $\varepsilon_1 = -\dfrac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}$ and $\varepsilon_2 = -\dfrac{2(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}$.

Substituting $\Delta x = r\cos\theta$, $\Delta y = r\sin\theta$, we find that ε_1 and ε_2 depend on θ and do not tend to zero for arbitrary θ, in the limit as $r \to 0$.

Example 2.13 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x - y}, & (x, y) \neq (1, -1) \\ 0, & (x, y) = (1, -1) \end{cases}$$

is continuous and differentiable at $(1, -1)$.

Solution We have

$$\lim_{(x,y)\to(1,-1)}\frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,-1)}(x + y) = 0 = f(1, -1).$$

Therefore, the function is continuous at $(1, -1)$.

The partial derivatives are given by

$$f_x(1,-1) = \lim_{\Delta x\to 0}\frac{f(1+\Delta x, -1) - f(1, -1)}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{(1+\Delta x)^2 - 1}{(1+\Delta x)+1} - 0\right] = \lim_{\Delta x\to 0}\frac{2+\Delta x}{2+\Delta x} = 1.$$

$$f_y(1,-1) = \lim_{\Delta y\to 0}\frac{f(1, -1+\Delta y) - f(1, -1)}{\Delta y} = \lim_{\Delta y\to 0}\frac{1}{\Delta y}\left[\frac{1-(-1+\Delta y)^2}{1-(-1+\Delta y)} - 0\right] = \lim_{\Delta y\to 0}\frac{2-\Delta y}{2-\Delta y} = 1.$$

Therefore, the first order partial derivatives exist at $(1, -1)$.

Now, we have

$$f_x(x, y) = \frac{(x-y)(2x) - (x^2 - y^2)(1)}{(x-y)^2} = \frac{x^2 - 2xy + y^2}{(x-y)^2} = \frac{(x-y)^2}{(x-y)^2},\ (x, y) \neq (1, -1)$$

and

$$f_x(x, y) = 1,\ (x, y) = (1, -1).$$

Since

$$\lim_{(x,y)\to(1,-1)} f_x(x, y) = \lim_{(x,y)\to(1,-1)}\frac{(x-y)^2}{(x-y)^2} = 1 = f_x(1, -1)$$

the partial derivative f_x is continuous at (1, –1). Also $f_y(1, -1)$ exists. Hence, $f(x, y)$ is differentiable at (1, –1).

Alternately, we can show that $\lim_{\Delta\rho \to 0} [(\Delta z - dz)/\Delta\rho] = 0$.

2.3.2 Approximation by Total Differentials

From Theorem 2.2, we have for a function $f(x, y)$ of two variables

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx f_x \,\Delta x + f_y \,\Delta y$$

or

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x \,\Delta x + f_y \,\Delta y \tag{2.28}$$

where the partial derivatives are evaluated at the given point (x, y). This result has applications in estimating errors in calculations.

Consider now a function of n variables $x_1, x_2, \ldots, x_n$. Let the function $z = f(x_1, x_2, \ldots, x_n)$ be differentiable at the point $P(x_1, x_2, \ldots, x_n)$. Let there be errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ in measuring the values of $x_1, x_2, \ldots, x_n$ respectively. Then, the computed value of z using the inexact values of the arguments will be obtained with an error

$$\Delta z = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) - f(x_1, x_2, \ldots, x_n). \tag{2.29}$$

When the errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ are small in magnitude, we obtain (using the Remark 6 (a), Eq. (2.27))

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) \approx f(x_1, x_2, \ldots, x_n) + f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n \tag{2.30}$$

where the partial derivatives are evaluated at the point $(x_1, x_2, \ldots, x_n)$. This is the generalization of the result for functions of two variables given in Eq. (2.28).

Since the partial derivatives and errors in arguments can be both positive and negative, we define the *absolute error* as (using Eq. (2.29))

$$|\Delta z| \approx |dz| = |df| = |f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n|.$$

Then,
$$|df| \le |f_{x_1}||\Delta x_1| + |f_{x_2}||\Delta x_2| + \ldots + |f_{x_n}||\Delta x_n| \tag{2.31}$$

gives the *maximum absolute error* in z. If $\max |\Delta x_i| \le \Delta x$, then we can write

$$|df| \le \Delta x\,[\,|f_{x_1}| + |f_{x_2}| + \ldots + |f_{x_n}|\,].$$

The expression $|df|/|f|$ is called the *maximum relative error* and $[\,|df|/|f|\,] \times 100$ is called the *percentage error*.

The maximum relative error can also be written as

$$\frac{|df|}{|f|} \le \left|\frac{\partial f/\partial x_1}{f}\right| |\Delta x_1| + \left|\frac{\partial f/\partial x_2}{f}\right| |\Delta x_2| + \ldots + \left|\frac{\partial f/\partial x_n}{f}\right| |\Delta x_n|$$

$$\le \left|\frac{\partial}{\partial x_1}[\ln |f|]\right| |\Delta x_1| + \left|\frac{\partial}{\partial x_2}[\ln |f|]\right| |\Delta x_2| + \ldots + \left|\frac{\partial}{\partial x_n}[\ln |f|]\right| |\Delta x_n|.$$

Example 2.14 Find the total increment and the total differential of the function $z = x + y + xy$ at the point (1, 2) for $\Delta x = 0.1$ and $\Delta y = -0.2$. Find the maximum absolute error and the maximum relative error.

Solution We are given that $f(x, y) = x + y + xy$, $(x, y) = (1, 2)$.

Therefore, $f(1, 2) = 5$, $f_x(1, 2) = 3$, $f_y(1, 2) = 2$. We have

$$\text{total increment} = f(x + \Delta x, y + \Delta y) - f(x, y)$$
$$= [(x + \Delta x) + (y + \Delta y) + (x + \Delta x)(y + \Delta y)] - [x + y + xy]$$
$$= \Delta x + \Delta y + x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y.$$

At the point (1, 2) with $\Delta x = 0.1$ and $\Delta y = -0.2$, we obtain

$$\text{total increment} = 0.1 - 0.2 + 1(-0.2) + 2(0.1) + (0.1)(-0.2) = -0.12$$

$$\text{total differential} = f_x(1, 2)\,\Delta x + f_y(1, 2)\,\Delta y = 3(0.1) + (2)(-0.2) = -0.1$$

$$\text{maximum absolute error} = |df| = \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y| = 3(0.1) + 2(0.2) = 0.7$$

$$\text{maximum relative error} = \frac{|df|}{|f|} = \frac{0.7}{5} = 0.14.$$

Example 2.15 Using differentials, find an approximate value of

(i) $f(4.1, 4.9)$, where $f(x, y) = \sqrt{x^3 + x^2 y}$,

(ii) $f(2.1, 3.2)$, where $f(x, y) = x^y$.

Solution

(i) Let $(x, y) = (4, 5)$, $\Delta x = 0.1$, $\Delta y = -0.1$. We have

$$f(x, y) = \sqrt{x^3 + x^2 y},\; f(4, 5) = 12,\; f_x(x, y) = \frac{3x^2 + 2xy}{2\sqrt{x^3 + x^2 y}},\; f_x(4, 5) = \frac{11}{3},$$

$$f_y(x, y) = \frac{x^2}{2\sqrt{x^3 + x^2 y}},\; f_y(4, 5) = \frac{2}{3}.$$

Therefore,

$$f(4.1, 4.9) \approx f(4, 5) + f_x(4, 5)\,\Delta x + f_y(4, 5)\,\Delta y$$
$$= 12 + (11/3)(0.1) + (2/3)(-0.1) = 12.3.$$

The exact value is $f(4.1, 4.9) = 12.3$

(ii) Let $(x, y) = (2, 3)$, $\Delta x = 0.1$, $\Delta y = 0.2$. We have

$$f(x, y) = x^y,\; f(2, 3) = 8,\; f_x(x, y) = yx^{y-1},\; f_x(2, 3) = 12,$$

$$f_y(x, y) = x^y \ln x,\; f_y(2, 3) = 8 \ln 2 = 2.54518.$$

Therefore, $f(2.1, 3.2) \approx f(2, 3) + f_x(2, 3)\,\Delta x + f_y(2, 3)\,\Delta y$

$$= 8 + 12(0.1) + (0.2)(5.54518) = 10.3090.$$

The exact value is $f(2.1, 3.2) = 10.7424.$

Therefore, $f(2.1, 3.2) = f(2, 3) + f_x(2, 3)\,\Delta x + f_y(2, 3)\,\Delta y$

$$= 8 + 12(0.1) + (2.408)(0.2) = 9.6816.$$

The exact value is $f(2.1, 3.2) = 10.7424$.

Example 2.16 Find the percentage error in the computed area of an ellipse when an error of 2% is made in measuring the semi major and semi minor axes.

Solution Let the major and minor axes of the ellipse be $2a$ and $2b$ respectively. The errors Δa and Δb in computing the lengths of the semi major and minor axes are

$$\Delta a = a(0.02) = 0.02\,a \quad \text{and} \quad \Delta b = b(0.02) = 0.02\,b.$$

The area of the ellipse is given by $A = \pi ab$. Therefore, we have the following:

Maximum absolute error in computing the area of ellipse is

$$|dA| = \left|\frac{\partial A}{\partial a}\right| |\Delta a| + \left|\frac{\partial A}{\partial b}\right| |\Delta b| = \pi b(0.02a) + \pi a(0.02b) = 0.04\,\pi ab.$$

Maximum relative error is

$$\left|\frac{dA}{A}\right| = (0.04\,\pi ab)\left(\frac{1}{\pi ab}\right) = 0.04.$$

$$\text{Percentage error} = \left|\frac{dA}{A}\right| \times 100 = 4\%.$$

2.3.3 Derivatives of Composite and Implicit Functions (*Chain Rule*)

Let $z = f(x, y)$ be a function of two independent variables x and y. Suppose that x and y are themselves functions of some independent variable t, say $x = \phi(t)$, $y = \psi(t)$. Then, $z = f[\phi(t), \psi(t)]$ is a composite function of the independent variable t. Now, assume that the partial derivatives f_x, f_y are continuous functions of x, y and $\phi(t)$, $\psi(t)$ are differentiable functions of t.

Let Δx, Δy and Δz be the increments respectively in x, y and z corresponding to the increment Δt in t. Then we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

Dividing both sides by Δt, we get

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}. \tag{2.32}$$

Now as $\Delta t \to 0$; $\Delta x \to 0$, $\Delta y \to 0$ and $\varepsilon_1\left(\frac{\Delta x}{\Delta t}\right) \to 0$, $\varepsilon_2\left(\frac{\Delta y}{\Delta t}\right) \to 0$. Therefore, taking limits on both sides in Eq. (2.32) as $\Delta t \to 0$, we obtain

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{2.33}$$

Now, let x and y be functions of two independent variables u and v, say $x = \phi(u, v)$, $y = \psi(u, v)$. Then, $z = f[\phi(u, v), \psi(u, v)]$ is a composite function of two independent variables u and v. Assume that the functions $f(x, y)$, $\phi(u, v)$, $\psi(u, v)$ have continuous partial derivatives with respect to their arguments. Now, consider v as a constant and give an increment Δu to u. Let $\Delta_u x$ and $\Delta_u y$ be the corresponding increments in x and y. Then, the increment Δz in z is given by (using Eq. (2.20))

$$\Delta z = \frac{\partial f}{\partial x} \Delta_u x + \frac{\partial f}{\partial y} \Delta_u y + \varepsilon_1 \Delta_u x + \varepsilon_2 \Delta_u y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta u \to 0$.

Dividing both sides by Δu, we get

$$\frac{\Delta z}{\Delta u} = \frac{\partial f}{\partial x} \frac{\Delta_u x}{\Delta u} + \frac{\partial f}{\partial y} \frac{\Delta_u y}{\Delta u} + \varepsilon_1 \frac{\Delta_u x}{\Delta u} + \varepsilon_2 \frac{\Delta_u y}{\Delta u}. \tag{2.34}$$

Taking limits on both sides in Eq. (2.34) as $\Delta u \to 0$, we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}. \tag{2.35}$$

Similarly, keeping u as constant and varying v, we obtain

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}. \tag{2.36}$$

The rules given in Eqs. (2.35) and (2.36) are called the *chain rules*. These rules can be easily extended to a function of n variables $z = f(x_1, x_2, \ldots, x_n)$. If the partial derivatives of f with respect to all its arguments are continuous and $x_1, x_2, \ldots, x_n$ are differentiable functions of some independent variable t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \ldots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}. \tag{2.37}$$

Example 2.17 Find df/dt at $t = 0$, where

(i) $f(x, y) = x \cos y + e^x \sin y$, $x = t^2 + 1$, $y = t^3 + t$.

(ii) $f(x, y, z) = x^3 + x z^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$.

Solution

(i) When $t = 0$, we get $x = 1$, $y = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (\cos y + e^x \sin y)(2t) + (-x \sin y + e^x \cos y)(3t^2 + 1).$$

Substituting $t = 0$, $x = 1$ and $y = 0$, we obtain $(df/dt) = e$.

(ii) When $t = 0$, we get $x = 1$, $y = 1$, $z = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

$$= (3x^2 + z^2 + yz)(e^t) + (3y^2 + xz)(-\sin t) + (2xz + xy)(3t^2).$$

Substituting $t = 0$, $x = 1$, $y = 1$, $z = 0$, we obtain $(df/dt) = 3$.

Example 2.18 If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Solution Using the chain rule, we obtain

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2e^{-2v}\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y}.$$

Therefore,

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

Change of variables

Suppose that $f(x, y)$ is a function of two independent variables x, y and x, y are functions of two new independent variables u, v given by $x = \phi(u, v)$, $y = \psi(u, v)$. By chain rule, we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

We want to determine $\partial f/\partial x$, $\partial f/\partial y$ in terms of $\partial f/\partial u$ and $\partial f/\partial v$. Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f/\partial x}{\dfrac{\partial f}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial f}{\partial v}\dfrac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\dfrac{\partial f}{\partial v}\dfrac{\partial x}{\partial u} - \dfrac{\partial f}{\partial u}\dfrac{\partial x}{\partial v}} = \frac{1}{\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}}.$$

The determinant

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

is called the *Jacobian* of the variables of transformation. Similarly, we write

$$\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u} = \frac{\partial(f, y)}{\partial(u, v)} = \begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

and

$$\frac{\partial f}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial(x, f)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial f/\partial u & \partial f/\partial v \end{vmatrix} = -\frac{\partial(f, x)}{\partial(u, v)}.$$

Hence, we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(u, v)}\right] \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(u, v)}\right]. \tag{2.38}$$

Similarly, if $f(x, y, z)$ is a function of three independent variables x, y, z and x, y, z are functions of three new independent variables u, v, w given by $x = F(u, v, w)$, $y = G(u, v, w)$, $z = H(u, v, w)$, then by chain rule, we have

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}.$$

Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f}{dx}=\frac{1}{J}\left[\frac{\partial(f,y,z)}{\partial(u,v,w)}\right]=\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

$$\frac{\partial f}{dy}=\frac{1}{J}\left[\frac{\partial(x,f,z)}{\partial(u,v,w)}\right]=-\frac{1}{J}\left[\frac{\partial(f,x,z)}{\partial(u,v,w)}\right]=-\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

$$\frac{\partial f}{dz}=\frac{1}{J}\left[\frac{\partial(x,y,f)}{\partial(u,v,w)}\right]=\frac{1}{J}\left[\frac{\partial(f,x,y)}{\partial(u,v,w)}\right]=\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\end{vmatrix} \tag{2.39}$$

where

$$J=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

is the Jacobian of the variables of transformation.

Example 2.19 If $z=f(x,y), x=r\cos\theta, y=r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Solution The variables of transformation are r and θ. We have

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

$$\frac{\partial(f,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos\theta\frac{\partial f}{\partial r}-\sin\theta\frac{\partial f}{\partial\theta}$$

$$\frac{\partial(f,x)}{\partial(r,\theta)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \partial x/\partial r & \partial x/\partial\theta\end{vmatrix}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \cos\theta & -r\sin\theta\end{vmatrix}=-r\sin\theta\frac{\partial f}{\partial r}-\cos\theta\frac{\partial f}{\partial\theta}.$$

Hence, using Eq. (2.38), we obtain

$$\frac{\partial f}{\partial x}=\frac{1}{J}\left[\frac{\partial(f,y)}{\partial(r,\theta)}\right]=\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}$$

$$\frac{\partial f}{\partial y}=-\frac{1}{J}\left[\frac{\partial(f,x)}{\partial(r,\theta)}\right]=\sin\theta\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}.$$

Squaring and adding, we obtain the required result.

Example 2.20(a) If $u=f(x,y,z)$ and $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2.$$

Solution The variables of transformation are r, θ and ϕ. We have

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}=r^2\sin\theta$$

$$\frac{\partial(f,x,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$=r^2\sin^2\theta\cos\phi\frac{\partial f}{\partial r}+r\sin\theta\cos\theta\cos\phi\frac{\partial f}{\partial\theta}-r\sin\phi\frac{\partial f}{\partial\phi}$$

$$\frac{\partial(f,x,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi\\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$= -r^2 \sin^2\theta \sin\phi \frac{\partial f}{\partial r} - r\sin\theta\cos\theta\sin\phi \frac{\partial f}{\partial \theta} - r\cos\phi \frac{\partial f}{\partial \phi}.$$

$$\frac{\partial(f,x,y)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial \theta & \partial f/\partial \phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\theta & r\sin\theta\cos\phi \end{vmatrix}$$

$$= r^2 \sin\theta\cos\theta \frac{\partial f}{\partial r} - r\sin^2\theta \frac{\partial f}{\partial \theta}.$$

Using Eq. (2.39), we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f,y,z)}{\partial(r,\theta,\phi)}\right] = \sin\theta\cos\phi \frac{\partial f}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial f}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f,x,z)}{\partial(r,\theta,\phi)}\right] = \sin\theta\sin\phi \frac{\partial f}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\frac{\partial f}{\partial z} = \frac{1}{J}\left[\frac{\partial(f,x,y)}{\partial(r,\theta,\phi)}\right] = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}.$$

Squaring and adding, we obtain the required result.

Remark 7

The variables of transformation $u = f(x, y, z)$, $v = g(x, y, z)$, $w = h(x, y, z)$ are functionally related if

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0,$$

that is, there exists a relationship between the variables u, v, w and the transformation is not independent.

Example 2.20(b) Show that the variables $u = x - y + z$, $v = x + y - z$, $w = x^2 + xz - xy$, are functionally related. Find the relationship between them.

Solution The Jacobian of transformation is given by

$$J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2x+z-y & -x & x \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 2x+z-y & -x & x \end{vmatrix} = 0.$$

Hence, the variables are related.

Now, $w = x(x - y + z) = xu$, and $u + v = 2x$. Therefore, $2w = u(u + v)$.

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2} + \frac{1}{1 + y^2/x^2}\left(\frac{1}{x}\right) = \frac{2y}{x^2 + y^2} + \frac{x}{x^2 + y^2} = \frac{2y + x}{x^2 + y^2}.$$

Therefore,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x - y}{2y + x} = \frac{y - 2x}{2y + x}, \quad y \neq -\frac{x}{2}.$$

Exercises 2.2

1. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

has partial derivatives $f_x(0, 0)$, $f_y(0, 0)$, but the partial derivatives are not continuous at $(0, 0)$.

2. Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x - y}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

possesses partial derivatives at $(0, 0)$, though it is not continuous at $(0, 0)$.

3. For the function

$$f(x, y) = \begin{cases} \dfrac{y(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

compute $f_x(0, y)$, $f_y(x, 0)$, $f_x(0, 0)$ and $f_y(0, 0)$, if they exist.

4. Show that the function $f(x, y) = \sqrt{x^2 + y^2}$ is not differentiable at $(0, 0)$.

5. Show that the function

$$f(x, y) = \begin{cases} (x^2 + y^2)\cos\left[\dfrac{1}{\sqrt{x^2 + y^2}}\right], & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is differentiable at $(0, 0)$ and that f_x, f_y are not continuous at $(0, 0)$. Does this result contradict Theorem 2.2?

Find the first order partial derivatives for the following functions at the specified point:

6. $f(x, y) = x^4 - x^2y^2 + y^4$ at $(-1, 1)$.
7. $f(x, y) = \ln(x/y)$ at $(2, 3)$.
8. $f(x, y) = x^2 e^{y/x}$ at $(4, 2)$.
9. $f(x, y) = x/\sqrt{x^2 + y^2}$ at $(6, 7)$.
10. $f(x, y) = \cot^{-1}(x + y)$ at $(1, 2)$.
11. $f(x, y) = \ln\left[\dfrac{\sqrt{x^2 + y^2} - x}{\sqrt{x^2 + y^2} + x}\right]$ at $(3, 4)$.
12. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ at $(2, 1, 2)$.
13. $f(x, y, z) = e^{x/y} + e^{z/y}$ at $(1, 1, 1)$.
14. $f(x, y, z) = (xy)^{\sin z}$ at $(3, 5, \pi/2)$.
15. $f(x, y, z) = \ln(x + \sqrt{y^2 + z^2})$ at $(2, 3, 4)$.

Find dw/dt in following problems.

16. $w = x^2 + y^2$, $x = (t^2 - 1)/t$, $y = t/(t^2 + 1)$ at $t = 1$.
17. $w = x^2 + y^2 + z^2$, $x = \cos t$, $y = \ln(t + 1)$, $z = e^t$ at $t = 0$.
18. $w = e^x \sin(y + 2z)$, $x = t$, $y = 1/t$, $z = t^2$.
19. $w = xy + yz + zx$, $x = t^2$, $y = te^t$, $z = te^{-t}$.
20. $w = z \ln y + y \ln z + xyz$, $x = \sin t$, $y = t^2 + 1$, $z = \cos^{-1} t$ at $t = 0$.

Verify the given results in the following problems:

21. If $z = f(ax + by)$, then $b\dfrac{\partial z}{\partial x} - a\dfrac{\partial z}{\partial y} = 0$.

22. If $z = \log[(x^2 - y^2)/(x^2 + y^2)]$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

23. If $u = f(x - y, y - z, z - x)$, then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

24. If $z = f(x, y)$, $x = r\cosh\theta$, $y = r\sinh\theta$, then

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

25. If $z = y + f(u)$, $u = \dfrac{x}{y}$, then $u\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$.

26. If $w = f(u, v)$, $u = \sqrt{x^2 + y^2}$, $v = \cot^{-1}(y/x)$, then

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2 + y^2}\left[(x^2 + y^2)\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

27. If $z = f(x, y)$, $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, where α is a constant, then

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

28. If $z = \ln(u^2 + v)$, $u = e^{x + y^2}$, $v = x + y^2$, then $2y\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.

29. If $w = \sqrt{x^2 + y^2 + z^2}$, $x = u\cos v$, $y = u\sin v$, $z = uv$, then

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1 + v^2}}$$

30. If $w = \sin^{-1} u$, $u = (x^2 + y^2 + z^2)/(x + y + z)$, then

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \tan w.$$

Check whether the variables in the following transformations are functionally related. If so, find the relationship between them.

31. $u = x^2 - y^2 - z^2$, $v = x^2 - y^2 + z^2$, $w = x^4 + y^4 + z^4 - 2x^2y^2$.
32. $u = x + 3z$, $v = x - y - z$, $w = y^2 + 16z^2 + 8yz$.
33. $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$, $w = x^3 + y^3 + z^3 - 3xyz$.
34. $u = (x + y)/(1 - xy)$, $v = \tan^{-1} x + \tan^{-1} y$, $x > 0, y > 0, xy < 1$.
35. $u = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$, $v = \sin^{-1} x + \sin^{-1} y$, $x \ge 0$, $y \ge 0$, $x^2 + y^2 \le 1$.

Using implicit differentiation, obtain the following:

36. $\dfrac{dy}{dx}$, when $x^y + y^x = \alpha$, α any constant, $x > 0, y > 0$.