29. $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, $\mathbf{h} = \begin{bmatrix} 25 \\ 13 \end{bmatrix} e^{5t}$.

30. In Problems 25 and 26, use the method of diagonalisation to find the solution of the systems.

5.7 Answers and Hints

Exercise 5.1

1. Constant coeff.
2. Variable coeff.
3. Constant coeff.
4. Variable coeff.
5. Variable coeff.
6. Variable coeff.
7. Any subinterval on $(-\infty, 0)$, $(0, \infty)$.
8. Any subinterval on $(-\infty, \infty)$.
9. Any subinterval on $(-\infty, 0)$, $(0, \infty)$.
10. Any subinterval on $[0, \infty)$.
11. Any subinterval on $(3, \infty)$.
12. Any subinterval on $(0, \infty)$.
13. Any subinterval on $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$.
14. $4m < x < 4(m + 1)$, $m = 0, 2, 4, \ldots$.
15. No, because the equation is not normal on any interval containing $x = 0$, Remark 1 is also not applicable.
16. $2x$. No, because the equation is not normal on any interval containing $x = 0$.
17. No, because $x = 0$ at which the equation is not normal is included in the interval $[-3, 3]$, even though the conditions are specified at $x = 2$.
21. $6x + 3 = (3/4)(2x) + (3/2)(3x + 2)$, linearly dependent.
22. Dependent, $9x^2 - x + 2 = 3(x^2 - x) + 2(3x^2 + x + 1)$.
23. Independent, no linear combination can be found, alternately $W = 14$.
24. $W = -16 \sin^6 x$, linearly indpendent.
25. $W = 1$, linearly independent.
26. Dependent, $W = 0$, $x \in I$. Alternately, $\cosh x = e^x - \sinh x$.
27. Linearly independent, $W = -4/x$.
28. Dependent, $W = 0$.
29. Linearly independent, $W = -4$.
30. Dependent, $\sinh x = \cosh x - e^{-x}$.
31. $W = -2 \tan^3 x$, linearly independent on $(0, \pi/2)$, $\left((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right)$, $n = 1, 2, \ldots$.
32. (i) Three, (ii) Three.
33. $W(y_1, y_2) = 2$, $y_3 = 2y_1 - y_2/2$.
34. $y_i'' = -(a_1/a_0)y_i' - (a_2/a_0)y_i$, $W(x) = y_1y_2' - y_2y_1'$. Differentiating $W(x)$ and substituting for y_i'', we obtain $a_0W'(x) + a_1W(x) = 0$. Finding the integrating factor we obtain the solution as given. The value of c depends on y_1, y_2.
35. Substitution shows that $\cos at$, $\sin at$ are solutions. $W = a \neq 0$. y_1, y_2 are linearly independent on any interval I. Using the Abel's formula we get $W = c$, where c can be taken as a : Yes.
36. Substitution shows that e^{2x} and xe^{2x} are solutions of the equation. $W = e^{4x} \neq 0$, y_1, y_2 are linearly independent on any interval I. Using Abel's formula we get $W = ce^{4x}$ which is same as the earlier value when $c = 1$.
37. Normal in $(0, \infty)$, $W = x^{1/2}$. $\{y_1, y_2\}$ forms a basis.
38. Normal in any I, $W = 3e^{4x}$. $\{y_1, y_2\}$ forms a basis.
39. Normal in $(0, \infty)$, $W = 2x$. $\{y_1, y_2\}$ forms a basis.

40. Normal in $(-\infty, \infty)$, $W = 20$. $\{y_1, y_2, y_3\}$ forms a basis.

41. Normal in $(-\infty, \infty)$, $W = e^{3x}$. $\{y_1, y_2, y_3\}$ forms a basis.

42. Normal in $(-\infty, \infty)$, $W = 12\sqrt{3}$. $\{y_1, y_2, y_3\}$ forms a basis.

43. Normal in $(0, \infty)$, $W = -2/x$. $\{y_1, y_2\}$ forms a basis.

44. $W(u, v) = (ad - bc)(y_1 y_2' - y_2 y_1')$. Since $y_1 y_2' - y_2 y_1' \neq 0$, $W(u, v) \neq 0$ if $ad - bc \neq 0$, (the determinant of the coefficient matrix of the transformation). Take $a = 1$, $b = 1$, $c = 1$, $d = -1$, $ad - bc = -2$, $u = e^{kx}$, $v = e^{-kx}$.

45. $W(y_1, y_2) \neq 0$. If for $x_0 \in I$, either $y_1(x_0), y_2(x_0)$ vanish or $y_1'(x_0), y_2'(x_0)$ vanish, then $W(y_1, y_2) = 0$.

46. Simplify $W(y, y_1, y_2)$ and substitute $y_i'' = -(ay_i' + by_i)$, $i = 1, 2$. We obtain

$$W(y, y_1, y_2) = (y'' + ay' + by)(y_1 y_2' - y_2 y_1') = 0.$$

47. At the given point $y_1(x_1) = y'(x_1) = 0$. Therefore, $y_1 \equiv 0$.

48. The differential equation is $W(y, y_1, y_2) = 0$, where $y_1 = e^{3x}$, $y_2 = e^{-2x}$, $y'' - y' - 6y = 0$.

49. $y'' + 2\alpha y' + (\alpha^2 + \omega^2)y = 0$.

50. $y'' - 10y' + 25y = 0$.

Exercise 5.2

1. $(7e^x - e^{4x})/3$.
2. $(3e^{2x} - e^{-2x})/2$.
3. $(1 + 5x)e^{-3x}$.
4. $\frac{1}{2}(5x^2 - (1/x^2))$.
5. $(3 + \ln x)x$.
6. $Ae^{2x} + Be^{-2x}$.
7. $Ae^{2x} + Be^{-x}$.
8. $Ae^x + Be^{-2x}$.
9. $Ae^{6x} + Be^{-2x}$.
10. $Ae^{m_1 x} + Be^{m_2 x}$, $m_1 = -2 + \sqrt{3}$, $m_2 = -2 - \sqrt{3}$.
11. $Ae^{2x} + Be^{x/4}$.
12. $Ae^{x/2} + Be^{-(5x)/2}$.
13. $(A + Bx)e^{-x}$.
14. $(A + Bx)e^{-\pi x}$.
15. $(A + Bx)e^{(2x)/3}$.
16. $(A + Bx)e^{-x/2}$.
17. $(A + Bx)e^{(2x)/5}$.
18. $A\cos 5x + B\sin 5x$.
19. $(A\cos x + B\sin x)e^{-2x}$.
20. $e^x(A\cos x + B\sin x)$.
21. $e^{x/2}(A\cos 2x + B\sin 2x)$.
22. $e^{3x}(A\cos 3x + B\sin 3x)$.
23. $A + Be^{-9x}$.
24. $e^{ax}(A\cos bx + B\sin bx)$.
25. $m = 3, -2$, ch. equation is $m^2 - m - 6 = 0$, diff. equation is $y'' - y' - 6y = 0$.
26. $m = 1/4, -3/4$, ch. equation is $16m^2 + 8m - 3 = 0$, diff. equation is $16y'' + 8y' - 3y = 0$.
27. $m = 0, -2$, ch. equation is $m(m + 2) = 0$, diff. equation is $y'' + 2y' = 0$.
28. $m = 2, 2$, ch. equation is $(m - 2)^2 = 0$, diff. equation is $y'' - 4y' + 4y = 0$.
29. $m = -1, -1$, ch. equation is $(m + 1)^2 = 0$, diff. equation is $y'' + 2y' + y = 0$.
30. $y'' + 9y = 0$.
31. $y'' + 2ay' + (a^2 + b^2)y = 0$.
32. $y'' - 10y' + 34y = 0$.
33. $e^x - e^{-x}$.
34. $e^{4x} + 3e^{-3x}$.
35. $e^x - e^{-2x}$.

36. $a \cos \sqrt{g}\, t$.

37. $e^{2x}(2 \cos x - 5 \sin x)$.

38. $e^{x/5}[\cos (x/5) - \sin (x/5)]$.

39. $((x/2) - 1)\, e^{-(3x)/2}$.

40. $xe^{-x/3}$.

41. $\cos 5x + B \sin 5x$, B arbitrary.

42. $[(2e^2 - 1)e^{-6x} - e^{6x}]/(e^2 - 1)$.

43. $e^{-x}(\cos x + \sin x)$.

44. $(Ax + B)e^{x/3}$, $A = e^{-2/3} - 1$, $B = 2 - e^{-2/3}$.

45. $(e^{x+2} - e^{3x})/(e^2 - 1)$.

48. (i) b = constant, (ii) $a(x) = b(x)$.

49. $(D + 4)(D + 1)y = 0$, set $(D + 1)y = v$ and $(D + 4)v = 0$; $v = A_1 e^{-4x}$, $y = Ae^{-4x} + Be^{-x}$.

50. $(2D + 1)(2D + 3)y = 0$, set $(2D + 3)y = v$ and $(2D + 1)v = 0$, $v = A_1 e^{-x/2}$, $y = Ae^{-x/2} + Be^{(-3x)/2}$.

51. $(2D + 3)(2D + 3)y = 0$, set $(2D + 3)y = v$, $(2D + 3)v = 0$, $v = A_1 e^{-(3x)/2}$, $y = (Ax + B)e^{-(3x)/2}$.

52. $(D + 3)(D + 3)y = 0$, set $(D + 3)y = v$, $(D + 3)v = 0$, $v = A_1 e^{-3x}$, $y = (Ax + B)e^{-3x}$.

53. $(D + 2)(D - 2)y = 0$, set $(D - 2)y = v$, $(D + 2)v = 0$, $v = A_1 e^{-2x}$, $y = Ae^{-2x} + Be^{2x}$.

54. $(3D + 1)(3D + 1)y = 0$, set $(3D + 1)y = v$, $(3D + 1)v = 0$, $v = A_1 e^{-x/3}$, $y = (Ax + B)e^{-x/3}$.

55. For oscillatory solutions, the discriminant of the characteristic equation should be less than zero. $|1 - c| < 2\sqrt{b}$, $1 - 2\sqrt{b} < c < 1 + 2\sqrt{b}$.

56. $\omega = n$, $y(x) = B_n \sin nx$, B_n arbitrary.

57. $y_n(x) = A_n \cos nx$, A_n arbitrary $y(x) = \sum_{n=1}^{\infty} y_n(x)$.

58. $y_n(x) = B_n \sin [(2n + 1)x/2]$, B_n arbitrary $y(x) = \sum_{n=1}^{\infty} y_n(x)$.

59. $y(x) = e^{px}(A'e^{qx} + B'e^{-qx}) = e^{px}[A \cosh qx + B \sinh qx]$.

60. (i) For $c^2 > 4mk$, both the characteristic roots $-p \pm q$ where $p = c/(2m)$ and $q = \sqrt{c^2 - 4mk}/(2m)$, are negative and $q < p$. Therefore, the solution $y(t) = e^{-pt}(Ae^{qt} + Be^{-qt}) \to 0$ as $t \to \infty$, that is, there exists a t_0 such that for $t > t_0$ the system is in equilibrium. $y = [av_0 e^{-pt} \sinh qt]/q$.

(ii) For $c^2 < 4mk$, the characteristic roots are $-p \pm iq$, where $p = c/(2m)$ and $q = \sqrt{4mk - c^2}/(2m)$ are complex. The solutions are oscillatory in this case. The solution is $y(t) = e^{-pt}(A \cos qt + B \sin qt)$. The oscillations are damped and they decay as $t \to \infty$. $y = (e^{-pt} v_0 \sin qt)/q$.

(iii) For $c^2 = 4mk$, the characteristic roots are repeated roots $-p$. The solution is $y(t) = (A + Bt)e^{-pt}$. $y = v_0 t e^{-pt}$.

61. $Ae^{3x} + Be^{-2x}$.

62. $Ae^{x} + Be^{-4x}$.

63. $u = x + 1/x$, $y_2 = 1 + x^2$, $Ax + B(1 + x^2)$.

64. $u = -\cot x$, $y_2 = -x^{-1/2} \cos x$, $x^{-1/2}(A \cos x + B \sin x)$.

65. $u = -e^{-x}(x^2 - 2x + 2)$, $y_2 = -(x^2 - 2x + 2)$, $Ae^{x} + B(x^2 - 2x + 2)$.

Exercise 5.3

1. $A + Be^{3x} + Ce^{-3x}$.
2. $Ae^{x/2} + Be^{2x} + Ce^{-3x}$.
3. $Ae^{x} + Be^{-x} + Ce^{2x/3}$.
4. $Ae^{2x} + Be^{-2x} + Ce^{3x} + De^{-3x}$.
5. $Ae^{x} + Be^{2x} + Ce^{-x/2} + De^{x/2}$.
6. $A + Be^{2x} + Ce^{-2x} + De^{-x}$.
7. $Ae^{x/4} + Be^{x/2} + Ce^{x} + De^{-x}$.
8. $Ae^{x/3} + Be^{-x/3} + Ce^{x/4} + De^{-x/4}$.

9. $A + (Bx + C)e^x$.
10. $Ae^{-2x} + (Bx + C)e^{-x}$.
11. $Ae^{-2x} + (Bx + C)e^{2x}$.
12. $(A + Bx + Cx^2)e^{x/3}$.
13. $A + Be^x + (Cx + D)e^{5x}$.
14. $A + (Bx^2 + Cx + D)e^x$.
15. $(Ax + B)e^{-x} + (Cx + D)e^{x/2}$.
16. $(Ax + B)e^{3x} + (Cx + D)e^{2x/3}$.
17. $A + B \cos x + C \sin x$.
18. $Ae^{2x} + B \cos 2x + C \sin 2x$.
19. $Ae^{-3x} + e^{-x}(B \cos x + C \sin x)$.
20. $Ae^x + e^{3x}(B \cos 2x + C \sin 2x)$.
21. $Ae^x + Be^{-x} + C \cos 3x + D \sin 3x$.
22. $Ae^x + Be^{-2x} + C \cos 4x + D \sin 4x$.
23. $A \cos 5x + B \sin 5x + C \cos (x/2) + D \sin (x/2)$.
24. $e^{2x}(A \cos x + B \sin x) + e^{-3x}(C \cos x + D \sin x)$.
25. $(A + Bx) \cos 5x + (C + Dx) \sin 5x$.
26. $(A + Bx) \cos x + (C + Dx) \sin x$.
27. $m = 0, 1, 3, y''' - 4y'' + 3y' = 0$.
28. $m = -1, \pm 5i, y''' + y'' + 25y' + 25y = 0$.
29. $m = -1, -1, 2, y''' - 3y' - 2y = 0$.
30. $m = 0, 0, 1, 3, y^{iv} - 4y''' + 3y'' = 0$.
31. $m = 2, 2, 2, -2, y^{iv} - 4y''' + 16y' - 16y = 0$.
32. $m = \pm 3, \pm 2i, y^{iv} - 5y'' - 36y = 0$.
33. $(3e^{3x} + 2e^{-2x} - 5e^x)/30$.
34. $(9e^x - 5e^{3x/2} + e^{-3x/2})/5$.
35. $(2 + x)e^x - e^{3x}$.
36. $(1 + x)e^{-x} + (2 - x)e^{2x}$.
37. $x + \cos x + \sin x$.
38. $\cos 2x + 2 \sin 2x - e^x$.
39. $e^x + e^{-x}(\cos x + 2 \sin x)$.
40. $1 + 2x + 3x^2 + e^{3x}$.
41. $A \sin \pi x$, A arbitrary.
42. $1 + 2 \sinh 6x + \cosh 6x$.
43. $2 \sin 2x + \sin 3x$.
44. $D_n \sin nx, \Sigma D_n \sin nx$.
45. $2 \cos 3x + \cos x$.

Exercise 5.4

1. $A(x) = -e^{2x}/8, B(x) = -e^{-2x}/8, y = c_1e^{-x} + c_2e^{3x} - (e^x/4)$.
2. $A(x) = -e^{-4x}/4, B(x) = (4x + 1)e^{-4x}/16, y = (c_1x + c_2) e^{2x} + e^{-2x}/16$.
3. $A(x) = \cos^3 x/3, B(x) = (\sin 3x + 3 \sin x)/12, y_p = (\cos x)/3, y = c_1 \cos 2x + c_2 \sin 2x + y_p$.
4. $A(x) = \ln | \cos x |, B(x) = x, y_p = \cos x \ln | \cos x | + x \sin x, y = c_1 \cos x + c_2 \sin x + y_p$.
5. $A(x) = -x, B(x) = \ln | \sin x |, y_p = \sin x \ln | \sin x | - x \cos x, y = c_1 \cos x + c_2 \sin x + y_p$.
6. $A(x) = \sin x - \ln | \sec x + \tan x |, B(x) = -\cos x, y_p = -\cos x \ln | \sec x + \tan x |$,
   $y = c_1 \cos x + c_2 \sin x + y_p$.
7. $A(x) = -x/2, B(x) = -e^{-2x}/4, y(x) = c_1e^x + c_2e^{3x} - (xe^x)/2$.
8. $A(x) = \dfrac{1}{4} \ln | \cos 2x |, B(x) = x/2, y_p = \dfrac{1}{4} \cos 2x \ln | \cos 2x | + \dfrac{1}{2} x \sin 2x$.
   $y(x) = c_1 \cos 2x + c_2 \sin 2x + y_p$.
9. $A(x) = (\cos 4x)/16, B(x) = (4x + \sin 4x)/16, y_p = (\cos 2x + 4x \sin 2x)/16$.
   $y(x) = c_1 \cos 2x + c_2 \sin 2x + (x \sin 2x)/4$.
10. $A(x) = \sin x + x \cos x, B(x) = -\cos x, y_p = -e^{-2x} \sin x, y(x) = (c_1x + c_2)e^{-2x} + y_p$.
11. $A(x) = -x, B(x) = \ln | x |, y_p = x [\ln | x | - 1] e^{-3x}, y(x) = (c_1x + c_2) e^{-3x} + y_p$.

12. $A(x) = (\cos 2x)/4$, $B(x) = (2x + \sin 2x)/4$, $y(x) = c_1e^{-x}\cos x + c_2e^{-x}\sin x + (xe^{-x}\sin x)/2$.

13. $g(x) = x$, $A(x) = x^2/4$, $B(x) = -x^4/8$, $y_p = x^3/8$, $y(x) = c_1x + (c_2/x) + y_p$.

14. $g(x) = \ln|x|$, $A(x) = [\ln|x|]^2/8$, $B(x) = -x^4[4\ln|x| - 1]/64$.
$y_p = x^2[8(\ln|x|)^2 - 4\ln|x| + 1]/64$, $y(x) = c_1x^2 + c_2/x^2 + y_p$.

15. $g(x) = 1/x^6$, $A(x) = [1 + 5\ln|x|]/(25x^5)$, $B(x) = -1/(5x^5)$,
$y_p = 1/(25x^4)$, $y(x) = c_1x + c_2x\ln|x| + y_p$.

16. $g(x) = x + (1/x)$, $A(x) = -[(x^2/2) + \ln|x|]$, $B(x) = x - (1/x)$,
$y_p = (x^3/2) - x(1 + \ln|x|)$, $y(x) = c_1x + c_2x^2 + y_p$.

17. $g(x) = 16e^{-2x}\operatorname{cosec}^2 2x$, $A(x) = 4\ln|\operatorname{cosec} 2x + \cot 2x|$, $B(x) = -4/\sin 2x$.
$y_p = 4e^{-2x}\cos 2x\ln|\operatorname{cosec} 2x + \cot 2x| - 4e^{-2x}$, $y(x) = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + y_p$.

18. $A(x) = (\ln|\sec 2x + \tan 2x|)/8$, $B(x) = -x/4$, $C(x) = (\ln|\cos 2x|)/8$,
$y(x) = c_1 + c_2\cos 2x + c_3\sin 2x - (x\cos 2x)/4 + (\sin 2x\ln|\cos 2x|)/8 + (\ln|\sec 2x + \tan 2x|)/8$.

19. $A(x) = x^2/4$, $B(x) = -x$, $C(x) = (\ln|x|)/2$,
$y(x) = (c_1 + c_2x + c_3x^2)e^{2x} + (x^2\ln|x|e^{2x})/2$.

20. $y_p = \dfrac{1}{k}\displaystyle\int_0^x g(t)[\sin kx\cos kt - \cos kx\sin kt]\,dt = \dfrac{1}{k}\int_0^x g(t)\sin[k(x-t)]\,dt$.

Exercise 5.5

1. $y_p = -(50x^2 - 30x + 69)/500$, $y_c = Ae^{-2x} + Be^{5x}$.

2. $y_p = (20 - 51x + 9x^2 - 9x^3)/27$, $y_c = Ae^{-x} + Be^{3x/2}$.

3. $y_p = (35e^x + 3e^{3x})/105$, $y_c = Ae^{x/2} + Be^{-x/2}$.

4. $y_p = (e^{-2x} - 7x - 14)/7$, $y_c = Ae^{-x} + Be^{x/3}$.

5. $y_p = -e^{-3x} + e^x/15$, $y_c = Ae^{-2x} + Be^{-4x}$.

6. $y_p = 3xe^{-x}$, $y_c = Ae^{-x} + Be^{-3x}$.

7. $y_p = -xe^{-2x} + e^x/3$, $y_c = Ae^{-2x} + Be^{x/2}$.

8. $y_p = 2xe^{3x} - xe^{-2x}$, $y_c = Ae^{-2x} + Be^{3x}$.

9. $y_p = 2xe^{x/3}$, $y_c = Ae^{-2x} + Be^{x/3}$.

10. $y_p = (2\sin x - \cos x)/5$, $y_c = Ae^{-x} + Be^{-2x}$.

11. $y_p = (\sin 3x - 5\cos 3x)/2$, $y_c = Ae^{2x} + Be^{-3x}$.

12. $y_p = 2(\sin 2x - \cos 2x)$, $y_c = Ae^x + Be^{-5x}$.

13. $y_p = x(-3\cos 5x + 5\sin 5x)$, $y_c = A\cos 5x + B\sin 5x$.

14. $y_p = -2x\cos 4x$, $y_c = A\cos 4x + B\sin 4x$.

15. $y_p = 4x^2e^{2x} + e^{3x}$, $y_c = (Ax + B)e^{2x}$.

16. $y_p = 3x^2e^{(x/2)}/4$, $y_c = (Ax + B)e^{x/2}$.

17. $y_p = 13x^2e^{-3x} + e^{2x}/5$, $y_c = (Ax + B)e^{-3x}$.

18. $y_p = e^x(\sin x - 2\cos x)/5$, $y_c = A\cos x + B\sin x$.

19. $y_p = -(xe^{-x}\cos 3x)/6$, $y_c = e^{-x}(A\cos 3x + B\sin 3x)$.

20. $y_p = 8xe^{2x}\sin x$, $y_c = e^{2x}(A\cos x + B\sin x)$.

21. $y_p = -3xe^{3x}\cos 2x/4$, $y_c = e^{3x}(A\cos 2x + B\sin 2x)$.
22. $r(x) = 3e^{-2x}(1 + \cos 2x)$, $y_p = e^{-2x}(c_1x^2 + c_2\cos 2x + c_3\sin 2x) = [3e^{-2x}(2x^2 - \cos 2x)]/4$, $y_c = (Ax + B)e^{-2x}$.
23. $r(x) = 3e^{-x}(3\sin x - \sin 3x)$, $y_p = e^{-x}[-45(\cos x + \sin x) + (\cos 3x + 3\sin 3x)]/10$, $y_c = Ae^{-x} + Be^{-2x}$.
24. $r(x) = 2(e^{3x} + e^{-3x})$, $y_p = (e^{-3x} + 12xe^{3x})/12$, $y_c = Ae^{x} + Be^{3x}$.
25. $y_p = -3xe^{-x}$, $y_c = Ae^{x} + Be^{-x} + Ce^{-4x}$.
26. $y_p = xe^{x} - 2x^2e^{-2x}$, $y_c = (Ax + B)e^{-2x} + Ce^{x}$.
27. $y_p = 6x^3e^{3x}$, $y_c = (Ax^2 + Bx + C)e^{3x}$.
28. $y_p = 2(\cos 2x - 2\sin 2x)/5$, $y_c = Ae^{x} + B\cos x + C\sin x$.
29. $y_p = -[2(x^2 + x) + x(\cos 2x + \sin 2x)]/2$, $y_c = Ae^{2x} + B\cos 2x + C\sin 2x$.
30. $y_p = -x\sin 4x/2$, $y_c = Ae^{4x} + Be^{-4x} + C\cos 4x + D\sin 4x$.
31. $y_p = -(x^4 + 25)$, $y_c = Ae^{x} + Be^{-x} + C\cos x + D\sin x$.
32. $y_p = x^2 - 2x$, $y_c = A + (Bx^2 + Cx + D)e^{-x}$.
33. $y_p = 3xe^{2x}$, $y_c = Ae^{2x} + Be^{-2x} + C\cos x + D\sin x$.
34. $y_p = -5x^3e^{-2x}$, $y_c = A + (Bx^2 + Cx + D)e^{-2x}$.
35. $y_p = -(x^3 + 6x^2)/12$, $y_c = Ax + B + Ce^{4x} + De^{-4x}$.

Exercise 5.6

1. $y = Ax^2 + B/x^2$.
2. $y = (A/x) + (B/x^2)$.
3. $y = Ax + B/x$.
4. $y = (A + B\ln x)x^{-1/3}$.
5. $y = (A + B\ln x)x^{-3/2}$.
6. $y = A\cos(\ln x/\sqrt{2}) + B\sin(\ln x/\sqrt{2})$.
7. $y = (A + B\ln x)/x$.
8. $y = x[A\cos(2\ln x) + B\sin(2\ln x)]$.
9. $y = x^{-1}[A\cos(3\ln x) + B\sin(3\ln x)]$.
10. $y = x^{1/3}[A\cos(\ln x) + B\sin(\ln x)]$.
11. $y = A + Bx + C\ln x$.
12. $y = [A + B\ln x + C\ln^2 x]x$.
13. $y = Ax + x^{-1}[B\cos(\ln x) + C\sin(\ln x)]$.
14. $y = (A/x) + (B/x^2) + (C/x^3)$.
15. $y = (A/x) + (B + C\ln x)x^2$.
16. $y = (A/x^2) + x[B\cos(4\ln x) + C\sin(4\ln x)]$.
17. $y = A + Bx + Cx^2 + D\ln x$.
18. $y = Ax^2 + (B/x^2) + C\cos(\ln x) + D\sin(\ln x)$.
19. $y = A\sqrt{x} + (B/\sqrt{x}) + C\cos(2\ln x) + D\sin(2\ln x)$.
20. $y = (A + B\ln x)x + (C + D\ln x)/x$.
21. $y = Ax^2 + (B/x) - x - 3$.
22. $y = Ax + Bx^3 + \ln x + 2$.
23. $y = Ax + (B/x^2) + 2x\ln x + 7$.
24. $y = Ax^2 + (B/x^3) + 3x^2\ln x$.
25. $y = A + (B/x) + [\sin(\ln x) - \cos(\ln x)]/2$.
26. $y = Ax + (B/x^5) + 2x(3\ln^2 x - \ln x)/3$.
27. $y = (A + B\ln x)x^{1/2} + 4\cos(\ln x) - 3\sin(\ln x)$.
28. $y = (A + B\ln x)x^2 + x^3$.
29. $y = (A + B\ln x)x^{-3/2} + 2\sin(\ln x) - \cos(\ln x)$.
30. $y = Ax + (B/x^2) - x[3\cos(\ln x) + \sin(\ln x)]/10$.
31. $y = (A/x) + Bx^4 - x^2 - \ln x + 3/4$.
32. $y = Ax + (B/x) + (C/x^5) + 2x^2$.

33. $y = Ax^2 + (B/x^2) + (C/x^3) - (3 \ln x)/x^2$.
34. $y = (A + B \ln x + C \ln^2 x)x^2 + 3x^3 - 8x$.
35. $y = (A + B \ln x)x^{1/2} + (C/x) + \sin(\ln x) + 7\cos(\ln x)$.
36. Set $3x + 1 = z$, $y = [A + B \ln(3x + 1)](3x + 1)^{1/3} + \frac{3}{2}(x - 1)$.
37. Set $x + 2 = z$, $y = A(x + 2) + (x + 2)^{1/2}[B \cos t + C \sin t] + 8(x + 2)^2 - 96(x + 2)\ln(x + 2) - 96$, where $t = \sqrt{3}\ln(x + 2)/2$.
38. $y = Ax + (B/x) + Cx^2 + (D/x^2) + 1/(4x^3)$.
39. $y = Ax^{3/2} + Bx^{-3/2} + (C + D \ln x)x + 2x^2 - 1/9$.
40. $y = A\cos(\ln x) + B\sin(\ln x) + C\cos(2\ln x) + D\sin(2\ln x) + 1/(20x^2)$.
41. $y = \frac{1}{4}\left(\sqrt{x} + \frac{1}{x}\right) + \frac{x}{2}$.
42. $y = 4(\ln x - 1)\sqrt{x} + \ln x + 4$.
43. $y = [7x - 10x^2 + 5x^3 + x \ln x]/2$.
44. $y = x[4\sin(\ln x) - 2\cos(\ln x)] + 3$.
45. $y = \frac{1}{x}[2\cos(3\ln x) + 3\sin(3\ln x) + \frac{x^2}{2}$.

Exercise 5.7

1. $Ae^{-x} + Be^{-4x} + e^{2x}$.
2. $Ae^{x} + Be^{-x} + e^{3x}$.
3. $Ae^{-x} + Be^{4x} + e^{5x} - (e^{x})/6$.
4. $e^{-x/2}[A\cos(\sqrt{7}x/2) + B\sin(\sqrt{7}x/2)] + \frac{4}{11}e^{x/2}$.
5. $e^{-3x/2}[A\cos(\sqrt{3}x/2) + B\sin(\sqrt{3}x/2)] + e^{x}$.
6. $(A + Bx)e^{x} + 4e^{2x} + (5e^{4x})/9$.
7. $(A + Bx)e^{x/3} + (e^{-x})/4$.
8. $(A + Bx)e^{3x} + 7x^2e^{3x}$.
9. $Ae^{2x} + Be^{-3x} + (xe^{2x})/5$.
10. $Ae^{2x} + Be^{-x/2} - e^{-x/2}(4x + 5x^2)/50$.
11. $Ae^{x} + Be^{-x} + [3e^{x}(x^2 - x)]/2$.
12. $Ae^{-2x} + Be^{-x/4} - \frac{1}{98}(7x^2 + 8x)e^{-2x}$.
13. $(A + Bx)e^{-x/3} + (x^2e^{-x/3})/18$.
14. $Ae^{x/2} + Be^{-4x} - e^{-4x}(9x^2 + 4x)/162$.
15. $Ae^{-x} + Be^{2x} + Ce^{-3x} - (e^{x})/2$.
16. $Ae^{x} + Be^{-2x} + Ce^{-x/2} + (e^{2x})/2$.
17. $Ae^{x} + Be^{-x} + Ce^{2x} + (e^{3x})/8$.
18. $(A + Bx + Cx^2)e^{2x} + 3x^3e^{2x}$.
19. $(A + Bx)e^{x} + Ce^{-x/2} + (8x^2e^{x})/3$.
20. $Ae^{2x} + Be^{-2x} + Ce^{-3x} - 3e^{-2x}(2x^2 - 3x)/4$.
21. $A\cos 4x + B\sin 4x + (\cos 2x)/12$.
22. $Ae^{x} + Be^{3x/2} + (\sin x + 5\cos x)/26$.
23. $Ae^{2x} + Be^{x/3} + (3\cos x - 4\sin x)/25$.
24. $Ae^{3x} + Be^{x/2} + (14\cos 2x - 5\sin 2x)/221$.
25. $e^{-x/2}[A\cos(\sqrt{3}x/2) + B\sin(\sqrt{3}x/2)] + 16\sin x$.
26. $e^{3x/4}[A\cos(x/4) + B\sin(x/4)] + 16(4\cos x - \sin x)/51$.
27. $A\cos 3x + B\sin 3x - (x\cos 3x)/6$.
28. $A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x) + (x\sin\sqrt{3}x)/(2\sqrt{3})$.

29. $e^{-x}(A \cos 2x + B \sin 2x) + (xe^{-x} \sin 2x)/4$.
30. $e^{2x}(A \cos x + B \sin x) - 12\, x \cos x\, e^{2x}$.
31. $e^{3x}(A \cos 2x + B \sin 2x) - 7x \cos 2x\, e^{3x}$.
32. $e^{x}[A \cos 3x + B \sin 3x + x\,(8 \sin 3x - 12 \cos 3x)/3]$.
33. $Ae^{3x} + B \cos x + C \sin x - 3x(\cos x + 3 \sin x)/10$.
34. $Ae^{x} + B \cos 3x + C \sin 3x - x(3 \cos 3x + \sin 3x)/2$.
35. $Ae^{2x} + e^{x}(B \cos 2x + C \sin 2x) - 6xe^{x}(2 \sin 2x - \cos 2x)/5$.
36. $Ae^{2x} + e^{x/2}(B \cos x + C \sin x) - 4xe^{x/2}(2 \cos x + 3 \sin x)/13$.
37. $A \cos x + B \sin x + C^* \cos 2x + D^* \sin 2x - 8x(\cos x + 2 \sin 2x)/3$.
38. $A \cos 5x + B \sin 5x + (225x^3 + 100x^2 - 54x - 8)/625$.
39. $(A + Bx)e^{-3x} + (12x^2 - 16x + 5)/27$.
40. $Ae^{-x} + Be^{3x} - (18x^2 + 30x - 8)/27$.
41. $Ae^{2x} + Be^{3x} + [(52x + 25)(\cos 2x - 5 \sin 2x) - 21(5 \cos 2x + \sin 2x)]/2704$.
42. $Ae^{x} + Be^{-2x} - [(25x^2 + 5x - 9)(3 \sin x + \cos x) + (35x + 12)(3 \cos x - \sin x)]/250$.
43. $Ae^{3x} + Be^{-2x} - e^{-2x}(5x^2 + 2x)/50$.
44. $Ae^{-3x} + Be^{-4x} + e^{x}(8 \sin 2x - 9 \cos 2x)/290$.
45. $Ae^{-x} + Be^{-3x} + e^{2x}(7 \cos x + 4 \sin x)/130$.
46. $e^{-3x/2}[A \cos p + B \sin p] + 4e^{x}(25 \cos p + 10\sqrt{7} \sin p)/1325,\ p = \sqrt{7}x/2$.
47. Write $xe^{x} \sin x = \text{Im}\,[xe^{(1+i)x}]$, $Ae^{-x} + Be^{-2x} + e^{x}[5(1 - x) \cos x + (5x - 2) \sin x]/50$.
48. Write $xe^{2x} \cos x = \text{Re}\,[xe^{(2+i)x}]$, $A \cos 3x + B \sin 3x + e^{2x}[(30x - 11) \cos x + (10x - 2) \sin x]/400$.
49. $Ae^{-x/2} + Be^{-3x/2} - e^{-x/2}[(x - 2) \cos x - (x + 1) \sin x]/8$.
50. $A \cos x + B \sin x + C^* \cos \sqrt{2}x + D^* \sin \sqrt{2}x - 4[9x^2 \cos x - (2x^3 - 51x) \sin x]/3$.
51. $y = Ae^{x/2} + Be^{3x},\quad B = 1/5$.
52. $\displaystyle\int e^{-mx} r(x)\,dx = \int e^{-mx}(D - m)y\,dx = e^{-mx}y$, or $\displaystyle y = e^{mx}\int e^{-mx} r(x)\,dx$.
53. Use the result

$$\frac{d}{dx}\int_a^b f(x, t)\,dt = f(x, b)\,\frac{db}{dx} - f(x, a)\,\frac{da}{dx} + \int_a^b \frac{\partial f}{\partial x}\,dt$$

$$\frac{dy}{dx} = \int_a^x r(t) \cos n(x - t)\,dt,\quad \frac{d^2y}{dx^2} = r(x) - n\int_a^b r(t) \sin n(x - t)\,dt = r(x) - n^2 y.$$

54. $D^m(x\,u) = xD^m u + mD^{m-1}u = xD^m u + \left[\dfrac{d}{dD}D^m\right]u \quad m = 1, 2, \ldots$

$$F(D)(x\,u) = x[a_0D^n + a_1D^{n-1} + \ldots + a_n]u + \frac{d}{dD}[a_0D^n + a_1D^{n-1} + \ldots + a_n]u$$

$$= xF(D)u + F'(D)u.$$

55. $F(D)(xv) = x\,F(D)v + F'(D)v$. Let $F(D)v = u$.

$F(D)[x\{F(D)\}^{-1}u] = xF(D)[F(D)]^{-1}u + F'(D)[F(D)]^{-1}u = xu + F'(D)[F(D)]^{-1}u$

$xu = F(D)[x\{F(D)\}^{-1}u] - F'(D)[F(D)]^{-1}u$

$[F(D)]^{-1}(xu) = x[F(D)]^{-1}u - F'(D)[F(D)]^{-2}u.$

56. When $F(m) = 0$, $F'(m) \neq 0$, $F(D) = (D - m)G(D)$ and $F'(m) = G(m)$.

$$[F(D)]^{-1} e^{mx} = \frac{xe^{mx}}{G(m)} = x\left(\frac{1}{F'(m)}\right)e^{mx},$$

When $F(m) = 0$, $F'(m) = 0$, $F''(m) \neq 0$, $F(D) = (D - m)^2G(D)$ and $F''(m) = 2G(m)$.

$$[F(D)]^{-1} e^{mx} = \frac{x^2}{2!}\frac{e^{mx}}{G(m)} = x^2\left(\frac{1}{F''(m)}\right)e^{mx}.$$

Problem 8: $F(3) = 0$, $F'(3) = 0$, $F''(3) = 2$, $y_p = 7x^2e^{3x}$.

Problem 9: $F(2) = 0$, $F'(2) = 5$, $y_p = (xe^{2x})/5$.

Problem 13: $F(-1/3) = 0$, $F'(-1/3) = 0$, $F''(-1/3) = 18$, $y_p = (x^2e^{-x/3})/18$.

Problem 19: $F(1) = 0$, $F'(1) = 0$, $F''(1) = 6$, $y_p = (8x^2e^x)/3$.

57. Note that $[F(D)]^{-1}$ can be written as

$$[F(D)]^{-1} = A_1(D - m_1)^{-1} + A_2(D - m_2)^{-1} + \ldots + A_n(D - m_n)^{-1}$$

$\left(\text{equivalent to writing in partial fractions of } \frac{1}{F(m)} \text{ as } \frac{1}{F(m)} = \frac{A_1}{m - m_1} + \frac{A_2}{m - m_2} + \ldots + \frac{A_n}{m - m_n}\right)$

Now apply the solution of Problem 52.

Problem 40: $y_p = \frac{1}{4}e^{3x}\int e^{-3x}(2x^2 + 6x)\,dx - \frac{1}{4}e^{-x}\int e^{x}(2x^2 + 6x)\,dx$

$= -\frac{1}{27}(18x^2 + 30x - 8).$

58. For forced damped oscillations, $c^2 < 4mk$, $y_c = e^{-ct/(2m)}.[A \cos dt + B \sin dt]$

$d = \sqrt{4mk - c^2}\,/(2m).$

For forced undamped oscillations, $y_c = A \cos(\sqrt{k/m}\,t) + B \sin(\sqrt{k/m}\,t)$.

$c \neq 0$, $y_p = F_0\,[(k - m\omega^2)\cos \omega t + c\omega \sin \omega t]/[(k - m\omega^2)^2 + c^2\omega^2]$

$c = 0$, $y_p = F_0 \cos \omega t/(k - m\omega^2)$.

Exercise 5.8

1. $y_1'' - 4y_1' + 3y_1 = 0$, $y_1 = Ae^t + Be^{3t}$, $y_2 = Be^{3t} - Ae^t$.
2. $y_1'' - 2y_1' - 8y_1 = 0$, $y_1 = Ae^{-2t} + Be^{4t}$, $y_2 = 3Be^{4t} - 3Ae^{-2t}$.
3. $y_1 = A \cos 3t + B \sin 3t$, $y_2 = 3(B \cos 3t - A \sin 3t)$.
4. $y_1 = Ae^{-t} + Be^{-4t}$, $y_2 = -(3Ae^{-t} + 6Be^{-4t})$.
5. $y_1 = Ae^t + Be^{-t} + 2 \cos t$, $y_2 = Be^{-t} - Ae^t + 6 \sin t$.
6. $y_1 = Ae^{5t} + Be^{-3t} + 4e^{-t}/3 + 8(15t - 2)/75$.

   $3y_2 = 4e^{-t} - 6Ae^{5t} + 2Be^{-3t} - 8(15t + 13)/75$.
7. $y_1 = Ae^{2t} + Be^{-2t} + 4e^t + e^{-t}$, $y_2 = -(5Ae^{2t} + Be^{-2t} + 10e^t + 2e^{-t})$.