Tuesday, October 27, 2020 9:45 AM

Maxima & Minima

$$z = f(x,y)$$

the point $(a,b) \longrightarrow (a+h, b+k)$

$f(x) \le f(a) \ \forall x \in$

$f(x) \ge f(a) \ \forall x \in$

① ✓ $f(a+h, b+k) \ge f(a,b) \rightarrow (a,b)$ relative local min

② ✓ $f(a+h, b+k) \le f(a,b) \longrightarrow$ relative max or local maxim

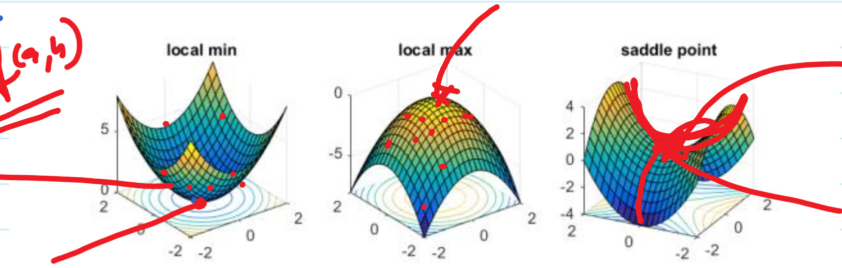
$(a+h, b+k)$ is a point in the nbd of (a,b)

\# A function may attain its minimum or maximum value on the boundary of region.

$f(a,b) \ge f(a+h, b+k)$

Local minim

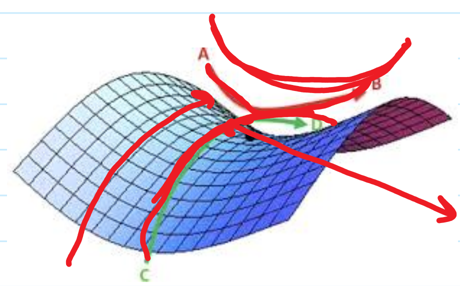
$f(a+h, b+k) \ge f(a,b)$



Min

Max

$f(a,b)$



Saddle points

let $z = f(x,y)$ be the given fxn

① Find $\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0$] solve the system simultaneously to get the critical points.

let $(a,b), (c,d), \ldots \ldots$

② find $r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y} \quad \& \quad t = \frac{\partial^2 f}{\partial y^2}$

\# At (a,b)

(5) (i) $rt - s^2 > 0$, & $r < 0$, then (a,b) is a point of maxima & Max value $= f(a,b)$

(ii) $rt - s^2 > 0$ & $r > 0$ then (a,b) is a point of minima & Min value $= f(a,b)$

(iii) $rt - s^2 < 0$, then (a,b) is a saddle point

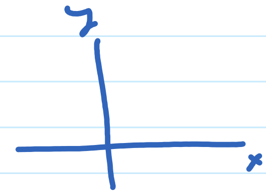
(iv) $rt - s^2 = 0$, the case is doubtful and needs further investigation

Q Find the relative maximum and minimum values of the function
$$f(x,y) = 2(x^2 - y^2) - x^4 + y^4$$
$[-2+1- = 1$

Sol (1) $\frac{\partial f}{\partial x} = 4x - 4x^3$ & $\frac{\partial f}{\partial y} = -4y + 4y^3$

(2) Put $\frac{\partial f}{\partial x} = 0 \Rightarrow 4x - 4x^3 = 0$ or $4x(1-x^2) = 0$

& $\frac{\partial f}{\partial y} = 0 \Rightarrow -4y + 4y^3 = 0$ or $-4y(1-y^2) = 0$



$\therefore$ $4x(1-x^2) = 0 \Rightarrow x = 0, x = -1, x = 1$

$-4y(1-y^2) = 0 \Rightarrow y = 0, y = -1, y = 1$

$\therefore$ Points are $(0,0), (0,-1), (0,1)$
$(-1,0), (-1,-1), (-1,1)$
$(1,0), (1,-1), (1,1)$

$r = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$

$rt - s^2 = 4(1-3x^2) \times -4(1-3y^2) - 0$
$= -16(1-3x^2)(1-3y^2)$

(i) At $(0,0)$, $rt - s^2 = -16 < 0$, $\therefore$ $(0,0)$ is a saddle point.

(ii) At $(0,-1)$, $rt-s^2 = 32>0$, & $r = 4>0$, $\therefore (0,-1)$ is a point of minima
& Min value $= f(0,-1) = -1$

(iii) At $(0,1)$, $rt-s^2 = 32>0$ & $r=4>0$, $\therefore (0,1)$ is a point of minima
& Min value $= f(0,1) = -1$

(iv) At $(-1,0)$, $rt-s^2 = 32>0$ & $r = -8<0$, $\therefore (-1,0)$ is a point of maxima
& Max value $= f(-1,0) = 1$

(v) At $(1,0)$, $rt-s^2 = 32>0$ & $r = -8<0$, $\therefore (1,0)$ is a point of maxima
& Max value $= f(1,0) = 1$

(vi) At $(-1,-1)$, $rt-s^2 = -64<0$ $\therefore (-1,-1)$ is a saddle point

(vii) At $(1,1)$, $rt-s^2 = -64<0$ $\therefore (1,1)$ ——————

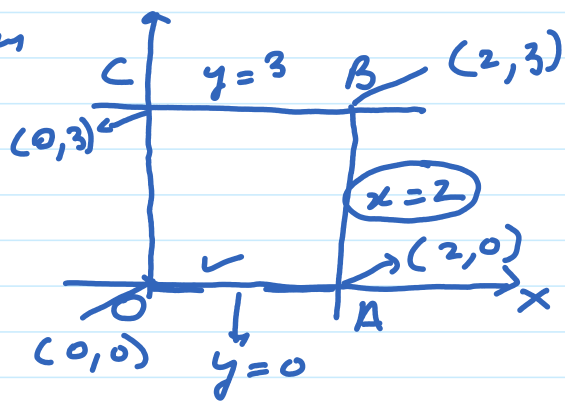
(viii) At $(-1,1)$, $rt-s^2 = -64<0$ $(-1,1)$ is also saddle point

(ix) At $(1,-1)$, $rt-s^2 = -64<0$ $(1,-1)$ is also saddle point

Q $\boxed{f(x,y) = 2(x^2-y^2) - x^4+y^4}$ $(-1,1)$

Q Find the absolute maximum & minimum values of
$f(x,y) = 4x^2+9y^2-8x-12y+4$
over the rectangle in the first quadrant bounded by the lines $x=2$, $y=3$ & the coordinate axes

Sol The function f attains max/minimum value at the critical points or on the boundary of the rectangle OABC



① $\frac{\partial f}{\partial x} = 8x-8$ & $\frac{\partial f}{\partial y} = 18y-12$

② Put $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$8x-8=0$ & $18y-12=0$
$x=1$ & $y = \frac{12}{18} = 2/3$

$\therefore$ Critical point $(1, 2/3)$

$\therefore$ Critical point $(1, 2/3)$

Now $r = \frac{\partial^2 f}{\partial x^2} = 8$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 18$

$\therefore rt - s^2 = 8 \times 18 - 0 = 144$

At $(1, 2/3)$, $rt - s^2 = 144 > 0$ & $r = 8 > 0$

$\therefore (1, 2/3)$ is a point of Minima & Min value $= f(1, 2/3)$
$= -4$

On the boundary OA, we have $y = 0$, $f(x,0) = 4x^2 - 8x + 4 = g(x)$

a function of single value

Here $g(x) = 4x^2 - 8x + 4$

$\frac{dg}{dx} = 8x - 8$

Put $\frac{dg}{dx} = 0 \Rightarrow 8x - 8 = 0 \Rightarrow \boxed{x = 1}$

$\therefore \frac{d^2g}{dx^2} = 8$

At $x = 1$, $\frac{d^2g}{dx^2} = 8 > 0$ $\therefore x = 1$ is point of min

Min value $= g(1) = f(1,0) = 0$

Also the corners are $O(0,0)$ & $A(2,0)$

$\therefore$ we have $f(0,0) = 4$ & $f(2,0) = 4$

On the boundary AB, $x = 2$, $f(2,y) = 9y^2 - 12y + 4 = h(y)$

Here $h(y) = 9y^2 - 12y + 4$

$\frac{dh}{dy} = 18y - 12$

Put $\frac{dh}{dy} = 0 \Rightarrow 18y - 12 = 0 \Rightarrow y = 2/3$

& $\frac{d^2h}{dy^2} = 18 > 0$

At $y = 2/3$, $\frac{d^2h}{dy^2} = 18 > 0$ $\therefore y = 2/3$ is point of minima

& Min value $= h(2/3) = f(2, 2/3)$
$= 0$

Also the corner is $A(2,3)$ $\therefore$

$f(2,3) = 49$

Also along the boundary BC, $y=3$

$\therefore f(x,3) = 4x^2 - 8x + 49 = g_1(x)$

$g_1(x) = 4x^2 - 8x + 49$

$\frac{dg_1}{dx} = 8x - 8$

Put $\frac{dg_1}{dx} = 0 \Rightarrow 8x - 8 = 0$

$\Rightarrow \boxed{x = 1}$

$\therefore \frac{d^2 g_1}{dx^2} = 8 > 0$

At $x=1$, $\frac{d^2 g_1}{dx^2} = 8 > 0$ $\therefore x = 1$ is a point of minima

& Min value $= g_1(1) = f(1,3) = 45$

Also at the corner $(0,3)$, $f(0,3) = 49$

Along the boundary CO, $x = 0$, $f(0,y) = 9y^2 - 12y + 4 = h(y)$
which is same as that of $h(y)$.

Absolute Min value $= -4$ at $(1, 2/3)$

Absolute Max value $= 49$ occurs at $(0,3)$ & $(2,3)$.

Conditional Maximum/Minimum →

$f(x_1, x_2, ---x_n)$ when the variables are not independent but connected by one or more constraints of the form

$\phi_i(x_1, x_2 --- x_n) = 0$, $i = 1, 2, --- k$

where generally $n > k$

Lagrange's Method of Multiplies

Max/Min $f(x,y,z)$ sub to the $\phi_1(x,y,z)$ & $\phi_2(x,y,z)$

(1) $F(x,y,z) = f(x,y,z) + \lambda_1 \phi_1(x,y,z) + \lambda_2 \phi_2(x,y,z)$

(2) $\left.\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0 \right] — \text{(A)}$

(3) Solve (*) simultaneously to get stationary points

Q Find the minimum value of $x^2+y^2+z^2$ subject to the condition $xyz=a^3$.

Sol Consider $F(x,y,z) = (x^2+y^2+z^2) + \lambda(xyz-a^3)$ — (1)

$f(x,y,z) = x^2+y^2+z^2$

$\phi(x,y,z) = xyz-a^3$

$F = f+\lambda\phi$ Lagrange's Mul

$$\frac{\partial F}{\partial x} = 2x + \lambda yz$$

$$\frac{\partial F}{\partial y} = 2y + \lambda xz$$

$$\frac{\partial F}{\partial z} = 2z + \lambda xy$$

Put $\frac{\partial F}{\partial x} = 0 \Rightarrow 2x+\lambda yz = 0$ or $x\lambda yz = -2x^2$ — (1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y+\lambda xz = 0$ or $y\lambda xz = -2y^2$ — (2)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z+\lambda xy = 0$ or $z\lambda xy = -2z^2$ — (3)

From (1) (2) & (3) $\boxed{-2x^2 = -2y^2 = -2z^2}$ or $\boxed{x^2=y^2=z^2}$ But the given condition is $\boxed{xyz=a^3}$

$x^2y^2z^2 = a^6$

$x^6 = a^6$

$x = \pm a$

$y = \pm a$

$\therefore$ The possible stationary points are $(a,-a,-a)$, (a,a,a), $(-a,-a,a)$ & $(-a,a,-a)$

At all the stationary points $f(x,y,z) = x^2+y^2+z^2$ is $3a^2$

$\boxed{x^2=y^2=z^2}$

$\boxed{xyz=a^3}$

$x^2y^2z^2 = a^6$

$x^2x^2x^2 = a^6$

$x^6 = a^6$

$\boxed{x=+a, \; x=-a}$

Now

A M of x^2,y^2,z^2 is $\frac{x^2+y^2+z^2}{3}$

& GM of x^2,y^2,z^2 is $(x^2y^2z^2)^{1/3}$

$= (a^6)^{1/3} = a^2$

Since A.M $\geq$ G.M

$\therefore \frac{x^2+y^2+z^2}{3} \geq a^2$ or $\boxed{x^2+y^2+z^2 \geq 3a^2}$

Hence, all the stationary points are the points of constrained minima & Minimum value at all the points is $3a^2$

Q Find the extreme values of $f(x,y,z) = 2x+3y+z$ such that $\underbrace{x^2+y^2=5}$, & $\underbrace{x+z=1}$

Sol Here $f(x,y,z) = 2x+3y+z$ & $\phi_1(x,y,z) = x^2+y^2-5$ &
$\phi_2(x,y,z) = x+z-1$

Consider $F(x,y,z) = f + \underbrace{\lambda_1}\phi_1 + \underbrace{\lambda_2}\phi_2$

$$F(x,y,z) = 2x+3y+z + \lambda_1(x^2+y^2-5) + \lambda_2(x+z-1)$$

Now, $\frac{\partial F}{\partial x} = 2 + 2\lambda_1 x + \lambda_2 = 0 \Rightarrow 2+2\lambda_1 x - 1 = 0 \Rightarrow 2\lambda_1 x = -1$
$x = -1/2\lambda_1$

$\frac{\partial F}{\partial y} = 3 + 2\lambda_1 y = 0 \Rightarrow 2\lambda_1 y = -3$ or $y = -3/2\lambda_1$

$\frac{\partial F}{\partial z} = 1 + \lambda_2 = 0 \Rightarrow \boxed{\lambda_2 = -1}$

Here $\lambda_2 = -1$, $x = -1/2\lambda_1$, $y = -3/2\lambda_1$

Sub these in $x^2+y^2=5$

$$\left(\frac{-1}{2\lambda_1}\right)^2 + \left(\frac{-3}{2\lambda_1}\right)^2 = 5 \text{ or } \frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5$$

or $\frac{10}{4\lambda_1^2} = 5$ or $4\lambda_1^2 = 2$
or $\lambda_1^2 = \frac{1}{2}$

$\therefore \lambda_1 = \pm\frac{1}{\sqrt{2}}$

$\therefore$ when $\lambda_1 = \frac{1}{\sqrt{2}}$, we get $x = \frac{-\sqrt{2}}{2}$, $y = \frac{-3\sqrt{2}}{2}$ & $z = 1 - x$
$= 1 + \frac{\sqrt{2}}{2}$
$= \frac{2+\sqrt{2}}{2}$

$\therefore$ Point is $\left(-\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right)$

& $f(x,y,z) = 2x+3y+z$
$= 2 \times -\frac{\sqrt{2}}{2} + 3 \times -\frac{3\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}$

$$= \frac{-2\sqrt{2} - 9\sqrt{2} + 2 + \sqrt{2}}{2} = \frac{2 - 10\sqrt{2}}{2}$$
$$= 1 - 5\sqrt{2}$$

Also when $\lambda_1 = -\frac{1}{\sqrt{2}}$ then $x = \frac{\sqrt{2}}{2}$, $y = \frac{3\sqrt{2}}{2}$, $z = 1 - x$
$$= 1 - \frac{\sqrt{2}}{2}$$
$$= \frac{2-\sqrt{2}}{2}$$

$\therefore$ point is $\left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\right)$

& $f(x,y,z) = 2x + 3y + z$
$$= \frac{2\sqrt{2}}{2} + \frac{3\times3\sqrt{2}}{2} + \frac{2-\sqrt{2}}{2}$$
$$= \frac{2\sqrt{2} + 9\sqrt{2} + 2 - \sqrt{2}}{2} = \frac{2 + 10\sqrt{2}}{2}$$
$$= 1 + 5\sqrt{2}$$

$\therefore$ Extreme values are $1 - 5\sqrt{2}$ & $1 + 5\sqrt{2}$ Ans

---

Q Find the limit of the following

① $\lim_{(x,y)\to(2,1)} 3x+4y$

$= 3\times2 + 4\times1 = \underline{10}$

② $\lim_{(x,y)\to(1,1)} x^2+2y$

$= (1)^2 + 2(1) = 3$

① Put x & y in the fun
② $\frac{0}{0}$ or $\frac{\infty}{\infty}$
③ limit along the path
if along the different paths we all get a finite same limit then limit exist

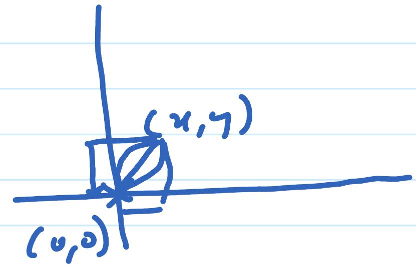
∞

Q $\lim_{(x,y)\to(0,0)} \left[\frac{xy}{\sqrt{x^2+y^2}}\right]$

$\left[\frac{0}{0}\right]$

Let the path is $y = mx$

Sol let the path is $y = mx$



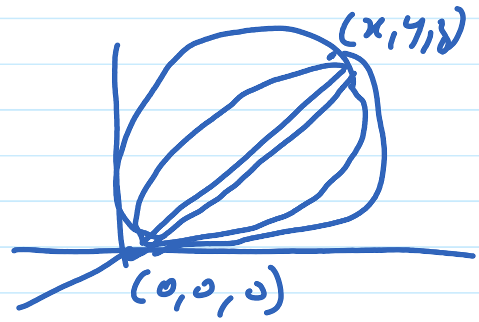
$$= \lim_{x \to 0} \frac{x \times mx}{\sqrt{x^2 + m^2 x^2}}$$

$$= \lim_{x \to 0} \frac{m x^{\cancel{2}}}{\cancel{x}\sqrt{1+m^2}}$$

$$= \lim_{x \to 0} x \cdot \frac{m}{\sqrt{1+m^2}}$$

$$= 0 \times \frac{m}{\sqrt{1+m^2}} = 0$$

Q $$\lim_{(x,y,z) \to (0,0,0)} \frac{xy + yz + zx}{\sqrt{x^2+y^2+z^2}}$$



Sol let the path by $y = mx$, $z = m_1 x$

$$\lim_{x \to 0} \frac{mx^2 + mm_1 x^2 + m_1 x^2}{\sqrt{x^2 + m^2 x^2 + m_1^2 x^2}}$$

$$= \lim_{x \to 0} \frac{x^{\cancel{2}}\left[m + mm_1 + m_1\right]}{\cancel{x}\sqrt{1 + m^2 + m_1^2}}$$

$$= \lim_{x \to 0} x \left[\frac{m + mm_1 + m_1}{\sqrt{1+m^2+m_1^2}}\right]$$

$$= 0 \times [\quad] = 0$$

Q $$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}$$ $\left[\frac{0}{0}\right]$

Take the path $y = mx$

$$\lim \frac{mx^2}{\ldots} = \lim_{x \to 0} \frac{\cancel{x^2} m}{\cancel{x^2}[1+m^2]}$$

$$\lim_{x\to 0} \frac{m x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x^2 m}{x^2[1+m^2]}$$

$$= \frac{m}{1+m^2}$$

which depends on 'm' therefore limit does not exist

Q $\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{y}{x}\right]$

$$= \lim_{x\to 0} \tan^{-1}\left[\frac{1}{x}\right]$$

If $\lim_{x\to 0^-} \tan^{-1}\left[\frac{1}{x}\right] = \lim_{x\to -\infty} \tan^{-1}[-\infty] = -\pi/2$

& $\lim_{x\to 0^+} \tan^{-1}\left[\frac{1}{x}\right] = \tan^{-1}[+\infty] = +\pi/2$

$\therefore \lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{y}{x}\right]$ does not exist

Q. $\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y}$

Choose the path $y = mx^2$, As $(x,y)\to(0,0)$, $x\to 0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y} = \lim_{x\to 0} \frac{x+\sqrt{m}\,x}{x^2+mx^2}$$

$$= \lim_{x\to 0} \frac{x\,[1+\sqrt{m}]}{x^2[1+m]}$$

$$= \lim_{x\to 0} \frac{1}{x} \frac{[1+\sqrt{m}]}{1+m}$$

$$= \infty$$

Since the limit is not finite, the limit does not exist.

Q $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$

Q $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$

✓ Yes exist
✓ No doesn't exist

Choose the path $y = mx^3$,
As $(x,y)\to(0,0)$, we get $x\to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2} = \lim_{x\to 0} \frac{x^3\, m x^3}{x^6+m^2x^6}$$

$$= \lim_{x\to 0} \frac{\cancel{x^6}\, m}{\cancel{x^6}[1+m^2]} = \underbrace{\frac{m}{1+m^2}}$$

which depends on m. For different values of m, we obtain different limits,
Hence limit does not exist.

$m=1$, $\frac{1}{2}$
$m=2$, $\frac{2}{5}$
$m=3$ $\frac{3}{10}$

Q Show that the following functions are continuous at the point $(0,0)$

(1) $$f(x,y) = \begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2}, & (x,y)\neq(0,0) \\ \underbrace{0}, & (x,y)=(0,0) \end{cases}$$

Sol

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^4+3y^4}{x^2+y^2}$$

① $\displaystyle\lim_{(x,y)\to(\underset{\sim}{0,0})} f(x,y)$
② $f(0,0)$
③ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Choose the path $y = mx$, As $(x,y)\to(0,0)$, we get $x\to 0$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{2x^4+3m^4x^4}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^{\cancel{4}\,2}[2+3m^4]}{\cancel{x^2}[1+m^2]}$$

$$= 0\times\frac{(2+3m^4)}{1+m^2}$$

✓ $x\to 0$
$\boxed{x\neq 0}$
✓ $(x,y)\to(0,0)$
$\overline{(x,y)}\neq(0,0)$

$= 0$

Also $f(0,0) = 0$

$\therefore \lim\limits_{(x,y)} f(x,y) = f(0,0)$

$\therefore f(x,y)$ is cts at $(0,0)$.

(ii) $f(x,y) = \begin{cases} \dfrac{2x(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Sol $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x(x^2-y^2)}{x^2+y^2}$

choose the path $y = mx$, As $(x,y)\to(0,0)$, we get $x \to 0$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{2x(x^2-m^2x^2)}{x^2+m^2x^2}$

$= \lim\limits_{x\to 0} \dfrac{2x^{\cancel{3}}(1-m^2)}{\cancel{x^2}(1+m^2)}$

$= 0 \times \dfrac{(1-m^2)}{1+m^2} = 0$

Also $f(0,0) = 0$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$\therefore f(x,y)$ is cts at $(0,0)$.

(iii) $f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y) \neq (0,0) \\ \dfrac{1}{2}, & (x,y) = (0,0) \end{cases}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}[2(x+2y)]}$

$(x,y)\to(0,0) \qquad (x,y)\to(0,0) \quad \tan^{-1}[\ldots]$

Put $x+2y=t$, As $(x,y)\to(0,0)$, we get $t\to 0$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} \frac{\sin^{-1} t}{\tan^{-1} 2t}$$

$$= \lim_{t\to 0} \frac{\frac{\sin^{-1} t}{t} \times \cancel{t}}{\frac{\tan^{-1} 2t}{2t} \times 2\cancel{t}}$$

$$= \frac{1}{1} \times \frac{1}{2} = \frac{1}{2}$$

Also $f(0,0) = 1/2$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$\therefore f(x,y)$ is cts at $(0,0)$

$$\lim_{(x,y)} f(x,y) = \lim_{x\to 0} \frac{\sin^{-1}(x(1+m))}{\tan^{-1}(2x(1+m))}$$

$$= \lim_{x\to 0} \frac{\frac{\sin^{-1}(x(1+m))}{x(1+m)} \times \cancel{x(1+m)}}{\frac{\tan^{-1}(2x(1+m))}{2x(1+m)} \times 2\cancel{x(1+m)}}$$

$$= \frac{1}{2} \ \#$$

Discuss the continuity of

Q $$f(x,y) = \begin{cases} \frac{x-y}{x+y}, & (x,y)\neq(0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

at the point $(0,0)$

(A) Cont at $(0,0)$

B discts at (0,0)

Sol $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

choose $y=mx$, As $(x,y)\to(0,0)$, we get $x\to 0$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x-mx}{x+mx}$

$= \lim_{x\to 0} \frac{\cancel{x}(1-m)}{\cancel{x}(1+m)}$

$= \frac{1-m}{1+m}$

which depends on 'm', $\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\therefore$ the given fun is discts at (0,0).

Dis the cont of

Q $f(x,y) = \begin{cases} \frac{x^2 - x\sqrt{y}}{x^2+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

at (0,0).

A $\to$ cts at (0,0)

B $\to$ discts at (0,0)

Sol $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 - x\sqrt{y}}{x^2+y}$

Choose the path $y=mx^2$, As $(x,y)\to(0,0)$, $x\to 0$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2 - x\sqrt{m}x}{x^2+mx^2}$

$= \lim_{x\to 0} \frac{\cancel{x^2}[1-\sqrt{m}]}{\cancel{x^2}[1+m]}$

$= \frac{1-\sqrt{m}}{1+m}$

which depends upon m, $\therefore \lim f(x,y)$ does not exist. Hence $f(x,y)$ is not cts at $(0,0)$.

Q Discuss the continuity of

$$f(x,y) = \begin{cases} \dfrac{x^2+xy+x+y}{x+y}, & (x,y) \neq (2,2) \\ 4, & (x,y) = (2,2) \end{cases}$$

at the point $(2,2)$.

Sol #
(A) cts at $(2,2)$
(B) discts at $(2,2)$

$$\lim_{(x,y)\to(2,2)} f(x,y) = \lim_{(x,y)\to(2,2)} \frac{x^2+xy+x+y}{x+y}$$

$$= \frac{4+4+2+2}{4}$$

$$= \frac{12}{4} = 3$$

or

$$\lim_{(x,y)\to(2,2)} \frac{x(x+y)+(x+y)}{x+y}$$

$$= \lim_{(x,y)\to(2,2)} \frac{\cancel{(x+y)}(x+1)}{\cancel{(x+y)}}$$

$$= \lim_{x\to 2} x+1 = \underline{3}$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 3 \neq f(2,2) = 4$$

$\therefore f(x,y)$ is discontinuous at $(2,2)$

Q Q $\lim_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$ $\left[\dfrac{0}{0}\right]$

Sol choose the path $y = mx$, As $(x,y)\to(0,0)$, we get $x\to 0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\sqrt{1+m^2}}$$

$$= \frac{1}{\sqrt{1+m^2}}$$

limit does not exist.