Chapter 15

Vector Differential and Integral Calculus

15.1 Introduction

In this chapter, we shall study the vector differential and integral calculus. We often call this study as vector analysis or vector field theory. We first introduce few concepts.

Scalar function A scalar function $f(x, y, z)$ is a function defined at each point in a certain domain D in space. Its value is real and depends only on the point $P(x, y, z)$ in space, but not on any particular coordinate system being used. For every point $(x, y, z) \in D, f$ has a real value. We say that a *scalar field f* is defined in D. For example, The distance function in 3-D space which defines the distance between the points $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$

$$f(P) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{15.1}$$

defines a scalar field. In this case, the domain D is the whole of the 3-D space.

Vector function A function $\mathbf{v} = \mathbf{v}(P) = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ defined at each point $P \in D$ is called a vector function. We say that a *vector field* is defined in D. In cartesian coordinates, we can write

$$\mathbf{v} = v_1(x, y, z)\,\mathbf{i} + v_2(x, y, z)\,\mathbf{j} + v_3(x, y, z)\,\mathbf{k}. \tag{15.2}$$

For example, the velocity field $\mathbf{v}(P)$ defined at any point P on a rotating body defines a vector field.

If the scalar and vector fields depend on time also, then we denote them as $f(P, t)$ and $\mathbf{v}(P, t)$, respectively. Both the fields are independent of the choice of the coordinate systems.

Level surfaces Let $f(x, y, z)$ be a single valued continuous scalar function defined at every point $P \in D$. Then $f(x, y, z) = c$, a constant, defines the equation of a surface and is called a level surface of the function. For different values of c, we obtain different surfaces, no two of which intersect. For example, if $f(x, y, z)$ represents temperature in a medium, then $f(x, y, z) = c$ represents a surface on which the temperature is a constant c. Such surfaces are called *isothermal* surfaces.

Example 15.1 Find the level surfaces of the scalar fields in space, defined by the following functions

(i) $f(x, y, z) = (x^2 + y^2 + z^2)$, (ii) $f(x, y, z) = z - \sqrt{x^2 + y^2}$.

Solution

(i) We find that $f(x, y, z) = c$ gives $x^2 + y^2 + z^2 = c$. Therefore, the level surfaces are spheres of radius $\sqrt{c}$.

(ii) We find that $f(x, y, z) = c$ gives $z - \sqrt{x^2 + y^2} = c$ or $x^2 + y^2 = (z - c)^2$. The level surfaces are cones.

15.2 Parametric Representations, Continuity and Differentiability of Vector Functions

Parametric representation of curves

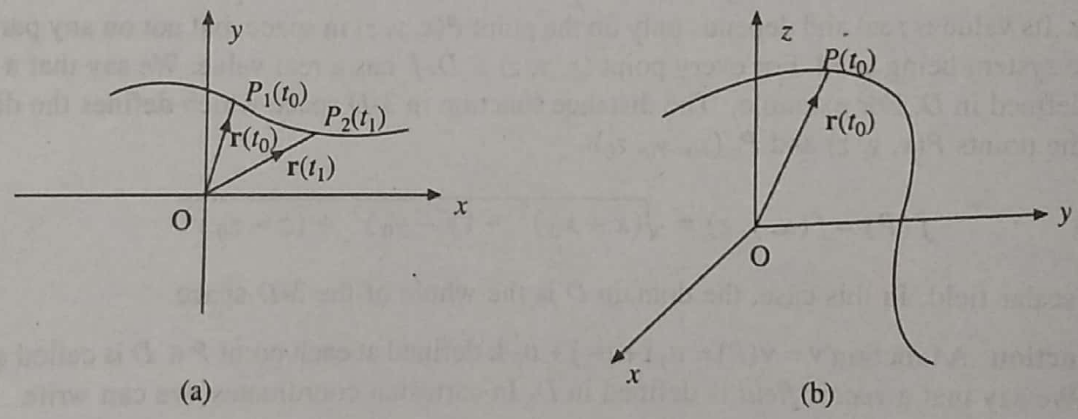
We recall that a curve C in the two dimensional x-y plane can be parametrised by $x = x(t)$, $y = y(t)$, $a \le t \le b$. Then, the position vector of a point P on the curve C can be written as

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}. \tag{15.3}$$

Therefore, the position vector of a point on a curve defines a vector function (Fig. 15.1a). Similarly, a three-dimensional curve or a space curve C can be parametrised as

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},\ a \le t \le b \tag{15.4}$$

(Fig. 15.1b).



$\mathbf{r}(t_0) = x(t_0)\,\mathbf{i} + y(t_0)\,\mathbf{j}$, $\quad\quad$ $\mathbf{r}(t_0) = x(t_0)\,\mathbf{i} + y(t_0)\,\mathbf{j} + z(t_0)\,\mathbf{k}$

Fig. 15.1. Position vector of a point.

The following are the parametric forms of some of the curves

Straight line $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} = (a_1 + tb_1)\mathbf{i} + (a_2 + tb_2)\mathbf{j} + (a_3 + tb_3)\mathbf{k}$ (15.5)

This represents the position vector of a point on the line L which passes through the point A with position vector $\mathbf{a}$ and has the direction of the vector $\mathbf{b}$. There are also alternate ways of writing the parametric form of a straight line. For example, consider the line L: $x + y = 1$ in the first quadrant (Fig. 15.2). We can write the parametric form as

$$\mathbf{r}(t) = t\,\mathbf{i} + (1 - t)\,\mathbf{j},\ 0 \le t \le 1.$$

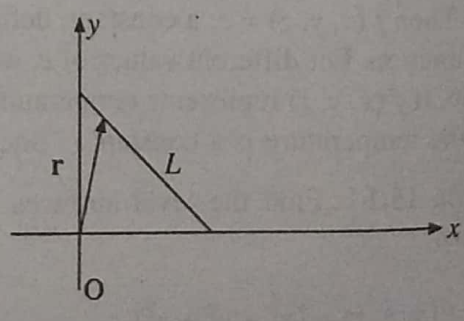


Fig. 15.2. Line $x + y = 1$.

Circle The parametric form of the circle $x^2 + y^2 = a^2$, can be written as

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j},\ 0 \le t \le 2\pi. \tag{15.6}$$

Ellipse The parametric form of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be written as

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + b \sin t\, \mathbf{j},\ 0 \le t \le 2\pi. \tag{15.7}$$

Parabola Consider the parabola as $y^2 = 4ax$. Then, we can take $y = t$ as a parameter and write the parametric form of the parabola as

$$\mathbf{r}(t) = (t^2/4a)\, \mathbf{i} + t\, \mathbf{j},\ -\infty < t < \infty. \tag{15.8}$$

Circle in a plane in 3-dimensions Consider the circle as $x^2 + y^2 = a^2$, $z = d$ which lies in the plane $z = d$. Then, we can write the parametric form as

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + d\, \mathbf{k}. \tag{15.9}$$

Helix The curve traced by the vector function

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + ct\, \mathbf{k},\ a > 0 \tag{15.10}$$

is called a *circular helix*. The curve lies on the cylinder $x^2 + y^2 = a^2$. If $c > 0$, it is shaped in the right-handed direction and if $c < 0$, it is shaped in the left-handed direction (Fig. 15.3).

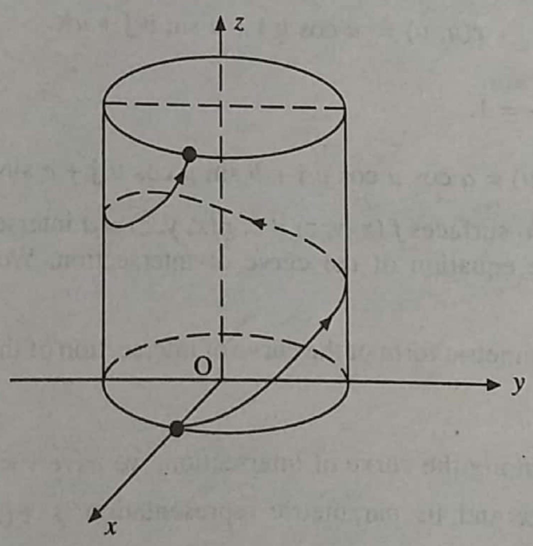


Fig. 15.3. Circular Helix.

The vector function

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + b \sin t\, \mathbf{j} + ct\, \mathbf{k},\quad a > 0,\ b > 0,\ c > 0 \tag{15.11}$$

describes an elliptical helix.

Parametric representation of surfaces

Parametric representation of surfaces can be done using two parameters. Let $f(x, y, z) = c$ or $g(x, y, z) = 0$ be the equation of a surface. Let an explicit representation of the surface be written as

$$z = h(x, y). \tag{15.12}$$

Then, if we set $x = u$, $y = v$, then the parametric form of the surface can be written as

$$\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + h(u, v)\,\mathbf{k}. \tag{15.13}$$

Alternately, we can choose u, v as two independent parameters and write $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$. Then the parametric representation of the surface can be written as

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k};\ (u, v) \in D. \tag{15.14}$$

The following are the parametric representations of some surfaces.

Cylinder $x^2 + y^2 = a^2$.

$$\mathbf{r}(u, v) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + v\mathbf{k}. \tag{15.15}$$

Sphere $x^2 + y^2 + z^2 = a^2$

$$\mathbf{r}(u, v) = a\cos u\cos v\,\mathbf{i} + a\sin u\cos v\,\mathbf{j} + a\sin v\,\mathbf{k},\ 0 \le u \le 2\pi,\ -\pi/2 \le v \le \pi/2. \tag{15.16}$$

Paraboloid of revolution $z = x^2 + y^2$.

$$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\mathbf{k}. \tag{15.17}$$

Cone of revolution $z^2 = x^2 + y^2$.

$$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\mathbf{k}. \tag{15.18}$$

Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

$$\mathbf{r}(u, v) = a\cos u\cos v\,\mathbf{i} + b\sin u\cos v\,\mathbf{j} + c\sin v\,\mathbf{k}. \tag{15.19}$$

Curve of intersection Two surfaces $f(x, y, z) = c$, $g(x, y, z) = d$ intersect along a curve. It is often possible to parametrise the equation of the curve of intersection. We illustrate the same in the following example.

Example 15.2 Find the parametric form of the curve of intersection of the plane $y = x$ and the surface $z = \sqrt{16 - x^2 - y^2}$.

Solution Let $x = t$. Then, along the curve of intersection, we have $y = t$ and $z = \sqrt{16 - 2t^2}$. The curve is in the plane $y = x$ and its parametric representation is $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{16 - 2t^2}\,\mathbf{k}$, $0 \le t \le 2\sqrt{2}$.

Limit, continuity and differentiability of vector functions

The concepts of limit, continuity and differentiability of calculus can easily be used for vector functions. Let the vector function be written in its parametric form, that is $\mathbf{v} = \mathbf{v}(t)$. In the cartesian system, we can write $\mathbf{v}(t) = v_1(t)\,\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\,\mathbf{k}$. Then, we define the following.

Limit The vector function $\mathbf{v}(t)$ has the limit $\boldsymbol{l}$ as $t \to a$, if $\mathbf{v}(t)$ is defined in some neighborhood of a, except possibly at $t = a$, and

$$\lim_{t \to a} |\,\mathbf{v}(t) - \boldsymbol{l}\,| = 0. \tag{15.20}$$

We write $\lim_{t\to a} \mathbf{v}(t) = \boldsymbol{l}$. In the cartesian system, this implies that limits of the component functions $v_1(t)$, $v_2(t)$ and $v_3(t)$ exist as $t \to a$ and

$$\lim_{t\to a} v_1(t) = l_1, \ \lim_{t\to a} v_2(t) = l_2, \ \lim_{t\to a} v_3(t) = l_3$$

where $\boldsymbol{l} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k}$.

Continuity A vector function $\mathbf{v}(t)$ is said to be continuous at $t = a$, if

(i) $\mathbf{v}(t)$ is defined in some neighborhood of a, (ii) $\lim_{t\to a} \mathbf{v}(t)$ exists, and (iii) $\lim_{t\to a} \mathbf{v}(t) = \mathbf{v}(a)$.
In cartesian system, this implies that $\mathbf{v}(t)$ is continuous at $t = a$, if and only if the component functions $v_1(t)$, $v_2(t)$ and $v_3(t)$ are continuous at $t = a$.

Differentiability A vector function $\mathbf{v}(t)$ is said to be differentiable at a point , if the limit

$$\lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$$

exists. If the limit exists, then we write it as $\mathbf{v}'(t)$ or as $d\mathbf{v}/dt$. In the cartesian system, this implies that the component functions $v_1(t)$, $v_2(t)$ and $v_3(t)$ are differentiable at a point t, that is the limits

$$\lim_{\Delta t\to 0} \frac{v_i(t+\Delta t) - v_i(t)}{\Delta t}, \quad i = 1, 2, 3 \text{ exist.}$$

Therefore,
$$\mathbf{v}'(t) = v_1'(t)\,\mathbf{i} + v_2'(t)\,\mathbf{j} + v_3'(t)\,\mathbf{k}. \tag{15.21}$$

Let $\mathbf{v}(t) = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be the parametric representation of a curve C. Then

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \frac{dx(t)}{dt}\mathbf{i} + \frac{dy(t)}{dt}\mathbf{j} + \frac{dz(t)}{dt}\mathbf{k}. \tag{15.22}$$

Geometric representation of r′(t) Let $\mathbf{r}'(t) \neq \mathbf{0}$. The vectors

$$\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t) \quad \text{and} \quad \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

are parallel. Now, let $\Delta t \to 0$. Then $\mathbf{r}(t+\Delta t) \to \mathbf{r}(t)$. If limit $\Delta\mathbf{r}/\Delta t$ exists as $\Delta t \to 0$, then the limiting position of this vector, that is $\lim_{\Delta t\to 0} (\Delta\mathbf{r}/\Delta t) = d\mathbf{r}/dt$ is the tangent line to the curve at the point P (Fig. 15.4). Therefore, $\mathbf{r}'(t)$ represents the tangent vector to the curve C.

The unit vector in the direction of the tangent is given by $\mathbf{r}'(t)/\,|\,\mathbf{r}'(t)\,|$.

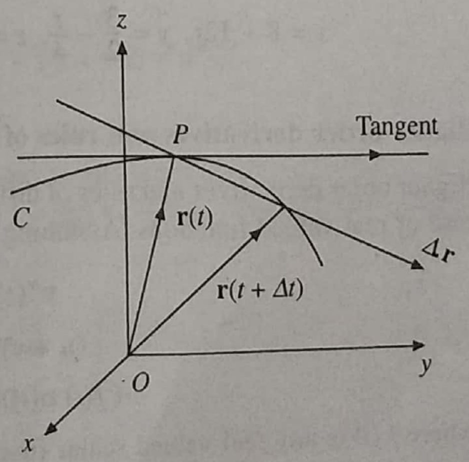


Fig. 15.4. Tangent vector.

Smooth curve Let $\mathbf{r}(t)$ denote the position vector of a point P on the curve C. Let $\mathbf{r}(t)$ have continuous first derivative, that is the component functions $x(t)$, $y(t)$ and $z(t)$ have continuous first derivatives. Let $\mathbf{r}'(t) \neq \mathbf{0}$, for all $t \in (a, b)$. Then, $\mathbf{r}(t)$ defines a smooth function on (a, b). The curve C traced by $\mathbf{r}(t)$ is called a *smooth curve*.

Example 15.3 Represent the parabola $y = 1 - 2x^2, -1 \le x \le 1$ in parametric form. Hence, find $\mathbf{r}'(0)$ and $\mathbf{r}'(1/\sqrt{2})$.

Solution Let $x = \sin t$. Then $y = 1 - 2\sin^2 t = \cos 2t$, and $-\pi/2 \le t \le \pi/2$. Hence,

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos 2t\,\mathbf{j}, \quad -\pi/2 \le t \le \pi/2.$$

For $x = 1/\sqrt{2}$, we get $t = \pi/4$. Therefore, $\mathbf{r}'(t) = \cos t\,\mathbf{i} - 2\sin 2t\,\mathbf{j}$, and $\mathbf{r}'(0) = \mathbf{i}$, $\mathbf{r}'(\pi/4) = (\mathbf{i}/\sqrt{2}) - 2\mathbf{j}$. The tangent line at $t = 0$ is parallel to the x-axis. (Note that $t = 0$ gives $x = 0, y = 1$ which is the vertex of the parabola.)

Example 15.4 Find the tangent vector to the curve whose parametric representation is $x = \cos t$, $y = \sin t$, $z = t$, $-\pi \le t \le \pi$. Hence, find the unit tangent vector.

Solution The position vector of a point on the curve is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$. Therefore, the tangent vector is $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$. We have $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. Hence, the unit tangent vector is $\hat{\mathbf{r}}'(t) = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})/\sqrt{2}$.

Example 15.5 Find the tangent vector to the curve whose parametric representation is $x = t^3$, $y = (t + 1)/t$, $z = t^2 + 1$, at $t = 2$. Hence, find the parametric representation of the tangent vector.

Solution The position vector of a point on the given curve is

$$\mathbf{r}(t) = t^3\mathbf{i} + \left(1 + \frac{1}{t}\right)\mathbf{j} + (t^2 + 1)\,\mathbf{k}, \quad t \ne 0.$$

Therefore, the tangent vector is

$$\mathbf{r}'(t) = (3t^2)\,\mathbf{i} - \frac{1}{t^2}\mathbf{j} + 2t\,\mathbf{k} \quad \text{and} \quad \mathbf{r}'(2) = 12\,\mathbf{i} - \frac{1}{4}\mathbf{j} + 4\mathbf{k}.$$

The position vector of the point at which $\mathbf{r}'(2)$ is the tangent is $\mathbf{r}(2) = 8\mathbf{i} + (3/2)\,\mathbf{j} + 5\mathbf{k}$. Therefore, we require the position vector of a point on the line passing through the point whose position vector is $\mathbf{r}(2)$ and has the direction of $\mathbf{r}'(2)$. Hence, parametric form of the line is given by (see Eq. (15.5))

$$x = 8 + 12t,\; y = \frac{3}{2} - \frac{t}{4},\; z = 5 + 4t, \text{ or } \mathbf{r}'(t) = \left(8, \frac{3}{2}, 5\right) + t\left(12, -\frac{1}{4}, 4\right).$$

Higher order derivatives and rules of differentiation

Higher order derivatives and rules of differentiation for vector functions have the same form as in the case of real valued functions. Assuming that the derivatives exist, we have the following results

$$\mathbf{v}''(t) = v_1''(t)\,\mathbf{i} + v_2''(t)\,\mathbf{j} + v_3''(t)\,\mathbf{k} \tag{15.23}$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}' \tag{15.24}$$

$$(f(t)\,\mathbf{u}(t))' = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t), \tag{15.25}$$

where $f(t)$ is any real valued scalar function.

$$(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t) \tag{15.26}$$

$$(\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t) \tag{15.27}$$

where • and × represent the dot and cross products, respectively. Note that the cross product of two vectors is not commutative.

Example 15.6 Find $\mathbf{v}'(t)$ in each of the following cases.

(i) $\mathbf{v}(t) = (\cos t + t^2)(t\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, (ii) $v(t) = (3t\,\mathbf{i} + 5t^2\mathbf{j} + 6\,\mathbf{k}) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k})$

(iii) $\mathbf{v}(t) = (t\,\mathbf{i} + e^t\mathbf{j} - t^2\mathbf{k}) \times (t^2\mathbf{i} + \mathbf{j} + t^3\mathbf{k})$

Solution

(i) Using Eq. (15.25), we obtain

$$\mathbf{v}'(t) = (\cos t + t^2)'(t\,\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + (\cos t + t^2)(t\,\mathbf{i} + \mathbf{j} + 2\mathbf{k})'$$
$$= (-\sin t + 2t)(t\,\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + (\cos t + t^2)(\mathbf{i})$$
$$= (3t^2 - t\sin t + \cos t)\,\mathbf{i} + (2t - \sin t)(\mathbf{j} + 2\mathbf{k})$$

(ii) Using Eq. (15.26) we obtain

$$v'(t) = (3t\,\mathbf{i} + 5t^2\,\mathbf{j} + 6\mathbf{k})' \cdot (t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}) + (3t\,\mathbf{i} + 5t^2\mathbf{j} + 6\mathbf{k}) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + t\,\mathbf{k})'$$
$$= (3\mathbf{i} + 10t\,\mathbf{j}) \cdot (t^2\mathbf{i} - 2t\,\mathbf{j} + t\,\mathbf{k}) + (3t\,\mathbf{i} + 5t^2\mathbf{j} + 6\mathbf{k}) \cdot (2t\,\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$= 3t^2 - 20t^2 + 6t^2 - 10t^2 + 6 = 6 - 21t^2.$$

(iii) $$\mathbf{v}'(t) = (t\,\mathbf{i} + e^t\mathbf{j} - t^2\mathbf{k}) \times (t^2\mathbf{i} + \mathbf{j} + t^3\mathbf{k})' + (t\mathbf{i} + e^t\mathbf{j} - t^2\mathbf{k})' \times (t^2\mathbf{i} + \mathbf{j} + t^3\mathbf{k})$$
$$= (t\,\mathbf{i} + e^t\mathbf{j} - t^2\mathbf{k}) \times (2t\mathbf{i} + 3t^2\mathbf{k}) + (\mathbf{i} + e^t\,\mathbf{j} - 2t\mathbf{k}) \times (t^2\mathbf{i} + \mathbf{j} + t^3\mathbf{k})$$
$$= [3t^2e^t\mathbf{i} - 5t^3\mathbf{j} - 2te^t\,\mathbf{k}] + [(t^3e^t + 2t)\mathbf{i} - 3t^3\,\mathbf{j} + (1 - t^2e^t)\mathbf{k}]$$
$$= [t^2e^t(3 + t) + 2t]\,\mathbf{i} - 8t^3\mathbf{j} + [1 - te^t\,(2 + t)]\,\mathbf{k}$$

Example 15.7 Prove that $[\mathbf{v}(t) \times \mathbf{v}'(t)]' = \mathbf{v}(t) \times \mathbf{v}''(t)$.

Solution Using Eq. (15.27), we obtain

$$[\mathbf{v}(t) \times \mathbf{v}'(t)]' = \mathbf{v}(t) \times [\mathbf{v}'(t)]' + \mathbf{v}'(t) \times \mathbf{v}'(t) = \mathbf{v}(t) \times \mathbf{v}''(t)$$

since the second term on the right hand side is a null vector.

Length of a space curve Let the curve C be represented in the parametric form as $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$. In cartesian system, we have $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. Then, the length of the curve is given by

$$\text{length} = l = \int_a^b [(x'(t))^2 + (y'(t))^2 + (z'(t))^2]^{1/2}\,dt = \int_a^b [\mathbf{r}'(t) \cdot \mathbf{r}'(t)]^{1/2}\,dt. \quad (15.28)$$

Note that the integrand is the norm of $\mathbf{r}'(t)$, that is

$$\| \mathbf{r}'(t) \| = [(x'(t))^2 + (y'(t))^2 + (z'(t))^2]^{1/2}$$

Then, we can write

$$l = \int_a^b \| \mathbf{r}'(t) \| \, dt \tag{15.29}$$

The notation $| \mathbf{r}'(t) |$ may also be used instead of $\| \mathbf{r}'(t) \|$.

Now, define the real valued function $s(t)$ as

$$s(t) = \int_a^t [(x'(\xi))^2 + (y'(\xi))^2 + (z'(\xi))^2]^{1/2} \, d\xi = \int_a^t \| \mathbf{r}'(\xi) \| \, d\xi \tag{15.30}$$

Then, $s(t)$ is the arc length of the curve from its initial point $(x(a), y(a), z(a))$ to an arbitrary point $(x(t), y(t), z(t))$ on the curve C. Therefore, $s(t)$ is the length function. Using Eq. (15.30), it is possible to solve for t as a function of s, that is $t = t(s)$. Then, the curve C can be parametrised in terms of the arc length s as

$$\mathbf{r}(s) = \mathbf{r}(t(s)) = x(t(s)) \, \mathbf{i} + y(t(s)) \, \mathbf{j} + z(t(s)) \, \mathbf{k} \tag{15.31}$$

Remark 1

If the position vector $\mathbf{r}(t)$ of a point on a curve C is expressed in terms of the arc length s, that is $\mathbf{r} = \mathbf{r}(s)$, then $\mathbf{r}'(s)$ is a unit vector.

Example 15.8 Find the length of the Helix traced by

$$\mathbf{r}(t) = a \cos t \, \mathbf{i} + a \sin t \, \mathbf{j} + ct \, \mathbf{k}, \quad a > 0, \quad 0 \le t \le 2\pi.$$

Solution We have $x(t) = a \cos t$, $y(t) = a \sin t$, $z(t) = ct$ and $x'(t) = -a \sin t$, $y'(t) = a \cos t$, $z'(t) = c$. Therefore,

$$s = \text{arc length} = \int_0^{2\pi} [a^2 \sin^2 t + a^2 \cos^2 t + c^2]^{1/2} \, dt = (2\pi)(a^2 + c^2)^{1/2}.$$

Example 15.9 In Example 15.8, express the position vector $\mathbf{r}(t)$ in terms of the arc length s.

Solution We have

$$s = \int_0^t (a^2 + c^2)^{1/2} \, dt = t(a^2 + c^2)^{1/2} \quad \text{or} \quad t = s/(a^2 + c^2)^{1/2}.$$

Therefore,

$$\mathbf{r}(s) = a \cos(s^*) \, \mathbf{i} + a \sin(s^*) \, \mathbf{j} + c \, s^* \, \mathbf{k}, \quad s^* = s/(a^2 + c^2)^{1/2}.$$

It can be verified that $| \mathbf{r}'(s) | = 1$.

15.2.1 Motion of a Body or Particle on a Curve

Suppose that a body or a particle moves along a curve C. Then, the position vector of the particle or the body at time t is given by $\mathbf{r}(t) = x(t) \, \mathbf{i} + y(t) \, \mathbf{j} + z(t) \, \mathbf{k}$. Assume that $x(t)$, $y(t)$ and $z(t)$ are twice differentiable. Let t vary over the interval $[a, b]$. Then, at time t the particle has travelled a distance $s(t)$ along the curve from the initial point $(x(a), y(a), z(a))$ to the point $(x(t), y(t), z(t))$. This distance is given by (see Eq. (15.30))

$$s(t) = \int_a^t [(x'(\xi))^2 + (y'(\xi))^2 + (z'(\xi))^2]^{1/2} \, d\xi. \tag{15.32}$$

Then, the vectors

$$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k} \tag{15.33}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + z''(t)\,\mathbf{k} \tag{15.34}$$

are called the *velocity* and *acceleration* of the particle respectively. The scalar quantity $|\mathbf{v}(t)|$ is called the *speed* of the particle, that is

$$|\mathbf{v}(t)| = [(x'(t))^2 + (y'(t))^2 + (z'(t))^2]^{1/2} \tag{15.35}$$

gives the speed of the particle. If $\mathbf{v}(t) \neq \mathbf{0}$, then the velocity vector is tangential to the curve of motion of the particle. Therefore, we can interpret that at any given instant of time, the particle is moving in the direction of the tangent to the curve. Comparing Eqs. (15.32) and (15.35), we find that speed is related to the arc length where $s'(t) = |\mathbf{v}(t)|$. Therefore, we may also write

$$s(t) = \int_a^t |\mathbf{v}(\xi)|\,d\xi. \tag{15.36}$$

Example 15.10 The position vector of a moving particle is given by $\mathbf{r}(t) = t^3\mathbf{i} + t\,\mathbf{j} + t^2\mathbf{k}$. Determine the velocity, speed and acceleration of the particle in the direction of the motion.

Solution We have $\mathbf{r}'(t) = 3t^2\mathbf{i} + \mathbf{j} + 2t\,k$ and $\mathbf{r}''(t) = 6t\mathbf{i} + 2\mathbf{k}$. Hence,

velocity $= \mathbf{v}(t) = \mathbf{r}'(t) = 3t^2\mathbf{i} + \mathbf{j} + 2t\,\mathbf{k}$, speed $= |\mathbf{v}(t)| = (9t^4 + 4t^2 + 1)^{1/2}$,

acceleration $= \mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\mathbf{k}$.

Exercise 15.1

Find the level surfaces of the scalar fields defined by the following functions.

1. $f = x + y + z$.
2. $f = x^2 + y^2 + z^2$.
3. $f = x^2 + y^2 - z$.
4. $f = x^2 + 9y^2 + 16z^2$.

Find the parametric representation of the straight line through the point P and has the direction $\mathbf{b}$ in the following problems.

5. $P(1, 2, 3)$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
6. $P(1, -1, 1)$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$.

Find the parametric representations of the following straight lines/curves. Use the indicated representation whereever given.

7. $x = y,\ y = z$.
8. $x + y + z = 3,\ y - z = 0$.
9. $x - y + z = 5,\ x - 2y + 3z = 3$.
10. $2x + y + 2z = 2,\ y + z = 0$.
11. $x = y^2 + z^2,\ y = z$.
12. $y^2 + z^2 = 9,\ x = 9 - y^2,\ y = 3 \sin t$.
13. $y = x^2 + z^2,\ y = 4,\ x = 2 \sin t$.
14. $(y - a)^2 + (z - a)^2 = 2a^2,\ x = 0,\ y - a = \sqrt{2}\, a \sin t$.

In the following problems, find the indicated derivative using the differentiation rules. Assume that all the given vector functions are differentiable.

15. $\mathbf{u}(t) = 5t^2\mathbf{i} + t\,\mathbf{j} + t^3\mathbf{k},\ f(t) = \sin t,\ [f(t)\,\mathbf{u}(t)]'$.
16. $\mathbf{u}(t) = [\sin(2t)\,\mathbf{i} - \cos(2t)\,\mathbf{j} + t\mathbf{k}],\ \mathbf{v}(t) = [\cos(2t)\,\mathbf{i} - \sin(2t)\,\mathbf{j} + t^2\mathbf{k}],\ [\mathbf{u}(t) \cdot \mathbf{v}(t)]'$.

17. $\mathbf{u}(t) = 6t^2\mathbf{i} - t\mathbf{j} + 3t^2\mathbf{k}$, $\mathbf{v}(t) = t\,\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$, $[\mathbf{u}(t) \times \mathbf{v}(t)]'$.
18. $\mathbf{u}(t) = (1 - t)\,\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$, $\mathbf{v}(t) = (1 + t)\,\mathbf{i} + e^t\,\mathbf{j} + t\,\mathbf{k}$, $[\mathbf{u}(t) \times \mathbf{v}(t)]'$.
19. $[t^2\mathbf{u}(t^2)]'$.
20. $[\mathbf{u}(at) + \mathbf{v}(a/t)]'$.
21. $[\mathbf{u}(t) \times \mathbf{u}''(t)]'$.
22. $[\mathbf{u}(t) \cdot (\mathbf{u}'(t) \times \mathbf{u}''(t))]'$.

In the following problems, find the parametric equation of the tangent line to the given curve at the indicated point.

23. $x = t$, $y = 2t^2$, $z = 3t^3$; $t = 2$.
24. $x = \sin t$, $y = \cos t$, $z = t$, $t = \pi/4$.
25. $x = t^2 - 1$, $y = t + 1$, $z = t/(t + 1)$, $t = 2$.
26. $x = t$, $y = e^t$, $z = 1$, $t = 1$.

Find the lengths of the following curves. In problems **29** to **31**, express $\mathbf{r}(t)$ as a function of the arc length.

27. $\mathbf{r}(t) = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}$, $0 \le t \le \pi/2$ (one cusp of hypocycloid).
28. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j} + \sqrt{3}\,t\,\mathbf{k}$, $0 \le t \le \pi$.
29. $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$.
30. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\,\mathbf{k}$, $-2\pi \le t \le 2\pi$.
31. $\mathbf{r}(t) = (t^2/2)\,\mathbf{i} + (t^3/3)\,\mathbf{k}$, $0 \le t \le 2$.
32. $\mathbf{r}(t) = t\,\mathbf{i} + (t^2/2)\,\mathbf{j}$, $0 \le t \le 1$.
33. Let $\mathbf{T}$ denote the unit tangent to the curve $x = 2t$, $y = 3t + 4$, $z = 3t$. Show that $d\mathbf{T}/dt = \mathbf{0}$. Interpret the result.
34. The position vector of a moving particle is $\mathbf{r}(t) = (\cos t + \sin t)\,\mathbf{i} + (\sin t - \cos t)\,\mathbf{j} + t\mathbf{k}$. Determine the velocity, speed and acceleration of the particle in the direction of the motion.
35. If a particle moves with constant speed c, then show that its acceleration vector $\mathbf{a}(t)$ is perpendicular to the velocity vector $\mathbf{v}(t)$.

15.3 Gradient of a Scalar Field and Directional Derivative

Let $f(x, y, z)$ be a real valued function defining a scalar field. To define the gradient of a scalar field, we first introduce a vector operator called *del* operator denoted by ∇. We define the vector differential operator in two and three dimensions as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \quad \text{and} \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

The *gradient* of a scalar field $f(x, y, z)$, denoted by ∇f or grad (f) is defined as

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}. \tag{15.37}$$

Note that the *del* operator ∇ operates on a scalar field and produces a vector field.

Example 15.11 Find the gradient of the following scalar fields

(i) $f(x, y) = y^2 - 4xy$ at $(1, 2)$,
(ii) $x^2y^2 + xy^2 - z^2$ at $(3, 1, 1)$.

Solution

(i) We have $\nabla f(x, y) = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y}\right)(y^2 - 4xy)$

$$= \mathbf{i}\frac{\partial}{\partial x}(y^2 - 4xy) + \mathbf{j}\frac{\partial}{\partial y}(y^2 - 4xy) = -4y\,\mathbf{i} + (2y - 4x)\,\mathbf{j}$$

At (1, 2), we obtain $\nabla f(x, y) = -8\,\mathbf{i}$.

(ii) We have $\nabla f(x, y, z) = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)(x^2y^2 + xy^2 - z^2)$

$$= (2xy^2 + y^2)\mathbf{i} + (2x^2y + 2xy)\,\mathbf{j} - 2z\mathbf{k}.$$

At (3, 1, 1) we obtain $\nabla f(x, y) = 7\mathbf{i} + 24\,\mathbf{j} - 2\,\mathbf{k}$.

Example 15.12 If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $|\,\mathbf{r}\,| = r$ and $\hat{\mathbf{r}} = \mathbf{r}/r$; then show that grad $(1/r) = -\,\hat{\mathbf{r}}/r^2$.

Solution We have $r^2 = x^2 + y^2 + z^2$. Therefore,

$$\text{grad}\left(\frac{1}{r}\right) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = \mathbf{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \mathbf{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \mathbf{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\left(\frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} + \frac{z}{r}\,\mathbf{k}\right) = -\frac{1}{r^2}\left(\frac{\mathbf{r}}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})/r$ is the unit vector.

Geometrical representation of the gradient Let $f(P) = f(x, y, z)$ be a differentiable scalar field. Let $f(x, y, z) = k$ be a level surface and $P_0(x_0, y_0, z_0)$ be a point on it. There are infinite number of smooth curves on the surface passing through the point P_0. Each of these curves has a tangent at P_0. The totality of all these tangent lines form a tangent plane to the surface at the point P_0. A vector normal to this plane at P_0 is called the *normal vector* to the surface at this point.

Consider now a smooth curve C on the surface passing through a point P on the surface. Let $x = x(t)$, $y = y(t)$, $z = z(t)$ be the parametric representation of the curve C. Any point P on C has the position vector $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = k.$$

Then, $\dfrac{d}{dt} f(x(t), y(t), z(t)) = 0$

By chain rule, we have $\dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt} = 0$

or $\left(\mathbf{i}\dfrac{\partial f}{\partial x} + \mathbf{j}\dfrac{\partial f}{\partial y} + \mathbf{k}\dfrac{\partial f}{\partial z}\right)\cdot\left(\mathbf{i}\dfrac{dx}{dt} + \mathbf{j}\dfrac{dy}{dt} + \mathbf{k}\dfrac{dz}{dt}\right) = 0$

or $\nabla f \cdot \mathbf{r}'(t) = 0$. (15.38)

Let $\nabla f(P) \neq \mathbf{0}$ and $\mathbf{r}'(t) \neq \mathbf{0}$. Now, $\mathbf{r}'(t)$ is a tangent vector to C at the point P and lies in the tangent plane to the surface at P. Hence, $\nabla f(P)$ is orthogonal to every tangent vector at P. Therefore, $\nabla f(P)$ is the vector normal to the surface $f(x, y, z) = k$ at the point P (Figs. 15.5 and 15.6).

Remark 2

In two variables, let $f(x, y) = k$ be a level curve C of a differentiable scalar field $f(x, y)$. Let $P_0(x_0, y_0)$ be a point on it. If this level curve is parametrised as $x = x(t)$, $y = y(t)$ then $x_0 = x(t_0)$ and $y_0 = y(t_0)$. Then, $\nabla f(x_0, y_0)$ is the normal vector to the curve C at the point P_0 (Fig. 15.7).

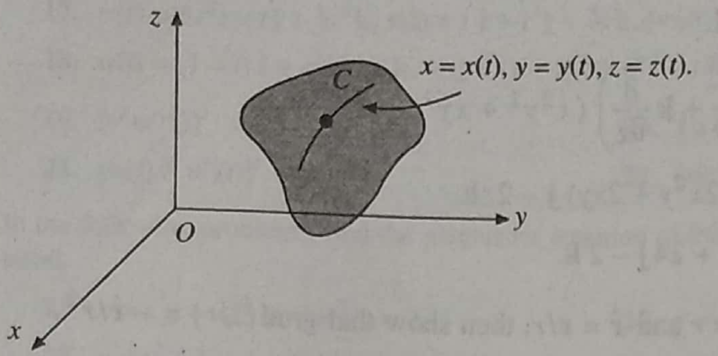


Fig. 15.5 A smooth curve on the surface.

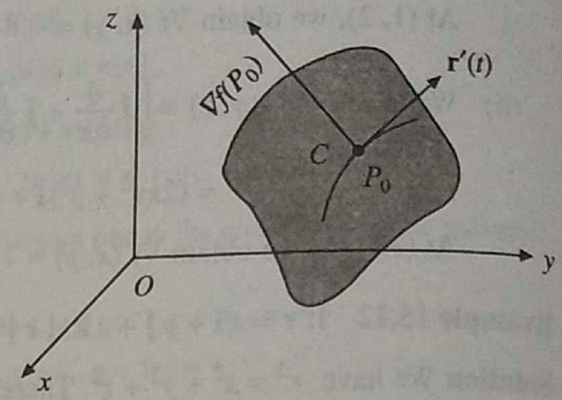


Fig. 15.6 Normal vector to a surface.

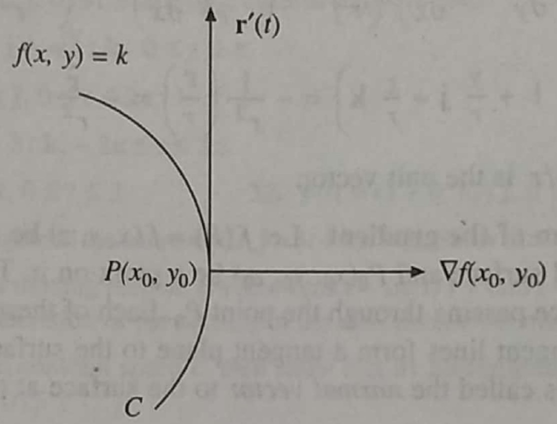


Fig. 15.7 Normal vector to a curve *C*

Remark 3 The unit normal vector is $\hat{\mathbf{n}}$ = grad f/|grad f|.

Remark 4

(a) The gradient vector (normal vector) at a point P_0 (x_0, y_0, z_0) on the surface $f(x, y, z) = k$ can be used to derive the equation of the tangent plane at the point P_0, to the surface. Let (x, y, z) be any point on the tangent plane. Then, $(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is a vector in the tangent plane. Hence, this vector is orthogonal to the gradient vector at P_0. Therefore,

$$\nabla f(P_0) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or
$$(x - x_0)\frac{\partial f}{\partial x}(P_0) + (y - y_0)\frac{\partial f}{\partial y}(P_0) + (z - z_0)\frac{\partial f}{\partial z}(P_0) = 0 \tag{15.39}$$

is the equation of the tangent plane at P_0.

(b) (i) Angle between two curves is the angle between their tangents at the common point.

(ii) Angle between two surfaces is the angle between their normals at the common point.

Example 15.13 Find a unit normal vector to the surface $xy^2 + 2yz = 8$ at the point $(3, -2, 1)$.

Solution Let $f(x, y, z) = xy^2 + 2yz = 8$. Then,

$$\frac{\partial f}{\partial x} = y^2, \quad \frac{\partial f}{\partial y} = 2xy + 2z \quad \text{and} \quad \frac{\partial f}{\partial z} = 2y.$$

Therefore, $$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = y^2\mathbf{i} + 2(xy + z)\mathbf{j} + 2y\mathbf{k}.$$

At (3, – 2, 1), we obtain the normal vector as $\nabla f(3, -2, 1) = 4\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}$. The unit normal vector at (3, –2, 1) is given by

$$\frac{(4\mathbf{i} - 10\,\mathbf{j} - 4\mathbf{k})}{\sqrt{16 + 100 + 16}} = \frac{2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}}{\sqrt{33}}.$$

Example 15.14 Find the normal vector and the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^2}$ at the point (3, 4, 5).

Solution Let $f(x, y, z) = z - \sqrt{x^2 + y^2} = 0$ be the surface. Then, the normal vector is given by

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(z - \sqrt{x^2 + y^2})$$

$$= -\frac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j} + \mathbf{k} = -\frac{x}{z}\,\mathbf{i} - \frac{y}{z}\,\mathbf{j} + \mathbf{k}, \quad (z \neq 0).$$

At (3, 4, 5), the normal vector is given by

$$\nabla f(3, 4, 5) = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} + \mathbf{k}$$

The tangent plane at the point (3, 4, 5) is given by

$$-\frac{3}{5}(x - 3) - \frac{4}{5}(y - 4) + (z - 5) = 0 \quad \text{or} \quad 3x + 4y - 5z = 0.$$

Example 15.15 Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point (1, 1, 1).

Solution The angle between two surfaces at a common point is the angle between their normals at that point. We have

$$f_1(x, y, z) = x \log z - y^2 + 1 = 0, \quad \nabla f_1(x, y, z) = (\log z)\,\mathbf{i} - 2y\mathbf{j} + (x/z)\,\mathbf{k}$$

$$\nabla f_1(1, 1, 1) = -2\mathbf{j} + \mathbf{k} = \mathbf{n}_1$$

$$f_2(x, y, z) = x^2 y + z - 2 = 0, \quad \nabla f_2(x, y, z) = 2xy\,\mathbf{i} + x^2\mathbf{j} + \mathbf{k}$$

$$\nabla f_2(1, 1, 1) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{n}_2$$

Therefore, $$\cos\theta = \left|\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right| = \frac{1}{\sqrt{5}\sqrt{6}} = \frac{1}{\sqrt{30}}, \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right).$$

Properties of Gradient Let f and g be any two differentiable scalar fields. Then,

$$\nabla(f + g) = \nabla f + \nabla g \tag{15.40}$$

$$\nabla(c_1 f + c_2 g) = c_1\nabla f + c_2\nabla g, \quad c_1, c_2 \text{ arbitrary constants} \tag{15.41}$$

$$\nabla(f g) = f\,\nabla g + g\,\nabla f \tag{15.42}$$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2},\ g \neq 0 \qquad (15.43)$$

Directional derivative

Let $f(P) = f(x, y, z)$ be a differentiable scalar field. Then, $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ denote the rates of change of f in the directions of x, y and z axis, respectively. If $f(x, y, z) = k$ is a level surface and P_0 is any point on it, then $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ at P_0 (x_0, y_0, z_0) denote the slopes of the tangent lines in the directions of **i**, **j** and **k** respectively. It is natural to give the definition of derivative in any direction which we shall call as the directional derivative.

Let $\hat{\mathbf{b}} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ be any unit vector. Let P_0 be any point P_0: $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

Then, the position vector of any point Q on the line passing through P_0 and in the direction of $\hat{\mathbf{b}}$ is given by

$$\mathbf{r} = \mathbf{a} + t\,\hat{\mathbf{b}} = (a_1 + b_1 t)\,\mathbf{i} + (a_2 + b_2 t)\,\mathbf{j} + (a_3 + b_3 t)\,\mathbf{k} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

That is, the point Q $(a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t)$ is on this line. Now, the vector from the point P_0 to Q is given by $t\,\hat{\mathbf{b}}$. Since $|\hat{\mathbf{b}}| = 1$, the distance from P_0 to Q is t. Then,

$$\frac{\partial f}{\partial t} = \lim_{t \to 0} \frac{f(Q) - f(P_0)}{t}$$

if it exists, is called the *directional derivative* of f at the point P_0 in the direction of $\hat{\mathbf{b}}$ (Fig. 15.8).

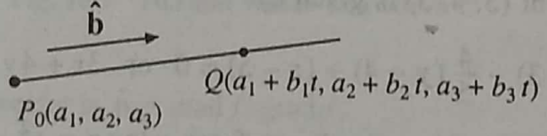


Fig. 15.8. Directional derivative.

Therefore, $\dfrac{\partial}{\partial t} f(x(t), y(t), z(t))$ is the rate of change of f with respect to the distance t.

We have (by chain rule)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \qquad (15.44)$$

where dx/dt, dy/dt and dz/dt are evaluated at $t = 0$, that is, at the point P_0. We write Eq. (15.44) as

$$\frac{\partial f}{\partial t} = \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) \cdot \left(\mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}\right) = \nabla f \cdot \frac{d\mathbf{r}}{dt}.$$

But $d\mathbf{r}/dt = \hat{\mathbf{b}}$ (a unit vector). Therefore, the directional derivative of f in the direction of $\hat{\mathbf{b}}$ is given by

$$\textit{directional derivative} = \nabla f \cdot \hat{\mathbf{b}} = \text{grad}\,(f) \cdot \hat{\mathbf{b}} \qquad (15.45)$$

which is denoted by $D_{\mathbf{b}}(f)$. Note that $\hat{\mathbf{b}}$ is a unit vector. If the direction is specified by a vector $\mathbf{u}$, then we have $\hat{\mathbf{b}} = \mathbf{u}/|\mathbf{u}|$. Some authors use s in place of t to denote length in the above derivation.

Remark 5

The length and direction of grad f is independent of the choice of the coordinate system.

Example 15.16 Find the directional derivative of $f(x, y, z) = xy^2 + 4xyz + z^2$ at the point (1, 2, 3) in the direction of $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$.

Solution We have

$$\nabla f = (y^2 + 4yz)\,\mathbf{i} + (2xy + 4xz)\,\mathbf{j} + (4xy + 2z)\,\mathbf{k}.$$

At the point (1, 2, 3), we have $\nabla f = 28\,\mathbf{i} + 16\,\mathbf{j} + 14\,\mathbf{k}$. The unit vector in the given direction is

$$\hat{\mathbf{b}} = (3\,\mathbf{i} + 4\mathbf{j} - 5\,\mathbf{k})/5\sqrt{2}.$$

Therefore, $D_{\mathbf{b}}(1, 2, 3) = \nabla f \cdot \mathbf{b} = \dfrac{1}{5\sqrt{2}}(28\,\mathbf{i} + 16\,\mathbf{j} + 14\,\mathbf{k}) \cdot (3\,\mathbf{i} + 4\,\mathbf{j} - 5\,\mathbf{k}) = \dfrac{78}{5\sqrt{2}}.$

Example 15.17 Find the directional derivative of $f(x, y) = x^2y^3 + xy$ at (2, 1), in the direction of a unit vector which makes an angle of $\pi/3$ with x-axis.

Solution We have

$$\nabla f = (2xy^3 + y)\,\mathbf{i} + (3x^2y^2 + x)\,\mathbf{j}, \quad \text{and} \quad \text{at } (2, 1), \quad \nabla f = 5\,\mathbf{i} + 14\,\mathbf{j}.$$

The unit vector is given by $\hat{\mathbf{b}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} = \dfrac{1}{2}\,\mathbf{i} + \dfrac{\sqrt{3}}{2}\,\mathbf{j}$, since $\theta = \pi/3$. Therefore,

$$\text{directional derivative} = (5\,\mathbf{i} + 14\,\mathbf{j}) \cdot \left(\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right) = \frac{5 + 14\sqrt{3}}{2}.$$

Maximum rate of change of a scalar field

Using the definition of scalar product, we have from Eq. (15.45)

$$D_{\mathbf{b}} f = \nabla f \cdot \hat{\mathbf{b}} = |\nabla f|\,|\hat{\mathbf{b}}| \cos\theta = |\nabla f| \cos\theta$$

since $|\hat{\mathbf{b}}| = 1$ and θ is the angle between the vectors ∇f and $\hat{\mathbf{b}}$. Since $-1 \le \cos\theta \le 1$, we have $-|\nabla f| \le D_{\mathbf{b}} f \le |\nabla f|$. Therefore, *the maximum value of the directional derivative is* $|\nabla f|$ *and it occurs when* $\theta = 0$, *that is,* $\hat{\mathbf{b}}$ *has the direction of* ∇f. *This direction is the direction of the normal vector. The minimum value of the directional derivative is* $-|\nabla f|$ *and it occurs when* $\theta = \pi$, *that is,* $\hat{\mathbf{b}}$ *and* ∇f *have opposite directions.* We may also say that the gradient vector ∇f points in the direction in which f increases most rapidly and $-\nabla f$ points in the direction in which f decreases most rapidly. Sometimes, the maximum rate of change is denoted by $\partial f/\partial n$, where $\mathbf{n}$ is the unit normal.

Conservative vector field

A vector field $\mathbf{v}$ is said to be conservative if the vector function can be written as the gradient of a scalar function f, that is, $\mathbf{v} = \nabla f$. In such a vector field, the work done in moving a particle from a point P to a point Q depends only on the points P and Q, and is independent of path along which the particle is displaced from P to Q. It may be noted that not every vector field is conservative (we shall have more discussion on conservative vector fields in section 15.5).

Example 15.18 Show that the vector field defined by the vector function $\mathbf{v} = xyz(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$ is conservative.

Solution If the given vector field is conservative, then it can be expressed as the gradient of a scalar function $f(x, y, z)$. Therefore,

$$\nabla f = \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) = \mathbf{v} = xyz\,(yz\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}).$$

Comparing, we obtain

$$\frac{\partial f}{\partial x} = xy^2z^2,\quad \frac{\partial f}{\partial y} = x^2yz^2,\quad \frac{\partial f}{\partial z} = x^2y^2z.$$

Integrating the first equation, we obtain $f(x, y, z) = \frac{1}{2}x^2y^2z^2 + g(y, z)$. Substituting in the second equation, we get

$$\frac{\partial f}{\partial y} = x^2yz^2 = x^2yz^2 + \frac{\partial g}{\partial y},\quad \text{or}\quad \frac{\partial g}{\partial y} = 0\quad \text{or}\quad g = g\,(z).$$

Substituting in the third equation, we get

$$\frac{\partial f}{\partial z} = x^2y^2z = x^2y^2z + \frac{dg}{dz},\quad \text{or}\quad \frac{dg}{dz} = 0\quad \text{or}\quad g = k, \text{ constant}.$$

Hence, $f\,(x, y, z) = \frac{1}{2}x^2y^2z^2 + k$. Therefore, there exists a scalar function $f\,(x, y, z)$ such that $\nabla f = \mathbf{v}$ and the vector field $\mathbf{v}$ is conservative.

Exercise 15.2

In problems **1** to **8**, compute the gradient of the scalar function and evaluate it at the given point.

1. $x^3 - 3x^2y^2 + y^3$, (1, 2).

2. $x\sin(yz) + y\sin(xz) + z\sin(xy)$, $(0, \pi/4, 1)$.

3. $\sin(xyz)$, $(1, -1, \pi)$.

4. $\ln(x^2 + y^2 + z^2)$, (3, – 4, 5).

5. $(x^2 + y^2 + z^2)^{1/2}$, (1, 1, 1).

6. $e^{xy}(x + y + z)$, (2, 1, 1).

7. $\ln(x + y + z)$, (1, 2, –1).

8. $x^3 + y^3 \sin 4y + z^2$, $(1, \pi/3, 1)$.

In problems **9** to **14**, find the normal vector and the unit normal vector to the given curve/surface at the indicated point.

9. $y^2 = 16x$, (4, 8).

10. $x^2 + y^2 = 25$, (3, 4)

11. $x^2 - y^2 = 12$, (4, 2).

12. $x^2 + 2y^2 + z^2 = 4$, (1, 1, 1).

13. $z = xy$, (–1, –2, 2).

14. $z^2 = x^2 - y^2$, $(2, 1, \sqrt{3})$.

Prove the following properties of gradient (f and g are scalar functions).

15. $\nabla(fg) = f\nabla g + g\nabla f$.

16. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, $g \neq 0$.

17. $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$.

18. $\nabla(g^m) = m g^{m-1}\nabla g$.

19. If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, show that $(\mathbf{u}\cdot\nabla)\,\mathbf{r} = \mathbf{u}$.

20. If $u = u(x, y, z, t)$, $x = x(t)$, $y = y(t)$, $z = z(t)$, show that

In problems **21** to **26**, find the directional derivative of the given scalar function at the given point in the indicated direction.

21. xyz, (1, 4, 3), in the direction of the line from (1, 2, 3) to (1, –1, – 3).

22. $\sqrt{xy^2 + 2x^2z}$, (2, – 2, 1), in the direction of negative z-axis

23. $x^2y - y^2z - xyz$, (1, –1, 0), in the direction $(\mathbf{i} - \mathbf{j} + 2\,\mathbf{k})$.

24. $(x^2 + y^2 + z^2)^{3/2}$, (–1, 1, 2), in the direction $\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

25. $x^2 + y^2 + 2z^2$, (1, 1, 2), in the direction of grad f.

26. $2x^2 + y^2 + z^2$, (1, 2, 3), in the direction of the line $x/3 = y/4 = z/5$.

In problems **27** to **29**, find a vector that gives the direction of maximum rate of increase. Find the maximum rate.

27. $e^{2y} \cos x$, at ($\pi/4$, 0).

28. $3x^2 + y^2 + 2z^2$, at (0, 1, 2).

29. $6xyz$, at (–1, 2, 1).

30. $x^2y^2z^2 + xz^2 + x^2y$, at (1, 2, –1).

In problems **31** to **34**, find a vector that gives the direction of minimum rate of increase. Find the minium rate.

31. $x^3 - xy^2 + y^3$, (– 2, 1).

32. $\tan(x^2 + y^2)$, $(\sqrt{\pi/3}, \sqrt{\pi/3})$.

33. $x^2 - y^2 + z^2$, (1, 2, 1).

34. $\sqrt{xy}\, e^z$, (4, 4, 1).

35. If $f(x, y) = x^2 - xy - y + y^2$, find all points where the directional derivative in the direction $\mathbf{b} = (\mathbf{i} + \sqrt{3}\,\mathbf{j})/2$ is zero.

36. It is given that $\nabla f(P) = 3\mathbf{i} + 4\mathbf{j}$. Find a unit vector $\hat{\mathbf{b}}$ such that (i) $D_{\mathbf{b}} f(P)$ is maximum and (ii) $D_{\mathbf{b}} f(P)$ is minimum.

37. Suppose $D_{\mathbf{b}} f(P) = 1$, $D_{\mathbf{u}} f(P) = 3$, $\mathbf{b} = (3\mathbf{i} + 4\,\mathbf{j})/5$, $\mathbf{u} = (4\mathbf{i} - 3\,\mathbf{j})/5$. Find $\nabla f(P)$.

38. Find the values of the constants a, b and c such that the maximum value of the directional derivative of $f(x, y, z) = axy^2 + byz + cx^2z^2$ at (1, –1, 1) is in the direction parallel to the axis of y and has magnitude 6.

39. The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (4, 4, 2) desires to fly in such a direction that it gets cooled faster. Find the direction in which it should fly.

In problems **40** to **45**, find the equation of the tangent plane to the graph of the equation at the given point.

40. $x^2 - 3y^2 - z^2 = 2$, (3, 1, 2).

41. $z = 16 - x^2 - y^2$, (1, 3, 6).

42. $xy + yz + zx = -1$, (1, –1, 2).

43. $z = 2e^{-x} \sin(2y)$, (0, $\pi/12$, 1).

44. $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, (x_0, y_0, z_0).

45. $(x^2/a^2) - (y^2/b^2) + (z^2/c^2) = 1$, (x_0, y_0, z_0).

In problems **46** to **49**, find a scalar function f such that $\mathbf{v} = \nabla f$.

46. $\mathbf{v} = xy(2yz\,\mathbf{i} + 2xz\,\mathbf{j} + xy\,\mathbf{k})$.

47. $\mathbf{v} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$.

48. $\mathbf{v} = 12x\,\mathbf{i} - 15y^2\mathbf{j} + \mathbf{k}$.

49. $\mathbf{v} = e^{xyz}(yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k})$.

50. Does there exist a function $f(x, y, z) \neq 0$ such that $\nabla f = \mathbf{0}$ for all (x, y, z)? If it exists, does the level surface $f(x, y, z) = k$ have a normal vector?

In problems **51** and **53**, find the angle between the two surfaces at the indicated point of intersection.

51. $z = x^2 + y^2$, $z = 2x^2 - 3y^2$, (2, 1, 5).

52. $x^2 + y^2 = 4$, $x^2 + y^2 + z^2 = 12$, (2, 2, – 2).

53. $x^2 + y^2 + z^2 = 9$, $z + 3 = x^2 + y^2$, (– 2, 1, 2).

In problems **54** and **55**, find the parametric equations for the normal line at the given point.

54. $z = 3x^2 - 2y^2$, (2, 1, 10).

55. $x^2 + 2y^2 + 4z^2 = 10$, (2, 1, –1).

15.4 Divergence and Curl of a Vector Field

In the previous section, we have defined the gradient operator which, when operated on a scalar field, produces a vector field. We shall now discuss two other important vector operations.

Let $\mathbf{v} = v_1(x, y, z)\,\mathbf{i} + v_2(x, y, z)\,\mathbf{j} + v_3(x, y, z)\,\mathbf{k}$ define a vector field.

Divergence of vector field v

Divergence of **v**, denoted by div **v**, is defined as the scalar field

$$\text{div } \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}. \tag{15.46}$$

We observe that div **v** can also be written in terms of the gradient operator as

$$\text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k})$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Note that $\nabla \cdot \mathbf{v}$ is just a notation for div **v** and it is not a scalar product in the usual sense, since $\nabla \cdot \mathbf{v} \neq \mathbf{v} \cdot \nabla$. In fact

$$\mathbf{v} \cdot \nabla = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z}$$

is a scalar operator.

Example 15.19 Find the divergence of the vector field

$$\mathbf{v} = (x^2y^2 - z^3)\,\mathbf{i} + 2xyz\,\mathbf{j} + e^{xyz}\,\mathbf{k}$$

Solution We have

$$\text{div } \mathbf{v} = \frac{\partial}{\partial x}(x^2y^2 - z^3) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(e^{xyz})$$

$$= 2xy^2 + 2xz + xye^{xyz}.$$

Curl of vector field v

Curl of a vector field **v**, denoted by curl **v**, is defined as the vector field

$$\text{curl } \mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}. \tag{15.47}$$

We observe that curl **v** can also be written in terms of the gradient operator as

$$\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{15.48}$$

Note that $\nabla \times \mathbf{v}$ is just a notation for curl **v** and it is not a vector product in the usual sense, since $\nabla \times \mathbf{v} \neq -\mathbf{v} \times \nabla$.

Sometimes, curl $\mathbf{v}$ is also written as

$$\text{curl } \mathbf{v} = \Sigma\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i}$$

where Σ denotes summation obtained by the cyclic rotation of the unit vectors **i, j, k,** the components v_1, v_2, v_3 and the independent variables x, y, z, respectively.

Example 15.20 Find the curl of the vector field

$$\mathbf{v} = (x^2y^2 - z^3)\,\mathbf{i} + 2xyz\,\mathbf{j} + e^{xyz}\,\mathbf{k}.$$

Solution We have

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y^2 - z^3 & 2xyz & e^{xyz} \end{vmatrix}$$
$$= \mathbf{i}\,(xze^{xyz} - 2xy) - \mathbf{j}(yze^{xyz} + 3z^2) + \mathbf{k}\,(2yz - 2x^2y).$$

There are two fundamental relations between the gradient, divergence and curl vectors. We prove these.

Curl of gradient Let f be a differentiable scalar field. Then

$$\text{curl (grad } f) = \mathbf{0} \quad \text{or} \quad \nabla \times (\nabla f) = \mathbf{0} \tag{15.49}$$

Proof. From the definition, we have

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) + \mathbf{j}\left(\frac{\partial^2 f}{\partial x\,\partial z} - \frac{\partial^2 f}{\partial x\,\partial z}\right) + \mathbf{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x\,\partial y}\right) = \mathbf{0}.$$

Divergence of curl Let $\mathbf{v}$ be a differentiable vector field. Then

$$\text{div (curl } \mathbf{v}) = 0 \quad \text{or} \quad \nabla \cdot (\nabla \times \mathbf{v}) = 0. \tag{15.50}$$

Proof From the definition, we have for $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$

$$\nabla\cdot(\nabla \times \mathbf{v}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left[\mathbf{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\right]$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = 0.$$

Example 15.21 Prove that div $(f\,\mathbf{v}) = f\,(\text{div } \mathbf{v}) + (\text{grad } f)\cdot\mathbf{v}$, where f is a scalar function.

Solution We have

$$\nabla\cdot(f\,\mathbf{v}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(f\,v_1\,\mathbf{i} + f\,v_2\,\mathbf{j} + f\,v_3\,\mathbf{k})$$
$$= \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)$$

$$= \left(v_1 \frac{\partial f}{\partial x} + f \frac{\partial v_1}{\partial x} \right) + \left(v_2 \frac{\partial f}{\partial y} + f \frac{\partial v_2}{\partial y} \right) + \left(v_3 \frac{\partial f}{\partial z} + f \frac{\partial v_3}{\partial z} \right)$$

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) + (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \cdot \left(\mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} \right)$$

$$= f(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot (\nabla f) = f(\nabla \cdot \mathbf{v}) + \nabla f \cdot \mathbf{v}.$$

Example 15.22 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\,\mathbf{k}$ and $r = |\,\mathbf{r}\,|$, show that div $(\mathbf{r}/r^3) = 0$.

Solution We have

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\frac{x}{r^3} \mathbf{i} + \frac{y}{r^3} \mathbf{j} + \frac{z}{r^3} \mathbf{k} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) = \frac{3}{r^3} - \frac{3}{r^4} \left(x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right).$$

Since $r^2 = x^2 + y^2 + z^2$, we obtain $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Therefore,

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = \frac{3}{r^3} - \frac{3}{r^5} (x^2 + y^2 + z^2) = \frac{3}{r^3} - \frac{3}{r^3} = 0.$$

Example 15.23 If $\mathbf{a}$ is a constant vector and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, show that curl $(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$.

Solution Let $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ where a_1, a_2 and a_3 are constants. We have

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (\mathbf{a} \times \mathbf{r})$$

$$= \mathbf{i} \times \frac{\partial}{\partial x} (\mathbf{a} \times \mathbf{r}) + \mathbf{j} \times \frac{\partial}{\partial y} (\mathbf{a} \times \mathbf{r}) + \mathbf{k} \times \frac{\partial}{\partial z} (\mathbf{a} \times \mathbf{r})$$

$$= \mathbf{i} \times \left(\mathbf{a} \times \frac{\partial \mathbf{r}}{\partial x} \right) + \mathbf{j} \times \left(\mathbf{a} \times \frac{\partial \mathbf{r}}{\partial y} \right) + \mathbf{k} \times \left(\mathbf{a} \times \frac{\partial \mathbf{r}}{\partial z} \right) \text{ (since } \mathbf{a} \text{ is a constant vector)}$$

$$= \mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k})$$

$$= (\mathbf{i} \cdot \mathbf{i})\, \mathbf{a} - (\mathbf{i} \cdot \mathbf{a})\, \mathbf{i} + (\mathbf{j} \cdot \mathbf{j}) \mathbf{a} - (\mathbf{j} \cdot \mathbf{a}) \mathbf{j} + (\mathbf{k} \cdot \mathbf{k}) \mathbf{a} - (\mathbf{k} \cdot \mathbf{a})\, \mathbf{k}$$

$$= 3\mathbf{a} - (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) = 2\mathbf{a}.$$

Remark 6

Let $\mathbf{v}$ denote the velocity of a fluid in a medium. If div $(\mathbf{v}) = 0$, then the fluid is said to be *incompressible*. In electromagnetic theory, if div $(\mathbf{v}) = 0$, then the vector field $\mathbf{v}$ is said to be *solenoidal*.

Remark 7

Some books use the word *rotation* in place of curl, that is, curl $(\mathbf{v})$ is written as rot $(\mathbf{v})$. In fluid mechanics, curl $(\mathbf{v})$ where $\mathbf{v}$ is the velocity, measures the vorticity of the fluid. If curl $(\mathbf{v}) = \mathbf{0}$, then $\mathbf{v}$ is said to be an *irrotational field*.

Physical interpretation of divergence We shall present an interpretation in fluid mechanics. Consider the flow of a compressible fluid of density $\rho(x, y, z, t)$, (density is mass per unit volume) and velocity

$$\mathbf{v}(x, y, z, t) = v_1(x, y, z, t)\,\mathbf{i} + v_2(x, y, z, t)\mathbf{j} + v_3(x, y, z, t)\,\mathbf{k}.$$

Therefore, the density and velocity vary from point to point and also with respect to time. Consider an infinitesimal volume element (parallelopiped of sides Δx, Δy, Δz) placed in the fluid as given in Fig. 15.9. The fluid enters the elemental volume through the faces and goes out from the other faces.

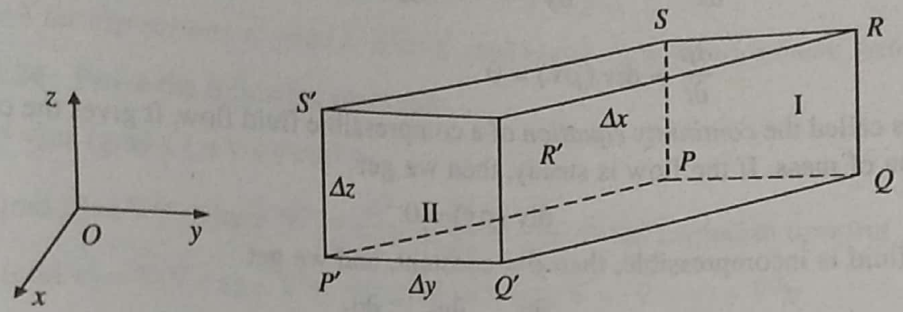


Fig. 15.9. Flow through a parallelopiped.

The face $PQRS$ is denoted as I and the face $P'\,Q'\,R'\,S'$ is denoted as II. Let us now compute the loss of fluid as it flows through the element in time Δt. We assume the following

(Volume of the fluid flowing through an element of surface area Δs in time Δt) $\approx$ (component of fluid velocity normal to the surface $\times$ area of surface $\Delta S \times \Delta t$).

The area of face I is $\Delta y\, \Delta z$ and the direction of the normal is $-\mathbf{i}$. Therefore, the mass of the fluid entering through the face I in time Δt, is approximately equal to

$$-(\rho v_1)(x, y, z, t)\, \Delta y\, \Delta z\, \Delta t.$$

The area of the face II is $\Delta y\, \Delta z$ and the direction of the normal is $\mathbf{i}$. Hence, the mass of the fluid leaving this face in time Δt is approximately equal to

$$(\rho v_1)\,(x + \Delta x, y, z, t)\, \Delta y\, \Delta z\, \Delta t.$$

Therefore, the approximate loss of mass as the fluid flows through the faces, perpendicular to the y-z plane, is

$$[(\rho v_1)\,(x + \Delta x, y, z, t) - (\rho v_1)\,(x, y, z, t)]\, \Delta y\, \Delta z\, \Delta t. \tag{15.51a}$$

Similarly, the approximate losses of mass through other faces of the elemental volume $\Delta V (= \Delta x\, \Delta y\, \Delta z)$, are

$$[(\rho v_2)\,(x, y + \Delta y, z, t) - (\rho v_2)\,(x, y, z, t)]\, \Delta x\, \Delta z\, \Delta t \tag{15.51b}$$

and

$$[(\rho v_3)\,(x, y, z + \Delta z, t) - (\rho v_3)\,(x, y, z, t)]\, \Delta x\, \Delta y\, \Delta t. \tag{15.51c}$$

Therefore, adding Eqs. (15.51a) to (15.51c), we obtain the total loss of mass of the fluid during the time Δt as

$$\Big[\frac{1}{\Delta x}\{(\rho v_1)\,(x + \Delta x, y, z, t) - (\rho v_1)\,(x, y, z, t)\} + \frac{1}{\Delta y}\{(\rho v_2)\,(x, y + \Delta y, z, t) - (\rho v_2)\,(x, y, z, t)\}$$

$$+ \frac{1}{\Delta z}\{(\rho v_3)\,(x, y, z + \Delta z, t) - (\rho v_3)\,(x, y, z, t)\}\Big]\, \Delta V\, \Delta t \tag{15.52a}$$

This loss of mass is due to the rate of change of density with respect to time and hence is equal to

$$-\frac{\partial \rho}{\partial t}\,\Delta V\,\Delta t. \tag{15.52b}$$

Equate the expressions in Eqs. (15.52a) and (15.52b), let $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$, $\Delta t \to 0$, and divide the resulting equation by $\Delta V\,\Delta t$. In the limit, we get

$$\frac{\partial}{\partial x}(\rho v_1) + \frac{\partial}{\partial y}(\rho v_2) + \frac{\partial}{\partial z}(\rho v_3) = -\frac{\partial \rho}{\partial t}$$

or

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \tag{15.53}$$

This equation is called the *continuity equation* of a compressible fluid flow. It gives the condition for the conservation of mass. If the flow is steady, then we get

$$\operatorname{div}(\rho \mathbf{v}) = 0.$$

Further, if the fluid is incompressible, then ρ = constant, and we get

$$\operatorname{div}(\mathbf{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0.$$

Therefore, when the fluid is incompressible, the divergence of the velocity vector vanishes. It states that the amount of fluid that enters and leaves a given volume is same, that is, there is no loss in the mass of the fluid.

Physical interpretation of curl Let a rigid body rotate with the uniform angular velocity $\Omega = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, about an axis l through the origin O (Fig. 15.10). Let the position vector of any point $P(x, y, z)$ on the rotating body be $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The tangential (linear) velocity $\mathbf{v}$ of the point $P(x, y, z)$ is given by

$$\begin{aligned}\mathbf{v} = \Omega \times \mathbf{r} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\\ &= (bz - cy)\,\mathbf{i} + (cx - az)\,\mathbf{j} + (ay - bx)\,\mathbf{k}.\end{aligned}$$

Now, $$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\Omega.$$

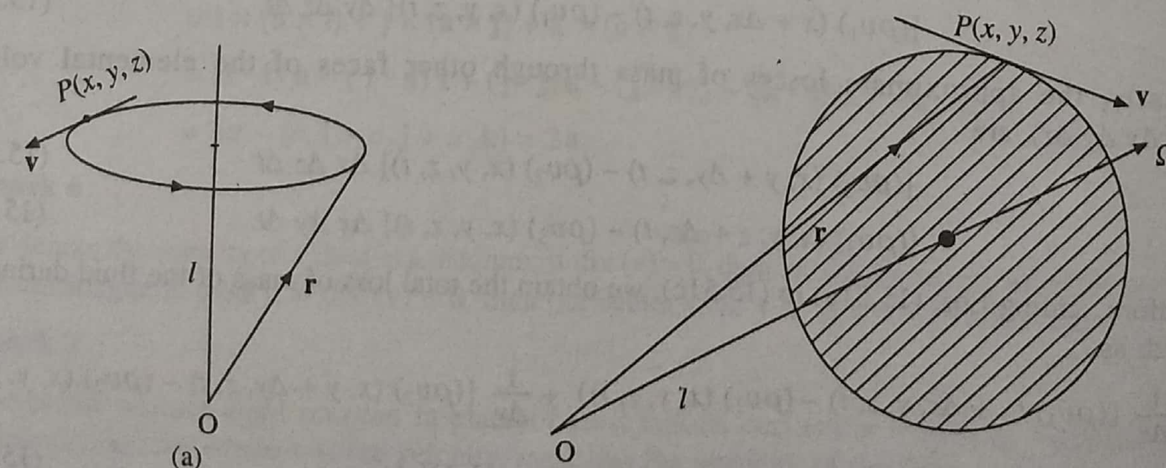


Fig. 15.10. Angular velocity.

Therefore, the angular velocity of the point $P(x, y, z)$ is given by $\Omega = (\text{curl } \mathbf{v}/2)$. Hence, the angular velocity of a uniformly rotating body is equal to one-half of the curl of the linear velocity. Because of this interpretation, the notation *rotation* or *rot* for curl is also used.

Remark 8

A force field **F** is said to be *conservative* if it is derivable from a potential function f, that is $\mathbf{F} = \text{grad } f$. Then, curl $(\mathbf{F})$ = curl (grad f) = **0**. Therefore, if **F** is conservative then curl $(\mathbf{F}) = \mathbf{0}$ and there exists a scalar potential function f such that $\mathbf{F} = \text{grad } f$.

See Appendix 3 for expressions of grad f, div($\mathbf{v}$), curl($\mathbf{v}$) *etc. in polar coordinate systems.*

Example 15.24 Prove the following identities

(i) curl $(f\mathbf{v}) = (\text{grad } f) \times \mathbf{v} + f \text{ curl } \mathbf{v}$

(ii) div (grad f) $= \nabla^2 f$, where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y} + \dfrac{\partial^2}{\partial z^2}$ is the Laplacian operator

(iii) curl (curl $\mathbf{v}$) $= \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$, or grad (div $\mathbf{v}$) $= \nabla \times (\nabla \times \mathbf{v}) + \nabla^2 \mathbf{v}$.

where f is a scalar function.

Solution We have

(i) curl $(f\mathbf{v}) = \nabla \times (f\mathbf{v}) = \nabla \times (f v_1 \mathbf{i} + f v_2 \mathbf{j} + f v_3 \mathbf{k})$

$$= \Sigma \left[\frac{\partial}{\partial y}(f v_3) - \frac{\partial}{\partial z}(f v_2) \right] \mathbf{i} = \Sigma \left[f\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + v_3 \frac{\partial f}{\partial y} - v_2 \frac{\partial f}{\partial z} \right] \mathbf{i}$$

$$= f\left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)\mathbf{k} \right]$$

$$+ \left[\left(v_3 \frac{\partial f}{\partial y} - v_2 \frac{\partial f}{\partial z} \right)\mathbf{i} + \left(v_1 \frac{\partial f}{\partial z} - v_3 \frac{\partial f}{\partial x} \right)\mathbf{j} + \left(v_2 \frac{\partial f}{\partial x} - v_1 \frac{\partial f}{\partial y} \right)\mathbf{k} \right]$$

$$= f(\text{curl } \mathbf{v}) + \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} \right) \times (\mathbf{i} v_1 + \mathbf{j} v_2 + \mathbf{k} v_3)$$

$$= f(\text{curl } \mathbf{v}) + (\text{grad } f) \times \mathbf{v}.$$

(ii)
$$\text{div (grad} f) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$$

(iii)
$$\text{curl (curl } \mathbf{v}) = \nabla \times (\nabla \times \mathbf{v}) = \left(\Sigma \mathbf{i}\frac{\partial}{\partial x} \right) \times \left[\Sigma \mathbf{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \right]$$

$$= \Sigma \mathbf{i} \left[\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) - \frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \right]$$

$$= \Sigma \mathbf{i} \left[\frac{\partial^2 v_2}{\partial y \partial x} + \frac{\partial^2 v_3}{\partial z \partial x} - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) \right]$$

$$= \Sigma \mathbf{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)-\left(\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

$$= \Sigma \mathbf{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)-\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z2}\right)\right]$$

$$= \left(\Sigma \mathbf{i}\frac{\partial}{\partial x}\right)(\nabla \cdot \mathbf{v}) - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\Sigma \mathbf{i}\, v_1)$$

$$= \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}.$$

Hence, grad (div $\mathbf{v}$) = curl (curl $\mathbf{v}$) + $\nabla^2\mathbf{v}$.

Exercise 15.3

In problems **1** to **8**, compute div ($\mathbf{v}$), curl ($\mathbf{v}$) and verify that div (curl $\mathbf{v}$) = 0.

1. $\mathbf{v} = x\,\mathbf{i} + 2y\,\mathbf{j} + z\,\mathbf{k}$.
2. $\mathbf{v} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$.
3. $\mathbf{v} = (x^2+y^2+z^2)^{3/2}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$.
4. $\mathbf{v} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.
5. $\mathbf{v} = xe^{-y}\,\mathbf{i} + 2ze^{-y}\,\mathbf{j} + xy^2\,\mathbf{k}$.
6. $\mathbf{v} = xyz\,\mathbf{i} + 2x^2y\,\mathbf{j} + (xz^2 - y^2z)\,\mathbf{k}$.
7. $\mathbf{v} = (x^2 - y^2)\,\mathbf{i} + 4xy\,\mathbf{j} + (x^2 - xy)\,\mathbf{k}$.
8. $\mathbf{v} = (x^2 + yz)\,\mathbf{i} + (y^2 + zx)\,\mathbf{j} + (z^2 + xy)\,\mathbf{k}$.

In problems **9** to **12**, compute grad f and verify that curl (grad f) = **0**.

9. $f(x, y, z) = x + y - 2z^2$.
10. $f(x, y, z) = x \sin(x + y + z)$.
11. $f(x, y, z) = e^{x+y+z}$.
12. $f(x, y, z) = 16x\,y^3z^2$.

Show that the vectors in problems **13**, **14** are solenoidal.

13. $(2x + 3y)\,\mathbf{i} + (x - y)\,\mathbf{j} - (x + y + z)\,\mathbf{k}$.
14. $e^{x+y-2z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$.
15. If $\mathbf{v} = -(x + y + 2)\,\mathbf{i} - 2\mathbf{j} + (x + y)\,\mathbf{k}$, show that $\mathbf{v} \cdot$ curl $\mathbf{v} = 0$.
16. If $f = x^2 + y^2 + z^2$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, show that div $(f\mathbf{r}) = 5f$.

In problems **17** to **25**, **a** is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then prove the given identities.

17. grad $(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$.
18. div $(\mathbf{a} \times \mathbf{r}) = 0$.
19. curl $(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$.
20. div $[(\mathbf{a} \cdot \mathbf{r})\,\mathbf{r}] = 4\,(\mathbf{a} \cdot \mathbf{r})$.
21. $\nabla \cdot [(\mathbf{r} \cdot \mathbf{r})\,\mathbf{a}] = 2(\mathbf{r} \cdot \mathbf{a})$.
22. $\mathbf{a} \times$ (curl $\mathbf{r}$) = **0**.
23. $\nabla \times \left[\frac{1}{r^3}(\mathbf{a} \times \mathbf{r})\right] = \frac{3}{r^5}(\mathbf{a} \cdot \mathbf{r})\mathbf{r} - \frac{\mathbf{a}}{r^3},\ r = |\mathbf{r}|$.
24. $\nabla \times (\mathbf{a} \times \mathbf{v}) = \mathbf{a}\,(\nabla \cdot \mathbf{v}) - (\mathbf{a} \cdot \nabla)\,\mathbf{v}$, **v** is any vector.
25. $\nabla \cdot (\mathbf{a} \times \mathbf{v}) = -\mathbf{a} \cdot (\nabla \times \mathbf{v})$, **v** is any vector.

In problems **26** to **29**, show that the vector field **v** is irrotational and find a scalar function $f(x, y, z)$ such that $\mathbf{v}$ = grad f.

26. $(y^2 - x^2 + y)\,\mathbf{i} + x\,(2y + 1)\,\mathbf{j}$.
27. $3x^2y^2z^4\,\mathbf{i} + 2x^3yz^4\,\mathbf{j} + 4x^3y^2z^3\,\mathbf{k}$.
28. $e^{xy}(y\,\mathbf{i} + x\,\mathbf{j}) + 2e^z\,\mathbf{k}$.
29. $\cos(x^2 + y^2 + z^2)\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.
30. Find the constants a, b and c such that $\mathbf{v} = (3x + ay + z)\,\mathbf{i} + (2x - y + bz)\,\mathbf{j} + (x + cy + z)\,\mathbf{k}$ is irrotational.

Assuming continuity of partial derivatives verify the identities given in problems **31** to **36**.

31. $\nabla \cdot (f\,\nabla g) = f\,\nabla^2 g + \nabla f \cdot \nabla g$.
32. $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$.
33. $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$.
34. $\nabla \cdot [(f\,\nabla g) \times (g\,\nabla f)] = 0$.
35. $\nabla \cdot (\nabla f \times \nabla g) = 0$.
36. $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\,\mathbf{u} - (\mathbf{u} \cdot \nabla)\,\mathbf{v} + (\nabla \cdot \mathbf{v})\,\mathbf{u} - (\nabla \cdot \mathbf{u})\,\mathbf{v}$.