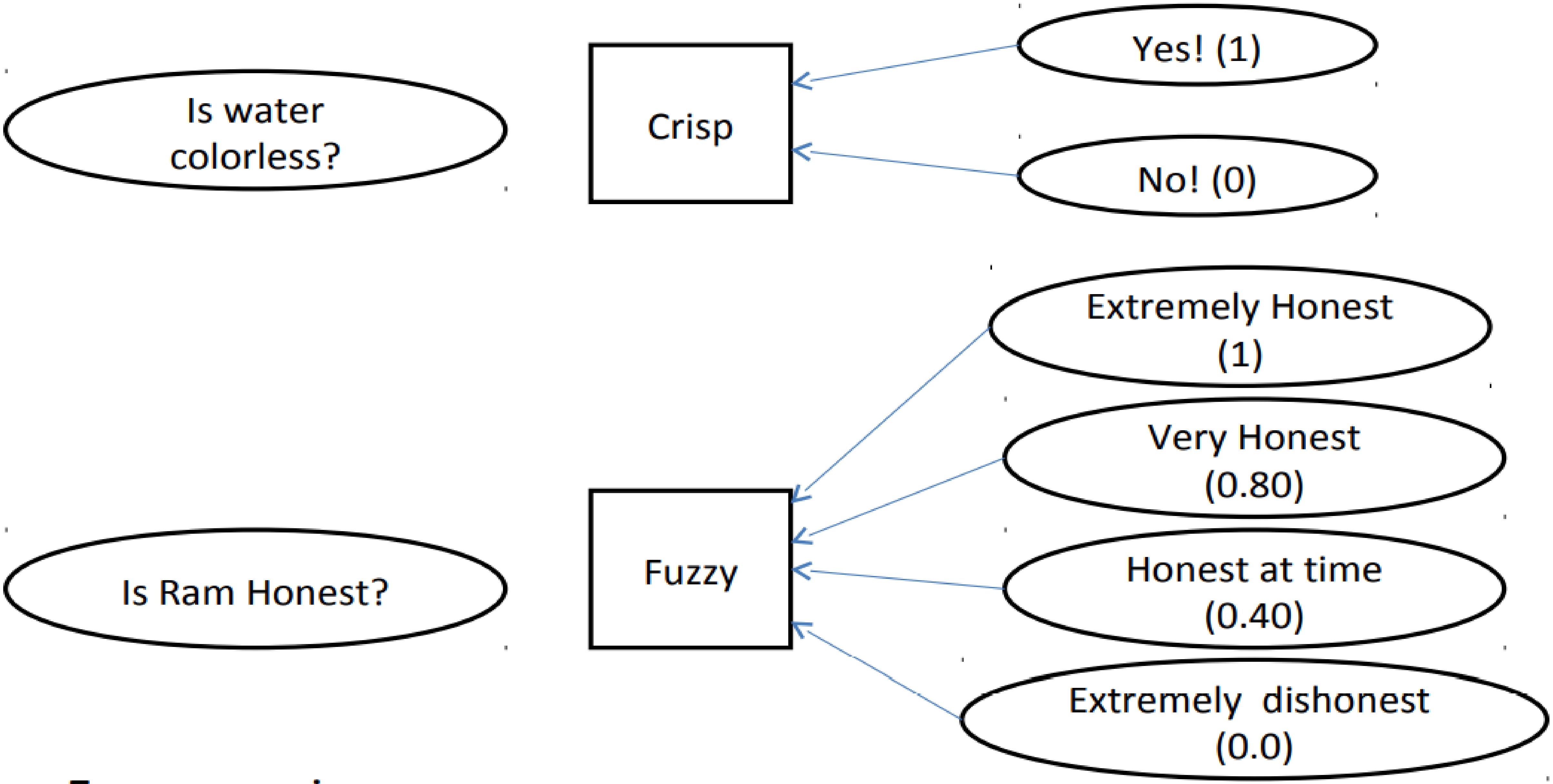
Fuzzy vs Crisp

Example



Fuzzy vs crips

FUZZY SET OPERATIONS

Let A and B be fuzzy sets in the universe of discourse U. For a given element x on the universe, the following function theoretic operations of union, intersection and complement are defined for fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ on U.

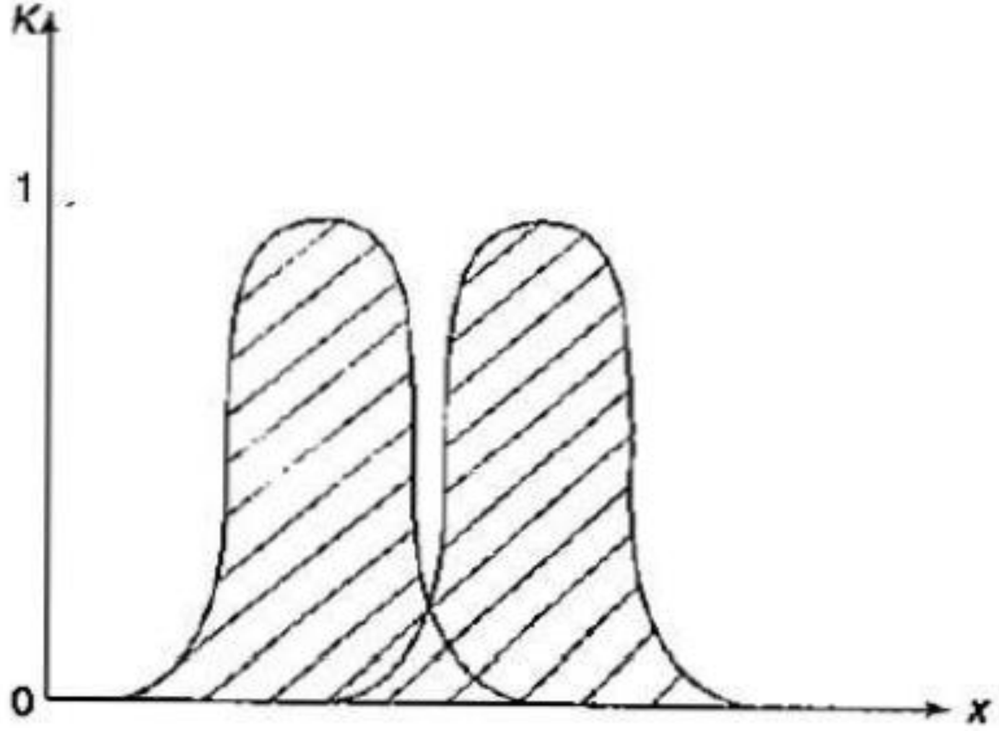
1) Union

The union of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, denoted by $\underset{\sim}{A} \cup \underset{\sim}{B}$, is defined as

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)] = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x) \quad \text{for all } x \in U$$

Where V indicates max operation.

The Venn diagram for union operation of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is shown in Figure below:

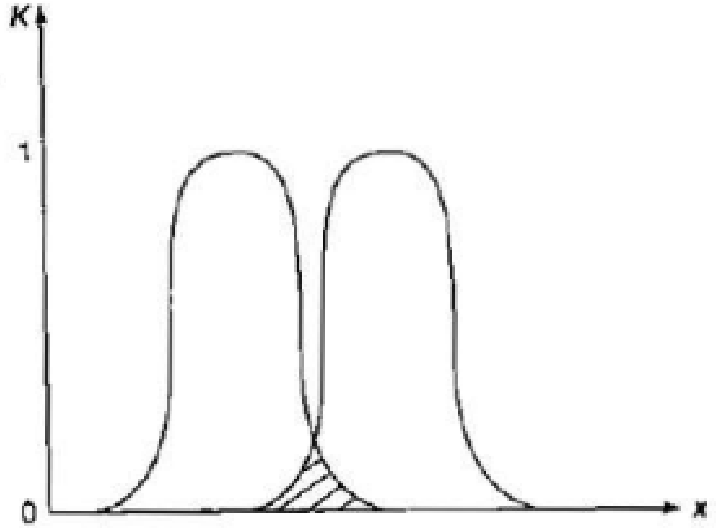


2) Intersection

The intersection of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, denoted by $\underset{\sim}{A} \cap \underset{\sim}{B}$, is defined by

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)] = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x) \quad \text{for all } x \in U$$

where ^ indicates min operator. The Venn diagram for intersection operation of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is shown in Figure below.

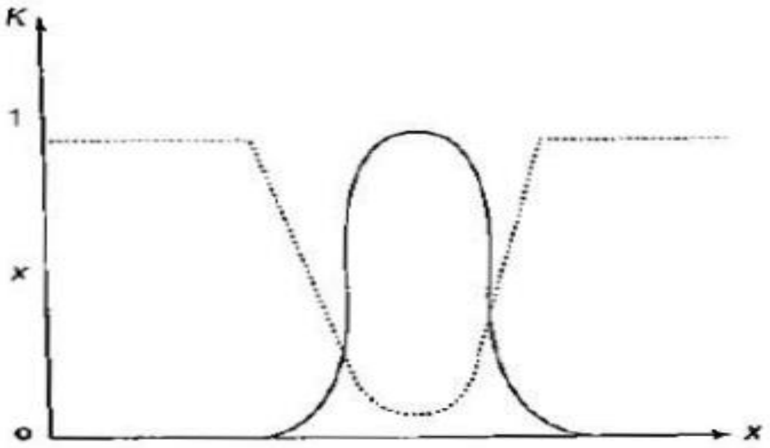


3) Complement

When $\mu_A(x) \in [0,1]$, the complement of $\underset{\sim}{A}$, denoted as $\overline{\underset{\sim}{A}}$ is defined by

$$\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x) \quad \text{for all } x \in U$$

The Venn diagram for complement operation of fuzzy set $\underset{\sim}{A}$ is shown in Figure below.



4) Algebraic sum

The algebraic sum $(\underset{\sim}{A}+\underset{\sim}{B})$ of fuzzy sets, fuzzy set $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined as

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$$

4) Algebraic sum

The algebraic sum $(\underset{\sim}{A}+\underset{\sim}{B})$ of fuzzy sets, fuzzy set $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined as

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x)=\mu_{\underset{\sim}{A}}(x)+\mu_{\underset{\sim}{B}}(x)-\mu_{\underset{\sim}{A}}(x)\cdot\mu_{\underset{\sim}{B}}(x)$$

5) Algebraic product

The algebraic product $(\underset{\sim}{A}\,.\,\underset{\sim}{B})$ of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined as

$$\mu_{\underset{\sim}{A}\cdot\underset{\sim}{B}}(x)=\mu_{\underset{\sim}{A}}(x)\cdot\mu_{\underset{\sim}{B}}(x)$$

FUZZY RELATIONS

Fuzzy relations relate elements of one universe (say *X)* to those of another universe (say Y) through the Cartesian product of the two universes. These can also be referred to as fuzzy sets defined on universal sets, which are Cartesian products.

A fuzzy relation is based on the concept that everything is related to some extent or unrelated.

A fuzzy relation is a fuzzy set defined on the Cartesian product of classical sets $\{X_1, X_2, \ldots : X_n\}$ where tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership $\mu_R(x_1, x_2, \ldots, x_n)$ within the relation. That is,

$$R(X_1, X_2, \ldots, X_n) = \int_{X_1 \times X_2 \times \cdots \times X_n} \mu_R(x_1, x_2, \ldots, x_n) | (x_1, x_2, \ldots, x_n), \quad x_i \in X_i$$

A fuzzy relation between two sets X and Y is called binary fuzzy relation and is denoted by *R(X,* Y). A binary relation *R(X,* Y) is referred to as bipartite graph when $X \neq Y$. The binary relation on a single set *X* is called directed graph or digraph. This relation occurs when X=Y and is denoted as *R(X,X)* or $R(X^2)$.

Fuzzy Matrix

Let

$$\underset{\sim}{X} = \{x_1, x_2, \ldots, x_n\} \quad \text{and} \quad \underset{\sim}{Y} = \{y_1, y_2, \ldots, y_m\}$$

Fuzzy relation *R(X,* Y) can be expressed by an *n* x *m* matrix as follows:

$$\underset{\sim}{R}(\underset{\sim}{X}, \underset{\sim}{Y}) = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & . & . & \mu_R(x_1, y_m) \\ \mu_R(x_2, y_1) & \mu_R(x_2, y_2) & . & . & \mu_R(x_2, y_m) \\ . & . & . & . & . \\ . & . & . & . & . \\ \mu_R(x_n, y_1) & \mu_R(x_n, y_2) & . & . & \mu_R(x_n, y_m) \end{bmatrix}$$

The matrix representing a fuzzy relation is called fuzzy matrix. A fuzzy relation *R* is a mapping from Cartesian space *X x Y* to the interval [0, 1].

Fuzzy Graph

A fuzzy graph is a graphical representation of a binary fuzzy relation. Each element in X and Y corresponds to a node in the fuzzy graph. The connection links are established between the nodes by the elements of $\underset{\sim}{X} \times \underset{\sim}{Y}$ with nonzero membership grades in *R(X, Y)*. The links may also be present in the form of arcs.

When $X \neq Y$, the link connecting the two nodes is an undirected binary graph called bipartite graph.

When X= *Y*, a node is connected to itself and directed links are used; in such a case, the fuzzy graph is called directed graph.

The domain of a binary fuzzy relation *R(X* ,Y) is the fuzzy set, *dom R(X,* Y), having the membership function as

$$\mu_{domain\, R}(x) = \max_{y \in Y} \mu_R(x, y) \quad \forall x \in X$$

Consider a universe *X= {x1,x2,x3, x4}* and the binary fuzzy relation on *X as*

$$\underset{\sim}{R}(X, X) = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 1 \end{bmatrix} \end{array}$$

OPERATIONS ON FUZZY RELATIONS

The basic operations on fuzzy sets also apply on fuzzy relations.

Let $\underset{\sim}{R}$ and $\underset{\sim}{S}$ be fuzzy relations on the Cartesian space X x Y. The operations that can be performed on these fuzzy relations are described below:

1. Union

$$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]$$

2. Intersection

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]$$

3. Complement

$$\mu_{\bar{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$$

4. Containment

$$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$$

5.Inverse

The inverse of a fuzzy relation R on X x Y is denoted by R^{-1}. It is a relation on Y x X defined by, $R^{-1}(y, x) = R(x, y)$ for all pairs $(y, x) \in Y \times X$.

6. Projection

For a fuzzy relation R(X,Y), let (R ↓Y) denote the projection of R onto Y. Then (R ↓Y) is a fuzzy relation in Y whose membership function is defined by:

3. Complement

$$\mu_{\bar{R}}(x, y) = 1 - \mu_{R}(x, y)$$

4. Containment

$$R \subset S \Rightarrow \mu_{R}(x, y) \leq \mu_{S}(x, y)$$

5.Inverse

The inverse of a fuzzy relation R on X x Y is denoted by R^{-1}. It is a relation on Y x X defined by, $R^{-1}(y, x) = R(x, y)$ for all pairs $(y, x) \in Y \times X$.

6. Projection

For a fuzzy relation R(X,Y), let (R ↓Y) denote the projection of R onto Y. Then (R ↓Y) is a fuzzy relation in Y whose membership function is defined by:

$$\mu_{[R \downarrow Y]}(x, y) = \max_{x} \mu_{R}(x, y)$$

Fuzzy Set Operation Examples

Given X to be the universe of discourse and Ã and to Ḃ be fuzzy sets with μA (x) and μB (x) are their respective membership function, the fuzzy set operations are as follows:

- Union: μA U B(x) = max (μA (x), μB (x))
- Intersection: μA ∩ B(x) = min (μA (x), μB (x))
- Complement: μA (x) =1- μA (x)

- Example:
- $A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$
- $B = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$
  - Union: $A \cup B = \{(x_1, 0.8), (x_2, 0.7), (x_3, 1)\}$

  Because $\mu_{A \cup B}(x_1) = \max(\mu_A(x_1), \mu_B(x_1))$

  $= \max(0.5, 0.8)$

  $= 0.8$

  $\mu_{A \cup B}(x_2) = 0.7$ and $\mu_{A \cup B}(x_3) = 1$

- A = {(x1 ,0.8),(x2 ,0.5),(x3 ,0.5)}
- B = {(x1 ,0.8),(x2 ,0.2),(x3 ,1)}

- Example:
- A = {(x1 ,0.5),(x2 ,0.7),(x3 ,0)}
- B = {(x1 ,0.8),(x2 ,0.2),(x3 ,1)}
  - Intersection: A ∩ B = {(x1 ,0.5),(x2 ,0.2),(x3 ,0)}

  Because μA ∩ B(x1) = min (μA (x1), μB (x1))

  = max(0.5,0.8)

  = 0.5

  μA ∩ B(x2) = 0.2 and μA ∩ B(x3) = 0

- Example:
- $A = \{(x1, 0.5), (x2, 0.7), (x3, 0)\}$
- Complement: $A^c = \{(x1, 0.5), (x2, 0.3), (x3, 1)\}$

Because $\mu A (x1) = 1 - \mu A (x1)$

$= 1 – 0.5$

$= 0.5$

$\mu A (x2) = 0.3$ and $\mu A (x3) = 1$

Fuzzy Logic

In crisp logic, the truth values acquired by propositions or predicates are 2-valued, namely True, False which may be treated numerically equivalent to (0, 1). However, in fuzzy logic, truth values are multivalued such as absolutely true, partly true, absolutely false, very true, and so on and are numerically equivalent to (0-1).

Fuzzy Propositions

- A fuzzy proposition is a statement which acquires a fuzzy truth value. Thus, given P’ to be a fuzzy proposition, T(P’) represents the truth value (0-1) attached to P. In its simplest form m, fuzzy propositions are associated with fuzzy sets. The fuzzy membership value associated with the fuzzy set A’ for P’ is treated as the fuzzy truth value T(P’).
- $T(P') = \mu_{A'}(x)$ where $0 \le \mu_{A'}(x) \le 1$

P’: Ram is honest.

T(P’) = 0.8, if P’ is partly true.

T(P’) = 1, if P’ is absolutely true.

Fuzzy Connectives

- Fuzzy logic similar to crisp logic supports the following connectives:

Negation : –

Disjunction : or

Conjunction : and

Implication : =>

Table 7.3 Fuzzy connectives

Symbol	Connective	Usage	Definition
$-$	Negation	$\bar{\bar{P}}$	$1 - T(\bar{P})$
$\vee$	Disjunction	$\bar{P} \vee \bar{Q}$	$\max(T(\bar{P}), T(\bar{Q}))$
$\wedge$	Conjunction	$\bar{P} \wedge \bar{Q}$	$\min(T(\bar{P}), T(\bar{Q}))$
$\Rightarrow$	Implication	$\bar{P} \Rightarrow \bar{Q}$	$\bar{\bar{P}} \vee \bar{Q} = \max(1 - T(\bar{P}), T(\bar{Q}))$

$\tilde{P}$ and $\tilde{Q}$ related by the '$\Rightarrow$' operator are known as antecedent and consequent respectively. Also, just as in crisp logic, here too, '$\Rightarrow$' represents the IF-THEN statement as

IF x is $\tilde{A}$ THEN y is $\tilde{B}$, and is equivalent to

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times Y) \tag{7.18}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), 1 - \mu_{\tilde{A}}(x)) \tag{7.19}$$

Also, for the compound implication IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$ the relation R is equivalent to

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{C}) \tag{7.20}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(y))) \tag{7.21}$$

Example

$\tilde{P}$: Mary is efficient, $T(\tilde{P}) = 0.8$

$\tilde{Q}$: Ram is efficient, $T(\tilde{Q}) = 0.65$

(i) $\bar{\tilde{P}}$: Mary is not efficient.

$$T(\bar{\tilde{P}}) = 1 - T(\tilde{P}) = 1 - 0.8 = 0.2$$

(ii) $\tilde{P} \wedge \tilde{Q}$: Mary is efficient and so is Ram.

$$\begin{aligned} T(\tilde{P} \wedge \tilde{Q}) &= \min(T(\tilde{P}), T(\tilde{Q})) \\ &= \min(0.8, 0.65) \\ &= 0.65 \end{aligned}$$

(iii) $T(\tilde{P} \vee \tilde{Q})$: Either Mary or Ram is efficient.

$$T(\tilde{P} \vee \tilde{Q}) = \max(T(\tilde{P}), T(\tilde{Q}))$$
$$= \max(0.8, 0.65)$$
$$= 0.8$$

(iv) $\tilde{P} \Rightarrow \tilde{Q}$: If Mary is efficient then so is Ram.

$$T(\tilde{P} \Rightarrow \tilde{Q}) = \max(1 - T(\tilde{P}), T(\tilde{Q}))$$
$$= \max(0.2, 0.65)$$
$$= 0.65$$

Fuzzy Quantifiers

- Just as in crisp logic where predicates are quantified by quantifiers, fuzzy logic propositions are also quantified by fuzzy quantifiers. There are two classes of fuzzy quantifiers such as:
  - Absolute quantifiers
  - Relative quantifiers

  While absolute quantifiers are defined over Ʀ, relative quantifiers are defined over (0-1)

Absolute quantifier	Relative quantifier
round about 250	almost
much greater than 6	about
some where around 20	most

Fuzzy Inference

- Fuzzy inference also referred to as approximate reasoning refers to computational procedures used for evaluating linguistic descriptions. The two important inferring procedures are
  - Generalized Modus Ponens (GMP)
  - Generalized Modus Tollens (GMT)

GMP is formally stated as

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \quad x \text{ is } \tilde{A}' \\ \hline & \quad y \text{ is } \tilde{B}' \end{array} \tag{7.22}$$

Here, $\tilde{A}$, $\tilde{B}$, $\tilde{A}'$ and $\tilde{B}'$ are fuzzy terms. Every fuzzy linguistic statement above the line is analytically known and what is below is analytically unknown.

To compute the membership function of $\tilde{B}'$, the max-min composition of fuzzy set A' with $\tilde{R}\,(x, y)$ which is the known implication relation (IF-THEN relation) is used. That is,

$$\tilde{B}' = \tilde{A}' \circ \tilde{R}(x,y) \tag{7.23}$$

In terms of membership function,

$$\mu_{\tilde{B}'}(y) = \max\left(\min\left(\mu_{\tilde{A}'}(x), \mu_{\tilde{R}}(x,y)\right)\right) \tag{7.24}$$

where $\mu_{\tilde{A}'}(x)$ is the membership function of $\tilde{A}'$, $\mu_{\tilde{R}}(x,y)$ is the membership function of the implication relation and $\mu_{\tilde{B}'}(y)$ is the membership function of $\tilde{B}'$

On the other hand, GMT has the form

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \quad y \text{ is } \tilde{B}' \\ \hline & \quad x \text{ is } \tilde{A}' \end{array}$$

The membership of $\tilde{A}'$ is computed on similar lines as

$$\tilde{A}' = \tilde{B}' \circ \tilde{R}(x,y)$$

In terms of membership function,

$$\mu_{\tilde{A}'}(x) = \max\left(\min\left(\mu_{\tilde{B}'}(y), \mu_{\tilde{R}}(x,y)\right)\right) \tag{7.25}$$

Apply the fuzzy Modus Ponens rule to deduce Rotation is quite slow given

(i) If the temperature is high then the rotation is slow.

(ii) The temperature is very high.

Let $\tilde{H}$ (High), $\tilde{VH}$ (Very High), $\tilde{S}$ (Slow) and $\tilde{QS}$ (Quite Slow) indicate the associate fuzzy sets as follows:

For $X = \{30, 40, 50, 60, 70, 80, 90, 100\}$, the set of temperatures and $Y = \{10, 20, 30, 40, 50, 60\}$ the set of rotations per minute,

$$\tilde{H} = \{(70, 1)\ (80, 1)\ (90, 0.3)\}$$

$$\tilde{VH} = \{(90, 0.9)\ (100, 1)\}$$

$$\tilde{QS} = \{(10, 1)\ (20, 0.8)\}$$

$$\tilde{S} = \{(30, 0.8)\ (40, 1)\ (50, 0.6)\}$$

$$\tilde{R}\,(x, y) = \max(\tilde{H} \times \tilde{S}, \bar{\tilde{H}} \times Y)$$

$$\tilde{H} \times \tilde{S} = \begin{array}{cc} & \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{array}{c} 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{H}} \times Y = \begin{array}{cc} & \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{array}{c} 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\bar{R}(x,y) = \begin{array}{c|cccccc} & 10 & 20 & 30 & 40 & 50 & 60 \\ \hline 30 & 1 & 1 & 1 & 1 & 1 & 1 \\ 40 & 1 & 1 & 1 & 1 & 1 & 1 \\ 50 & 1 & 1 & 1 & 1 & 1 & 1 \\ 60 & 1 & 1 & 1 & 1 & 1 & 1 \\ 70 & 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 80 & 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 90 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 100 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

To deduce Rotation is quite slow we make use of the composition rule

$$\bar{Q}S = V\bar{H} \circ \bar{R}(x,y)$$

$$= [0\ 0\ 0\ 0\ 0\ 0\ 0.9\ 1] \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= [1\ 1\ 1\ 1\ 1\ 1]$$

Fuzzy Rule Based System

- Fuzzy linguistic descriptions are formal representations of systems made through fuzzy IF-THEN rules. They encode knowledge about a system in statements of the form-

IF (a set of conditions) are satisfied THEN (a set of consequents) can be inferred. Fuzzy IF-THEN rules are coded in the form.

$$\text{IF } (x_1 \text{ is } \tilde{A}_1 . x_2 \text{ is } \tilde{A}_2, \ldots, x_n \text{ is } \tilde{A}_n) \text{ THEN } (y_1 \text{ is } \tilde{B}_1, y_2 \text{ is } \tilde{B}_2, \ldots, y_n \text{ is } \tilde{B}_n).$$

where linguistic variables x_i, y_j take the values of fuzzy sets A_i and B_j respectively.

Example

If there is heavy rain and strong winds
then there must be severe flood warning.

Here, heavy, strong, and severe are fuzzy sets qualifying the variables rain, wind, and flood warning respectively.

A collection of rules referring to a particular system is known as a *fuzzy rule base*. If the conclusion C to be drawn from a rule base R is the conjunction of all the individual consequents C_i of each rule, then

$$C = C_1 \cap C_2 \cap \ldots \cap C_n$$

where

$$\mu_C(y) = \min(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y$$

where Y is the universe of discourse.

On the other hand, if the conclusion C to be drawn from a rule base R is the disjunction of the individual consequents of each rule, then

$$C = C_1 \cup C_2 \cup C_3 \ldots \cup C_n$$

where

$$\mu_C(y) = \max(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y$$

Defuzzification

- In many situations, for a system whose output is fuzzy, it is easier to take a crisp decision if the output is represented as a single scalar quantity. This conversion of a fuzzy set to single crisp value is called defuzzification and is the reverse process of fuzzification

Defuzzification Methods

- Centroid Method
  - Also known as the center of gravity or the center of area method, it obtains the center of area (x*) occupied by the fuzzy set. It is given by the expression

$$x^* = \frac{\int \mu(x)\, x\, dx}{\int \mu(x)\, dx}$$

for a continuous membership function, and

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)}$$

for a discrete membership function.

Here, n represents the number of elements in the sample, x_i's are the elements, and $\mu(x_i)$ is its membership function.

- Centre of Sums
  - In the centroid method, the overlapping area is counted once whereas in center of sums, the overlapping arca is counted twice. COS builds the resultant membership function by taking the algebraic sum of outputs from each of the contributing fuzzy sets A'_1, A'_2., etc. The Defuzzified value x * is given by

$$x^* = \frac{\sum_{i=1}^{N} x_i \cdot \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)}$$

Here n is the number of fuzzy sets and N the number of fuzzy variables. COS is actually the most commonly used defuzzification method. It can be implemented easily and leads to rather fast inference cycles.

This method considers values with maximum membership. There are different maxima methods with different conflict resolution strategies for multiple maxima.

- First of Maxima Method (FOM)
- Last of Maxima Method (LOM)
- Mean of Maxima Method (MOM)

- Mean of maxima
  - One simple way of defuzzifying the output is to take the crisp value with the highest degree of membership. In cases with more than one element having the maximum value, the mean value of the maxima is taken. The equation of the defuzzified value x* is given by

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|}$$

where $M = \{x_i \mid \mu(x_i)$ is equal to the *height* of fuzzy set$\}$
$|M|$ is the cardinality of the set M. In the continuous case, M could be defined as

$$M = \{x \in [-c, c] \mid \mu(x) \text{ is equal to the height of the fuzzy set}\}$$

In such a case, the *mean of maxima* is the arithmetic average of mean values of all intervals contained in M including zero length intervals.

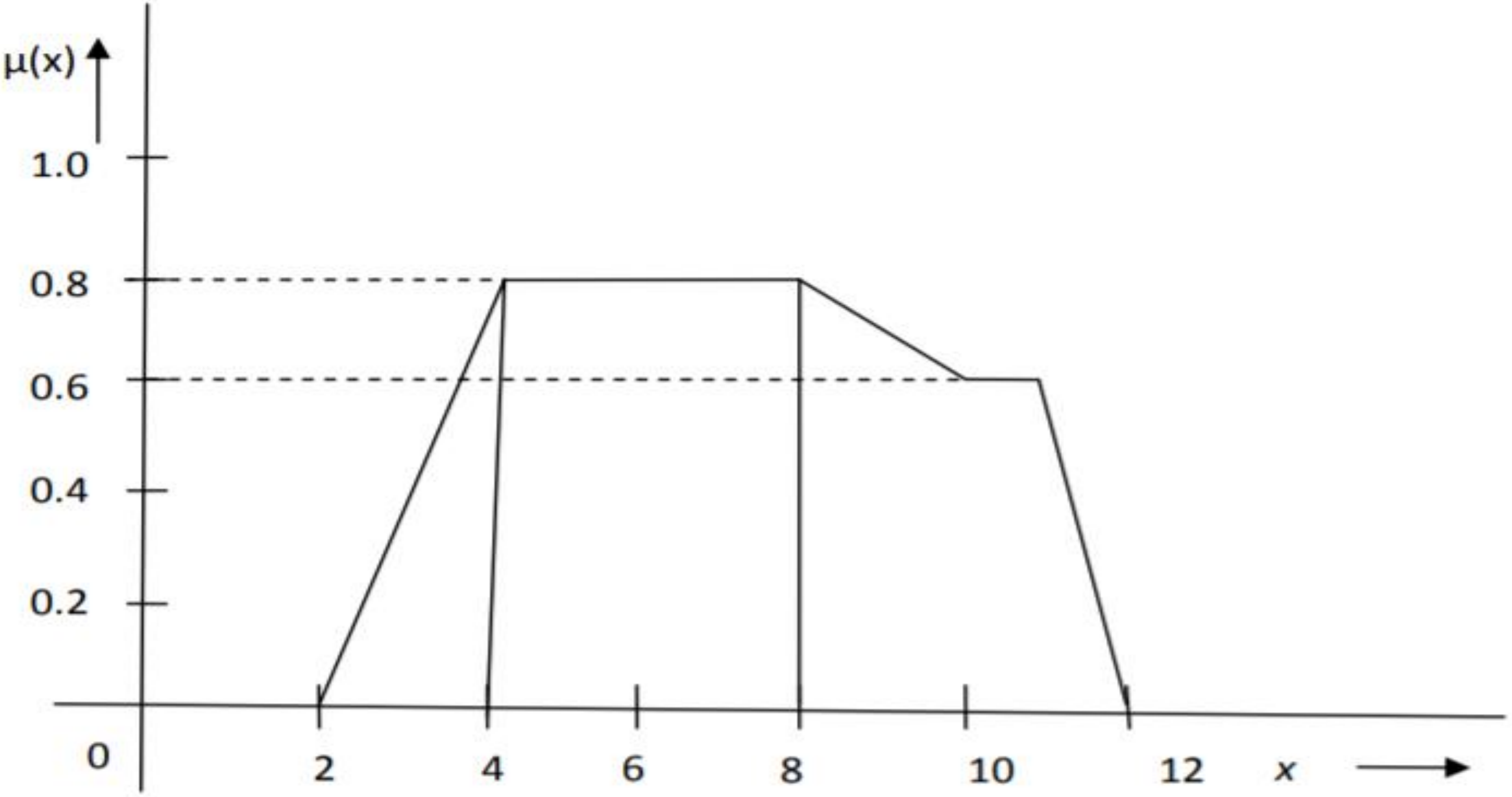
The *height* of a fuzzy set A, i.e. $h(A)$ is the largest membership grade obtained by any element in that set.

- ## First of Maxima Method (FOM)

This method determines the smallest value of the domain with maximum membership value.

Example:

The defuzzified value x^* of the given fuzzy set will be x^*=4.



Last of Maxima Method (LOM)

Determine the largest value of the domain with maximum membership value.

In the example given for FOM, the defuzzified value for LOM method will be $x^*= 8$

Mean of Maxima Method (MOM)

In this method, the defuzzified value is taken as the element with the highest membership values. When there are more than one element having maximum membership values, the mean value of the maxima is taken.
Let A be a fuzzy set with membership function $\mu_A(x)$ defined over $x \in X$, where X is a universe of discourse. The defuzzified value is let say x^* of a fuzzy set and is defined as,

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|},$$

Here, $M = \{x_i |\ \mu_A(x_i)$ is equal to the height of the fuzzy set A$\}$ and $|M|$ is the cardinality of the set M.

Example

In the example as shown in Fig. , x = 4, 6, 8 have maximum membership values and hence |M| = 3

According to MOM method, $x^* = \frac{\sum_{x_i \in M} x_i}{|M|}$

Now the defuzzified value x^* will be $x^* = \frac{4+6+8}{3} = \frac{18}{3} = 6.$