CSE408
All pairs shortest path

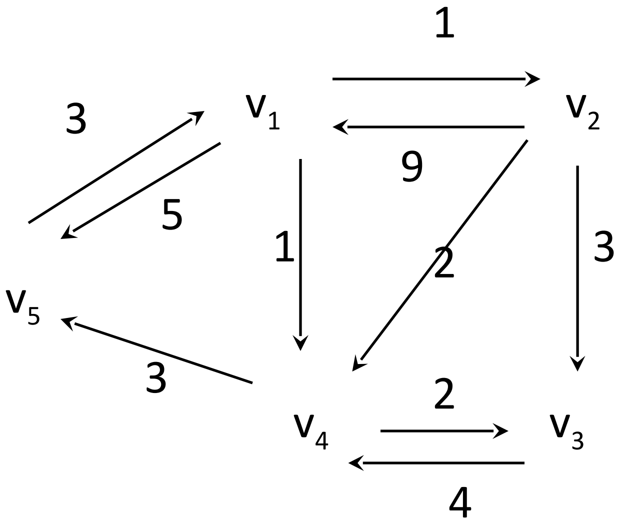
Lecture #29

All pairs shortest path

- *The problem:* find the shortest path between every pair of vertices of a graph
- *The graph*: may contain negative edges but no negative cycles
- *A representation*: a weight matrix where
  W(i,j)=0 if i=j.
  W(i,j)=∞ if there is no edge between i and j.
  W(i,j)="weight of edge"
- Note: we have shown principle of optimality applies to shortest path problems

The weight matrix and the graph

	1	2	3	4	5
1	0	1	∞	1	5
2	9	0	3	2	∞
3	∞	∞	0	4	∞
4	∞	∞	2	0	3
5	3	∞	∞	∞	0



The subproblems

- How can we define the shortest distance $d_{i,j}$ in terms of “smaller” problems?
- One way is to restrict the paths to only include vertices from a restricted subset.
- Initially, the subset is empty.
- Then, it is incrementally increased until it includes all the vertices.

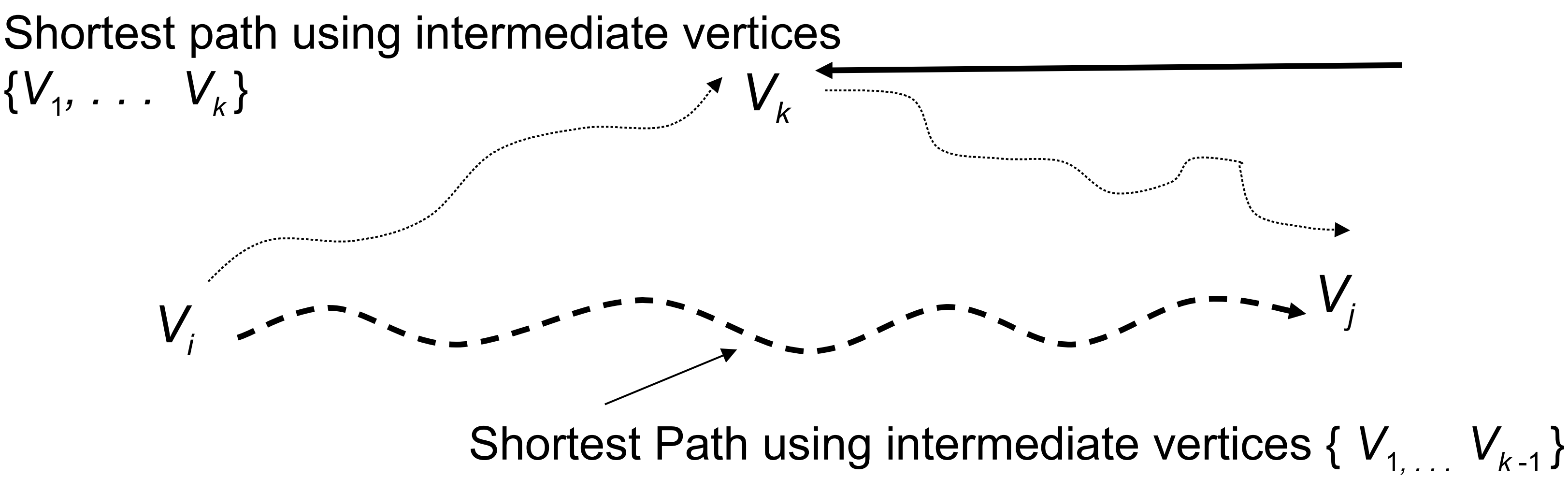
The subproblems

- Let $D^{(k)}[i,j]$=weight of a shortest path from v_i to v_j using only vertices from $\{v_1,v_2,\ldots,v_k\}$ as intermediate vertices in the path
  - $D^{(0)}=W$
  - $D^{(n)}=D$ which is the goal matrix
- How do we compute $D^{(k)}$ from $D^{(k\text{-}1)}$?

The Recursive Definition:

Case 1: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1,v_2,\ldots,v_k\}$ as intermediate vertices does not use v_k. Then $D^{(k)}[i,j]= D^{(k-1)}[i,j]$.

Case 2: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1,v_2,\ldots,v_k\}$ as intermediate vertices does use v_k. Then $D^{(k)}[i,j]= D^{(k-1)}[i,k]+ D^{(k-1)}[k,j]$.



The recursive definition

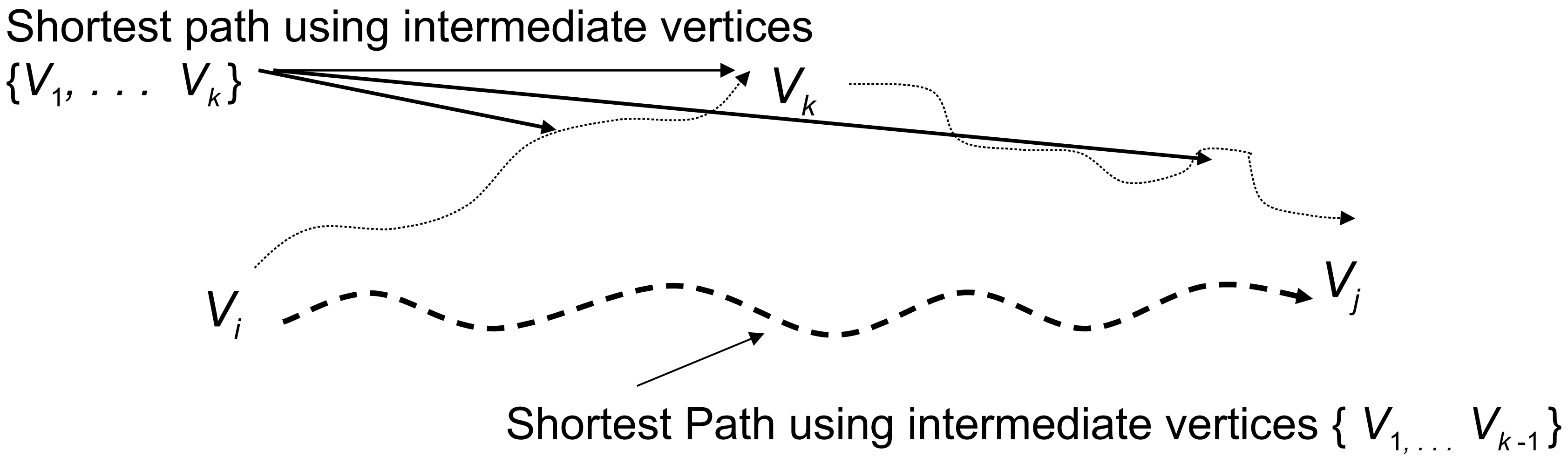
- **Since**

  $D^{(k)}[i,j] = D^{(k-1)}[i,j]$ **or**

  $D^{(k)}[i,j] = D^{(k-1)}[i,k] + D^{(k-1)}[k,j]$.

  We conclude:

  $D^{(k)}[i,j] = \min\{ D^{(k-1)}[i,j], D^{(k-1)}[i,k] + D^{(k-1)}[k,j] \}$.



The pointer array P

- **Used to enable finding a shortest path**
- **Initially the array contains 0**

- **Each time that a shorter path from *i* to *j* is found the *k* that provided the minimum is saved (highest index node on the path from *i* to *j*)**

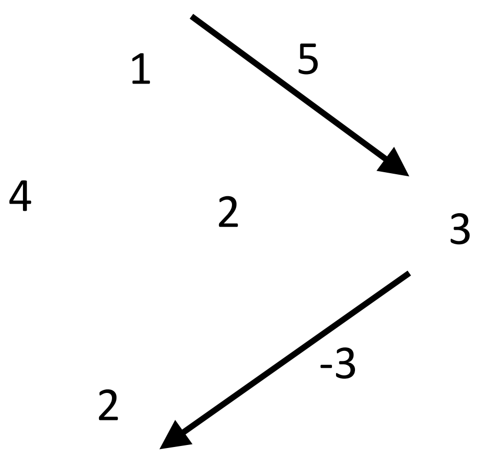
- **To print the intermediate nodes on the shortest path a recursive procedure that print the shortest paths from *i* and *k*, and from *k* to *j* can be used**

Floyd's Algorithm Using n+1 *D* matrices

Floyd//Computes shortest distance between all pairs of //nodes, and saves P to enable finding shortest paths

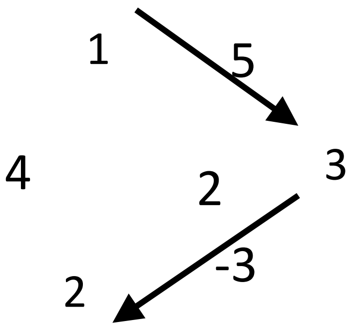
1. $D^0 \leftarrow W$ // initialize *D* array to *W* [ ]
2. $P \leftarrow 0$ // initialize P array to [0]
3. **for** $k \leftarrow 1$ **to** n
4. **do for** $i \leftarrow 1$ **to** n
5. **do for** $j \leftarrow 1$ **to** n
6. **if** $(D^{k-1}[i, j] > D^{k-1}[i, k] + D^{k-1}[k, j])$
7. **then** $D^k[i, j] \leftarrow D^{k-1}[i, k] + D^{k-1}[k, j]$
8. $P[i, j] \leftarrow k;$
9. **else** $D^k[i, j] \leftarrow D^{k-1}[i, j]$

Example



$$W = D^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \infty \\ 3 & \infty & -3 & 0 \end{array}$$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$



D^0 =

	1	2	3
1	0	4	5
2	2	0	∞
3	∞	-3	0

k = 1

Vertex 1 can be intermediate node

D^1 =

	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

$D^1[2,3] = \min(D^0[2,3], D^0[2,1]+D^0[1,3])$

$= \min (\propto, 7)$

$= 7$

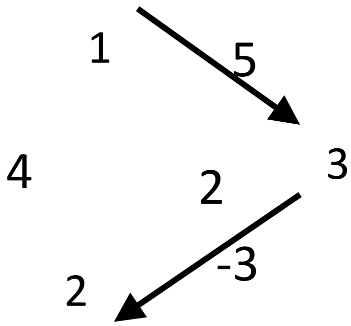
P =

	1	2	3
1	0	0	0
2	0	0	1
3	0	0	0

$D^1[3,2] = \min(D^0[3,2], D^0[3,1]+D^0[1,2])$

$= \min (-3,\propto)$

$= -3$



D¹ =

	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

k = 2
Vertices 1, 2 can be intermediate

D² =

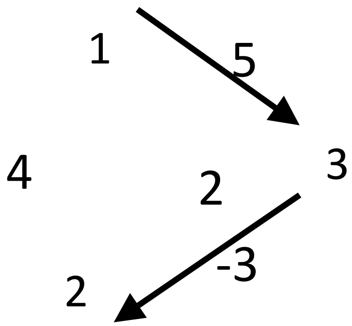
	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

$D^2[1,3] = \min(D^1[1,3], D^1[1,2]+D^1[2,3])$
$= \min (5, 4+7)$
$= 5$

P =

	1	2	3
1	0	0	0
2	0	0	1
3	2	0	0

$D^2[3,1] = \min(D^1[3,1], D^1[3,2]+D^1[2,1])$
$= \min (\propto, -3+2)$
$= -1$



$$D^2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & 7 \\ 3 & -1 & -3 & 0 \end{array}$$

k = 3
Vertices 1, 2, 3 can be intermediate

$$D^3 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 2 & 5 \\ 2 & 2 & 0 & 7 \\ 3 & -1 & -3 & 0 \end{array}$$

$$\begin{aligned} D^3[1,2] &= \min(D^2[1,2],\ D^2[1,3]+D^2[3,2]\ ) \\ &= \min(4,\ 5+(-3)) \\ &= 2 \end{aligned}$$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 \end{array}$$

$$\begin{aligned} D^3[2,1] &= \min(D^2[2,1],\ D^2[2,3]+D^2[3,1]\ ) \\ &= \min(2,\ 7+(-1)) \\ &= 2 \end{aligned}$$

Floyd's Algorithm: Using 2 D matrices

Floyd

1. $D \leftarrow W$ // initialize D array to W [ ]
2. $P \leftarrow 0$ // initialize P array to [0]
3. **for** $k \leftarrow 1$ **to** n

// Computing D' from D

4. **do for** $i \leftarrow 1$ **to** n
5. **do for** $j \leftarrow 1$ **to** n
6. **if** $(D[i, j] > D[i, k] + D[k, j])$
7. **then** $D'[i, j] \leftarrow D[i, k] + D[k, j]$
8. $P[i, j] \leftarrow k$;
9. **else** $D'[i, j] \leftarrow D[i, j]$
10. ***Move D' to D.***

Can we use only one D matrix?

- *D*[*i,j*] depends only on elements in the *k*th column and row of the distance matrix.
- We will show that the *k*th row and the *k*th column of the distance matrix are unchanged when D^k is computed
- This means *D* can be calculated *in-place*

The main diagonal values

- **Before we show that *k*th row and column of *D* remain unchanged we show that the main diagonal remains 0**

- $D^{(k)}[\,j,j\,] = \min\{\,D^{(k-1)}[\,j,j\,],\ \ D^{(k-1)}[\,j,k\,] + D^{(k-1)}[\,k,j\,]\,\}$
  $= \min\{\,0,\ \ D^{(k-1)}[\,j,k\,] + D^{(k-1)}[\,k,j\,]\,\}$
  $= 0$

- **Based on which assumption?**

The *k*th column

- *k*th column of D^k is equal to the *k*th column of D^{k-1}

- *Intuitively true* - a path from i to k will not become shorter by adding k to the allowed subset of intermediate vertices

- For all i, $D^{(k)}[i,k] =$
  $= \min\{ D^{(k-1)}[i,k],\ D^{(k-1)}[i,k]+ D^{(k-1)}[k,k] \}$
  $= \min \{ D^{(k-1)}[i,k], D^{(k-1)}[i,k]+0 \}$
  $= D^{(k-1)}[i,k]$

The *k*th row

- *k*th row of D^k is equal to the *k*th row of D^{k-1}

For all j, $D^{(k)}[k,j] =$

$= \min\{ D^{(k-1)}[k,j], D^{(k-1)}[k,k] + D^{(k-1)}[k,j] \}$

$= \min\{ D^{(k-1)}[k,j], 0+D^{(k-1)}[k,j] \}$

$= D^{(k-1)}[k,j]$

Floyd's Algorithm using a single *D*

Floyd

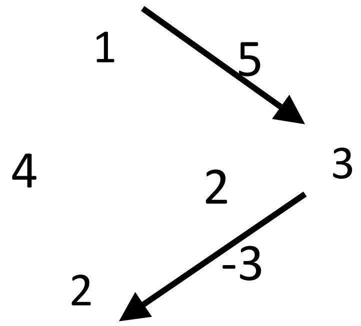
1. $D \leftarrow W$ // initialize *D* array to *W* [ ]
2. $P \leftarrow 0$ // initialize P array to [0]
3. **for** $k \leftarrow 1$ **to** n
4. **do for** $i \leftarrow 1$ **to** n
5. **do for** $j \leftarrow 1$ **to** n
6. **if** $(D[i, j] > D[i, k] + D[k, j])$
7. **then** $D[i, j] \leftarrow D[i, k] + D[k, j]$
8. $P[i, j] \leftarrow k;$

Printing intermediate nodes on shortest path from q to r

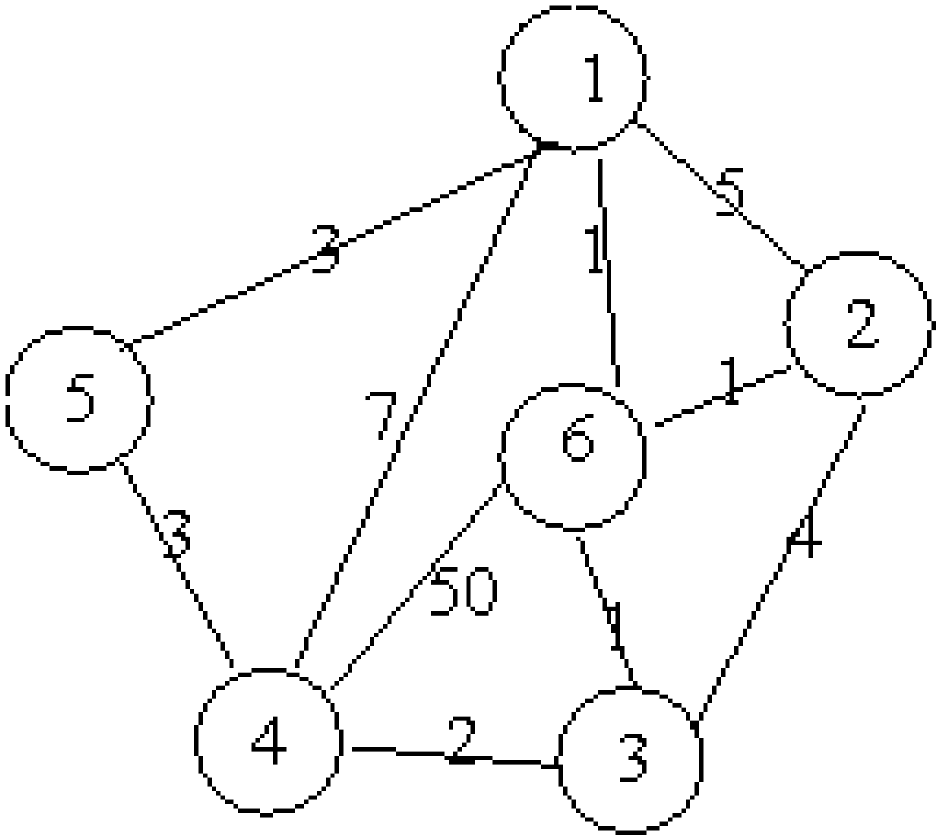
```
path(index q, r)
    if (P[ q, r ]!=0)
        path(q, P[q, r])
        println( "v"+ P[q, r])
        path(P[q, r], r)
      return;
    //no intermediate nodes
    else return
```

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 \end{array}$$

Before calling path check D[q, r] < ∞, and print node q, after the call to path print node r



Example



The final distance matrix and P

$D^6 =$

	1	2	3	4	5	6
1	0	2(6)	2(6)	4(6)	3	1
2	2(6)	0	2(6)	4(6)	5(6)	1
3	2(6)	2(6)	0	2	5(4)	1
4	4(6)	4(6)	2	0	3	3(3)
5	3	5(6)	5(4)	3	0	4(1)
6	1	1	1	3(3)	4(1)	0

The values in parenthesis are the non zero P values.

The call tree for Path(1, 4)

Path(1, 4)

Path(1, 6)

P(1, 6)=0

Print
v6

Path(6, 4)

Path(6, 3)

P(6, 3)=0

Print
v3

Path(3, 4)

P(3, 4)=0

The intermediate nodes on the shortest path from 1 to 4 are v6, v3.
The shortest path is v1, v6, v3, v4.

Thank You !!!