Exercise 2.1

Using the δ-ε approach, establish the following limits.

1. $\displaystyle\lim_{(x,y)\to(1,1)} (x^2 + y^2 - 1) = 1.$

2. $\displaystyle\lim_{(x,y)\to(2,1)} (x^2 + 2x - y^2) = 7$

3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2+1} = 0.$

4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0.$

5. $\displaystyle\lim_{(x,y)\to(0,0)} \left[y + x\cos\left(\frac{1}{y}\right)\right] = 0.$

6. $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\frac{1}{xy} =$

...rmine the following limits if they exist.

7. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$.

8. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3-y^3}{x-y}$.

9. $\lim\limits_{(x,y)\to(a,0)} \left(1+\dfrac{x}{y}\right)^{y}$.

10. $\lim\limits_{(x,y)\to(0,0)} \cot^{-1}\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$.

11. $\lim\limits_{(x,y)\to(0,1)} \dfrac{(y-1)\tan^2 x}{x^2(y^2-1)}$.

12. $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)\sin y}{y\ln x}$.

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x-y}{x^2+y^2}$.

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$.

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^3+y^3}$.

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^2}{(x^4+y^2)^2}$.

17. $\lim\limits_{(x,y,z)\to(0,0,0)} \log\left(\dfrac{z}{xy}\right)$.

18. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+z}{x+y+z^2}$.

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy^2z^2}{x^4+y^4+z^8}$.

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x(x+y+z)}{x^2+y^2+z^2}$.

...scuss the continuity of the following functions at the given points.

21. $f(x,y)=\begin{cases} \dfrac{(x-y)^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

22. $f(x,y)=\begin{cases} \dfrac{1}{1+e^{1/x}}+y^2, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

23. $f(x,y)=\begin{cases} \dfrac{e^{xy}}{x^2+1}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

24. $f(x,y)=\begin{cases} \dfrac{x^2+y^2}{xy}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

25. $f(x,y)=\begin{cases} \dfrac{x^2+y^2}{\tan xy}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

26. $f(x,y)=\begin{cases} \dfrac{x^2-2xy+y^2}{x-y}, & (x,y)\neq(1,-1) \\ 0, & (x,y)=(1,-1) \end{cases}$

at (1, −1).

27. $f(x,y)=\begin{cases} \dfrac{xy(x-y)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

28. $f(x,y)=\begin{cases} \dfrac{x^4y^4}{(x^2+y^4)^3}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

29. $f(x,y)=\begin{cases} \dfrac{\sin\sqrt{|xy|}-\sqrt{|xy|}}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

at (0, 0).

30. $f(x,y)=\begin{cases} \dfrac{2x^2+y^2}{3+\sin x}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

31. $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0 \quad, & (x, y) = (0, 0) \end{cases}$

at $(0, 0)$.

32. $f(x, y) = \begin{cases} \dfrac{x^5 - y^5}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad, & (x, y) = (0, 0) \end{cases}$

at $(0, 0)$.

33. $f(x, y) = \begin{cases} \dfrac{x^2 y}{1 + x}, & x \neq -1 \\ y \quad, & (x, y) = (-1, \alpha) \end{cases}$

at $(-1, \alpha)$.

34. $f(x, y, z) = \begin{cases} \dfrac{xyz}{x^2 + y^2 + z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0 \quad, & (x, y, z) = (0, 0, 0) \end{cases}$

at $(0, 0, 0)$.

35. $f(x, y, z) = \begin{cases} \dfrac{2xy}{x^2 - 3z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0 \quad, & (x, y, z) = (0, 0, 0) \end{cases}$

at $(0, 0, 0)$.

2.3 Partial Derivatives

$= 8$ and the coordinate planes

$\iiint\limits_T xe^{...}z\,dx\,dy\,dz$, T: Region bounded by $x + y^2 + z^2 = 1$

$\iiint\limits_T x^2y\,dx\,dy\,dz$, T: Region bounded by $\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1$

Answers and Hints

Exercise 2.1

1. $|f(x, y) - 1| = |(x-1)^2 + (y-1)^2 + 2(x-1) + 2(y-1)|$
$\qquad \le |x-1|^2 + |y-1|^2 + 2|x-1| + 2|y-1| < \varepsilon$

   (i) if $|x - 1| < \delta$, $|y - 1| < \delta$ is used, we get $2\delta^2 + 4\delta < \varepsilon$ or $\delta < [\sqrt{(\varepsilon + 2)/2} - 1]$

   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/6$

   (iii) if $(x-1)^2 + (y-1)^2 < \delta^2$ and $|x - 1| < \delta$, $|y - 1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon + 4} - 2$

2. $|f(x, y) - 7| = |(x-2)^2 - (y-1)^2 + 8(x-2) - 2(y-1)|$
$\qquad \le |x-2|^2 + |y-1|^2 + 8|x-2| + 2|y-1| < \varepsilon$

   (i) if $|x - 2| < \delta$, $|y - 1| < \delta$ is used, we get $2\delta^2 + 8\delta < \varepsilon$, or $\delta < \sqrt{(\varepsilon + 8)/2} - 2$

   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/10$

   (iii) if $(x-2)^2 + (y-1)^2 < \delta^2$ and $|x - 2| < \delta$, $|y - 1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon + 16} - 4$

3. $\left|\frac{x+y}{x^2+y^2+1}\right| \le |x + y| \le |x| + |y| \le 2\sqrt{x^2 + y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.

4. Let $x = r\cos\theta$, $y = r\sin\theta$. Therefore

   $\left|\frac{x^3+y^3}{x^2+y^2}\right| \le |r(\cos^3\theta + \sin^3\theta)| \le 2r < \varepsilon$. Take $\delta < \varepsilon/2$.

5. $|f(x, y) - 0| \le |x| + |y| \le 2\sqrt{x^2 + y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.

6. $|f(x, y) - 0| \le x^2 + y^2 < \varepsilon$. Take $\delta < \sqrt{\varepsilon}$

7. Choose the path $y = mx$. Limit does not exist.

8. Factorize and cancel $x - y$; 1.

9. $[1 + (x/y)]^x = [[1 + (x/y)]^{y/x}]^{x^2/y}$; e^0.

10. 0

11. 1/2

12. Limit does not exist.

13. 1

14. Limit does not exist.

15. Let $x = r\cos\theta$, $y = r\sin\theta$; $\frac{1}{r}\left(\frac{\cos^2\theta}{\cos^3\theta + \sin^3\theta}\right) \to \infty$ as $r \to 0$. Limit does not exist.

16. Choose the path $y = mx^2$. Limit does not exist.

17. Choose the path $z = x^2$, $y = mx$. Limit does not exist.

18. Choose the path $y = mx$, $z = mx$. Limit does not exist.

19. Choose the path $z = \sqrt{x}$, $y = mx$. Limit does not exist.

20. Choose the path $z = 0$, $y = mx$. Limit does not exist.

21. Choose the path $y = mx$. Discontinuous.

22. Limit is 0 for $x > 0$ and 1 for $x < 0$. Discontinuous.

23. Discontinuous.

24. Choose the path $y = mx$. Discontinuous.

25. Choose the path $y = mx$. Discontinuous.

26. Cancel $(x - y)$. Discontinuous.

27. Let $x = r\cos\theta$, $y = r\sin\theta$. Continuous.

28. Choose the path $y^2 = mx$. Discontinuous.

29. Since $x^2 + y^2 \geq 2|x||y|$, we have $\frac{1}{\sqrt{x^2+y^2}} \leq \frac{1}{\sqrt{2|xy|}}$. Therefore, $|f(x, y)| \leq \frac{\left|\sin\sqrt{|xy|} - \sqrt{|xy|}\right|}{\sqrt{2}\sqrt{|xy|}}$. Continuous.

30. Since $2 \leq 3 + \sin x \leq 4$, we have $[1/(3 + \sin x)] \leq 1/2$. Therefore, $|f(x, y)| \leq [(2x^2 + y^2)/2] \leq x^2 + y^2$. Continuous.

31. The function is not defined along the path $y = -x$. Discontinuous.

32. $\left|\frac{x^5 - y^5}{x^2 + y^2}\right| \leq \frac{|x|^5 + |y|^5}{x^2 + y^2} \leq \frac{(x^2+y^2)^{5/2} + (x^2+y^2)^{5/2}}{x^2 + y^2}$. Continuous.

33. Function is unbounded in any neighborhood of $x = -1$. Discontinuous.

34. Since $|x|, |y|, |z|$ are all $\leq \sqrt{x^2+y^2+z^2}$, $|f| \leq \sqrt{x^2+y^2+z^2}$. Continuous.

35. The function is unbounded along $x = \sqrt{3}z$. Discontinuous.

xercise 2.2

1. $f_x(0, 0) = 0$, $f_y(0, 0) = 0$. For $(x, y) \neq (0, 0)$, find f_x, f_y and choose the path $y = mx$. The limits do not exist as $(x, y) \to (0, 0)$.

2. $f(x, y)$ is unbounded as $(x, y) \to (0, 0)$, for example along $x = y$; $f_x(0, 0) = 1$, $f_y(0, 0) = -1$.

3. $f_x(0, 0) = 0$, $f_y(0, 0) = -1$, $f_x(0, y) = 0$, $f_y(x, 0) = 1$.

4. $f_x(0, 0) = 1$, $f_y(0, 0) = 1$, $dz = \Delta x + \Delta y$, $\lim [(\Delta z - dz)/\Delta\rho]$ ...