Mathematical Expectation.

6·1. Mathematical Expectation. Let X be a random variable (r.v.) with p.d.f. (p.m.f.) $f(x)$. Then its mathematical expectation, denoted by $E(X)$ is given by :

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \quad (\text{ for continuous } r.v.)$$

$$= \sum_{x} x\, f(x), \quad (\text{ for discrete } r.v.)$$

provided the righthand integral or series is absolutely convergent, *i.e.*,

$$\int_{-\infty}^{\infty} |x f(x)|\, dx = \int_{-\infty}^{\infty} |x| f(x)\, dx < \infty$$

or

$$\sum_{x} |x\, f(x)| = \sum_{x} |x|\, f(x) < \infty$$

Theorem 6·1. *If X and Y are random variables then*

$$E(X+Y)=E(X)+E(Y),$$

provided all the expectations exist.

Theorem 6·1(a). *The mathematical expectation of the sum of n random variables is equal to the sum of their expectations, provided all the expectations exist.*

Symbolically, if $X_1, X_2, \ldots, X_n$ *are random variables then*

$$E(X_1+X_2+\ldots+X_n)=E(X_1)+E(X_2)+\ldots+E(X_n) \qquad \ldots(6\cdot13)$$

or

$$E\left(\sum_{i=1}^{n} X_i\right)=\sum_{i=1}^{n} E(X_i), \qquad \ldots(6\cdot13a)$$

if all the expectations exist.

Theorem 6·2. *If X and Y are independent random variables, then*

$$E(XY) = E(X) \cdot E(Y)$$

Theorem 6·2(a). *The mathematical expectation of the product of a number of independent random variables is equal to the product of their expectations. Symbolically, if $X_1, X_2, ..., X_n$ are n independent random variables, then*

$$\left.\begin{aligned} E(X_1 X_2 \ldots X_n) &= E(X_1)\,E(X_2) \ldots E(X_n) \\ \text{i.e.,} \quad E\left(\prod_{i=1}^{n} X_i\right) &= \prod_{i=1}^{n} E(X_i) \end{aligned}\right\} \quad \ldots(6\cdot16)$$

provided all the expectations exist.

Theorem 6·3. *If X is a random variable and 'a' is constant, then*

(i) $E[a\Psi(X)] = a\,E[\Psi(X)]$

(ii) $E[\Psi(X) + a] = E[\Psi(X)] + a,$

where $\Psi(X)$, a function of X, is a r.v. and all the expectations exist.

Theroem 6·4. *If X is a random variable and a and b are constants, then*

$$E(aX + b) = a\,E(X) + b \qquad \text{...(6·22)}$$

provided all the expectations exist.

If $b = 0$, then we get

$$E(aX) = a \cdot E(X)$$

Taking $a = 1,\ b = -\overline{X} = -E(X)$, we get

$$E(X - \overline{X}) = 0$$

6·5. Expectation of a Linear Combination of Random Variables

Let $X_1, X_2, \ldots, X_n$ be any n random variables and if $a_1, a_2, \ldots, a_n$ are any n constants, then

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i) \qquad \ldots(6\cdot25)$$

provided all the expectations exist.

Theorem 6·5 (a). *If $X \geq 0$ then $E(X) \geq 0$.*

Theorem 6·5 (b). *Let X and Y be two random variables such that $Y \leq X$ then*

$$E(Y) \leq E(X),$$

provided the expectations exist.

Theorem 6·6. $|E(X)| \leq E|X|$,

provided the expectations exist.

Theorem 6·8. *If X is a random variable, then*

$$V(aX+b) = a^2 V(X),$$

where a and b are constants.

Example 6·1. *Let X be a random variable with the following probability distribution :*

x	:	-3	6	9
$P_r(X = x)$	:	$1/6$	$1/2$	$1/3$

Find $E(X)$ and $E(X^2)$ and using the laws of expectation, evaluate $E(2X+1)^2$.

Solution. $E(X) = \Sigma x \,.\, p(x)$

$$= (-3) \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = \frac{11}{2}$$

$$E(X^2) = \Sigma x^2 p(x)$$

$$= 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = \frac{93}{2}$$

$$\therefore \quad E(2X+1)^2 = E[4X^2 + 4X + 1] = 4E(X^2) + 4E(X) + 1$$

$$= 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1 = 209$$

Example 6·2. **(a)** *Find the expectation of the number on a die when thrown.*

(b) *Two unbiased dice are thrown. Find the expected values of the sum of numbers of points on them.*

Solution. *(a)* Let X be the random variable representing the number on a die when thrown. Then X can take any one of the values 1,2,3,..., 6 each with equal probability 1/6. Hence

$$E(X) = \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \frac{1}{6}\times 3 + \ldots + \frac{1}{6}\times 6$$

$$= \frac{1}{6}(1+2+3+\ldots+6) = \frac{1}{6}\times\frac{6\times 7}{2} = \frac{7}{2} \qquad \ldots(*)$$

(b) The probability function of X (the sum of numbers obtained on two dice), is

Value of X : x	2	3	4	5	6	7	.....	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	.....	2/36	1/36

$$E(X) = \sum_i p_i x_i$$

$$= 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36}$$

$$+ 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$$

$$= \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$$

$$= \frac{1}{36}\times 252 = 7$$

Aliter. Let X_i be the number obtained on the ith dice ($i = 1, 2$) when thrown. Then the sum of the number of points on two dice is given by

$$S = X_1 + X_2$$

$$\Rightarrow \quad E(S) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7 \qquad \text{[On using (*)]}$$

Example 6·5. *A coin is tossed until a head appears. What is the expectation of the number of tosses required ?*

Solution. Let X denote the number of tosses required to get the first head. Then X can materialise in the following ways :

$$\therefore \qquad E(X) = \sum_{x=1}^{\infty} x\, p(x)$$

Event	x	*Probability* $p(x)$
H	1	$1/2$
TH	2	$1/2 \times 1/2 = 1/4$
TTH	3	$1/2 \times 1/2 \times 1/2 = 1/8$
$\vdots$	$\vdots$	$\vdots$

$$= 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16} + \ldots \qquad \ldots(*)$$

This is an arithmetic-geometric series with ratio of GP being $r = 1/2$.

Let $\qquad S = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots$

Then $\qquad \frac{1}{2} S = \qquad \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + \ldots$

$\therefore \quad (1 - \frac{1}{2}) S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$\Rightarrow \qquad \frac{1}{2} S = \frac{1/2}{1 - (1/2)} = 1$

[Since the sum of an infinite G.P. with first term a and common ratio r (< 1) is $a/(1 - r)$]

$\Rightarrow \qquad S = 2$

Hence, substituting in (*), we get

$$E(X) = 2$$

Example 6·6. *What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success in each trial ?*

Solution. Let the random variable X denote the number of failures preceding the first success. Then X can take the values 0, 1, 2, ..., ∞. We have

$$p(x) = P(X=x) = P[x \text{ failures precede the first success}] = q^x p$$

where $q = 1 - p$ is the probability of failure in a trial. Then by def.

$$E(X) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x \,.\, q^x p = pq \sum_{x=1}^{\infty} x\, q^{x-1}$$

$$= pq\,[1 + 2q + 3q^2 + 4q^3 + ...]$$

Now $1 + 2q + 3q^2 + 4q^3 + \ldots$ is an infinite arithmetic-geometric series.

Let $\qquad S = 1 + 2q + 3q^2 + 4q^3 + \ldots$

$$qS = \quad q + 2q^2 + 3q^3 + \ldots$$

$\therefore \qquad (1-q)S = 1 + q + q^2 + q^3 + \ldots = \dfrac{1}{1-q}$

$\Rightarrow \qquad S = \dfrac{1}{(1-q)^2}$

$\therefore \qquad 1 + 2q + 3q^2 + 4q^3 + \ldots = \dfrac{1}{(1-q)^2}$

Hence $\qquad E(X) = \dfrac{pq}{(1-q)^2} = \dfrac{pq}{p^2} = \dfrac{q}{p}$

Example 6·8. *Let variate X have the distribution*

$$P(X=0) = P(X=2) = p\,;\; P(X=1) = 1-2p, \quad \text{for } 0 \le p \le \tfrac{1}{2}.$$

For what p is the Var (X) a maximum ?

Solution. Here the r.v. X takes the values 0, 1 and 2 with respective probabilities p, $1 - 2p$ and p, $0 \le p \le \frac{1}{2}$.

$$\therefore \quad E(X) = 0 \times p + 1 \times (1 - 2p) + 2 \times p = 1$$

$$E(X^2) = 0 \times p + 1^2 \times (1 - 2p) + 2^2 \times p = 1 + 2p$$

$$\therefore \quad Var(X) = E(X^2) - [E(X)]^2 = 2p \; ; \; 0 \le p \le \tfrac{1}{2}$$

Obviously $Var(X)$ is maximum when $p = \frac{1}{2}$, and

$$[Var(X)]_{max} = 2 \times \tfrac{1}{2} = 1$$