Objective

- To differentiate functions using the power rule, constant rule, constant multiple rule, and sum and difference rules.

The Derivative is …

- Used to find the "slope" of a function at a point.
- Used to find the "slope of the tangent line" to the graph of a function at a point.
- Used to find the "instantaneous rate of change" of a function at a point.
- Computed by finding the limit of the difference quotient as Δx approaches 0. (Limit Definition)

Rules for Differentiation

- <u>Differentiation</u> is the process of computing the derivative of a function.

You may be asked to:

- Differentiate.
- Derive.
- Find the derivative of…

Rules for Differentiation

- Working with the definition of the derivative is important because it helps you really understand what the derivative means.

The Power Rule

$$\frac{d}{dx}[x^N] = Nx^{N-1}, \quad N \text{ is any real number}$$

$$\frac{d}{dx}[x] = 1$$

The Constant Rule

$$\frac{d}{dx}[c] = 0, \quad c \text{ is a constant}$$

- The derivative of a constant function is zero.

The Constant Multiple Rule

$$\frac{d}{dx}[c(f(x))] = c(f'(x)), \quad c \text{ is a constant}$$

- The derivative of a constant times a function is equal to the constant times the derivative of the function.

The Sum and Difference Rules

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

The derivative of a sum is the sum of the derivatives.

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

The derivative of a difference is the difference of the derivatives.

Constant Rule

- Find the derivative of:

$$f(x) = 7$$

$$f'(x) = 0$$

$$y = -3$$

$$\frac{dy}{dx} = 0 \quad \text{or} \quad y' = 0$$

Power Rule

- Differentiate:

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$g(x) = x^{100}$$

$$g'(x) = 100x^{99}$$

$$y = x^9$$

$$\frac{dy}{dx} = 9x^8$$

Constant Multiple Rule

- Find the derivative of:

$$y = 2x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = 2\left(\tfrac{1}{3} x^{-\frac{2}{3}} \right)$$

$$\frac{dy}{dx} = \frac{2}{3x^{\frac{2}{3}}}$$

Constant Multiple Rule

- Find the derivative of:

$$f(x) = \frac{4x^2}{5} = \frac{4}{5}x^2$$

$$f'(x) = \tfrac{4}{5}\left(2x\right)$$

$$f'(x) = \frac{8}{5}x$$

Constant Multiple Rule

- Find the derivative of:

$$g(x) = 5x^7$$

$$g'(x) = 35x^6$$

Rewriting Before Differentiating

Function	Rewrite	Differentiate	Simplify
$f(x)=\frac{5}{2x^3}$	$f(x)=\frac{5}{2}x^{-3}$	$f'(x)=\frac{5}{2}(-3x^{-4})$	$f'(x)=-\frac{15}{2x^4}$

Rewriting Before Differentiating

Function	Rewrite	Differentiate	Simplify
$g(x) = \frac{7}{3x^{-2}}$	$g(x) = \frac{7}{3}x^2$	$g'(x) = \frac{7}{3}(2x)$	$g'(x) = \frac{14}{3}x$

Rewriting Before Differentiating

Function	Rewrite	Differentiate	Simplify
$h(x)=\sqrt{x}$	$h(x)=x^{1/2}$	$h'(x)=\frac{1}{2}x^{-1/2}$	$h'(x)=\frac{1}{2x^{1/2}}$

Rewriting Before Differentiating

Function	Rewrite	Differentiate	Simplify
$j(x)=\dfrac{1}{2\sqrt[3]{x^2}}$	$j(x)=\dfrac{1}{2x^{2/3}}$ $j(x)=\dfrac{1}{2}x^{-2/3}$	$j'(x)=\dfrac{1}{2}\left(-\dfrac{2}{3}x^{-5/3}\right)$	$j'(x)=-\dfrac{1}{3x^{5/3}}$

Sum & Difference Rules

- Differentiate:

$$f(x) = 5x^2 + 7x - 6$$

$$f'(x) = 10x + 7$$

$$g(x) = 4x^6 - 3x^5 - 10x^2 + 5x + 16$$

$$g'(x) = 24x^5 - 15x^4 - 20x + 5$$

Conclusion

- Notations for the derivative:

$$f'(x) \qquad y' \qquad \frac{dy}{dx}$$

- The derivative of a constant is zero.
- To find the derivative of $f(x) = x^N$
  1. Pull a copy of the exponent out in front of the term.
  2. Subtract one from the exponent.

The Hessian matrix

The "**Hessian matrix**" of a multivariable function $f(x, y, z, \dots)$, which different authors write as $\mathbf{H}(f)$, $\mathbf{H}f$, or $\mathbf{H}_f$, organizes all second partial derivatives into a matrix:

$$\mathbf{H}f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} & \cdots \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} & \cdots \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Let $f(x, y) = x^3y^3 - xy$.

What is the Hessian of f?

Choose 1 answer:

(A) $$\begin{bmatrix} 3x^2y^3 - y & 3x^3y^2 - x \\ 6xy^3 & 6x^3y \end{bmatrix}$$

(B) $$\begin{bmatrix} 6x^3y & 9x^2y^2 - 1 \\ 9x^2y^2 - 1 & 6xy^3 \end{bmatrix}$$

(C) $$\begin{bmatrix} 6xy^3 & 9x^2y^2 - 1 \\ 9x^2y^2 - 1 & 6x^3y \end{bmatrix}$$

(D) $$\begin{bmatrix} 9x^2y^2 - 1 & 6xy^3 \\ 6x^3y & 9x^2y^2 - 1 \end{bmatrix}$$

The **Jacobian matrix** is a matrix composed of the first-order partial derivatives of a multivariable function.

The formula for the Jacobian matrix is the following:

$$f(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_m)$$

$$J_f = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Therefore, Jacobian matrices will always have as many rows as vector components $(f_1, f_2, \ldots, f_m)$, and the number of columns will match the number of variables $(x_1, x_2, \ldots, x_n)$ of the function.

MULTI-VARIABLE CHAIN RULE

Suppose that $z = f(x, y)$, where x and y themselves depend on one or more variables. Multivariable Chain Rules allow us to differentiate z with respect to any of the variables involved:

> Let $x = x(t)$ and $y = y(t)$ be differentiable at t and suppose that $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and
>
> $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

$z = x^2y - y^2$ where x and y are parametrized as $x = t^2$ and $y = 2t$.

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$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2xy)(2t) + (x^2 - 2y)(2) \\ &= (2t^2 \cdot 2t)(2t) + \left((t^2)^2 - 2(2t)\right)(2) \\ &= 8t^4 + 2t^4 - 8t \\ &= 10t^4 - 8t.\end{aligned}$$

ALTERNATE SOLUTION

Since $x(t) = t^2$ and $y(t) = 2t$,

$$\begin{aligned} z &= x^2y - y^2 \\ &= \left(t^2\right)^2(2t) - (2t)^2 \\ &= 2t^5 - 4t^2. \end{aligned}$$

We can now compute $\frac{dz}{dt}$ directly!

$$\frac{dz}{dt} = 10t^4 - 8t,$$

We now suppose that x and y are both multivariable functions.

Let $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v) and suppose that $z = f(x, y)$ is differentiable at the point $(x(u, v), y(u, v))$. Then $f(x(u, v), y(u, v))$ has first-order partial derivatives at (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Let $z = e^{x^2 y}$, where
$x(u, v) = \sqrt{uv}$ and
$y(u, v) = 1/v$. Then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Let $z = e^{x^2 y}$, where
$x(u,v) = \sqrt{uv}$ and
$y(u,v) = 1/v$. Then

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \left(2xye^{x^2y}\right)\left(\frac{\sqrt{v}}{2\sqrt{u}}\right) + \left(x^2e^{x^2y}\right)(0) \\
&= 2\sqrt{uv}\cdot\frac{1}{v}e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot\frac{\sqrt{v}}{2\sqrt{u}} + (\sqrt{uv})^2\cdot e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot(0) \\
&= e^u + 0 \\
&= e^u \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \left(2xye^{x^2y}\right)\left(\frac{\sqrt{u}}{2\sqrt{v}}\right) + \left(x^2e^{x^2y}\right)\left(-\frac{1}{v^2}\right) \\
&= 2\sqrt{uv}\cdot\frac{1}{v}e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot\frac{\sqrt{u}}{2\sqrt{v}} + (\sqrt{uv})^2e^{(\sqrt{uv})^2\cdot\frac{1}{v}}\cdot\left(-\frac{1}{v^2}\right) \\
&= \frac{u}{v}e^u - \frac{u}{v}e^u \\
&= 0.
\end{aligned}$$

ALTERNATE SOLUTION

Since $x = \sqrt{uv}$ and $y = \frac{1}{v}$,

$$\begin{aligned} z &= e^{x^2 y} \\ &= e^{(\sqrt{uv})^2\left(\frac{1}{v}\right)} \\ &= e^u. \end{aligned}$$

Then, differentiating z with respect to u and v, respectively,

$$\frac{\partial z}{\partial u} = e^u$$
$$\frac{\partial z}{\partial v} = 0$$

Key Concepts

- Let $x = x(t)$ and $y = y(t)$ be differentiable at t and suppose that $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

- Let $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v) and suppose that $z = f(x, y)$ is differentiable at the point $(x(u, v), y(u, v))$. Then $f(x(u, v), y(u, v))$ has first-order partial derivatives at (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

1. Given the following information use the Chain Rule to determine $\dfrac{dz}{dt}$.

$$z = \cos\left(y x^2\right) \qquad x = t^4 - 2t, \quad y = 1 - t^6$$

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$$z = \cos\left(y\,x^2\right) \qquad x = t^4 - 2t, \quad y = 1 - t^6$$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= \left[-2xy\sin\left(yx^2\right)\right]\left[4t^3 - 2\right] + \left[-x^2\sin\left(yx^2\right)\right]\left[-6t^5\right] \\
&= \boxed{-2\left(t^4 - 2t\right)\left(1 - t^6\right)\left(4t^3 - 2\right)\sin\left(\left(1 - t^6\right)\left(t^4 - 2t\right)^2\right) + 6t^5\left(t^4 - 2t\right)^2\sin\left(\left(1 - t^6\right)\left(t^4 - 2t\right)^2\right)}
\end{aligned}$$

2. Given the following information use the Chain Rule to determine $\frac{dw}{dt}$.

$$w = \frac{x^2 - z}{y^4} \qquad x = t^3 + 7, \quad y = \cos(2t), \quad z = 4t$$

2. Given the following information use the Chain Rule to determine $\frac{dw}{dt}$.

$$w = \frac{x^2 - z}{y^4} \qquad x = t^3 + 7, \quad y = \cos(2t), \quad z = 4t$$

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \left[\frac{2x}{y^4}\right]\left[3t^2\right] + \left[\frac{-4\left(x^2 - z\right)}{y^5}\right]\left[-2\sin(2t)\right] + \left[-\frac{1}{y^4}\right][4] \\
&= \boxed{\frac{6t^2\left(t^3+7\right)}{\cos^4(2t)} + \frac{8\sin(2t)\left(\left(t^3+7\right)^2 - 4t\right)}{\cos^5(2t)} - \frac{4}{\cos^4(2t)}}
\end{aligned}$$

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$$z = x^2 y^4 - 2y \qquad\qquad y = \sin\left(x^2\right)$$

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$$\begin{aligned}
\frac{dz}{dx} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \\
&= \left[2xy^4\right] + \left[4x^2y^3 - 2\right]\left[2x\cos\left(x^2\right)\right] \\
&= \boxed{2x\sin^4\left(x^2\right) + 2x\left(4x^2\sin^3\left(x^2\right) - 2\right)\cos\left(x^2\right)}
\end{aligned}$$

4. Given the following information use the Chain Rule to determine $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

$$z = x^{-2}y^{6} - 4x \qquad x = u^{2}v, \quad y = v - 3u$$

4. Given the following information use the Chain Rule to determine $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

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$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \left[-2x^{-3}y^{6} - 4\right]\left[2uv\right] + \left[6x^{-2}y^{5}\right]\left[-3\right] \\
&= \boxed{2uv\left(-2u^{-6}v^{-3}(v-3u)^{6} - 4\right) - 18u^{-4}v^{-2}(v-3u)^{5}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \left[-2x^{-3}y^{6} - 4\right]\left[u^{2}\right] + \left[6x^{-2}y^{5}\right]\left[1\right] \\
&= \boxed{u^{2}\left(-2u^{-6}v^{-3}(v-3u)^{6} - 4\right) + 6u^{-4}v^{-2}(v-3u)^{5}}
\end{aligned}$$

5. Given the following information use the Chain Rule to determine z_t and z_p .

$$z = 4y\sin(2x) \qquad x = 3u - p, \quad y = p^2 u, \quad u = t^2 + 1$$

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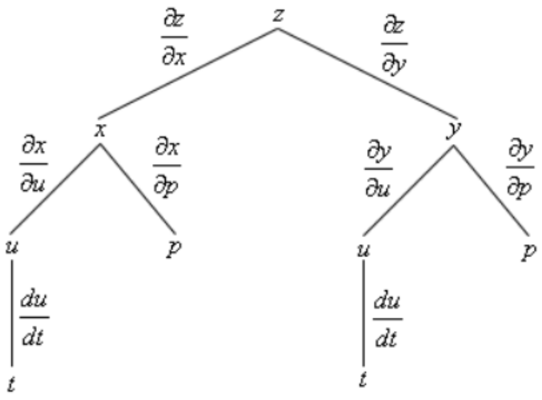

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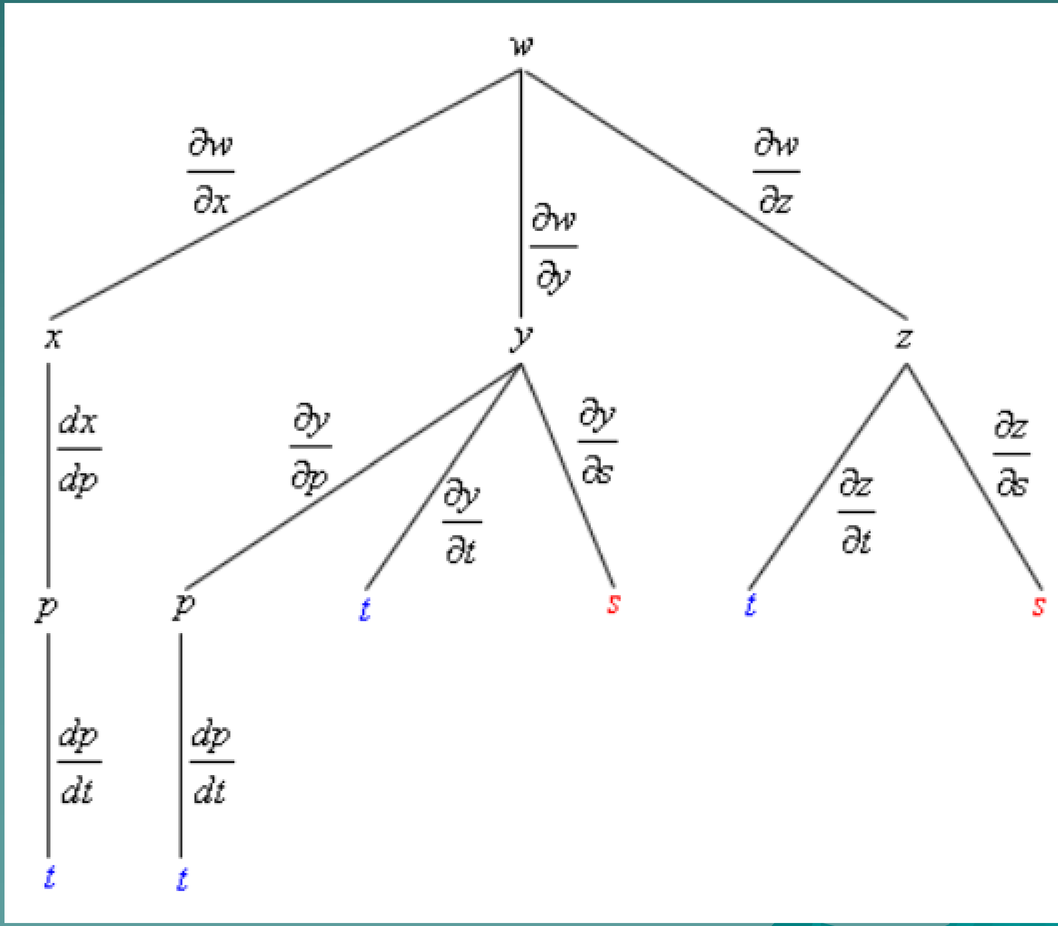
$$z_t = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} \qquad z_p = \frac{\partial z}{\partial p} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p}$$

$$\begin{aligned} z_t &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} \\ &= [8y\cos(2x)]\,[3]\,[2t] + [4\sin(2x)]\left[p^2\right][2t] \\ &= \boxed{48ty\cos(2x) + 8tp^2\sin(2x)} \end{aligned}$$

$$\begin{aligned} z_p &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p} \\ &= [8y\cos(2x)]\,[-1] + [4\sin(2x)]\,[2pu] \\ &= \boxed{-8y\cos(2x) + 8pu\sin(2x)} \end{aligned}$$

6. Given the following information use the Chain Rule to determine $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

$$w = \sqrt{x^2 + y^2} + \frac{6z}{y} \qquad x = \sin(p), \quad y = p + 3t - 4s, \quad z = \frac{t^3}{s^2}, \quad p = 1 - 2t$$



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$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dp}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{dx}{dp}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= \left[\frac{x}{\sqrt{x^2+y^2}}\right][\cos(p)][-2] + \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right][1][-2] + \\
&\qquad + \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right][3] + \left[\frac{6}{y}\right]\left[\frac{3t^2}{s^2}\right] \\
&= \boxed{\frac{-2x\cos(p)}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2} + \frac{18t^2}{ys^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= \left[\frac{y}{\sqrt{x^2+y^2}} - \frac{6z}{y^2}\right][-4] + \left[\frac{6}{y}\right]\left[-\frac{2t^3}{s^3}\right] \\
&= \boxed{\frac{-4y}{\sqrt{x^2+y^2}} + \frac{24z}{y^2} - \frac{12t^3}{ys^3}}
\end{aligned}$$