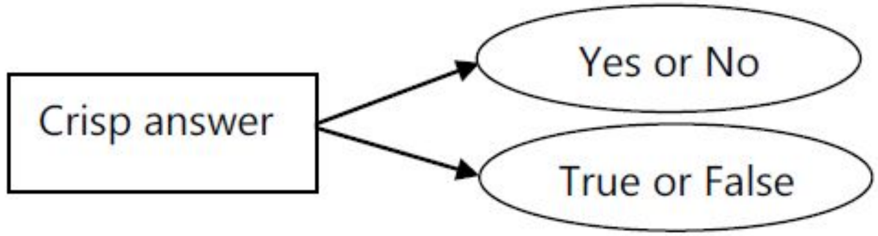
# Fuzzy Logic

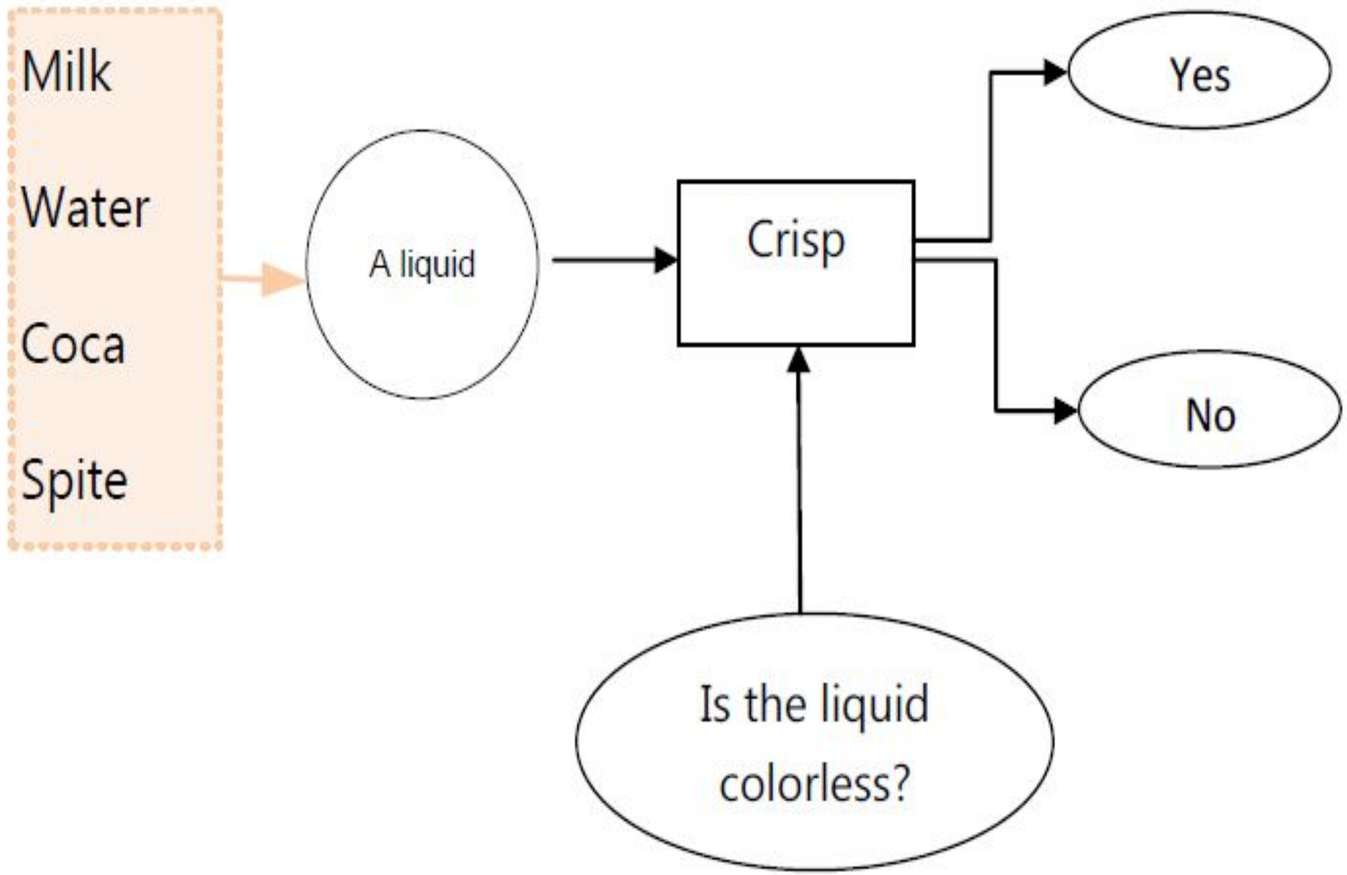
# Introduction

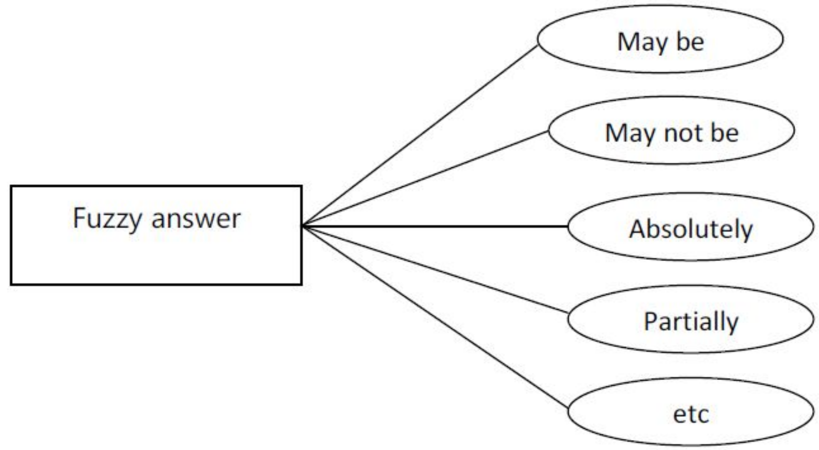
- Introduced in 1965 by Lofti A. Zadeh, a professor in the University of California, Berkeley
- Got accepted as emerging technology since 1980s
- Dictionary meaning of **fuzzy** is not clear, noisy etc.
- Antonym of fuzzy is **crisp**
- Having a fundamental trade-off between precision and cost which can be called "**principle of incompatibility**".
- Generalizes classical two valued logic for reasoning uncertainty.
- Helps us to tackle the uncertainty and vagueness associated with the event.

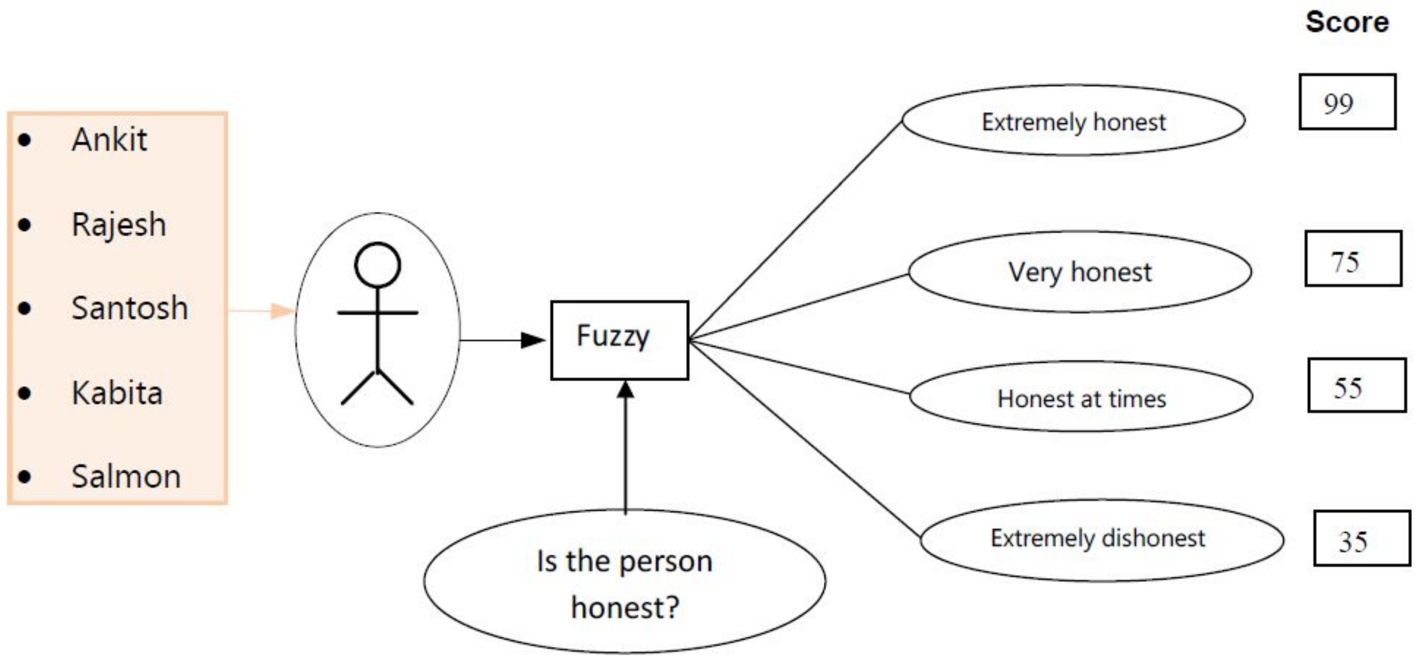
- The factors influencing humans learning ability are:
  - Generalization
  - Association (memory mapping)
  - Information loss (memory loss or forgetfulness).

- **Undecidability**
  - Ambiguity that originates from the inability to distinguish between various states of an event is termed as **undecidability.**
- **Probability Theory versus Possibility Theory**
  - Possibility measures the degree of ease for a variable to take a value, while probability measures the likelihood for a variable to take a value.
  - So they deal with two different types of uncertainty.
  - **Possibility theory handles imprecision and probability theory handles likelihood of occurrence.**
  - fuzzy set theory can define set membership as a possibility distribution.
  - Also, fuzzy logic measures the degree to which an outcome belongs to an event while probability measures the likely hood of event to occur.

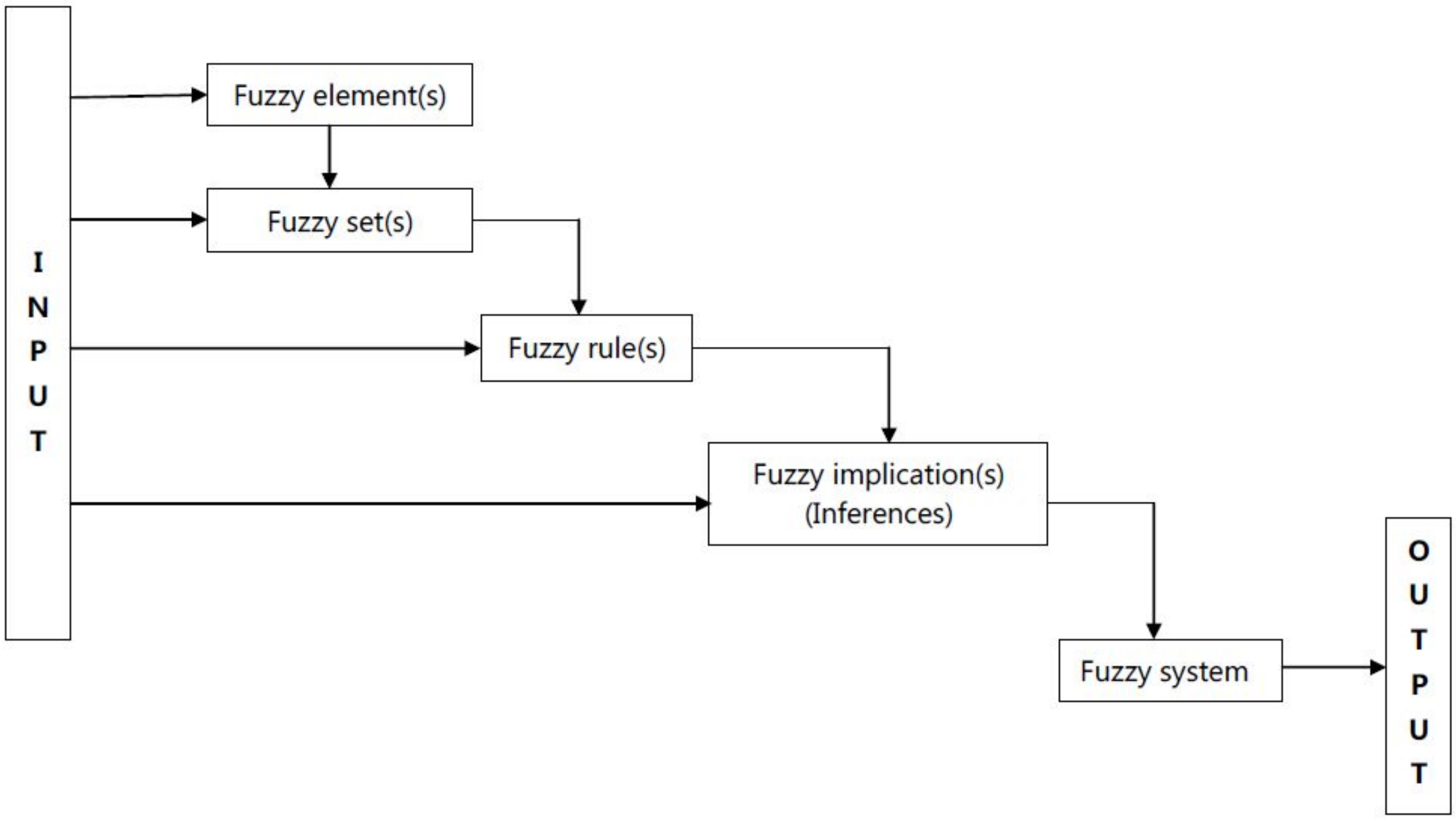

Crisp answer
Yes or No
True or False


Milk
Water
Coca
Spite
A liquid
Crisp
Yes
No
Is the liquid colorless?


Fuzzy answer
May be
May not be
Absolutely
Partially
etc


Score
Ankit
Rajesh
Santosh
Kabita
Salmon
Fuzzy
Is the person honest?
Extremely honest
99
Very honest
75
Honest at times
55
Extremely dishonest
35

# Concept of Fuzzy System



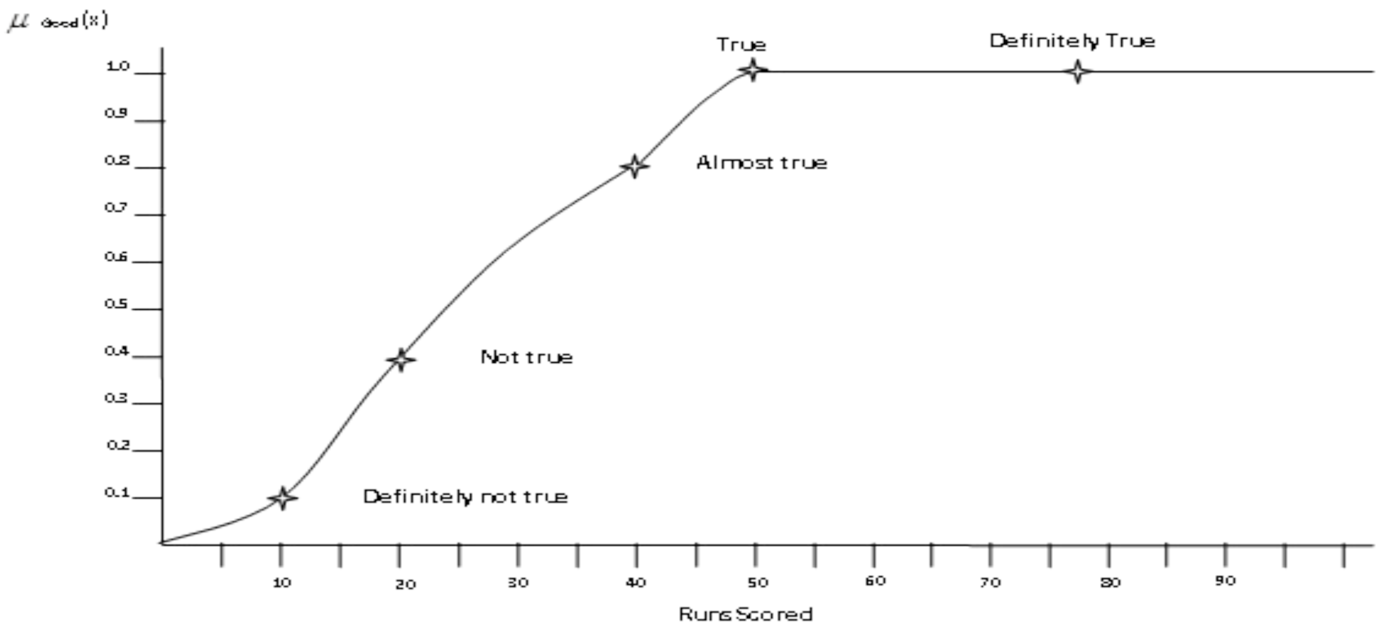
# Classical Sets and Fuzzy Set

- 'Crisp set' is a collection of distinct (precisely defined) elements. In classical set theory, a crisp set can be a superset containing other crisp sets.
- Fuzziness is a property of language. Its main source is the imprecision involved in defining and using symbols.
- **Representation of a Classical Set**
  - Classical set as already mentioned is a collection of objects of any kind. They can be represented as list method, rule method and characteristic function method.

| Method | Examples |
|---|---|
| List method | N={1,2,3,...100} |
| Listing of objects in a set parenthesis. | A={Madras, Madurai, Roorkee, Mumbai} |
| Rule method | N={x│f(x)} |
| The set is defined by property or rule satisfied by every | N= {f (3)} = {3, 6, 9.....} |
| objects within the set parenthesis. | A={Densely populated city} |
| Characteristic function method | A= {Pigeon, Horse, Trichy, 5, 6} |
| The set is defined by a function termed as characteristic | x A implies that x is a member of set A |
| function denoted as μ . The characteristic function declares | x A implies that x is not a member of A. |
| whether the element is a member of the set or not | |

# Representation of a Fuzzy Set

- Representation of a physical variable as a continuous curve is called **fuzzy set or membership function**.
- The numerical value is termed as **'Degree of membership' or 'Truth function'.**



Let us discuss about fuzzy set.

X = All students in IT60108.

S = All **Good students**.

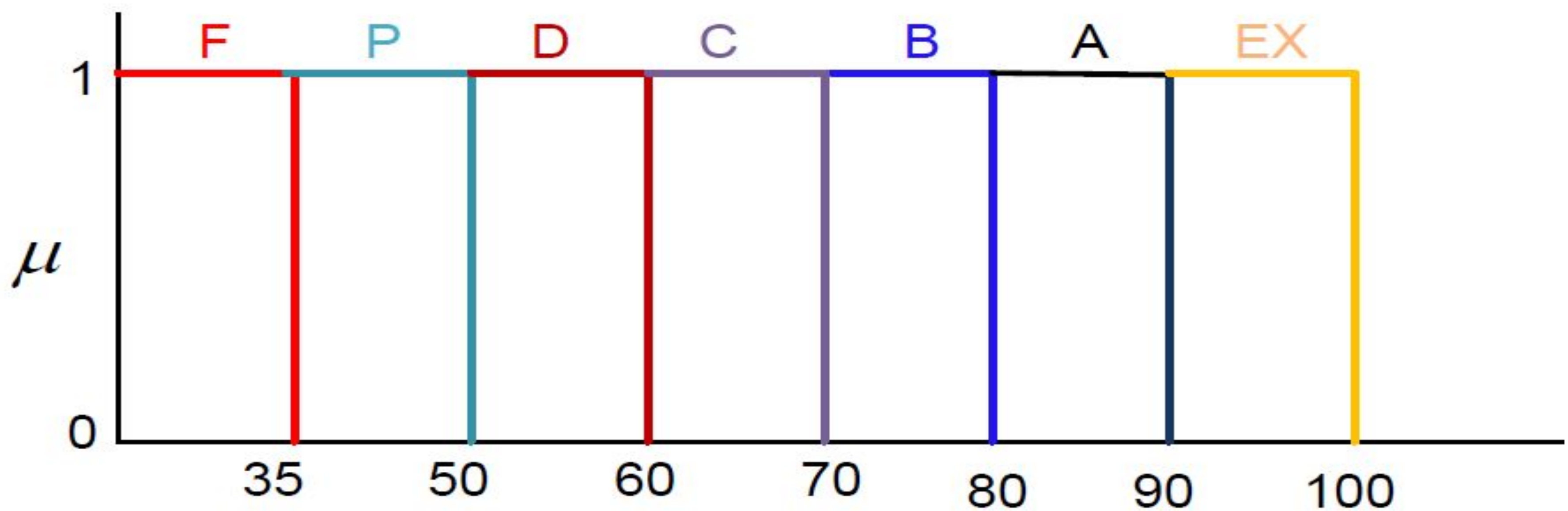
$S = \{ (s, g) \mid s \in X \}$ and g(s) is a measurement of goodness of the student *s*.

**Example:**

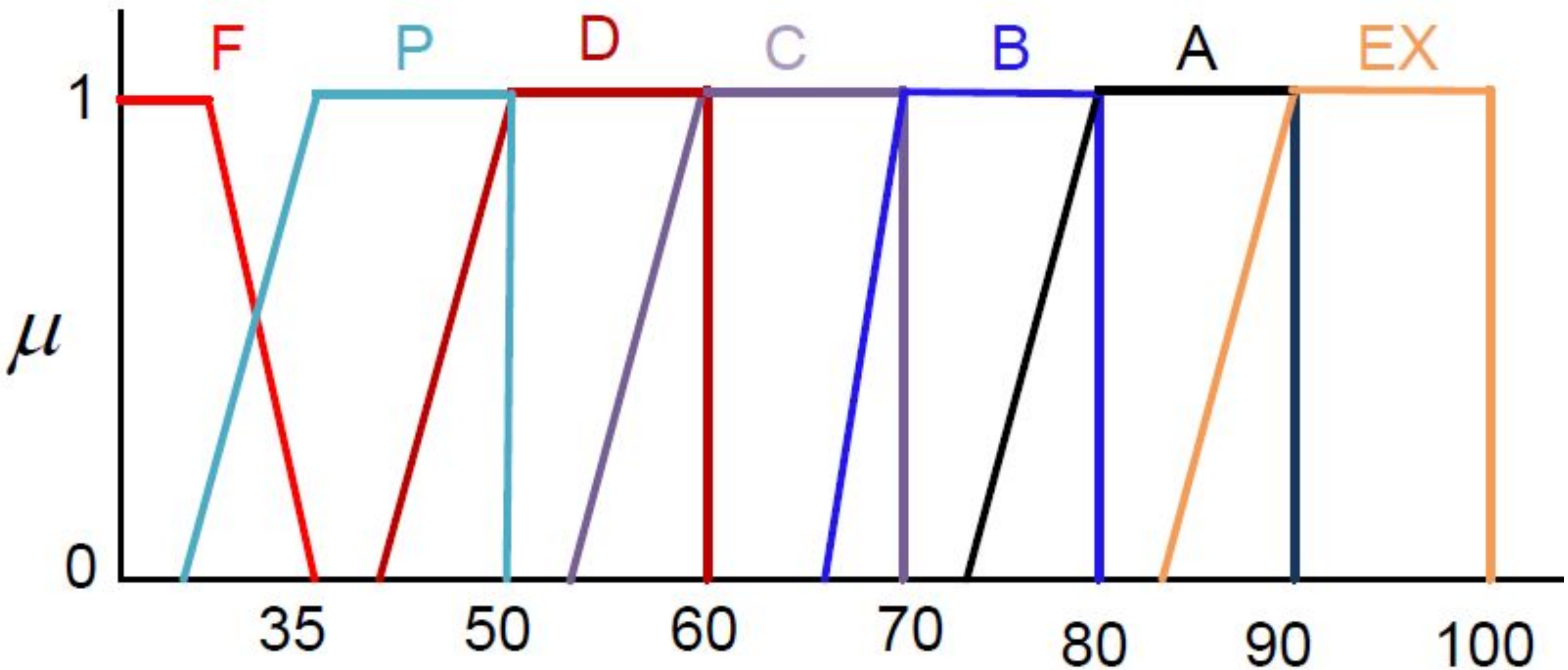
S = { (Rajat, 0.8), (Kabita, 0.7), (Salman, 0.1), (Ankit, 0.9) } etc.

# Example: Course evaluation in a crisp way

1. EX = Marks $\geq$ 90
2. A = 80 $\leq$ Marks $<$ 90
3. B = 70 $\leq$ Marks $<$ 80
4. C = 60 $\leq$ Marks $<$ 70
5. D = 50 $\leq$ Marks $<$ 60
6. P = 35 $\leq$ Marks $<$ 50
7. F = Marks $<$ 35



# Example: Course evaluation in a fuzzy way

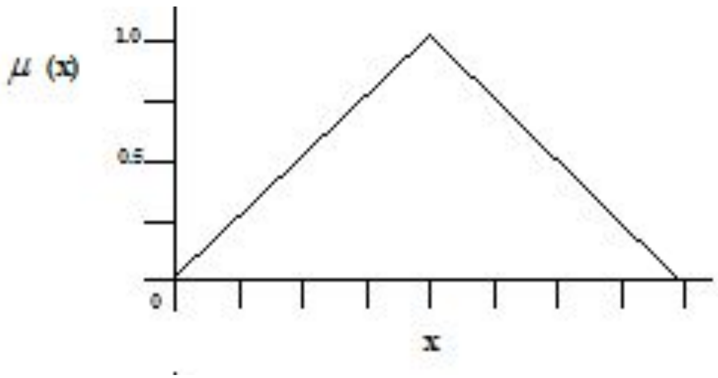


# Basic Properties of Fuzzy Sets

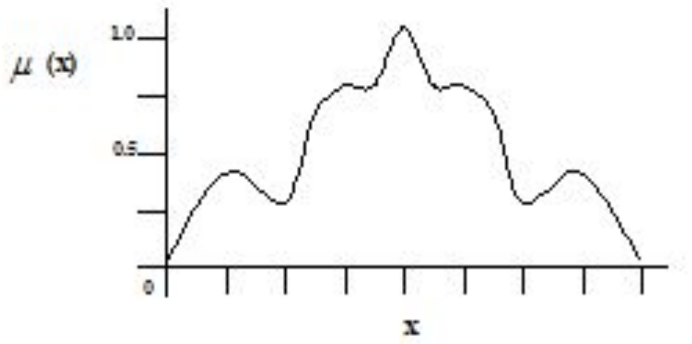
- The complete range of the model variable is called **‘universe of discourse’**.
- Universe of discourse is associated with the system variable (score) and not with a particular fuzzy set (Good score, Average score or Bad score).
- The range of values covered by a particular fuzzy set is termed as **domain of fuzzy set.**
- The domain of fuzzy set is set of elements whose degree of membership in the fuzzy set is greater than zero.

# Non-convex and Convex Fuzzy Sets

- Non-convex fuzzy sets are fuzzy sets, in which membership grade alternately increases and decreases on the domain.
- Fuzzy sets in which membership grades do not alternately increases and decreases are called convex fuzzy set.




Convex fuzzy set




Non-convex fuzzy set

# Membership function

- If X is a universe of discourse and x 2 X, then a fuzzy set A in X is defined as a set of ordered pairs, that is

$$A = \{(x, \mu_A(x)) | x \in X\}$$

where $\mu_A(x)$ is called the membership function for the fuzzy set A.

**Example:**

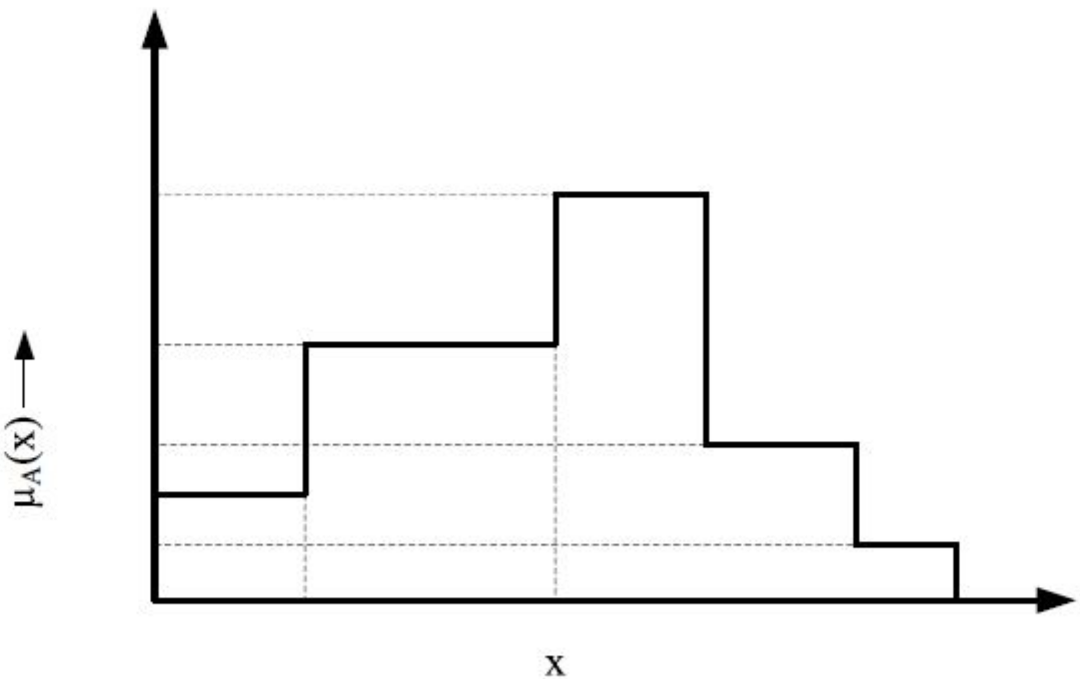
X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

# Membership function with discrete membership values

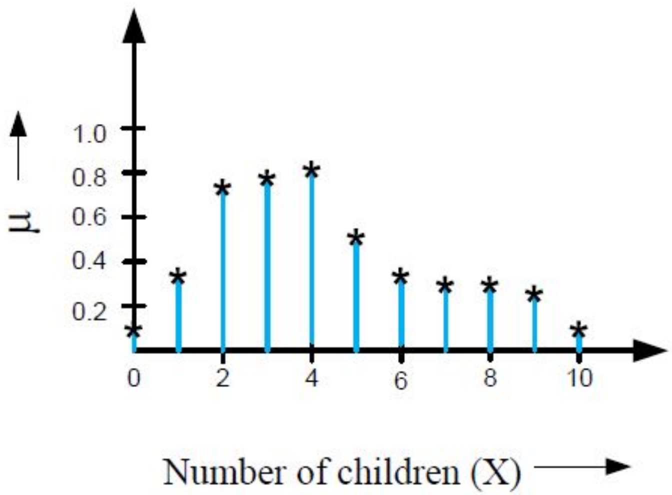
- The membership values may be of discrete values.




A fuzzy set with discrete values of $\mu$

# Example

Either elements or their membership values (or both) also may be of discrete values.



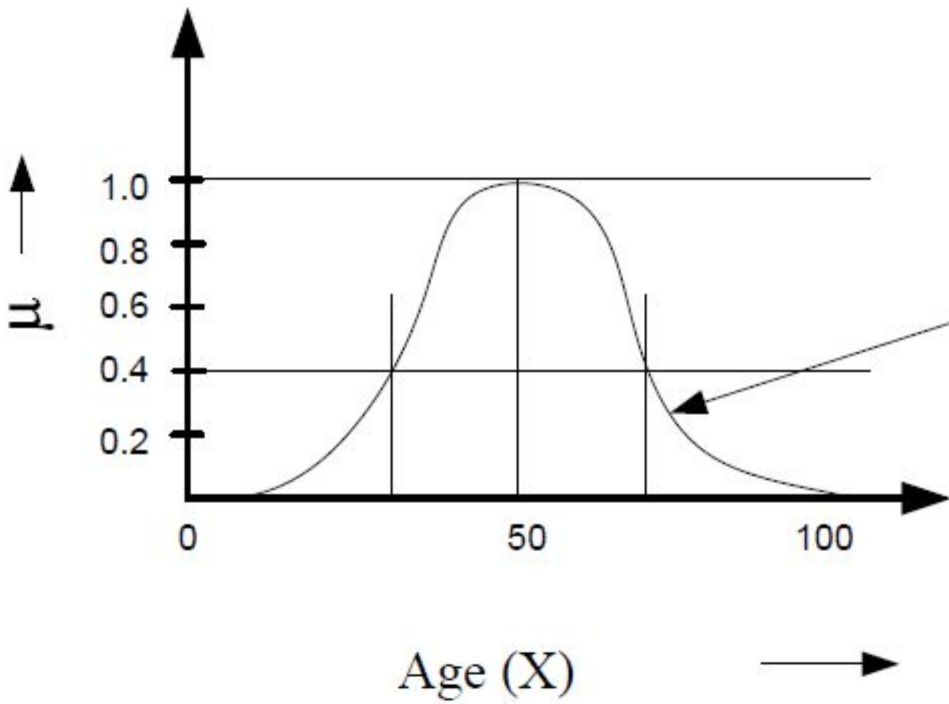

A = “Happy family”

A ={(0,0.1),(1,0.30),(2,0.78)……(10,0.1)}

Note : X = discrete value

How you measure happiness ??

# Membership function with continuous membership values




$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

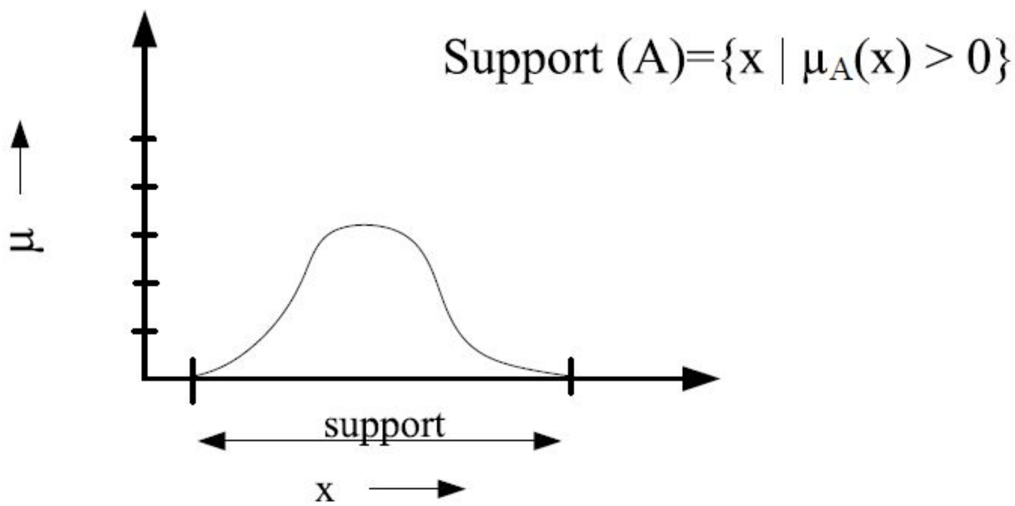
B = "Middle aged"

$$B=\{(x,\mu_B(x))\}$$

Note : x = real value
= $R^+$

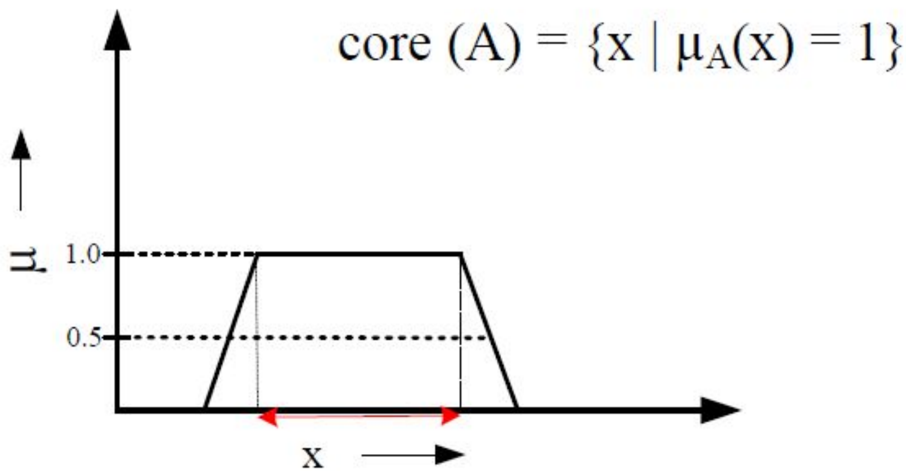
# Fuzzy terminologies: Support

**Support**: The support of a fuzzy set $A$ is the set of all points $x \in X$ such that $\mu_A(x) > 0$

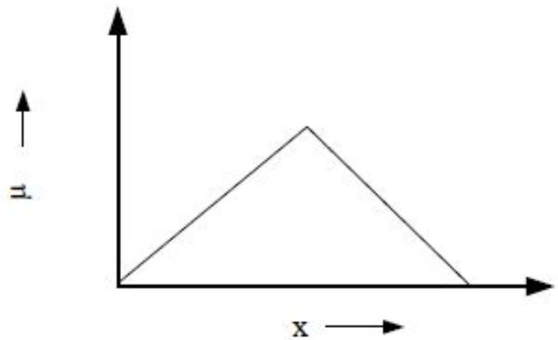


# Fuzzy terminologies: Core

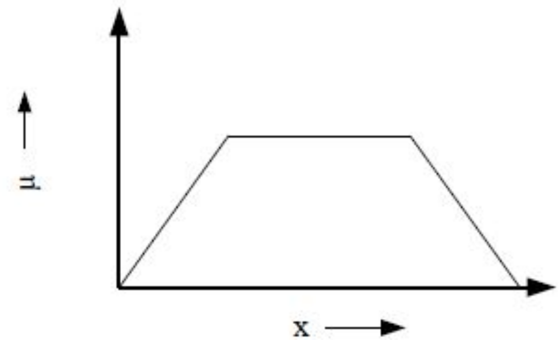
**Core**: The core of a fuzzy set $A$ is the set of all points $x$ in $X$ such that $\mu_A(x) = 1$



# Fuzzy membership functions




< triangular >

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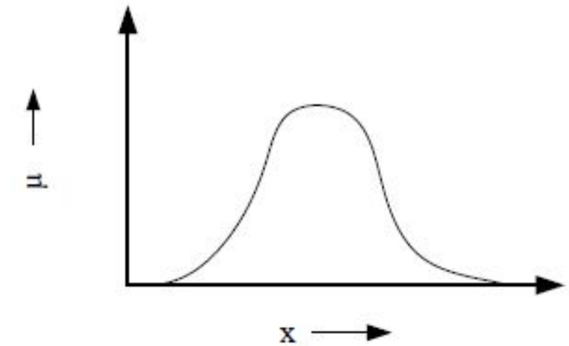
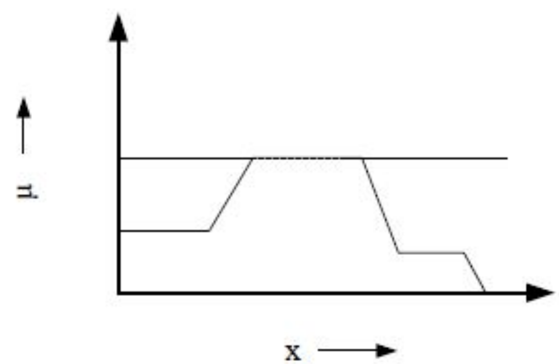



< curve >

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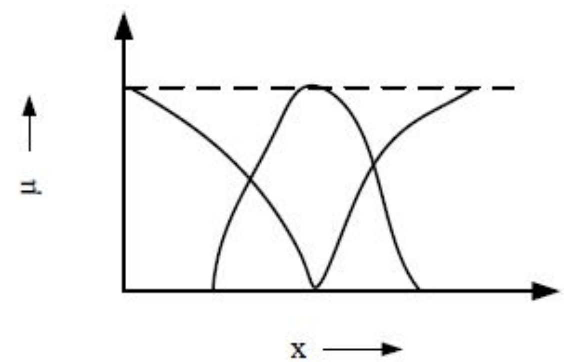


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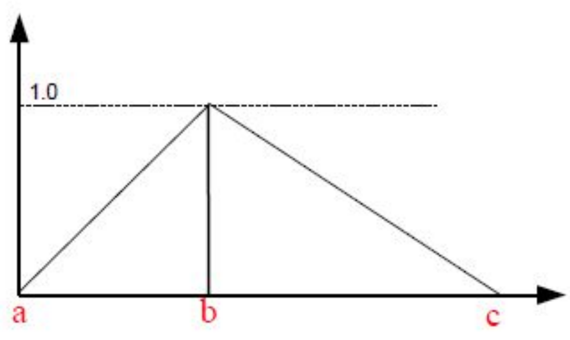
# Triangular MFs

- A triangular MF is specified by three parameters {a; b; c} and can be formulated as follows.

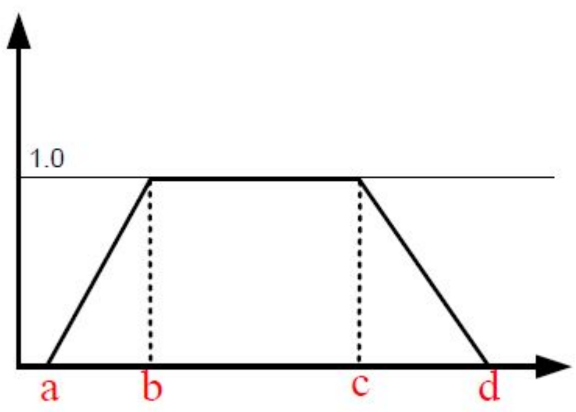
$$triangle(x; a, b, c) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ \frac{c-x}{c-b} & \text{if } b \le x \le c \\ 0 & \text{if } c \le x \end{cases} \quad (1)$$



# Fuzzy MFs: Trapezoidal

A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

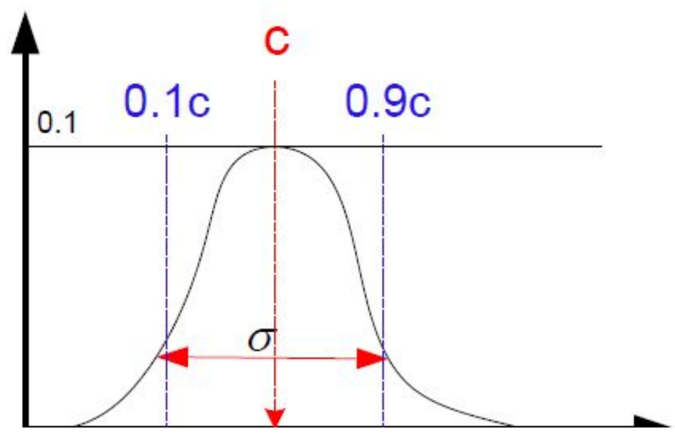
$$trapeziod(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases} \tag{2}$$



# Fuzzy MFs: Gaussian

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

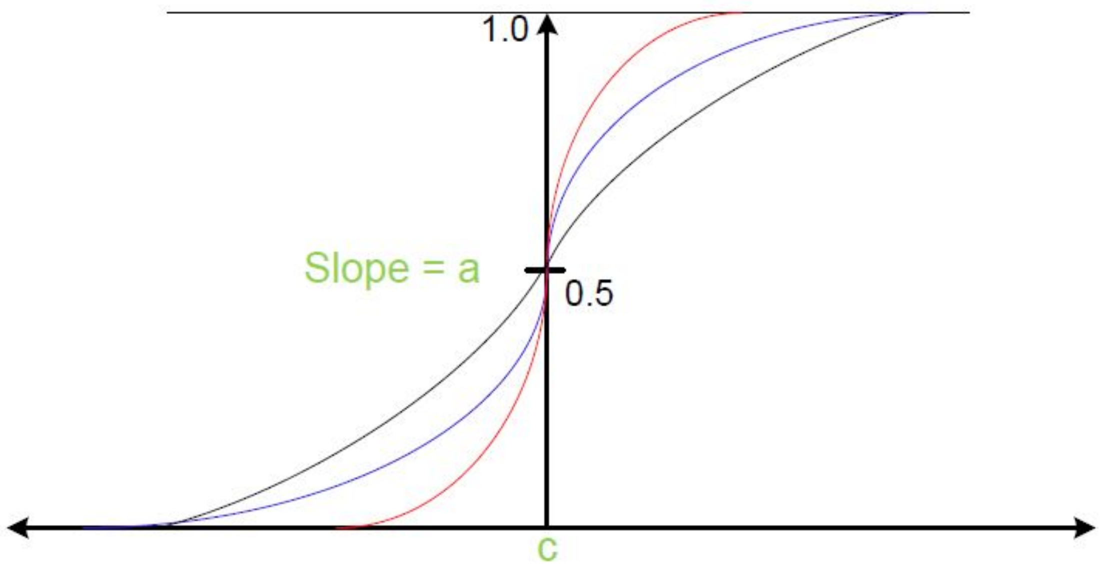
$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$



# Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$ ; where $c$ = crossover point and $a$ = slope at $c$;

$$\text{Sigmoid}(x;a,c) = \frac{1}{1+e^{-[\frac{a}{x-c}]}}$$



# Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks $\leq$ 90

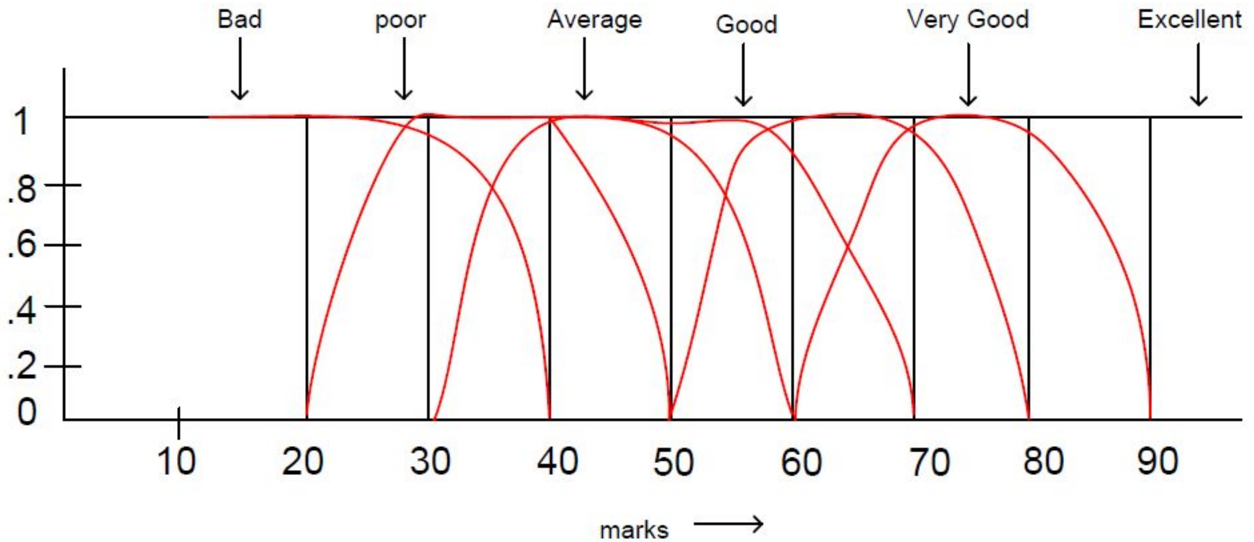
Very good = 75 $\leq$ Marks $\leq$ 90

Good = 60 $\leq$ Marks $\leq$ 75

Average = 50 $\leq$ Marks $\leq$ 60

Poor = 35 $\leq$ Marks $\leq$ 50

Bad= Marks $\leq$ 35



# Fuzzy Set Operations

- Some of the important set operations that can be carried out using fuzzy sets are:
  - Intersection of fuzzy sets
  - Union of fuzzy sets
  - Complement of fuzzy sets

- **Intersection of Fuzzy Sets**
  - Intersection of two fuzzy sets contains the elements that are common to both the sets and is also equivalent to logical AND operation.
  - In fuzzy logic, this operation is performed by taking the minimum of the truth membership functions.

$$\mu_A(x) \wedge \mu_B(x) = \min\{\mu_A(x), \mu_B(x)\}$$

- **Union of Fuzzy Sets**
  - Union of two fuzzy sets contains all the elements belonging to both the sets and is equivalent to logical OR operation.
  - In fuzzy logic, this operation is performed by taking the maximum of the truth membership functions.

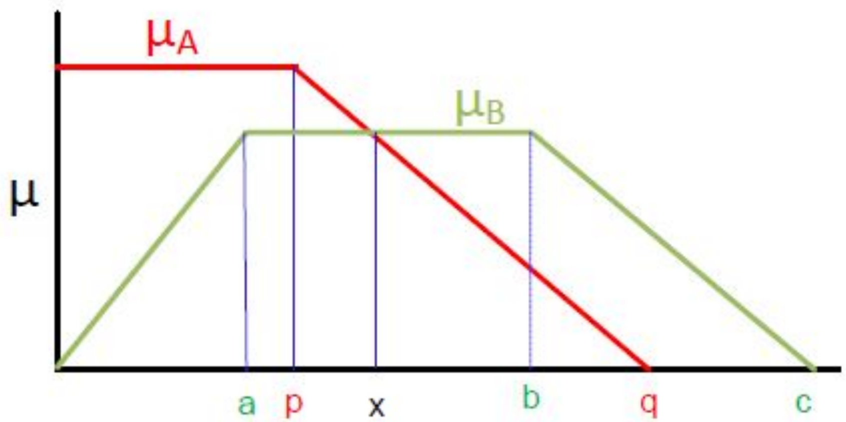
$$\mu_A(x) \vee \mu_B(x) = \max\{\mu_A(x), \mu_B(x)\}$$

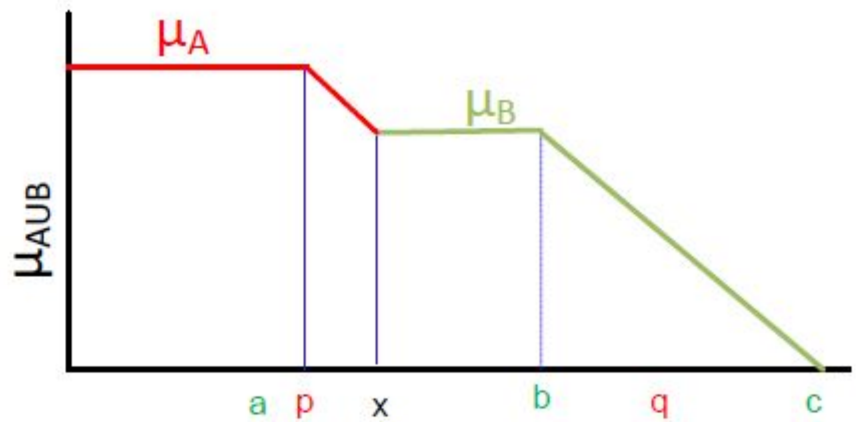
- **Complement of Fuzzy Sets**
  - The complement of a fuzzy set consists of the elements from universe which are not present in the set. This is equivalent to logical NOT operation.
  - In fuzzy logic, this operation is performed by subtracting the membership value from 1.

$$\mu_{\tilde{A}}(x) = 1 - \mu_A(x)$$

# Example: Union

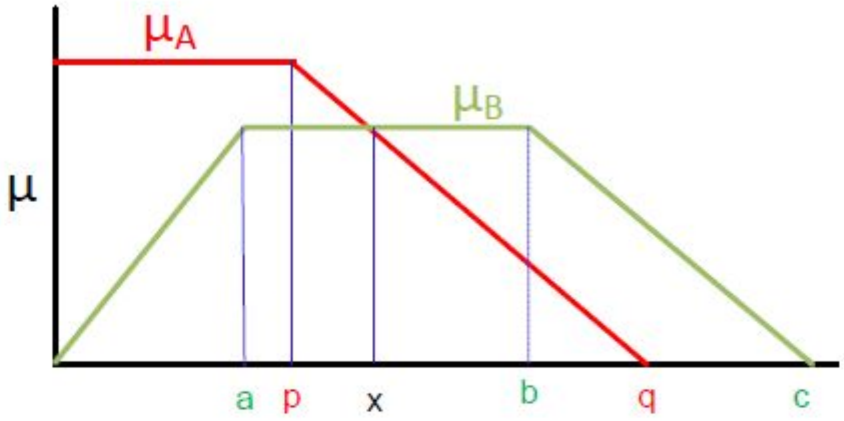
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$

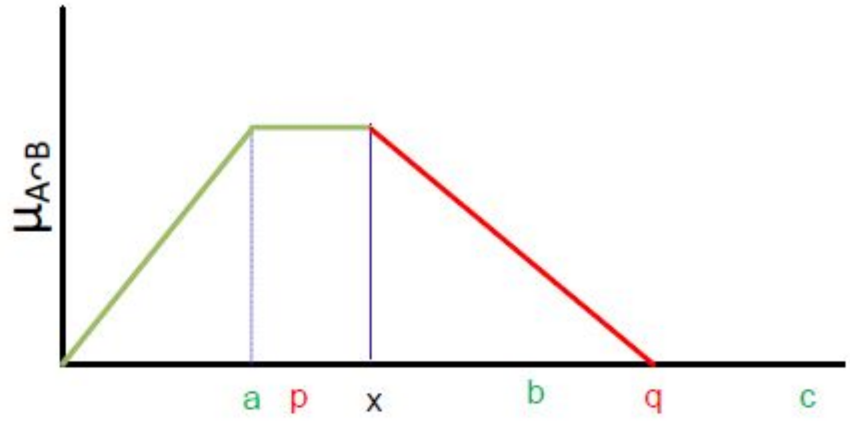


# Example: Intersection

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\} \text{ and}$$
$$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$$
$$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$$

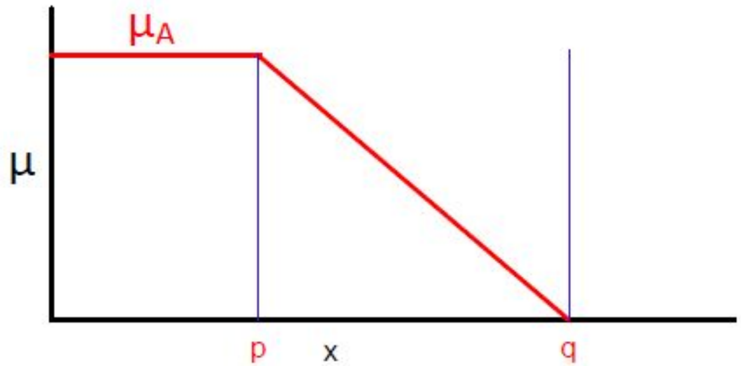


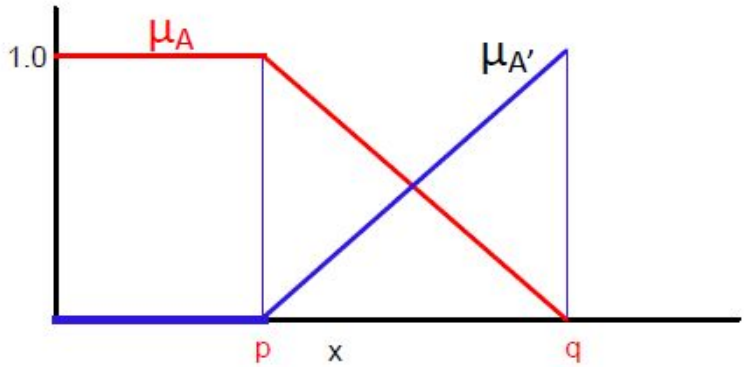



# Example: Complement

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$$

$$C = A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$$



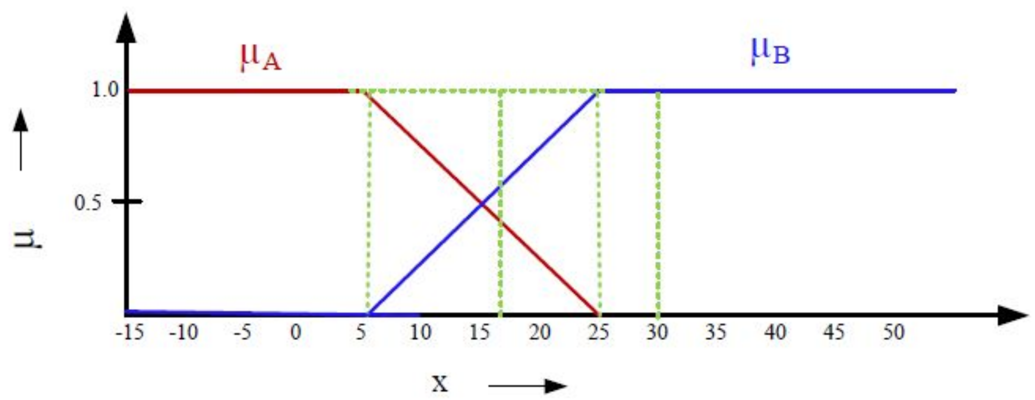



# A real-life example

Two fuzzy sets $A$ and $B$ with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

$A$ = **Cold climate** with $\mu_A(x)$ as the MF.

$B$ = **Hot climate** with $\mu_B(x)$ as the M.F.




Here, $X$ being the universe of discourse representing entire range of temperatures.

What are the fuzzy sets representing the following?

1. **Not cold climate**
2. **Not hold climate**
3. **Extreme climate**
4. **Pleasant climate**

Note: Note that "Not cold climate" $\neq$ "Hot climate" and vice-versa.

Answer would be the following.

1. **Not cold climate**

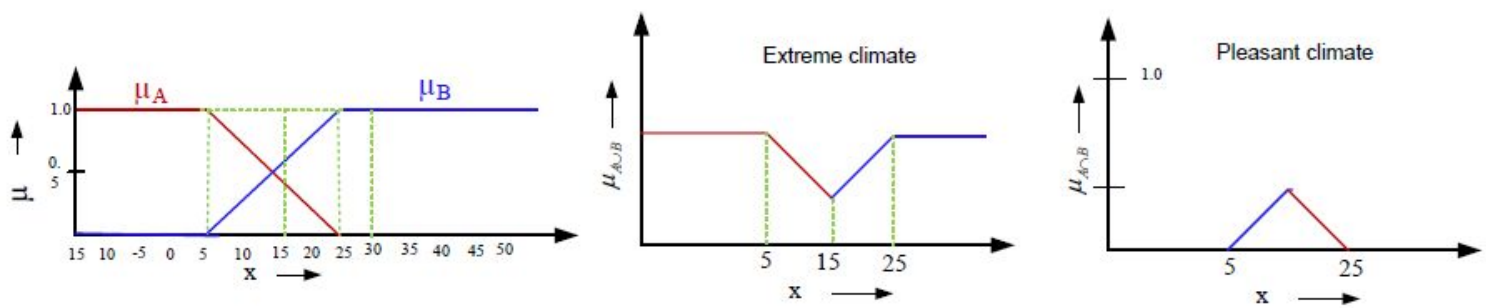
   $\overline{A}$ with $1 - \mu_A(x)$ as the MF.
2. **Not hot climate**

   $\overline{B}$ with $1 - \mu_B(x)$ as the MF.
3. **Extreme climate**

   $A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.
4. **Pleasant climate**

   $A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



# Important Terminologies in Fuzzy Set Operations

- **Empty Fuzzy Set:**
  - A fuzzy set A is said to be an empty set if it has no members and its membership function is zero everywhere in its universe of discourse U.

$$A \equiv \varnothing \text{ if } \mu_A(x) = 0, \ \forall_x \in U$$

- **Normal Fuzzy Set:**
- A fuzzy set A is said to be normal if it has a membership function that includes at least one singleton equal to unity in its universe of discourse U

$$\mu_A (x_o) = 1$$

- **Equal Fuzzy Sets**
  - If the membership functions of any two fuzzy sets A and B are equal everywhere in the universe of discourse, then the two fuzzy sets are said to be equal thereby satisfying the equation.

$$A \equiv B \text{ if } \mu_A (x) = \mu_B (x)$$

- **Fuzzy Set Support**
  - A fuzzy set A is supported only if the crisp set of all $x \in U$ such that the membership values are non-zero.

$$\mu_A(x) > 0$$

- **Fuzzy Set Support**
  - The product of the fuzzy set A and B produces a new fuzzy set, with its membership function value equal to the algebraic product of the membership function of A and B.

$$\mu_{A \cdot B}(x) \equiv \mu_A(x) . \mu_B(x)$$

- **Fuzzy Set Multiplication by a Crisp Number**
  - The membership function of a fuzzy set A is multiplied by the crisp number 'b' to obtain a new fuzzy set whose membership function $\mu_{bA}(x)$.

$$\mu_{bA}(x) \equiv b . \mu_A(x)$$

- **Power of a Fuzzy Set**
  - If $\alpha$ is set as the power of a fuzzy set A, then a new fuzzy set $A^{\alpha}$ has a membership function value

$$\mu_{A^{\alpha}}(x) \equiv [\mu_A(x)]^{\alpha}$$

# Properties of Fuzzy Sets

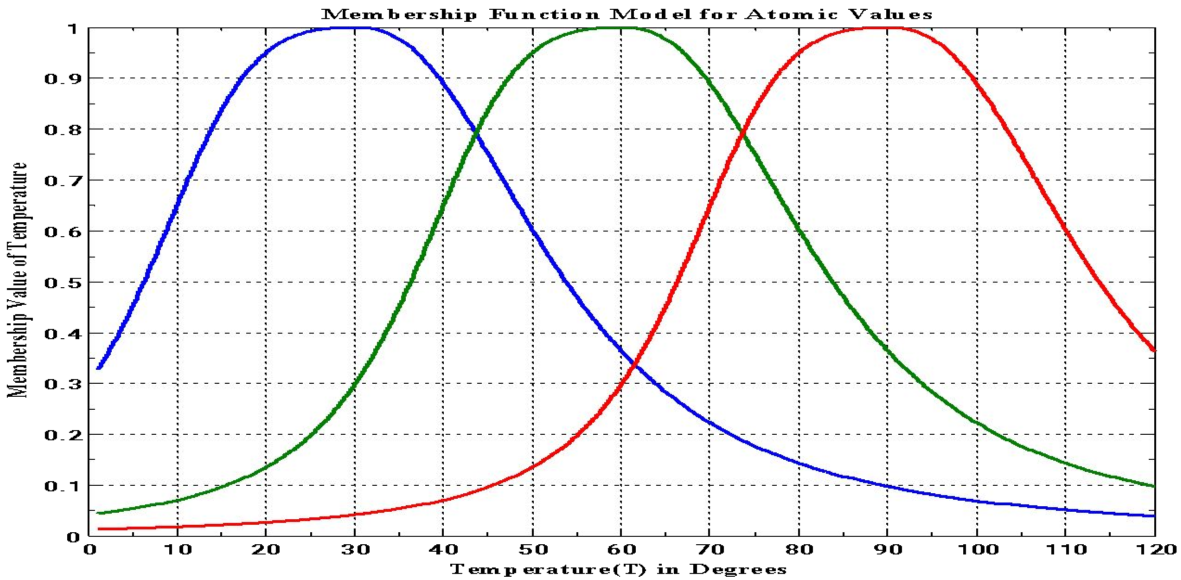
| | | |
|---|---|---|
| Double negation law | $\overline{(\overline{A})} = A$ | |
| Idempotency | $A \cup A = A$ | $A \cap A = A$ |
| Commutativity | $A \cup B = B \cup A$ | $A \cap B = B \cap A$ |
| Associative Property | $(A \cup B) \cup C = A \cup (B \cup C)$ | $(A \cap B) \cap C = A \cap (B \cap C)$ |
| Distributive Property | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ |
| Adsorption | $A \cap (A \cup B) = A$ | $A \cup (A \cap B) = A$ |
| De Morgan's laws | $\overline{\mathrm{A} \cap \mathrm{B}} = \overline{\mathrm{A}} \cup \overline{\mathrm{B}}$ | $\overline{\mathrm{A} \cup \mathrm{B}} = \overline{\mathrm{A}} \cap \overline{\mathrm{B}}$ |

| | | |
|---|---|---|
| Law of Contradiction | $A \cap \overline{B} \neq \varnothing$ | |
| Law of Excluded Middle | $A \cup \overline{A} \neq U$ | |
| Intersection of a Fuzzy Set with an Empty Set | $A \cap \varnothing = \varnothing$ | $\mu_A(x) \wedge 0 = 0$ |
| Union of a Fuzzy Set with an Empty Set | $A \cup \varnothing = A$ | $\mu_A(x) \vee 0 = \mu_A(x)$ |
| Intersection of a Fuzzy Set with Universe of Discourse | $A \cap U = A$ | $\mu_A(x) \wedge 1 = \mu_A(x)$ |
| Union of a Fuzzy Set with Universe of Discourse | $A \cup U = U$ | $\mu_A(x) \vee 1 = 1$ |

Some of the properties that are not valid for fuzzy sets but valid for crisp sets are

# Natural Language and Fuzzy Interpretations

- A collection of these primary terms will form phrases, of our natural language. Examples of some atomic terms are slow, medium, young, beautiful, angry, cold, temperature etc.
- The collection of atomic terms can form compound terms. Examples of compound terms are very cold, medium speed, young lady, fairly beautiful picture, etc .



# Linguistic Modifiers

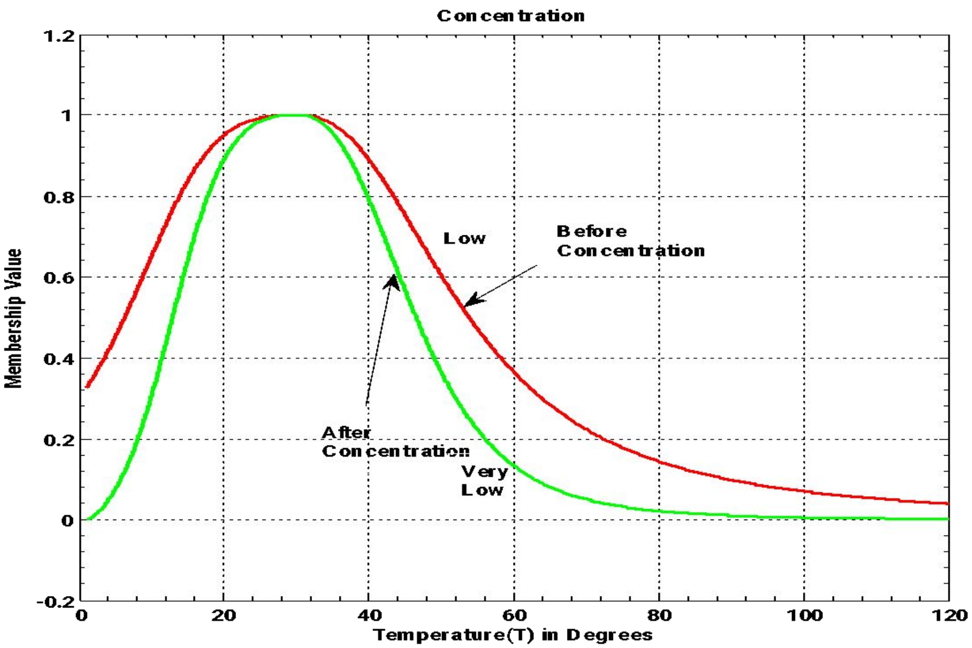
| Primary Terms with Modifiers | Modification | Membership Function |
|---|---|---|
| Very | $\delta^{2}$ | $\int_{I} \frac{[\mu_{\delta}(x)]^{2}}{x}$ |
| Very–_very | $\delta^{4}$ | $\int_{I} \frac{[\mu_{\delta}(x)]^{4}}{x}$ |
| More or less Slightly | $\delta^{\frac{1}{2}}$ | $\int_{I} \frac{[\mu_{\delta}(x)]^{\frac{1}{2}}}{x}$ |
| Plus | $\delta^{1.25}$ | $\int_{I} \frac{[\mu_{\delta}(x)]^{1.25}}{x}$ |
| Minus | $\delta^{0.75}$ | $\int_{I} \frac{[\mu_{\delta}(x)]^{0.75}}{x}$ |
| Over | $1-\delta,\ x \geq x_{max}$<br>$0,\quad x < x_{max}$ | $1 - \int_{I} \frac{\mu_{\delta}(x)}{x},\ x \geq x_{min}$<br>$0,\quad x > x_{min}$ |
| Under | $1-\delta,\ x \leq x_{min}$<br>$0,\quad x > x_{min}$ | $1 - \int_{I} \frac{\mu_{\delta}(x)}{x},\ x \leq x_{max}$<br>$0,\quad x < x_{max}$ |
| Indeed | $2\delta^{2},\quad 0 \leq \delta \leq 0.5$<br>$1-2[1-\delta]^{2},\quad 0.5 \leq \delta \leq 1.0$ | $2\int_{I}\left[\frac{\mu_{\delta}(x)}{x}\right]^{2},\quad 0 \leq \int_{I}\left[\frac{\mu_{\delta}(x)}{x}\right] \leq 0.5$<br>$1-2\left[1-\int_{I}\left[\frac{\mu_{\delta}(x)}{x}\right]\right]^{2},\ 0.5 \leq \int_{I}\left[\frac{\mu_{\delta}(x)}{x}\right] \leq 1.0$ |

# Linguistic Modifiers

- **Fuzzy Concentration**
  - Concentrations tend to concentrate or reduces the fuzziness of the elements in a fuzzy set by reducing the degree of membership of all elements that are only ‘‘partly’’ in the set.

$$\mu_A(\text{Low}, T) = \frac{1}{1 + 0.0005(T - 30)^{2.4}}$$

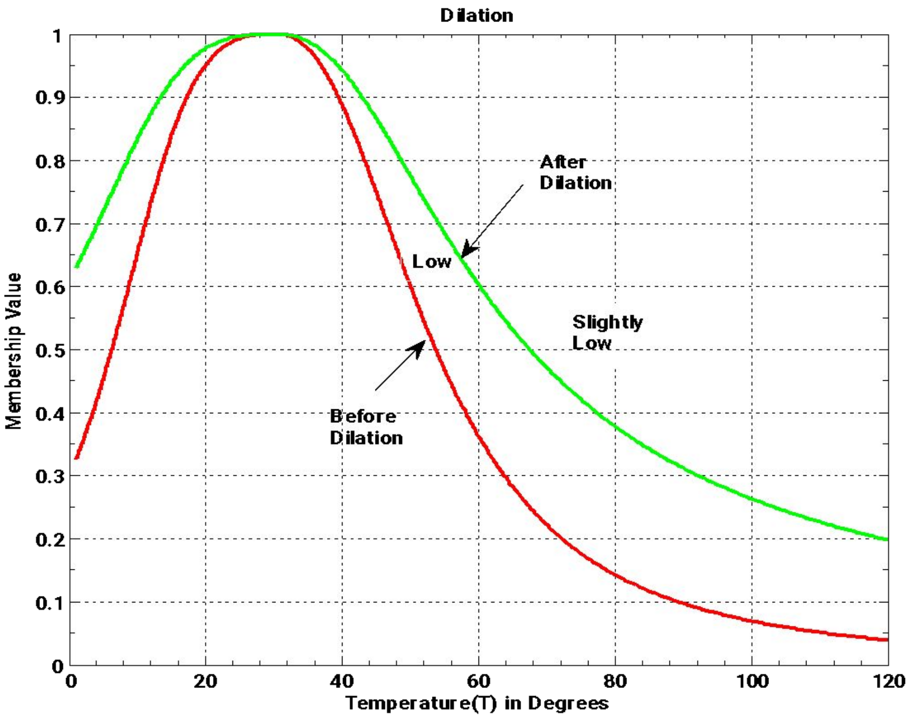
$$\mu_A(\text{Very Low}, T) = \left[\frac{1}{1 + 0.0005(T - 30)^{2.4}}\right]^2$$



- **Fuzzy Dilation**
  - Dilation tend to dilate or increase the fuzziness of the elements in a fuzzy set by increasing the degree of membership of all elements that are only ‘‘partly’’ in the set.

$$\mu_A(\text{Low}, T) = \frac{1}{1 + 0.0005(T - 30)^{2.4}}$$

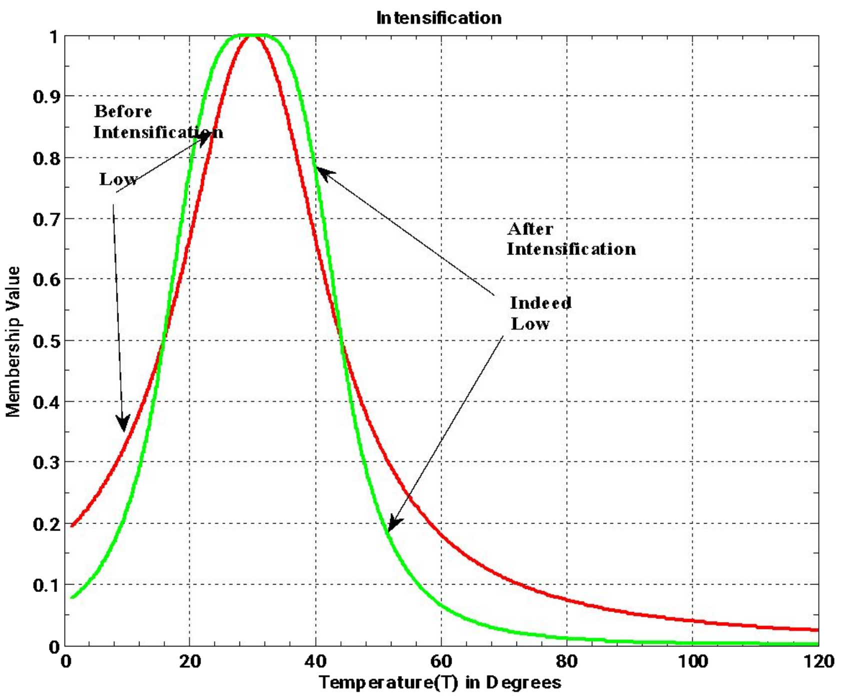
$$\mu_A(\text{Slightly Low}, T) = \left[\frac{1}{1 + 0.0005(T - 30)^{2.4}}\right]^{0.5}$$



- **Fuzzy Intensification:**
  - This property of the linguistic modifier is found to have a property which is a combination of both fuzzy concentration and fuzzy intensification.

$$\mu_A(\text{Low}, T) = \frac{1}{1 + 0.0005(T-30)^{2.4}}$$

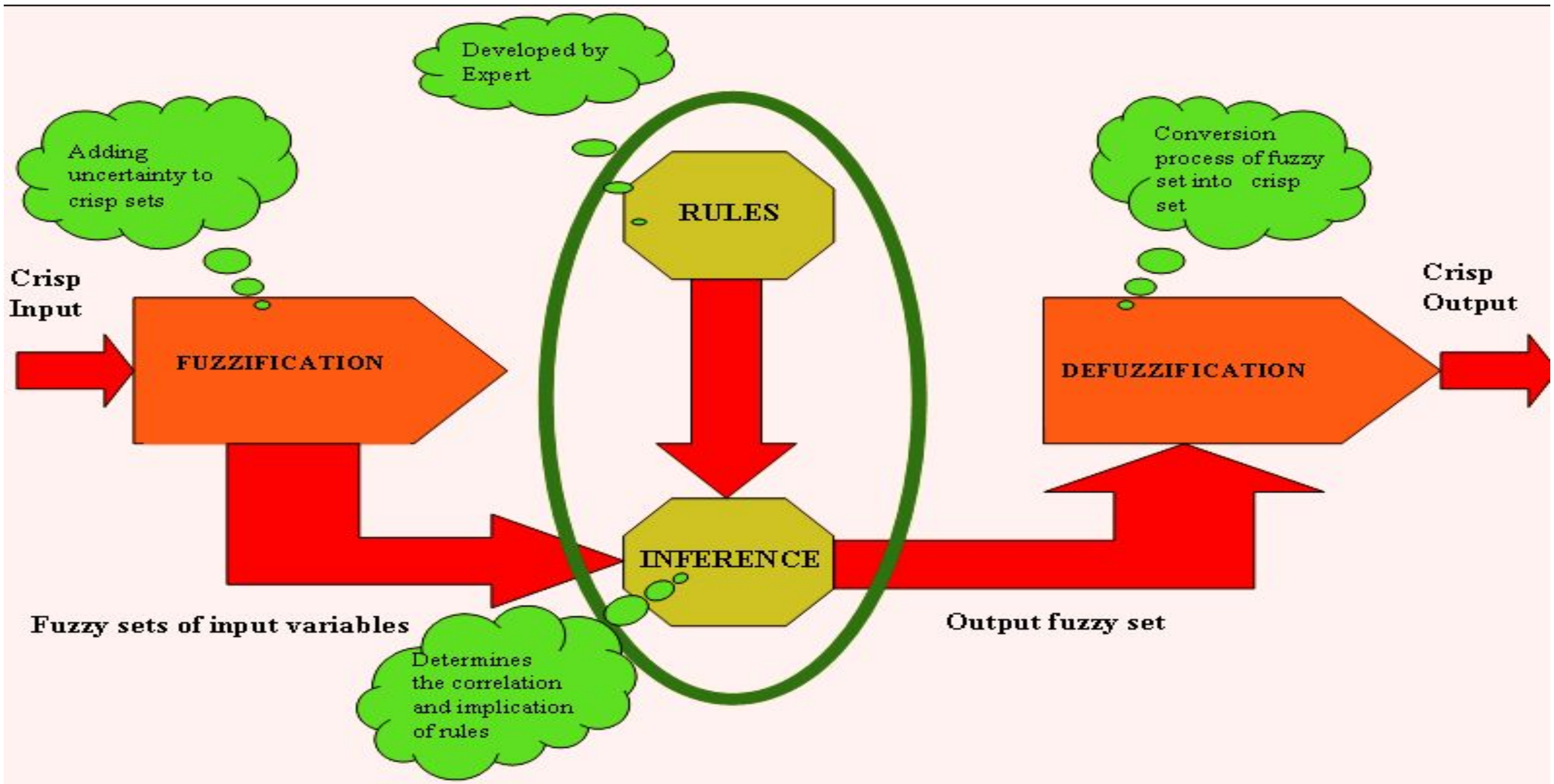
$$\mu_A(\text{Indeed Low}, T) = \begin{cases} 2 \times \left[\dfrac{1}{1+0.005(T-30)^{2.0}}\right]^2, & 0 \le \left[\dfrac{1}{1+0.005(T-30)^{2.0}}\right] \le 0.5 \\ 1 - 2 \times \left[\dfrac{1}{1+0.005(T-30)^{2.0}}\right]^2, & 0.5 \le \left[\dfrac{1}{1+0.005(T-30)^{2.0}}\right] \le 1.0 \end{cases}$$



# Logical Operations using Linguistic Modifiers

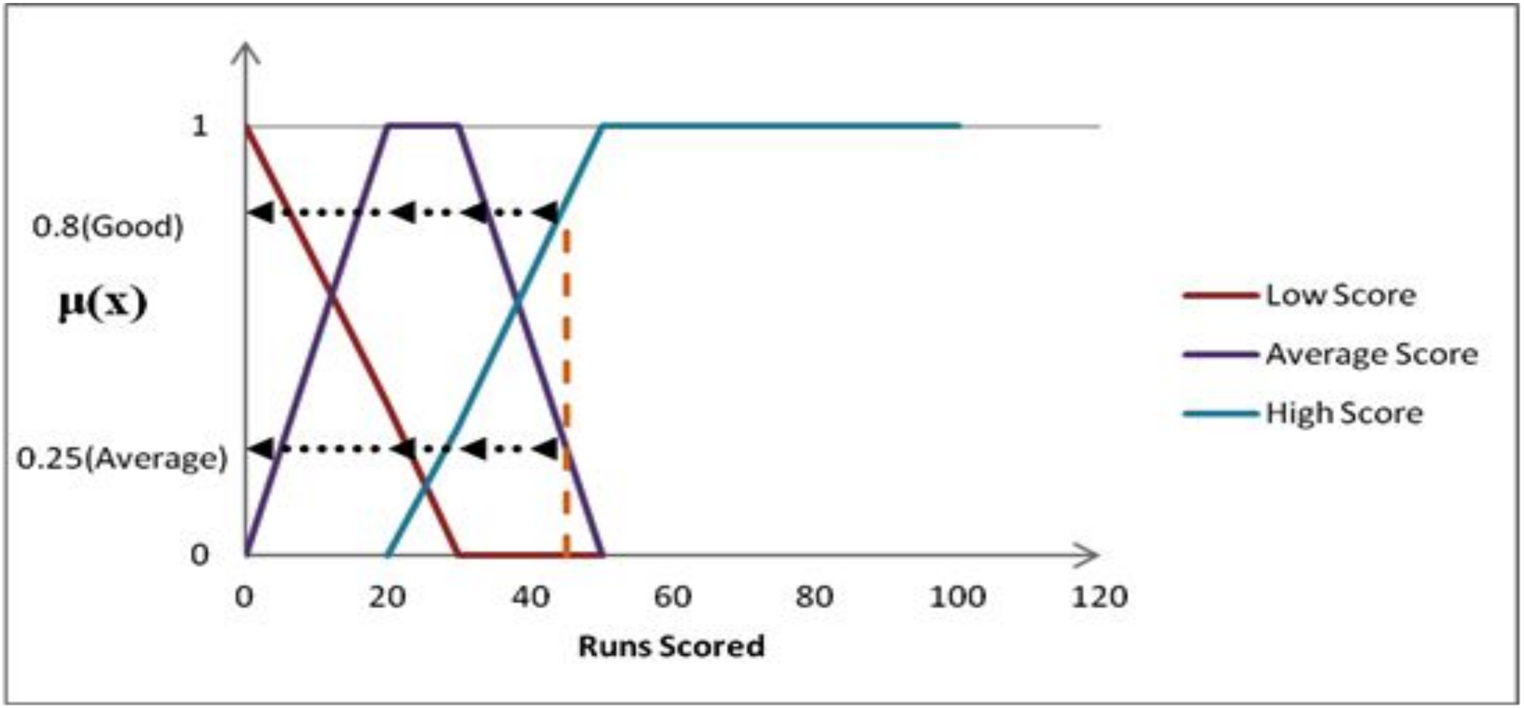
- It is possible to perform logical operations using the linguistic modifiers, however the order of precedence need to be followed to avoid confusion. This problem can be resolved by properly using the parentheses by the rule that use “association to the right”.
- The precedence of operation should be “NOT” in the First, “AND” in the second and “OR” in the third.
- For example, if we have a compound terms “plus very minus very low”, then following the rule “association to the right” while using the parentheses we get “Plus (very (minus (very (low))))”.

# Structure of Fuzzy Inference System



# Fuzzification

- The process of converting the crisp values to fuzzy variable is called 'fuzzification'.



# Fuzzy Connectives

- The linguistic variables are combined by various connectives such as negation, disjunction, conjunction and implication etc

| Symbol | Connective | Usage | Definition |
|---|---|---|---|
| - | Negation | $\overline{\mathrm{P}}$ | $1-T(P)$ |
| $\vee$ | Disjunction | $\mathrm{P} \vee \mathrm{Q}$ | $\max(\mathrm{T(P)}, \mathrm{T(PP)}$ |
| $\wedge$ | Conjunction | $\mathrm{P} \wedge \mathrm{Q}$ | $\min(\mathrm{T(P)}, \mathrm{T(Q)})$ |
| $\Longrightarrow$ | Implication | $\mathrm{P} \Rightarrow \mathrm{Q}$ | $\overline{\mathrm{P}} \vee \mathrm{Q} = \max(1-\mathrm{T(P)}, \mathrm{T(Q)})$ |

# Output of a fuzzy System

The output of a fuzzy system can be a single fuzzy set or union of two or more fuzzy sets.

To understand the second concept, let us consider a fuzzy system with $n$-rules.

$R_1$: If $x$ is $A_1$ then $y$ is $B_1$
$R_2$: If $x$ is $A_2$ then $y$ is $B_2$
.........................................
.........................................
$R_n$: If $x$ is $A_n$ then $y$ is $B_n$

In this case, the output $y$ for a given input $x = x_1$ is possibly $B = B_1 \cup B_2 \cup .....B_n$

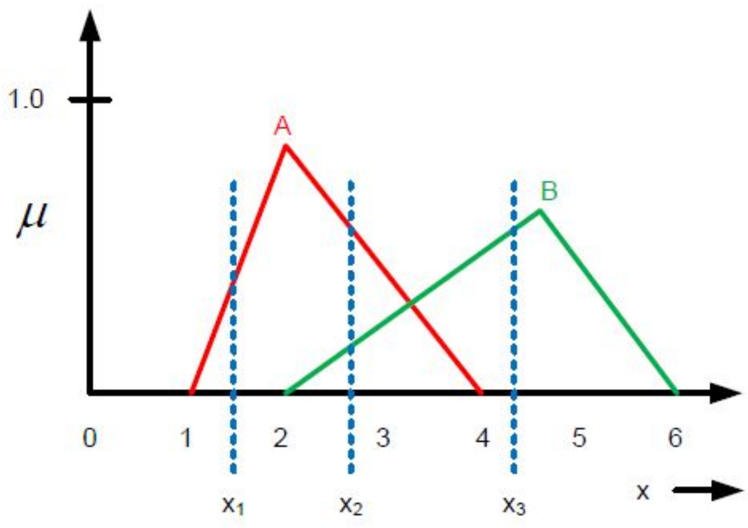
# Output of a fuzzy System

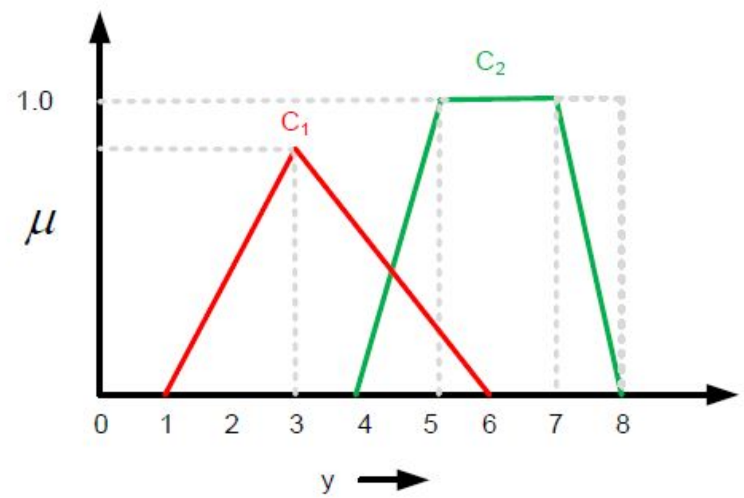
Suppose, two rules $R_1$ and $R_2$ are given as follows:

1. $R_1$: If $x$ is $A_1$ then $y$ is $C_1$
2. $R_2$: If $x$ is $A_2$ then $y$ is $C_2$

Here, the output fuzzy set $C = C_1 \cup C_2$.

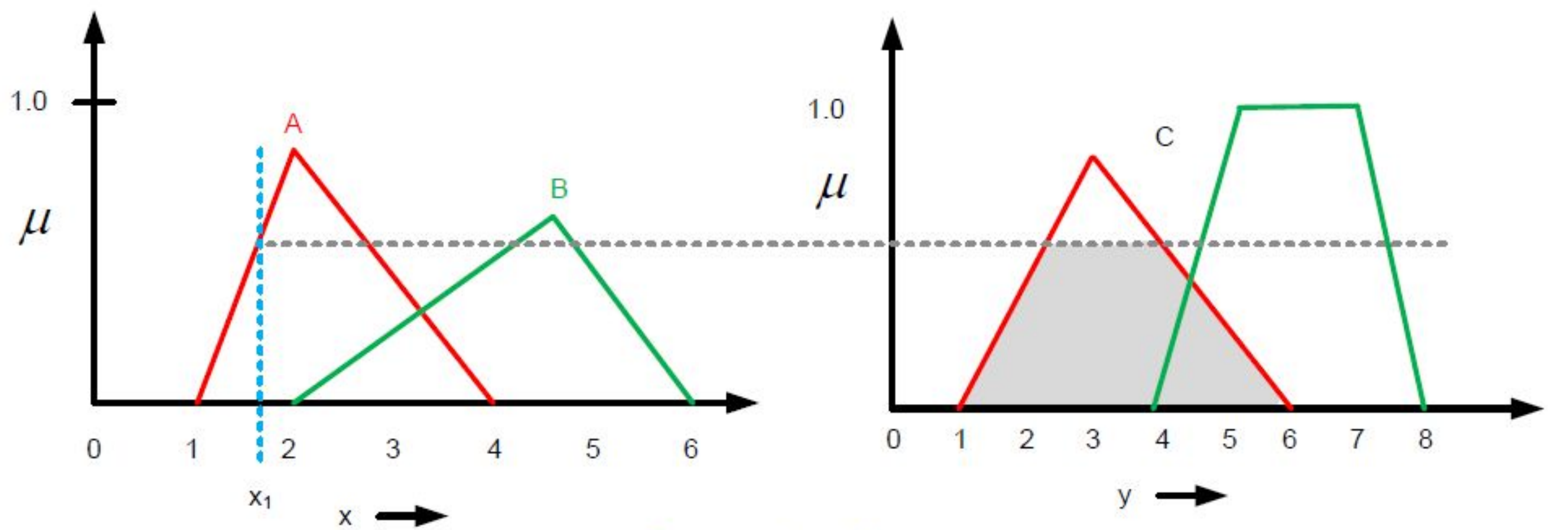
For instance, let us consider the following:



# Output fuzzy set : Illustration

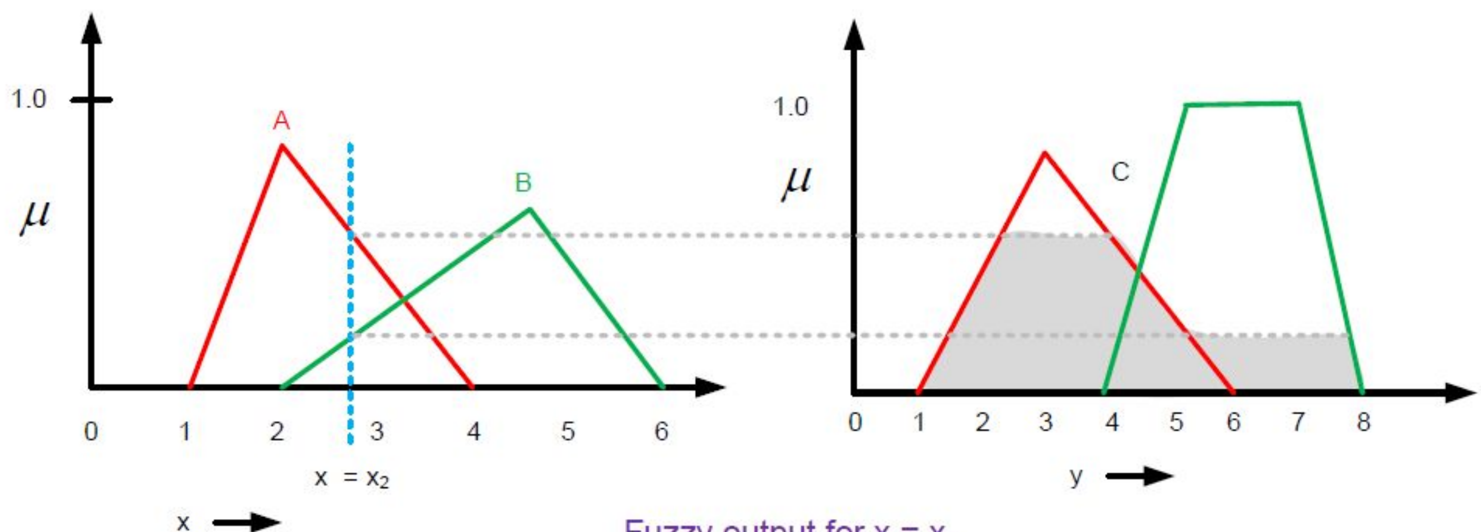
The fuzzy output for $x = x_1$ is shown below.




Fuzzy output for x = $x_1$

# Output fuzzy set : Illustration

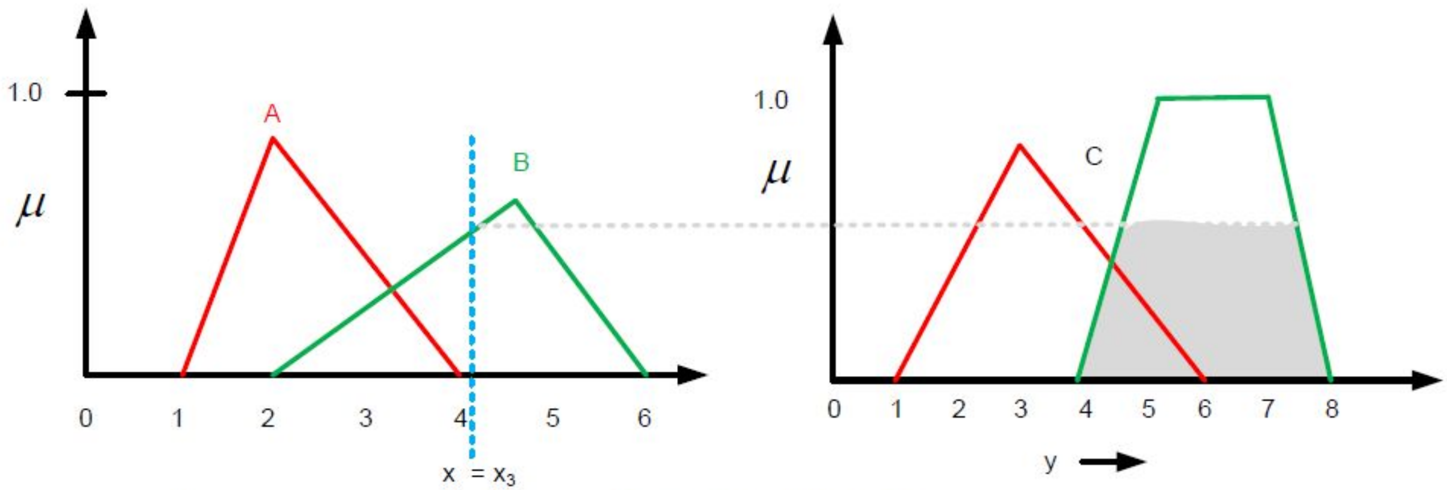
The fuzzy output for $x = x_2$ is shown below.




Fuzzy output for x = $x_2$

# Output fuzzy set : Illustration

The fuzzy output for $x = x_3$ is shown below.

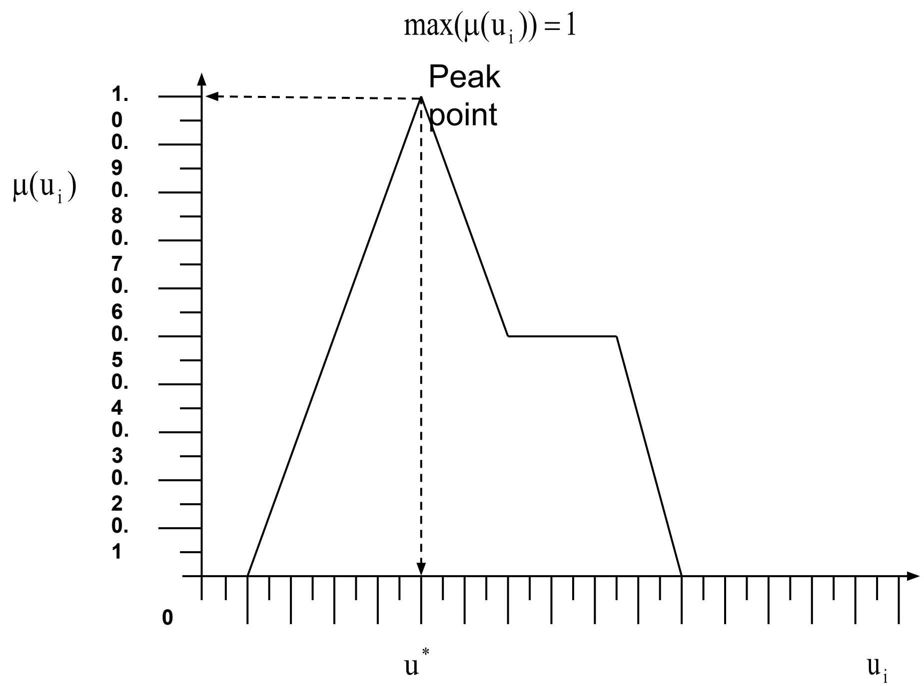



Fuzzy output for $x = x_3$

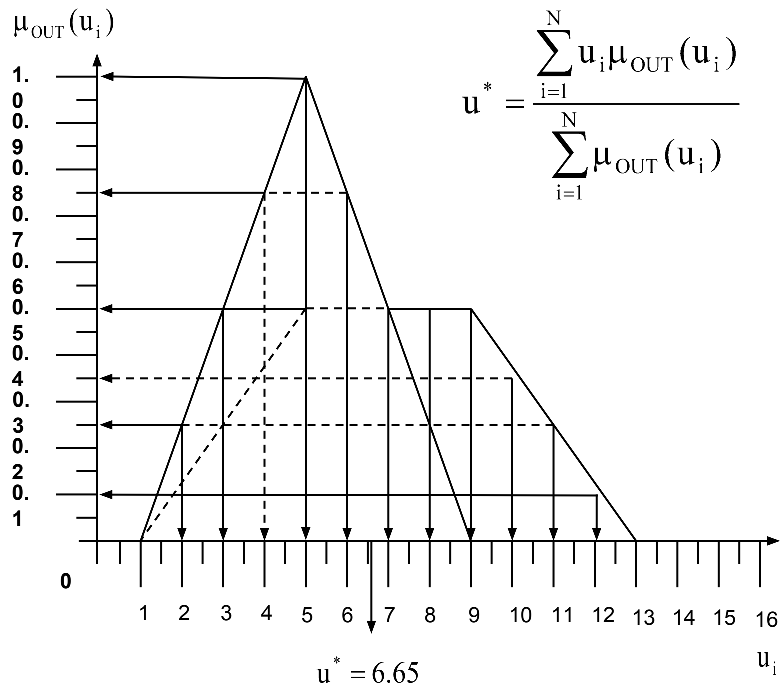
# Defuzzification

- Defuzzification is the process of converting the fuzzy set into a crisp value.
  - Max-membership (Height method)
  - Centre of Area (COA) or Center of Gravity or Centroid Method
  - Weighted Average Method
  - Mean of Maxima (Middle of maxima)
  - Centre of Sums
  - Centre of Largest Area (COLA)
  - First of Maxima
  - Last of Maxima

# Max-membership (Height method)

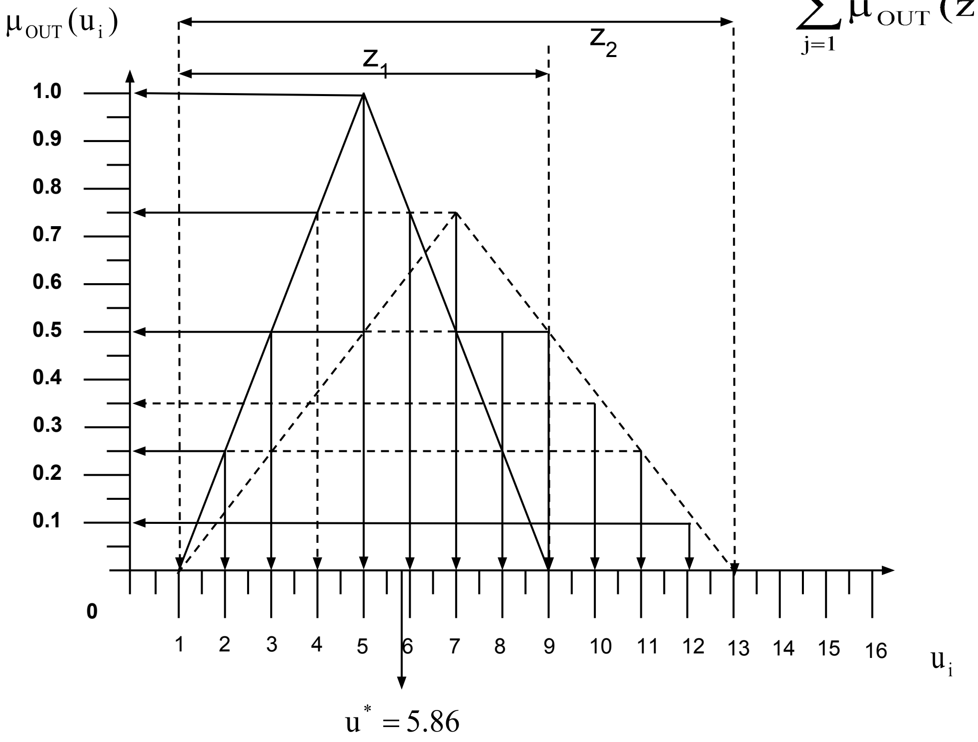


# Centre of Area (COA) Defuzzification



# Weighted Average Method

$$u^* = \frac{\sum_{j=1}^{Z} \bar{z}_j \mu_{OUT}(\bar{z}_j)}{\sum_{j=1}^{Z} \mu_{OUT}(\bar{z}_j)}$$




**(d) Mean of Maxima (MOM)**

$$u^* = \frac{1}{M} \sum_{m=1}^{M} u_m$$

**(e) Centre of Sums (COS)**

$$u^* = \frac{\sum_{i=1}^{N} u_i \sum_{k=1}^{n} \mu_{C'_k}(u_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{C'_k}(u_i)}$$

# MoM : Example 1

Suppose, a fuzzy set **Young** is defined as follows:

Young = $\{$(15,0.5), (20,0.8), (25,0.8), (30,0.5), (35,0.3) $\}$

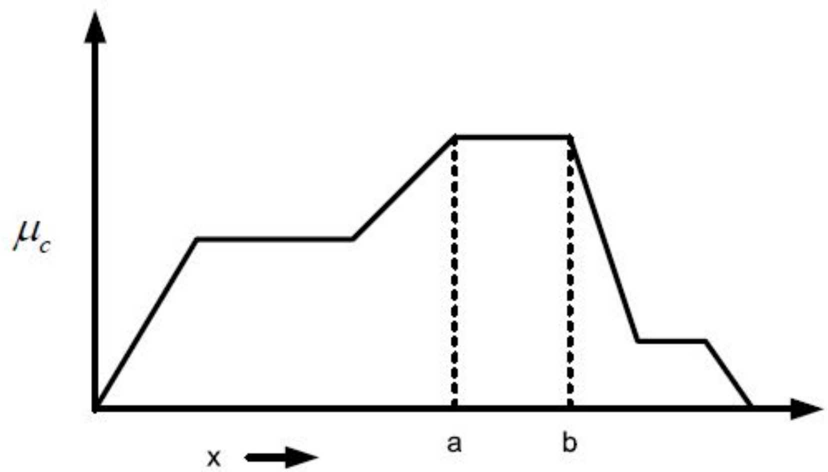
Then the crisp value of **Young** using MoM method is

$x^* = \frac{20+25}{2} = 22.5$

Thus, a person of 22.5 years old is treated as young!

# MoM : Example 2

What is the crisp value of the fuzzy set using MoM in the following case?



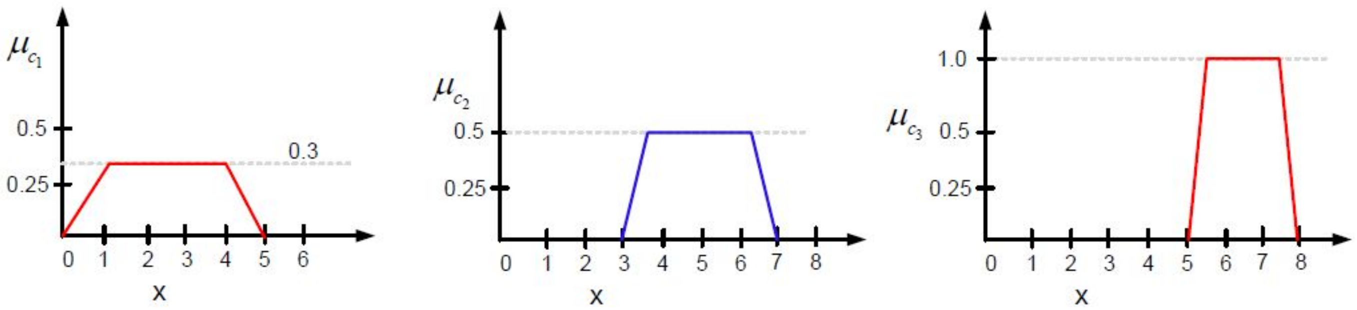

$$X^* = \frac{a+b}{2}$$

**Note:**

- Thus, MoM is also synonymous to middle of maxima.

# COS Example

Consider the three output fuzzy sets as shown in the following plots:




In this case, we have

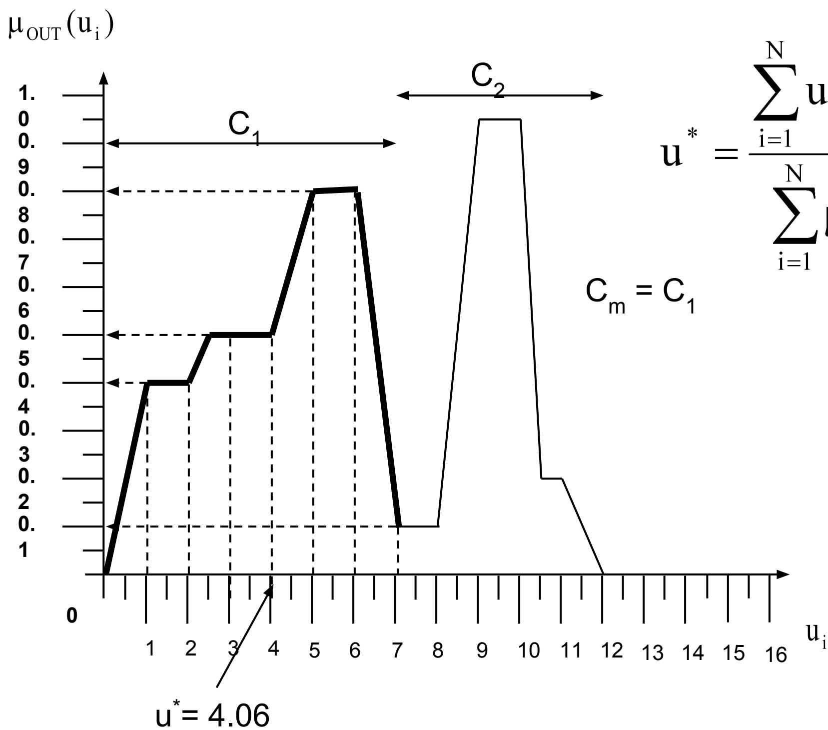
$$A_{c_1} = \frac{1}{2} \times 0.3 \times (3+5), x_1 = 2.5$$

$$A_{c_2} = \frac{1}{2} \times 0.5 \times (4+2), x_2 = 5$$

$$A_{c_3} = \frac{1}{2} \times 1 \times (3+1), x_3 = 6.5$$

Thus, $x^* = \frac{\frac{1}{2}\times 0.3\times(3+5)\times 2.5+\frac{1}{2}\times 0.5\times(4+2)\times 5+\frac{1}{2}\times 1\times(3+1)\times 6.5}{\frac{1}{2}\times 0.3\times(3+5+\frac{1}{2}\times 0.5\times(4+2)+\frac{1}{2}\times 1\times(3+1)} = 5.00$

## (f) Centre of Largest Area (COLA)




$$u^* = \frac{\sum_{i=1}^{N} u_i \mu_{C_m}(u_i)}{\sum_{i=1}^{N} \mu_{C_m}(u_i)}$$

$C_m = C_1$

## (g) First of Maxima

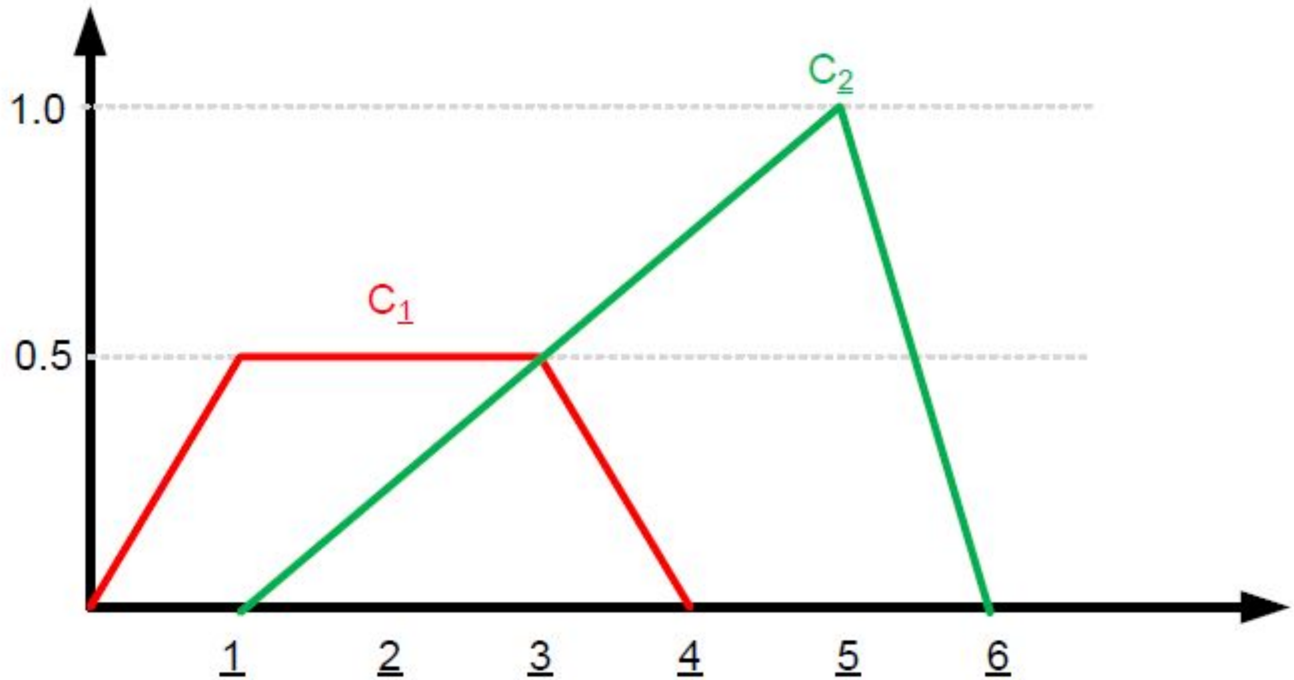
$$\max(\mu_{C_m}(u)) \geq \max(\mu_{C_i}(u)), i = 1 \ldots N_C$$

$$u^* = \min(u_i)$$

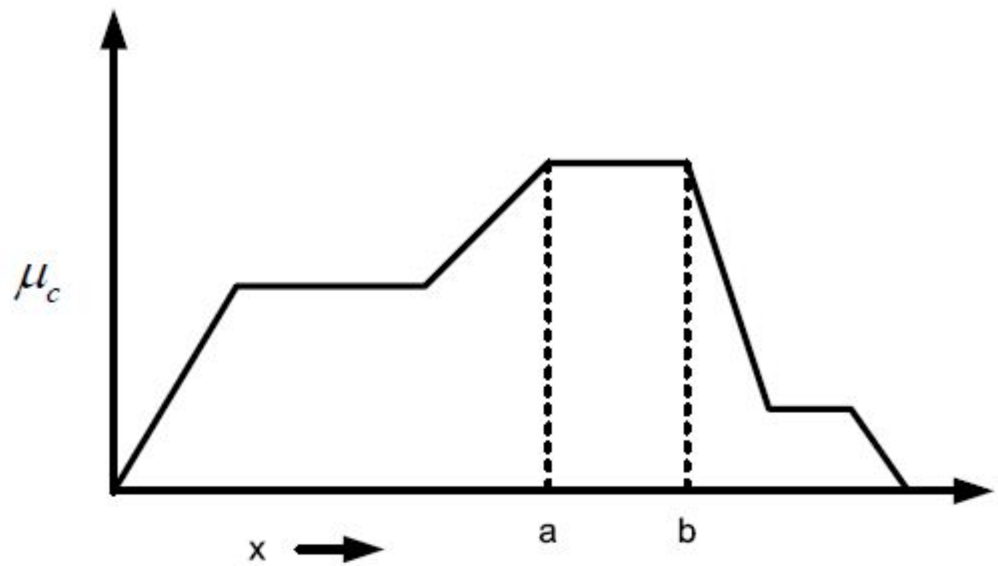
## (h) Last of Maxima

$$u^* = \max(u_i)$$

# Find the crisp value using first of maxima



# Find the crisp value using last of maxima



# Thanks