Tuesday, October 06, 2020 9:42 AM

Rolle's Theorem

If a function $f(x)$ is defined in $[a, b]$ is such that

(1) $f(x)$ is continuous in $[a, b]$

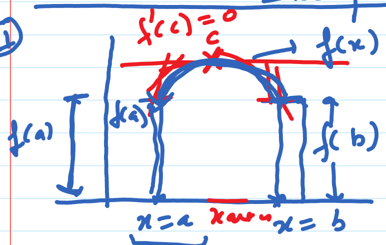
(2) $f(x)$ is derivable in (a, b)

(3) $f(a) = f(b)$

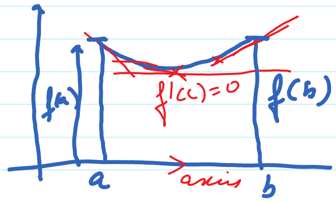
then there exists at least one point (one real nr) $c \in (a, b)$ s.t $f'(c) = 0$

Geometrical Interpretation

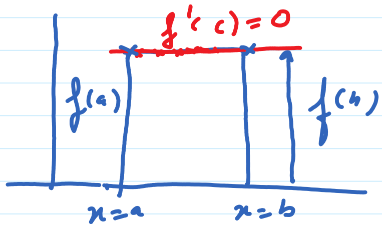
(1)

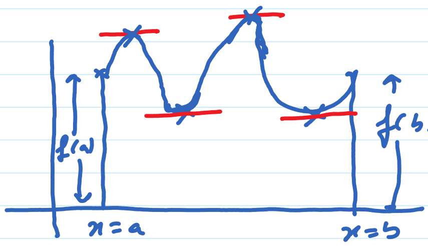


$\boxed{f(a) = f(b)}$



(2)





$$y = f(x)$$
$$\frac{dy}{dx} = f'(x)$$

$$\frac{dy}{dx} = \tan\theta$$
$$\tan\theta = 0 \Rightarrow \theta = 0$$
$$\frac{dy}{dx} = 0 \quad \text{or} \quad f'(x) = 0$$
$$f'(c) = 0$$
$$\underline{a < c < b}$$

Q Verify Rolle's Theorem for the function
$$f(x) = \frac{8x^2}{3} - 2x, \quad x \in \left[0, \frac{3}{4}\right]$$

Sol
$$f(x) = \frac{8x^2}{3} - 2x$$

(I) Since $f(x)$ is a polynomial

$\therefore f(x)$ is cts in $\left[0, \frac{3}{4}\right]$

(II) $f'(x) = \frac{16x}{3} - 2$, which exists in $\left(0, \frac{3}{4}\right)$

$\therefore f(x)$ is derivable in $\left(0, \frac{3}{4}\right)$

(III)
$$f(0) = 0 - 0 = 0$$
$$f\left(\tfrac{3}{4}\right) = \frac{\cancel{8}}{\cancel{3}} \times \frac{\cancel{9}^{3}}{\cancel{16}_{2}} - \cancel{2} \times \frac{3}{\cancel{4}_{2}} = 0$$

$\therefore f(x)$ satisfies all the conditions of Rolle's Theorem

$\therefore \exists$ at least one ~~rate~~ real nr $c \in (a, b)$ s.t
$$f'(c) = 0$$

$\therefore \ \frac{16}{3}c - 2 = 0 \Rightarrow c = \frac{3}{8} \in \left(0, \frac{3}{4}\right)$

$\therefore$ Rolle's Theorem is verified

Q Discuss the applicability of Rolle's Theorem

$$f(x) = \sin^4 x + \cos^4 x \quad \text{in } \left[0, \pi/2\right]$$

Sol

$$f(x) = \sin^4 x + \cos^4 x$$
$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$$
$$= 1 - \frac{1}{2}(2\sin x \cos x)^2$$

$$f(x) = \underbrace{1}_{} - \underbrace{\frac{1}{2}\underbrace{\sin^2 2x}_{}}_{} \quad \text{in } \left[0, \frac{\pi}{2}\right]$$

① $\sin^2 x$ & constant fxn are all continuous in $[0, \pi/2]$

$\therefore f(x)$ is continuous in $[0, \pi/2]$

② $f'(x) = 0 - \frac{1}{2} \times 2\sin 2x \times 2\cos 2x$

$= -2\sin 2x \cos 2x$

$= -\sin 4x$, which exists in $(0, \pi/2)$

$\therefore f(x)$ is derivable in $(0, \pi/2)$

③ $f(0) = 1 - \frac{1}{2} \times 0 = 1$

$f(\pi/2) = 1 - \frac{1}{2}\sin^2 2 \times \frac{\pi}{2} = 1$

$\therefore f(0) = f(\pi/2)$

$\therefore$ All the conditions of Rolle's Theorem are satisfied

$\therefore \exists$ at least one $c \in \underbrace{(0, \pi/2)}_{}$ s.t. $f'(c) = 0$

$\Rightarrow -\sin 4c = 0$

or $\sin 4c = 0 = \sin n\pi$

$\Rightarrow 4c = 0, \pi, 2\pi, 3\pi, 4\pi, -----$

$\Rightarrow c = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3}{4}\pi, \pi, .----$

Now $c = \frac{\pi}{4} \in (0, \pi/2)$

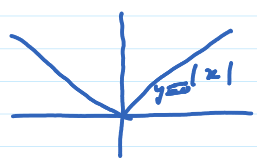
$\therefore$ Rolle's Theorem is verified

Q Discuss the applicability of Rolle's Theorem to $f(x) = |x|$ in the interval $[-1, 1]$

Sol

$f(x)$ is not diff at $x = 0 \in (-1, 1)$

$\therefore$ Rolle's Theorem is not applicable



$\rightarrow f(x) = |x| \rightarrow x = 0$

L.H.D $= Lt \quad \frac{f(x) - f(0)}{} = Lt \quad \frac{|x| - 0}{}$

$$L.H.D = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x| - 0}{x}$$

$$= \lim_{h \to 0} \frac{|0-h|}{0-h} = -1$$

$$R.H.D = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = +1$$

$L.H.D \neq R.H.S$

$\therefore f(x) = |x|$ is not derivable at $x = 0$.

Q It is given that for the function f given by $f(x) = x^3 + bx^2 + ax$, $x \in [1, 3]$, Rolle's Theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the value of a & b

Sol

$$f(x) = x^3 + bx^2 + ax$$

$$f'(x) = 3x^2 + 2bx + a$$

$$\therefore f'(c) = 3c^2 + 2bc + a$$

Now Rolle's Theorem holds $\therefore f'(c) = 0$

i.e. $3c^2 + 2bc + a = 0$

$$c = \frac{-2b \pm \sqrt{4b^2 - 12a}}{6}$$

$$= \frac{-2b \pm 2\sqrt{b^2 - 3a}}{6}$$

$$= -\frac{b}{3} \pm \frac{\sqrt{b^2 - 3a}}{3}$$

Here $2 + \frac{1}{\sqrt{3}} = -\frac{b}{3} + \frac{\sqrt{b^2 - 3a}}{3}$ $\left[\because c = 2 + \frac{1}{\sqrt{3}}\right]$

$\Rightarrow -\frac{b}{3} = 2 \Rightarrow \boxed{b = -6}$

and $\frac{\sqrt{b^2 - 3a}}{3} = \frac{1}{\sqrt{3}}$

$\Rightarrow \boxed{a = 11}$

Lagrange's Mean Value Theorem

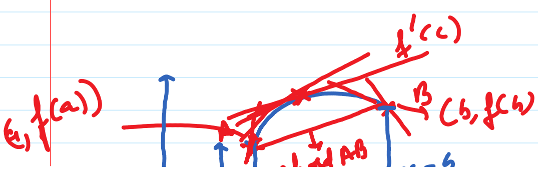
If $f(x)$ is

① Continuous in $[a, b]$

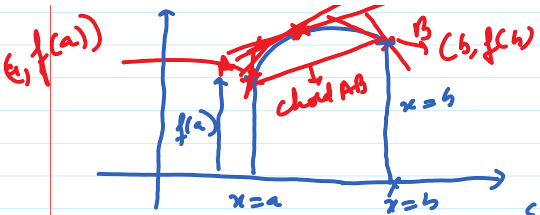
② differentiable in (a, b)

then $\exists$ at least one real number $c \in (a, b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

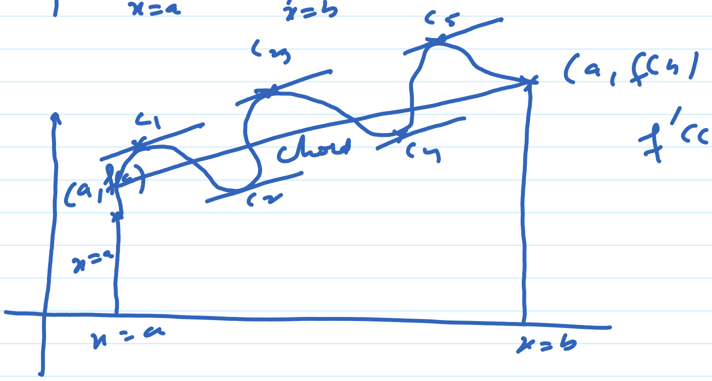


slope of chord AB $= \frac{f(b) - f(a)}{b - a}$



slope of chord AB $= \dfrac{f(b)-f(a)}{b-a}$

$$f'(c) = \frac{f(b)-f(c)}{b-a}$$



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

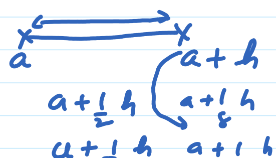
<u>Second form</u> $f(x)$ on $[a, b]$

If $b = a+h$, then since $a < c < b$,

✓ $c = \underline{a + \theta h}$, $\underline{0 < \theta < 1}$

$f(x)$ on $[a, a+h]$
$f(x)$ is diff on $(a, a+h)$

$\therefore$ (i) $f(x)$ is cts on $[a, a+h]$
(ii) $f(x)$ diff on $(a, a+h)$



$\therefore$ $\exists$ at least one number θ $(0 < \theta < 1)$

s.t $$f'(a+\theta h) = \frac{f(a+h) - f(a)}{\cancel{a}+h-\cancel{a}}$$

$$\Rightarrow h f'(a+\theta h) = f(a+h) - f(a)$$

or $$\boxed{f(a+h) = f(a) + h f'(a+\theta h)}$$

Q Examine the applicability of L.M.V in interval $[1, 4]$ of the function $f(x) = x^2 - 4x - 3$.

Sol ① $f(x)$ is cts in $[1, 4]$
② $f(x)$ is diff in $(1, 4)$

$f(x)$ satisfies all the conditions of L.M.V
$\therefore$ $\exists$ at least one $c \in (1, 4)$ s.t

$$f'(c) = \frac{f(4) - f(1)}{3}$$

$f'(x) = 2x - 4$

or $$2c - 4 = \frac{-3+6}{3}$$

$$2c - 4 = 1$$

or $c = \frac{5}{2} \in (1, 4)$

$\therefore$ LMV Theorem is verified.

Q Discuss the applicability of LMV Theorem for the function $f(x) = |x|$ on $[-1, 1]$.

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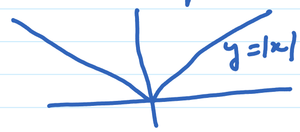


Sol ① $f(x)$ is cts $[-1,1]$

② $L.H.D \longrightarrow -1$

$RHD \longrightarrow +1$] Incorrect

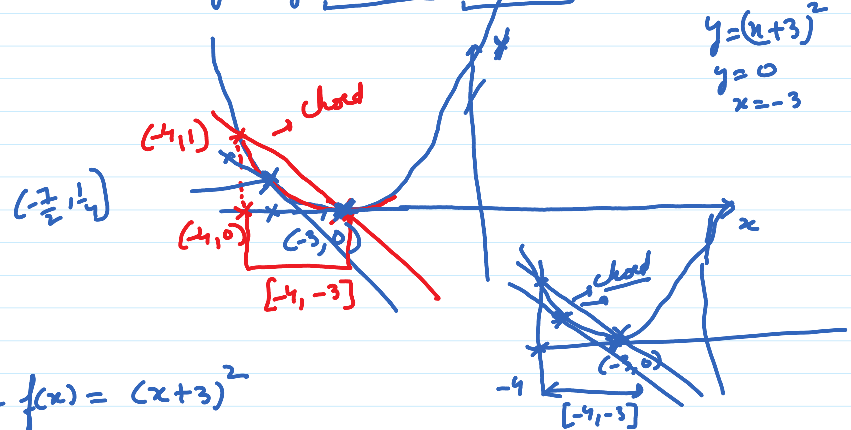
Here $f(x)$ is not diff at $x=0$

$\therefore$ LMV Theorem is not applicable

Q Find a point on the parabola $y=(x+3)^2$ where the tangent is parallel to the chord joining $(-3,0)$ & $(-4,1)$.

Sol $f(x)=(x+3)^2$

$y=(x+3)^2$
$y=0$
$x=-3$



Here $y=f(x)=(x+3)^2$

or $f(x)=x^2+6x+9$ —①

Here $f(x)=x^2+6x+9$ be defined on the interval $[-4,-3]$

① $f(x)$ is cts on $[-4,-3]$

② $f'(x)=2x+6$, which exists in $(-4,-3)$

$f(x)$ is derivable in $(-4,-3)$

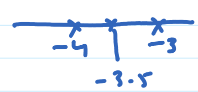
$\therefore$ All the conditions of LMV Theorem are satisfied

$\therefore \exists$ at least one $x\in(-4,-3)$ s.t

$f(x)=(x+3)^2$

$$f'(x)=\frac{f(-3)-f(-4)}{-3-(-4)}$$

$$2x+6=\frac{0-1}{-3+4}$$



$\therefore 2x+6=-1$

or $2x=-7 \Rightarrow x=-\frac{7}{2}\in(-4,-3)$

$\therefore$ when $x=-\frac{7}{2}$, $y=\left(-\frac{7}{2}+3\right)^2=\frac{1}{4}$

$\therefore$ required point on the parabola is $\left(-\frac{7}{2},\frac{1}{4}\right)$

Q In the Mean value Theorem $f(b)-f(a)=(b-a)f'(c) \rightarrow$ Determine c lying between a & b if $f(x)=x(x-1)(x-2)$, $a=0$ & $b=1/2$

Sol $f(x)=x(x-1)(x-2)$

$f(x) = x^3 - 3x^2 + 2x$

$f'(x) = \underline{3x^2 - 6x + 2}$

It is given that $f(b) - f(a) = (b-a) f'(c)$

$f(\tfrac{1}{2}) - f(0) = (\tfrac{1}{2} - 0)[3c^2 - 6c + 2]$

$\Rightarrow \frac{3}{8} - 0 = \frac{1}{2}(3c^2 - 6c + 2)$

or $12c^2 - 24c + 5 = 0$ #

on solving $c = 1.764,\ 0.236$

Hence $c = 0.236 \in (0, \tfrac{1}{2})$

Q. Prove that (if $0 < a < b < 1$); $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$

Hence deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}. \ \#$$

Sol. let $f(x) = \tan^{-1} x$ be defined in $[a, b]$

(1) $f(x)$ is cts in $[a, b]$

(2) $f'(x) = \frac{1}{1+x^2}$ exists in (a, b)

By Mean value Theorem $f'(c) = \frac{f(b) - f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}, \quad a < c < b \ \text{—(1)}$$

Here $a < c < b$

$\therefore a^2 < c^2 < b^2$

$\Rightarrow 1 + a^2 < 1 + c^2 < 1 + b^2$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

or $\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$ —(2)

using (1) & (2)

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

Multiply by $b - a > 0$, we get

$$\boxed{\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}} \ \#$$

let $b = 4/3$ & $a = 1$

$$\frac{\frac{4}{3} - 1}{1 + (\frac{4}{3})^2} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{\frac{4}{3} - 1}{1 + 1^2}$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\boxed{\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}}$$ proved

Cauchy Mean Value Theorem

If (i) $f(x)$ & $g(x)$ be continuous in $[a,b]$
(ii) $f'(x)$ & $g'(x)$ exist in (a,b)
(iii) $g'(x) \neq 0$ for any value of x in (a,b)

then there is at least one value of c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{g(b)-g(a)}{b-a} = g'(c)$$

Sol Consider

$$\phi(x) = f(x) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(x)$$

① $\phi(x)$ is cont in $[a,b]$
② $\phi(x)$ is diff in (a,b)
③ $$\phi(a) = f(a) - \frac{(f(b)-f(a))}{g(b)-g(a)} g(a) =$$

$$\phi(b) = f(b) - \frac{(f(b)-f(a))}{g(b)-g(a)} g(b) =$$

$\therefore$ $\phi(a) = \phi(b)$

Here $\phi(x)$ satisfies all the conditions of Rolle's Theorem

$\therefore$ $\exists$ at least one $c \in (a,b)$ s.t.

$$\phi'(c) = 0$$

i.e. $$f'(c) - \frac{(f(b)-f(a))}{g(b)-g(a)} g'(c) = 0$$

or $$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$ C.M.V. Thm

Q Verify Cauchy Mean value Theorem for the functions $f(x) = e^x$ & $g(x) = e^{-x}$ in the interval (a,b)

Sol ① $f(x) = e^x$ & $g(x) = e^{-x}$ are both continuous in $[a,b]$
② $f'(x) = e^x$, $g'(x) = -e^{-x}$ both exists in (a,b)

$\therefore$ By Cauchy's Mean value Theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\therefore \quad \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

or $$\frac{(e^b - e^a)}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^c \cdot e^c$$

or $$e^b - e^a \times e^b \cdot e^a = -e^{2c}$$

$$\frac{e^a - e^b}{\dots} \quad \text{or} \quad -\frac{(e^b - e^a)}{e^b - e^a} \times e^{a+b} = -e^{2c} \qquad e^{a+b} = e^{2c}$$

or $e^{2c} = e^{a+b} \Rightarrow 2c = a+b$

or $c = \frac{a+b}{2} \in (a,b)$

$\therefore$ Cauchy's Mean value Theorem is verified

Taylor's Theorem (Generalised mean value therem)

If I) $f(x)$ and its first $(n-1)$ derivatives be continuous in $[a, a+h]$, and (II) $f^n(x)$ exists for every value of x in $(a, a+h)$, then there is at least one number $\theta (0 < \theta < 1)$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a+\theta h),$$

$R_n \to 0$

which is called Taylor's theorem with Lagrange's form of remainder, the remainder R_n being $\frac{h^n}{n!}f^n(a+\theta h)$.

$f(a+h) = f(a) + h f'(a+\theta h)$

$f \to [a, a+h]$

$f(a+h) = f(a) + h f'(c)$

① $f(x)$ cts, $f'(x)$ cts, $f''(x)$, ..., $f^{n-1}(x)$ cts

② $f^n(x)$ — exist

Cor I: If $n = 1$

$$\boxed{f(a+h) = f(a) + h f'(a+\theta h)}$$

Second form of LMV

If $n = 2$

$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a+\theta h)$

Cor 2: If $a = 0$

$f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(0) + \text{———} + \frac{h^n}{n!} f^n(\theta h)$

Put $h = x$

or $f(x) = f(0) + x f'(0) + \cdots + \frac{x^n}{n!} f^n(\theta x) \longrightarrow$ Maclaurin's Thm

$R_n = \frac{h^n}{n!} f^n(a+\theta h)$

If $n \to \infty$, $R_n \to 0$

① $f(a+h) = f(a) + h f'(a+h\theta) \longrightarrow \boxed{\text{LMV 2}}$

Maclaurin's Series

Maclaurin's Theorem Lagrange's form of remainder.

If $a = 0$ and $h = x$,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \frac{h^n}{n!}f^n(\theta x),$$

$n \to \infty$, $R_n = \frac{h^n f^n(\theta x)}{n!} \to 0$

Imp ① Find the Maclaurian's theorem with Lagrange's form of remainder for $f(x) = \cos x$

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \frac{h^n}{n!}f^n(\theta x),$

① *Find the Maclaurian's theorem with Lagrange's form of remainder for* $f(x) = \cos x$

Sol

$f(x) = \cos x \quad = \cos x$

$f'(x) = -\sin x \quad = \cos\left[\frac{\pi}{2} + x\right]$

$f''(x) = -\cos x \quad = \cos\left[2\cdot\frac{\pi}{2} + x\right]$

$f'''(x) = \sin x \quad = \cos\left[3\frac{\pi}{2} + x\right]$

$f^{iv}(x) = \cos x \quad = \cos\left[4\cdot\frac{\pi}{2} + x\right]$

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$\vdots$

$f^n(x) = \cos\left[n\cdot\frac{\pi}{2} + x\right]$

$\therefore \quad f^n(0) = \cos\left(n\frac{\pi}{2}\right)$

Take $n = 2m$ $\quad f^{(2n)}(0) = \cos\left[2m\cdot\frac{\pi}{2}\right] = \cos n\pi = (-1)^n$

If $n = 2m+1$ $\quad f^{(2n+1)}(0) = \cos\left[(2m+1)\frac{\pi}{2}\right] = 0.$

Sub these values in the Maclaurian's Theorem with Lagrange's form of remainder, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{2n}}{2n!}f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

$$\cos x = 1 + x\cdot 0 + \frac{x^2}{2!}(-1) + 0 + \cdots + \frac{x^{2n}}{2n!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}\cos\left[(2n+1)\frac{\pi}{2} + \theta x\right]$$

$$= 1 - \frac{x^2}{2!} + \cdots + (-1)^n\frac{x^{2n}}{2n!} + \cos\left[(2n+1)\frac{\pi}{2} + \theta x\right]\frac{x^{2n+1}}{(2n+1)!}$$

$$= 1 - \frac{x^2}{2!} + \cdots + (-1)^n\frac{x^{2n}}{2n!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\sin\theta x$$

Ans

$\therefore \cos\left[\left(n\pi + \left(\frac{\pi}{2} + \theta x\right)\right)\right] = (-1)^n\cos\left(\frac{\pi}{2} + \theta x\right)$

$= (-1)^n \times -\sin\theta x = (-1)^{n+1}\sin\theta x$

If $f(x) = \log(1+x), x > 0$ *using Maclaurin's theorem, show that for* $0 < \theta < 1,$

\# $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}.$ $\qquad$ \# $\frac{1}{(1+\theta x)^3} < 1$

Deduce that $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ *for* $x > 0.$

Sol $\quad f(x) = \log(1+x), \quad x > 0$

Sol $\quad$ By Maclaurin's theorem with Remainder R_3, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \quad —①$$

let $f(x) = \log(1+x), \qquad f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}, \qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$

$f'''(x) = \frac{+2}{(1+x)^3} \qquad f'''(\theta x) = \frac{2}{[1+\theta x]^3}$

Sub these values in ①, we get

$$f(x) = 0 + x\,1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}\times\frac{2}{(1+\theta x)^3} \qquad 3! = 3\times2\times1$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \quad \text{Ans}$$

$f(x) = 0 + x\,1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}\,\frac{2}{(1+\theta x)^3}$ $\qquad 3! = 3\times2\times1$

$= x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$ Ans — (2)

Now $x>0$, $0<\theta<1$ $\qquad \boxed{\begin{matrix}1>0\\2>0\\2>0\end{matrix}}$

or $x>0$, $\theta>0$

$\Rightarrow \theta x > 0$

$1+\theta x > 1+0$

or $1+\theta x > 1$ $\qquad 2>0$

Cubing both sides $(1+\theta x)^3 > 1^3$ $\qquad 6>0$

or $(1+\theta x)^3 > 1$ $\qquad 2\times6>0$

on Taking reciprocal, we get

$$\boxed{\frac{1}{(1+\theta x)^3} < 1} \quad — (*)$$

Using (*) in (2), we get

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}.1$$

$$\therefore \boxed{\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}} \quad \text{Ans}$$

let $n\to\infty$, $R_n\to 0$

Expansions of functions

Maclaurin's Series → If $f(x)$ can be expanded as an infinite series, then

$$f(x) = f(0) + x^1 f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \infty \quad —(1)$$

Imp Q. Using Maclaurin's series, expand $\tan x$ upto the terms containing x^5.

Sol #

$f(x) = \tan x$ $\qquad f(0) = 0$

$f'(x) = \sec^2 x = 1 + \tan^2 x$ $\qquad f'(0) = 1+0 = 1$

$f''(x) = 2\tan x \sec^2 x$

$= 2\tan x[1+\tan^2 x]$ $\qquad f''(0) = 0$

$= 2\tan x + 2\tan^3 x$

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$

$= 2[1+\tan^2 x] + 6\tan^2 x[1+\tan^2 x]$

$f'''(x) = 2 + 8\tan^2 x + 6\tan^4 x$ $\qquad f'''(0) = 2$

$f^{iv}(x) = 0 + 16\tan x\sec^2 x + 24\tan^3 x \sec^2 x$

$= 16\tan x[1+\tan^2 x] + 24\tan^3 x[1+\tan^2 x]$

$= 16\tan x + 40\tan^3 x + 24\tan^5 x$ $\qquad f^{iv}(0) = 0$

$f^{v}(x) = 16\sec^2 x + 120\tan^2 x \sec^2 x + 120\tan^4 x\sec^2 x$ $\qquad f^{v}(0) = 16$

Sub the values in the Maclaurin's series, we get

$f(x) = 0 + x\times1 + x^2\times0 + \frac{x^3}{3!}\times2 + \frac{x^4}{4!}\times0 + \frac{x^5}{5!}\times16 + \cdots$

Sub the values in the Maclaurin's series, we get

$$f(x) = 0 + x \times 1 + \frac{x^2}{2!}\times 0 + \frac{x^3}{3!}\times 2 + \frac{x^4}{4!}\times 0 + \frac{x^5}{5!}\times 16 + \cdots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots\cdots \infty$$

or $$\boxed{\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots \infty}$$ CTM

$\frac{1}{5.4.3.2.1}$ x^{+6}

$$\boxed{\frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots}$$

(2) Expansions by use of known series

148

$= 16\tan x(1+\tan^2 x) + 24\tan^3 x(1+\tan^2 x)$

$= 16\tan x + 40\tan^3 x + 24\tan^5 x$

$f^{v}(0) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x.$

and so on.

Substituting the values of $f(0)$, $f'(0)$, etc. in the Maclaurin's series, we get

$$\tan x = 0 + x.1 + \frac{x^2}{2!} + \frac{x^3}{3!}.2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.16 + \ldots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

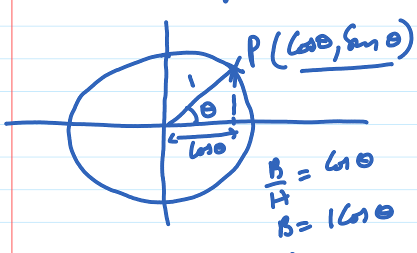
(2) Expansion by use of known series. When the expansion of a function is required only ... terms, it is often convenient to employ the following well-known series :

1. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$ $\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{(2n+1)}}{(2n+1)!}$
2. $\sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \ldots$ $\sum_{n=0}^{\infty} \frac{\theta^{(2n+1)}}{(2n+1)!}$
3. $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$ $\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{2n!}$
4. $\cosh\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots$ $\sum_{n=0}^{\infty} \frac{\theta^{2n}}{2n!}$
5. $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \ldots$
6. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$
7. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
8. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ $= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$
9. $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right)$ $= -\sum_{n=1}^{\infty} \frac{x^n}{n}$
10. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$ → Binomial series expansion

Example 4.21. Expand $e^{\sin x}$ by Maclaurin's series or otherwise up to ...

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}, \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

Circular Triangle

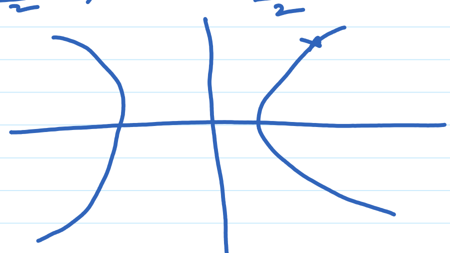


$\frac{B}{H} = \cos\theta$

$B = 1\cos\theta$

$\frac{P}{H} = \sin\theta$

$P = \sin\theta$



Points on the hyperbola are $\sinh\theta$ & $\cosh\theta$

$(a-b)^2 = a^2 + b^2 - 2ab$

Q Expand $e^{\sin x}$ by Maclaurian series or otherwise upto the terms containing x^4

Sol $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots \infty$$

$$e^{\sin x} = 1 + \sin x + \frac{1}{2!}\sin^2 x + \frac{1}{3!}\sin^3 x + \frac{1}{4!}\sin^4 x + \cdots \infty$$

$$e^{\sin x} = 1 + \sin x + \frac{1}{2!}\sin^2 x + \frac{1}{3!}\sin^3 x + \frac{1}{4!}\sin^4 x + \cdots \infty$$

$$= 1 + \left[x - \frac{x^3}{3!} + \cdots\right] + \frac{1}{2!}\left[x - \frac{x^3}{3!} + \cdots\right]^2 + \frac{1}{3!}\left[x - \frac{x^3}{3!} + \cdots\right]^3 + \frac{1}{4!}\left[x - \frac{x^3}{3!} + \cdots\right]^4 + \cdots$$

$$= 1 + \left[x - \frac{x^3}{3!} + \cdots\right] + \frac{1}{2!}\left[x^2 - \frac{2}{3!}x^4 + \cdots\right] + \frac{1}{3!}\left[x^3 + \cdots\right] + \frac{1}{4!}\left[x^4 + \cdots\right]$$

$$= 1 + x + \left[\frac{1}{2!}\right]x^2 + \left[\frac{-1}{3!} + \frac{1}{3!}\right]x^3 + \left[\frac{1}{2!} + \frac{-2}{3!} + \frac{1}{4!}\right]x^4 + \cdots \infty$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{x^4}{8} + \cdots \infty \quad \text{Ans}$$

Q Expand $\log(1+\sin^2 x)$ in powers of x as far the term in x^6.

Sol $\sin^2 x = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]^2$

$$= \left[x - \left[\frac{x^3}{6} - \frac{x^5}{120} + \cdots\right]\right]^2$$

$$= x^2 - 2x\left[\frac{x^3}{6} - \frac{x^5}{120} + \cdots\right] + \left[\frac{x^3}{6} - \frac{x^5}{120} + \cdots\right]^2$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{120} + \frac{x^6}{36} + \cdots \qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \cdots = t \text{ (say)}$$

$$\therefore \log(1+\sin^2 x) = \log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

$$\therefore \log(1+\sin^2 x) = \left[x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \cdots\right] - \frac{1}{2}\left[x^2 - \frac{x^4}{3} + \boxed{\frac{2}{45}x^6} + \cdots\right]^2 + \frac{1}{3}\left[x^2 - \frac{x^4}{3} + \cdots\right]^3 - \cdots \#$$

$$= \left[x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \cdots\right] - \frac{1}{2}\left[x^4 - \frac{2}{3}x^6 + \cdots\right] + \frac{1}{3}\left[x^6 + \cdots\right] + \cdots$$

$$= x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 + \cdots \quad \text{Ans}$$

Taylor's Series If $f(x+h)$ can be expanded as an infinite series then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \infty$$

Cor ① Take $x = a$ & $h = x - a$

$$f(a+x-a) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \infty$$

$$\text{or } f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \infty$$

$\boxed{\text{Taylor series in the powers of } (x-a)}$ #

Cor ② If $a = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \cdots \infty$$

→ Maclaurin's Series

Q Expand $\log_e x$ in power of $(x-1)$ and hence evaluate $\boxed{\log_e 1.1}$ correct to 4 decimal places.

Sol

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \frac{(x-a)^3}{\lfloor 3}f'''(a) + \frac{(x-a)^4}{\lfloor 4}f^{iv}(a) + \cdots \infty$$

$$\therefore f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{\lfloor 2}f''(1) + \frac{(x-1)^3}{\lfloor 3}f'''(1) + \frac{(x-1)^4}{\lfloor 4}f^{iv}(1) + \cdots \infty \quad -(1)$$

$f(x) = \log_e x$: $f(1) = \log_e 1 = 0$

$f'(x) = \frac{1}{x}$ $f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $f'''(1) = 2$

$\frac{4}{4\times3\times2}$

$f^{iv}(x) = \frac{-6}{x^3}$ $f^{iv}(1) = -6$

sub these values in (1), we get

$$f(x) = 0 + (x-1)\times 1 + \frac{(x-1)^2}{2!}\times -1 + \frac{(x-1)^3}{\lfloor 3}\times 2 + \frac{(x-1)^4}{\lfloor 4}\times -6 + \cdots \infty$$

$$\therefore \log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \infty \quad -(*)$$

Put $x = 1.1$, then $x-1 = 1.1-1 = 0.1$

$\therefore$ from (*)

$$\log_e x = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots \infty$$

$$= 0.0953 \text{ approx}$$

MCQ

OR

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$$

$$\log_e x = \log(1+(x-1))$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \infty$$

$$(uv)_1 = u_1 v + u v_1$$

Leibnitz's Theorem for the n^{th} derivative of the product of two functions

If u, v are two functions of x possessing derivatives of n^{th} order then

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n$$

$$\boxed{\begin{array}{l} {}^nC_0 = 1 \\ {}^nC_n = 1 \end{array}}$$

Q If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)\underbrace{y_{n+2}}_{n+2} - (2n+1)x\underbrace{y_{n+1}} - (n^2-a^2)\underbrace{y_n} = 0$

Sol

$$y = e^{a\sin^{-1}x}$$

Sol $y = e^{a\sin^{-1}x}$

$y_1 = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}}$

or $y_1 = \frac{ya}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = ya$

SBS $(1-x^2) y_1^2 = y^2 a^2$

Diff again w.r.t., we get

$(1-x^2) \times 2y_1 y_2 + y_1^2(-2x) = a^2 2yy_1$

or $2y_1(1-x^2)y_2 - 2xy_1^2 - a^2 2y_1 y = 0$

Divide throughout by $2y_1 \neq 0$

$(1-x^2)y_2 - xy_1 - a^2 y = 0$ —— (1)

Diff (1) n times by Leibnitz Theorem, we get

$\left[{}^nC_0\, y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2)\right]$

$- \left[{}^nC_0\, y_{n+1}x + {}^nC_1\, y_n(1)\right] - a^2 y_n = 0$

$\left| {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} \right.$

or $(1-x^2)y_{n+2} + \frac{\lfloor n}{\lfloor 1 \lfloor n-1}(-2x)y_{n+1} + \frac{\lfloor n}{\lfloor 2 \lfloor n-2}(-2 y_n)$

$- y_{n+1}x - \frac{\lfloor n}{\lfloor 1 \lfloor n-1} y_n - a^2 y_n = 0$

$\left| \lfloor n = n \lfloor n-1 \right.$

or $(1-x^2)y_{n+2} + -2nxy_{n+1} - n(n-1)y_n$

$- y_{n+1}x - ny_n - a^2 y_n = 0$

or $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + \left[-n^2 + \cancel{n} - \cancel{n} - a^2\right]y_n = 0$

or $\boxed{(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0}$

$(y_2)_n$

y_{n+2}