Friday, October 16, 2020 9:56 AM

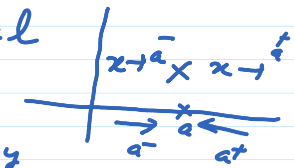
Partial Differentiation

$y = f(x) \longrightarrow x$ value

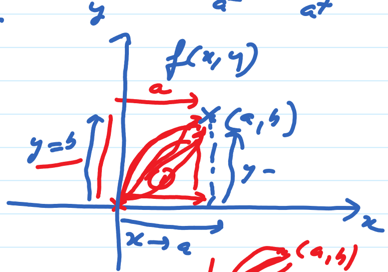
$Z = f(x, y) \rightarrow Z$ is a function of two variables

$Z = f(x_1, x_2, x_3 - - - x_n) \rightarrow Z$ is a fun of n variables

$\lim_{x\to a} f(x)$ $\qquad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = l$

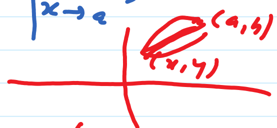


$\lim_{(x,y)\to(a,b)} f(x,y) \longrightarrow$ when this limit will exists



$\lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\}$, $\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right]$

Apart from this $\lim_{(x,y)\to(a,b)} f(x,y)$ should give the same value along all the paths



Q $f(x,y) = \frac{x-y}{x+y}$ Find $\lim_{(x,y)\to(0,0)} f(x,y)$

$y = x$ - - - -
$y = 2x$ $\quad \boxed{y = mx}$
$y = 3x$

$y = mx$
$y = mx^2$
$y = mx^3$

Sol $\lim_{(x,y)\to(0,0)} \boxed{\frac{x-y}{x+y}}$

let the path along where $(x,y) \to (0,0)$ is $y = mx$ $\quad$ m is have diff value

$\lim_{x\to 0} \frac{x - mx}{x + mx}$

$= \lim_{x\to 0} \frac{\cancel{x}(1-m)}{\cancel{x}(1+m)} = \frac{1-m}{1+m} \rightarrow$? No unique

which is not unique $\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Q $\lim_{y\to 0}\left\{\lim_{x\to 0} \frac{x-y}{x+y}\right\} = \lim_{y\to 0}\left[\frac{-\cancel{y}}{\cancel{y}}\right] = -1$

$\lim_{x\to 0}\left\{\lim_{y\to 0} \frac{x-y}{x+y}\right\} = \lim_{x\to 0}\left[\frac{\cancel{x}}{\cancel{x}}\right] = +1$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

Q $\boxed{\lim_{x\to a} f(x) = f(a)}$

\# $\left[\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)\right. \longrightarrow$ Continuous fun

Q \# $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

y const

$$z = f(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h} \text{ or } f_x(x,y)$$

x – const

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k) - f(x,y)}{k} \text{ or } f_y(x,y)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[f_x(x,y)\right] = \lim_{h\to 0} \frac{f_x(x+h, y) - f_x(x,y)}{h} \text{ or } f_{xx}(x,y)$$

$\frac{\partial}{\partial y}$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial y}\left[f_x\right] = \lim_{k\to 0} \frac{f_x(x, y+k) - f_x(x,y)}{k}$$

$$\frac{\partial^2 z}{\partial y^2} =$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

$$p \equiv \frac{\partial z}{\partial x}$$

$$q \equiv \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 z}{\partial y \partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial f(x,y)}{\partial x} \text{ or } f_x(x,y)$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \text{ or } f_{xx}$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} \text{ or } f_{yy}(x,y)$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} \text{ or } f_{yx}(x,y)$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \text{ or } f_{xy}(x,y)$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f(x,y)}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[f_y(x,y)\right] = \lim_{h\to 0} \frac{f_y(x+h, y) - f_y(x,y)}{h}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f(x,y)}{\partial x}\right]$$

$$= \frac{\partial}{\partial y}\left[f_x(x,y)\right] = \lim_{k\to 0} \frac{f_x(x, y+k) - f_x(x,y)}{k}$$

$$\left[\frac{\partial z}{\partial x}\right]_{y-\text{const}}, \quad \left[\frac{\partial z}{\partial y}\right]_{x-\text{const}}$$

$\left[\frac{\partial z}{\partial x}\right]_{y-\text{Const}}$, $\left[\frac{\partial z}{\partial y}\right]_{x-\text{Const}}$

$z = x^2y + y$

$\frac{\partial z}{\partial x} = 2xy + 0 = 2xy$

$\boxed{\frac{\partial z}{\partial y} = x^2+1}$

Q Find the first & second order derivative of

$$z = x^3 + y^3 + 3axy \quad —(1)$$

Sol Diff (1) partially w.r.t x & y respectively

$\frac{\partial z}{\partial x} = 3x^2 + 0 + 3ay = 3x^2 + 3ay$

$\frac{\partial^2 z}{\partial x^2} = 6x + 0 = 6x$

$\frac{\partial^2 z}{\partial y \partial x} = 0 + 3a = 3a$

$\frac{\partial z}{\partial y} = 3y^2 + 3ax$

$\frac{\partial^2 z}{\partial y^2} = 6y$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 3a$

or $\frac{\partial^2 z}{\partial x \partial y} = 3a$

Q If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

S.T $\frac{\partial^2 u}{\partial x \partial y} = \boxed{\frac{\partial^2 u}{\partial y \partial x}}$

Sol $u = x^2 \tan^{-1} y/x - y^2 \tan^{-1}(x/y)$

$\frac{\partial u}{\partial x} = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2} + 2x \tan^{-1} y/x - y^2 \times \frac{1}{1+\frac{x^2}{y^2}} \times \frac{1}{y}$

$= \cancel{x^2} \cdot \frac{x^2}{x^2+y^2} \times \frac{-y}{\cancel{x^2}} + 2x \tan^{-1} y/x - y^2 \times \frac{y^2}{y^2+x^2} \times \frac{1}{\cancel{y}}$

$\frac{\partial u}{\partial x} = -\frac{x^2 y}{x^2+y^2} + 2x \tan^{-1}\frac{y}{x} - \frac{y^3}{y^2+x^2} \quad —(1)$

Diff (1) partially w.r.t 'y', we get

$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[-\frac{x^2 y}{x^2+y^2} + 2x \tan^{-1}\frac{y}{x} - \frac{y^3}{y^2+x^2}\right]$

$= -x^2\left(\frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right) + 2x\left[\frac{\partial}{\partial y}\tan^{-1}(y/x)\right] - \frac{\partial}{\partial y}\left[\frac{y^3}{y^2+x^2}\right]$

$= -x^2\left[\frac{(x^2+y^2)1 - y \times 2y}{(x^2+y^2)^2}\right] + 2x\left[\frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} \cdot 1\right] - \left[\frac{(y^2+x^2) \times 3y^2 - y^3 \times 2y}{(y^2+x^2)^2}\right]$

$= -x^2\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] + 2\cancel{x} \times \frac{x^2}{x^2+y^2} \times \frac{1}{\cancel{x}} - \left[\frac{3y^4+3x^2y^2-2y^4}{(y^2+x^2)^2}\right]$

$= \frac{-x^2[x^2-y^2]}{(x^2+y^2)^2} + \frac{2x^2}{x^2+y^2} - \frac{(y^4+3x^2y^2)}{(x^2+y^2)^2}$ #

$\boxed{\text{L.H.S}}$ is exercise

$$\overline{(x^2+y^2)^2} \qquad \overline{x^2+y^2} \qquad \overline{(x^2+y^2)^2}$$

$\boxed{L.H.S}$ is exercise

Q. If $v = (x^2+y^2+z^2)^{-1/2}$

PT $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} = ?$

Sol Here $v = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial v}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \times \cancel{2}x$$

$$= -x(x^2+y^2+z^2)^{-3/2} \quad -①$$

Diff ① partially w.r.t 'x', we get

$$\frac{\partial^2 v}{\partial x^2} = -\left[x \times \left(-\frac{3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-5/2} \cancel{2x} + (x^2+y^2+z^2)^{-3/2} \times 1\right]$$

$$= -\left[-3x^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-5/2}\right)$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad -①$$

on the similar lines

$$\frac{\partial^2 v}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad -②$$

& $$\frac{\partial^2 v}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad -③$$

on adding ①, ② & ③.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 3(x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2\right]^1 - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0 \text{ Ans}$$

(a) 1
(b) 3
✓(c) 0
(d) None of these

②

Q. If $u = \log(x^3+y^3+z^3-3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}$$

Sol $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \quad -①$$

Diff (A) partially w.r.t 'x'

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \quad -(1)$$

Diff (4) partially w.r.t 'x'

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times 3x^2-3yz$$

Similarly $$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} \times 3y^2-3xz$$

and $$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \times 3z^2-3xy$$

on adding

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-3xy-3yz-3xz}{x^3+y^3+z^3-3xyz}. \quad \text{\# and}$$

$$= \frac{3\left[x^2+y^2+z^2-\cancel{xy-yz-xz}\right]}{(x+y+z)(x^2+y^2+z^2\cancel{-xy-yz-xz})}$$

$$= \frac{3}{x+y+z} \quad \#$$

$\therefore$ from (1)

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{3}{x+y+z}\right]+\frac{\partial}{\partial y}\left[\frac{3}{x+y+z}\right]+\frac{\partial}{\partial z}\left[\frac{3}{x+y+z}\right]$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2} \quad \text{Ans}$$

Q If $\theta = t^n e^{-\frac{r^2}{4t}}$, what value of n will make

$$\# \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\theta}{\partial t}\right] = \frac{\partial\theta}{\partial t} \; ? \quad -(2)$$

Sol Here $\theta = t^n e^{-\frac{r^2}{4t}}$ — (1) $\qquad \theta \rightarrow r \,\&\, t$

Diff (1) partially w.r.t 't', we get

$$\frac{\partial\theta}{\partial t} = t^n \cdot e^{-\frac{r^2}{4t}} \times +\frac{r^2}{4t^2} + n t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\left.\frac{\partial\theta}{\partial t} = \frac{r^2}{4} t^{n-2} e^{-\frac{r^2}{4t}} + n t^{n-1} e^{-r^2/4t}\right] -(1)$$

$$r^2\frac{\partial\theta}{\partial t} = \frac{r^4}{4} t^{n-2} e^{-\frac{r^2}{4t}} + n r^2 t^{n-1} e^{-r^2/4t}$$

Diff it partially w.r.t 'r', we get

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial\theta}{\partial t}\right] = \frac{t^{n-2}}{4}\frac{\partial}{\partial r} r^4 e^{-r^2/4t} + n t^{n-1}\frac{\partial}{\partial r} r^2 e^{-r^2/4t}$$

diff it partially w.r.t. r, we get

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial \theta}{\partial t}\right] = \frac{t^{n-2}}{4}\frac{\partial}{\partial r} r^4 e^{-r^2/4t} + n t^{n-1}\frac{\partial}{\partial r} r^2 e^{-r^2/4t}$$

$$= \frac{t^{n-2}}{4}\left[r^4 e^{-r^2/4t} \times \frac{-2r}{4t} + 4r^3 e^{-r^2/4t}\right] + n t^{n-1}\left[r^2 e^{-r^2/4t} \times \frac{-2r}{4t} + 2r e^{-r^2/4t}\right]$$

$$= -\frac{1}{8} r^5 t^{n-3} e^{-r^2/4t} + t^{n-2} r^3 e^{-r^2/4t} - n t^{n-2} r^3 e^{-r^2/4t} + 2rn t^{n-1} e^{-r^2/4t}$$

Multiply both sides with $\frac{1}{r^2}$, we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial \theta}{\partial t}\right] = -\frac{1}{8} r^3 t^{n-3} e^{-r^2/4t} + t^{n-2} r e^{-r^2/4t} - n t^{n-2} r e^{-r^2/4t} + \frac{2}{r} n t^{n-1} e^{-r^2/4t}$$

$$= \left[-\frac{r^3}{8} t^{n-3} + t^{n-2} r - n t^{n-2} r + \frac{2}{r} n t^{n-1}\right] e^{-r^2/4t}$$

$$= \left[-\frac{r^2}{8} t^{n-3} + (1-n) t^{n-2} r + \frac{2}{r} n t^{n-1}\right] e^{-r^2/4t}$$

Sub these values in (*), we get

$$\left[-\frac{r^2}{8} t^{n-3} + (1-n) t^{n-2} r + \frac{2}{r} n t^{n-1}\right] e^{-r^2/4t} = \left[\frac{r^2}{4} t^{n-2} + n t^{n-1}\right] e^{-r^2/4t}$$

$$\Rightarrow -\frac{r^2}{8} t^{n-3} + (1-n) t^{n-2} r + \frac{2}{r} n t^{n-1} = \frac{r^2}{4} t^{n-2} + n t^{n-1}$$

or $-\frac{r^2}{8} t^{n-3} + (1-n) t^{n-2} r + \frac{2}{r} n t^{n-1} - \frac{r^2}{4} t^{n-2} - n t^{n-1} = 0$

Q. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

a^x

Sol. Here $u = x^y$

$$\frac{\partial u}{\partial y} = x^y \cdot \log x$$

Now diff it partially w.r.t. 'x', we get

$$\frac{\partial^2 u}{\partial x \partial y} = x^y \times \frac{1}{x} + y x^{y-1} \cdot \log x$$

$$= x^{y-1} + y x^{y-1} \log x$$

diff it again partially w.r.t 'x', we get

$$\frac{\partial^3 u}{\partial x^2 \partial y} = (y-1)x^{y-2} + y\left[x^{y-1} \times \frac{1}{x} + (y-1)x^{y-2}\log x\right]$$

$$\boxed{= (y-1)x^{y-2} + y x^{y-2} + y(y-1)x^{y-2}\log x} \quad \text{--- (2)}$$

Now $u = x^y$

Diff it w.r.t 'x', we get

$$\frac{\partial u}{\partial x} = y x^{y-1}$$

Diff it w.r.t 'y', we get

$$\frac{\partial^2 u}{\partial y \partial x} = y\, x^{y-1}\log x + x^{y-1} \cdot 1$$

Diff it w.r.t 'x', we get

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = y(y-1)x^{y-2}\log x + y x^{y-1} \times \frac{1}{x} + (y-1)x^{y-2}$$

$$= y(y-1)x^{y-2}\log x + y x^{y-2} + (y-1)x^{y-2} \quad \text{--- (3)}$$

From (2) & (3)

$$\boxed{\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}}$$

which variable to be treated as constant

Q $u = f(r,\theta)$ $\quad \left(\frac{\partial u}{\partial r}\right)_\theta$ or $\left(\frac{\partial u}{\partial \theta}\right)_r \longrightarrow$ constant

as const

Q If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$ prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Sol $u \to f(r)$ $\qquad$ Here $x = r\cos\theta$, $y = r\sin\theta$

SBS & adding

$$\boxed{x^2 + y^2 = r^2}$$

or $r^2 = x^2 + y^2$

for dividing $\frac{y}{x} = \tan\theta$ or $\theta = \tan^{-1} y/x$

$$\boxed{u \to r \to x, y}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} \quad \text{--- (1)}$$

Diff it again w.r.t 'x', we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r) \cdot \frac{\partial r}{\partial x}\right]$$

$$= f''(r) \cdot \frac{\partial r}{\partial x} \cdot \frac{\partial r}{\partial x} + f'(r) \cdot \frac{\partial^2 r}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\quad \frac{\partial r}{\partial x}\right]$$

$$= f''(r)\cdot\frac{\partial r}{\partial x}\cdot\frac{\partial r}{\partial x} + f'(r)\cdot\frac{\partial^2 r}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2} \quad \text{— (2)}$$

$$\text{IIly} \quad \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2} \quad \text{— (3)}$$

on adding

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right] \quad \text{— (5)}$$

Now $r^2 = x^2+y^2$ $\quad\Big|\quad r^2 = x^2+y^2$

$$2r\frac{\partial r}{\partial x} = 2x \quad \& \quad 2r\frac{\partial r}{\partial y} = 2y$$

$$\text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \& \quad \frac{\partial r}{\partial y} = y/r$$

$$\text{also} \quad \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{x}{r}\right] = \frac{r\frac{\partial x}{\partial x} - x\frac{\partial r}{\partial x}}{r^2} \quad \& \quad \frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}\left[y/r\right]$$

$$= \frac{r - x\times\frac{x}{r}}{r^2} \qquad\qquad = \frac{r - y\cdot\partial r/\partial y}{r^2}$$

$$= \frac{r^2-x^2}{r^3} \qquad\qquad = \frac{r^2-y^2}{r^3}$$

Sub these values in (5), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right] + f'(r)\left[\frac{r^2-x^2}{r^3}+\frac{r^2-y^2}{r^3}\right]$$

$$= f''(r)\left[\frac{x^2+y^2}{r^2}\right] + f'(r)\left[\frac{2r^2-(x^2+y^2)}{r^3}\right]$$

$$= f''(r)\left[\frac{r^2}{r^2}\right] + f'(r)\left[\frac{2r^2-r^2}{r^3}\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)} \quad \text{Ans}$$

Homogeneous functions

A function $f(x,y)$ is said to be a homogeneous function of degree 'n' in x,y if it can be expressed as

$$x^n\phi(y/x) \quad \text{or} \quad y^n\phi_1(x/y)$$

$$\text{or} \quad \boxed{f(\lambda x,\lambda y) = \lambda^n f(x,y)}$$

Ex

$$f(x,y) = \frac{x^3+y^3}{x-y}$$

$$= \frac{x^3\left[1+\frac{y^3}{x^3}\right]}{x\left[1-\frac{y}{x}\right]} = x^2\left(\frac{\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[1-\frac{y}{x}\right]}\right)$$

$$= \frac{x^{3}\left[1+\frac{y^3}{x^3}\right]}{x\left[1-y/x\right]} = x^2\frac{\left[1+\left(\frac{y}{x}\right)^3\right]}{(1-y/x)}$$

$$= x^2\,\phi(y/x)$$

$\therefore$ $f(x,y)$ is a homogenous fun of degree 2

or $$f(\lambda x,\lambda y) = \frac{(\lambda x)^3+(\lambda y)^3}{\lambda x-\lambda y} = \frac{\lambda^3x^3+\lambda^3y^3}{\lambda[x-y]}$$

$$= \lambda^3\frac{[x^3+y^3]}{\lambda[x-y]}$$

$$= \lambda^2 f(x,y)$$

$\Rightarrow$ $f(x,y)$ is a homog fun of degree '2'

imp

Euler's Theorem $\rightarrow$ If z is a homogenous function of degree 'n' wrt x & y then

$$\boxed{x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz}$$

Proof: let z be a homogenous function of degree n in x & y

$$\therefore \quad z = x^n f(y/x) \quad -①$$

Diff ① partially w r t x & y, we get

$$\frac{\partial z}{\partial x} = nx^{n-1}f(y/x) + x^n f'(y/x)\times -y/x^2$$

$$= nx^{n-1}f(y/x) - x^{n-2}y\,f'(y/x) \quad -②$$

& $$\frac{\partial z}{\partial y} = x^n f'(y/x)\times\frac{1}{x}$$

$$= x^{n-1}f'(y/x) \quad -③$$

Multiply ② by x & ③ by y and on adding, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = nx^n f(y/x) - x^{n-1}y\cancel{f'(y/x)} + x^{n-1}y\cancel{f'(y/x)}$$

$$= nz$$

$$\boxed{\therefore\ x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz}$$

Q Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2u\log u$

where $\log u = \frac{x^3+y^3}{3x+4y}$

Sol let $z = \log u$ — (1)

$\therefore z = \frac{x^3+y^3}{3x+4y}$

$= \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[3+\frac{4y}{x}\right]} = x^2\phi(y/x)$

$\therefore z$ is a homogeneous func of degree '2'

$\therefore$ By Euler's thm

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$\Rightarrow x\frac{\partial}{\partial x}\log u + y\frac{\partial}{\partial y}\log u = 2\log u$$

$$\Rightarrow x\frac{1}{u}\frac{\partial u}{\partial x} + y\frac{1}{u}\frac{\partial u}{\partial y} = 2\log u$$

or $$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u}$$

#

Imp

Q If z is a homogeneous function of degree n in x & y then show that $\partial \to$ partial

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Sol

Here z is a homogeneous function of degree n then By Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad — (1)$$

Diff (1) partially with respect to x & y respectively

$$\frac{\partial}{\partial x}\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}[nz]$$

or $$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\partial y} = n\frac{\partial z}{\partial x} \quad — (2)$$

&

$$\frac{\partial}{\partial y}\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial y}[nz]$$

or $$x\frac{\partial^2 z}{\partial y\partial x} + y\cdot\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = n\frac{\partial z}{\partial y} \quad — (3)$$

Multiply (2) by x & (3) by 'y' & on adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + xy\frac{\partial^2 z}{\partial x\partial y} + xy\frac{\partial^2 z}{\partial y\partial x} + y^2\frac{\partial^2 z}{\partial y^2} + y\frac{\partial z}{\partial y} =$$

$$nx\frac{\partial z}{\partial x} + ny\frac{\partial z}{\partial y}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - \left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$$=(n-1)\left(\underbrace{x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}}\right)$$

$$=(n-1)(nz)$$

or $\boxed{x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial y^2}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z}$ CTM

Q If $u=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$

& $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}$$

Sol Here $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ → Not a hom

$u(x,y)=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$u(\lambda x,\lambda y)=$

$\sin^{-1}\left[\frac{\lambda x+\lambda y}{\sqrt{\lambda}\sqrt{x}+\sqrt{\lambda}\sqrt{y}}\right]$

$\sin^{-1}\lambda^{1/2}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

or $\sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

let $w=\sin u$ — (1)

$\therefore\ w=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Here w is a homogeneous function of degree '$1/2$'

$\therefore$ By Euler's Theorem

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=\frac{1}{2}w$$

$$x\frac{\partial}{\partial x}\sin u+y\frac{\partial}{\partial y}\sin u=\frac{1}{2}\sin u$$

$$\Rightarrow\ x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

$$\text{or}\ \cos u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{2}\sin u$$

or $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u}$ — (1)

Diff (1) w.r.t. x & y respectively and we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\frac{\partial}{\partial x}\tan u$$

$x \times$ or $\left\{x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}\right.$ — (2)

and $y\times\left[x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}\right.$ — (3)

Multiply (2) by x & (3) by y & on adding, we get

Multiply (2) by x & (3) by y & on adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} + xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} =$$

$$\frac{1}{2}\sec^2 u\, x\frac{\partial u}{\partial x} + \frac{1}{2}\sec^2 u\, y\frac{\partial u}{\partial y}$$

or

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] - \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= \left[\frac{1}{2}\sec^2 u - 1\right]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= \left[\frac{1}{2\cos^2 u} - 1\right]\left[\frac{1}{2}\tan u\right]$$

$$= \left[\frac{1 - 2\cos^2 u}{2\cos^2 u}\right]\frac{1}{2}\frac{\sin u}{\cos u}$$

$\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$

$$= -\frac{[2\cos^2 u - 1]}{2\cos^2 u} \times \frac{1}{2}\frac{\sin u}{\cos u}$$

$$= -\frac{\cos 2u \sin u}{4\cos^3 u}$$

Total Derivative

If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$

Chain Rule

$u \to x, y \to t$

$u = f(x, y)$

$u + \delta u = f(x + \delta x, y + \delta y)$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \quad (*)$$

If $u \to x, y, z \to t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} \quad (*)$$

Q Differentiation of Implicit function $\rightarrow$ If $\boxed{f(x, y) = c}$ be an implicit relation between x & y which defines as a differentiable function of x, then

$f \to x, y \to x$

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\text{or} \quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

or $\quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

or $\quad$ # $\boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$ Ans

Q Given $u = \sin(x/y)$, $x = e^t$ & $y = t^2$, find $\frac{du}{dt}$ as a function of t, verify your result by direct substitution

Sol $\quad u \to x, y \to t$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \cos\left(\frac{x}{y}\right)\times\frac{1}{y}\times e^t + \cos\left(\frac{x}{y}\right)\times -\frac{x}{y^2}\times 2t$$

$$= \cos\left[\frac{e^t}{t^2}\right]\times\frac{1}{t^2}\times e^t + \cos\left(\frac{e^t}{t^2}\right)\times -\frac{e^t}{t^{4\,3}}\times 2\cancel{t}$$

$$= \frac{e^t}{t^2}\cos\left[\frac{e^t}{t^2}\right]\left[1 - \frac{2}{t}\right] = e^t\cos\left(\frac{e^t}{t^2}\right)\frac{(t-2)}{t^3}$$

$$u = \sin\left[\frac{x}{y}\right] = \sin\left[\frac{e^t}{t^2}\right]$$

$$\therefore \frac{du}{dt} = \cos\left[\frac{e^t}{t^2}\right]\times\frac{d}{dt}\left[\frac{e^t}{t^2}\right]$$

$$= \cos\left[\frac{e^t}{t^2}\right]\left[\frac{t^2e^t - 2te^t}{t^4}\right]$$

$$= \cos\left(\frac{e^t}{t^2}\right)\frac{e^t(t-2)}{t^3} \quad \#$$

Ans

Q If $\boxed{u = x\log xy}$ where $\boxed{x^3+y^3+3xy=1}$, find $\frac{du}{dx}$

Implicit function

Sol $\quad u \to x, y \to x$

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \qquad \text{—(1)}$$

Now let $f(x,y) = x^3+y^3+3xy-1$, we have

$f(x,y) = c$

$df = 0$

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$

$$\# \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2+3y}{3y^2+3x}$$

$$= -\frac{x^2+y}{y^2+x}$$

Also $u = x \log xy$

$$\frac{\partial u}{\partial x} = x \times \frac{1}{xy} \times y + \log xy = 1 + \log xy$$

& $$\frac{\partial u}{\partial y} = x \times \frac{1}{\cancel{x}y} \times \cancel{x} = x/y$$

Sub these values in (1), we get

$$\frac{du}{dx} = (1+\log xy) + \left(\frac{x}{y}\right) \times -\left(\frac{x^2+y}{y^2+x}\right) \quad \text{Ans}$$

Q If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = ?$

Sol let $v = \frac{y-x}{xy}$, & $w = \frac{z-x}{xz}$

$= \frac{1}{x} - \frac{1}{y}$ & $= \frac{1}{x} - \frac{1}{z}$

so that $u = u(v, w)$

$u \to \underbrace{v, w}_{} \to \underbrace{x, y, z}_{}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left[\frac{-1}{x^2}\right] + \frac{\partial u}{\partial w}\left(\frac{-1}{x^2}\right) \quad -(1)$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left[\frac{+1}{y^2}\right] + \frac{\partial u}{\partial w}\left[0\right] \quad -(2)$$

& $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}\left[0\right] + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right) \quad -(3)$$

Multiply (1) by x^2, (2) by y^2 & (3) by z^2 & on adding

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0 \quad \text{Ans}$$

(A) 0 (d) 3
(B) 1
(C) 2

Q If $u = F(x-y, y-z, z-x)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = ?$$

Sol $u = F(x-y, y-z, z-x)$

let $x - y = r$, $y - z = s$, $z - x = t$

$u = F(r, s, t)$ where $r = x-y$, $s = y-z$ & $t = z-x$

$u \to \underbrace{r, s, t}_{} \to x, y \,\&\, z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1) \quad -(1)$$

Also
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} = \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial t}(0) \quad -(2)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}=\underbrace{\frac{\partial u}{\partial r}(-1)}+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}(0) \quad -(2)$$

Sub.

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\underbrace{\frac{\partial u}{\partial t}(1)} \quad -(3)$$

on adding (1), (2) & (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Q. If $z=f(x,y)$ and $\underline{x=e^u\cos v}$, $\underline{y=e^u\sin v}$,

\# Prove that (i) $x\frac{\partial z}{\partial v}+y\frac{\partial z}{\partial u}=e^{2u}\frac{\partial z}{\partial y}$

(ii) $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$

Sol

$$z \longrightarrow x,y \longrightarrow u,v$$

$y\Big[\ \because \frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}\left[e^u\cos v\right]+\frac{\partial z}{\partial y}\left[e^u\sin v\right] \quad -(1)$

$x\Big[$ Also $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}\left(-e^u\sin v\right]+\frac{\partial z}{\partial y}\left(e^u\cos v\right) \quad -(2)$

Multiply (1) by y & (2) by x & on adding we get

$$x\frac{\partial z}{\partial v}+y\frac{\partial z}{\partial u}=-x(e^u\sin v)\frac{\partial z}{\partial x}+xe^u\cos v\frac{\partial z}{\partial y}$$
$$+ye^u\cos v\frac{\partial z}{\partial x}+ye^u\sin v\frac{\partial z}{\partial y}$$

$$=-\cancel{\underline{(e^u\cos v)(e^u\sin v)}\frac{\partial z}{\partial x}}+(e^u\cos v)^2\frac{\partial z}{\partial y}$$
$$+\cancel{\underline{(e^u\sin v)(e^u\cos v)\frac{\partial z}{\partial x}}}+(e^u\sin v)^2\frac{\partial z}{\partial y}$$

$$=e^{2u}\cos^2 v\frac{\partial z}{\partial y}+e^{2u}\sin^2 v\frac{\partial z}{\partial y}$$
$$=e^{2u}\left[\cos^2 v+\sin^2 v\right]\frac{\partial z}{\partial y}$$
$$=e^{2u}\frac{\partial z}{\partial y}$$

Squaring & adding (1) & (2), we get

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=\left[e^u\cos v\frac{\partial z}{\partial x}+e^u\sin v\frac{\partial z}{\partial y}\right]^2+$$
$$\left[-e^u\sin v\frac{\partial z}{\partial x}+e^u\cos v\frac{\partial z}{\partial y}\right]^2$$

$$=e^{2u}\cos^2 v\left(\frac{\partial z}{\partial x}\right)^2+e^{2u}\sin^2 v\left(\frac{\partial z}{\partial y}\right)^2+\cancel{2e^{2u}\sin v\cos v\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}$$
$$+e^{2u}\sin^2 v\left(\frac{\partial z}{\partial x}\right)^2+e^{2u}\cos^2 v\left(\frac{\partial z}{\partial y}\right)^2-\cancel{2e^{2u}\sin v\cos v\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}}$$

$$=e^{2u}\left(\frac{\partial z}{\partial x}\right)^2+e^{2u}\left(\frac{\partial z}{\partial y}\right)^2=e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$

$$= e^{2u}\left(\frac{\partial z}{\partial x}\right)^2 + e^{2u}\left(\frac{\partial z}{\partial y}\right)^2 = e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

Jacobians

If u and v are functions of two independent variables x & y, then the determinant

$$\begin{array}{l} 1^{st} \longrightarrow \\ 2^{nd} \longrightarrow \end{array} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called Jacobian of u, v w.r.t x, y.

and it can be written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(u,v)$ or $J\left(\frac{u,v}{x,y}\right)$.

(ii) $u, v, w \longrightarrow x, y, z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Change of variables

Suppose $f(x,y)$ is a function of two independent variables x & y and x, y are functions of two new independent variables u, v given by $x = \phi(u,v)$, $y = \psi(u,v)$. By chain rule.

$$f \longrightarrow x, y \longrightarrow u, v$$

$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$

& $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$

or $\frac{\partial x}{\partial u}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial y}{\partial u}\cdot\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial u}$ —(1)

& $\frac{\partial x}{\partial v}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial y}{\partial v}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial v}$ —(2)

By Crammer Rule

$\frac{\partial f}{\partial x} = \frac{\Delta_1}{\Delta}$ & $\frac{\partial f}{\partial y} = \frac{\Delta_2}{\Delta}$

Crammer Rule

$ax + by = c$
$a_1x + b_1y = c_1$

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$

Here $\Delta = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \cdots \end{vmatrix}$

Here $\Delta = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ ✓

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$

$\Delta = \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}$

$\Delta_1 = \begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}$

$$= \frac{\partial(x,y)}{\partial(u,v)} \equiv J$$

Also $\Delta_1 = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(f,y)}{\partial(u,v)}$

Also $\Delta_2 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial f}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial f}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{vmatrix}$

$$= - \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}$$

$$= - \frac{\partial(f,x)}{\partial(u,v)}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\Delta_1}{\Delta} = \frac{\frac{\partial(f,y)}{\partial(u,v)}}{J} = \frac{1}{J}\frac{\partial(f,y)}{\partial(u,v)}$$

$$\& \quad \frac{\partial f}{\partial y} = \frac{\Delta_2}{\Delta} = \frac{-\frac{\partial(f,x)}{\partial(u,v)}}{J} = -\frac{1}{J}\frac{\partial(f,y)}{\partial(u,v)}$$

where $J = \frac{\partial(u,v)}{\partial(x,y)}$

Remark $f \to x, y \to u, v$

\# ✓ $\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f,y)}{\partial(u,v)}$ & $\frac{\partial f}{\partial y} = -\frac{1}{J}\frac{\partial(f,x)}{\partial(u,v)}$] #

Ex If $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Sol Here $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\boxed{J = r}$$

$\partial(f,x)$ ... $\frac{\partial f}{}$ $\frac{\partial f}{}$ ... $\frac{\partial f}{}$ $\frac{\partial f}{}$

$$\boxed{J = r}$$

$$\frac{\partial(f,x)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \cos\theta & -r\sin\theta \end{vmatrix}$$

$$= -r\sin\theta \frac{\partial f}{\partial r} - \cos\theta \frac{\partial f}{\partial \theta}$$

Also $\frac{\partial(f,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos\theta \frac{\partial f}{\partial r} - \sin\theta \frac{\partial f}{\partial \theta}$

Now $\frac{\partial f}{\partial x} = \frac{1}{J} \frac{\partial(f,y)}{\partial(r,\theta)} = \frac{1}{r}\left[r\cos\theta \frac{\partial f}{\partial r} - \sin\theta \frac{\partial f}{\partial \theta}\right]$

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \quad \text{— (A)}$$

Also $\frac{\partial f}{\partial y} = -\frac{1}{J}\frac{\partial(f,x)}{\partial(r,\theta)} = -\frac{1}{r}\left[-r\sin\theta \frac{\partial f}{\partial r} - \cos\theta \frac{\partial f}{\partial \theta}\right]$

$$\frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \quad \text{— (B)}$$

On squaring (A) & (B) & on adding

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \cos^2\theta\left(\frac{\partial f}{\partial r}\right)^2 + \frac{\sin^2\theta}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 - \cancel{\frac{2\cos\theta\sin\theta}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial \theta}}$$
$$+ \sin^2\theta\left(\frac{\partial f}{\partial r}\right)^2 + \frac{\cos^2\theta}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \cancel{\frac{2\cos\theta\sin\theta}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial \theta}}$$

$$= \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 \quad \text{RHS}$$

#

$$f - x, y, z \longrightarrow u, v, w$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{J}\frac{\partial(f,y,z)}{\partial(u,v,w)} \\ \frac{\partial f}{\partial y} &= -\frac{1}{J}\frac{\partial(f,x,z)}{\partial(u,v,w)} \\ \frac{\partial f}{\partial z} &= \frac{1}{J}\frac{\partial(f,x,y)}{\partial(u,v,w)} \end{aligned}\right\} \quad \text{where } J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

<u>Note</u> The variable of transformation $u = f(x,y,z)$, $v = g(x,y,z)$

$w = h(x,y,z)$ are functionally related if

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

that is, $\exists$ a relationship between the variables u, v, w and the transformation is not independent.

Q. Show that the variables $u = x - y + z$, $v = x + y - z$, $w = x^2 + xz - xy$ are functionally related. Find the relationship between them.

Sol

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2x+z-y & -x & x \end{vmatrix}$$

0 Yes
No

$$= 0$$

Hence, the variables are related

$$w = x(x+z-y) = xu$$

Now
$$u = x - y + z$$
$$v = x + y - z$$
$$u + v = 2x \quad \text{or} \quad x = \frac{u+v}{2}$$

$$\therefore w = xu = \frac{(u+v)}{2} u \quad \text{or} \quad \boxed{2w = (u+v)u}$$

Q. If $u(x,y) = x^2 \tan^{-1} y/x - y^2 \tan^{-1} x/y$, $x > 0$, $y > 0$, then evaluate $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Sol $f \to x, y \to (n)$

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = \boxed{n(n-1) f}$$

Here
$$u(\lambda x, \lambda y) = \lambda^2 x^2 \tan^{-1}(y/x) - \lambda^2 y^2 \tan^{-1}\left(\frac{x}{y}\right)$$
$$= \lambda^2 u(x,y)$$

$\therefore$ By Euler's Thm

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2(2-1)u$$
$$= 2u$$

Q. Let $u = \frac{x^3 + y^3}{x+y}$, $(x,y) \neq (0,0)$. Then evaluate

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}$$

Sol $u \to$ is a homogeneous fun of degree '2'

$\therefore$ By Euler's Theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

Diff it partially w.r.t 'x', we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \frac{\partial u}{\partial x}$$

or $$\boxed{x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} = 0}$$

Q let $f(x,y)$ and $g(x,y)$ be two homogeneous functions of degree m & n respectively where $m \neq 0$ let $h = f+g$

If $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = 0$, then show that $f = \alpha g$ for some scalar α

Proof Here f & g are homogeneous fns of degree m & n

$\therefore$ By Euler's Theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf$$

& $$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng$$

on adding $$x\left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right] + y\left[\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right] = mf + ng$$

or $$x\frac{\partial}{\partial x}(f+g) + y\frac{\partial}{\partial y}[f+g] = mf+ng$$

let $h = f+g$

$$\therefore \quad x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = mf + ng$$

If $$x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = 0$$

then $$mf + ng = 0$$

$$\Rightarrow mf = -ng$$

$$\Rightarrow f = \left(\frac{-n}{m}\right) g$$

$$\Rightarrow \boxed{f = \alpha g} \quad \text{where } \alpha = \frac{-n}{m} \text{ is some scalar}$$

change of variable — Spherical polar coordinates

Q If $u = f(x,y,z)$ & $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial \phi}\right)^2$$

Sol
$$f \to x,y,z \to r,\theta \,\&\, \phi$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$z = r\ldots$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi \right] + r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi \right]$$

$$= \cos\theta\; r^2\sin\theta\cos\theta\,[1] + r\sin\theta\; r\sin^2\theta\,[1]$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta\,[1] = \underline{r^2\sin\theta}$$

$$\frac{\partial(f,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$\boxed{y = r\sin\theta\sin\phi}$

$z = r\cos\theta$

$$= \cos\theta \begin{vmatrix} \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + r\sin\theta \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \phi} \\ \sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$= \cos\theta\left[r\sin\theta\cos\phi\frac{\partial f}{\partial \theta} - r\cos\theta\sin\phi\frac{\partial f}{\partial \phi} \right] + r\sin\theta\left[r\sin\theta\cos\phi\frac{\partial f}{\partial r} - \sin\theta\sin\phi\frac{\partial f}{\partial \phi} \right]$$

$$= r\sin\theta\cos\theta\cos\phi\frac{\partial f}{\partial \theta} - r\cos^2\theta\sin\phi\frac{\partial f}{\partial \phi} + r^2\sin^2\theta\cos\phi\frac{\partial f}{\partial r} - r\sin^2\theta\sin\phi\frac{\partial f}{\partial \phi}$$

$$= r^2\sin^2\theta\cos\phi\frac{\partial f}{\partial r} + r\sin\theta\cos\theta\cos\phi\frac{\partial f}{\partial \theta} - r\sin\phi\frac{\partial f}{\partial \phi}$$

Also

$$\frac{\partial(f,x,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = -r^2\sin^2\theta\sin\phi\frac{\partial f}{\partial r} - r\sin\theta\cos\theta\sin\phi\frac{\partial f}{\partial \theta} - r\cos\phi\frac{\partial f}{\partial \phi}$$

$$\frac{\partial(f,x,y)}{\partial(r,\theta,\phi)} = -r^2\sin\theta\cos\theta\frac{\partial f}{\partial r} - r\sin^2\theta\frac{\partial f}{\partial \theta}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{1}{\ }\,\frac{\partial(f,y,z)}{\ } =$$

$\partial(r, \theta, \phi)$

$$\frac{\partial \phi}{\partial y} = -\frac{1}{J} \frac{\partial(f, x, z)}{\partial(r, \theta, \phi)} =$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{J} \frac{\partial(f, x, y)}{\partial(r, \theta, y)} =$$