STOKE'S THEOREM* (*Relation between line and surface integrals*)

If S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$ *be any continuously differentiable vector point function, then*

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S curl\, \mathbf{F} \cdot \mathbf{N} ds$$

where $\mathbf{N} = \cos \alpha \mathbf{I} + \cos \beta \mathbf{J} + \cos \gamma \mathbf{K}$ *is a unit external normal at any point of S.*

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* Named after an Irish mathematician *Sir George Gabriel Stokes* (1819–1903) who became professor in Cambridge. His important contributions are to infinite series, geodesy and theory of viscous fluids.

Verify Stoke's theorem for $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}$ *taken around the rectangle bounded by the lines* $x = \pm a$, $y = 0$, $y = b$.

Solution. Let $ABCD$ be the given rectangle as shown in Fig. 8.16.

$$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = \int_{AB} \mathbf{F}\cdot d\mathbf{R} + \int_{BC} \mathbf{F}\cdot d\mathbf{R} + \int_{CD} \mathbf{F}\cdot d\mathbf{R} + \int_{DA} \mathbf{F}\cdot d\mathbf{R}$$

and
$$\mathbf{F}\cdot d\mathbf{R} = [(x^2+y^2)\mathbf{I} - 2xy\mathbf{J}]\cdot(\mathbf{I}dx + \mathbf{J}dy) = (x^2+y^2)dx - 2xydy$$

Along AB, $x = a$ (*i.e.*, $dx = 0$) and y varies from 0 to b.

$$\therefore \qquad \int_{AB} \mathbf{F}\cdot d\mathbf{R} = -2a\int_0^b y\,dy = -2a\cdot\frac{b^2}{2} = -ab^2.$$

Similarly,
$$\int_{BC} \mathbf{F}\cdot d\mathbf{R} = \int_a^{-a}(x^2+b^2)dx = -\frac{2a^3}{3} - 2ab^2\cdot$$

$$\int_{CD} \mathbf{F}\cdot d\mathbf{R} = 2a\int_b^0 y\,dy = -ab^2$$

and
$$\int_{DA} \mathbf{F}\cdot d\mathbf{R} = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3}\cdot$$

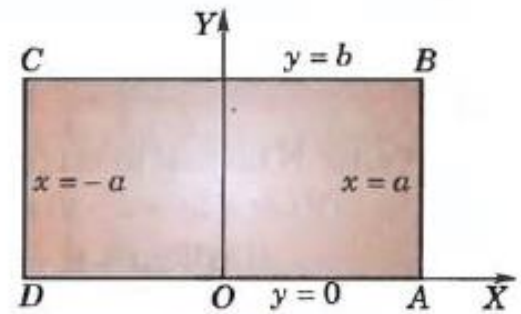

Fig. 8.16

Thus
$$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = -4ab^2 \qquad ...(i)$$

Also since $\quad$ curl $\mathbf{F} = -4\mathbf{K}y$

$$\therefore \qquad \int_S \text{curl}\,\mathbf{F}\cdot\mathbf{N}ds = \int_0^b\int_{-a}^{a} -4\mathbf{K}y\cdot\mathbf{K}\,dxdy = -4\int_0^b\int_{-a}^{a} y\,dxdy$$

$$= -4\int_0^b |x|_{-a}^{a}\,y\,dy = -8a\left|\frac{y^2}{2}\right|_0^b = -4ab^2 \qquad ...(ii)$$

Hence Stoke's theorem is verified from the equality of (*i*) and (*ii*).

Verify Stoke's theorem for the vector field $\mathbf{F} = (2x - y)\,\mathbf{I} - yz^2\,\mathbf{J} - y^2z\,\mathbf{K}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy-plane.

Solution. The projection of the upper half of given sphere on the xy-plane ($z = 0$) is the circle $c[x^2 + y^2 = 1]$ (Fig. 8.17).

$$\oint_c \mathbf{F}.\,d\mathbf{R} = \oint_c [(2x - y)dx - yz^2\,dy - y^2\,zdz] = \oint_c (2x - y)dx \qquad [z = 0 \text{ in the } xy\text{-plane}]$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta) \qquad [\text{Putting } x = \cos\theta, y = \sin\theta]$$

$$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\,d\theta = \times 0 + 4\int_0^{\pi/2} \sin^2\theta\,d\theta = \pi. \qquad ...(i)$$

Now
$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{I} + 0\,\mathbf{J} + \mathbf{K} = \mathbf{K}$$

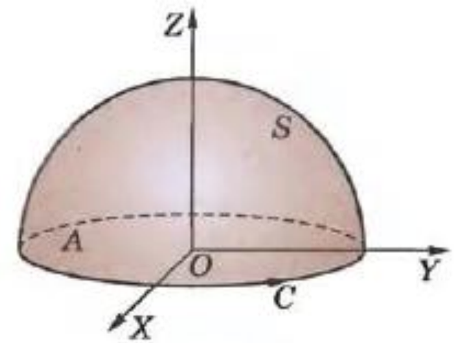

Fig. 8.17

$$\therefore \quad \int \text{curl}\,\mathbf{F}.\,\mathbf{N}ds = \int_S \mathbf{K}.\mathbf{N}\,ds = \int_A \mathbf{K}.\mathbf{N}\frac{dxdy}{|\mathbf{N}.\mathbf{K}|}$$

where A is the projection of S on xy-plane and $ds = dxdy/\mathbf{N}.\mathbf{K}$

$$= \int_A dx\,dy = \text{area of circle } C = \pi \qquad ...(ii)$$

Hence, the Stokes theorem is verified from the equality of (i) and (ii).

Uses Stoke's theorem evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ *where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

Solution. Here $\mathbf{F} = (x + y)\,\mathbf{I} + (2x - z)\,\mathbf{J} + (y + z)\,\mathbf{K}$

$$\therefore \quad \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\mathbf{I} + \mathbf{K}$$

Also equation of the plane through A, B, C (Fig. 8.18) is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \text{ or } 3x + 2y + z = 6$$

Vector **N** normal to this plane is

$$\nabla\,(3x + 2y + z - 6) = 3\mathbf{I} + 2\mathbf{J} + \mathbf{K}$$

$$\therefore \quad \hat{\mathbf{N}} = \frac{3\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{(9 + 4 + 1)}} = \frac{1}{\sqrt{14}}(3\mathbf{I} + 2\mathbf{J} + \mathbf{K})$$

Hence $\int_C [(x + y)dx + (2x - z)dy + (y + z)dz] = \int_C \mathbf{F} \cdot d\mathbf{R}$

$$= \int_S \text{curl } \mathbf{F} \cdot \hat{\mathbf{N}}\, ds \qquad \text{where } S \text{ is the triangle } ABC$$

$$= \int_S (2\mathbf{I} + \mathbf{K}) \cdot \left(\frac{3\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{14}}\right) ds = \frac{1}{\sqrt{14}}(6 + 1)\int_S ds$$

$$= \frac{7}{\sqrt{14}}\,(\text{Area of } \Delta ABC) = \frac{7}{\sqrt{14}} \cdot 3\sqrt{14} = 21.$$

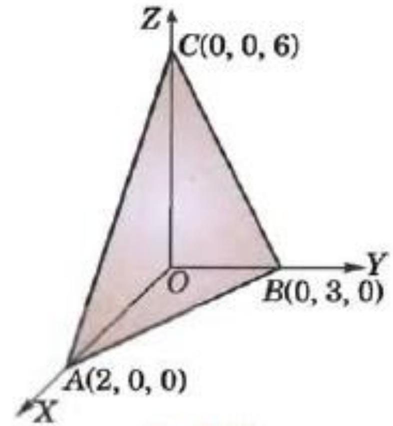

Fig. 8.18

Apply Stoke's theorem to evaluate $\int_C (ydx + zdy + xdz)$ where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

Solution. The curve C is evidently a circle lying in the plane $x + z = a$, and having $A(a, 0, 0)$, $B(0, 0, a)$ as the extremities of the diameter (Fig. 8.19).

$$\therefore \quad \int_C (y\,dx + z\,dy + x\,dz) = \int_C (y\mathbf{I} + z\mathbf{J} + x\mathbf{K}) \,.\, d\mathbf{R}$$

$$= \int_S \text{curl}\,(y\mathbf{I} + z\mathbf{J} + x\mathbf{K}) \,.\, \mathbf{N}ds$$

where S is the circle on AB as diameter and $\mathbf{N} = \frac{1}{\sqrt{2}}\mathbf{I} + \frac{1}{\sqrt{2}}\mathbf{K}$

$$= \int_S -(\mathbf{I} + \mathbf{J} + \mathbf{K}) \,.\, \left(\frac{1}{\sqrt{2}}\mathbf{I} + \frac{1}{\sqrt{2}}\mathbf{K}\right) ds$$

$$= -\frac{2}{\sqrt{2}}\int_S ds = -\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}.$$

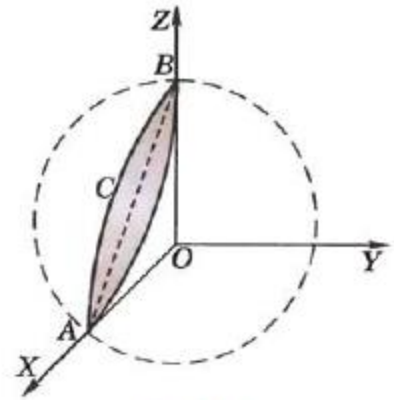

Fig. 8.19

If S be any closed surface, prove that $\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$.

Solution. Cut open the surface S by any plane and let S_1, S_2 denote its upper and lower portions. Let C be the common curve bounding both these portions.

$$\therefore \qquad \int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{R} - \int_C \mathbf{F} \cdot d\mathbf{R} = 0,$$

on applying Stoke's theorem. The second integral is negative because it is traversed in a direction opposite to that of the first.

GAUSS DIVERGENCE THEOREM* (*Relation between surface and volume integrals*)

If $\mathbf{F}$ is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F} \,.\, \mathbf{N} ds = \int_E div\, \mathbf{F}\, dv$$

where $\mathbf{N}$ *is the unit external normal vector.*

Verify Divergence theorem for $\mathbf{F} = (x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$ *taken over the rectangular parallelepiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Solution. As $\quad \operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$

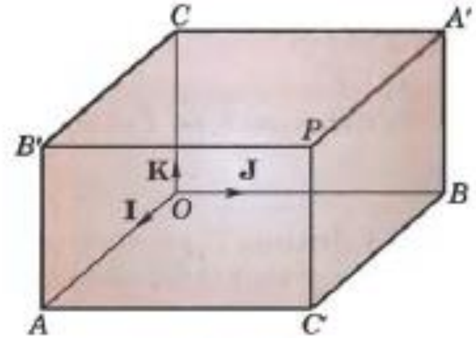

Fig. 8.21

$$\therefore \quad \int_R \operatorname{div} \mathbf{F}\, dv = 2\int_0^c \int_0^b \int_0^a (x + y + z)\, dxdydz$$

$$= 2\int_0^c dz \int_0^b dy\left(\frac{a^2}{2} + ya + za\right)$$

$$= 2\int_0^c dz\left(\frac{a^2}{2}b + \frac{ab^2}{2} + abz\right)$$

$$= 2\left(\frac{a^2 b}{2}c + \frac{ab^2}{2}c + ab\frac{c^2}{2}\right)$$

$$= abc\,(a + b + c) \qquad \text{...}(i)$$

Also $\quad \int_S \mathbf{F}.\mathbf{N}ds = \int_{S_1} \mathbf{F}.\mathbf{N}ds + \int_{S_2} \mathbf{F}.\mathbf{N}ds + ... + \int_{S_6} \mathbf{F}.\mathbf{N}ds$

where S_1 in the face $OAC'B$, S_2 the face $CB'PA'$, S_3 the face $OBA'C$, S_4 the face $AC'PB'$, S_5 the face $OCB'A$ and S_6 the face $BAP'C'$ (Fig. 8.21).

Now $\quad \int_{S_1} \mathbf{F}.\mathbf{N}ds = \int_{S_1} \mathbf{F}.(-\mathbf{K})\, ds = -\int_0^b \int_0^a (0 - xy)\, dxdy = \frac{a^2 b^2}{4}$

$$\int_{S_2} \mathbf{F}.\mathbf{N}ds = \int_{S_2} \mathbf{F}.\,\mathbf{K}ds = \int_0^b \int_0^a (c^2 - xy)\, dxdy = abc^2 - \frac{a^2 b^2}{4}$$

Similarly, $\quad \int_{S_3} \mathbf{F}.\mathbf{N}ds = \frac{b^2 c^2}{4}, \int_{S_4} \mathbf{F}.\,\mathbf{N}ds = a^2 bc - \frac{b^2 c^2}{4},$

$$\int_{S_5} \mathbf{F}.\mathbf{N}ds = \frac{c^2 a^2}{4} \text{ and } \int_{S_6} \mathbf{F}.\,\mathbf{N}ds = ab^2 c - \frac{c^2 a^2}{4}$$

Thus $\quad \int_S \mathbf{F}.\mathbf{N}ds = abc(a + b + c) \qquad \text{...}(ii)$

Hence the theorem is verified from the equality of (i) and (ii).

Evaluate $\int_S \mathbf{F} \cdot \mathbf{ds}$ *where* $\mathbf{F} = 4x\mathbf{I} - 2y^2\mathbf{J} + z^2\mathbf{K}$ *and* S *is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.

Solution. By divergence theorem,

$$\int_S \mathbf{F} \,.\, d\mathbf{s} = \int_V div\, \mathbf{F}\, dv$$

$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) \right] dv$$

$$= \iiint_V ((4 - 4y + 2z)\, dxdydz$$

$$= \int_{-2}^{2} \int_{\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \int_0^3 (4 - 4y + 2z)\, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left| 4z - 4yz + z^2 \right|_0^3 dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (12 - 12y + 9)\, dydx$$

$$= \int_{-2}^{2} \left| 21y - 6y^2 \right|_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{(4 - x^2)}\, dx = 84 \int_0^2 \sqrt{(4 - x^2)}\, dx = 84 \left| \frac{x\sqrt{(4 - x^2)}}{2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right|_0^2 = 84\pi.$$

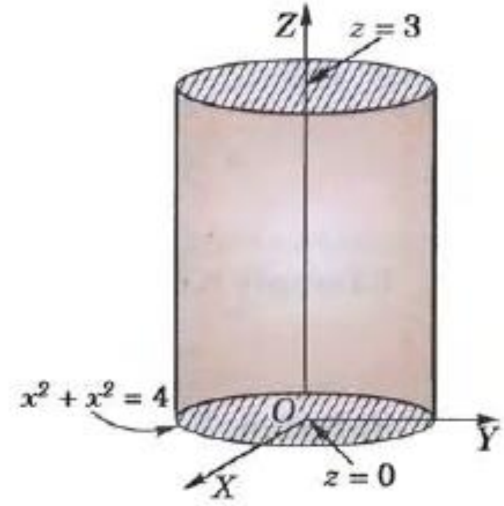

Fig. 8.22

Evaluate $\int_S (yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}) \,.\, d\mathbf{S}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.*

Solution. The surface of the region V: $OABC$ is piecewise smooth (Fig. 8.23) and is comprised of four surfaces (*i*) S_1 – circular quadrant OBC in the yz-plane,

(*ii*) S_2 – circular quadrant OCA in the zx-plane,

(*iii*) S_3 – circular quadrant OAB in the xy-plane,

and (*iv*) S–surface ABC of the sphere in the first octant.

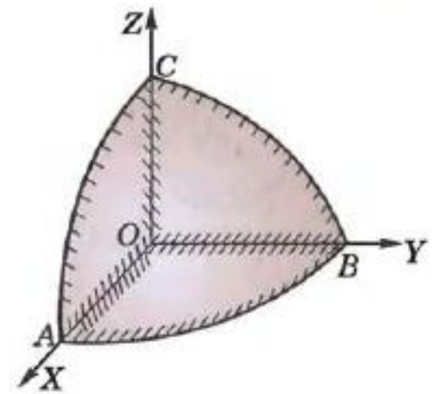

Fig. 8.23

Also $\quad \mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$

By Divergence theorem,

$$\int_V div\,\mathbf{F}\,dv = \int_{S_1} \mathbf{F}.\,d\mathbf{S} + \int_{S_2} \mathbf{F}.\,d\mathbf{S} + \int_{S_3} \mathbf{F}.\,d\mathbf{S} + \int_S \mathbf{F}.\,d\mathbf{S} \quad ...(1)$$

Now $\quad div\,\mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.$

For the surface S_1, $x = 0$

$$\therefore \quad \int_{S_1} \mathbf{F}.\,d\mathbf{S} = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} (yz\mathbf{I}).(-\,dydz\mathbf{I}) = -\int_0^a \int_0^{\sqrt{(a^2-y^2)}} yzdydz = = -\frac{a^4}{8}$$

Thus (1) becomes $0 = -\frac{3a^4}{8} + \int_S \mathbf{F}.\,d\mathbf{S}$ whence $\int_S \mathbf{F}.\,d\mathbf{S} = 3a^4/8.$