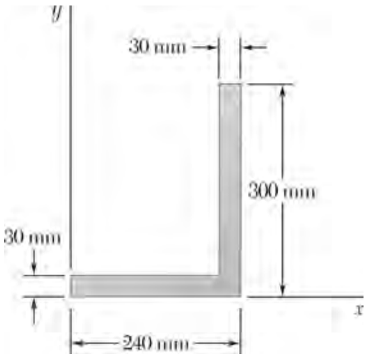
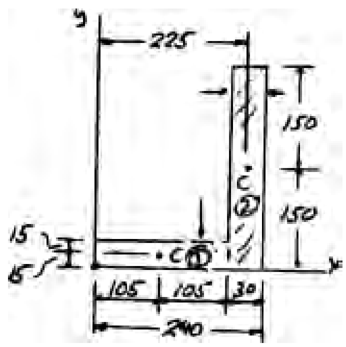
CHAPTER 5

	**PROBLEM 5.1** Locate the centroid of the plane area shown.

SOLUTION

Dimensions in mm



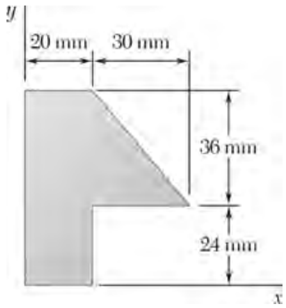
	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	6300	105	15	0.66150×10^6	0.094500×10^6
2	9000	225	150	2.0250×10^6	1.35000×10^6
Σ	15300			2.6865×10^6	1.44450×10^6

Then

$$\bar{X} = \frac{\Sigma\,\bar{x}A}{\Sigma A} = \frac{2.6865\times 10^6}{15300} \qquad \bar{X} = 175.6 \text{ mm} \blacktriangleleft$$

$$\bar{Y} = \frac{\Sigma\,\bar{y}A}{\Sigma A} = \frac{1.44450\times 10^6}{15300} \qquad \bar{Y} = 94.4 \text{ mm} \blacktriangleleft$$

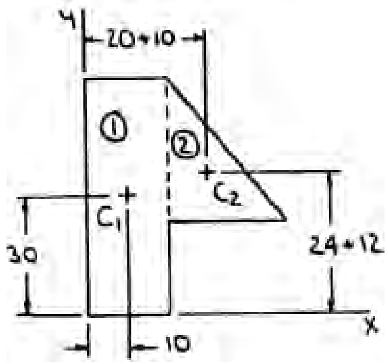
PROBLEM 5.2



Locate the centroid of the plane area shown.

SOLUTION

Dimensions in mm



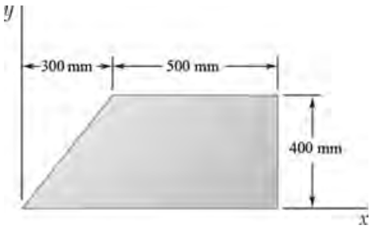
	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	1200	10	30	12000	36000
2	540	30	36	16200	19440
Σ	1740			28200	55440

Then

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{28200}{1740} \qquad \bar{X} = 16.21 \text{ mm} \blacktriangleleft$$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{55440}{1740} \qquad \bar{Y} = 31.9 \text{ mm} \blacktriangleleft$$

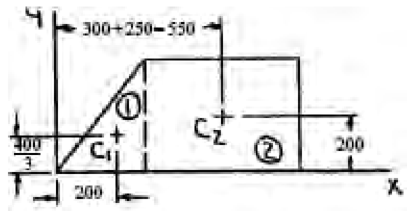
PROBLEM 5.3



Locate the centroid of the plane area shown.

SOLUTION

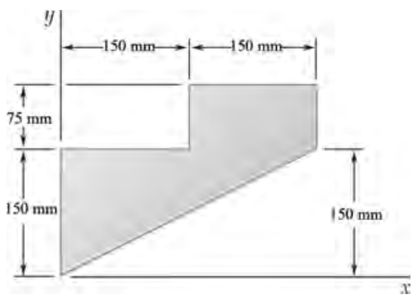
Dimensions in mm



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{2}\times 300\times 400 = 60000$	200	400/3	12 10^6	8 10^6
2	$500\times 400 = 200000$	550	200	110 10^6	40 10^6
Σ	260000			122 10^6	48 10^6

Then $$\bar{X} = \frac{\Sigma\,\bar{x}A}{\Sigma A} = \frac{122\times 10^6}{260\times 10^3} = 469.23$$ $\bar{X} = 469$ mm ◀

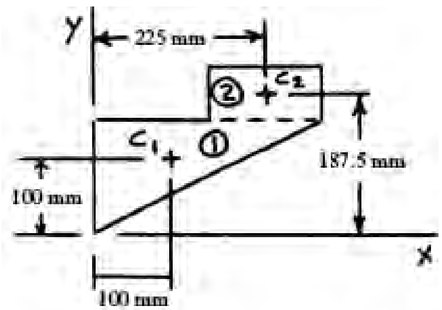
$$\bar{Y} = \frac{\Sigma\,\bar{y}A}{\Sigma A} = \frac{48\times 10^6}{260\times 10^3} = 184.62$$ $\bar{Y} = 184.6$ mm ◀



PROBLEM 5.4

Locate the centroid of the plane area shown.

SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{2}(300)(150) = 22500$	100	100	$2.25 \quad 10^6$	$2.25 \quad 10^6$
2	$(150)(75) = 11250$	225	187.5	$2.53125 \quad 10^6$	$2.109375 \quad 10^6$
Σ	33750			$4.78125 \quad 10^6$	$4.359375 \quad 10^6$

Then

$$\bar{X}A = \Sigma\bar{x}A$$

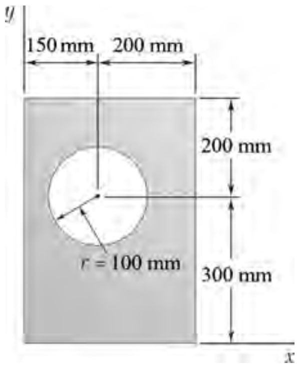
$$\bar{X}(33750) = 4.78125\times10^6$$

$\bar{X} = 141.7$ mm ◀

$$\bar{Y}A = \Sigma\bar{y}A$$

$$\bar{Y}(33750) = 4.359375\times10^6$$

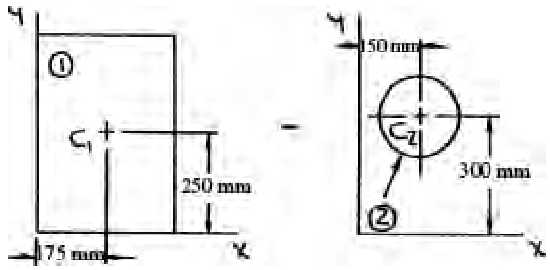
$\bar{Y} = 129.2$ mm ◀



PROBLEM 5.5

Locate the centroid of the plane area shown.

SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$350 \times 500 = 175 \times 10^3$	175	250	30.625 10^6	43.75 10^6
2	$-\pi(100)^2 = -31.4159 \times 10^3$	150	300	-4.7124 10^6	-9.4248 10^6
Σ	143.5841 10^3			25.9126 10^6	34.3252 10^6

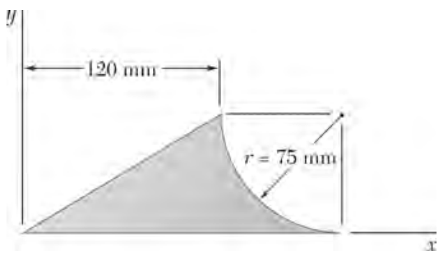
Then

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{25.9126 \times 10^6}{143.5841 \times 10^3}$$

$\bar{X} = 180.5$ mm ◀

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{34.3252 \times 10^6}{143.5841 \times 10^3}$$

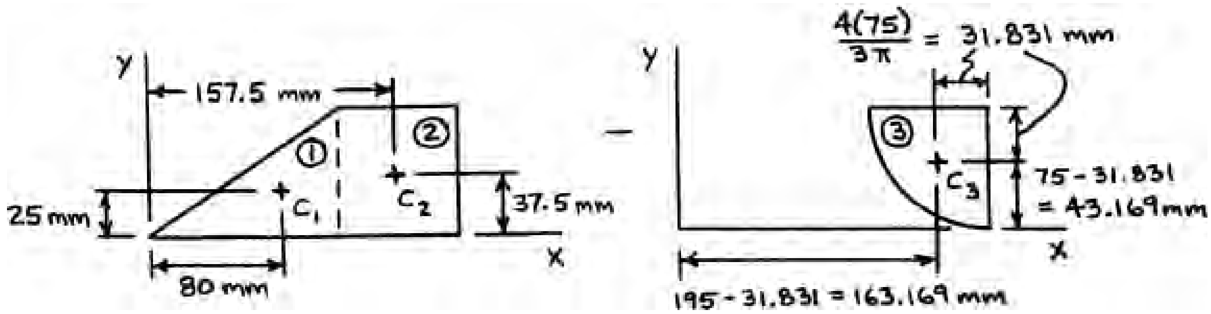
$\bar{Y} = 239$ mm ◀



PROBLEM 5.6

Locate the centroid of the plane area shown.

SOLUTION

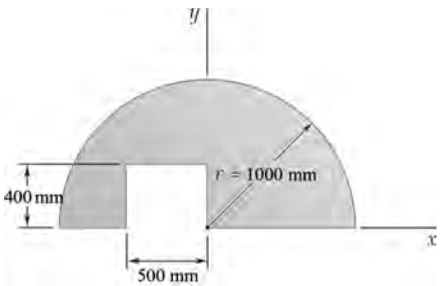


	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{2}(120)(75) = 4500$	80	25	360×10^3	112.5×10^3
2	$(75)(75) = 5625$	157.5	37.5	885.94×10^3	210.94×10^3
3	$-\frac{\pi}{4}(75)^2 = -4417.9$	163.169	43.169	-720.86×10^3	-190.716×10^3
Σ	5707.1			525.08×10^3	132.724×10^3

Then $\bar{X}A = \Sigma \bar{x} A \qquad \bar{X}(5707.1) = 525.08 \times 10^3 \qquad \bar{X} = 92.0 \text{ mm}$ ◀

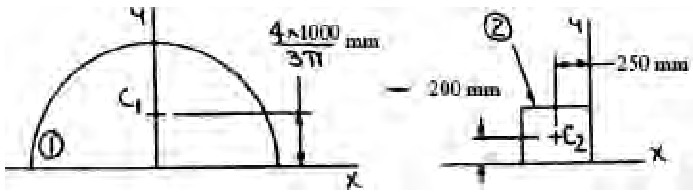
$\bar{Y}A = \Sigma \bar{y} A \qquad \bar{Y}(5707.1) = 132.724 \times 10^3 \qquad \bar{Y} = 23.3 \text{ mm}$ ◀

PROBLEM 5.7



Locate the centroid of the plane area shown.

SOLUTION

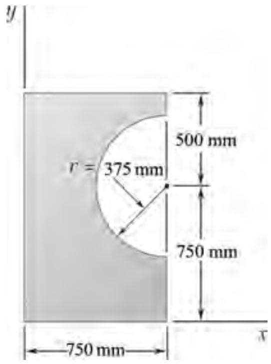


	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{\pi}{2}(1000)^2 = 1.57080 \times 10^6$	0	424.413	0	666.668 10^6
2	$-500 \times 400 = -0.2 \times 10^6$	–250	200	50 10^6	–40 10^6
Σ	1.3708 10^6			50 10^6	626.668 10^6

Then

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{50 \times 10^6}{1.3708 \times 10^6} \qquad \bar{X} = 36.5 \text{ mm} \blacktriangleleft$$

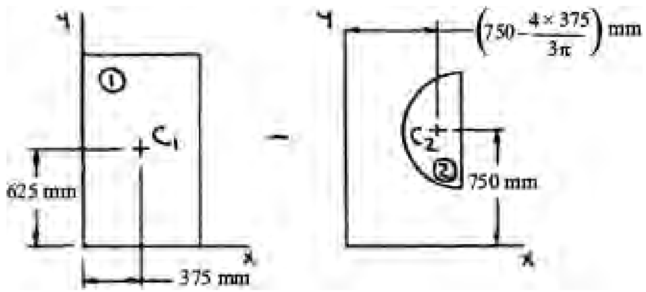
$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{626.668 \times 10^6}{1.3708 \times 10^6} \qquad \bar{Y} = 457 \text{ mm} \blacktriangleleft$$



PROBLEM 5.8

Locate the centroid of the plane area shown.

SOLUTION

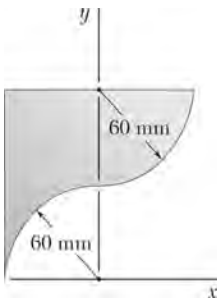


	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$750 \times 1200 = 0.9375 \times 10^6$	375	625	$351.5625 \quad 10^6$	$585.9375 \quad 10^6$
2	$-\dfrac{\pi}{2}(375)^2 = -0.22089 \times 10^6$	590.845	750	$-130.512 \quad 10^6$	$-165.6675 \quad 10^6$
Σ	$0.71661 \quad 10^6$			$221.0505 \quad 10^6$	$420.27 \quad 10^6$

Then

$$\bar{X} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{221.0505 \times 10^6}{0.71661 \times 10^6} \qquad \bar{X} = 308 \text{ mm} \blacktriangleleft$$

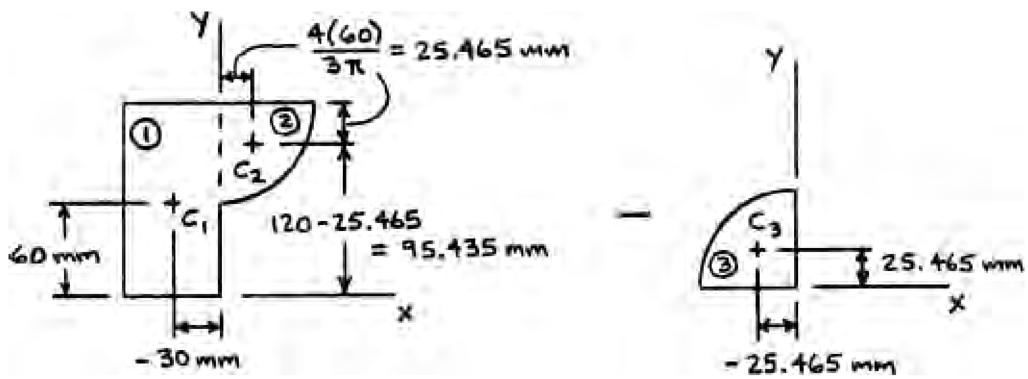
$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{420.27 \times 10^6}{0.71661 \times 10^6} \qquad \bar{Y} = 586 \text{ mm} \blacktriangleleft$$



PROBLEM 5.9

Locate the centroid of the plane area shown.

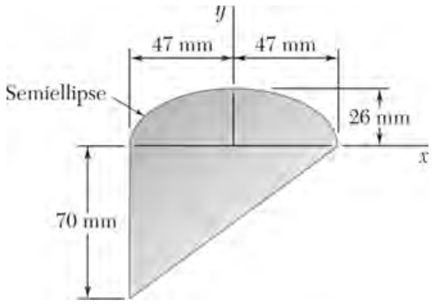
SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$(60)(120) = 7200$	–30	60	-216×10^3	432×10^3
2	$\frac{\pi}{4}(60)^2 = 2827.4$	25.465	95.435	72.000×10^3	269.83×10^3
3	$-\frac{\pi}{4}(60)^2 = -2827.4$	–25.465	25.465	72.000×10^3	-72.000×10^3
Σ	7200			-72.000×10^3	629.83×10^3

Then $\bar{X}A = \Sigma\bar{x}A \qquad \bar{X}(7200) = -72.000\times10^3 \qquad \bar{X} = -10.00 \text{ mm}$ ◀

$\bar{Y}A = \Sigma\bar{y}A \qquad \bar{Y}(7200) = 629.83\times10^3 \qquad \bar{Y} = 87.5 \text{ mm}$ ◀

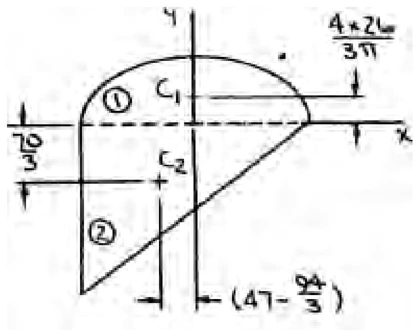


PROBLEM 5.10

Locate the centroid of the plane area shown.

SOLUTION

Dimensions in mm



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{\pi}{2} \times 47 \times 26 = 1919.51$	0	11.0347	0	21181
2	$\frac{1}{2} \times 94 \times 70 = 3290$	–15.6667	–23.333	–51543	–76766
Σ	5209.5			–51543	–55584

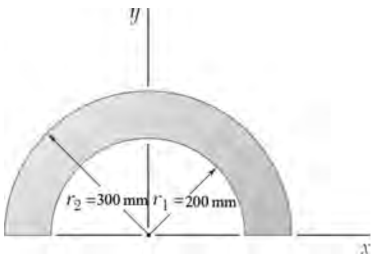
Then

$$\bar{X} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{-51543}{5209.5}$$

$\bar{X} = -9.89$ mm ◄

$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{-55584}{5209.5}$$

$\bar{Y} = -10.67$ mm ◄

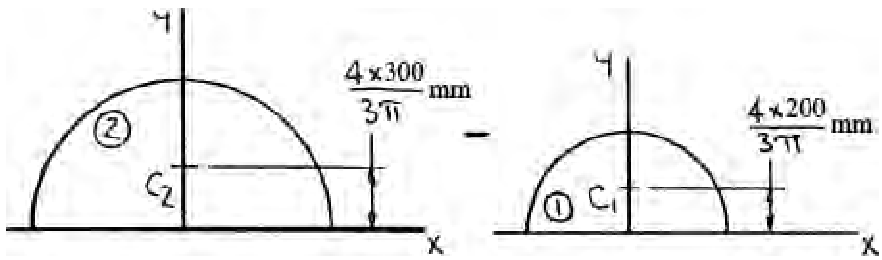


PROBLEM 5.11

Locate the centroid of the plane area shown.

SOLUTION

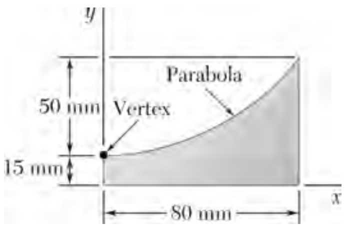
First note that symmetry implies $\bar{X} = 0$ ◀



	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$-\dfrac{\pi(200)^2}{2} = -20000\pi$	$\dfrac{800}{3\pi}$	$-5.3333\quad 10^6$
2	$\dfrac{\pi(300)^2}{2} = 45,000\pi$	$400/\pi$	$18\quad 10^6$
Σ	$78.5398\quad 10^3$		$12.6667\quad 10^6$

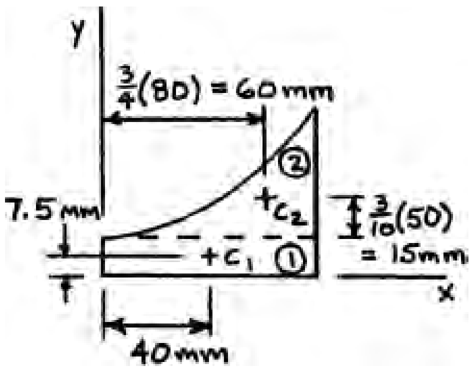
Then
$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{12.667\times 10^6}{78.5398\times 10^3}$$
or $\bar{Y} = 161.3$ mm ◀

PROBLEM 5.12



Locate the centroid of the plane area shown.

SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$(15)(80) = 1200$	40	7.5	48×10^3	9×10^3
2	$\frac{1}{3}(50)(80) = 1333.33$	60	30	80×10^3	40×10^3
Σ	2533.3			128×10^3	49×10^3

Then

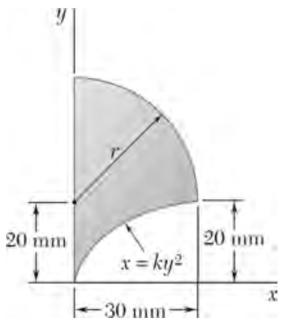
$$\bar{X}A = \Sigma \bar{x}A$$

$$\bar{X}(2533.3) = 128 \times 10^3 \qquad \bar{X} = 50.5 \text{ mm} \blacktriangleleft$$

$$\bar{Y}A = \Sigma \bar{y}A$$

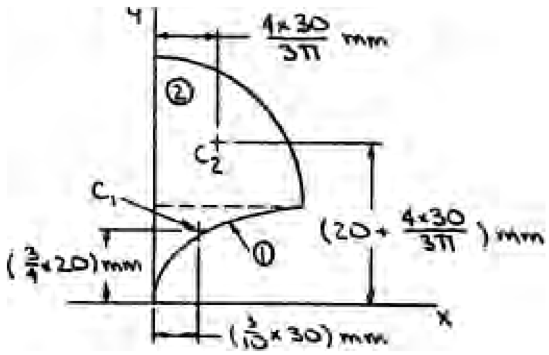
$$\bar{Y}(2533.3) = 49 \times 10^3 \qquad \bar{Y} = 19.34 \text{ mm} \blacktriangleleft$$

PROBLEM 5.13



Locate the centroid of the plane area shown.

SOLUTION

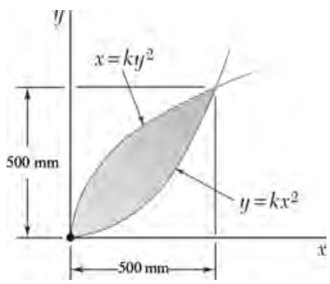


	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{3}\times 30\times 20 = 200$	9	15	1800	3000
2	$\frac{\pi}{4}(30)^2 = 706.86$	12.7324	32.7324	9000.0	23137
Σ	906.86			10800	26137

Then

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{10800}{906.86} \qquad \bar{X} = 11.91 \text{ mm} \blacktriangleleft$$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{26137}{906.86} \qquad \bar{Y} = 28.8 \text{ mm} \blacktriangleleft$$

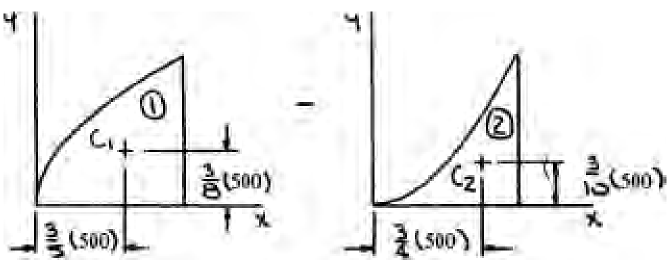


PROBLEM 5.14

Locate the centroid of the plane area shown.

SOLUTION

Dimensions in mm



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{2}{3}\times(500)(500)=\frac{0.5\times10^6}{3}$	300	187.5	50 10^6	31.25 10^6
2	$\frac{-1}{3}(500)(500)=\frac{-0.25\times10^6}{3}$	375	150	–31.25 10^6	–12.5 10^6
Σ	$\frac{0.25\times10^6}{3}$			18.75 10^6	18.75 10^6

Then

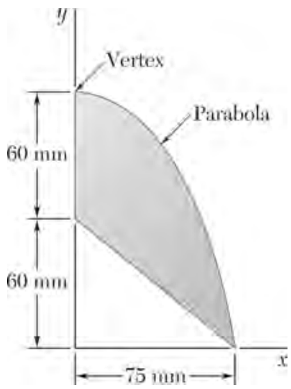
$$\bar{X}=\frac{\Sigma\bar{x}A}{\Sigma A}=\frac{18.75\times10^6}{\left(\frac{0.25\times10^6}{3}\right)}$$

$\bar{X}=225$ mm ◀

$$\bar{Y}=\frac{\Sigma\bar{y}A}{\Sigma A}=\frac{18.75\times10^6}{\left(\frac{0.25\times10^6}{3}\right)}$$

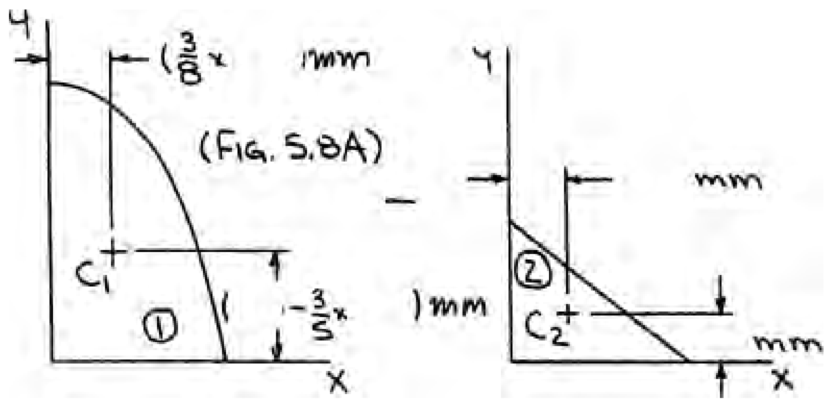
$\bar{Y}=225$ mm ◀

PROBLEM 5.15



Locate the centroid of the plane area shown.

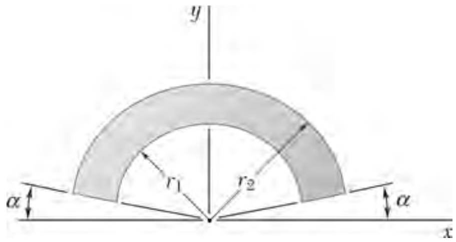
SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{2}{3}(75)(120) = 6000$	28.125	48	168750	288000
2	$-\frac{1}{2}(75)(60) = -2250$	25	20	–56250	–45000
Σ	3750			112500	243000

Then
$$\bar{X}\Sigma A = \Sigma \bar{x}A$$
$$\bar{X}(3750 \text{ mm}^2) = 112500 \text{ mm}^3 \qquad \text{or} \quad \bar{X} = 30.0 \text{ mm} \blacktriangleleft$$

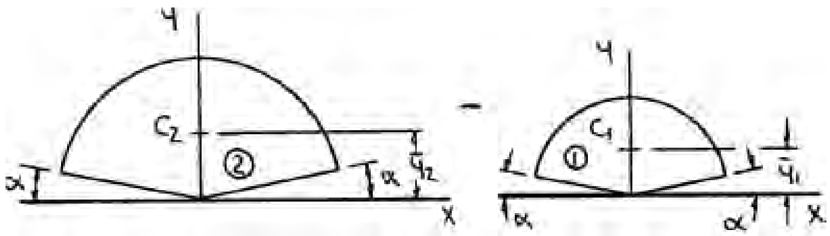
and
$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$
$$\bar{Y}(3750 \text{ mm}^2) = 243000 \qquad \text{or} \quad \bar{Y} = 64.8 \text{ mm} \blacktriangleleft$$



PROBLEM 5.16

Determine the y coordinate of the centroid of the shaded area in terms of r_1, r_2, and α.

SOLUTION



First, determine the location of the centroid.

From Figure 5.8A: $$\bar{y}_2 = \frac{2}{3}r_2\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\left(\frac{\pi}{2}-\alpha\right)} \qquad A_2 = \left(\frac{\pi}{2}-\alpha\right)r_2^2$$

$$= \frac{2}{3}r_2\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}$$

Similarly $$\bar{y}_1 = \frac{2}{3}r_1\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)} \qquad A_1 = \left(\frac{\pi}{2}-\alpha\right)r_1^2$$

Then $$\Sigma\bar{y}A = \frac{2}{3}r_2\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}\left[\left(\frac{\pi}{2}-\alpha\right)r_2^2\right] - \frac{2}{3}r_1\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}\left[\left(\frac{\pi}{2}-\alpha\right)r_1^2\right]$$

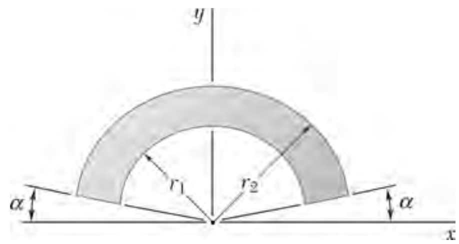
$$= \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

and $$\Sigma A = \left(\frac{\pi}{2}-\alpha\right)r_2^2 - \left(\frac{\pi}{2}-\alpha\right)r_1^2$$

$$= \left(\frac{\pi}{2}-\alpha\right)\left(r_2^2 - r_1^2\right)$$

Now $$\bar{Y}\Sigma A = \Sigma\,\bar{y}A$$

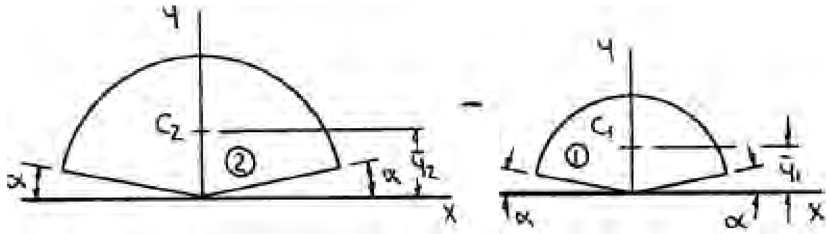
$$\bar{Y}\left[\left(\frac{\pi}{2}-\alpha\right)\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha \qquad \bar{Y} = \frac{2}{3}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)\left(\frac{2\cos\alpha}{\pi - 2\alpha}\right) \blacktriangleleft$$



PROBLEM 5.17

Show that as r_1 approaches r_2, the location of the centroid approaches that for an arc of circle of radius $(r_1 + r_2)/2$.

SOLUTION



First, determine the location of the centroid.

From Figure 5.8A:

$$\bar{y}_2 = \frac{2}{3} r_2 \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_2 = \left(\frac{\pi}{2} - \alpha\right) r_2^2$$

$$= \frac{2}{3} r_2 \frac{\cos \alpha}{\left(\frac{\pi}{2} - \alpha\right)}$$

Similarly

$$\bar{y}_1 = \frac{2}{3} r_1 \frac{\cos \alpha}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_1 = \left(\frac{\pi}{2} - \alpha\right) r_1^2$$

Then

$$\Sigma \bar{y} A = \frac{2}{3} r_2 \frac{\cos \alpha}{\left(\frac{\pi}{2} - \alpha\right)} \left[\left(\frac{\pi}{2} - \alpha\right) r_2^2\right] - \frac{2}{3} r_1 \frac{\cos \alpha}{\left(\frac{\pi}{2} - \alpha\right)} \left[\left(\frac{\pi}{2} - \alpha\right) r_1^2\right]$$

$$= \frac{2}{3}\left(r_2^3 - r_1^3\right) \cos \alpha$$

and

$$\Sigma A = \left(\frac{\pi}{2} - \alpha\right) r_2^2 - \left(\frac{\pi}{2} - \alpha\right) r_1^2$$

$$= \left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)$$

Now

$$\bar{Y} \Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}\left[\left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right) \cos \alpha$$

$$\bar{Y} = \frac{2}{3}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)\left(\frac{2 \cos \alpha}{\pi - 2\alpha}\right)$$

PROBLEM 5.17 (Continued)

Using Figure 5.8B, $\bar{Y}$ of an arc of radius $\frac{1}{2}(r_1 + r_2)$ is

$$\bar{Y} = \frac{1}{2}(r_1 + r_2)\frac{\sin(\frac{\pi}{2} - \alpha)}{(\frac{\pi}{2} - \alpha)}$$

$$= \frac{1}{2}(r_1 + r_2)\frac{\cos\alpha}{(\frac{\pi}{2} - \alpha)} \qquad (1)$$

Now
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r_2 - r_1)\left(r_2^2 + r_1 r_2 + r_1^2\right)}{(r_2 - r_1)(r_2 + r_1)}$$

$$= \frac{r_2^2 + r_1 r_2 + r_1^2}{r_2 + r_1}$$

Let
$$r_2 = r + \Delta$$
$$r_1 = r - \Delta$$

Then
$$r = \frac{1}{2}(r_1 + r_2)$$

and
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r + \Delta)^2 + (r + \Delta)(r - \Delta)(r - \Delta)^2}{(r + \Delta) + (r - \Delta)}$$

$$= \frac{3r^2 + \Delta^2}{2r}$$

In the limit as $\Delta \to 0$ (i.e., $r_1 = r_2$), then

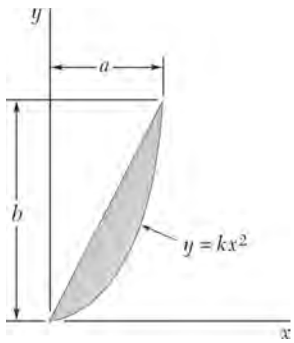
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{3}{2}r$$

$$= \frac{3}{2} \times \frac{1}{2}(r_1 + r_2)$$

So that
$$\bar{Y} = \frac{2}{3} \times \frac{3}{4}(r_1 + r_2)\frac{\cos\alpha}{\frac{\pi}{2} - \alpha} \qquad \text{or} \quad \bar{Y} = (r_1 + r_2)\frac{\cos\alpha}{\pi - 2\alpha} \blacktriangleleft$$

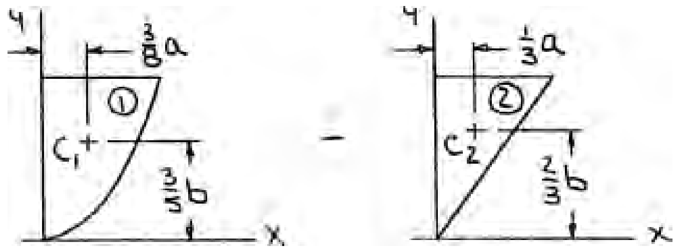
which agrees with Equation (1).

PROBLEM 5.18



For the area shown, determine the ratio a/b for which $\bar{x} = \bar{y}$.

SOLUTION



	A	$\bar{x}$	$\bar{y}$	$\bar{x}A$	$\bar{y}A$
1	$\frac{2}{3}ab$	$\frac{3}{8}a$	$\frac{3}{5}b$	$\frac{a^2b}{4}$	$\frac{2ab^2}{5}$
2	$-\frac{1}{2}ab$	$\frac{1}{3}a$	$\frac{2}{3}b$	$-\frac{a^2b}{6}$	$-\frac{ab^2}{3}$
Σ	$\frac{1}{6}ab$			$\frac{a^2b}{12}$	$\frac{ab^2}{15}$

Then $$\bar{X}\,\Sigma A = \Sigma\,\bar{x}A$$

$$\bar{X}\left(\frac{1}{6}ab\right) = \frac{a^2b}{12}$$

or $$\bar{X} = \frac{1}{2}a$$

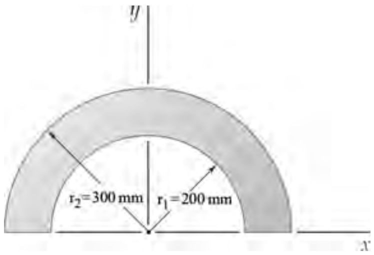
$$\bar{Y}\,\Sigma A = \Sigma\,\bar{y}A$$

$$\bar{Y}\left(\frac{1}{6}ab\right) = \frac{ab^2}{15}$$

or $$\bar{Y} = \frac{2}{5}b$$

Now $$\bar{X} = \bar{Y} \Rightarrow \quad \frac{1}{2}a = \frac{2}{5}b$$

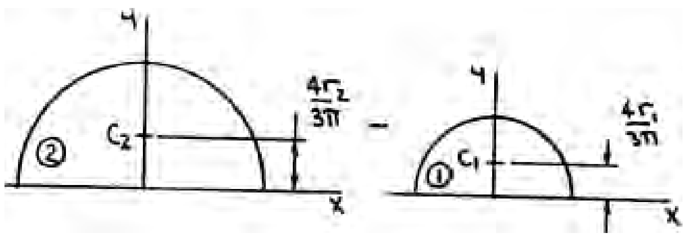
or $\frac{a}{b} = \frac{4}{5}$ ◀



PROBLEM 5.19

For the semiannular area of Problem 5.11, determine the ratio r_2/r_1 so that $\bar{y} = 3r_1/4$.

SOLUTION



	A	$\bar{Y}$	$\bar{Y}A$
1	$-\frac{\pi}{2}r_1^2$	$\frac{4r_1}{3\pi}$	$-\frac{2}{3}r_1^3$
2	$\frac{\pi}{2}r_2^2$	$\frac{4r_2}{3\pi}$	$\frac{2}{3}r_2^3$
Σ	$\frac{\pi}{2}\left(r_2^2 - r_1^2\right)$		$\frac{2}{3}\left(r_2^3 - r_1^3\right)$

Then $$\bar{Y}\ \Sigma A = \Sigma\,\bar{y}A$$

or $$\frac{3}{4}r_1 \times \frac{\pi}{2}\left(r_2^2 - r_1^2\right) = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

$$\frac{9\pi}{16}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] = \left(\frac{r_2}{r_1}\right)^3 - 1$$

Let $$p = \frac{r_2}{r_1}$$

$$\frac{9\pi}{16}[(p+1)(p-1)] = (p-1)(p^2 + p + 1)$$

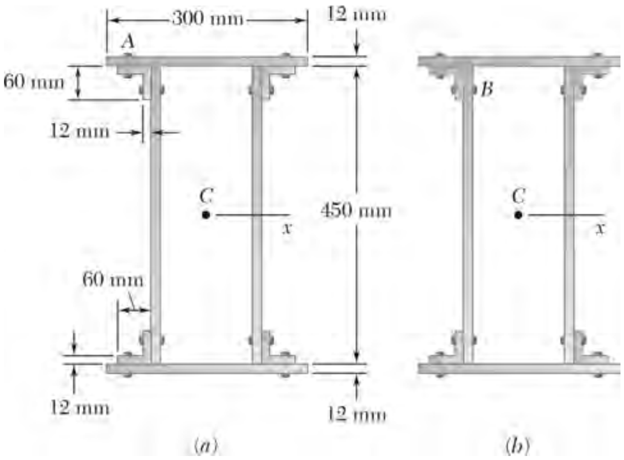
or $$16p^2 + (16 - 9\pi)p + (16 - 9\pi) = 0$$

PROBLEM 5.19 (Continued)

Then $$p = \frac{-(16-9\pi) \pm \sqrt{(16-9\pi)^2 - 4(16)(16-9\pi)}}{2(16)}$$

or $$p = -0.5726$$
$$p = 1.3397$$

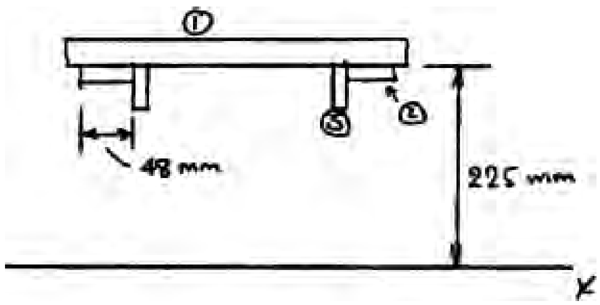
Taking the positive root $$\frac{r_2}{r_1} = 1.340 \blacktriangleleft$$



PROBLEM 5.20

A composite beam is constructed by bolting four plates to four 60 60 12-mm angles as shown. The bolts are equally spaced along the beam, and the beam supports a vertical load. As proved in mechanics of materials, the shearing forces exerted on the bolts at *A* and *B* are proportional to the first moments with respect to the centroidal *x* axis of the red shaded areas shown, respectively, in Parts (*a*) and (*b*) of the figure. Knowing that the force exerted on the bolt at *A* is 280 N, determine the force exerted on the bolt at *B*.

SOLUTION



From the problem statement: *F* is proportional to Q_x.

Therefore: $\dfrac{F_A}{(Q_x)_A} = \dfrac{F_B}{(Q_x)_B}$, or $F_B = \dfrac{(Q_x)_B}{(Q_x)_A} F_A$

For the first moments: $(Q_x)_A = \left(225 + \dfrac{12}{2}\right)(300 \times 12)$

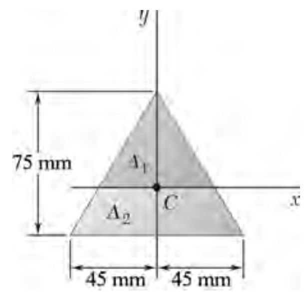
$$= 831600 \text{ mm}^3$$

$$(Q_x)_B = (Q_x)A + 2\left(225 - \frac{12}{2}\right)(48 \times 12) + 2(225 - 30)(12 \times 60)$$

$$= 1364688 \text{ mm}^3$$

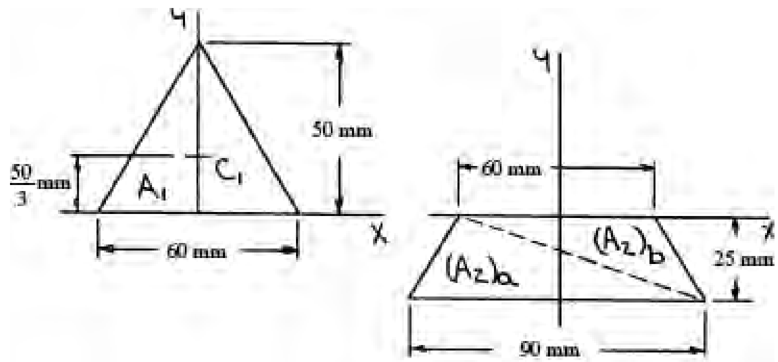
Then $F_B = \dfrac{1364688}{831600}(280 \text{ N})$ or $F_B = 459 \text{ N}$ ◀

PROBLEM 5.21



The horizontal x axis is drawn through the centroid C of the area shown, and it divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x axis, and explain the results obtained.

SOLUTION



Note that $\quad Q_x = \Sigma \bar{y} A$

Then $\quad (Q_x)_1 = \left(\frac{50}{3}\text{ mm}\right)\left(\frac{1}{2}\times 60\times 50\right)\text{mm}^2 \quad$ or $\quad (Q_x)_1 = 25000\text{ mm}^3$ ◀

and
$$(Q_x)_2 = \left(-\frac{2}{3}\times 25\text{ mm}\right)\left(\frac{1}{2}\times 90\times 25\right)\text{mm}^2 + \left(-\frac{1}{3}\times 25\text{ mm}\right)\left(\frac{1}{2}\times 60\times 25\right)\text{mm}^2$$

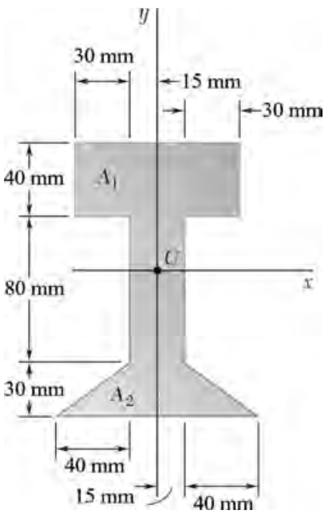
or $\quad (Q_x)_2 = -25000\text{ mm}^3$ ◀

Now $\quad Q_x = (Q_x)_1 + (Q_x)_2 = 0$

This result is expected since x is a centroidal axis $(\text{thus } \bar{y} = 0)$

and $\quad Q_x = \Sigma \bar{y} A = \bar{Y}\Sigma A (\bar{y} = 0 \Rightarrow Q_x = 0)$

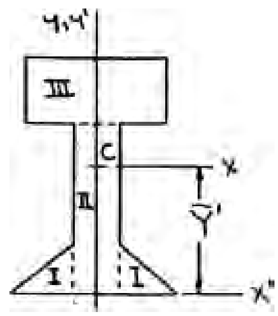
PROBLEM 5.22



The horizontal x axis is drawn through the centroid C of the area shown, and it divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x axis, and explain the results obtained.

SOLUTION

First determine the location of the centroid C. We have



	A, mm^2	$\bar{y}'$, mm	$\bar{y}'A$, mm^3
I	$2\left(\frac{1}{2}\times 40\times 30\right)=1200$	10	12000
II	$30\times 110=3300$	55	181500
III	$90\times 40=3600$	130	468000
Σ	8100		661500

Then $$\bar{Y}'\,\Sigma A=\Sigma y'A$$
$$\bar{Y}'(8100)=661500$$

or $$\bar{Y}'=81.667\text{ mm}$$

Now $$Q_x=\Sigma\bar{y}_1 A$$

Then $$(Q_x)_1=\left[\frac{1}{2}(110-81.667)\text{mm}\right][(30)(110-81.667)]\text{ mm}^2$$
$$+[(130-81.667)\text{mm}][(90)(40)]\text{ mm}^2$$ or $(Q_x)_1=186040\text{ mm}^3$ ◀

and $$(Q_x)_2=-\left[\frac{1}{2}(81.667\text{ mm})\right][(30)(81.667)]\text{mm}^2$$
$$-[(81.667-10)\text{mm}]\times 2\left[\left(\frac{1}{2}\times 40\times 30\right)\text{mm}^2\right]$$ or $(Q_x)_2=-186040\text{ mm}^3$ ◀

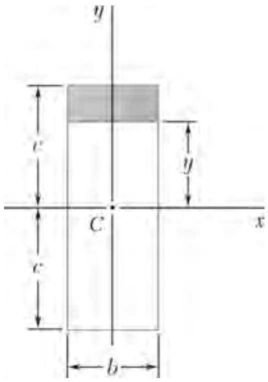
PROBLEM 5.22 (Continued)

Now $$Q_x = (Q_x)_1 + (Q_x)_2 = 0$$

This result is expected since x is a centroidal axis $(\text{thus}\, \bar{Y} = 0)$

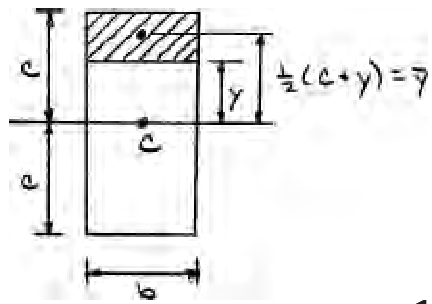
and $$Q_x = \Sigma \bar{y} A = \bar{Y} \Sigma A \quad (\bar{Y} = 0 \Rightarrow Q_x = 0)$$

PROBLEM 5.23



The first moment of the shaded area with respect to the x axis is denoted by Q_x. (a) Express Q_x in terms of b, c, and the distance y from the base of the shaded area to the x axis. (b) For what value of y is O_x maximum, and what is that maximum value?

SOLUTION



Shaded area:

$$A = b(c-y)$$

$$Q_x = \bar{y}A$$

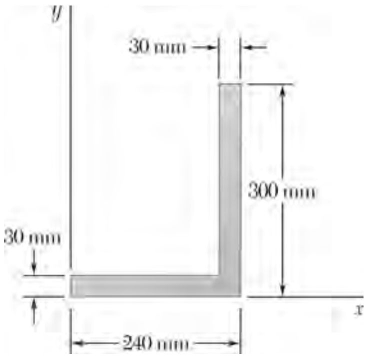
$$= \frac{1}{2}(c+y)[b(c-y)]$$

(a) $$Q_x = \frac{1}{2}b(c^2 - y^2)$$ ◀

(b) For $Q_{\max}$: $$\frac{dQ}{dy} = 0 \quad \text{or} \quad \frac{1}{2}b(-2y) = 0$$ $y = 0$ ◀

For $y = 0$: $$(Q_x) = \frac{1}{2}bc^2$$ $(Q_x) = \frac{1}{2}bc^2$ ◀

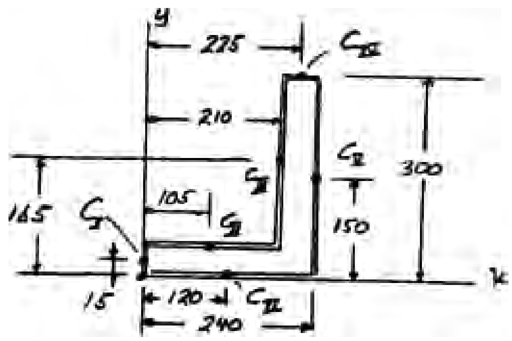
PROBLEM 5.24



A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

Perimeter of Figure 5.1



	L	$\bar{x}$	$\bar{y}$	$\bar{x}L$, mm^2	$\bar{y}L$, mm^2
I	30	0	15	0	0.45×10^3
II	210	105	30	22.05×10^3	6.3×10^3
III	270	210	165	56.7×10^3	44.55×10^3
IV	30	225	300	6.75×10^3	9×10^3
V	300	240	150	72×10^3	45×10^3
VII	240	120	0	28.8×10^3	0
Σ	1080			186.3×10^3	105.3×10^3

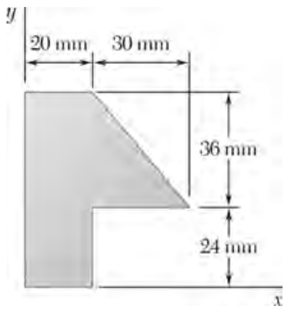
$$\bar{X}\Sigma L = \Sigma \bar{x} L$$

$$\bar{X}(1080 \text{ mm}) = 186.3\times10^3 \text{ mm}^2 \qquad \bar{X} = 172.5 \text{ mm} \blacktriangleleft$$

$$\bar{Y}\Sigma L = \Sigma \bar{y} L$$

$$\bar{Y}(1080 \text{ mm}) = 105.3\times10^3 \text{ mm}^2 \qquad \bar{Y} = 97.5 \text{ mm} \blacktriangleleft$$

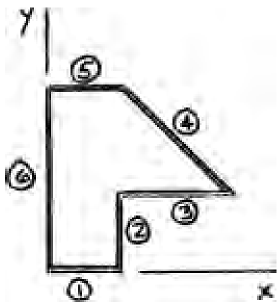
PROBLEM 5.25



A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

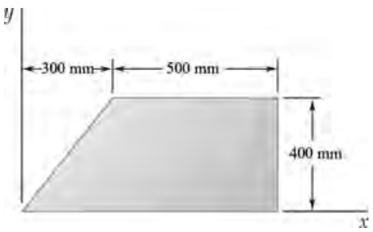
First note that because wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.



	L, mm	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}L$, mm^2	$\bar{y}L$, mm^2
1	20	10	0	200	0
2	24	20	12	480	288
3	30	35	24	1050	720
4	46.861	35	42	1640.14	1968.16
5	20	10	60	200	1200
6	60	0	30	0	1800
Σ	200.86			3570.1	5976.2

Then $\bar{X}\Sigma L = \Sigma \bar{x}L \quad \bar{X}(200.86) = 3570.1$ $\qquad \bar{X} = 17.77 \text{ mm}$ ◀

$\bar{Y}\Sigma L = \Sigma \bar{y}L \quad \bar{Y}(200.86) = 5976.2$ $\qquad \bar{Y} = 29.8 \text{ mm}$ ◀

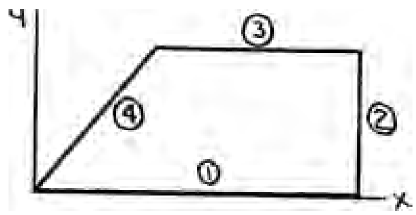


PROBLEM 5.26

A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

First note that because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.



	L, mm	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}L$, mm^2	$\bar{y}L$, mm^2
1	800	400	0	320 10^3	0
2	400	800	200	320 10^3	80 10^3
3	500	550	400	275 10^3	200 10^3
4	$\sqrt{(300)^2+(400)^2}=500$	150	200	75 10^3	100 10^3
Σ	2200			990 10^3	380 10^3

Then

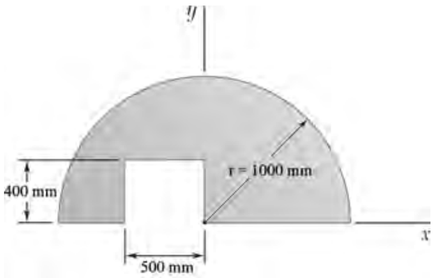
$$\bar{X}\,\Sigma L = \Sigma \bar{x} L$$

$$\bar{X}(2200) = 990 \times 10^3 \qquad \text{or} \quad \bar{X} = 450 \text{ mm} \blacktriangleleft$$

and

$$\bar{Y}\,\Sigma L = \Sigma \bar{y} L$$

$$\bar{Y}(2200) = 380 \times 10^3 \qquad \text{or} \quad \bar{Y} = 172.7 \text{ mm} \blacktriangleleft$$

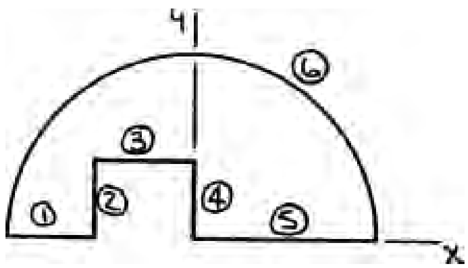


PROBLEM 5.27

A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

First note that because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.

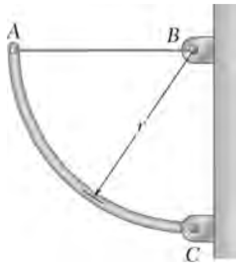


$$\bar{Y}_6 = \frac{2}{\pi}(1000 \text{ mm})$$

	L, mm	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}L$, mm^2	$\bar{y}L$, mm^3
1	500	–750	0	-375×10^3	0
2	400	–500	200	-200×10^3	80×10^3
3	500	–250	400	-125×10^3	200×10^3
4	400	0	200	0	80×10^3
5	1000	500	0	500×10^3	0
6	$\pi(1000) = 1000\pi$	0	$\frac{2000}{\pi}$	0	2000×10^3
Σ	5941.59			-200×10^3	2360×10^3

Then $$\bar{X} = \frac{\Sigma\bar{x}L}{\Sigma L} = \frac{-200\times10^3}{5941.59}$$ $\bar{X} = -33.7$ mm ◀

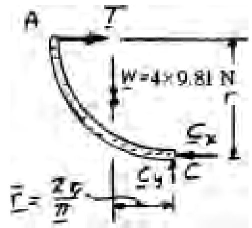
$$\bar{Y} = \frac{\Sigma\bar{y}L}{\Sigma L} = \frac{2360\times10^3}{5941.59}$$ $\bar{Y} = 397$ mm ◀



PROBLEM 5.28

A uniform circular rod of weight 4 kg and radius 250 mm is attached to a pin at C and to the cable AB. Determine (a) the tension in the cable, (b) the reaction at C.

SOLUTION



For quarter circle $\quad \bar{r} = \dfrac{2r}{\pi}$

(a) $\quad \stackrel{+}{\circlearrowleft}\Sigma M_C = 0{:}\quad W\left(\dfrac{2r}{\pi}\right) - Tr = 0$

$$T = W\left(\frac{2}{\pi}\right) = (4\times 9.81)\text{N}\left(\frac{2}{\pi}\right) = 24.981\text{ N}$$

$T = 25.0$ N ◀

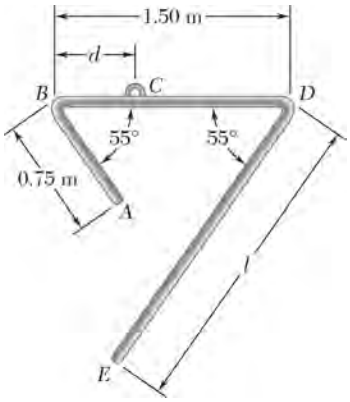
(b) $\quad \xrightarrow{+}\Sigma F_x = 0{:}\quad T - C_x = 0 \quad 24.981\text{ N} - C_x = 0$

$\mathbf{C}_x = 25.0$ N ←

$+\uparrow\Sigma F_y = 0{:}\quad C_y - W = 0 \qquad C_y - 4\times 9.81\text{ N} = 0 \Rightarrow C_y = 39.24\text{ N}$

$\mathbf{C}_y = 39.2$ N ↑

$\mathbf{C} = 46.5$ N ⦪ 57.5° ◀

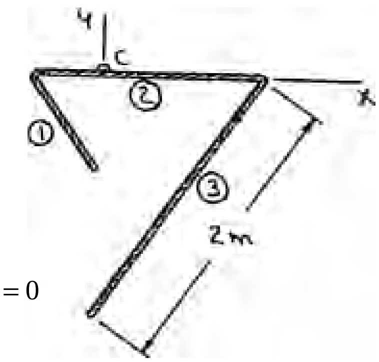


PROBLEM 5.29

Member *ABCDE* is a component of a mobile and is formed from a single piece of aluminum tubing. Knowing that the member is supported at *C* and that $l = 2$ m, determine the distance *d* so that portion *BCD* of the member is horizontal.

SOLUTION

First note that for equilibrium, the center of gravity of the component must lie on a vertical line through *C*. Further, because the tubing is uniform, the center of gravity of the component will coincide with the centroid of the corresponding line. Thus, $\bar{X} = 0$



So that

$$\Sigma \bar{x} L = 0$$

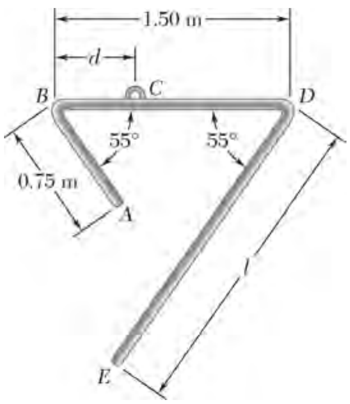
Then

$$-\left(d - \frac{0.75}{2}\cos 55°\right)\text{m} \times (0.75\text{ m})$$
$$+(0.75 - d)\text{m} \times (1.5\text{ m})$$
$$+\left[(1.5 - d)\text{m} - \left(\frac{1}{2} \times 2\text{ m} \times \cos 55°\right)\right] \times (2\text{ m}) = 0$$

or

$$(0.75 + 1.5 + 2)d = \left[\frac{1}{2}(0.75)^2 - 2\right]\cos 55° + (0.75)(1.5) + 3$$

or $d = 0.739$ m ◀

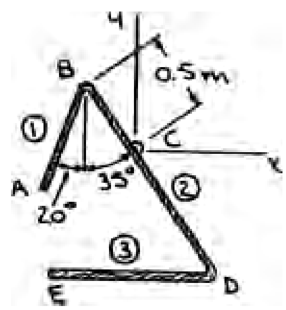


PROBLEM 5.30

Member *ABCDE* is a component of a mobile and is formed from a single piece of aluminum tubing. Knowing that the member is supported at *C* and that *d* is 0.50 m, determine the length *l* of arm *DE* so that this portion of the member is horizontal.

SOLUTION

First note that for equilibrium, the center of gravity of the component must lie on a vertical line through *C*. Further, because the tubing is uniform, the center of gravity of the component will coincide with the centroid of the corresponding line. Thus,



$$\bar{X} = 0$$

So that $$\Sigma \bar{x} L = 0$$

or
$$-\left(\frac{0.75}{2}\sin 20° + 0.5\sin 35°\right)\text{m} \times (0.75\text{ m})$$
$$+(0.25\text{ m} \times \sin 35°) \times (1.5\text{ m})$$
$$+\left(1.0 \times \sin 35° - \frac{l}{2}\right)\text{m} \times (l\text{ m}) = 0$$

or
$$\underbrace{-0.096193}_{(\bar{x}L)_{AB} + (\bar{x}L)_{BD}} + \underbrace{\left(\sin 35° - \tfrac{l}{2}\right) l}_{(\bar{x}L)_{DE}} = 0$$

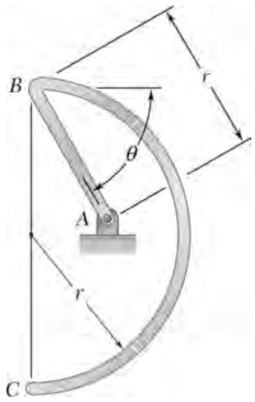
The equation implies that the center of gravity of *DE* must be to the right of *C*.

Then $$l^2 - 1.14715 l + 0.192386 = 0$$

or
$$l = \frac{1.14715 \pm \sqrt{(-1.14715)^2 - 4(0.192386)}}{2}$$

or $$l = 0.204\text{ m}$$ or $l = 0.943$ m ◀

Note that $\sin 35° - \frac{1}{2}l > 0$ for both values of l so both values are acceptable.

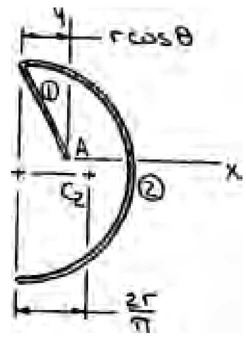


PROBLEM 5.31

The homogeneous wire *ABC* is bent into a semicircular arc and a straight section as shown and is attached to a hinge at *A*. Determine the value of θ for which the wire is in equilibrium for the indicated position.

SOLUTION

First note that for equilibrium, the center of gravity of the wire must lie on a vertical line through *A*. Further, because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line. Thus,



$$\bar{X} = 0$$

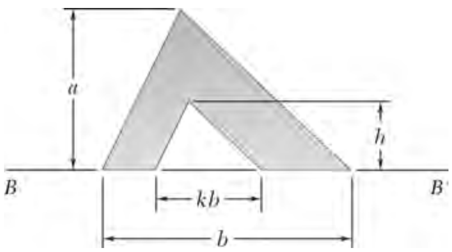
So that $$\Sigma \bar{x} L = 0$$

Then $$\left(-\frac{1}{2} r \cos\theta\right)(r) + \left(\frac{2r}{\pi} - r\cos\theta\right)(\pi r) = 0$$

or $$\cos\theta = \frac{4}{1+2\pi}$$

$$= 0.54921$$

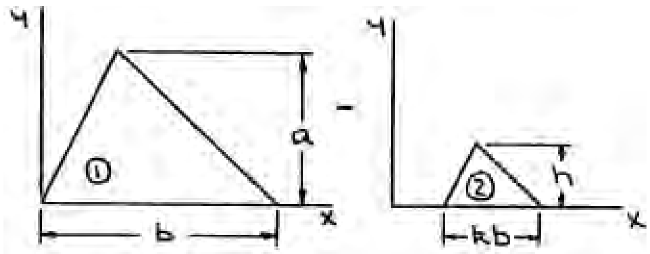
or $\theta = 56.7°$ ◀



PROBLEM 5.32

Determine the distance h for which the centroid of the shaded area is as far above line BB' as possible when (a) $k = 0.10$, (b) $k = 0.80$.

SOLUTION



	A	$\bar{y}$	$\bar{y}A$
1	$\frac{1}{2}ba$	$\frac{1}{3}a$	$\frac{1}{6}a^2b$
2	$-\frac{1}{2}(kb)h$	$\frac{1}{3}h$	$-\frac{1}{6}kbh^2$
Σ	$\frac{b}{2}(a-kh)$		$\frac{b}{6}(a^2-kh^2)$

Then

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}\left[\frac{b}{2}(a-kh)\right] = \frac{b}{6}(a^2-kh^2)$$

or

$$\bar{Y} = \frac{a^2-kh^2}{3(a-kh)} \qquad (1)$$

and

$$\frac{d\bar{Y}}{dh} = \frac{1}{3}\frac{-2kh(a-kh)-(a^2-kh^2)(-k)}{(a-kh)^2} = 0$$

or

$$2h(a-kh)-a^2+kh^2 = 0 \qquad (2)$$

Simplifying Eq. (2) yields

$$kh^2-2ah+a^2 = 0$$

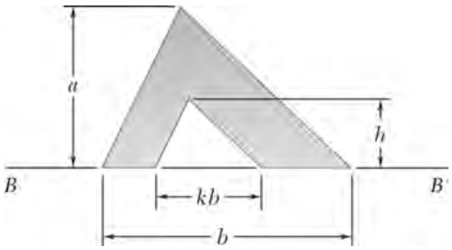
PROBLEM 5.32 (Continued)

Then
$$h = \frac{2a \pm \sqrt{(-2a)^2 - 4(k)(a^2)}}{2k}$$
$$= \frac{a}{k}\left[1 \pm \sqrt{1-k}\right]$$

Note that only the negative root is acceptable since $h < a.$ Then

(*a*)
$$k = 0.10$$
$$h = \frac{a}{0.10}\left[1 - \sqrt{1-0.10}\right] \qquad \text{or} \quad h = 0.513a \blacktriangleleft$$

(*b*)
$$k = 0.80$$
$$h = \frac{a}{0.80}\left[1 - \sqrt{1-0.80}\right] \qquad \text{or} \quad h = 0.691a \blacktriangleleft$$



PROBLEM 5.33

Knowing that the distance h has been selected to maximize the distance $\bar{y}$ from line BB' to the centroid of the shaded area, show that $\bar{y} = 2h/3$.

SOLUTION

See solution to Problem 5.32 for analysis leading to the following equations:

$$\bar{Y} = \frac{a^2 - kh^2}{3(a - kh)} \tag{1}$$

$$2h(a - kh) - a^2 + kh^2 = 0 \tag{2}$$

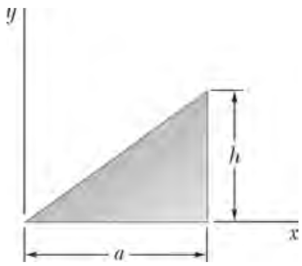
Rearranging Eq. (2) (which defines the value of h which maximizes $\bar{Y}$) yields

$$a^2 - kh^2 = 2h(a - kh)$$

Then substituting into Eq. (1) (which defines $\bar{Y}$)

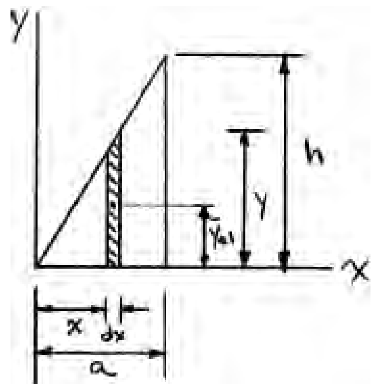
$$\bar{Y} = \frac{1}{3(a - kh)} \times 2h(a - kh) \qquad \text{or} \quad \bar{Y} = \frac{2}{3}h \blacktriangleleft$$

PROBLEM 5.34



Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION



$$\frac{y}{x} = \frac{h}{a}$$

$$y = \frac{h}{a}x$$

$$\overline{x}_{EL} = x$$

$$\overline{y}_{EL} = \frac{1}{2}y$$

$$dA = ydx$$

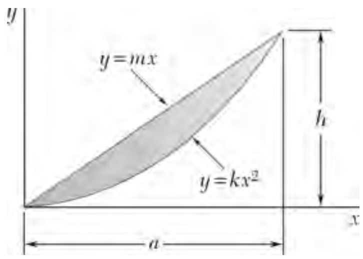
$$A = \int_0^a ydx = \int_0^a \left(\frac{h}{a}x\right)dx = \frac{1}{2}ah$$

$$\int \overline{x}_{EL}dA = \int xydx = \int_0^a x\left(\frac{h}{a}x\right)dx = \frac{h}{a}\left[\frac{x^3}{3}\right]_0^a = \frac{1}{3}ha^2$$

$$\int \overline{y}_{EL}dA = \int_0^a \left(\frac{1}{2}y\right)ydx = \frac{1}{2}\int_0^a \left(\frac{h}{a}x\right)^2 dx = \frac{1}{2}\frac{h^2}{b^2}\left[\frac{x^3}{3}\right]_0^a = \frac{1}{6}h^2a$$

$$\overline{x}A = \int \overline{x}_{EL}dA: \quad \overline{x}\left(\frac{1}{2}ah\right) = \frac{1}{3}ha^2 \qquad \overline{x} = \frac{2}{3}a \blacktriangleleft$$

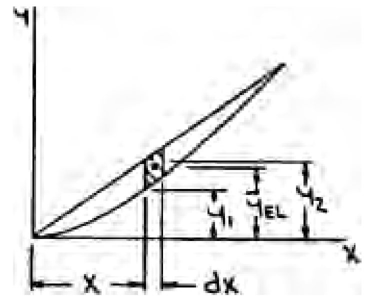
$$\overline{y}A = \int \overline{y}_{EL}dA: \quad \overline{y}\left(\frac{1}{2}ah\right) = \frac{1}{6}h^2a \qquad \overline{y} = \frac{1}{3}h \blacktriangleleft$$



PROBLEM 5.35

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION



At $(a,\ h)$ $\qquad y_1: \quad h = ka^2$

or $\qquad k = \dfrac{h}{a^2}$

$\qquad y_2: \quad h = ma$

or $\qquad m = \dfrac{h}{a}$

Now $\qquad \overline{x}_{EL} = \overline{x}$

$$\overline{y}_{EL} = \frac{1}{2}(y_1 + y_2)$$

and $\qquad dA = (y_2 - y_1)dx = \left[\dfrac{h}{a}x - \dfrac{h}{a^2}x^2\right]dx$

$$= \frac{h}{a^2}(ax - x^2)dx$$

Then $\qquad A = \int dA = \int_0^a \dfrac{h}{a^2}(ax - x^2)dx = \dfrac{h}{a^2}\left[\dfrac{a}{2}x^2 - \dfrac{1}{3}x^3\right]_0^a = \dfrac{1}{6}ah$

and $\qquad \int \overline{x}_{EL}dA = \int_0^a x\left[\dfrac{h}{a^2}(ax - x^2)dx = \dfrac{h}{a^2}\left[\dfrac{a}{3}x^3 - \dfrac{1}{4}x^4\right]_0^a = \dfrac{1}{12}a^2h\right.$

$$\int \overline{y}_{EL}dA = \int \frac{1}{2}(y_1 + y_2)[(y_2 - y_1)dx] = \int \frac{1}{2}\left(y_2^2 - y_1^2\right)dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{h^2}{a^2}x^2 - \frac{h^2}{a^4}x^4\right)dx$$

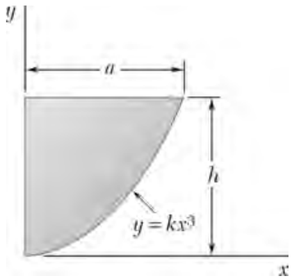
$$= \frac{1}{2}\frac{h^2}{a^4}\left[\frac{a^2}{3}x^3 - \frac{1}{5}x^5\right]_0^a$$

$$= \frac{1}{15}ah^2$$

PROBLEM 5.35 (Continued)

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{1}{6}ah\right) = \frac{1}{12}a^2h \qquad \bar{x} = \frac{1}{2}a \blacktriangleleft$$

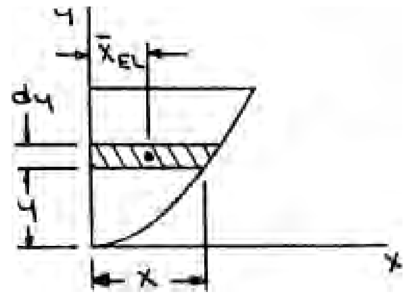
$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{1}{6}ah\right) = \frac{1}{15}ah^2 \qquad \bar{y} = \frac{2}{5}h \blacktriangleleft$$



PROBLEM 5.36

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION



For the element (EL) shown

At $\quad x = a, \quad y = h: \quad h = ka^3 \quad \text{or} \quad k = \dfrac{h}{a^3}$

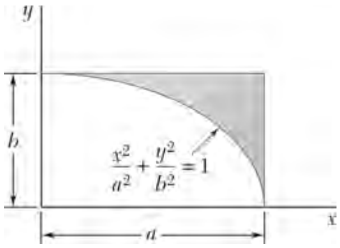
Then $\quad x = \dfrac{a}{h^{1/3}} y^{1/3}$

Now
$$dA = x\,dy = \frac{a}{h^{1/3}} y^{1/3}\,dy$$
$$\bar{x}_{EL} = \frac{1}{2}x = \frac{1}{2}\frac{a}{h^{1/3}} y^{1/3}$$
$$\bar{y}_{EL} = y$$

Then
$$A = \int dA = \int_0^h \frac{a}{h^{1/3}} y^{1/3} dy = \frac{3}{4}\frac{a}{h^{1/3}}\left(y^{4/3}\right)\Big|_0^h = \frac{3}{4}ah$$

and
$$\int \bar{x}_{EL}\,dA = \int_0^h \frac{1}{2}\frac{a}{h^{1/3}} y^{1/3}\left(\frac{a}{h^{1/3}} y^{1/3} dy\right) = \frac{1}{2}\frac{a}{h^{2/3}}\left(\frac{3}{5}y^{5/3}\right)\Big|_0^h = \frac{3}{10}a^2h$$
$$\int \bar{y}_{EL}\,dA = \int_0^h y\left(\frac{a}{h^{1/3}} y^{1/3} dy\right) = \frac{a}{h^{1/3}}\left(\frac{3}{7}y^{7/3}\right)\Big|_0^h = \frac{3}{7}ah^2$$

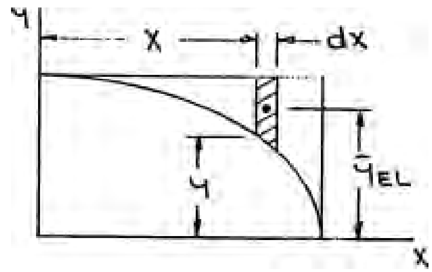
Hence
$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{3}{2}ah\right) = \frac{3}{10}a^2h \qquad \bar{x} = \frac{2}{5}a \blacktriangleleft$$
$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{3}{4}ah\right) = \frac{3}{7}ah^2 \qquad \bar{y} = \frac{4}{7}h \blacktriangleleft$$



PROBLEM 5.37

Determine by direct integration the centroid of the area shown.

SOLUTION



For the element (EL) shown $\quad y = \frac{b}{a}\sqrt{a^2 - x^2}$

and

$$dA = (b - y)dx$$

$$= \frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx$$

$$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}(y + b)$$

$$= \frac{b}{2a}\left(a + \sqrt{a^2 - x^2}\right)$$

Then

$$A = \int dA = \int_0^a \frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx$$

To integrate, let $\quad x = a\sin\theta: \quad \sqrt{a^2 - x^2} = a\cos\theta, \ dx = a\cos\theta\, d\theta$

Then

$$A = \int_0^{\pi/2} \frac{b}{a}(a - a\cos\theta)(a\cos\theta d\theta)$$

$$= \frac{b}{a}\left[a^2\sin\theta - a^2\left(\frac{\theta}{2} + \sin\frac{2\theta}{4}\right)\right]_0^{\pi/2}$$

$$= ab\left(1 - \frac{\pi}{4}\right)$$

and

$$\int \bar{x}_{EL}\, dA = \int_0^a x\left[\frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx\right]$$

$$= \frac{b}{a}\left[\left(\frac{a}{2}x^2 + \frac{1}{3}(a^2 - x^2)^{3/2}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{6}a^3 b$$

PROBLEM 5.37 (Continued)

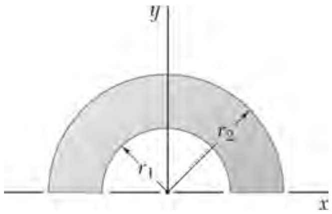
$$\int \bar{y}_{EL}\, dA = \int_0^a \frac{b}{2a}\left(a+\sqrt{a^2-x^2}\right)\left[\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx\right]$$

$$= \frac{b^2}{2a^2}\int_0^a (x^2)\ dx = \frac{b^2}{2a^2}\left(\frac{x^3}{3}\right)\Bigg|_0^a$$

$$= \frac{1}{6}ab^2$$

$$\bar{x}A = \int \bar{x}_{EL}\, dA: \quad \bar{x}\left[ab\left(1-\frac{\pi}{4}\right)\right] = \frac{1}{6}a^2 b \qquad \text{or } \bar{x} = \frac{2a}{3(4-\pi)} \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}\, dA: \quad \bar{y}\left[ab\left(1-\frac{\pi}{4}\right)\right] = \frac{1}{6}ab^2 \qquad \text{or } \bar{y} = \frac{2b}{3(4-\pi)} \blacktriangleleft$$



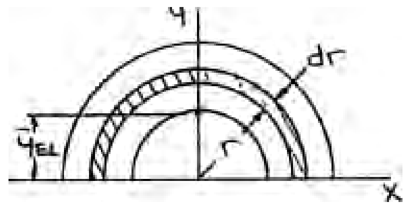
PROBLEM 5.38

Determine by direct integration the centroid of the area shown.

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

For the element (EL) shown



$$\bar{y}_{EL} = \frac{2r}{\pi} \quad \text{(Figure 5.8B)}$$

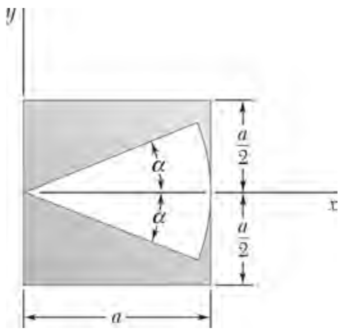
$$dA = \pi r dr$$

Then $$A = \int dA = \int_{r_1}^{r_2} \pi r\, dr = \pi\left(\frac{r^2}{2}\right)\Bigg|_{r_1}^{r_2} = \frac{\pi}{2}\left(r_2^2 - r_1^2\right)$$

and $$\int \bar{y}_{EL} dA = \int_{r_1}^{r_2} \frac{2r}{\pi}(\pi r dr) = 2\left(\frac{1}{3}r^3\right)\Bigg|_{r_1}^{r_2} = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

So $$\bar{y}A = \int \bar{y}_{EL} dA: \quad \bar{y}\left[\frac{\pi}{2}\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

or $\bar{y} = \dfrac{4}{3\pi}\dfrac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$ ◀

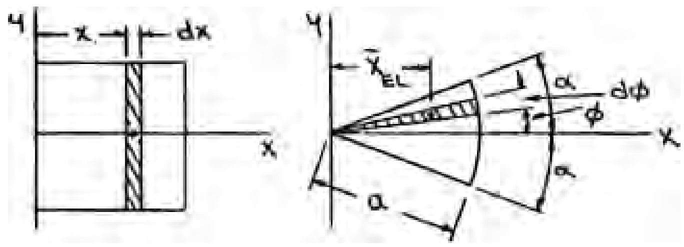


PROBLEM 5.39

Determine by direct integration the centroid of the area shown.

SOLUTION

First note that symmetry implies $\bar{y} = 0$ ◀



$$dA = adx \qquad\qquad dA = \frac{1}{2}a(ad\phi)$$

$$\bar{x}_{EL} = x \qquad\qquad \bar{x}_{EL} = \frac{2}{3}a\cos\phi$$

Then

$$A = \int dA = \int_0^a adx - \int_{-\alpha}^{\alpha} \frac{1}{2}a^2 d\phi$$

$$= a[x]_0^a - \frac{a^2}{2}[\phi]_{\alpha}^{\alpha} = a^2(1-\alpha)$$

and

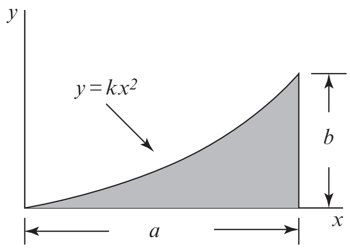
$$\int \bar{x}_{EL}dA = \int_0^a x(adx) - \int_{-\alpha}^{\alpha} \frac{2}{3}a\cos\phi\left(\frac{1}{2}a^2 d\phi\right)$$

$$= a\left[\frac{x^2}{2}\right]_0^a - \frac{1}{3}a^3[\sin\phi]_{-\alpha}^{\alpha}$$

$$= a^3\left(\frac{1}{2} - \frac{2}{3}\sin\alpha\right)$$

$$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}[a^2(1-\alpha)] = a^3\left(\frac{1}{2} - \frac{2}{3}\sin\alpha\right)$$

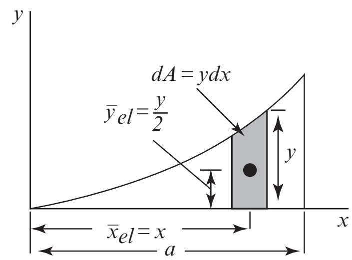
or $\bar{x} = \dfrac{3 - 4\sin\alpha}{6(1-\alpha)}a$ ◀



PROBLEM 5.40

Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION



Determination of the Constant *k*

$$k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad \text{or} \quad x = \frac{b}{b^{1/2}}y^{1/2}$$

Vertical Differential Element

$$A = \int dA = \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a = \frac{ab}{3}$$

$$Q_y = \int \overline{x}_{el}\,dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx = \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2b}{4}$$

Since $Q_y = \overline{x}A$, we have

$$\overline{x}A = \int \overline{x}_{el}\,dA \qquad \overline{x}\frac{ab}{3} = \frac{a^2b}{4} \qquad \overline{x} = \frac{3}{4}a \ ◀$$

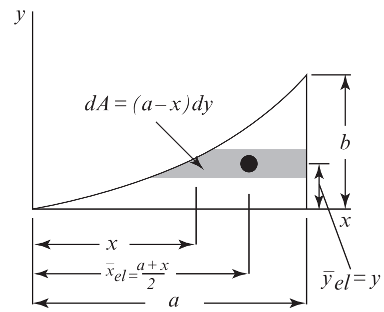
$$Q_x = \int \overline{y}_{el}\,dA = \int \frac{y}{2}y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx = \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Since $Q_x = \overline{y}A$, we have

$$\overline{y}A = \int \overline{y}_{el}\,dA \qquad \overline{y}\frac{ab}{3} = \frac{ab^2}{10} \qquad \overline{y} = \frac{3}{10}b \ ◀$$

Alternate Method:

Horizontal Differential Element. The same results can be obtained by considering a horizontal element. The first moments of the area are

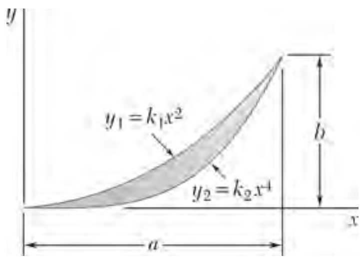


$$Q_y = \int \bar{x}_{el}\, dA = \int \frac{a+x}{2}(a-x)\,dy = \int_0^b \frac{a^2 - x^2}{2}\,dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b}y\right)dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\, dA = \int y(a-x)\,dy = \int y\left(a - \frac{a}{b^{1/2}}y^{1/2}\right)dy$$

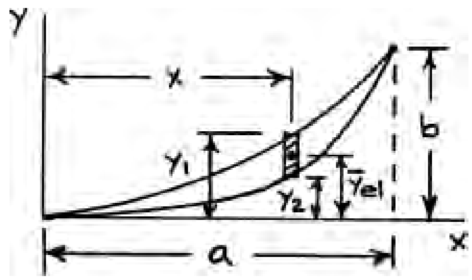
$$= \int_0^b \left(ay - \frac{a}{b^{1/2}}y^{3/2}\right)dy = \frac{ab^2}{10}$$



PROBLEM 5.41

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



$$y_1 = k_1x^2 \quad \text{but} \quad b = k_1a^2 \quad y_1 = \frac{b}{a^2}x^2$$

$$y_2 = k_2x^4 \quad \text{but} \quad b = k_2a^4 \quad y_2 = \frac{b}{a^4}x^4$$

$$dA = (y_2 - y_1)dx = \frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}(y_1 + y_2)$$

$$= \frac{b}{2a^2}\left(x^2 + \frac{x^4}{a^2}\right)$$

$$A = \int dA = \frac{b}{a^2}\int_0^a \left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^3}{3} - \frac{x^5}{5a^2}\right]_0^a$$

$$= \frac{2}{15}ba$$

$$\int \bar{x}_{EL}dA = \int_0^a x \times \frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b}{a^2}\int_0^a \left(x^3 - \frac{x^5}{a^2}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^4}{4} - \frac{x^6}{6a^2}\right]_0^a$$

$$= \frac{1}{12}a^2b$$

PROBLEM 5.41 (Continued)

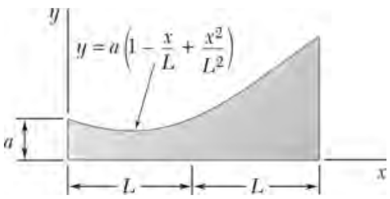
$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{b}{2a^2}\left(x^2 + \frac{x^4}{a^2}\right)\frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b^2}{2a^4}\int_0^a \left(x^4 - \frac{x^8}{a^4}\right)dx$$

$$= \frac{b^2}{2a^4}\left[\frac{x^5}{5} - \frac{x^9}{9a^4}\right]_0^a = \frac{2}{45}ab^2$$

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{2}{15}ba\right) = \frac{1}{12}a^2 b \qquad \bar{x} = \frac{5}{8}a \blacktriangleleft$$

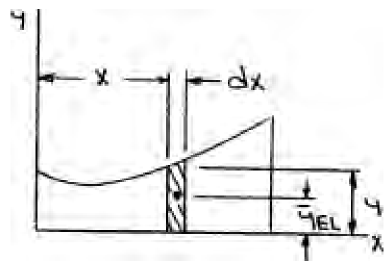
$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{2}{15}ba\right) = \frac{2}{45}ab^2 \qquad \bar{y} = \frac{1}{3}b \blacktriangleleft$$



PROBLEM 5.42

Determine by direct integration the centroid of the area shown.

SOLUTION



We have

$$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)$$

$$dA = ydx = a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx$$

Then

$$A = \int dA = \int_0^{2L} a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx = a\left[x - \frac{x^2}{2L} + \frac{x^3}{3L^2}\right]_0^{2L}$$

$$= \frac{8}{3}aL$$

and

$$\int \bar{x}_{EL}dA = \int_0^{2L} x\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right] = a\left[\frac{x^2}{2} - \frac{x^3}{3L} + \frac{x^4}{4L^2}\right]_0^{2L}$$

$$= \frac{10}{3}aL^2$$

$$\int \bar{y}_{EL}dA = \int_0^{2L} \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right]$$

$$= \frac{a^2}{2}\int_0^{EL}\left(1 - 2\frac{x}{L} + 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right)dx$$

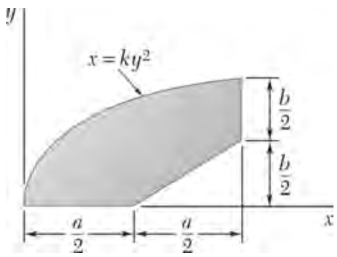
$$= \frac{a^2}{2}\left[x - \frac{x^2}{L} + \frac{x^3}{L^2} - \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^{2L}$$

$$= \frac{11}{5}a^2L$$

Hence

$$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\frac{8}{3}aL\right) = \frac{10}{3}aL^2 \qquad \bar{x} = \frac{5}{4}L \blacktriangleleft$$

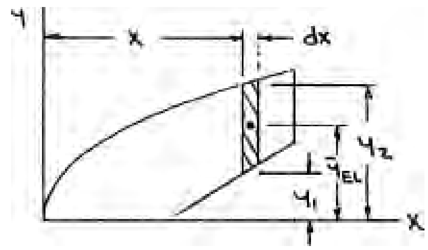
$$\bar{y}A = \int \bar{y}_{EL}dA: \quad \bar{y}\left(\frac{1}{8}a\right) = \frac{11}{5}a^2 \qquad \bar{y} = \frac{33}{40}a \blacktriangleleft$$



PROBLEM 5.43

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



For y_2 at $\quad x = a,\ y = b: \quad a = kb^2 \quad \text{or} \quad k = \dfrac{a}{b^2}$

Then $\quad y_2 = \dfrac{b}{\sqrt{a}} x^{1/2}$

Now $\quad \bar{x}_{EL} = x$

and for $\quad 0 \le x \le \dfrac{a}{2}: \quad \bar{y}_{EL} = \dfrac{y_2}{2} = \dfrac{b}{2}\dfrac{x^{1/2}}{\sqrt{a}}$

$$dA = y_2 dx = b\frac{x^{1/2}}{\sqrt{a}}dx$$

For $\quad \dfrac{a}{2} \le x \le a: \quad \bar{y}_{EL} = \dfrac{1}{2}(y_1 + y_2) = \dfrac{b}{2}\left(\dfrac{x}{a} - \dfrac{1}{2} + \dfrac{x^{1/2}}{\sqrt{a}}\right)$

$$dA = (y_2 - y_1)dx = b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx$$

Then $\quad A = \int dA = \int_0^{a/2} b\dfrac{x^{1/2}}{\sqrt{a}}dx + \int_{a/2}^{a} b\left(\dfrac{x^{1/2}}{\sqrt{a}} - \dfrac{x}{a} + \dfrac{1}{2}\right)dx$

$$= \frac{b}{\sqrt{a}}\left[\frac{2}{3}x^{3/2}\right]_0^{a/2} + b\left[\frac{2}{3}\frac{x^{3/2}}{\sqrt{a}} - \frac{x^2}{2a} + \frac{1}{2}x\right]_{a/2}^{a}$$

$$= \frac{2}{3}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{3/2} + (a)^{3/2} - \left(\frac{a}{2}\right)^{3/2}\right]$$

$$+ b\left\{-\frac{1}{2a}\left[(a^2) - \left(\frac{a}{2}\right)^2\right] + \frac{1}{2}\left[(a) - \left(\frac{a}{2}\right)\right]\right\}$$

$$= \frac{13}{24}ab$$

PROBLEM 5.43 (Continued)

and
$$\int \bar{x}_{EL}\,dA = \int_0^{a/2} x\left(b\frac{x^{1/2}}{\sqrt{a}}\,dx\right) + \int_{a/2}^{a} x\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)\right]dx$$

$$= \frac{b}{\sqrt{a}}\left[\frac{2}{5}x^{5/2}\right]_0^{a/2} + b\left[\frac{2}{5}\frac{x^{5/2}}{\sqrt{a}} - \frac{x^3}{3a} + \frac{x^4}{4}\right]_{a/2}^{a}$$

$$= \frac{2}{5}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{5/2} + (a)^{5/2} - \left(\frac{a}{2}\right)^{5/2}\right]$$

$$+ b\left\{-\frac{1}{3a}\left[(a)^3 - \left(\frac{a}{2}\right)^3\right] + \frac{1}{4}\left[(a)^2 - \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{71}{240}a^2 b$$

$$\int \bar{y}_{EL}\,dA = \int_0^{a/2} \frac{b}{2}\frac{x^{1/2}}{\sqrt{a}}\left[b\frac{x^{1/2}}{\sqrt{a}}\,dx\right]$$

$$+ \int_{a/2}^{a} \frac{b}{2}\left(\frac{x}{a} - \frac{1}{2} + \frac{x^{1/2}}{\sqrt{a}}\right)\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx\right]$$

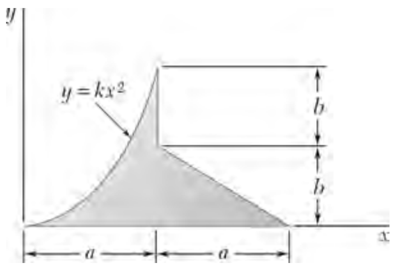
$$= \frac{b^2}{2a}\left[\frac{1}{2}x^2\right]_0^{a/2} + \frac{b^2}{2}\left[\left(\frac{x^2}{2a} - \frac{1}{3a}\left(\frac{x}{a} - \frac{1}{2}\right)^3\right)\right]_{a/2}^{a}$$

$$= \frac{b}{4a}\left[\left(\frac{a}{2}\right)^2 + (a)^2 - \left(\frac{a}{2}\right)^2\right] - \frac{b^2}{6a}\left(\frac{a}{2} - \frac{1}{2}\right)^3$$

$$= \frac{11}{48}ab^2$$

Hence
$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{13}{24}ab\right) = \frac{71}{240}a^2 b \qquad \bar{x} = \frac{17}{130}a = 0.546a \ \blacktriangleleft$$

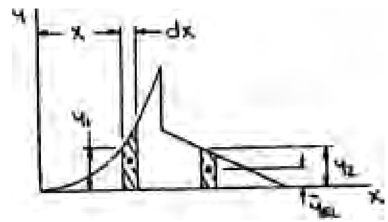
$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{13}{24}ab\right) = \frac{11}{48}ab^2 \qquad \bar{y} = \frac{11}{26}b = 0.423b \ \blacktriangleleft$$



PROBLEM 5.44

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



For y_1 at $\quad x = a, \quad y = 2b \quad 2b = ka^2 \quad$ or $\quad k = \dfrac{2b}{a^2}$

Then $\quad y_1 = \dfrac{2b}{a^2}x^2$

By observation $\quad y_2 = -\dfrac{b}{a}(x + 2b) = b\left(2 - \dfrac{x}{a}\right)$

Now $\quad \bar{x}_{EL} = x$

and for $0 \le x \le a$: $\quad \bar{y}_{EL} = \dfrac{1}{2}y_1 = \dfrac{b}{a^2}x^2 \quad$ and $\quad dA = y_1 dx = \dfrac{2b}{a^2}x^2 dx$

For $a \le x \le 2a$: $\quad \bar{y}_{EL} = \dfrac{1}{2}y_2 = \dfrac{b}{2}\left(2 - \dfrac{x}{a}\right) \quad$ and $\quad dA = y_2 dx = b\left(2 - \dfrac{x}{a}\right)dx$

Then

$$A = \int dA = \int_0^a \frac{2b}{a^2}x^2 dx + \int_a^{2a} b\left(2 - \frac{x}{a}\right)dx$$

$$= \frac{2b}{a^2}\left[\frac{x^3}{3}\right]_0^a + b\left[-\frac{a}{2}\left(2 - \frac{x}{a}\right)^2\right]_0^{2a} = \frac{7}{6}ab$$

and

$$\int \bar{x}_{EL} dA = \int_0^a x\left(\frac{2b}{a^2}x^2 dx\right) + \int_a^{2a} x\left[b\left(2 - \frac{x}{a}\right)dx\right]$$

$$= \frac{2b}{a^2}\left[\frac{x^4}{4}\right]_0^a + b\left[x^2 - \frac{x^3}{3a}\right]_0^{2a}$$

$$= \frac{1}{2}a^2 b + b\left\{\left[(2a)^2 - (a)^2\right] + \frac{1}{3a}\left[(2a^2) - (a)^3\right]\right\}$$

$$= \frac{7}{6}a^2 b$$

PROBLEM 5.44 (Continued)

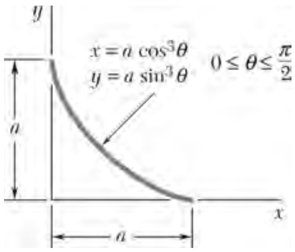
$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{b}{a^2}x^2\left[\frac{2b}{a^2}x^2 dx\right] + \int_0^{2a} \frac{b}{2}\left(2-\frac{x}{a}\right)\left[b\left(2-\frac{x}{a}\right)dx\right]$$

$$= \frac{2b^2}{a^4}\left[\frac{x^5}{5}\right]_0^a + \frac{b^2}{2}\left[-\frac{a}{3}\left(2-\frac{x}{a}\right)^3\right]_a^{2a}$$

$$= \frac{17}{30}ab^2$$

Hence $\quad \bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{7}{6}ab\right) = \frac{7}{6}a^2b \qquad \bar{x} = a$ ◄

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{7}{6}ab\right) = \frac{17}{30}ab^2 \qquad \bar{y} = \frac{17}{35}b \text{ ◄}$$

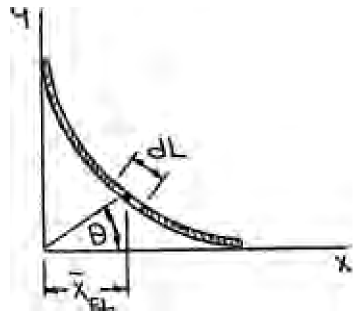


PROBLEM 5.45

A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid.

SOLUTION

First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line



Now $\quad \bar{x}_{EL} = a\cos^3\theta \quad$ and $\quad dL = \sqrt{dx^2 + dy^2}$

where $\quad x = a\cos^3\theta: \quad dx = -3a\cos^2\theta\sin\theta d\theta$

$$y = a\sin^3\theta: \quad dy = 3a\sin^2\theta\cos\theta d\theta$$

Then
$$\begin{aligned} dL &= [(-3a\cos^2\theta\sin\theta d\theta)^2 + (3a\sin^2\theta\cos\theta d\theta)^2]^{1/2} \\ &= 3a\cos\theta\sin\theta(\cos^2\theta + \sin^2\theta)^{1/2} d\theta \\ &= 3a\cos\theta\sin\theta d\theta \end{aligned}$$

$$L = \int dL = \int_0^{\pi/2} 3a\cos\theta\sin\theta d\theta = 3a\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{3}{2}a$$

and
$$\int \bar{x}_{EL} dL = \int_0^{\pi/2} a\cos^3\theta(3a\cos\theta\sin\theta d\theta)$$

$$= 3a^2\left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi/2} = \frac{3}{5}a^2$$

Hence $\quad \bar{x}L = \int \bar{x}_{EL} dL: \quad \bar{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2 \qquad \bar{x} = \frac{2}{5}a$ ◄

Alternative Solution

$$x = a\cos^3\theta \Rightarrow \cos^2\theta = \left(\frac{x}{a}\right)^{2/3}$$

$$y = a\sin^3\theta \Rightarrow \sin^2\theta = \left(\frac{y}{a}\right)^{2/3}$$

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1 \quad \text{or} \quad y = (a^{2/3} - x^{2/3})^{3/2}$$

Then $$\frac{dy}{dx} = (a^{2/3} - x^{2/3})^{1/2}(-x^{-1/3})$$

Now $$\bar{x}_{EL} = x$$

and $$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$dx = \left\{1 + \left[(a^{2/3} - x^{2/3})^{1/2}(-x^{-1/3})\right]^2\right\}^{1/2} dx$$

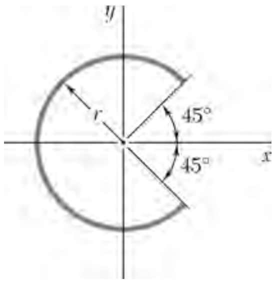
Then $$L = \int dL = \int_0^a \frac{a^{1/3}}{x^{1/3}} dx = a^{1/3}\left[\frac{3}{2}x^{2/3}\right]_0^a = \frac{3}{2}a$$

and $$\int \bar{x}_{EL} dL = \int_0^a x\left(\frac{a^{1/3}}{x^{1/3}} dx\right) = a^{1/3}\left[\frac{3}{5}x^{5/3}\right]_0^a = \frac{3}{5}a^2$$

Hence $$\bar{x}L = \int \bar{x}_{EL} dL: \quad \bar{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2$$

$$\bar{x} = \frac{2}{5}a \blacktriangleleft$$

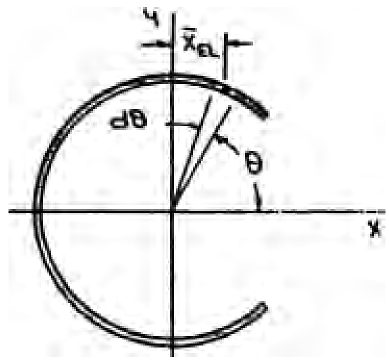
PROBLEM 5.46



A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid.

SOLUTION

First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line



Now $\qquad \bar{x}_{EL} = r\cos\theta \quad \text{and} \quad dL = rd\theta$

Then $\qquad L = \int dL = \int_{\pi/4}^{7\pi/4} rd\theta = r[\theta]_{\pi/4}^{7\pi/4} = \frac{3}{2}\pi r$

and

$$\int \bar{x}_{EL}dL = \int_{\pi/4}^{7\pi/4} r\cos\theta(rd\theta)$$

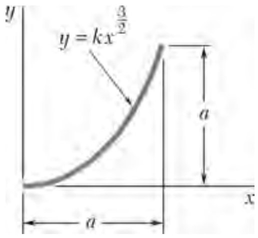
$$= r^2[\sin\theta]_{\pi/4}^{7\pi/4}$$

$$= r^2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= -r^2\sqrt{2}$$

Thus $\qquad \bar{x}L = \int \bar{x}dL: \quad \bar{x}\left(\frac{3}{2}\pi r\right) = -r^2\sqrt{2}$

$$\bar{x} = -\frac{2\sqrt{2}}{3\pi}r \blacktriangleleft$$

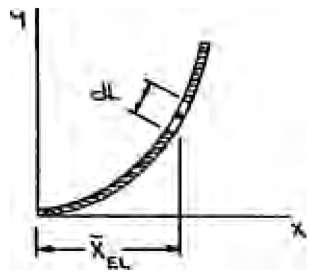


PROBLEM 5.47*

A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid. Express your answer in terms of a.

SOLUTION

First note that because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.



We have at $\quad x = a, \quad y = a$

$$a = ka^{3/2} \quad \text{or} \quad k = \frac{1}{\sqrt{a}}$$

Then $$y = \frac{1}{\sqrt{a}} x^{3/2}$$

and $$\frac{dy}{dx} = \frac{3}{2\sqrt{a}} x^{1/2}$$

Now $$\bar{x}_{EL} = x$$

and
$$\begin{aligned} dL &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \left[1 + \left(\frac{3}{2\sqrt{a}} x^{1/2}\right)^2\right]^{1/2} dx \\ &= \frac{1}{2\sqrt{a}} \sqrt{4a + 9x}\; dx \end{aligned}$$

Then
$$\begin{aligned} L = \int dL &= \int_0^a \frac{1}{2\sqrt{a}} \sqrt{4a + 9x}\, dx \\ &= \frac{1}{2\sqrt{a}} \left[\frac{2}{3} \times \frac{1}{9} (4a + 9x)^{3/2}\right]_0^a \\ &= \frac{a}{27}[(13)^{3/2} - 8] \\ &= 1.43971a \end{aligned}$$

and $$\int \bar{x}_{EL}\, dL = \int_0^a x \left[\frac{1}{2\sqrt{a}} \sqrt{4a + 9x}\, dx\right]$$

PROBLEM 5.47* (Continued)

Use integration by parts with

$$u = x \qquad dv = \sqrt{4a+9x}\;dx$$

$$du = dx \qquad v = \frac{2}{27}(4a+9x)^{3/2}$$

Then

$$\int \bar{x}_{EL}\,dL = \frac{1}{2\sqrt{a}}\left\{\left[x\times\frac{2}{27}(4a+9x)^{3/2}\right]_0^a - \int_0^a \frac{2}{27}(4a+9x)^{3/2}\,dx\right\}$$

$$= \frac{(13)^{3/2}}{27}a^2 - \frac{1}{27\sqrt{a}}\left[\frac{2}{45}(4a+9x)^{5/2}\right]_0^a$$

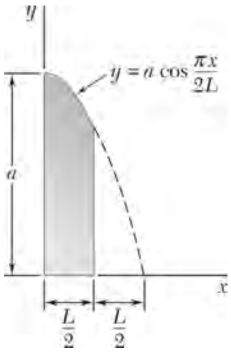
$$= \frac{a^2}{27}\left\{(13)^{3/2} - \frac{2}{45}[(13)^{5/2} - 32]\right\}$$

$$= 0.78566a^2$$

$$\bar{x}L = \int x_{EL}\,dL: \quad \bar{x}(1.43971a) = 0.78566a^2$$

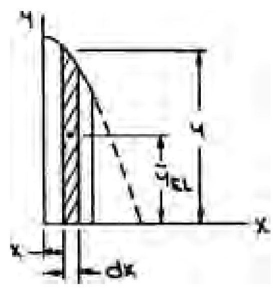
or $\bar{x} = 0.546a$ ◀

PROBLEM 5.48*



Determine by direct integration the centroid of the area shown.

SOLUTION



We have $$\bar{x}_{EL} = x$$

$$y_{EL} = \frac{1}{2}y = \frac{a}{2}\cos\frac{\pi x}{2L}$$

and $$dA = ydx = a\cos\frac{\pi x}{2L}dx$$

Then $$A = \int dA = \int_0^{L/2} a\cos\frac{\pi x}{2L}dx$$

$$= a\left[\frac{2L}{\pi}\sin\frac{\pi x}{2L}\right]_0^{L/2}$$

$$= \frac{\sqrt{2}}{\pi}aL$$

and $$\int x_{EL}dA = \int x\left(a\cos\frac{\pi x}{2L}dx\right)$$

Use integration by parts with $$u = x \qquad dv = \cos\frac{\pi x}{2L}dx$$

$$du = dx \qquad v = \frac{2L}{\pi}\sin\frac{\pi x}{2L}$$

Then $$\int \bar{x}\cos\frac{\pi x}{2L}dx = \frac{2L}{\pi}\times\sin\frac{\pi x}{2L} - \int\frac{2L}{\pi}\sin\frac{\pi x}{2L}dx$$

$$= \frac{2L}{\pi}\left(x\sin\frac{\pi x}{2L} + \frac{2L}{\pi}\cos\frac{\pi x}{2L}\right)$$

PROBLEM 5.48* (Continued)

$$\int x_{EL}\,dA = a\frac{2L}{\pi}\left[x\,\sin\frac{\pi x}{2L}+\frac{2L}{\pi}\cos\frac{\pi x}{2L}\right]_0^{L/2}$$

$$= a\frac{2L}{\pi}\left[\left(\frac{L}{2\sqrt{2}}+\frac{\sqrt{2}}{\pi}L\right)-\frac{2L}{\pi}\right]$$

$$= 0.106374aL^2$$

Also

$$\int \overline{y}_{EL}\,dA = \int_0^{L/2}\frac{a}{2}\cos\frac{\pi x}{2L}\left(a\,\cos\frac{\pi x}{2L}dx\right)$$

$$= \frac{a^2}{2}\left[\frac{x}{2}+\frac{\sin\frac{\pi x}{L}}{\frac{2\pi}{L}}\right]_0^{L/2} = \frac{a^{-2}}{2}\left(\frac{L}{4}+\frac{L}{2\pi}\right)$$

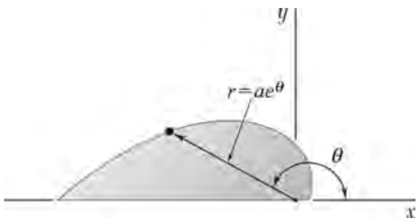
$$= 0.20458a^2L$$

$$\overline{x}A = \int \overline{x}_{EL}dA: \quad \overline{x}\left(\frac{\sqrt{2}}{\pi}aL\right) = 0.106374aL^2$$

or $\overline{x} = 0.236L$ ◀

$$\overline{y}A = \int \overline{y}_{EL}dA: \quad \overline{y}\left(\frac{\sqrt{2}}{\pi}aL\right) = 0.20458a^2L$$

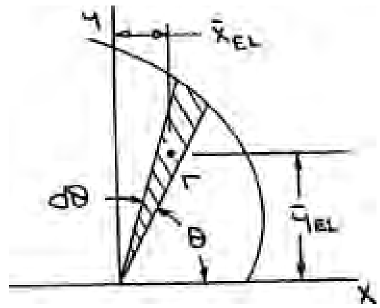
or $\overline{y} = 0.454a$ ◀



PROBLEM 5.49*

Determine by direct integration the centroid of the area shown.

SOLUTION



We have $$\bar{x}_{EL} = \frac{2}{3}r\cos\theta = \frac{2}{3}ae^{\theta}\cos\theta$$

$$\bar{y}_{EL} = \frac{2}{3}r\sin\theta = \frac{2}{3}ae^{\theta}\sin\theta$$

and $$dA = \frac{1}{2}(r)(rd\theta) = \frac{1}{2}a^2e^{2\theta}d\theta$$

Then $$A = \int dA = \int_0^{\pi}\frac{1}{2}a^2e^{2\theta}d\theta = \frac{1}{2}a^2\left[\frac{1}{2}e^{2\theta}\right]_0^{\pi}$$

$$= \frac{1}{4}a^2(e^{2\pi}-1)$$

$$= 133.623a^2$$

and $$\int \bar{x}_{EL}dA = \int_0^{\pi}\frac{2}{3}ae^{\theta}\cos\theta\left(\frac{1}{2}a^2e^{2\theta}d\theta\right)$$

$$= \frac{1}{3}a^3\int_0^{\pi}e^{3\theta}\cos\theta d\theta$$

To proceed, use integration by parts, with

$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta}d\theta$$

$$dv = \cos\theta\, d\theta \quad \text{and} \quad v = \sin\theta$$

Then $$\int e^{3\theta}\cos\theta d\theta = e^{3\theta}\sin\theta - \int \sin\theta(3e^{3\theta}d\theta)$$

Now let $$u = e^{3\theta} \quad \text{then} \quad du = 3e^{3\theta}d\theta$$

$$dv = \sin\theta d\theta, \quad \text{then} \quad v = -\cos\theta$$

Then $$\int e^{3\theta}\sin\theta d\theta = e^{3\theta}\sin\theta - 3\left[-e^{3\theta}\cos\theta - \int(-\cos\theta)(3e^{3\theta}d\theta)\right]$$

PROBLEM 5.49* (Continued)

So that
$$\int e^{3\theta}\cos\theta d\theta = \frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)$$

$$\int x_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)\right]_0^\pi$$

$$= \frac{a^3}{30}(-3e^{3\pi} - 3) = -1239.26a^3$$

Also
$$\int \overline{y}_{EL}dA = \int_0^\pi \frac{2}{3}ae^\theta \sin\theta\left(\frac{1}{2}a^2e^{2\theta}d\theta\right)$$

$$= \frac{1}{3}a^3\int_0^\pi e^{3\theta}\sin\theta d\theta$$

Using integration by parts, as above, with

$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta}d\theta$$

$$dv = \int \sin\theta\, d\theta \quad \text{and} \quad v = -\cos\theta$$

Then
$$\int e^{3\theta}\sin\theta d\theta = -e^{3\theta}\cos\theta - \int(-\cos\theta)(3e^{3\theta}d\theta)$$

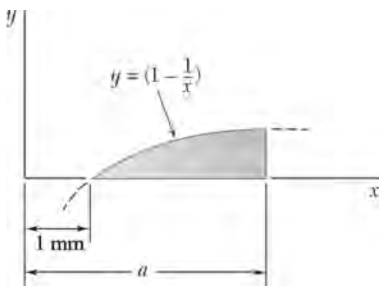
So that
$$\int e^{3\theta}\sin\theta d\theta = \frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)$$

$$\int \overline{y}_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)\right]_0^\pi$$

$$= \frac{a^3}{30}(e^{3\pi} + 1) = 413.09a^3$$

Hence
$$\overline{x}A = \int x_{EL}dA: \quad \overline{x}(133.623a^2) = -1239.26a^3 \qquad \text{or } \overline{x} = -9.27a \blacktriangleleft$$

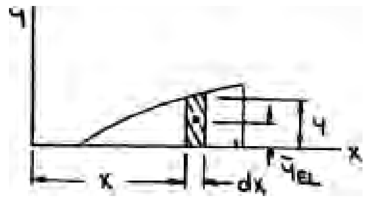
$$\overline{y}A = \int \overline{y}_{EL}dA: \quad \overline{y}(133.623a^2) = 413.09a^3 \qquad \text{or } \overline{y} = 3.09a \blacktriangleleft$$



PROBLEM 5.50

Determine the centroid of the area shown when $a = 2\,\text{mm}$.

SOLUTION



We have $\quad \bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{1}{x}\right)$$

and $\quad dA = y\,dx = \left(1 - \frac{1}{x}\right)dx$

Then $\quad A = \int dA = \int_1^a \left(1 - \frac{1}{x}\right)\frac{dx}{2} = [x - \ln x]_1^a = (a - \ln a - 1)$

and $\quad \int \bar{x}_{EL}\,dA = \int_1^a x\left[\left(1 - \frac{1}{x}\right)dx\right] = \left[\frac{x^2}{2} - x\right]_1^a = \left(\frac{a^2}{2} - a + \frac{1}{2}\right)$

$$\int \bar{y}_{EL}\,dA = \int_1^a \frac{1}{2}\left(1 - \frac{1}{x}\right)\left[\left(1 - \frac{1}{x}\right)dx\right] = \frac{1}{2}\int_1^a \left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{2}\left[x - 2\ln x - \frac{1}{x}\right]_1^a = \frac{1}{2}\left(a - 2\ln a - \frac{1}{a}\right)$$

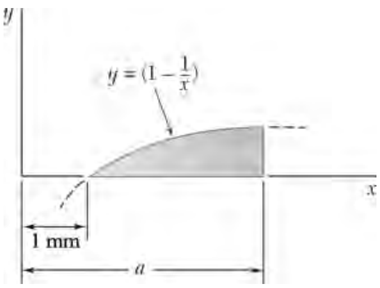
$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x} = \frac{\frac{a^2}{2} - a + \frac{1}{2}}{a - \ln a - 1}$$

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y} = \frac{a - 2\ln a - \frac{1}{a}}{2(a - \ln a - 1)}$$

Find: $\bar{x}$ and $\bar{y}$ when $a = 2$

We have $\quad \bar{x} = \dfrac{\frac{1}{2}(2)^2 - 2 + \frac{1}{2}}{2 - \ln 2 - 1} \qquad$ or $\quad \bar{x} = 1.629$ ◄

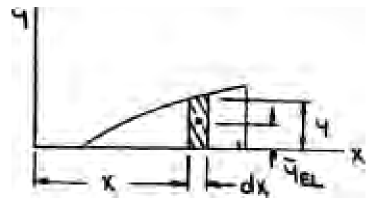
and $\quad \bar{y} = \dfrac{2 - 2\ln 2 - \frac{1}{2}}{2(2 - \ln 2 - 1)} \qquad$ or $\quad \bar{y} = 0.1853$ ◄



PROBLEM 5.51

Determine the value of a for which the ratio $\bar{x}/\bar{y}$ is 9.

SOLUTION



We have $\bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{1}{x}\right)$$

and $$dA = ydx = \left(1 - \frac{1}{x}\right)dx$$

Then $$A = \int dA = \int_1^a \left(1 - \frac{1}{x}\right)\frac{dx}{2} = [x - \ln x]_1^a$$

$$= (a - \ln a - 1)$$

and $$\int \bar{x}_{EL} dA = \int_1^a x\left[\left(1 - \frac{1}{x}\right)dx\right] = \left[\frac{x^2}{2} - x\right]_1^a$$

$$= \left(\frac{a^2}{2} - a + \frac{1}{2}\right)$$

$$\int \bar{y}_{EL} dA = \int_1^a \frac{1}{2}\left(1 - \frac{1}{x}\right)\left[\left(1 - \frac{1}{x}\right)dx\right] = \frac{1}{2}\int_1^a \left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{2}\left[x - 2\ln x - \frac{1}{x}\right]_1^a$$

$$= \frac{1}{2}\left(a - 2\ln a - \frac{1}{a}\right)$$

$$\bar{x}A = \int \bar{x}_{EL} dA: \quad \bar{x} = \frac{\frac{a^2}{2} - a + \frac{1}{2}}{a - \ln a - 1}$$

$$\bar{y}A = \int \bar{y}_{EL} dA: \quad \bar{y} = \frac{a - 2\ln a - \frac{1}{a}}{2(a - \ln a - 1)}$$

PROBLEM 5.51 (Continued)

Find: a so that $\frac{\bar{x}}{\bar{y}} = 9$

We have
$$\frac{\bar{x}}{\bar{y}} = \frac{\bar{x}A}{\bar{y}A} = \frac{\int \bar{x}_{EL}\,dA}{\int \bar{y}_{EL}\,dA}$$

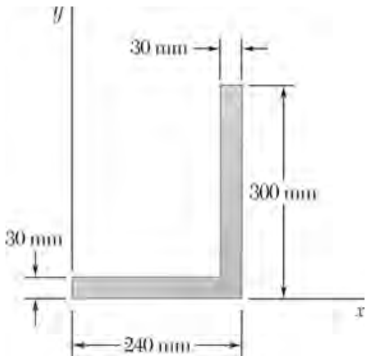
Then
$$\frac{\frac{1}{2}a^2 - a + \frac{1}{2}}{\frac{1}{2}\left(a - 2\ln a - \frac{1}{a}\right)} = 9$$

or
$$a^3 - 11a^2 + a + 18a\ln a + 9 = 0$$

Using trial and error or numerical methods and ignoring the trivial solution $a = 1$, find

$a = 1.901$ and $a = 3.74$ ◀

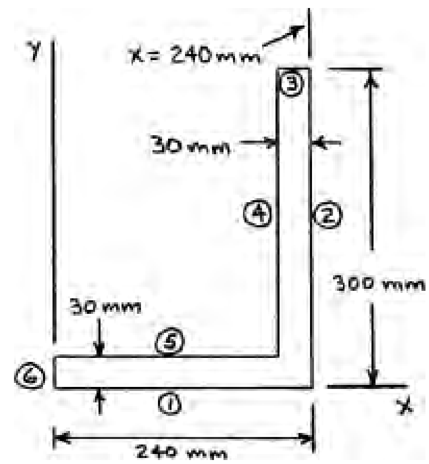
PROBLEM 5.52



Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.1 about (*a*) the line $x = 240$ mm, (*b*) the y axis.

Problem 5.1 Locate the centroid of the plane area shown.

SOLUTION



From the solution to Problem 5.1 we have

$$A = 15.3\times 10^3\,\text{mm}^2$$
$$\Sigma\,\bar{x}A = 2.6865\times 10^6\,\text{mm}^3$$
$$\Sigma\,\bar{y}A = 1.4445\times 10^6\,\text{mm}^3$$

Applying the theorems of pappus-Guldinus we have

(*a*) Rotation about the line $x = 240$ mm

$$\text{Volume} = 2\pi(240 - \bar{x})A$$
$$= 2\pi(240A - \Sigma\,\bar{x}A)$$
$$= 2\pi[240(15.3\times 10^3) - 2.6865\times 10^6] \qquad \text{Volume} = 6.19\times 10^6\,\text{mm}^3 \blacktriangleleft$$

$$\text{Area} = 2\pi \bar{X}_{\text{line}} L = 2\pi\,\Sigma(\bar{x}_{\text{line}})L$$
$$= 2\pi(\bar{x}_1 L_1 + \bar{x}_3 L_3 + \bar{x}_4 L_4 + \bar{x}_5 L_5 + \bar{x}_6 L_6)$$

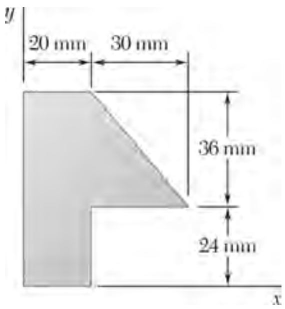
where $\bar{x}_1, \ldots, \bar{x}_6$ are measured with respect to line $x = 240$ mm

$$\text{Area} = 2\pi[(120)(240) + (15)(30) + (30)(270) + (135)(210) + (240)(30)] \qquad \text{Area} = 458\times 10^3\,\text{mm}^2 \blacktriangleleft$$

(*b*) Rotation about the y axis

$$\text{Volume} = 2\pi \bar{X}_{\text{area}} A = 2\pi(\Sigma\,\bar{x}A)$$
$$= 2\pi(2.6865\times 10^6\,\text{mm}^3) \qquad \text{Volume} = 16.88\times 10^6\,\text{mm}^3 \blacktriangleleft$$

$$\text{Area} = 2\pi \bar{X}_{\text{line}} L = 2\pi\,\Sigma(\bar{x}_{\text{line}})L$$
$$= 2\pi(\bar{x}_1 L_1 + \bar{x}_2 L_2 + \bar{x}_3 L_3 + \bar{x}_4 L_4 + \bar{x}_5 L_5)$$
$$= 2\pi[(120)(240) + (240)(300) + (225)(30) + (210)(270) + (105)(210)] \qquad \text{Area} = 1.171\times 10^6\,\text{mm}^2 \blacktriangleleft$$

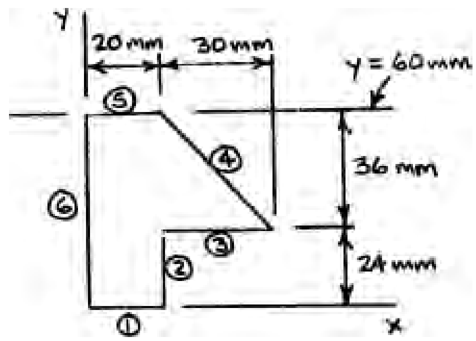


PROBLEM 5.53

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.2 about (*a*) the line $y = 60$ mm, (*b*) the y axis.

Problem 5.2 Locate the centroid of the plane area shown.

SOLUTION



From the solution to Problem 5.2 we have

$$A = 1740 \text{ mm}^2$$
$$\Sigma\,\bar{x}A = 28200 \text{ mm}^3$$
$$\Sigma\,\bar{y}A = 55440 \text{ mm}^3$$

Applying the theorems of pappus-Guldinus we have

(*a*) Rotation about the line $y = 60$ mm

$$\begin{aligned}\text{Volume} &= 2\pi(60-\bar{y})A\\ &= 2\pi(60A-\Sigma\,\bar{y}A)\\ &= 2\pi[60(1740)-55440]\end{aligned}$$

$\text{Volume} = 308\times10^3\,\text{mm}^3$ ◀

$$\begin{aligned}\text{Area} &= 2\pi\bar{Y}_{\text{line}}\\ &= 2\pi\,\Sigma(\bar{y}_{\text{line}})L\\ &= 2\pi(\bar{y}_1L_1+\bar{y}_2L_2+\bar{y}_3L_3+\bar{y}_4L_4+\bar{y}_6L_6)\end{aligned}$$

where $\bar{y}_1,\ldots,\bar{y}_6$ are measured with respect to line $y = 60$ mm

$$\text{Area} = 2\pi\left[(60)(20)+(48)(24)+(36)(30)+(18)\sqrt{(30)^2+(36)^2}+(30)(60)\right]$$

$\text{Area} = 38.2\times10^3\,\text{mm}^2$ ◀

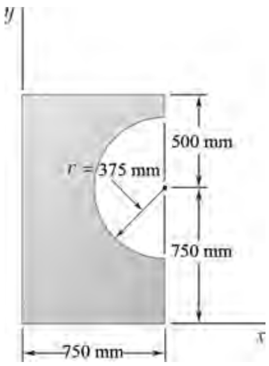
(*b*) Rotation about the y axis

$$\text{Volume} = 2\pi\bar{X}_{\text{area}}A = 2\pi(\Sigma\,\bar{x}A) = 2\pi(28200\text{ mm}^3)$$

$\text{Volume} = 177.2\times10^6\,\text{mm}^3$ ◀

$$\begin{aligned}\text{Area} &= 2\pi\bar{X}_{\text{line}}L = 2\pi\,\Sigma(\bar{x}_{\text{line}})L = 2\pi(\bar{x}_1L_1+\bar{x}_2L_2+\bar{x}_3L_3+\bar{x}_4L_4+\bar{x}_5L_5)\\ &= 2\pi\left[(10)(20)+(20)(24)+(35)(30)+(35)\sqrt{(30)^2+(36)^2}+(10)(20)\right]\end{aligned}$$

$\text{Area} = 22.4\times10^3\,\text{mm}^2$ ◀

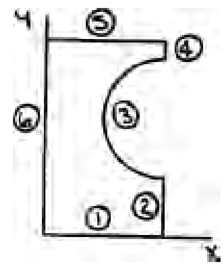


PROBLEM 5.54

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.8 about (*a*) the *x* axis, (*b*) the *y* axis.

Problem 5.8 Locate the centroid of the plane area shown.

SOLUTION



From the solution to Problem 5.8 we have

$$A = 0.71661\times 10^6\ \text{mm}^2$$
$$\Sigma\bar{x}A = 221.0505\times 10^6\ \text{mm}^3$$
$$\Sigma\bar{y}A = 420.27\times 10^6\ \text{mm}^3$$

Applying the theorems of Pappus-Guldinus we have

(*a*) Rotation about the *x* axis:

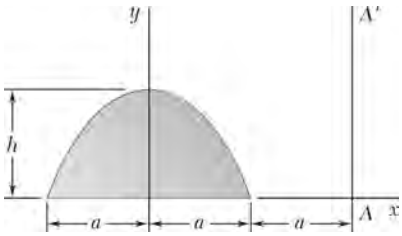
$$\text{Volume} = 2\pi\bar{Y}_{area}A = 2\pi\Sigma\bar{y}A$$
$$= 2\pi(420.27\times 10^6\ \text{mm}^3)$$
$$= 26406\times 10^9\ \text{mm}^3 \qquad \text{or} \quad \text{Volume} = 2.64\times 10^9\ \text{mm}^3 \blacktriangleleft$$

$$\text{Area} = 2\pi\bar{Y}_{line}A$$
$$= 2\pi\,\Sigma(\bar{y}_{line})A$$
$$= 2\pi(\bar{y}_2L_2 + \bar{y}_3L_3 + \bar{y}_4L_4 + \bar{y}_5L_5 + \bar{y}_6L_6)$$
$$= 2\pi[(187.5)(375) + (750)(\pi\times 375) + (1187.5)(125)$$
$$+ (1250)(750) + (625)(1250)] \qquad \text{or} \quad \text{Area} = 17.73\times 10^6\ \text{mm}^2 \blacktriangleleft$$

(*b*) Rotation about the *y* axis

$$\text{Volume} = 2\pi\bar{X}_{area}A = 2\pi\Sigma\bar{x}A$$
$$= 2\pi(221.0505\times 10^6\text{mm}^3) \qquad \text{or} \quad \text{Volume} = 1.389\times 10^9\ \text{mm}^3 \blacktriangleleft$$

$$\text{Area} = 2\pi\bar{X}_{line}L = 2\pi\,\Sigma(\bar{x}_{line})L$$
$$= 2\pi(\bar{x}_1L_1 + \bar{x}_2L_2 + \bar{x}_3L_3 + \bar{x}_4L_4 + \bar{x}_5L_5)$$
$$= 2\pi\left[(375)(750) + (750)(375) + \left(750 - \frac{2\times 375}{\pi}\right)(\pi\times 375) + (750)(125) + (375)(750)\right]$$
$$\text{or} \quad \text{Area} = 9.67\times 10^6\ \text{mm}^2 \blacktriangleleft$$



PROBLEM 5.55

Determine the volume of the solid generated by rotating the parabolic area shown about (*a*) the *x* axis, (*b*) the axis AA'.

SOLUTION

First, from Figure 5.8a we have $A = \frac{4}{3}ah$

$$\bar{y} = \frac{2}{5}h$$

Applying the second theorem of Pappus-Guldinus we have

(*a*) Rotation about the *x* axis:

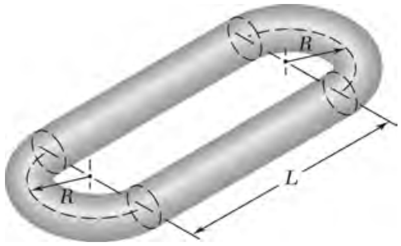
$$\text{Volume} = 2\pi\bar{y}A$$

$$= 2\pi\left(\frac{2}{5}h\right)\left(\frac{4}{3}ah\right) \qquad \text{or Volume} = \frac{16}{15}\pi ah^2 \blacktriangleleft$$

(*b*) Rotation about the line AA':

$$\text{Volume} = 2\pi(2a)A$$

$$= 2\pi(2a)\left(\frac{4}{3}ah\right) \qquad \text{or Volume} = \frac{16}{3}\pi a^2 h \blacktriangleleft$$



PROBLEM 5.56

Determine the volume and the surface area of the chain link shown, which is made from a 6-mm-diameter bar, if $R = 10$ mm and $L = 30$ mm.

SOLUTION

The area A and circumference C of the cross section of the bar are

$$A = \frac{\pi}{4}d^2 \quad \text{and} \quad C = \pi d.$$

Also, the semicircular ends of the link can be obtained by rotating the cross section through a horizontal semicircular arc of radius R. Now, applying the theorems of Pappus-Guldinus, we have for the volume V:

$$V = 2(V_{\text{side}}) + 2(V_{\text{end}})$$
$$= 2(AL) + 2(\pi RA)$$
$$= 2(L + \pi R)A$$

or

$$V = 2[30\text{ mm} + \pi(10\text{ mm})]\left[\frac{\pi}{4}(6\text{ mm})^2\right]$$
$$= 3470\text{ mm}^3$$

or $V = 3470\text{ mm}^3$ ◀

For the area A:

$$A = 2(A_{\text{side}}) + 2(A_{\text{end}})$$
$$= 2(CL) + 2(\pi RC)$$
$$= 2(L + \pi R)C$$

or

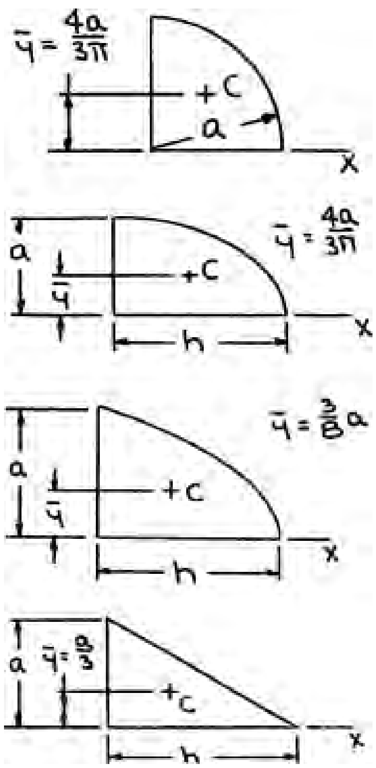
$$A = 2[30\text{ mm} + \pi(10\text{ mm})][\pi(6\text{ mm})]$$
$$= 2320\text{ mm}^2$$

or $A = 2320\text{ mm}^2$ ◀

PROBLEM 5.57

Verify that the expressions for the volumes of the first four shapes in Figure 5.21 on Page 253 are correct.

SOLUTION



Following the second theorem of Pappus-Guldinus, in each case a specific generating area A will be rotated about the x axis to produce the given shape. Values of $\bar{y}$ are from Figure 5.8a.

(1) Hemisphere: the generating area is a quarter circle

We have $V = 2\pi \bar{y} A = 2\pi \left(\frac{4a}{3\pi}\right)\left(\frac{\pi}{4}a^2\right)$ or $V = \frac{2}{3}\pi a^3$ ◀

(2) Semiellipsoid of revolution: the generating area is a quarter ellipse

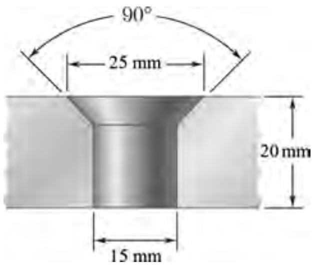
We have $V = 2\pi \bar{y} A = 2\pi \left(\frac{4a}{3\pi}\right)\left(\frac{\pi}{4}ha\right)$ or $V = \frac{2}{3}\pi a^2 h$ ◀

(3) Paraboloid of revolution: the generating area is a quarter parabola

We have $V = 2\pi \bar{y} A = 2\pi \left(\frac{3}{8}a\right)\left(\frac{2}{3}ah\right)$ or $V = \frac{1}{2}\pi a^2 h$ ◀

(4) Cone: the generating area is a triangle

We have $V = 2\pi \bar{y} A = 2\pi \left(\frac{a}{3}\right)\left(\frac{1}{2}ha\right)$ or $V = \frac{1}{3}\pi a^2 h$ ◀

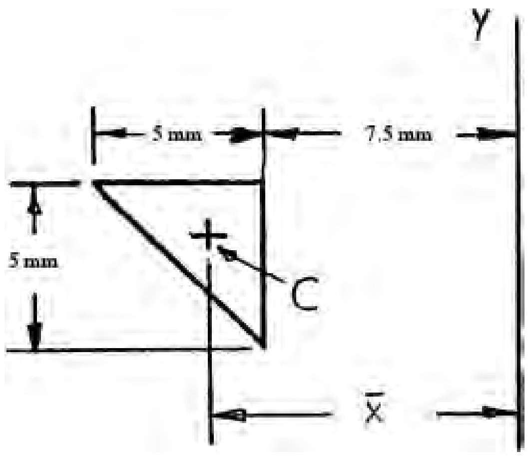


PROBLEM 5.58

A 15 mm-diameter hole is drilled in a piece of 20 mm-thick steel; the hole is then countersunk as shown. Determine the volume of steel removed during the countersinking process.

SOLUTION

The required volume can be generated by rotating the area shown about the *y* axis. Applying the second theorem of Pappus-Guldinus, we have



$$V = 2\pi \bar{x} A$$

$$= 2\pi\left[7.5 + \frac{1}{3}(5)\text{ mm}\right]\times\left[\frac{1}{2}\times 5\text{ mm}\times 5\text{ mm}\right]$$

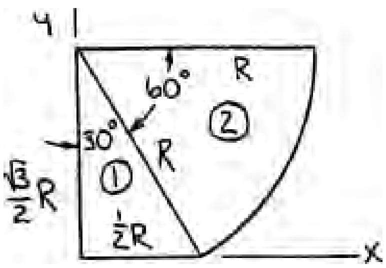
$V = 720\text{ mm}^3$ ◀

PROBLEM 5.59

Determine the capacity, in liters, of the punch bowl shown if $R = 250$ mm.

SOLUTION

The volume can be generated by rotating the triangle and circular sector shown about the *y* axis. Applying the second theorem of Pappus-Guldinus and using Figure 5.8a, we have

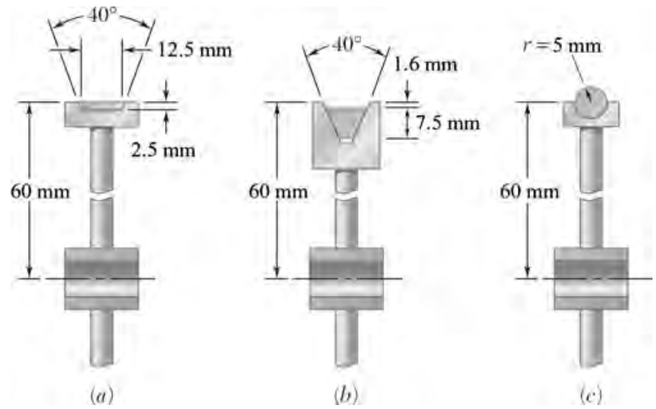


$$V = 2\pi\bar{x}A = 2\pi\Sigma\bar{x}A$$
$$= 2\pi(\bar{x}_1A_1 + \bar{x}_2A_2)$$
$$= 2\pi\left[\left(\frac{1}{3}\times\frac{1}{2}R\right)\left(\frac{1}{2}\times\frac{1}{2}R\times\frac{\sqrt{3}}{2}R\right) + \left(\frac{2R\sin 30°}{3\times\frac{\pi}{6}}\right)\left(\frac{\pi}{6}R^2\right)\right]$$
$$= 2\pi\left(\frac{R^3}{16\sqrt{3}} + \frac{R^3}{2\sqrt{3}}\right)$$
$$= \frac{3\sqrt{3}}{8}\pi R^3$$
$$= \frac{3\sqrt{3}}{8}\pi(0.25\text{ m})^3$$
$$= 0.031883\text{ m}^3$$

Since $\qquad 10^3\, l = 1\text{ m}^3$

$$V = 0.031883\text{ m}^3 \times \frac{10^3\, l}{1\text{ m}^3}$$

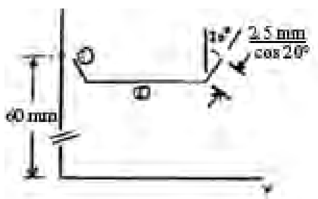
$V = 31.9\, l$ ◀



PROBLEM 5.60

Three different drive belt profiles are to be studied. If at any given time each belt makes contact with one-half of the circumference of its pulley, determine the *contact area* between the belt and the pulley for each design.

SOLUTION



Applying the first theorem of Pappus-Guldinus, the contact area A_C of a belt is given by:

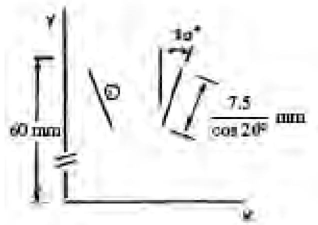
$$A_C = \pi \bar{y} L = \pi \Sigma \bar{y} L$$

where the individual lengths are the lengths of the belt cross section that are in contact with the pulley.

(*a*) $A_C = \pi[2(\bar{y}_1 L_1) + \bar{y}_2 L_2]$

$$\pi\left\{2\left[\left(60 - \frac{2.5}{2}\right)\text{mm}\right]\left[\frac{2.5\,\text{mm}}{\cos 20°}\right] + [(60 - 2.5)\,\text{mm}](12.5\,\text{mm})\right\}$$

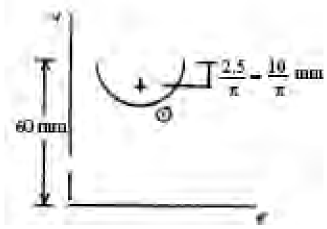
or $A_C = 3240\,\text{mm}^2$ ◀



(*b*) $A_C = \pi[2(\bar{y}_1 L_1)]$

$$= 2\pi\left[\left(60 - 1.6 - \frac{7.5}{2}\right)\text{mm}\right]\left(\frac{7.5\text{ mm}}{\cos 20°}\right)$$

or $A_C = 2740\text{ mm}^2$ ◀



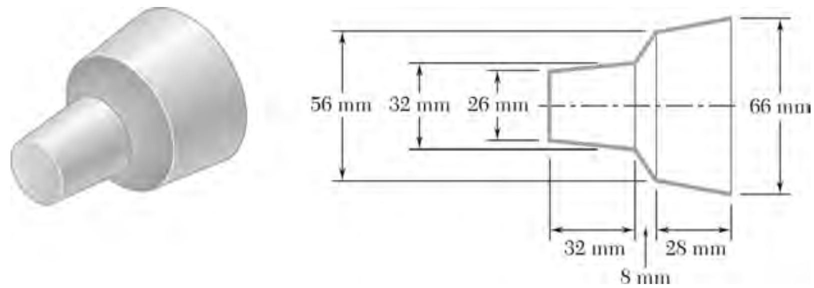
(*c*) $A_C = \pi[2(\bar{y}_1 L_1)]$

$$= \pi\left[\left(60 - \frac{2(5)}{\pi}\right)\text{mm}\right][\pi(5\,\text{mm})]$$

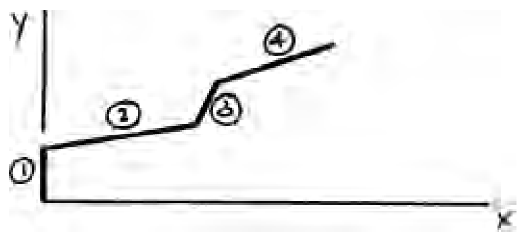
or $A_C = 2800\,\text{mm}^2$ ◀

PROBLEM 5.61

The aluminum shade for the small high-intensity lamp shown has a uniform thickness of 1 mm. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass of the shade.



SOLUTION



The mass of the lamp shade is given by

$$m = \rho V = \rho A t$$

where A is the surface area and t is the thickness of the shade. The area can be generated by rotating the line shown about the x axis. Applying the first theorem of Pappus Guldinus we have

$$A = 2\pi \bar{y} L = 2\pi \Sigma \bar{y} L$$
$$= 2\pi(\bar{y}_1 L_1 + \bar{y}_2 L_2 + \bar{y}_3 L_3 + \bar{y}_4 L_4)$$

or

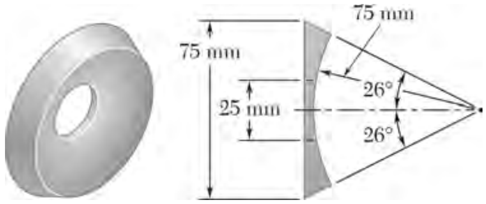
$$A = 2\pi\left[\frac{13 \text{ mm}}{2}(13 \text{ mm}) + \left(\frac{13+16}{2}\right)\text{mm} \times \sqrt{(32 \text{ mm})^2 + (3 \text{ mm})^2}\right.$$
$$+\left(\frac{16+28}{2}\right)\text{mm} \times \sqrt{(8 \text{ mm})^2 + (12 \text{ mm})^2}$$
$$\left.+\left(\frac{28+33}{2}\right)\text{mm} \times \sqrt{(28 \text{ mm})^2 + (5 \text{ mm})^2}\right]$$
$$= 2\pi(84.5 + 466.03 + 317.29 + 867.51)$$
$$= 10903.4 \text{ mm}^2$$

Then

$$m = \rho A t$$
$$= (2800 \text{ kg/m}^3)(10.9034 \times 10^{-3} \text{ m}^2)(0.001 \text{ m})$$

or

$$m = 0.0305 \text{ kg} \blacktriangleleft$$



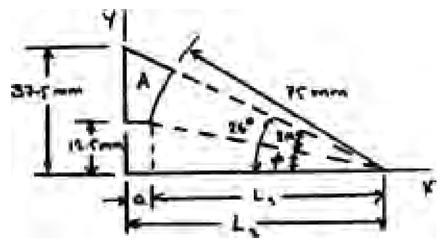
PROBLEM 5.62

The escutcheon (a decorative plate placed on a pipe where the pipe exits from a wall) shown is cast from brass. Knowing that the density of brass is 8470 kg/m³, determine the mass of the escutcheon.

SOLUTION

The mass of the escutcheon is given by $m = (\text{density})V$, where V is the volume. V can be generated by rotating the area A about the x-axis.

From the figure:



$$L_1 = \sqrt{75^2 - 12.5^2} = 73.9510 \text{ m}$$

$$L_2 = \frac{37.5}{\tan 26°} = 76.8864 \text{ mm}$$

$$a = L_2 - L_1 = 2.9324 \text{ mm}$$

$$\phi = \sin^{-1}\frac{12.5}{75} = 9.5941°$$

$$\alpha = \frac{26° - 9.5941°}{2} = 8.2030° = 0.143168 \text{ rad}$$

Area A can be obtained by combining the following four areas:

Applying the second theorem of Pappus-Guldinus and using Figure 5.8a, we have

$$V = 2\pi \bar{y} A = 2\pi \Sigma \bar{y} A$$

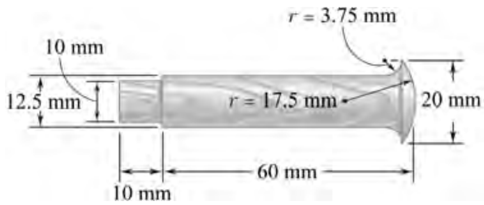
Seg.	A, mm²	$\bar{y}$, mm	$\bar{y}_A$, mm³
1	$\frac{1}{2}(76.886)(37.5) = 1441.61$	$\frac{1}{3}(37.5) = 12.5$	18020.1
2	$-\alpha(75)^2 = -805.32$	$\frac{2(75)\sin\alpha}{3\alpha}\sin(\alpha + \phi) = 15.2303$	−12265.3
3	$-\frac{1}{2}(73.951)(12.5) = -462.19$	$\frac{1}{3}(12.5) = 4.1667$	−1925.81
4	$-(2.9354)(12.5) = -36.693$	$\frac{1}{2}(12.5) = 6.25$	−229.33
Σ			3599.7

PROBLEM 5.62 (Continued)

Then

$$V = 2\pi \Sigma \bar{y} A$$
$$= 2\pi(3599.7 \text{ mm}^3)$$
$$= 22618 \text{ mm}^3$$
$$m = (\text{density})V$$
$$= (8470 \text{ kg/m}^3)(22.618 \times 10^{-6} \text{ m}^3)$$
$$= 0.191574 \text{ kg}$$

or $m = 0.1916$ kg ◀



PROBLEM 5.63

A manufacturer is planning to produce 20,000 wooden pegs having the shape shown. Determine how many gallons of paint should be ordered, knowing that each peg will be given two coats of paint and that one liter of paint covers 9 m².

SOLUTION

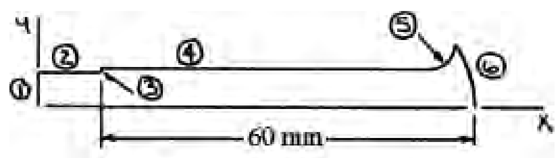
The number of gallons of paint needed is given by

$$\text{Number of liters} = (\text{Number of pegs})(\text{Surface area of 1 peg})\left(\frac{1\ \text{liter}}{9\,\text{m}^2}\right)(2\ \text{coats})$$

or $$\text{Number of liters} = 4444.44\,A_s \quad (A_s \sim \text{m}^2)$$

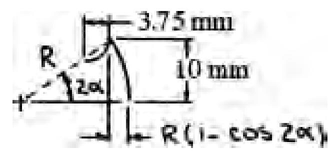
where A_s is the surface area of one peg. A_s can be generated by rotating the line shown about the x axis. Using the first theorem of Pappus-Guldinus and Figures 5.8b,

We have



$$R = 17.5\,\text{mm}$$

$$\sin 2\alpha = \frac{10\,\text{mm}}{17.5\,\text{mm}}$$



or $$2\alpha = 34.850^\circ \quad \alpha = 17.425^\circ$$

$$A_s = 2\pi \bar{Y} L = 2\pi \Sigma \bar{y} L$$

	L, mm	$\bar{y}$, mm	$\bar{y}L$, mm^2
1	5	2.5	12.5
2	10	5	50
3	1.25	$\frac{5+6.25}{2} = 5.625$	7.03125
4	$60 - 17.5(1 - \cos 34.850) - 3.75 = 53.111$	6.25	331.944
5	$\frac{\pi}{2} \times 3.75 = 5.8905$	$10 - \frac{2 \times 3.75}{\pi} = 7.6127$	44.843
6	$2\alpha(17.5)$	$\frac{17.5 \sin 17.425^\circ}{\alpha} \times \sin 17.425^\circ$	54.926

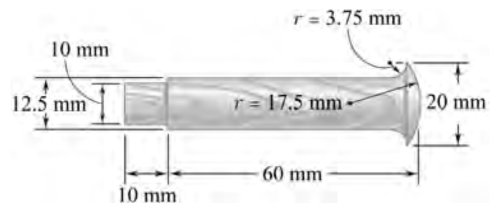
$$\Sigma \bar{y} L = 501.244\,\text{mm}^2$$

PROBLEM 5.63 (Continued)

Then $$A_s = 2\pi(501.244)\,\text{mm}^2 = 3149.41\,\text{mm}^2$$
$$= 3149.41\times 10^{-6}\,\text{m}^2$$

Finally $$\text{Number of liters} = 4444.44\times 3149.41\times 10^{-6}$$
$$= 13.997\,\text{liters}$$

Order 14 liters ◀

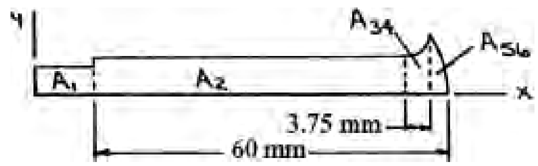


PROBLEM 5.64

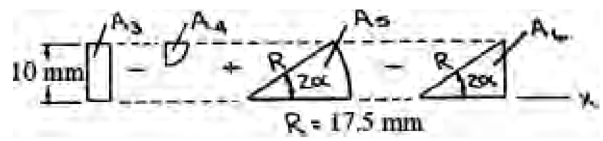
The wooden peg shown is turned from a dowel 20 mm in diameter and 80 mm long. Determine the percentage of the initial volume of the dowel that becomes waste.

SOLUTION

To obtain the solution it is first necessary to determine the volume of the peg. That volume can be generated by rotating the area shown about the *x* axis.



The generating area is next divided into six components as indicated



$$\sin 2\alpha = \frac{10}{17.5}$$

or $$2\alpha = 34.850° \qquad \alpha = 17.425°$$

Applying the second theorem of Pappus-Guldinus and then using Figure 5.8a, we have

$$V_{PEG} = 2\pi \bar{Y} A = 2\pi \Sigma \bar{y} A$$

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$10 \times 5 = 50$	2.5	125
2	$[60 - 17.5(1 - \cos 34.850°) - 3.75]$ $\times (6.25) = 331.946$	3.125	1037331
3	3.75×10	5	187.5
4	$-\frac{\pi}{4}(3.75)^2 = -11.045$	$10 - \frac{4 \times 3.75}{3\pi} = 8.4085$	−92.871
5	$\alpha(17.5)^2$	$\frac{2 \times 17.5 \sin 17.425°}{3\alpha} \times \sin 17.425°$	320.40
6	$-\frac{1}{2}(17.5 \cos 34.850°)(10) = -71.8069$	$\frac{1}{3}(10) = 3.3333$	−239.356

$$\Sigma \bar{y} \text{A} = 1338.004 \text{ mm}^3$$

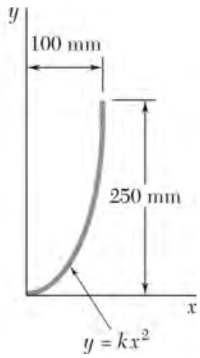
PROBLEM 5.64 (Continued)

Then $$V_{peg} = 2\pi(1338.004 \text{ mm}^3) = 8406.93 \text{ mm}^3$$

Now $$V_{dowel} = \frac{\pi}{4}(\text{diameter})^2 (\text{length}) = \frac{\pi}{4}(20 \text{ mm})^2 (80 \text{ mm}) = 25132.74 \text{ mm}^3$$

Then $$\% \text{ Waste} = \frac{V_{waste}}{V_{dowel}} \times 100\% = \frac{V_{dowel} - V_{peg}}{V_{dowel}} \times 100\% = \left(1 - \frac{8406.93}{25132.74}\right) \times 100\%$$

or $\% \text{ Waste} = 66.5\%$ ◀



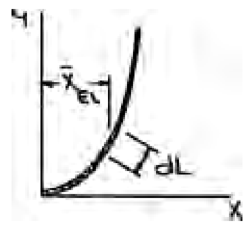
PROBLEM 5.65*

The shade for a wall-mounted light is formed from a thin sheet of translucent plastic. Determine the surface area of the outside of the shade, knowing that it has the parabolic cross section shown.

SOLUTION

First note that the required surface area A can be generated by rotating the parabolic cross section through π radians about the y axis. Applying the first theorem of Pappus-Guldinus we have

$$A = \pi \bar{x} L$$



Now at $\quad x = 100 \text{ mm}, \quad y = 250 \text{ mm}$

$$250 = k(100)^2 \quad \text{or} \quad k = 0.025 \text{ mm}^{-1}$$

and $\quad x_{EL} = x$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $\quad \dfrac{dy}{dx} = 2kx$

Then $\quad dL = \sqrt{1 + 4k^2x^2}\, dx$

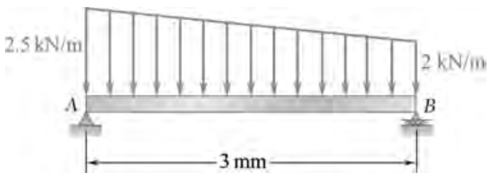
We have $\quad xL = \int x_{EL}\, dL = \int_0^{100} x\left(\sqrt{1 + 4k^2x^2 dx}\right)$

$$xL = \left[\frac{1}{3}\frac{1}{4k^2}(1 + 4k^2x^2)^{3/2}\right]_0^{100}$$

$$= \frac{1}{12}\frac{1}{(0.025)^2}\left\{[1 + 4(0.025)^2(100)^2]^{3/2} - (1)^{3/2}\right\}$$

$$= 17543.3 \text{ mm}^2$$

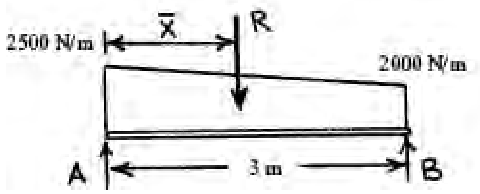
Finally $\quad A = \pi(17543.3 \text{ mm}^2) \qquad$ or $\quad A = 55.1 \times 10^3 \text{ mm}^2$ ◄

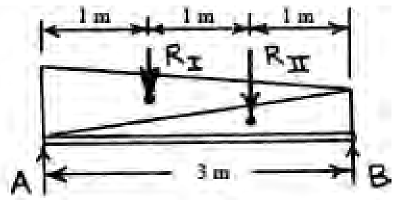


PROBLEM 5.66

For the beam and loading shown, determine (*a*) the magnitude and location of the resultant of the distributed load, (*b*) the reactions at the beam supports.

SOLUTION





$$R_{\mathrm{I}} = \frac{1}{2}(2500\ \mathrm{N/m})(3\,\mathrm{m}) = 3750\ \mathrm{N}$$

$$R_{\mathrm{II}} = \frac{1}{2}(2000\ \mathrm{N/m})(3\,\mathrm{m}) = 3000\ \mathrm{N}$$

$$R = R_{\mathrm{I}} + R_{\mathrm{II}} = 3750 + 3000 = 6750\ \mathrm{N}$$

$$\bar{X}R = \Sigma\bar{X}R: \quad \bar{X}(6750) = (1)(3750) + (2)(3000) \quad \bar{X} = 1.4444\ \mathrm{m}$$

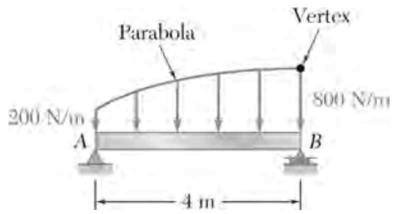
(*a*) $\mathbf{R} = 6750\ \mathrm{N}\downarrow \quad \bar{X} = 1.444\ \mathrm{m}$ ◀

(*b*) Reactions: $+\circlearrowleft \Sigma M_A = 0: \; B(3\,\mathrm{m}) - (6750\ \mathrm{N})(1.4444\ \mathrm{m}) = 0$

$B = 3249.99\ \mathrm{N}$ $\qquad \mathbf{B} = 3250\ \mathrm{N}\uparrow$ ◀

$+\uparrow \Sigma F_y = 0: \quad A + 3249.99\ \mathrm{N} - 6750\ \mathrm{N} = 0$

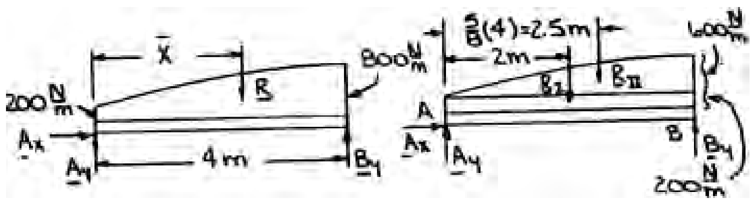
$A = 3500.01\ \mathrm{N}$ $\qquad \mathbf{A} = 3500\ \mathrm{N}\uparrow$ ◀



PROBLEM 5.67

For the beam and loading shown, determine (*a*) the magnitude and location of the resultant of the distributed load, (*b*) the reactions at the beam supports.

SOLUTION



(*a*) We have

$$R_{\mathrm{I}} = (4\text{ m})(200\text{ N/m}) = 800\text{ N}$$

$$R_{\mathrm{II}} = \frac{2}{3}(4\text{ m})(600\text{ N/m}) = 1600\text{ N}$$

Then ΣF_y: $\quad -R = -R_{\mathrm{I}} - R_{\mathrm{II}}$

or $\quad R = 800 + 1600 = 2400\text{ N}$

and ΣM_A: $\quad -\bar{X}(2400) = -2(800) - 2.5(1600)$

or $\quad \bar{X} = \dfrac{7}{3}\text{ m}$

$$\mathbf{R} = 2400\text{ N}\downarrow \quad \bar{X} = 2.33\text{ m} \blacktriangleleft$$

(*b*) Reactions $\quad \overset{+}{\rightarrow} \Sigma F_x = 0: \quad A_x = 0$

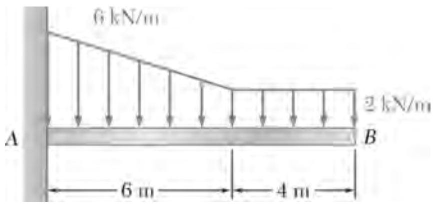
$$+\circlearrowleft \Sigma M_A = 0: \quad (4\text{ m})B_y - \left(\frac{7}{3}\text{ m}\right)(2400\text{ N}) = 0$$

or $\quad B_y = 1400\text{ N}$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 1400\text{ N} - 2400\text{ N} = 0$$

or $\quad A_y = 1000\text{ N}$

$$\mathbf{A} = 1000\text{ N}\uparrow \quad \mathbf{B} = 1400\text{ N}\uparrow \blacktriangleleft$$

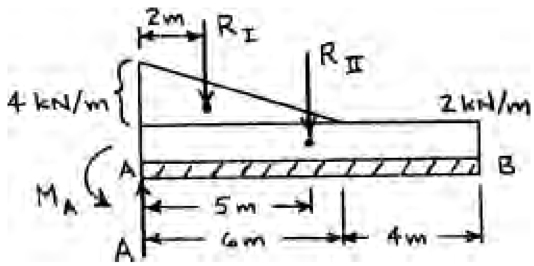


PROBLEM 5.68

Determine the reactions at the beam supports for the given loading.

SOLUTION

$$R_{\mathrm{I}} = \frac{1}{2}(4\text{ kN/m})(6\text{ m})$$
$$= 12\text{ kN}$$
$$R_{\mathrm{II}} = (2\text{ kN/m})(10\text{ m})$$
$$= 20\text{ kN}$$

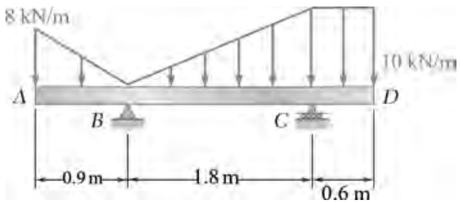


$$+\uparrow \Sigma F_y = 0: \quad A - 12\text{ kN} - 20\text{ kN} = 0$$

$\mathbf{A} = 32.0\text{ kN}\uparrow$ ◀

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A - (12\text{ kN})(2\text{ m}) - (20\text{ kN})(5\text{ m}) = 0$$

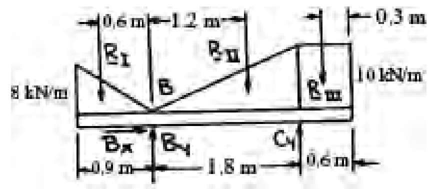
$\mathbf{M}_A = 124.0\text{ kN}\cdot\text{m} \circlearrowleft$ ◀



PROBLEM 5.69

Determine the reactions at the beam supports for the given loading.

SOLUTION



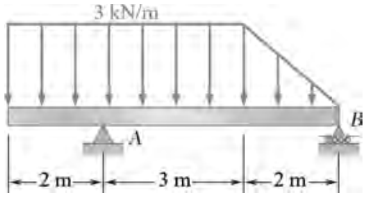
We have
$$R_{\mathrm{I}} = \frac{1}{2}(0.9\,\mathrm{m})(8\,\mathrm{kN/m}) = 3.6\,\mathrm{kN}$$
$$R_{\mathrm{II}} = \frac{1}{2}(1.8\,\mathrm{m})(10\,\mathrm{kN/m}) = 9.0\,\mathrm{kN}$$
$$R_{\mathrm{III}} = (0.6\,\mathrm{m})(10\,\mathrm{kN/m}) = 6\,\mathrm{kN}$$

Then
$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x = 0$$
$$\circlearrowleft + \ \Sigma M_B = 0: \quad (0.6\,\mathrm{m})(3.6\,\mathrm{kN}) - (1.2\,\mathrm{m})(9\mathrm{kN}) + (1.8\,\mathrm{m})C_y - (2.1\,\mathrm{m})(6\,\mathrm{kN}) = 0$$

or
$$C_y = 11.8\,\mathrm{kN} \qquad \mathbf{C} = 11.8\,\mathrm{kN}\uparrow \blacktriangleleft$$
$$+\uparrow \Sigma F_y = 0: \quad -3.6\,\mathrm{kN} + B_y - 9.0\,\mathrm{kN} + 11.8\,\mathrm{kN} - 6\,\mathrm{kN} = 0$$

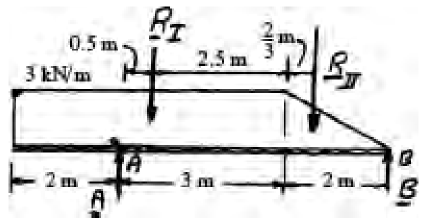
or
$$B_y = 6.8\,\mathrm{kN} \qquad \mathbf{B} = 6.8\,\mathrm{kN}\uparrow \blacktriangleleft$$

PROBLEM 5.70



Determine the reactions at the beam supports for the given loading.

SOLUTION



$$R_{\text{I}} = (3\,\text{kN/m})(5\text{ m})$$
$$R_{\text{I}} = 15\,\text{kN}$$
$$R_{\text{II}} = \frac{1}{2}(3\,\text{kN/m})(2\,\text{m})$$
$$R_{\text{II}} = 3\,\text{kN}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -(15\,\text{kN})(0.5\text{ m}) - (3\,\text{kN})(3\text{ m} + 2/3\,\text{m}) + B(5\,\text{m}) = 0$$

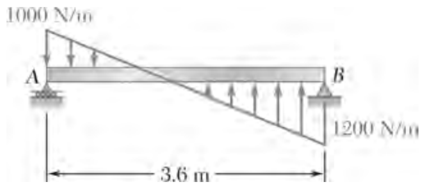
$$B = 3.7\,\text{kN}$$

$\mathbf{B} = 3.7\,\text{kN}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad A + 3.7\,\text{kN} - 15\,\text{kN} - 3\,\text{kN} = 0$$

$$A = 14.3\,\text{kN}$$

$\mathbf{A} = 14.3\,\text{kN}\uparrow$ ◀

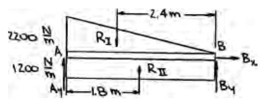


PROBLEM 5.71

Determine the reactions at the beam supports for the given loading.

SOLUTION

First replace the given loading with the loading shown below. The two loadings are equivalent because both are defined by a linear relation between load and distance and the values at the end points are the same.



We have $$R_{\mathrm{I}} = \frac{1}{2}(3.6 \text{ m})(2200 \text{ N/m}) = 3960 \text{ N}$$

$$R_{\mathrm{II}} = (3.6 \text{ m})(1200 \text{ N/m}) = 4320 \text{ N}$$

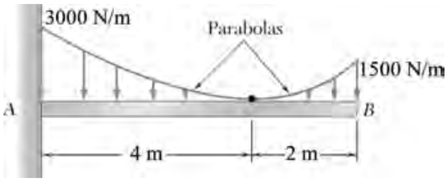
Then $$\xrightarrow{+} \Sigma F_x = 0: \quad B_x = 0$$

$$\circlearrowleft + \; \Sigma M_B = 0: \quad -(3.6 \text{ m})A_y + (2.4 \text{ m})(3960 \text{ N}) - (1.8 \text{ m})(4320 \text{ N}) = 0$$

or $$A_y = 480 \text{ N}$$ $\mathbf{A} = 480 \text{ N} \uparrow$ ◀

$$+\!\uparrow \Sigma F_y = 0: \quad 480 \text{ N} - 3960 \text{ N} + 4320 + B_y = 0$$

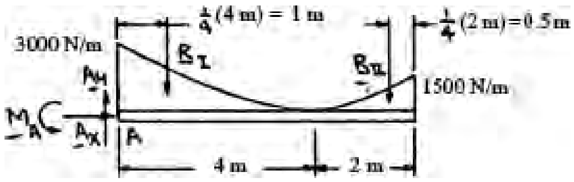
or $$B_y = -840 \text{ N}$$ $\mathbf{B} = 840 \text{ N} \downarrow$ ◀



PROBLEM 5.72

Determine the reactions at the beam supports for the given loading.

SOLUTION



We have $$R_{\mathrm{I}} = \frac{1}{3}(4\,\mathrm{m})(3000\ \mathrm{N/m}) = 4000\ \mathrm{N}$$

$$R_{\mathrm{II}} = \frac{1}{3}(2\,\mathrm{m})(1500\ \mathrm{N/m}) = 1000\ \mathrm{N}$$

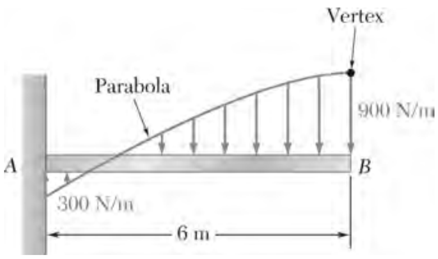
Then $$\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 4000\ \mathrm{N} - 1000\ \mathrm{N} = 0$$

or $$A_y = 5000\ \mathrm{N} \qquad \mathbf{A} = 5000\ \mathrm{N}\uparrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A - (1\,\mathrm{m})(4000\,\mathrm{N}) - (5.5\,\mathrm{m})(1000\,\mathrm{N}) = 0$$

or $$M_A = 9500\ \mathrm{N}\cdot\mathrm{m} \qquad \mathbf{M}_A = 9500\ \mathrm{N}\cdot\mathrm{m}\circlearrowleft \blacktriangleleft$$

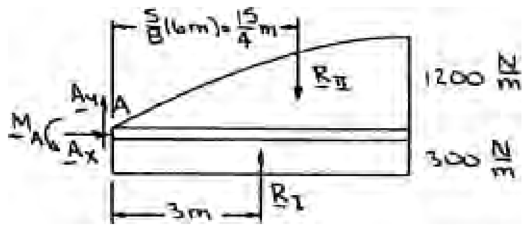


PROBLEM 5.73

Determine the reactions at the beam supports for the given loading.

SOLUTION

First replace the given loading with the loading shown below. The two loadings are equivalent because both are defined by a parabolic relation between load and distance and the values at the end points are the same.



We have $R_{\mathrm{I}} = (6\text{ m})(300\text{ N/m}) = 1800\text{ N}$

$$R_{\mathrm{II}} = \frac{2}{3}(6\text{ m})(1200\text{ N/m}) = 4800\text{ N}$$

Then $\xrightarrow{+} \Sigma F_x = 0$: $A_x = 0$

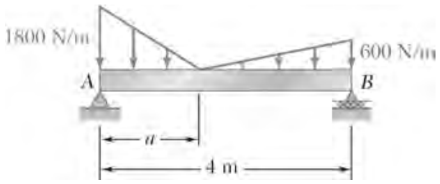
$$+\!\uparrow \Sigma F_y = 0: \quad A_y + 1800\text{ N} - 4800\text{ N} = 0$$

or $A_y = 3000\text{ N}$ $\quad\quad$ $\mathbf{A} = 3000\text{ N}\uparrow$ ◀

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A + (3\text{ m})(1800\text{ N}) - \left(\frac{15}{4}\text{ m}\right)(4800\text{ N}) = 0$$

or $M_A = 12.6\text{ kN}\cdot\text{m}$ $\quad\quad$ $\mathbf{M}_A = 12.6\text{ kN}\cdot\text{m} \circlearrowleft$ ◀

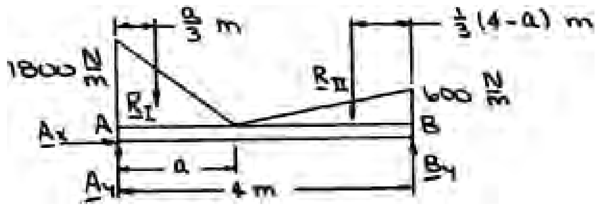
PROBLEM 5.74



Determine (*a*) the distance *a* so that the vertical reactions at supports *A* and *B* are equal, (*b*) the corresponding reactions at the supports.

SOLUTION

(*a*)



We have $$R_{\text{I}} = \frac{1}{2}(a\text{ m})(1800\text{ N/m}) = 900a\text{ N}$$

$$R_{\text{II}} = \frac{1}{2}\left[(4-a)\text{m}\right](600\text{ N/m}) = 300(4-a)\text{N}$$

Then $$+\uparrow \Sigma F_y = 0: \quad A_y - 900a - 300(4-a) + B_y = 0$$

or $$A_y + B_y = 1200 + 600a$$

Now $$A_y = B_y \Rightarrow A_y = B_y = 600 + 300a(\text{N}) \qquad (1)$$

Also $$+\circlearrowleft \Sigma M_B = 0: \quad -(4\text{ m})A_y + \left[\left(4 - \frac{a}{3}\right)\text{m}\right]\left[(900a)\text{N}\right]$$

$$+\left[\frac{1}{3}(4-a)\text{m}\right]\left[300(4-a)\text{N}\right] = 0$$

or $$A_y = 400 + 700a - 50a^2 \qquad (2)$$

Equating Eqs. (1) and (2) $$600 + 300a = 400 + 700a - 50a^2$$

or $$a^2 - 8a + 4 = 0$$

Then $$a = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(4)}}{2}$$

or $a = 0.53590$ m $\qquad a = 7.4641$ m

Now $a \leq 4\text{ m} \Rightarrow \qquad a = 0.536$ m ◀

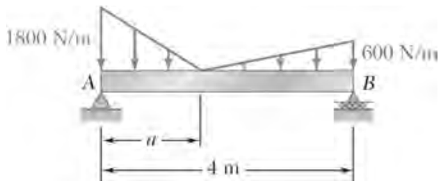
PROBLEM 5.74 (Continued)

(b) We have $\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$

Eq. (1)

$$A_y = B_y$$
$$= 600 + 300(0.53590)$$
$$= 761\text{ N}$$

$\mathbf{A} = \mathbf{B} = 761\text{ N}\uparrow$ ◀

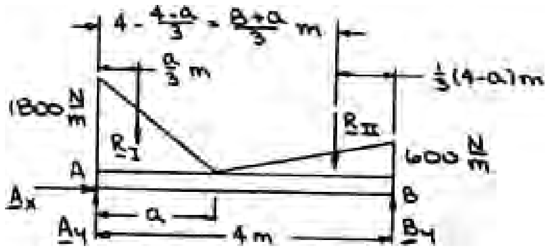


PROBLEM 5.75

Determine (*a*) the distance *a* so that the reaction at support *B* is minimum, (*b*) the corresponding reactions at the supports.

SOLUTION

(*a*)



We have $R_{\text{I}} = \frac{1}{2}(a \text{ m})(1800 \text{ N/m}) = 900a \text{ N}$

$$R_{\text{II}} = \frac{1}{2}[(4-a)\text{m}](600 \text{ N/m}) = 300(4-a) \text{ N}$$

Then $+\circlearrowleft \Sigma M_A = 0: \quad -\left(\frac{a}{3}\text{m}\right)(900a \text{ N}) - \left(\frac{8+a}{3}\text{m}\right)\left[300(4-a)\text{N}\right] + (4 \text{ m})B_y = 0$

or $B_y = 50a^2 - 100a + 800 \qquad (1)$

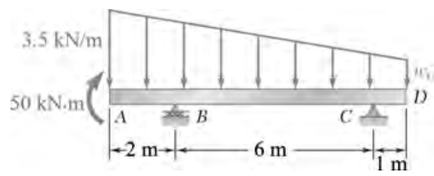
Then $\frac{dB_y}{da} = 100a - 100 = 0$ or $a = 1.000 \text{ m}$ ◀

(*b*) Eq. (1) $B_y = 50(1)^2 - 100(1) + 800 = 750 \text{ N}$ $\mathbf{B} = 750 \text{ N}\uparrow$ ◀

and $\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 900(1)\text{N} - 300(4-1)\text{N} + 750 \text{ N} = 0$$

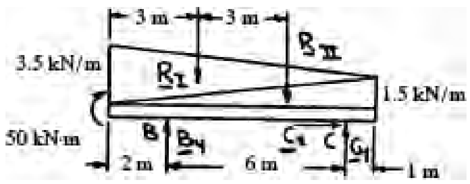
or $A_y = 1050 \text{ N}$ $\mathbf{A} = 1050 \text{ N}\uparrow$ ◀



PROBLEM 5.76

Determine the reactions at the beam supports for the given loading when $\omega_0 = 1.5$ kN/m.

SOLUTION



We have

$$R_{\text{I}} = \frac{1}{2}(9\,\text{m})(3.5\text{ kN/m}) = 15.75\text{ kN}$$

$$R_{\text{II}} = \frac{1}{2}(9\,\text{m})(1.5\text{ kN/m}) = 6.75\text{ kN}$$

Then

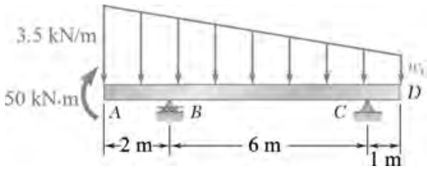
$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x = 0$$

$$\circlearrowleft + \; \Sigma M_B = 0: \quad -(50\text{ kN}\cdot\text{m}) - (1\text{m}) - (15.75\text{ kN}) - (4\,\text{m})(6.75\text{ kN}) + (6\,\text{m})C_y = 0$$

or $$C_y = 15.4583\,\text{kN} \qquad \mathbf{C} = 15.46\,\text{kN} \uparrow \blacktriangleleft$$

$$+\!\uparrow \Sigma F_y = 0: \quad B_y - 15.75\,\text{kN} - 6.75\,\text{kN} + 15.4583\,\text{kN} = 0$$

or $$B_y = 7.0417\,\text{kN} \qquad \mathbf{B} = 7.04\,\text{kN} \uparrow \blacktriangleleft$$

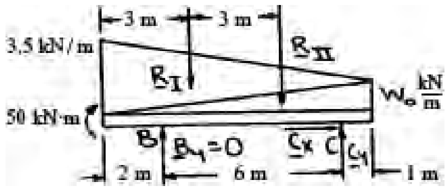


PROBLEM 5.77

Determine (*a*) the distributed load ω_0 at the end *D* of the beam *ABCD* for which the reaction at *B* is zero, (*b*) the corresponding reaction at *C*.

SOLUTION

(*a*)



We have $$R_{\text{I}} = \frac{1}{2}(9\,\text{m})(3.5\text{ kN/m}) = 15.75\text{ kN}$$

$$R_{\text{II}} = \frac{1}{2}(9\,\text{m})(\omega_0\text{kN/m}) = 4.5\ \omega_0\ \text{kN}$$

Then $$+\circlearrowleft \Sigma M_C = 0{:}\quad -(50\,\text{kN}\cdot\text{m}) + (5\,\text{m})(15.75\text{ kN}) + (2\,\text{m})(4.5\omega_0\ \text{kN}) = 0$$

$$\omega_0 = -3.194\,\text{kN/m}$$

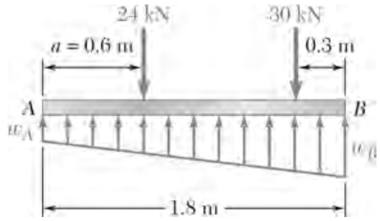
or $$\omega_0 = -3.194\,\text{kN/m}\uparrow \blacktriangleleft$$

(*b*) $$\xrightarrow{+} \Sigma F_x = 0{:}\quad C_x = 0$$

$$+\uparrow \Sigma F_y = 0{:}\quad -15.75 + (4.5)(3.194) + C_y = 0$$

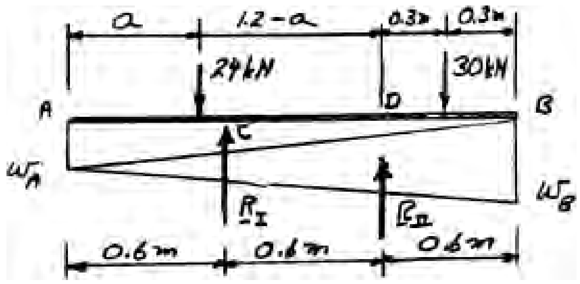
or $$C_y = 1.377\text{ kN} \qquad \mathbf{C} = 1.377\,\text{kN}\uparrow \blacktriangleleft$$

PROBLEM 5.78



The beam *AB* supports two concentrated loads and rests on soil that exerts a linearly distributed upward load as shown. Determine the values of ω_A and ω_B corresponding to equilibrium.

SOLUTION



$$R_{\mathrm{I}} = \frac{1}{2}\omega_A(1.8 \text{ m}) = 0.9\omega_A$$

$$R_{\mathrm{II}} = \frac{1}{2}\omega_B(1.8 \text{ m}) = 0.9\omega_B$$

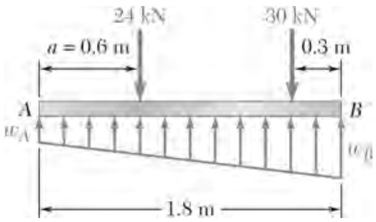
$$+\circlearrowleft \Sigma M_D = 0: \quad (24 \text{ kN})(1.2 - a) - (30 \text{ kN})(0.3 \text{ m}) - (0.9\omega_A)(0.6 \text{ m}) = 0 \qquad (1)$$

For $\quad a = 0.6 \text{ m}: \quad 24(1.2 - 0.6) - (30)(0.3) - 0.54\,\omega_a = 0$

$$14.4 - 9 - 0.54\omega_A = 0 \qquad \omega_A = 10.00 \text{ kN/m} \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -24 \text{ kN} - 30 \text{ kN} + 0.9(10 \text{ kN/m}) + 0.9\omega_B = 0 \qquad \omega_B = 50.0 \text{ kN/m} \blacktriangleleft$$

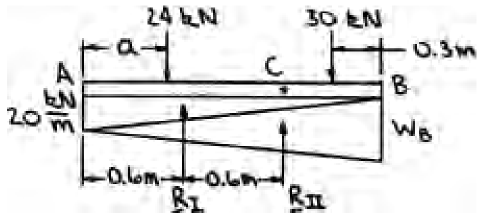
PROBLEM 5.79



For the beam and loading of Problem 5.78, determine (*a*) the distance *a* for which ω_A = 20 kN/m, (*b*) the corresponding value of ω_B.

Problem 5.78 The beam *AB* supports two concentrated loads and rests on soil that exerts a linearly distributed upward load as shown. Determine the values of ω_A and ω_B corresponding to equilibrium.

SOLUTION



We have $R_{\mathrm{I}} = \dfrac{1}{2}(1.8\text{ m})(20\text{ kN/m}) = 18\text{ kN}$

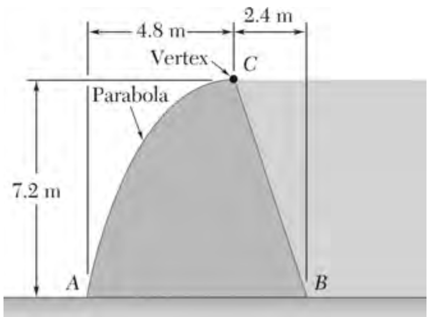
$$R_{\mathrm{II}} = \frac{1}{2}(1.8\text{ m})(\omega_B\text{ kN/m}) = 0.9\omega_B\text{ kN}$$

(*a*) $+\circlearrowleft \Sigma M_C = 0$: $(1.2 - a)\text{m} \times 24\text{ kN} - 0.6\text{ m} \times 18\text{ kN} - 0.3\text{ m} \times 30\text{ kN} = 0$

or $a = 0.375\text{ m}$ ◀

(*b*) $+\uparrow \Sigma F_y = 0$: $-24\text{ kN} + 18\text{ kN} + (0.9\omega_B)\text{ kN} - 30\text{ kN} = 0$

or $\omega_B = 40.0\text{ kN/m}$ ◀

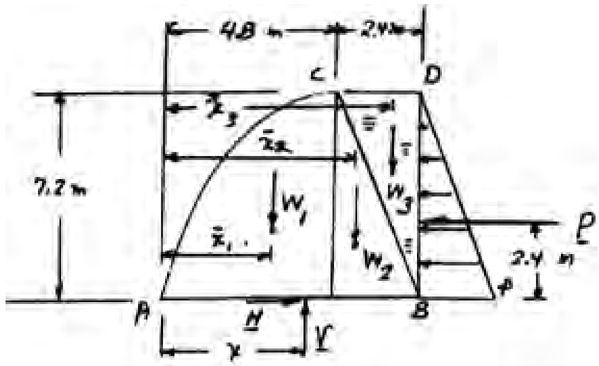


PROBLEM 5.80

The cross section of a concrete dam is as shown. For a 1-m-wide dam section determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the point of application of the resultant of Part *a*, (*c*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam.

SOLUTION

(*a*) Consider free body made of dam and triangular section of water shown. (Thickness = 1 m)



$$p = (7.2\,\text{m})(10^3\,\text{kg/m}^3)(9.81\,\text{m/s}^2)$$

$$W_1 = \frac{2}{3}(4.8\text{ m})(7.2\text{ m})(1\text{ m})(2.4\times 10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 542.5\text{ kN}$$

$$W_2 = \frac{1}{2}(2.4\text{ m})(7.2\text{ m})(1\text{ m})(2.4\times 10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 203.4\text{ kN}$$

$$W_3 = \frac{1}{2}(2.4\text{ m})(7.2\text{ m})(1\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 84.8\text{ kN}$$

$$P = \frac{1}{2}Ap = \frac{1}{2}(7.2\text{ m})(1\text{ m})(7.2\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)$$
$$= 254.3\text{ kN}$$

$\xrightarrow{+} \Sigma F_x = 0: \quad H - 254.3\text{ kN} = 0$ $\qquad$ $\mathbf{H} = 254\text{ kN} \rightarrow$ ◀

$+\uparrow \Sigma F_y = 0: \quad V - 542.5 - 203.4 - 84.8 = 0$

$V = 830.7\text{ kN}$ $\qquad$ $\mathbf{V} = 831\text{ kN} \uparrow$ ◀

PROBLEM 5.80 (Continued)

(*b*)
$$\bar{x}_1 = \frac{5}{8}(4.8 \text{ m}) = 3 \text{ m}$$
$$\bar{x}_2 = 4.8 + \frac{1}{3}(2.4) = 5.6 \text{ m}$$
$$\bar{x}_3 = 4.8 + \frac{2}{3}(2.4) = 6.4 \text{ m}$$

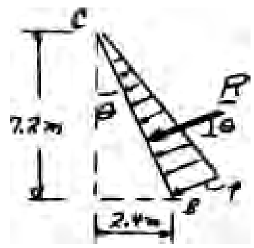
$$+\circlearrowleft \Sigma M_A = 0: \quad xV - \Sigma \bar{x}W + P(2.4 \text{ m}) = 0$$

$$x(830.7 \text{ kN}) - (3 \text{ m})(542.5 \text{ kN}) - (5.6 \text{ m})(203.4 \text{ kN}) - (6.4 \text{ m})(84.8 \text{ kN}) + (2.4 \text{ m})(254.3 \text{ kN}) = 0$$
$$x(830.7) - 1627.5 - 1139.0 - 542.7 + 610.3 = 0$$
$$x(830.7) - 2698.9 = 0$$

$x = 3.25$ m (To right of A) ◀

(*c*) <u>Resultant on face *BC*</u>

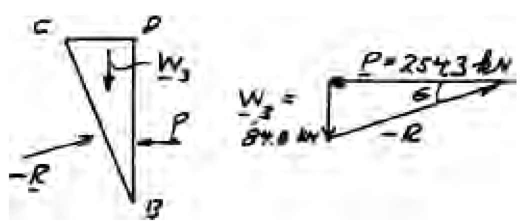
Direct computation:



$$P = \rho g h = (10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(7.2 \text{ m})$$
$$P = 70.63 \text{ kN/m}^2$$
$$BC = \sqrt{(2.4)^2 + (7.2)^2} = 7.589 \text{ m}$$
$$\theta = 18.43°$$
$$R = \frac{1}{2}PA = \frac{1}{2}(70.63 \text{ kN/m}^2)(7.589 \text{ m})(1 \text{ m})$$

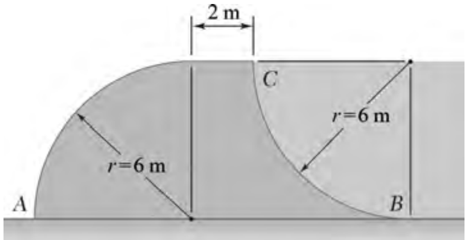
$\mathbf{R} = 268$ kN 18.43° ◀

Alternate computation: Use free body of water section *BCD*.



$-\mathbf{R} = 268$ kN 18.43°

$\mathbf{R} = 268$ kN 18.43° ◀

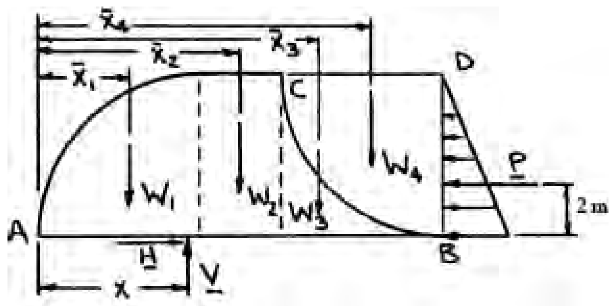


PROBLEM 5.81

The cross section of a concrete dam is as shown. For a 0.3 m-wide dam section determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the point of application of the resultant of Part *a*, (*c*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam.

SOLUTION

The free body shown consists of a 6.3 m thick section of the dam and the quarter circular section of water above the dam.



Note:

$$\bar{x}_1 = \left(6 - \frac{4\times 6}{3\pi}\right)\text{m}$$

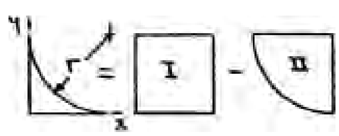
$$= 3.4535\,\text{m}$$

$$\bar{x}_2 = (6+1)\,\text{m} = 7\ \text{m}$$

$$\bar{x}_4 = \left(14 - \frac{4\times 6}{3\pi}\right)\text{m}$$

$$= 11.4535\,\text{m}$$

For area 3 first note.



	a	$\bar{x}$
I	r^2	$\frac{1}{2}r$
II	$-\frac{\pi}{4}r^2$	$r - \frac{4r}{3\pi}$

Then

$$\bar{x}_3 = 8\ \text{m} + \left[\frac{\frac{1}{2}(6)(6)^2 + \left(6 - \frac{4\times 6}{3\pi}\right)\left(-\frac{\pi}{4}\times 6^2\right)}{(6)^2 - \frac{\pi}{4}(6)^2}\right]\text{m}$$

$$= (8 + 1.3402)\,\text{m} = 9.3402\ \text{m}$$

PROBLEM 5.81 (Continued)

(*a*) Now $$W = \Sigma V$$

So that $$W_1 = \left(2400 \times 9.81\ \text{N/m}^3\right)\left[\frac{\pi}{4}(6\ \text{m})^2(0.3\,\text{m})\right] = 199,707\ \text{N}$$

$$W_2 = (2400 \times 9.81\ \text{N/m}^3)[(2\,\text{m})(6\,\text{m})(0.3\,\text{m})] = 84,758.4\ \text{N}$$

$$W_3 = (2400 \times 9.81\ \text{N/m}^3)\left[\left(6^2 - \frac{\pi}{4} \times 6^2\right)\text{m}^2 \times (0.3\,\text{m})\right] = 54,567.9\ \text{N}$$

$$W_4 = (100 \times 9.81\ \text{N/m}^3)\left[\frac{\pi}{4}(6\ \text{m})^2(0.3\,\text{m})\right] = 83211.4\ \text{N}$$

Also $$P = \frac{1}{2}Ap = \frac{1}{2}[(6\ \text{m})(0.3\,\text{m})][(1000 \times 9.81\ \text{N/m}^3)(6\,\text{m})] = 52,974\ \text{N}$$

Then $\xrightarrow{+} \Sigma F_x = 0\!: \quad H - 52,974\ \text{N} = 0$ or $\mathbf{H} = 53.0\ \text{kN} \rightarrow$ ◀

$+\!\uparrow \Sigma F_y = 0\!: \quad V - 199,707\ \text{N} - 84,758.4\ \text{N} - 54567.9\,\text{N} - 83211.4\ \text{N} = 0$

or $V = 422,244.7\ \text{N}$ $\mathbf{V} = 422\,\text{kN} \uparrow$ ◀

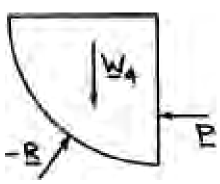
(*b*) We have $\circlearrowleft + \Sigma M_A = 0\!: \quad x(422,244.7\ \text{N}) - (3.4535\,\text{m})(199,707\ \text{N}) - (7\ \text{m})(84,758.4\ \text{N})$
$-(9.3402\,\text{m})(54,567.9\ \text{N}) - (11.4535\,\text{m})(83211.4\ \text{N})$
$+(2\,\text{m})(52,974\ \text{N}) = 0$

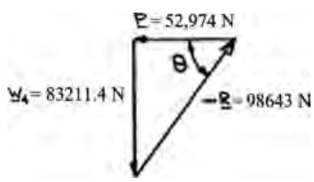
or $$422,244.7x - 689,688 - 593,309 - 509,675 - 953,062 + 105,948 = 0$$

or $x = 6.25\ \text{m}$ ◀

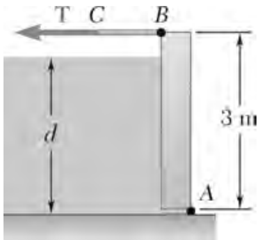
(*c*) Consider water section *BCD* as the free body

We have $$\Sigma\mathbf{F} = 0$$





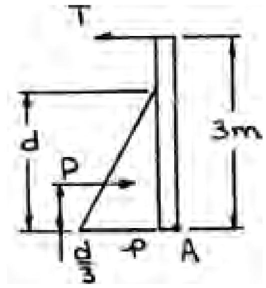
Then $-\mathbf{R} = 98.6\ \text{kN} \measuredangle\ 57.5°$ or $\mathbf{R} = 98.6\ \text{kN}\ \measuredangle\ 57.5°$ ◀



PROBLEM 5.82

The 3 × 4-m side *AB* of a tank is hinged at its bottom *A* and is held in place by a thin rod *BC*. The maximum tensile force the rod can withstand without breaking is 200 kN, and the design specifications require the force in the rod not to exceed 20 percent of this value. If the tank is slowly filled with water, determine the maximum allowable depth of water *d* in the tank.

SOLUTION



Consider the free-body diagram of the side.

We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g d)$$

Now $$\circlearrowleft + \Sigma M_A = 0: \quad hT - \frac{d}{3}P = 0$$

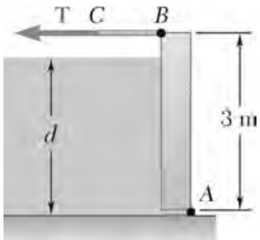
where $$h = 3 \text{ m}$$

Then for d_{max}.

$$(3 \text{ m})(0.2 \times 200 \times 10^3 \text{ N}) - \frac{d_{max}}{3}\left[\frac{1}{2}(4 \text{ m} \times d_{max}) \times (10^3 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2 \times d_{max})\right] = 0$$

or $$120 \text{ N}\cdot\text{m} - 6.54 d_{max}^3 \text{ N/m}^2 = 0$$

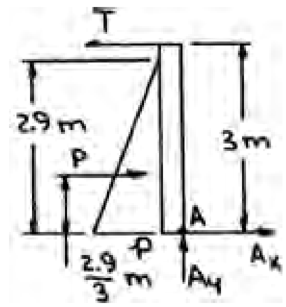
or $$d_{max} = 2.64 \text{ m} \blacktriangleleft$$



PROBLEM 5.83

The 3 × 4-m side of an open tank is hinged at its bottom A and is held in place by a thin rod BC. The tank is to be filled with glycerine, whose density is 1263 kg/m³. Determine the force **T** in the rod and the reactions at the hinge after the tank is filled to a depth of 2.9 m.

SOLUTION



Consider the free-body diagram of the side.

We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g d)$$

$$= \frac{1}{2}[(2.9\text{ m})(4\text{ m})]\,[(1263\text{ kg/m}^3)(9.81\text{ m/s}^2)(2.9\text{ m})]$$

$$= 208.40\text{ kN}$$

Then $$+\!\uparrow \Sigma F_y = 0\colon\quad A_y = 0$$

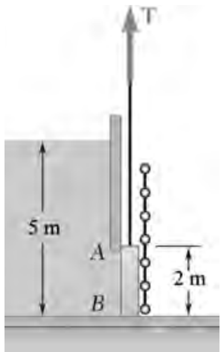
$$\circlearrowleft\!+\ \Sigma M_A = 0\colon\quad (3\text{ m})T - \left(\frac{2.9}{3}\text{ m}\right)(208.4\text{ kN}) = 0$$

or $$T = 67.151\text{ kN} \qquad \mathbf{T} = 67.2\text{ kN} \leftarrow \blacktriangleleft$$

$$\xrightarrow{+}\ \Sigma F_x = 0\colon\quad A_x + 208.40\text{ kN} - 67.151\text{ kN} = 0$$

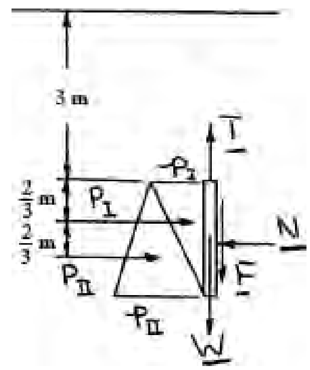
or $$A_x = -141.249\text{ kN} \qquad \mathbf{A} = 141.2\text{ kN} \leftarrow \blacktriangleleft$$

PROBLEM 5.84



The friction force between a 2×2-m square sluice gate AB and its guides is equal to 10 percent of the resultant of the pressure forces exerted by the water on the face of the gate. Determine the initial force needed to lift the gate if it weighs 5 kN.

SOLUTION



Consider the free-body diagram of the gate.

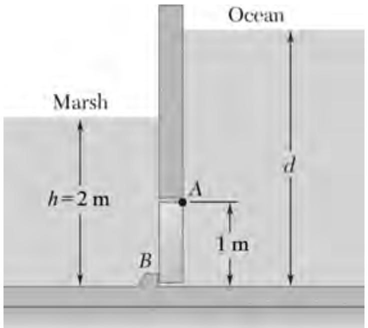
Now $$P_{\mathrm{I}} = \frac{1}{2}Ap_{\mathrm{I}} = \frac{1}{2}[(2\times 2)\ \mathrm{m}^2][(1000\times 9.81\ \mathrm{N/m}^3)(3\mathrm{m})]$$
$$= 58{,}860\ \mathrm{N}$$

$$P_{\mathrm{II}} = \frac{1}{2}Ap_{\mathrm{II}} = \frac{1}{2}[(2\times 2)\,\mathrm{m}^2][(1000\times 9.81\ \mathrm{N/m}^3)(5\ \mathrm{m})]$$
$$= 98{,}100\ \mathrm{N}$$

Then $$F = 0.1P = 0.1(P_{\mathrm{I}} + P_{\mathrm{II}})$$
$$= 0.1(58{,}860 + 98{,}100)\ \mathrm{N}$$
$$= 15696\ \mathrm{N}$$

Finally $$+\uparrow \Sigma F_y = 0: \quad T - 15696\ \mathrm{N} - 5000\ \mathrm{N} = 0 \Rightarrow T = 20696\ \mathrm{N}$$

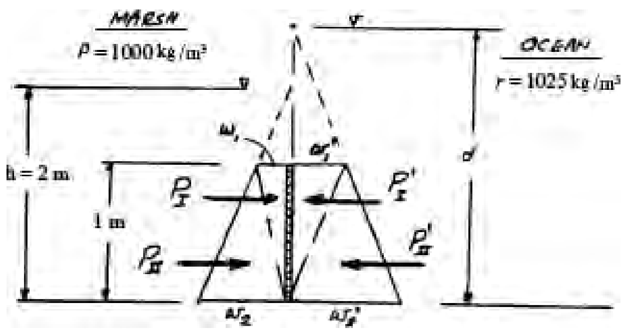
or $\mathbf{T} = 20.7\ \mathrm{kN}\uparrow$ ◀



PROBLEM 5.85

A freshwater marsh is drained to the ocean through an automatic tide gate that is 1.5 m wide and 1 m high. The gate is held by hinges located along its top edge at *A* and bears on a sill at *B*. If the water level in the marsh is $h = 2\,\text{m}$, determine the ocean level d for which the gate will open. (Density of salt water 1025 kg/m^3.)

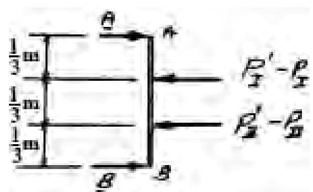
SOLUTION



Since gate is 1.5 m wide $\quad w = (1.5\text{m})p = 1.5\ \rho g$ (depth)

Thus:

$$w_1 = 1.5\ \rho g(h-1)$$
$$w_2 = 1.5\ \rho gh$$
$$w_1' = 1.5\ \rho' g(d-1)$$
$$w_2' = 1.5\ \rho' gd$$



$$P_{\text{I}}' - P_{\text{I}} = \frac{1}{2}(1\text{m})(w_1' - w_1)$$

$$= \frac{1}{2}(1\text{m})[1.5\rho' g(d-1) - 1.5\rho g(h-1)] = 0.75g\rho'(d-1) - 0.75g\ \rho(h-1)$$

$$P_{\text{II}}' - P_{\text{II}} = \frac{1}{2}(1\text{m})(w_2' - w_2)$$

$$= \frac{1}{2}(1\text{m})[1.5\rho' gd - 1.5\rho gh] = 0.75g\rho' d - 0.75g\ \rho h$$

$$\circlearrowleft + \Sigma M_A = 0: \quad (1\text{m})B - \left(\frac{1}{3}\text{ m}\right)(P_{\text{I}}' - P_{\text{I}}) - \left(\frac{2}{3}\text{ m}\right)(P_{\text{II}}' - P_{\text{II}}) = 0$$

$$B = \frac{1}{3}(P_{\text{I}}' - P_{\text{I}}) + \frac{2}{3}(P_{\text{II}}' - P_{\text{II}})$$

$$= \frac{1}{3}[0.75\rho' g(d-1) - 0.75\rho g(h-1)] + \frac{2}{3}[0.75\rho' gd - 0.75\rho gh]$$

$$= 0.25\rho' g(d-1) - 0.25\rho g(h-1) + 0.5\rho' gd - 0.5\rho gh$$

$$B = 0.75\rho' gd - 0.25\rho' g - 0.75\rho gh + 0.25\rho g$$

With $B = 0$ and $h = 2\,\text{m}$: $0 = 0.25\rho' g(3d-1) - 0.25\rho g(3h-1)$

$$(3d-1) = \frac{\rho(3h-1)}{\rho'}$$

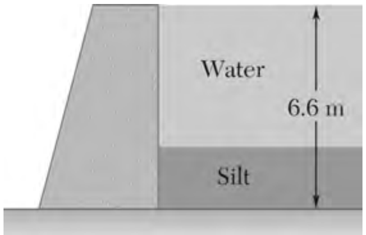
Data:

$$\rho' = 1025\text{ kg/m}^3$$

$$\rho = 1000\text{ kg/m}^3$$

$$3d - 1 = \frac{1000}{1025}(5) \Rightarrow d = 1.9593\,\text{m}$$

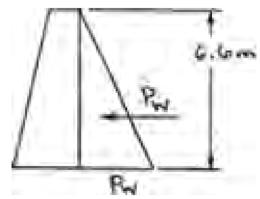
$d = 1.959$ m ◀



PROBLEM 5.86

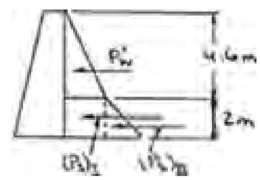
The dam for a lake is designed to withstand the additional force caused by silt that has settled on the lake bottom. Assuming that silt is equivalent to a liquid of density $\rho_s = 1.76 \times 10^3$ kg/m^3 and considering a 1-m-wide section of dam, determine the percentage increase in the force acting on the dam face for a silt accumulation of depth 2 m.

SOLUTION



First, determine the force on the dam face without the silt.

We have
$$P_w = \frac{1}{2}Ap_w = \frac{1}{2}A(\rho g h)$$
$$= \frac{1}{2}[(6.6 \text{ m})(1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(6.6 \text{ m})]$$
$$= 213.66 \text{ kN}$$



Next, determine the force on the dam face with silt

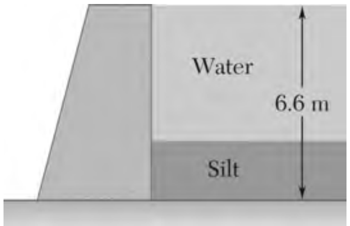
We have
$$P_w' = \frac{1}{2}[(4.6 \text{ m})(1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4.6 \text{ m})]$$
$$= 103.790 \text{ kN}$$
$$(P_s)_{\text{I}} = [(2.0 \text{ m})(1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4.6 \text{ m})]$$
$$= 90.252 \text{ kN}$$
$$(P_s)_{\text{II}} = \frac{1}{2}[(2.0 \text{ m})(1 \text{ m})][(1.76 \times 10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(2.0 \text{ m})]$$
$$= 34.531 \text{ kN}$$

Then
$$P' = P_w' + (P_s)_{\text{I}} + (P_s)_{\text{II}} = 228.57 \text{ kN}$$

The percentage increase, % inc., is then given by

$$\%\text{ inc.} = \frac{P' - P_w}{P_w} \times 100\%$$
$$= \frac{(228.57 - 213.66)}{213.66} \times 100\%$$
$$= 6.9874\%$$

$$\%\text{ inc.} = 6.98\% \blacktriangleleft$$



PROBLEM 5.87

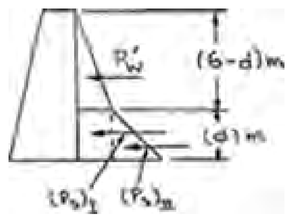
The base of a dam for a lake is designed to resist up to 120 percent of the horizontal force of the water. After construction, it is found that silt (that is equivalent to a liquid of density $\rho_s = 1.76 \times 10^3$ kg/m^3) is settling on the lake bottom at the rate of 12 mm/year. Considering a 1-m-wide section of dam, determine the number of years until the dam becomes unsafe.

SOLUTION

From Problem 5.86, the force on the dam face before the silt is deposited, is $P_w = 213.66$ kN. The maximum allowable force P_{allow} on the dam is then:

$$P_{\text{allow}} = 1.2P_w = (1.5)(213.66 \text{ kN}) = 256.39 \text{ kN}$$

Next determine the force P' on the dam face after *a* depth *d* of silt has settled



We have

$$P'_w = \frac{1}{2}[(6.6 - d)\text{m} \times (1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(6.6 - d)\text{m}]$$
$$= 4.905(6.6 - d)^2 \text{ kN}$$
$$(P_s)_{\text{I}} = [d(1 \text{ m})][(10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(6.6 - d)\text{m}]$$
$$= 9.81\,(6.6d - d^2)\text{kN}$$
$$(P_s)_{\text{II}} = \frac{1}{2}[d(1 \text{ m})][(1.76 \times 10^3 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(d)\text{m}]$$
$$= 8.6328d^2 \text{ kN}$$

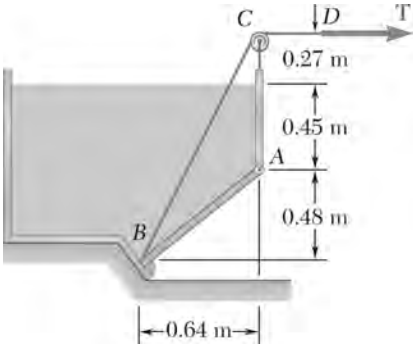
$$P' = P'_w + (P_s)_{\text{I}} + (P_s)_{\text{II}} = [4.905(43.560 - 13.2000d + d^2) + 9.81(6.6d - d^2) + 8.6328d^2]\text{kN}$$
$$= [3.7278d^2 + 213.66]\text{kN}$$

Now required that $P' = P_{\text{allow}}$ to determine the maximum value of *d*.

$$(3.7278d^2 + 213.66)\text{kN} = 256.39 \text{ kN}$$

or $$d = 3.3856 \text{ m}$$

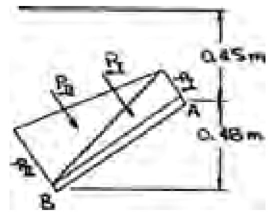
Finally $$3.3856 \text{ m} = 12 \times 10^{-3} \frac{\text{m}}{\text{year}} \times N$$ or $N = 282$ years ◀



PROBLEM 5.88

A 0.5 × 0.8-m gate *AB* is located at the bottom of a tank filled with water. The gate is hinged along its top edge *A* and rests on a frictionless stop at *B*. Determine the reactions at *A* and *B* when cable *BCD* is slack.

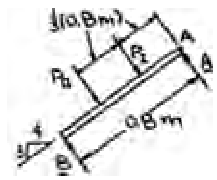
SOLUTION



First consider the force of the water on the gate.

We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g h)$$

so that $$P_{\mathrm{I}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.45\text{ m})]$$
$$= 882.9\text{ N}$$



$$P_{\mathrm{II}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.93\text{ m})]$$
$$= 1824.66\text{ N}$$

Reactions at *A* and *B* when $T = 0$

We have

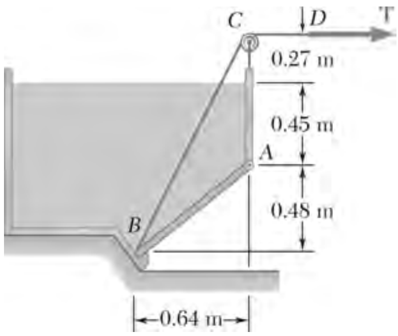
$$\circlearrowleft^{+}\ \Sigma M_A = 0: \quad \frac{1}{3}(0.8\text{ m})(882.9\text{ N}) + \frac{2}{3}(0.8\text{ m})(1824.66\text{ N}) - (0.8\text{ m})B = 0$$

or $$B = 1510.74\text{ N}$$

or $$\mathbf{B} = 1511\text{ N} \measuredangle 53.1° \blacktriangleleft$$

$$+\nwarrow \Sigma F = 0: \quad A + 1510.74\text{ N} - 882.9\text{ N} - 1824.66\text{ N} = 0$$

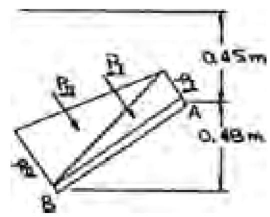
or $$\mathbf{A} = 1197\text{ N} \measuredangle 53.1° \blacktriangleleft$$



PROBLEM 5.89

A 0.5×0.8-m gate *AB* is located at the bottom of a tank filled with water. The gate is hinged along its top edge *A* and rests on a frictionless stop at *B*. Determine the minimum tension required in cable *BCD* to open the gate.

SOLUTION

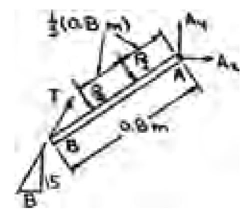


First consider the force of the water on the gate.

We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g h)$$

so that $$P_{\mathrm{I}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.45\text{ m})]$$
$$= 882.9\text{ N}$$

$$P_{\mathrm{II}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.93\text{ m})]$$
$$= 1824.66\text{ N}$$



T to open gate

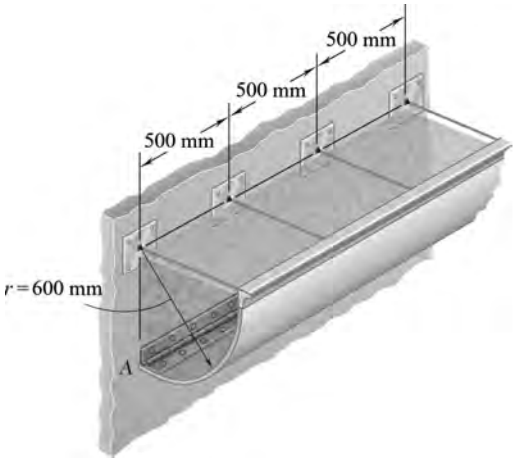
First note that when the gate begins to open, the reaction at $B \to 0$.

Then $$\circlearrowleft \Sigma M_A = 0\text{:}\quad \frac{1}{3}(0.8\text{ m})(882.9\text{ N}) + \frac{2}{3}(0.8\text{ m})(1824.66\text{ N})$$
$$-(0.45 + 0.27)\text{m} \times \left(\frac{8}{17}T\right) = 0$$

or $$235.44 + 973.152 - 0.33882T = 0$$

or $$\mathbf{T} = 3570\text{ N} \blacktriangleleft$$

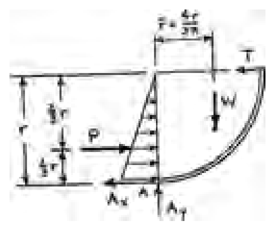
PROBLEM 5.90



A long trough is supported by a continuous hinge along its lower edge and by a series of horizontal cables attached to its upper edge. Determine the tension in each of the cables, at a time when the trough is completely full of water.

SOLUTION

Consider free body consisting of 500 mm length of the trough and water



$$l = 500\,\text{mm} \quad \text{length of free body}$$

$$W = \rho g v = \rho g\left[\frac{\pi}{4}r^2 l\right]$$

$$P_A = \rho g r$$

$$P = \frac{1}{2}P_A r l = \frac{1}{2}(\rho g r) r l = \frac{1}{2}\rho g r^2 l$$

$$+\circlearrowleft \Sigma M_A = 0: \quad Tr - Wr - P\left(\frac{1}{3}r\right) = 0$$

$$Tr - \left(\rho g\frac{\pi}{4}r^2 l\right)\left(\frac{4r}{3\pi}\right) - \left(\frac{1}{2}\rho g r^2 l\right)\left(\frac{1}{3}r\right) = 0$$

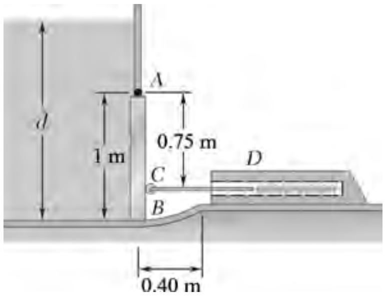
$$T = \frac{1}{3}\rho g r^2 l + \frac{1}{6}\rho g r^2 l = \frac{1}{2}\rho g r^2 l$$

Data: $\rho = 1000\ \text{kg/m}^3 \quad r = \frac{600}{12}\,\text{mm} = 0.6\,\text{m} \quad l = 500\,\text{mm} = 0.5\,\text{m}$

Then $T = \frac{1}{2}\times 1000 \times 9.81\,\text{N/m}^3 \times (0.6\,\text{m})^2(0.5\,\text{m})$

$= 882.9\ \text{N}$

$T = 883\ \text{N}$ ◀

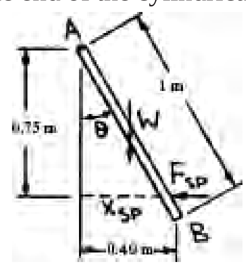


PROBLEM 5.91

A $1 \times 0.5\,\text{m}$ gate is hinged at *A* and is held in position by rod *CD*. End *D* rests against a spring whose constant is 24 kN/m. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod *CD* on the gate remains horizontal, determine the minimum depth of water *d* for which the bottom *B* of the gate will move to the end of the cylindrical portion of the floor.

SOLUTION

First determine the forces exerted on the gate by the spring and the water when *B* is at the end of the cylindrical portion of the floor



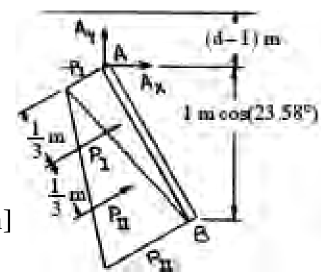
We have $\sin\theta = \dfrac{0.4}{1} = 0.4 \quad \theta = 23.58°$

Then $x_{SP} = (0.75\,\text{m})\,\tan(23.58°)$

and $F_{SP} = kx_{SP}$

$= 24\text{ kN/m} \times 0.75\,\text{m} \times \tan 23.58° = 7.8565\text{ kN}$

$= 7856.5\text{ N}$



Assume $d \geq 1\,\text{m}$

We have $P = \dfrac{1}{2}Ap = \dfrac{1}{2}(\rho g h)$

Then $P_{\text{I}} = \dfrac{1}{2}[(1\,\text{m})(0.5\,\text{m})] \times [(1000 \times 9.81\text{ N/m}^3)(d-1)\text{m}]$

$= 2452.5(d-1)\text{ N}$

$P_{\text{II}} = \dfrac{1}{2}[(1\,\text{m})(0.5\,\text{m})] \times [(1000 \times 9.81\text{ N/m}^3)(d - 1 + \cos 23.58°)]$

$= 2452.5(d - 0.0835)\text{N}$

For $d_{\min}$ so that gate opens, $W = 0$

Using the above free-body diagrams of the gate, we have

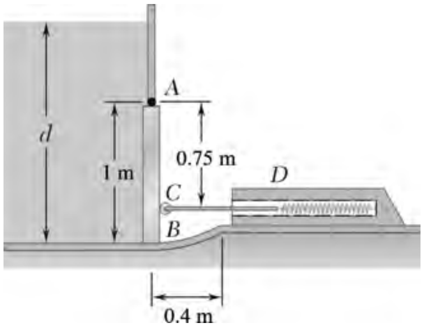
$$+\circlearrowleft \Sigma M_A = 0; \quad \left(\frac{1}{3}\text{m}\right)(2452.5)(d-1) + \left(\frac{2}{3}\text{m}\right)(2452.5)(d - 0.0835) - 0.75\,\text{m}(7856.5) = 0$$

or $(817.5d - 817.5) + (1635d - 136.5225) - 5892.375 = 0$

or $d = 2.7916\,\text{m}$

$d \geq 1\,\text{m} \Rightarrow$ assumption correct $\qquad d = 2.79\text{ m}$ ◀

PROBLEM 5.92



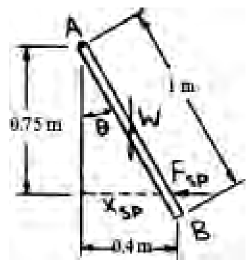
Solve Problem 5.91 if the gate is 500 kg.

Problem 5.91 A $1\times 0.5\,\text{m}$ gate is hinged at *A* and is held in position by rod *CD*. End *D* rests against a spring whose constant is 24 kN/m. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod *CD* on the gate remains horizontal, determine the minimum depth of water *d* for which the bottom *B* of the gate will move to the end of the cylindrical portion of the floor.

SOLUTION

First determine the forces exerted on the gate by the spring and the water when *B* is at the end of the cylindrical portion of the floor

We have $\sin\theta = \dfrac{0.4}{1} = 0.4 \quad \theta = 23.58°$



Then $x_{SP} = (0.75\,\text{m})\,\tan(23.58°)$

and $F_{SP} = kx_{SP}$

$= 24\text{ kN/m}\times 0.75\text{m}\times\tan 23.58° = 7.8565\,\text{kN}$

$= 7856.5\,\text{N}$

Assume $d \geq 1\,\text{m}$

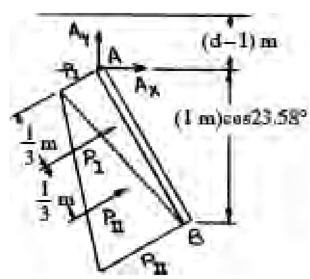
We have $P = \dfrac{1}{2}Ap = \dfrac{1}{2}A(\rho gh)$

Then $P_{\text{I}} = \dfrac{1}{2}[(1\text{m})(0.5\,\text{m})]\times[(1000\times 9.81\text{ N/m}^3)(d-1)\,\text{m}]$

$= 2452.5(d-1)\text{N}$

$P_{\text{II}} = \dfrac{1}{2}[(1\text{m})(0.5\text{m})]\times[(1000\times 9.81\,\text{N/m}^3)(d-1+\cos 23.58°)]$

$= 2452.5(d-0.0835)\text{N}$



For $d_{\min}$ so that gate opens, $W = 500\times 9.81\,\text{N}$

Using the above free-body diagrams of the gate, we have

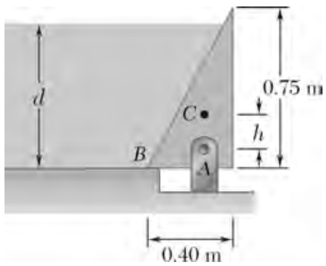
$$+\circlearrowleft \Sigma M_A = 0: \quad \left(\frac{1}{3}\text{ m}\right)(2452.5)(d-1) + \left(\frac{2}{3}\text{ m}\right)(2452.5)(d-0.0835)$$

$$-0.75\,\text{m}(7856.5) - (0.2\,\text{m})(500\times 9.81) = 0$$

or $(817.5d - 817.5) + (1635d - 136.5225) - 5892.375 - 981 = 0$

or $d = 3.1916\text{ m}$

$d \geq 1\,\text{m} \Rightarrow$ assumption correct $\quad d = 3.19\text{ m}$ ◄

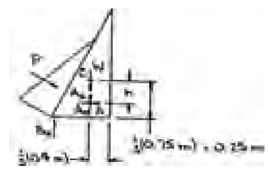


PROBLEM 5.93

A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at *A* and rests on a frictionless support at *B*. The pin is located at a distance $h = 0.10$ m below the center of gravity *C* of the gate. Determine the depth of water *d* for which the gate will open.

SOLUTION

First note that when the gate is about to open (clockwise rotation is impending), $B_y \to 0$ and the line of action of the resultant **P** of the pressure forces passes through the pin at *A*. In addition, if it is assumed that the gate is homogeneous, then its center of gravity *C* coincides with the centroid of the triangular area. Then

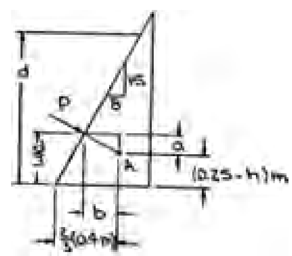


$$a = \frac{d}{3} - (0.25 - h)$$

and

$$b = \frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)$$

Now



$$\frac{a}{b} = \frac{8}{15}$$

so that

$$\frac{\frac{d}{3} - (0.25 - h)}{\frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)} = \frac{8}{15}$$

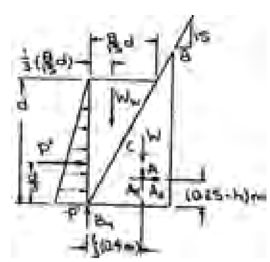
Simplifying yields

$$\frac{289}{45}d + 15h = \frac{70.6}{12} \qquad (1)$$

<u>Alternative solution</u>

Consider a free body consisting of a 1-m thick section of the gate and the triangular section *BDE* of water above the gate.

Now



$$P' = \frac{1}{2}Ap' = \frac{1}{2}(d \times 1\text{ m})(\rho g d)$$

$$= \frac{1}{2}\rho g d^2 \quad \text{(N)}$$

$$W' = \rho g V = \rho g\left(\frac{1}{2} \times \frac{8}{15}d \times d \times 1\text{ m}\right)$$

$$= \frac{4}{15}\rho g d^2 \quad \text{(N)}$$

PROBLEM 5.93 (Continued)

Then with $B_y = 0$ (as explained above), we have

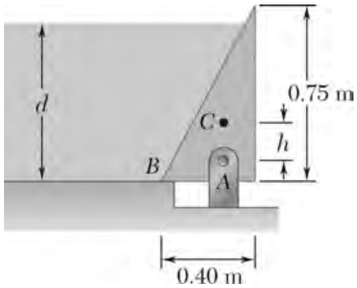
$$+\circlearrowleft \Sigma M_A = 0: \quad \left[\frac{2}{3}(0.4) - \frac{1}{3}\left(\frac{8}{15}d\right)\right]\left(\frac{4}{15}\rho g d^2\right) - \left[\frac{d}{3} - (0.25 - h)\right]\left(\frac{1}{2}\rho g d^2\right) = 0$$

Simplifying yields $$\frac{289}{45}d + 15h = \frac{70.6}{12}$$

as above.

Find d, $$h = 0.10 \text{ m}$$

Substituting into Eq. (1) $$\frac{289}{45}d + 15(0.10) = \frac{70.6}{12}$$ or $d = 0.683 \text{ m}$ ◀



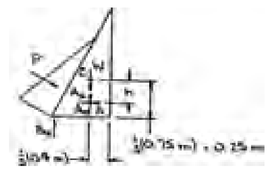
PROBLEM 5.94

A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at *A* and rests on a frictionless support at *B*. Determine the distance *h* if the gate is to open when $d = 0.75$ m.

SOLUTION

First note that when the gate is about to open (clockwise rotation is impending), $B_y \to 0$ and the line of action of the resultant **P** of the pressure forces passes through the pin at *A*. In addition, if it is assumed that the gate is homogeneous, then its center of gravity *C* coincides with the centroid of the triangular area. Then

$$a = \frac{d}{3} - (0.25 - h)$$

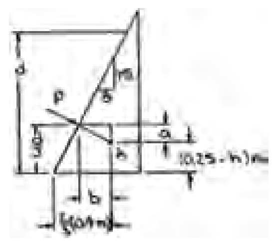


and

$$b = \frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)$$

Now

$$\frac{a}{b} = \frac{8}{15}$$



so that

$$\frac{\frac{d}{3} - (0.25 - h)}{\frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)} = \frac{8}{15}$$

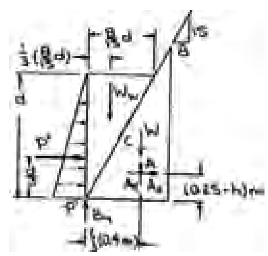
Simplifying yields

$$\frac{289}{45}d + 15h = \frac{70.6}{12} \qquad (1)$$

<u>Alternative solution</u>

Consider a free body consisting of a 1-m thick section of the gate and the triangular section *BDE* of water above the gate.

Now



$$P' = \frac{1}{2}Ap' = \frac{1}{2}(d \times 1\text{ m})(\rho g d)$$

$$= \frac{1}{2}\rho g d^2 \quad (\text{N})$$

$$W' = \rho g V = \rho g\left(\frac{1}{2} \times \frac{8}{15}d \times d \times 1\text{ m}\right)$$

$$= \frac{4}{15}\rho g d^2 \quad (\text{N})$$

PROBLEM 5.94 (Continued)

Then with $B_y = 0$ (as explained above), we have

$$\circlearrowleft + \Sigma M_A = 0: \quad \left[\frac{2}{3}(0.4) - \frac{1}{3}\left(\frac{8}{15}d\right)\right]\left(\frac{4}{15}\rho g d^2\right) - \left[\frac{d}{3} - (0.25 - h)\right]\left(\frac{1}{2}\rho g d^2\right) = 0$$

Simplifying yields $\qquad \dfrac{289}{45}d + 15h = \dfrac{70.6}{12}$

as above.

Find h, $\qquad d = 0.75\text{ m}$

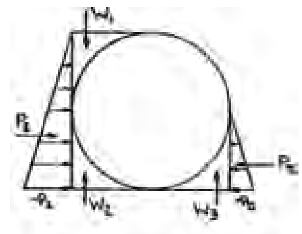
Substituting into Eq. (1) $\qquad \dfrac{289}{45}(0.75) + 15h = \dfrac{70.6}{12}$ $\qquad$ or $h = 0.0711\text{ m}$ ◀

PROBLEM 5.95

A 600 mm diameter drum of height 750 mm is placed on its side to act as a dam in a 750 mm-wide freshwater channel. Knowing that the drum is anchored to the sides of the channel, determine the resultant of the pressure forces acting on the drum.

SOLUTION

Consider the elements of water shown. The resultant of the weights of water above each section of the drum and the resultants of the pressure forces acting on the vertical surfaces of the elements is equal to the resultant hydrostatic force acting on the drum. Then



$$P_{\mathrm{I}} = \frac{1}{2}Ap_{\mathrm{I}} = \frac{1}{2}A(\rho gh)$$

$$= \frac{1}{2}\left[0.6\,\text{m}\times 0.75\,\text{m}\right]\times\left[1000\times 9.81\times 0.6\ \text{N/m}^2\right]$$

$$= 1324.35\ \text{N}$$

$$P_{\mathrm{II}} = \frac{1}{2}Ap_{\mathrm{II}} = \frac{1}{2}A(\rho gh)$$

$$= \frac{1}{2}\left[0.3\,\text{m}\times 0.75\,\text{m}\right]\times\left[1000\times 9.81\ \text{N/m}^3\times 0.3\,\text{m}\right]$$

$$= 331.0875\ \text{N}$$

$$W_1 = \rho g V_1 = (1000\times 9.81\ \text{N/m}^3)\left[(0.3)^2\,\text{m}^2 - \frac{\pi}{4}(0.3)^2\,\text{m}^2\right](0.75\,\text{m}) = 142.104\ \text{N}$$

$$W_2 = \rho g V_2 = (1000\times 9.81\ \text{N})\left[(0.3)^2\,\text{m}^2 + \frac{\pi}{4}(0.3)^2\,\text{m}^2\right](0.75\,\text{m}) = 1182.246\ \text{N}$$

$$W_3 = \rho g V_3 = (1000\times 9.81\ \text{N})\left[\frac{\pi}{4}(0.3)^2\,\text{m}^2\right](0.75\,\text{m}) = 520.071\,\text{N}$$

Then $\xrightarrow{+} \Sigma F_x$: $R_x = (1324.35 - 331.0875)\ \text{N} = 993.26\ \text{N}$

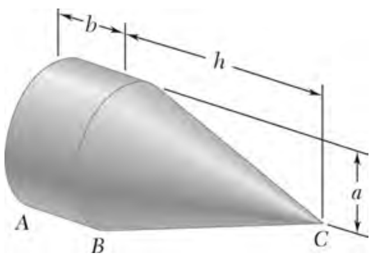
$+\!\uparrow \Sigma F_y$: $R_y = (-142.104 + 1182.246 + 520.071)\ \text{N} = 1560.21\,\text{N}$

Finally $R = \sqrt{R_x^2 + R_y^2} = 1849.546\,\text{N}$

$$\tan\theta = \frac{R_y}{R_x}$$

$$\theta = 57.5°$$

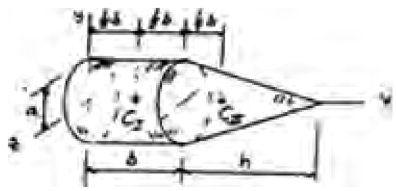
$\mathbf{R} = 1850\ \text{N}\ \measuredangle\ 57.5°$ ◄



PROBLEM 5.96

Determine the location of the centroid of the composite body shown when (a) $h = 2b$, (b) $h = 2.5b$.

SOLUTION



	V	$\bar{x}$	$\bar{x}V$
Cylinder I	$\pi a^2 b$	$\frac{1}{2}b$	$\frac{1}{2}\pi a^2 b^2$
Cone II	$\frac{1}{3}\pi a^2 h$	$b+\frac{1}{4}h$	$\frac{1}{3}\pi a^2 h\left(b+\frac{1}{4}h\right)$

$$V = \pi a^2\left(b+\frac{1}{3}h\right)$$

$$\Sigma\bar{x}V = \pi a^2\left(\frac{1}{2}b^2+\frac{1}{3}hb+\frac{1}{12}h^2\right)$$

(a) For $h = 2b$:

$$V = \pi a^2\left[b+\frac{1}{3}(2b)\right] = \frac{5}{3}\pi a^2 b$$

$$\Sigma\bar{x}V = \pi a^2\left[\frac{1}{2}b^2+\frac{1}{3}(2b)b+\frac{1}{12}(2b)^2\right]$$

$$= \pi a^2 b^2\left[\frac{1}{2}+\frac{2}{3}+\frac{1}{3}\right] = \frac{3}{2}\pi a^2 b^2$$

$$\bar{X}V = \Sigma\bar{x}V: \quad \bar{X}\left(\frac{5}{3}\pi a^2 b\right) = \frac{3}{2}\pi a^2 b^2 \qquad \bar{X} = \frac{9}{10}b$$

Centroid is $\frac{1}{10}b$ to left of base of cone ◀

PROBLEM 5.96 (Continued)

(*b*) For $h = 2.5b$: $$V = \pi a^2 \left[b + \frac{1}{3}(2.5b) \right] = 1.8333\pi a^2 b$$

$$\Sigma \bar{x} V = \pi a^2 \left[\frac{1}{2} b^2 + \frac{1}{3}(2.5b)b + \frac{1}{12}(2.5b)^2 \right]$$

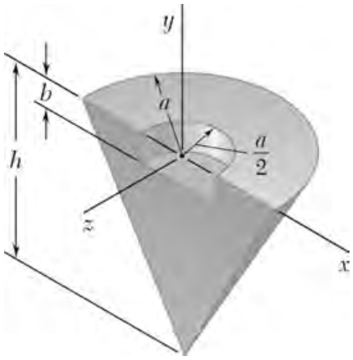
$$= \pi a^2 b^2 [0.5 + 0.8333 + 0.52083]$$

$$= 1.85416\pi a^2 b^2$$

$$\bar{X} V = \Sigma \bar{x} V: \quad \bar{X}(1.8333\pi a^2 b) = 1.85416\pi a^2 b^2 \qquad \bar{X} = 1.01136b$$

Centroid is 0.01136b to right of base of cone ◄

Note: Centroid is at base of cone for $h = \sqrt{6}b = 2.449b$

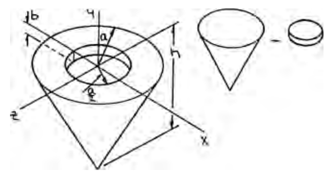


PROBLEM 5.97

Determine the y coordinate of the centroid of the body shown.

SOLUTION

First note that the values of $\bar{Y}$ will be the same for the given body and the body shown below. Then



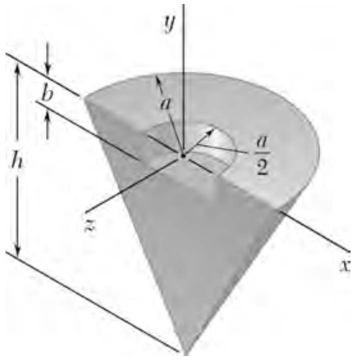
	V	$\bar{y}$	$\bar{y}V$
Cone	$\frac{1}{3}\pi a^2 h$	$-\frac{1}{4}h$	$-\frac{1}{12}\pi a^2 h^2$
Cylinder	$-\pi\left(\frac{a}{2}\right)^2 b = -\frac{1}{4}\pi a^2 b$	$-\frac{1}{2}b$	$\frac{1}{8}\pi a^2 b^2$
Σ	$\frac{\pi}{12}a^2(4h-3b)$		$-\frac{\pi}{24}a^2(2h^2-3b^2)$

We have $$\bar{Y}\Sigma V = \Sigma \bar{y}V$$

Then $$\bar{Y}\left[\frac{\pi}{12}a^2(4h-3b)\right] = -\frac{\pi}{24}a^2(2h^2-3b^2)$$

or $\bar{Y} = -\dfrac{2h^2-3b^2}{2(4h-3b)}$ ◀

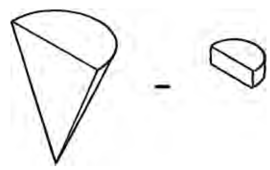
PROBLEM 5.98



Determine the z coordinate of the centroid of the body shown. (*Hint:* Use the result of Sample Problem 5.13.)

SOLUTION

First note that the body can be formed by removing a "half-cylinder" from a "half-cone," as shown.



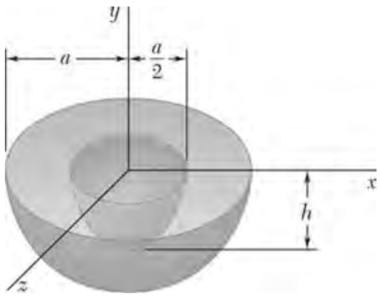
	V	$\bar{z}$	$\bar{z}V$
Half-Cone	$\frac{1}{6}\pi a^2 h$	$-\frac{a}{\pi}$ *	$-\frac{1}{6}a^3 h$
Half-Cylinder	$-\frac{\pi}{2}\left(\frac{a}{2}\right)^2 b = -\frac{\pi}{8}a^2 b$	$-\frac{4}{3\pi}\left(\frac{a}{2}\right) = -\frac{2a}{3\pi}$	$\frac{1}{12}a^3 b$
Σ	$\frac{\pi}{24}a^2(4h-3b)$		$-\frac{1}{12}a^3(2h-b)$

From Sample Problem 5.13

We have $$\bar{Z}\Sigma V = \Sigma \bar{z}V$$

Then $$\bar{Z}\left[\frac{\pi}{24}a^2(4h-3b)\right] = -\frac{1}{12}a^3(2h-b)$$

or $\bar{Z} = -\frac{a}{\pi}\left(\frac{4h-2b}{4h-3b}\right)$ ◀



PROBLEM 5.99

The composite body shown is formed by removing a semiellipsoid of revolution of semimajor axis h and semiminor axis $a/2$ from a hemisphere of radius a. Determine (a) the y coordinate of the centroid when $h = a/2$, (b) the ratio h/a for which $\bar{y} = -0.4a$.

SOLUTION

	V	$\bar{y}$	$\bar{y}V$
Hemisphere	$\frac{2}{3}\pi a^3$	$-\frac{3}{8}a$	$-\frac{1}{4}\pi a^4$
Semiellipsoid	$-\frac{2}{3}\pi\left(\frac{a}{2}\right)^2 h = -\frac{1}{6}\pi a^2 h$	$-\frac{3}{8}h$	$+\frac{1}{16}\pi a^2 h^2$

Then
$$\Sigma V = \frac{\pi}{6}a^2(4a-h)$$
$$\Sigma \bar{y}V = -\frac{\pi}{16}a^2(4a^2-h^2)$$

Now
$$\bar{Y}\Sigma V = \Sigma \bar{y}V$$

So that
$$\bar{Y}\left[\frac{\pi}{6}a^2(4a-h)\right] = -\frac{\pi}{16}a^2(4a^2-h^2)$$

or
$$\bar{Y}\left(4-\frac{h}{a}\right) = -\frac{3}{8}a\left[4-\left(\frac{h}{a}\right)^2\right] \qquad (1)$$

(a)
$$\bar{Y} = ? \quad \text{when} \quad h = \frac{a}{2}$$

Substituting $\frac{h}{a} = \frac{1}{2}$ into Eq. (1)
$$\bar{Y}\left(4-\frac{1}{2}\right) = -\frac{3}{8}a\left[4-\left(\frac{1}{2}\right)^2\right]$$

or
$$\bar{Y} = -\frac{45}{112}a \qquad \bar{Y} = -0.402a \blacktriangleleft$$

PROBLEM 5.99 (Continued)

(*b*) $$\frac{h}{a} = ? \quad \text{when} \quad \bar{Y} = -0.4a$$

Substituting into Eq. (1)

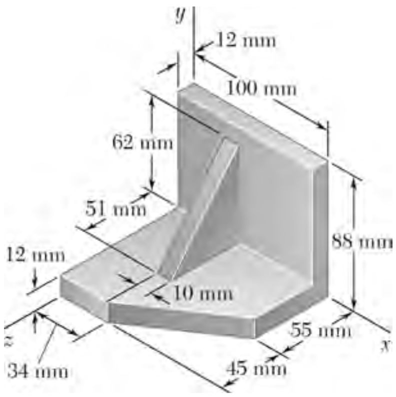
$$(-0.4a)\left(4 - \frac{h}{a}\right) = -\frac{3}{8}a\left[4 - \left(\frac{h}{a}\right)^2\right]$$

or $$3\left(\frac{h}{a}\right)^2 - 3.2\left(\frac{h}{a}\right) + 0.8 = 0$$

Then $$\frac{h}{a} = \frac{3.2 \pm \sqrt{(-3.2)^2 - 4(3)(0.8)}}{2(3)}$$

$$= \frac{3.2 \pm 0.8}{6}$$

or $\dfrac{h}{a} = \dfrac{2}{5}$ and $\dfrac{h}{a} = \dfrac{2}{3}$ ◀

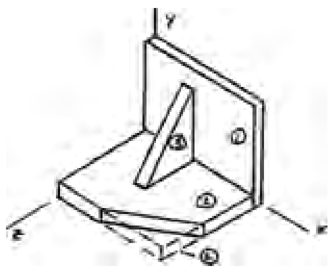


PROBLEM 5.100

For the stop bracket shown, locate the x coordinate of the center of gravity.

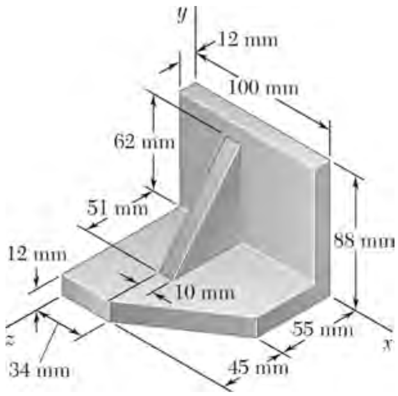
SOLUTION

Assume that the bracket is homogeneous so that its center of gravity coincides with the centroid of the volume.



	V, mm^3	$\bar{x}$, mm	$\bar{x}V$, mm^4
1	$(100)(88)(12) = 105600$	50	5280000
2	$(100)(12)(88) = 105600$	50	5280000
3	$\frac{1}{2}(62)(51)(10) = 15810$	39	616590
4	$-\frac{1}{2}(66)(45)(12) = -17820$	$34 + \frac{2}{3}(66) = 78$	-1389960
Σ	209190		9786600

Then $$\bar{X} = \frac{\bar{x}V}{V} = \frac{9786600}{209190}\text{ mm}$$ or $\bar{X} = 46.8$ mm ◀

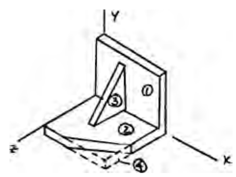


PROBLEM 5.101

For the stop bracket shown, locate the z coordinate of the center of gravity.

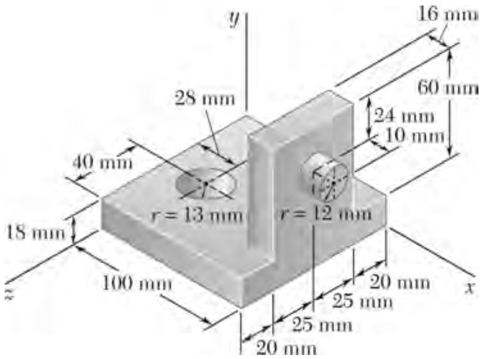
SOLUTION

Assume that the bracket is homogeneous so that its center of gravity coincides with the centroid of the volume.



	V, mm^3	$\bar{z}$, mm	$\bar{z}V$, mm^4
1	$(100)(88)(12) = 105600$	6	633600
2	$(100)(12)(88) = 105600$	$12 + \frac{1}{2}(88) = 56$	5913600
3	$\frac{1}{2}(62)(51)(10) = 15810$	$12 + \frac{1}{3}(51) = 29$	458490
4	$-\frac{1}{2}(66)(45)(12) = -17820$	$55 + \frac{2}{3}(45) = 85$	-1514700
	209190		5491000

Then $$\bar{Z} = \frac{\bar{z}V}{V} = \frac{5491000}{209190}\text{ mm}$$ or $\bar{Z} = 26.2$ mm ◄

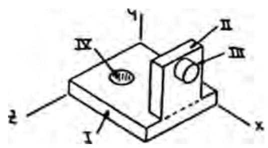


PROBLEM 5.102

For the machine element shown, locate the *y* coordinate of the center of gravity.

SOLUTION

First assume that the machine element is homogeneous so that its center of gravity will coincide with the centroid of the corresponding volume.

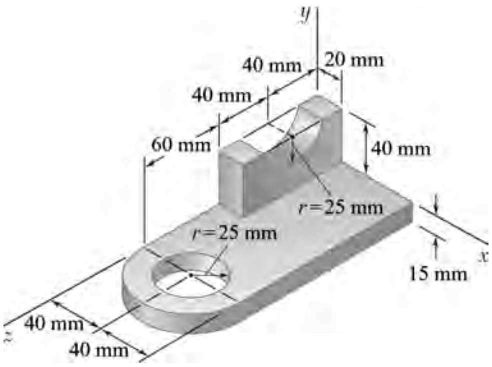


	V, mm^3	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4
I	$(100)(18)(90) = 162000$	50	9	8100000	1458000
II	$(16)(60)(50) = 48000$	92	48	4416000	2304000
III	$\pi(12)^2(10) = 4523.9$	105	54	475010	244290
IV	$-\pi(13)^2(18) = -9556.7$	28	9	–267590	–86010
	204967.2			12723420	3920280

We have $\bar{Y}\Sigma V = \Sigma \bar{y}V$

$$\bar{Y}(204967.2 \text{ mm}^3) = 3920280 \text{ mm}^4$$

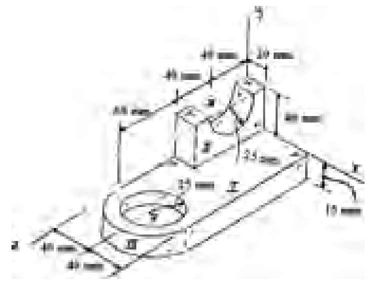
or $\bar{Y} = 19.13$ mm ◄



PROBLEM 5.103

For the machine element shown, locate the y coordinate of the center of gravity.

SOLUTION



For half cylindrical hole: (III)

$$r = 25 \text{ mm}$$

$$\bar{y}_{\text{III}} = 40 - \frac{4(25)}{3\pi}$$

$$= 29.3897 \text{ mm}$$

For half cylindrical plate: (IV)

$$r = 40 \text{ mm}$$

$$\bar{z}_{\text{IV}} = 140 + \frac{4(40)}{3\pi} = 156.977 \text{ mm}$$

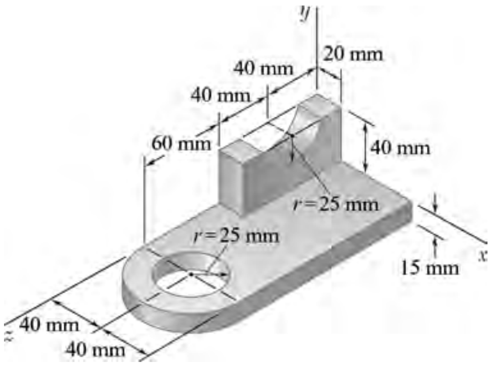
		V, mm^3	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{y}V$, mm^4	$\bar{z}V$, mm^4
I	Rectangular plate	$(140)(80)(15) = 168\times10^3$	-7.5	70	-1.26×10^6	11.76×10^6
II	Rectangular plate	$(80)(40)(20) = 64\times10^3$	20	40	1.28×10^6	2.56×10^6
III	− (Half cylinder)	$-\frac{\pi}{2}(25)^2(20) = -19.635\times10^3$	29.3897	40	-0.577×10^6	-0.7854×10^6
IV	(Half cylinder)	$\frac{\pi}{2}(40)^2(15) = 37.699\times10^3$	-7.5	156.977	-0.2827×10^6	5.9179×10^6
V	−(Cylinder)	$-\pi(25)^2(15) = -29.452\times10^3$	-7.5	140	0.2209×10^6	-4.1233×10^6
		220.612×10^3			-0.6188×10^6	15.3292×10^6

$$\bar{Y}\Sigma V = \Sigma\bar{y}V$$

$$\bar{Y}(220.612\times10^3\,\text{mm}^3) = -0.6188\times10^6\,\text{mm}^4$$

$$\bar{Y} = -2.8049\,\text{mm}$$

$\bar{Y} = -2.80$ mm ◀



PROBLEM 5.104

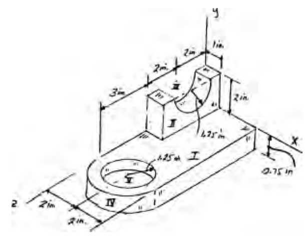
For the machine element shown, locate the z coordinate of the center of gravity.

SOLUTION

For half cylindrical hole: (III)

$$r = 25 \text{ mm}$$

$$\bar{y}_{\text{III}} = 40 - \frac{4(25)}{3\pi} = 29.3897 \text{ mm}$$



For half cylindrical plate: (IV) $r = 40$ mm

$$\bar{z}_{\text{IV}} = 140 + \frac{4(40)}{3\pi} = 156.977 \text{ mm}$$

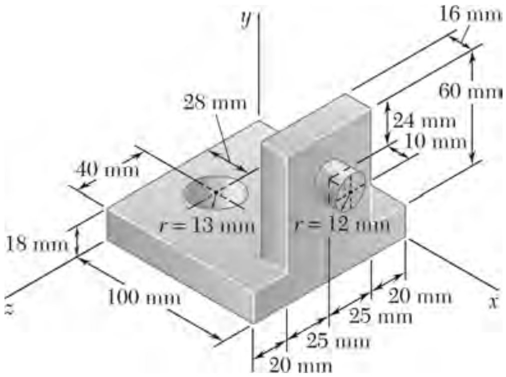
		V, mm^3	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{y}V$, mm^4	$\bar{z}V$, mm^4
I	Rectangular plate	$(140)(80)(15) = 168\times10^3$	-7.5	70	-1.26×10^6	11.76×10^6
II	Rectangular plate	$(80)(40)(20) = 64\times10^3$	20	40	1.28×10^6	2.56×10^6
III	− (Half cylinder)	$-\frac{\pi}{2}(25)^2(20) = -19.635\times10^3$	29.3897	40	-0.577×10^6	-0.7854×10^6
IV	(Half cylinder)	$\frac{\pi}{2}(40)^2(15) = 37.699\times10^3$	-7.5	156.977	-0.2827×10^6	5.9179×10^6
V	− (Cylinder)	$-\pi(25)^2(15) = -29.452\times10^3$	-7.5	140	0.2209×10^6	-4.1233×10^6
		220.612×10^3			-0.6188×10^6	15.3292×10^6

Now $\bar{Z}\Sigma V = \bar{z}V$

$$\bar{Z}(220.612\times10^3 \text{ mm}^3) = 15.3292\times10^6 \text{ mm}^4$$

$$\Rightarrow \bar{Z} = 69.4849 \text{ mm}$$

$\bar{Z} = 69.5$ mm ◀

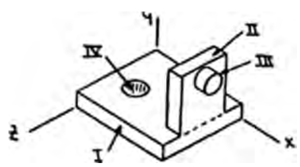


PROBLEM 5.105

For the machine element shown, locate the *x* coordinate of the center of gravity.

SOLUTION

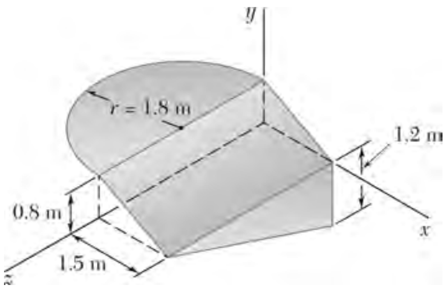
First assume that the machine element is homogeneous so that its center of gravity will coincide with the centroid of the corresponding volume.



	V, mm^3	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4
I	$(100)(18)(90) = 162000$	50	9	8100000	1458000
II	$(16)(60)(50) = 48000$	92	48	4416000	2304000
III	$\pi(12)^2(10) = 4523.9$	105	54	475010	244290
IV	$-\pi(13)^2(18) = -9556.7$	28	9	–267590	–86010
	204967.2			12723420	3920280

We have $$\bar{X}\Sigma V = \Sigma\bar{x}V$$

$$\bar{X}(204967.2\text{ mm}^3) = 12723420\text{ mm}^4$$ $\bar{X} = 62.1\text{ mm}$ ◄

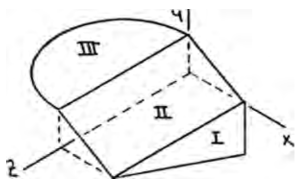


PROBLEM 5.106

Locate the center of gravity of the sheet-metal form shown.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the form will coincide with the centroid of the corresponding area.



$$\bar{y}_{\mathrm{I}} = -\frac{1}{3}(1.2) = -0.4 \text{ m}$$

$$\bar{z}_{\mathrm{I}} = \frac{1}{3}(3.6) = 1.2 \text{ m}$$

$$\bar{x}_{\mathrm{III}} = -\frac{4(1.8)}{3\pi} = -\frac{2.4}{\pi} \text{ m}$$

	A, m^2	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}A$, m^3	$\bar{y}A$, m^3	$\bar{z}A$, m^3
I	$\frac{1}{2}(3.6)(1.2) = 2.16$	1.5	0.4	1.2	3.24	-0.864	2.592
II	$(3.6)(1.7) = 6.12$	0.75	0.4	1.8	4.59	2.448	11.016
III	$\frac{\pi}{2}(1.8)^2 = 5.0894$	$-\frac{2.4}{\pi}$	0.8	1.8	-3.888	4.0715	9.1609
Σ	13.3694				3.942	5.6555	22.769

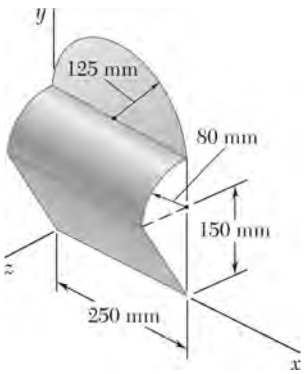
We have

$\bar{X}\Sigma V = \Sigma \bar{x} V$: $\bar{X}(13.3694 \text{ m}^2) = 3.942 \text{ m}^3$ or $\bar{X} = 0.295$ m ◄

$\bar{Y}\Sigma V = \Sigma \bar{y} V$: $\bar{Y}(13.3694 \text{ m}^2) = 5.6555 \text{ m}^3$ or $\bar{Y} = 0.423$ m ◄

$\bar{Z}\Sigma V = \Sigma \bar{z} V$: $\bar{Z}(13.3694 \text{ m}^2) = 22.769 \text{ m}^3$ or $\bar{Z} = 1.703$ m ◄

PROBLEM 5.107

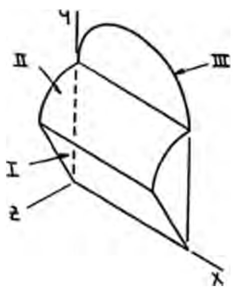


Locate the center of gravity of the sheet-metal form shown.

SOLUTION

First assume that the sheet metal is homogeneous so that the center of gravity of the form will coincide with the centroid of the corresponding area. Now note that symmetry implies

$$\bar{X} = 125 \text{ mm} \blacktriangleleft$$



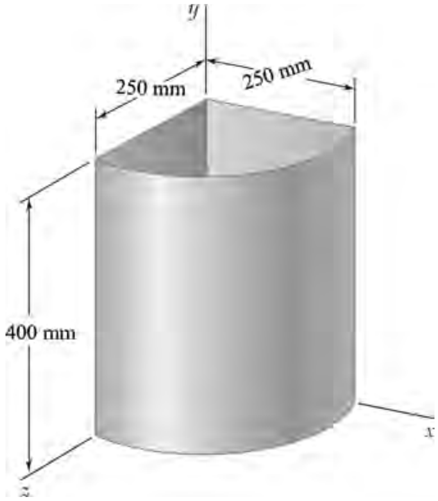
$$\bar{y}_{\text{II}} = 150 + \frac{2 \times 80}{\pi}$$
$$= 200.93 \text{ mm}$$
$$\bar{z}_{\text{II}} = \frac{2 \times 80}{\pi}$$
$$= 50.930 \text{ mm}$$
$$\bar{y}_{\text{III}} = 230 + \frac{4 \times 125}{3\pi}$$
$$= 283.05 \text{ mm}$$

	A, mm^2	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{y}A$, mm^3	$\bar{z}A$, mm^3
I	$(250)(170) = 42500$	75	40	3187500	1700000
II	$\frac{\pi}{2}(80)(250) = 31416$	200.93	50930	6312400	1600000
III	$\frac{\pi}{2}(125)^2 = 24544$	283.05	0	6947200	0
Σ	98460			16447100	3300000

We have

$$\bar{Y}\Sigma A = \Sigma \bar{y} A: \quad \bar{Y}(98460 \text{ mm}^2) = 16447100 \text{ mm}^3 \qquad \text{or } \bar{Y} = 1670 \text{ mm} \blacktriangleleft$$

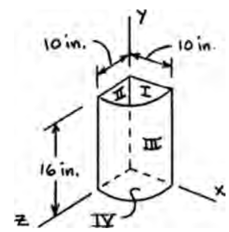
$$\bar{Z}\Sigma A = \Sigma \bar{z} A: \quad \bar{Z}(98460 \text{ mm}^2) = 3.300 \times 10^6 \text{ mm}^3 \qquad \text{or } \bar{Z} = 33.5 \text{ mm} \blacktriangleleft$$



PROBLEM 5.108

A wastebasket, designed to fit in the corner of a room, is 400 mm high and has a base in the shape of a quarter circle of radius 250 mm Locate the center of gravity of the wastebasket, knowing that it is made of sheet metal of uniform thickness.

SOLUTION



By symmetry: $\bar{X} = \bar{Z}$

For III (Cylindrical surface) $\bar{x} = \dfrac{2r}{\pi} = \dfrac{2(250)}{\pi} = 159.155 \text{ mm}$

$$A = \frac{\pi}{2} rh = \frac{\pi}{2}(250)(400) = 157.080 \times 10^3 \text{ mm}^2$$

For IV (Quarter-circle bottom) $\bar{x} = \dfrac{4r}{3\pi} = \dfrac{4(250)}{3\pi} = 106.103 \text{ mm}$

$$A = \frac{\pi}{4} r^2 = \frac{\pi}{4}(250)^2 49.0874 \times 10^3 \text{ mm}^2$$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
I	$(250)(400) = 100 \times 10^3$	125	200	12.5×10^6	20×10^6
II	$(250)(400) = 100 \times 10^3$	0	200	0	20×10^6
III	157.680×10^3	159.155	200	25×10^6	31.416×10^6
IV	49.0874×10^3	106.103	0	5.2083×10^6	0
Σ	406.1674×10^3			42.7083×10^6	71.416×10^6

$\bar{X}\Sigma A = \Sigma \bar{x} A$: $\bar{X}(406.1674 \times 10^3 \text{ mm}^2) = 42.7083 \times 10^6 \text{ mm}^3$

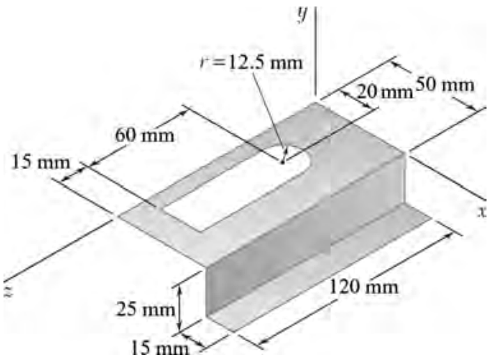
$$\bar{X} = 105.149 \text{ mm}$$

$\bar{X} = \bar{Z} = 1051 \text{ mm}$ ◀

$\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\bar{Y}(406.1674 \times 10^3 \text{ mm}^2) = 71.416 \times 10^6 \text{ mm}^3$

$$\bar{Y} = 175.829 \text{ mm}$$

$\bar{Y} = 175.8 \text{ mm}$ ◀

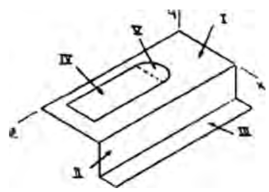


PROBLEM 5.109

A mounting bracket for electronic components is formed from sheet metal of uniform thickness. Locate the center of gravity of the bracket.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the bracket coincides with the centroid of the corresponding area. Then (see diagram)



$$\bar{z}_V = 45 - \frac{4(12.5)}{3\pi} = 39.6948 \text{ mm}$$

$$A_V = -\frac{\pi}{2}(12.5)^2 = -245.437 \text{ mm}^2$$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3	$\bar{z}A$, mm^3
I	(50)(120) = 6000	25	0	60	150×10^3	0	360×10^3
II	(25)(120) = 3000	50	–12.5	60	150×10^3	-37.5×10^3	180×10^3
III	(15)(120) = 1800	57.5	–25	60	103.5×10^3	-45×10^3	108×10^3
IV	–(25)(60) = –1500	20	0	75	-30×10^3	0	-112.5×10^3
V	–245.437	20	0	39.6948	-4.909×10^3	0	-9.7426×10^3
Σ	9054.563				368.591×10^3	-82.5×10^3	525.7574×10^3

We have

$$\bar{X}\Sigma A = \Sigma \bar{x} A$$

$$\bar{X}(9054.563 \text{ mm}^2) = 368.591 \times 10^3 \text{ mm}^3 \qquad \text{or } \bar{X} = 40.7 \text{ mm} \blacktriangleleft$$

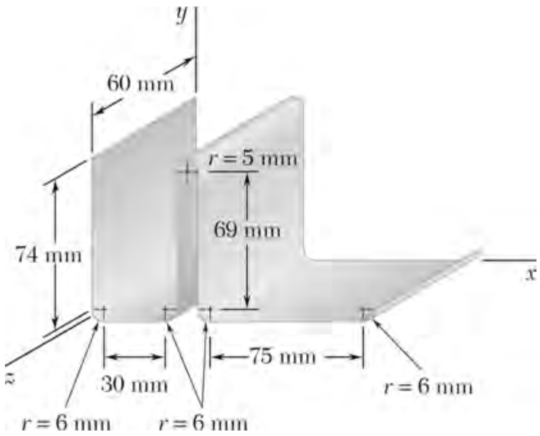
$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}(9054.563 \text{ mm}^2) = -82.5 \times 10^3 \text{ mm}^3 \qquad \text{or } \bar{Y} = -9.11 \text{ mm} \blacktriangleleft$$

$$\bar{Z}\Sigma A = \Sigma \bar{z} A$$

$$\bar{Z}(9054.563 \text{ mm}^2) = 525.7574 \times 10^3 \text{ mm}^3 \qquad \text{or } \bar{Z} = 58.1 \text{ mm} \blacktriangleleft$$

PROBLEM 5.110

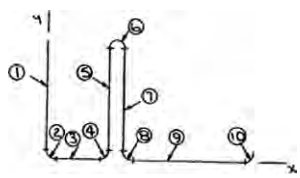


A thin sheet of plastic of uniform thickness is bent to form a desk organizer. Locate the center of gravity of the organizer.

SOLUTION

First assume that the plastic is homogeneous so that the center of gravity of the organizer will coincide with the centroid of the corresponding area. Now note that symmetry implies

$$\bar{Z} = 30.0 \text{ mm} \blacktriangleleft$$



$$\bar{x}_2 = 6 - \frac{2\times 6}{\pi} = 2.1803 \text{ mm}$$

$$\bar{x}_4 = 36 + \frac{2\times 6}{\pi} = 39.820 \text{ mm}$$

$$\bar{x}_8 = 58 - \frac{2\times 6}{\pi} = 54.180 \text{ mm}$$

$$\bar{x}_{10} = 133 + \frac{2\times 6}{\pi} = 136.820 \text{ mm}$$

$$\bar{y}_2 = \bar{y}_4 = \bar{y}_8 = \bar{y}_{10} = 6 - \frac{2\times 6}{\pi} = 2.1803 \text{ mm}$$

$$\bar{y}_6 = 75 + \frac{2\times 5}{\pi} = 78.183 \text{ mm}$$

$$A_2 = A_4 = A_8 = A_{10} = \frac{\pi}{2}\times 6\times 60 = 565.49 \text{ mm}^2$$

$$A_6 = \pi\times 5\times 60 = 942.48 \text{ mm}^2$$

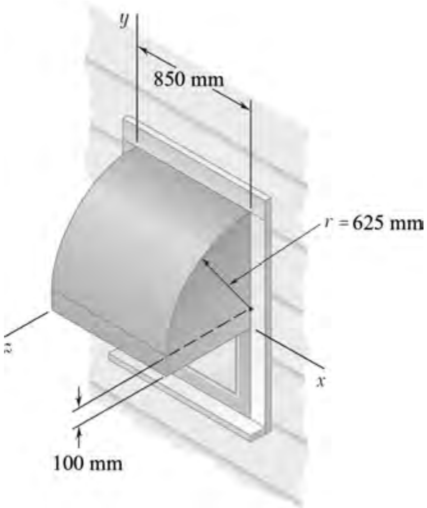
PROBLEM 5.110 (Continued)

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$(74)(60) = 4440$	0	43	0	190920
2	565.49	2.1803	2.1803	1233	1233
3	$(30)(60) = 1800$	21	0	37800	0
4	565.49	39.820	2.1803	22518	1233
5	$(69)(60) = 4140$	42	40.5	173880	167670
6	942.48	47	78.183	44297	73686
7	$(69)(60) = 4140$	52	40.5	215280	167670
8	565.49	54.180	2.1803	30638	1233
9	$(75)(60) = 4500$	95.5	0	429750	0
10	565.49	136.820	2.1803	77370	1233
Σ	22224.44			1032766	604878

We have $\bar{X}\Sigma A = \Sigma \bar{x} A$: $\bar{X}(22224.44 \text{ mm}^2) = 1032766 \text{ mm}^3$ or $\bar{X} = 46.5$ mm ◀

$\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\bar{Y}(22224.44 \text{ mm}^2) = 604878 \text{ mm}^3$ or $\bar{Y} = 27.2$ mm ◀

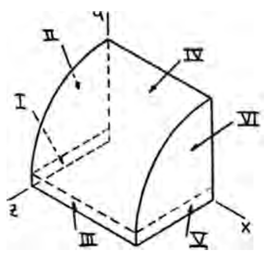
PROBLEM 5.111



A window awning is fabricated from sheet metal of uniform thickness. Locate the center of gravity of the awning.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the awning coincides with the centroid of the corresponding area.



$$\bar{y}_{\text{II}} = \bar{y}_{\text{VI}} = 100 + \frac{(4)(625)}{3\pi} = 365.258 \text{ mm}$$

$$\bar{z}_{\text{II}} = \bar{z}_{\text{VI}} = \frac{(4)(625)}{3\pi} = 265.258 \text{ mm}$$

$$\bar{y}_{\text{IV}} = 100 + \frac{(2)(625)}{\pi} = 497.887 \text{ mm}$$

$$\bar{z}_{\text{IV}} = \frac{(2)(625)}{\pi} = 397.887 \text{ mm}$$

$$A_{\text{II}} = A_{\text{VI}} = \frac{\pi}{4}(625)^2 = 306.796 \times 10^3 \text{ mm}^2$$

$$A_{\text{IV}} = \frac{\pi}{2}(625)(850) = 834.486 \times 10^3 \text{ mm}^2$$

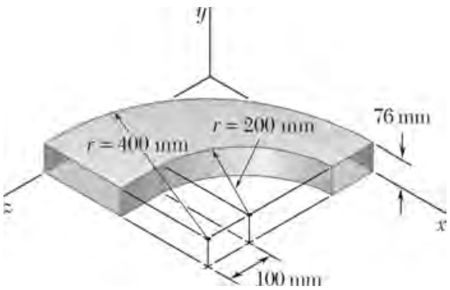
	A, mm^2	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{y}A$, mm^3	$\bar{z}A$, mm^3
I	$(100)(625) = 62.5 \times 10^3$	50	312.5	3.125×10^6	19.53125×10^6
II	306.796×10^3	365.258	265.258	112.060×10^6	81.380×10^6
III	$(100)(850) = 85 \times 10^3$	50	625	4.25×10^6	53.125×10^6
IV	834.486×10^3	497.887	397.887	415.480×10^6	332.031×10^6
V	$(100)(625) = 62.5 \times 10^3$	50	312.5	3.125×10^6	19.53125×10^6
VI	306.796×10^3	365.258	265.258	112.060×10^6	81.380×10^6
Σ	1658.078×10^3			650.1×10^6	586.9785×10^6

PROBLEM 5.111 (Continued)

Now, symmetry implies $\bar{X} = 425 \text{ mm}$ ◀

and $\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\bar{Y}(1658.078 \times 10^3 \text{ mm}^2) = 650.1 \times 10^6 \text{ mm}^3$ or $\bar{Y} = 392 \text{ mm}$ ◀

$\bar{Z}\Sigma A = \Sigma \bar{z} A$: $\bar{Z}(1658.078 \times 10^3 \text{ mm}^2) = 586.9785 \times 10^6 \text{ mm}^3$ or $\bar{Z} = 354 \text{ mm}$ ◀



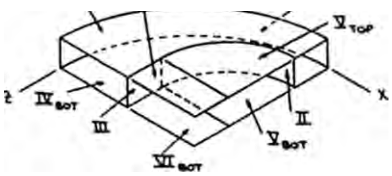
PROBLEM 5.112

An elbow for the duct of a ventilating system is made of sheet metal of uniform thickness. Locate the center of gravity of the elbow.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the duct coincides with the centroid of the corresponding area. Also, note that the shape of the duct implies

$$\bar{Y} = 38.0 \text{ mm} \blacktriangleleft$$



Note that $\bar{x}_{\mathrm{I}} = \bar{z}_{\mathrm{I}} = 400 - \frac{2}{\pi}(400) = 145.352 \text{ mm}$

$$\bar{x}_{\mathrm{II}} = 400 - \frac{2}{\pi}(200) = 272.68 \text{ mm}$$

$$\bar{z}_{\mathrm{II}} = 300 - \frac{2}{\pi}(200) = 172.676 \text{ mm}$$

$$\bar{x}_{\mathrm{IV}} = \bar{z}_{\mathrm{IV}} = 400 - \frac{4}{3\pi}(400) = 230.23 \text{ mm}$$

$$\bar{x}_{\mathrm{V}} = 400 - \frac{4}{3\pi}(200) = 315.12 \text{ mm}$$

$$\bar{z}_{\mathrm{V}} = 300 - \frac{4}{3\pi}(200) = 215.12 \text{ mm}$$

Also note that the corresponding top and bottom areas will contribute equally when determining $\bar{x}$ and $\bar{z}$.

Thus

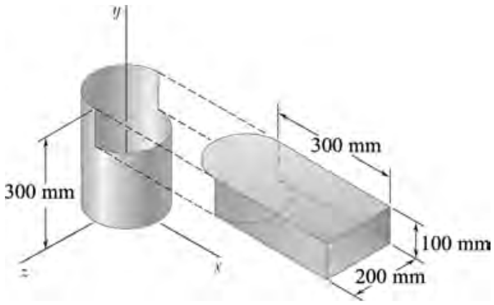
	A, mm^2	$\bar{x}$, mm	$\bar{z}$, mm	$\bar{x}A$, mm^3	$\bar{z}A$, mm^3
I	$\frac{\pi}{2}(400)(76) = 47752$	145.352	145.352	6940850	6940850
II	$\frac{\pi}{2}(200)(76) = 23876$	272.68	172.676	6510510	4122810
III	$100(76) = 7600$	200	350	1520000	2660000
IV	$2\left(\frac{\pi}{4}\right)(400)^2 = 251327$	230.23	230.23	57863020	57863020
V	$-2\left(\frac{\pi}{4}\right)(200)^2 = -62832$	315.12	215.12	–19799620	–13516420
VI	$-2(100)(200) = -40000$	300	350	–12000000	–14000000
Σ	227723			41034760	44070260

PROBLEM 5.112 (Continued)

We have $\bar{X}\Sigma A = \Sigma \bar{x} A$: $\bar{X}(227723 \text{ mm}^2) = 41034760 \text{ mm}^3$ or $\bar{X} = 180.2 \text{ mm}$ ◄

$\bar{Z}\Sigma A = \Sigma \bar{z} A$: $\bar{Z}(227723 \text{ mm}^2) = 44070260 \text{ mm}^3$ or $\bar{Z} = 193.5 \text{ mm}$ ◄

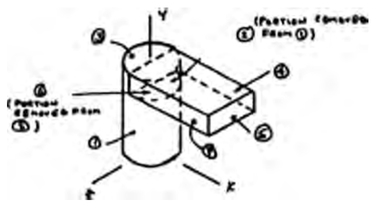
PROBLEM 5.113



A 200-mm-diameter cylindrical duct and a 100 × 200 mm rectangular duct are to be joined as indicated. Knowing that the ducts were fabricated from the same sheet metal, which is of uniform thickness, locate the center of gravity of the assembly.

SOLUTION

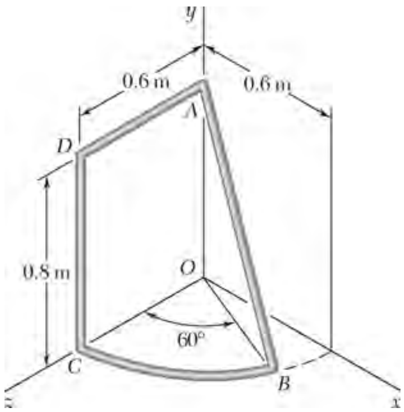
Assume that the body is homogeneous so that its center of gravity coincides with the centroid of the area.



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\pi(200)(300) = 188.496 \times 10^3$	0	150	0	28.2744×10^6
2	$-\dfrac{\pi}{2}(200)(100) = -31.4159 \times 10^3$	$\dfrac{2(100)}{\pi} = \dfrac{200}{\pi}$	250	-2×10^6	-7.8540×10^6
3	$\dfrac{\pi}{2}(100)^2 = 15.708 \times 10^3$	$-\dfrac{4(100)}{3\pi} = -\dfrac{400}{3\pi}$	300	-0.66667×10^6	4.7124×10^6
4	$(200)(300) = 60 \times 10^3$	150	300	9×10^6	18×10^6
5	$(200)(300) = 60 \times 10^3$	150	200	9×10^6	12×10^6
6	$-\dfrac{\pi}{2}(100)^2 = -15.708 \times 10^3$	$\dfrac{4(100)}{3\pi} = \dfrac{400}{3\pi}$	200	-0.66667×10^6	-3.1416×10^6
7	$(100)(300) = 30 \times 10^3$	150	250	4.5×10^6	7.5×10^6
8	$(100)(300) = 30 \times 10^3$	150	250	4.5×10^6	7.5×10^6
	337.0801×10^3			23.6667×10^6	66.9912×10^6

Then $$\bar{X} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{23.6667 \times 10^6}{337.0801 \times 10^3} \text{ mm}$$ or $\bar{X} = 70.2$ mm ◀

$$\bar{Y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{66.9912 \times 10^6}{337.0801 \times 10^3} \text{ mm}$$ or $\bar{Y} = 198.7$ mm ◀

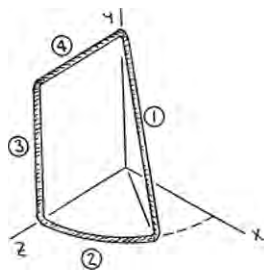


PROBLEM 5.114

A thin steel wire of uniform cross section is bent into the shape shown. Locate its center of gravity.

SOLUTION

First assume that the wire is homogeneous so that its center of gravity will coincide with the centroid of the corresponding line.



$$\bar{x}_1 = 0.3\sin 60° = 0.15\sqrt{3}\text{ m}$$

$$\bar{z}_1 = 0.3\cos 60° = 0.15\text{ m}$$

$$\bar{x}_2 = \left(\frac{0.6\sin 30°}{\frac{\pi}{6}}\right)\sin 30° = \frac{0.9}{\pi}\text{ m}$$

$$\bar{z}_2 = \left(\frac{0.6\sin 30°}{\frac{\pi}{6}}\right)\cos 30° = \frac{0.9}{\pi}\sqrt{3}\text{ m}$$

$$L_2 = \left(\frac{\pi}{3}\right)(0.6) = (0.2\pi)\text{ m}$$

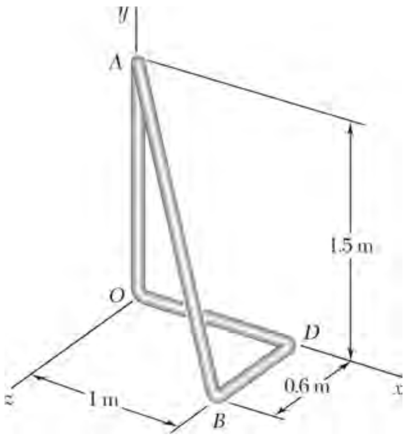
	L, m	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}L$, m^2	$\bar{y}L$, m^2	$\bar{z}L$, m^2
1	1.0	$0.15\sqrt{3}$	0.4	0.15	0.25981	0.4	0.15
2	0.2π	$\frac{0.9}{\pi}$	0	$\frac{0.9\sqrt{3}}{\pi}$	0.18	0	0.31177
3	0.8	0	0.4	0.6	0	0.32	0.48
4	0.6	0	0.8	0.3	0	0.48	0.18
Σ	3.0283				0.43981	1.20	1.12177

We have $\bar{X}\Sigma L = \Sigma\bar{x}L$: $\bar{X}(3.0283\text{ m}) = 0.43981\text{ m}^2$ or $\bar{X} = 0.1452$ m ◄

$\bar{Y}\Sigma L = \Sigma\bar{y}L$: $\bar{Y}(3.0283\text{ m}) = 1.20\text{ m}^2$ or $\bar{Y} = 0.396$ m ◄

$\bar{Z}\Sigma L = \Sigma\bar{z}L$: $\bar{Z}(3.0283\text{ m}) = 1.12177\text{ m}^2$ or $\bar{Z} = 0.370$ m ◄

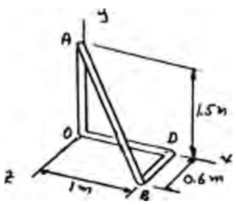
PROBLEM 5.115



Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

SOLUTION

Uniform rod



$$AB^2 = (1\text{ m})^2 + (0.6\text{ m})^2 + (1.5\text{ m})^2$$

$$AB = 1.9\text{ m}$$

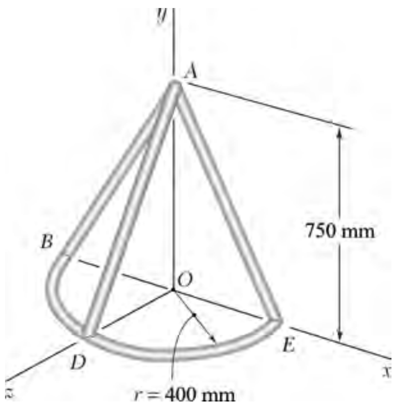
	L, m	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}L$, m^2	$\bar{y}L$, m^2	ΣL, m^2
AB	1.9	0.5	0.75	0.3	0.95	1.425	0.57
BD	0.6	1.0	0	0.3	0.60	0	0.18
DO	1.0	0.5	0	0	0.50	0	0
OA	1.5	0	0.75	0	0	1.125	0
Σ	5.0				2.05	2.550	0.75

$\bar{X}\Sigma L = \Sigma \bar{x} L: \quad \bar{X}(5.0\text{ m}) = 2.05\text{ m}^2$ $\qquad \bar{X} = 0.410\text{ m}$ ◀

$\bar{Y}\Sigma L = \Sigma \bar{y} L: \quad \bar{Y}(5.0\text{ m}) = 2.55\text{ m}^2$ $\qquad \bar{Y} = 0.510\text{ m}$ ◀

$\bar{Z}\Sigma L = \Sigma \bar{z} L: \quad \bar{Z}(5.0\text{ m}) = 0.75\text{ m}^2$ $\qquad \bar{Z} = 0.1500\text{ m}$ ◀

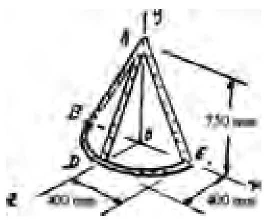
PROBLEM 5.116



Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

SOLUTION

By symmetry: $\bar{X} = 0$ ◀



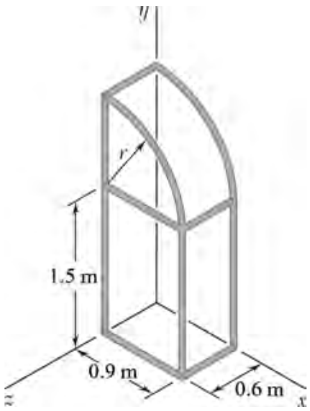
	L, mm	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{y}L$, mm^2	$\bar{z}L$, mm^2
AB	$\sqrt{(750)^2+(400)^2}=850$	375	0	318.75×10^3	0
AD	$\sqrt{(750)^2+(400)^2}=850$	375	200	318.75×10^3	170×10^3
AE	$\sqrt{(750)^2+(400)^2}=850$	375	0	318.75×10^3	0
BDE	$\pi(400)=1256.637$ mm	0	$\frac{2(400)}{\pi}=\frac{800}{\pi}$	0	320×10^3
Σ	3806.637			956.25×10^3	490×10^3

$$\bar{Y}\Sigma L = \Sigma \bar{y} L: \quad \bar{Y}(3806.637 \text{ mm}) = 956.25\times10^3 \text{ mm}^2$$

$$\bar{Y} = 251.21 \text{ mm} \qquad \bar{Y} = 251 \text{ mm} ◀$$

$$\bar{Z}\Sigma L = \Sigma \bar{z} L: \quad \bar{Z}(3806.637 \text{ mm}) = 490\times10^3 \text{ mm}^2$$

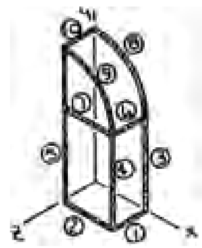
$$\bar{Z} = 128.723 \text{ mm} \qquad \bar{Z} = 128.7 \text{ mm} ◀$$



PROBLEM 5.117

The frame of a greenhouse is constructed from uniform aluminum channels. Locate the center of gravity of the portion of the frame shown.

SOLUTION



First assume that the channels are homogeneous so that the center of gravity of the frame will coincide with the centroid of the corresponding line.

$$\bar{x}_8 = \bar{x}_9 = \frac{2 \times 0.9}{\pi} = \frac{1.8}{\pi}\text{ m}$$

$$\bar{y}_8 = \bar{y}_9 = 1.5 + \frac{2 \times 0.9}{\pi} = 2.07296\text{ m}$$

	L, m	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}L$, m^2	$\bar{y}L$, m^2	$\bar{z}L$, m^2
1	0.6	0.9	0	0.3	0.54	0	0.18
2	0.9	0.45	0	0.6	0.405	0	0.54
3	1.5	0.9	0.75	0	1.35	1.125	0
4	1.5	0.9	0.75	0.6	1.35	1.125	0.9
5	2.4	0	1.2	0.6	0	2.88	1.44
6	0.6	0.9	1.5	0.3	0.54	0.9	0.18
7	0.9	0.45	1.5	0.6	0.405	1.35	0.54
8	$\frac{\pi}{2} \times 0.9 = 1.4137$	$\frac{1.8}{\pi}$	2.07296	0	0.81	2.9305	0
9	$\frac{\pi}{2} \times 0.9 = 1.4137$	$\frac{1.8}{\pi}$	2.07296	0.6	0.81	2.9305	0.84822
10	0.6	0	2.4	0.3	0	1.44	0.18
Σ	11.8274				6.21	14.681	4.80822

We have

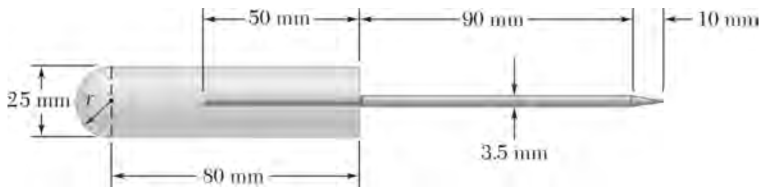
$\bar{X}\Sigma L = \Sigma \bar{x} L$: $\quad \bar{X}(11.8274\text{ m}) = 6.21\text{ m}^2$ $\quad$ or $\bar{X} = 0.525\text{ m}$ ◀

$\bar{Y}\Sigma L = \Sigma \bar{y} L$: $\quad \bar{Y}(11.8274\text{ m}) = 14.681\text{ m}^2$ $\quad$ or $\bar{Y} = 1.241\text{ m}$ ◀

$\bar{Z}\Sigma L = \Sigma \bar{z} L$: $\quad \bar{Z}(11.8274\text{ m}) = 4.80822\text{ m}^2$ $\quad$ or $\bar{Z} = 0.407\text{ m}$ ◀

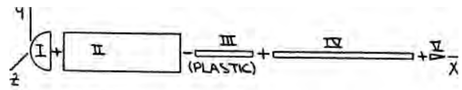
PROBLEM 5.118

A scratch awl has a plastic handle and a steel blade and shank. Knowing that the density of plastic is 1030 kg/m^3 and of steel is 7860 kg/m^3, locate the center of gravity of the awl.



SOLUTION

First, note that symmetry implies $\bar{Y} = \bar{Z} = 0$ ◀



$$\bar{x}_{\text{I}} = \frac{5}{8}(12.5 \text{ mm}) = 7.8125 \text{ mm}$$

$$W_{\text{I}} = (1030 \text{ kg/m}^3)\left(\frac{2\pi}{3}\right)(0.0125 \text{ m})^3$$

$$= 4.2133 \times 10^{-3} \text{ kg}$$

$$\bar{x}_{\text{II}} = 52.5 \text{ mm}$$

$$W_{\text{II}} = (1030 \text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.025 \text{ m})^2(0.08 \text{ m})$$

$$= 40.448 \times 10^{-3} \text{ kg}$$

$$\bar{x}_{\text{III}} = 92.5 \text{ mm} - 25 \text{ mm} = 67.5 \text{ mm}$$

$$W_{\text{III}} = -(1030 \text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.0035 \text{ m})^2(.05 \text{ m})$$

$$= -0.49549 \times 10^{-3} \text{kg}$$

$$\bar{x}_{\text{IV}} = 182.5 \text{ mm} - 70 \text{ mm} = 112.5 \text{ mm}$$

$$W_{\text{IV}} = (7860 \text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.0035 \text{ m})^2(0.14 \text{ m})^2 = 10.5871 \times 10^{-3} \text{ kg}$$

$$\bar{x}_{\text{V}} = 182.5 \text{ mm} + \frac{1}{4}(10 \text{ mm}) = 185 \text{ mm}$$

$$W_{\text{V}} = (7860 \text{ kg/m}^3)\left(\frac{\pi}{3}\right)(0.00175 \text{ m})^2(0.01 \text{ m}) = 0.25207 \times 10^{-3} \text{ kg}$$

PROBLEM 5.118 (Continued)

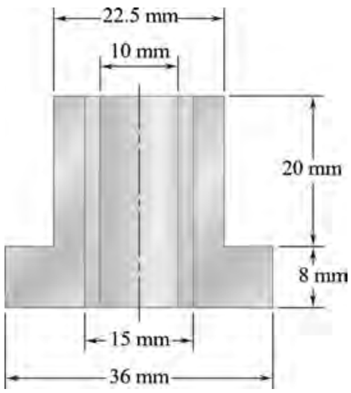
	W, kg	$\bar{x}$, mm	$\bar{x}W$, kg·mm
I	4.123×10^{-3}	7.8125	32.916×10^{-3}
II	40.948×10^{-3}	52.5	2123.5×10^{-3}
III	-0.49549×10^{-3}	67.5	-33.447×10^{-3}
IV	10.5871×10^{-3}	112.5	1191.05×10^{-3}
V	0.25207×10^{-3}	185	46.633×10^{-3}
Σ	55.005×10^{-3}		3360.7×10^{-3}

We have $\bar{X}\Sigma W = \Sigma\bar{x}W: \quad \bar{X}(55.005\times10^{-3}\text{ kg}) = 3360.7\times10^{-3}\text{ kg}\cdot\text{mm}$

or $$\bar{X} = 61.1\text{ mm} \blacktriangleleft$$

(From the end of the handle)

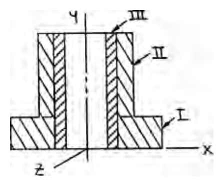
PROBLEM 5.119



A bronze bushing is mounted inside a steel sleeve. Knowing that the ρ_{br}= 8800 kg/m^3 and of steel is ρ_{steel} = 7860 kg/m^3, determine the location of the center of gravity of the assembly.

SOLUTION

First, note that symmetry implies $\bar{X} = \bar{Z} = 0$ ◀



Now $W = (\rho g)V$

$$\bar{y}_{\mathrm{I}} = 4 \text{ mm} \quad W_{\mathrm{I}} = (7860 \times 9.81 \text{ N/m}^3)\left\{\left(\frac{\pi}{4}\right)\left[\left(36^2 - 15^2\right)\text{mm}^2\right]\left(8 \text{ mm} \times 10^{-9} \text{ m}^3/\text{mm}^3\right)\right\} = 0.518873 \text{ N}$$

$$\bar{y}_{\mathrm{II}} = 18 \text{ mm} \quad W_{\mathrm{II}} = (7860 \times 9.81 \text{ N/m}^3)\left\{\left(\frac{\pi}{4}\right)\left[\left(22.5^2 - 15^2\right)\text{mm}^2\right]\left(20 \text{ mm} \times 10^{-9} \text{m}^3/\text{mm}^3\right)\right\} = 0.340647 \text{ N}$$

$$\bar{y}_{\mathrm{III}} = 14 \text{ mm} \quad W_{\mathrm{III}} = (8800 \times 9.81 \text{ N/m}^3)\left\{\left(\frac{\pi}{4}\right)\left[\left(15^2 - 10^2\right)\text{mm}^2\right]\left(28 \text{ mm} \times 10^{-9} \text{m}^3/\text{mm}^3\right)\right\} = 0.237306 \text{ N}$$

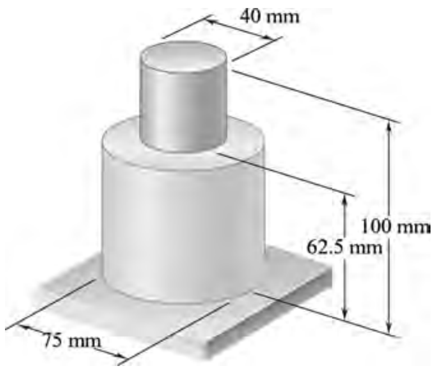
We have $\bar{Y}\Sigma W = \Sigma \bar{y} W$

$$\bar{Y} = \frac{(4 \text{ mm})(0.518873 \text{ N}) + (18 \text{ mm})(0.340647 \text{ N}) + (14 \text{ mm})(0.237306 \text{ N})}{(0.518873 + 0.340647 + 0.237306) \text{ N}} = 10.512 \text{ mm}$$

or $\bar{Y} = 10.51 \text{ mm}$ ◀

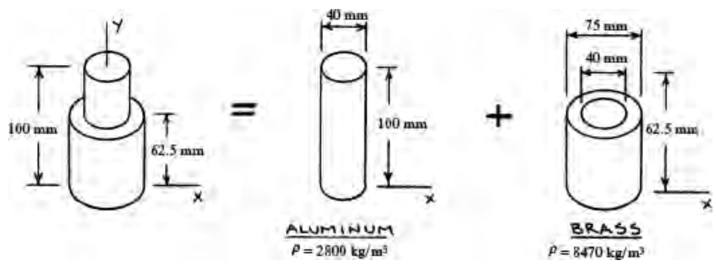
(above base)

PROBLEM 5.120



A brass collar, of length 62.5 mm, is mounted on an aluminum rod of length 100 mm. Locate the center of gravity of the composite body. (Densities: brass = 8470 kg/m^3, aluminum = 2800 kg/m^3.)

SOLUTION



Aluminum rod: $W = \rho_{\text{Al}} gV$

$$= 2800 \times 9.81\ \text{N/m}^3 \left[\frac{\pi}{4} \times (40\ \text{mm})^2 (100\ \text{mm})\right] \times 10^{-9}\ \text{m}^3/\text{mm}^3$$

$$= 3.4517\ \text{N}$$

Brass collar: $W = \rho_{\text{brass}} gV$

$$= 8470 \times 9.81\ \text{N/m}^3\ \frac{\pi}{4}\left[(75\ \text{mm})^2 - (40\ \text{mm})^2\right] \times 62.5\ \text{mm} \times 10^{-9}\ \text{m}^3/\text{mm}^3$$

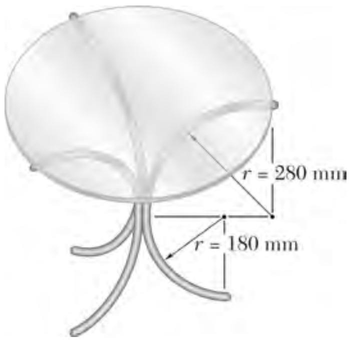
$$= 16.4168\ \text{N}$$

Component	W(N)	$\bar{y}$(mm)	$\bar{y}W$(N · mm)
Rod	3.4517	50	172.585
Collar	16.4168	31.25	513.025
	19.8685		685.61

$$\bar{Y}\Sigma W = \Sigma \bar{y} W: \quad \bar{Y}(19.8685\text{N}) = 685.61\text{N} \cdot \text{mm}$$

$$\bar{Y} = 34.507\ \text{mm} \qquad \bar{Y} = 34.5\ \text{mm} \blacktriangleleft$$

PROBLEM 5.121

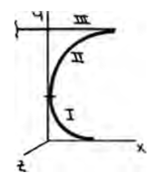


The three legs of a small glass-topped table are equally spaced and are made of steel tubing, which has an outside diameter of 24 mm and a cross-sectional area of 150 mm^2. The diameter and the thickness of the table top are 600 mm and 10 mm, respectively. Knowing that the density of steel is 7860 kg/m^3 and of glass is 2190 kg/m^3, locate the center of gravity of the table.

SOLUTION

First note that symmetry implies $\bar{X} = \bar{Z} = 0$ ◀

Also, to account for the three legs, the masses of components I and II will each bex multiplied by three



$$\bar{y}_{\text{I}} = 12 + 180 - \frac{2 \times 180}{\pi} = 77.408 \text{ mm} \qquad m_{\text{I}} = \rho_{ST} V_{\text{I}} = 7860 \text{ kg/m}^3 \times (150 \times 10^{-6} \text{ m}^2) \times \frac{\pi}{2}(0.180 \text{ m}) = 0.33335 \text{ kg}$$

$$\bar{y}_{\text{II}} = 12 + 180 + \frac{2 \times 280}{\pi} = 370.25 \text{ mm} \qquad m_{\text{II}} = \rho_{ST} V_{\text{II}} = 7860 \text{ kg/m}^3 \times (150 \times 10^{-6} \text{ m}^2) \times \frac{\pi}{2}(0.280 \text{ m}) = 0.51855 \text{ kg}$$

$$\bar{y}_{\text{III}} = 24 + 180 + 280 + 5 = 489 \text{ mm} \qquad m_{\text{III}} = \rho_{GL} V_{\text{III}} = 2190 \text{ kg/m}^3 \times \frac{\pi}{4}(0.6 \text{ m})^2 \times (0.010 \text{ m}) = 6.1921 \text{ kg}$$

	m, kg	$\bar{y}$, mm	$\bar{y}m$, kg·mm
I	3(0.33335)	77.408	77.412
II	3(0.51855)	370.25	515.98
III	6.1921	489	3027.9
Σ	8.7478		3681.3

We have $Y \Sigma m = \Sigma \bar{y} m$: $\bar{Y}(8.7478 \text{ kg}) = 3681.3 \text{ kg} \cdot \text{mm}$

or $\bar{Y} = 420.8 \text{ mm}$

The center of gravity is 421 mm ◀

(above the floor)

PROBLEM 5.122

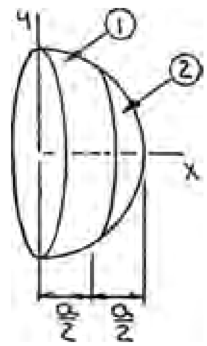
Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A hemisphere

SOLUTION

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$



The equation of the generating curve is $x^2 + y^2 = a^2$ so that $r^2 = a^2 - x^2$ and then

$$dV = \pi(a^2 - x^2)dx$$

Component 1

$$V_1 = \int_0^{a/2} \pi(a^2 - x^2)dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_0^{a/2}$$

$$= \frac{11}{24}\pi a^3$$

and

$$\int_1 \bar{x}_{EL} dV = \int_0^{a/2} x\left[\pi(a^2 - x^2)dx\right]$$

$$= \pi\left[a^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^{a/2}$$

$$= \frac{7}{64}\pi a^4$$

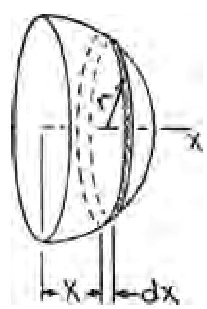
Now $\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{11}{24}\pi a^3\right) = \frac{7}{64}\pi a^4$ or $\bar{x}_1 = \frac{21}{88}a$ ◀

Component 2

$$V_2 = \int_{a/2}^{a} \pi(a^2 - x^2)dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{a/2}^{a}$$

$$= \pi\left\{\left[a^2(a) - \frac{a^3}{3}\right] - \left[a^2\left(\frac{a}{2}\right) - \frac{\left(\frac{a}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{5}{24}\pi a^3$$



PROBLEM 5.122 (Continued)

and $$\int_2 \bar{x}_{EL}\,dV = \int_{a/2}^{a} x\left[\pi(a^2 - x^2)dx\right] = \pi\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{a/2}^{a}$$

$$= \pi\left\{\left[a^2\frac{(a)^2}{2} - \frac{(a)^4}{4}\right] - \left[a^2\frac{\left(\frac{a}{2}\right)^2}{2} - \frac{\left(\frac{a}{2}\right)^4}{4}\right]\right\}$$

$$= \frac{9}{64}\pi a^4$$

Now $$\bar{x}_2 V_2 = \int_2 \bar{x}_{EL}\,dV: \quad \bar{x}_2\left(\frac{5}{24}\pi a^3\right) = \frac{9}{64}\pi a^4$$

or $\bar{x}_2 = \dfrac{27}{40}a$ ◀

PROBLEM 5.123

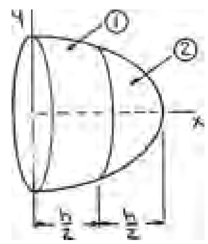
Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A semiellipsoid of revolution

SOLUTION

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$



The equation of the generating curve is $\dfrac{x^2}{h^2} + \dfrac{y^2}{a^2} = 1$ so that

$$r^2 = \frac{a^2}{h^2}(h^2 - x^2)$$

and then $$dV = \pi \frac{a^2}{h^2}(h^2 - x^2)dx$$

Component 1 $$V_1 = \int_0^{h/2} \pi \frac{a^2}{h^2}(h^2 - x^2)dx = \pi \frac{a^2}{h^2}\left[h^2 x - \frac{x^3}{3}\right]_0^{h/2}$$

$$= \frac{11}{24}\pi a^2 h$$

and $$\int_1 \bar{x}_{EL} dV = \int_0^{h/2} x\left[\pi \frac{a^2}{h^2}(h^2 - x^2)dx\right]$$

$$= \pi \frac{a^2}{h^2}\left[h^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^{h/2}$$

$$= \frac{7}{64}\pi a^2 h^2$$

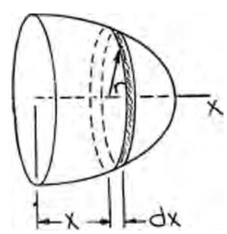
Now $$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{11}{24}\pi a^2 h\right) = \frac{7}{64}\pi a^2 h^2$$

or $\bar{x}_1 = \dfrac{21}{88}h$ ◀

Component 2 $$V_2 = \int_{h/2}^{h} \pi \frac{a^2}{h^2}(h^2 - x^2)dx = \pi \frac{a^2}{h^2}\left[h^2 x - \frac{x^3}{3}\right]_{h/2}^{h}$$

$$= \pi \frac{a^2}{h^2}\left\{\left[h^2(h) - \frac{(h)^3}{3}\right] - \left[h^2\left(\frac{h}{2}\right) - \frac{\left(\frac{h}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{5}{24}\pi a^2 h$$



PROBLEM 5.123 (Continued)

and
$$\int_2 \overline{x}_{EL}\, dV = \int_{h/2}^{h} x\left[\pi \frac{a^2}{h^2}(h^2 - x^2)dx\right]$$
$$= \pi \frac{a^2}{h^2}\left[h^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_{h/2}^{h}$$
$$= \pi \frac{a^2}{h^2}\left\{\left[h^2 \frac{(h)^2}{2} - \frac{(h)^4}{4}\right] - \left[h^2 \frac{\left(\frac{h}{2}\right)^2}{2} - \frac{\left(\frac{h}{2}\right)^4}{4}\right]\right\}$$
$$= \frac{9}{64}\pi a^2 h^2$$

Now
$$\overline{x}_2 V_2 = \int_2 \overline{x}_{EL}\, dV: \quad \overline{x}_2\left(\frac{5}{24}\pi a^2 h\right) = \frac{9}{64}\pi a^2 h^2$$
or $\overline{x}_2 = \frac{27}{40}h$ ◀

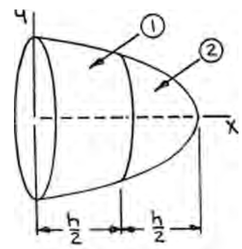
PROBLEM 5.124

Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A paraboloid of revolution

SOLUTION

Choose as the element of volume a disk of radius *r* and thickness *dx*. Then



$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

The equation of the generating curve is $x = h - \frac{h}{a^2}y^2$ so that $r^2 = \frac{a^2}{h}(h - x)$

and then $$dV = \pi \frac{a^2}{h}(h - x)dx$$

Component 1

$$V_1 = \int_0^{h/2} \pi \frac{a^2}{h}(h - x)dx$$

$$= \pi \frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_0^{h/2}$$

$$= \frac{3}{8}\pi a^2 h$$

and

$$\int_1 \bar{x}_{EL} dV = \int_0^{h/2} = x\left[\pi \frac{a^2}{h}(h - x)dx\right]$$

$$= \pi \frac{a^2}{h}\left[h\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{h/2} = \frac{1}{12}\pi a^2 h^2$$

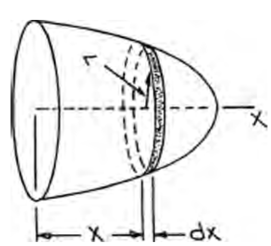
Now $$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{3}{8}\pi a^2 h\right) = \frac{1}{12}\pi a^2 h^2$$ or $\bar{x}_1 = \frac{2}{9}h$ ◀

Component 2

$$V_2 = \int_{h/2}^{h} \pi \frac{a^2}{h}(h - x)dx = \pi \frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_{h/2}^{h}$$

$$= \pi \frac{a^2}{h}\left\{\left[h(h) - \frac{(h)^2}{2}\right] - \left[h\left(\frac{h}{2}\right) - \frac{\left(\frac{h}{2}\right)^2}{2}\right]\right\}$$

$$= \frac{1}{8}\pi a^2 h$$



PROBLEM 5.124 (Continued)

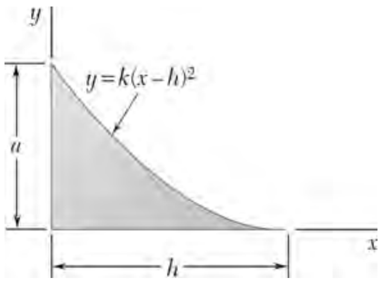
and $$\int_2 \overline{x}_{EL}\,dV = \int_{h/2}^{h} x\left[\pi\frac{a^2}{h}(h-x)dx\right] = \pi\frac{a^2}{h}\left[h\frac{x^2}{2}-\frac{x^3}{3}\right]_{h/2}^{h}$$

$$= \pi\frac{a^2}{h}\left\{\left[h\frac{(h)^2}{2}-\frac{(h)^3}{3}\right]-\left[h\frac{\left(\frac{h}{2}\right)^2}{2}-\frac{\left(\frac{h}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{1}{12}\pi a^2 h^2$$

Now $$\overline{x}_2 V_2 = \int_2 \overline{x}_{EL}\,dV: \quad \overline{x}_2\left(\frac{1}{8}\pi a^2 h\right) = \frac{1}{12}\pi a^2 h^2$$

or $\overline{x}_2 = \dfrac{2}{3}h$ ◀



PROBLEM 5.125

Locate the centroid of the volume obtained by rotating the shaded area about the x axis.

SOLUTION

First note that symmetry implies $\quad \overline{y} = 0$ ◀

and $\quad \overline{z} = 0$ ◀

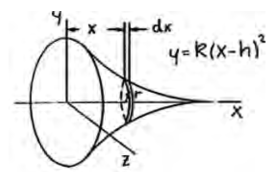
We have $\quad y = k(X - h)^2$

at $\quad x = 0, \quad y = a: \quad a = k(-h)^2$

or $\quad k = \dfrac{a}{h^2}$

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad \overline{X}_{EL} = x$$



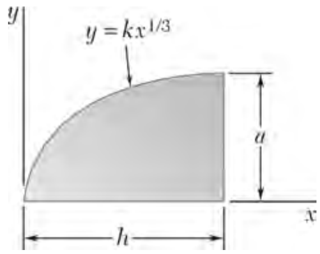
Now $\quad r = \dfrac{a}{h^2}(x-h)^2$

so that $\quad dV = \pi \dfrac{a^2}{h^4}(x-h)^4 dx$

Then
$$V = \int_0^h \pi \frac{a^2}{h^4}(x-h)^4 dx = \frac{\pi}{5}\frac{a^2}{h^4}\Big[(x-h)^5\Big]_0^h$$
$$= \frac{1}{5}\pi a^2 h$$

and
$$\int \overline{x}_{EL} dV = \int_0^h x\left[\pi \frac{a^2}{h^4}(x-h)^4 dx\right]$$
$$= \pi \frac{a^2}{h^4}\int_0^h (x^5 - 4hx^4 + 6h^2x^3 - 4h^3x^2 + h^4x)dx$$
$$= \pi \frac{a^2}{h^4}\left[\frac{1}{6}x^6 - \frac{4}{5}hx^5 + \frac{3}{2}h^2x^4 - \frac{4}{3}h^3x^3 + \frac{1}{2}h^4x^2\right]_0^h$$
$$= \frac{1}{30}\pi a^2 h^2$$

Now $\quad \overline{x}V = \int \overline{x}_{EL} dV: \quad \overline{x}\left(\dfrac{\pi}{5}a^2 h\right) = \dfrac{\pi}{30}a^2h^2 \qquad$ or $\quad \overline{x} = \dfrac{1}{6}h$ ◀



PROBLEM 5.126

Locate the centroid of the volume obtained by rotating the shaded area about the x axis.

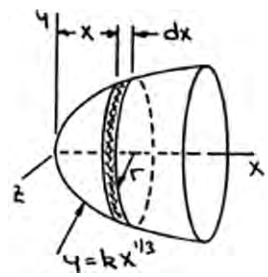
SOLUTION

First note that symmetry implies $\qquad \bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad x_{EL} = x$$



Now $\qquad r = kx^{1/3}$

so that $\qquad dV = \pi k^2 x^{2/3} dx$

at $x = h,\ y = a$: $\qquad a = kh^{1/3}$

or $\qquad k = \dfrac{a}{h^{1/3}}$

Then $\qquad dV = \pi \dfrac{a^2}{h^{2/3}} x^{2/3} dx$

and

$$V = \int_0^h \pi \frac{a^2}{h^{2/3}} x^{2/3} dx$$

$$= \pi \frac{a^2}{h^{2/3}} \left[\frac{3}{5} x^{5/3} \right]_0^h$$

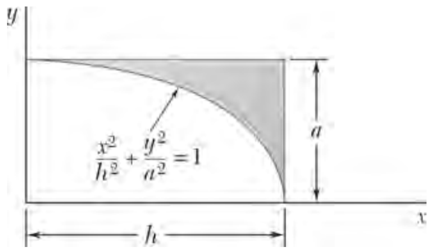
$$= \frac{3}{5} \pi a^2 h$$

Also

$$\int \bar{x}_{EL} dV = \int_0^h x = \left(\pi \frac{a^2}{h^{2/3}} x^{2/3} dx \right) = \pi \frac{a^2}{h^{2/3}} \left[\frac{3}{8} x^{8/3} \right]_0^h$$

$$= \frac{3}{8} \pi a^2 h^2$$

Now $\qquad \bar{x}V = \int \bar{x} dV: \qquad \bar{x}\left(\dfrac{3}{5} \pi a^2 h \right) = \dfrac{3}{8} \pi a^2 h^2 \qquad$ or $\bar{x} = \dfrac{5}{8} h$ ◀



PROBLEM 5.127

Locate the centroid of the volume obtained by rotating the shaded area about the line $x = h$.

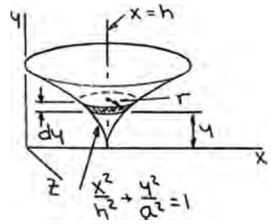
SOLUTION

First, note that symmetry implies $\bar{x} = h$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dy, \quad \bar{y}_{EL} = y$$



Now $x^2 = \frac{h^2}{a^2}(a^2 - y^2)$ so that $r = h - \frac{h}{a}\sqrt{a^2 - y^2}$

Then $$dV = \pi \frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

and $$V = \int_0^a \pi \frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

Let $$y = a \sin\theta \Rightarrow dy = a\cos\theta\, d\theta$$

Then
$$\begin{aligned}
V &= \pi \frac{h^2}{a^2}\int_0^{\pi/2}\left(a - \sqrt{a^2 - a^2\sin^2\theta}\right)^2 a\cos\theta\, d\theta \\
&= \pi \frac{h^2}{a^2}\int_0^{\pi/2}\left[a^2 - 2a(a\cos\theta) + (a^2 - a^2\sin^2\theta)\right] a\cos\theta\, d\theta \\
&= \pi a h^2 \int_0^{\pi/2}(2\cos\theta - 2\cos^2\theta - \sin^2\theta\cos\theta)\, d\theta \\
&= \pi a h^2\left[2\sin\theta - 2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} \\
&= \pi a h^2\left[2 - 2\left(\frac{\frac{\pi}{2}}{2}\right) - \frac{1}{3}\right] \\
&= 0.095870\pi a h^2
\end{aligned}$$

PROBLEM 5.127 (Continued)

and $$\int \bar{y}_{EL}\,dV = \int_0^a y\left[\pi\frac{h^2}{a^2}\left(a-\sqrt{a^2-y^2}\right)^2 dy\right]$$

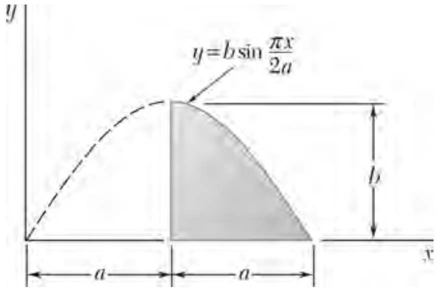
$$= \pi\frac{h^2}{a^2}\int_0^a\left(2a^2y - 2ay\sqrt{a^2-y^2} - y^3\right)dy$$

$$= \pi\frac{h^2}{a^2}\left[a^2y^2 + \frac{2}{3}a(a^2-y^2)^{3/2} - \frac{1}{4}y^4\right]_0^a$$

$$= \pi\frac{h^2}{a^2}\left\{\left[a^2(a)^2 - \frac{1}{4}a^4\right] - \left[\frac{2}{3}a(a^2)^{3/2}\right]\right\}$$

$$= \frac{1}{12}\pi a^2h^2$$

Now $$\bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}(0.095870\pi ah^2) = \frac{1}{12}\pi a^2h^2$$ or $\bar{y} = 0.869a$ ◀



PROBLEM 5.128*

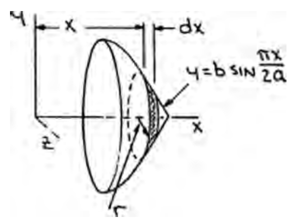
Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the x axis.

SOLUTION

First, note that symmetry implies $\qquad \bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx.



Then
$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

Now
$$r = b \sin \frac{\pi x}{2a}$$

so that
$$dV = \pi b^2 \sin^2 \frac{\pi x}{2a} dx$$

Then
$$\begin{aligned}
V &= \int_a^{2a} \pi b^2 \sin^2 \frac{\pi x}{2a} dx \\
&= \pi b^2 \left[\frac{x}{2} - \frac{\sin \frac{\pi x}{2a}}{2 \frac{\pi}{a}} \right]_a^{2a} \\
&= \pi b^2 \left[\left(\tfrac{2a}{2} \right) - \left(\tfrac{a}{2} \right) \right] \\
&= \frac{1}{2} \pi a b^2
\end{aligned}$$

and
$$\int \bar{x}_{EL} dV = \int_a^{2a} x \left(\pi b^2 \sin^2 \frac{\pi x}{2a} dx \right)$$

Use integration by parts with

$$u = x \qquad dV = \sin^2 \frac{\pi x}{2a}$$

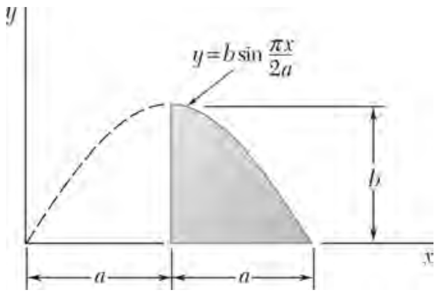
$$du = dx \qquad V = \frac{x}{2} - \frac{\sin \frac{\pi x}{a}}{\frac{2\pi}{a}}$$

PROBLEM 5.128* (Continued)

Then
$$\int \bar{x}_{EL}\,dV = \pi b^2\left\{\left[x\left(\frac{x}{2}-\frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{x}{2}-\frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}\right)dx\right\}$$
$$= \pi b^2\left\{\left[2a\left(\frac{2a}{2}\right)-a\left(\frac{a}{2}\right)\right]-\left[\frac{1}{4}x^2+\frac{a^2}{2\pi^2}\cos\frac{\pi x}{a}\right]_a^{2a}\right\}$$
$$= \pi b^2\left\{\left(\frac{3}{2}a^2\right)-\left[\frac{1}{4}(2a)^2+\frac{a^2}{2\pi^2}-\frac{1}{4}(a)^2+\frac{a^2}{2\pi^2}\right]\right\}$$
$$= \pi a^2 b^2\left(\frac{3}{4}-\frac{1}{\pi^2}\right)$$
$$= 0.64868\pi a^2 b^2$$

Now
$$\bar{x}V = \int \bar{x}_{EL}\,dV: \quad \bar{x}\left(\frac{1}{2}\pi a b^2\right) = 0.64868\pi a^2 b^2$$

or $\bar{x} = 1.297a$ ◀



PROBLEM 5.129*

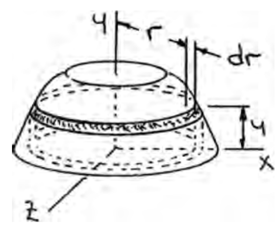
Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the *y* axis. (*Hint*: Use a thin cylindrical shell of radius *r* and thickness *dr* as the element of volume.)

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a cylindrical shell of radius *r* and thickness *dr*.



Then $$dV = (2\pi r)(y)(dr), \quad \bar{y}_{EL} = \frac{1}{2}y$$

Now $$y = b\sin\frac{\pi r}{2a}$$

so that $$dV = 2\pi br\sin\frac{\pi r}{2a}dr$$

Then $$V = \int_a^{2a} 2\pi br\sin\frac{\pi r}{2a}dr$$

Use integration by parts with

$$u = rd \qquad dv = \sin\frac{\pi r}{2a}dr$$

$$du = dr \qquad v = -\frac{2a}{\pi}\cos\frac{\pi r}{2a}$$

Then $$V = 2\pi b\left\{\left[(r)\left(-\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)dr\right\}$$

$$= 2\pi b\left\{-\frac{2a}{\pi}[(2a)(-1)] + \left[\frac{4a^2}{\pi^2}\sin\frac{\pi r}{2a}\right]_a^{2a}\right\}$$

$$V = 2\pi b\left(\frac{4a^2}{\pi} - \frac{4a^2}{\pi^2}\right)$$

$$= 8a^2b\left(1 - \frac{1}{\pi}\right)$$

$$= 5.4535a^2b$$

PROBLEM 5.129* (Continued)

Also
$$\int \bar{y}_{EL}\,dV = \int_a^{2a}\left(\tfrac{1}{2}b\sin\frac{\pi r}{2a}\right)\left(2\pi br\sin\frac{\pi r}{2a}dr\right)$$
$$= \pi b^2\int_a^{2a} r\sin^2\frac{\pi r}{2a}dr$$

Use integration by parts with

$$u = r \qquad dv = \sin^2\frac{\pi r}{2a}dr$$
$$du = dr \qquad v = \frac{r}{2} - \frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}$$

Then
$$\int \bar{y}_{EL}\,dV = \pi b^2\left\{\left[(r)\left(\frac{r}{2}-\frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{r}{2}-\frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}\right)dr\right\}$$
$$= \pi b^2\left\{\left[(2a)\left(\frac{2a}{2}\right)-(a)\left(\frac{a}{2}\right)\right]-\left[\frac{r^2}{4}+\frac{a^2}{2\pi^2}\cos\frac{\pi r}{a}\right]_a^{2a}\right\}$$
$$= \pi b^2\left\{\frac{3}{2}a^2-\left[\frac{(2a)^2}{4}+\frac{a^2}{2\pi^2}-\frac{(a)^2}{4}+\frac{a^2}{2\pi^2}\right]\right\}$$
$$= \pi a^2b^2\left(\frac{3}{4}-\frac{1}{\pi^2}\right)$$
$$= 2.0379a^2b^2$$

Now $\quad \bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}(5.4535a^2b) = 2.0379a^2b^2 \qquad$ or $\quad \bar{y} = 0.374b$ ◀

PROBLEM 5.130*

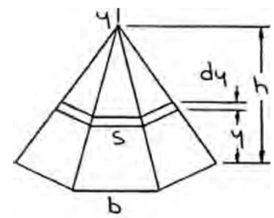
Show that for a regular pyramid of height h and n sides $(n = 3, 4, \ldots)$ the centroid of the volume of the pyramid is located at a distance $h/4$ above the base.

SOLUTION

Choose as the element of a horizontal slice of thickness dy. For any number N of sides, the area of the base of the pyramid is given by

$$A_{\text{base}} = kb^2$$

where $k = k(N)$; see note below. Using similar triangles, have



$$\frac{s}{b} = \frac{h-y}{h}$$

or $$s = \frac{b}{h}(h-y)$$

Then $$dV = A_{\text{slice}}dy = ks^2dy = k\frac{b^2}{h^2}(h-y)^2 dy$$

and $$V = \int_0^h k\frac{b^2}{h^2}(h-y)^2 dy = k\frac{b^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h$$

$$= \frac{1}{3}kb^2h$$

Also $$\bar{y}_{EL} = y$$

So then $$\int \bar{y}_{EL}dV = \int_0^h y\left[k\frac{b^2}{h^2}(h-y)^2 dy\right] = k\frac{b^2}{h^2}\int_0^h (h^2y - 2hy^2 + y^3)\,dy$$

$$= k\frac{b^2}{h^2}\left[\frac{1}{2}h^2y^2 - \frac{2}{3}hy^3 + \frac{1}{4}y^4\right]_0^h = \frac{1}{12}kb^2h^2$$

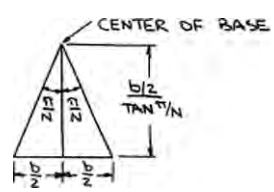
Now $$\bar{y}V = \int \bar{y}_{EL}dV: \quad \bar{y}\left(\frac{1}{3}kb^2h\right) = \frac{1}{12}kb^2h^2$$

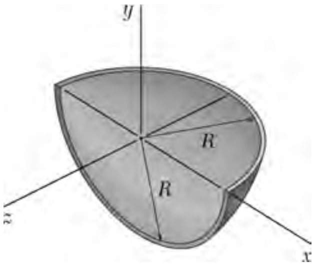
or $y = \dfrac{1}{4}h$ Q.E.D. ◀

Note $$A_{\text{base}} = N\left(\frac{1}{2}\times b \times \frac{\frac{b}{2}}{\tan\frac{\pi}{N}}\right)$$

$$= \frac{N}{4\tan\frac{\pi}{N}}b^2$$

$$= k(N)b^2$$





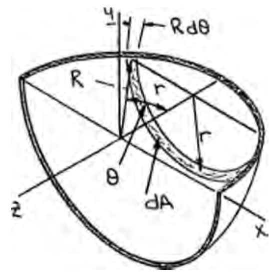
PROBLEM 5.131

Determine by direct integration the location of the centroid of one-half of a thin, uniform hemispherical shell of radius R.

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

The element of area dA of the shell shown is obtained by cutting the shell with two planes parallel to the xy plane. Now



$$dA = (\pi r)(R d\theta)$$

$$\bar{y}_{EL} = -\frac{2r}{\pi}$$

where $r = R\sin\theta$

so that $dA = \pi R^2 \sin\theta d\theta$

$$\bar{y}_{EL} = -\frac{2R}{\pi}\sin\theta$$

Then
$$A = \int_0^{\pi/2} \pi R^2 \sin\theta d\theta = \pi R^2[-\cos\theta]_0^{\pi/2}$$
$$= \pi R^2$$

and
$$\int \bar{y}_{EL} dA = \int_0^{\pi/2} \left(-\frac{2R}{\pi}\sin\theta\right)(\pi R^2 \sin\theta d\theta)$$
$$= -2R^3\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$
$$= -\frac{\pi}{2}R^3$$

Now $\bar{y}A = \int \bar{y}_{EL} dA$: $\quad \bar{y}(\pi R^2) = -\frac{\pi}{2}R^3$ or $\bar{y} = -\frac{1}{2}R$ ◀

Symmetry implies $\bar{z} = \bar{y}$ $\qquad \bar{z} = -\frac{1}{2}R$ ◀

PROBLEM 5.132

The sides and the base of a punch bowl are of uniform thickness t. If $t << R$ and R = 250 mm, determine the location of the center of gravity of (a) the bowl, (b) the punch.

SOLUTION

(a) Bowl

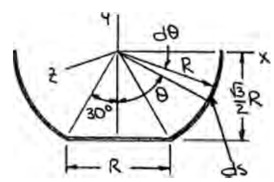
First note that symmetry implies $\qquad \bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

for the coordinate axes shown below. Now assume that the bowl may be treated as a shell; the center of gravity of the bowl will coincide with the centroid of the shell. For the walls of the bowl, an element of area is obtained by rotating the arc ds about the y axis. Then

$$dA_{\text{wall}} = (2\pi R \sin\theta)(Rd\theta)$$

and
$$(\bar{y}_{EL})_{\text{wall}} = -R\cos\theta$$



Then
$$A_{\text{wall}} = \int_{\pi/6}^{\pi/2} 2\pi R^2 \sin\theta d\theta$$
$$= 2\pi R^2[-\cos\theta]_{\pi/6}^{\pi/2}$$
$$= \pi\sqrt{3}R^2$$

and
$$\bar{y}_{\text{wall}} A_{\text{wall}} = \int (\bar{y}_{EL})_{\text{wall}}\, dA$$
$$= \int_{\pi/6}^{\pi/2} (-R\cos\theta)(2\pi R^2 \sin\theta d\theta)$$
$$= \pi R^3[\cos^2\theta]_{\pi/6}^{\pi/2}$$
$$= -\frac{3}{4}\pi R^3$$

By observation
$$A_{\text{base}} = \frac{\pi}{4}R^2, \quad \bar{y}_{\text{base}} = -\frac{\sqrt{3}}{2}R$$

Now
$$\bar{y}\Sigma A = \Sigma \bar{y} A$$

or
$$\bar{y}\left(\pi\sqrt{3}R^2 + \frac{\pi}{4}R^2\right) = -\frac{3}{4}\pi R^3 + \frac{\pi}{4}R^2\left(-\frac{\sqrt{3}}{2}R\right)$$

or
$$\bar{y} = -0.48763R \quad R = 250 \text{ mm}$$

$\bar{y} = -121.9$ mm ◀

PROBLEM 5.132 (Continued)

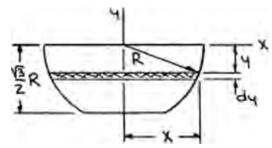
(*b*) Punch

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

and that because the punch is homogeneous, its center of gravity will coincide with the centroid of the corresponding volume. Choose as the element of volume a disk of radius *x* and thickness *dy*. Then

$$dV = \pi x^2 dy, \quad \bar{y}_{EL} = y$$



Now $x^2 + y^2 = R^2$

so that $dV = \pi(R^2 - y^2)dy$

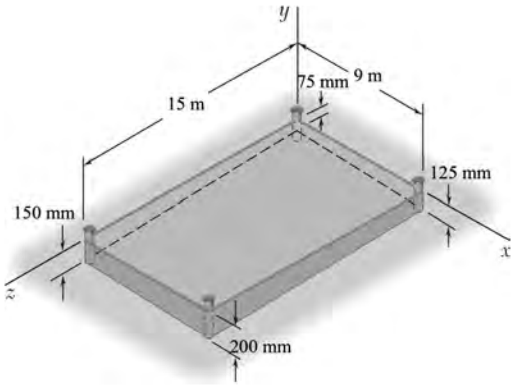
Then
$$V = \int_{-\sqrt{3}/2R}^{0} \pi(R^2 - y^2)\,dy$$
$$= \pi\left[R^2 y - \frac{1}{3}y^3\right]_{-\sqrt{3}/2R}^{0}$$
$$= -\pi\left[R^2\left(-\frac{\sqrt{3}}{2}R\right) - \frac{1}{3}\left(-\frac{\sqrt{3}}{2}R\right)^3\right] = \frac{3}{8}\pi\sqrt{3}R^3$$

and
$$\int \bar{y}_{EL} dV = \int_{-\sqrt{3}/2R}^{0} (y)\left[\pi\left(R^2 - y^2\right)dy\right]$$
$$= \pi\left[\frac{1}{2}R^2 y^2 - \frac{1}{4}y^4\right]_{-\sqrt{3}/2R}^{0}$$
$$= -\pi\left[\frac{1}{2}R^2\left(-\frac{\sqrt{3}}{2}R\right)^2 - \frac{1}{4}\left(-\frac{\sqrt{3}}{2}R\right)^4\right] = -\frac{15}{64}\pi R^4$$

Now
$$\bar{y}V = \int \bar{y}_{EL} dV: \quad \bar{y}\left(\frac{3}{8}\pi\sqrt{3}R^3\right) = -\frac{15}{64}\pi R^4$$

or
$$\bar{y} = -\frac{5}{8\sqrt{3}}R \quad R = 250 \text{ mm}$$

$\bar{y} = -90.2$ mm ◀



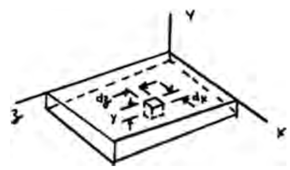
PROBLEM 5.133

After grading a lot, a builder places four stakes to designate the corners of the slab for a house. To provide a firm, level base for the slab, the builder places a minimum of 75 mm of gravel beneath the slab. Determine the volume of gravel needed and the x coordinate of the centroid of the volume of the gravel. (*Hint*: The bottom surface of the gravel is an oblique plane, which can be represented by the equation $y = a + bx + cz$.)

SOLUTION

The centroid can be found by integration. The equation for the bottom of the gravel is:

$y = a + bx + cz,$ where the constants a, b, and c can be determined as follows:



For $x = 0,$ and $z = 0$: $\quad y = -75$ mm, and therefore

$$-\frac{75}{1000}\text{ m} = a, \quad \text{or} \quad a = -0.075\text{ m}$$

For $x = 90$ m, and $z = 0$: $\quad y = -125\text{ mm}, = -0.125\text{ m}$ and therefore

$$-0.125\text{ m} = -0.075 + b(9\text{ m}), \qquad \text{or} \quad b = -\frac{1}{180}$$

For $x = 0,$ and $z = 15\text{ m}$: $\quad y = -150\text{ mm} = -0.15\text{ m},$ and therefore

$$-0.15\text{m} = -0.075\text{ m} + C(15\text{ m}), \qquad \text{or} \quad C = -\frac{1}{200}$$

Therefore:

$$y = -0.075\text{ m} - \frac{1}{180}x - \frac{1}{200}z$$

Now

$$\bar{x} = \frac{\int x_{EL}\, dV}{V}$$

A volume element can be chosen as:

$$dV = |y|\, dxdz$$

or

$$dV = \left[0.075 + \frac{1}{180}x + \frac{1}{200}z\right] dx\, dz$$

and

$$\bar{x}_{EL} = x$$

PROBLEM 5.133 (Continued)

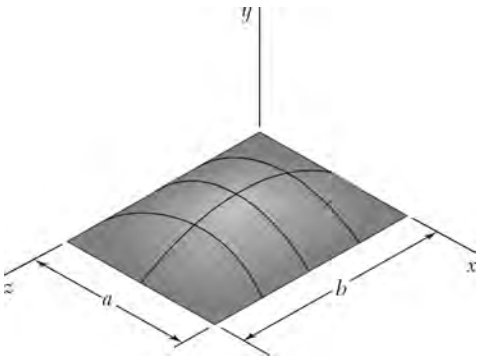
Then
$$\int x_{EL}\,dV = \int_0^{15}\int_0^9 x\left(0.075+\frac{1}{180}x+\frac{1}{200}z\right)dx\,dz$$
$$=\int_0^{15}\left[0.075\,\frac{x^2}{2}+\frac{1}{540}x^3+\frac{zx^2}{400}\right]_0^9 dz$$
$$=\int_0^{15}\left(\frac{351}{80}+\frac{81}{400}z\right)dz$$
$$=\left[\frac{351}{80}z+\frac{81}{800}z^2\right]_0^{15}$$
$$=\frac{1053}{16}+\frac{729}{32}=\frac{2835}{32}$$
$$=88.593\,\text{m}^4$$

The volume is:
$$V\int dV = \int_0^{15}\int_0^9\left(0.075+\frac{1}{180}x+\frac{1}{200}z\right)dx\,dz$$
$$=\int_0^{15}\left[0.075x+\frac{x^2}{360}+\frac{z}{200}x\right]_0^9 dz$$
$$=\int_0^{15}\left(\frac{9}{10}+\frac{9z}{200}\right)dz$$
$$=\left[\frac{9z}{10}+\frac{9}{400}z^2\right]_0^{15}$$
$$\frac{297}{16}=18.5625\text{ m}^3$$

Then
$$\bar{x}=\frac{\int \bar{x}_{EL}\,dV}{V}=\frac{88.593\,\text{m}^4}{18.5625\,\text{m}^3}=4.7727\,\text{m}$$

Therefore: $V = 18.56\text{ m}^3$ ◄

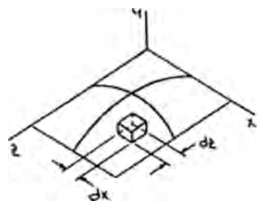
$\bar{x} = 4.77\text{ m}$ ◄



PROBLEM 5.134

Determine by direct integration the location of the centroid of the volume between the *xz* plane and the portion shown of the surface $y = 16h(ax - x^2)(bz - z^2)/a^2b^2$.

SOLUTION



First note that symmetry implies $\bar{x} = \dfrac{a}{2}$ ◀

$$\bar{z} = \frac{b}{2} \quad ◀$$

Choose as the element of volume a filament of base $dx \times dz$ and height y. Then

$$dV = y\,dx\,dz, \quad \bar{y}_{EL} = \frac{1}{2}y$$

or $$dV = \frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)\,dx\,dz$$

Then $$V = \int_0^b \int_0^a \frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)\,dx\,dz$$

$$V = \frac{16h}{a^2b^2}\int_0^b (bz - z^2)\left[\frac{a}{z}x^2 - \frac{1}{3}x^3\right]_0^a dz$$

$$= \frac{16h}{a^2b^2}\left[\frac{a}{2}(a^2) - \frac{1}{3}(a)^3\right]\left[\frac{b}{2}z^2 - \frac{1}{3}z^3\right]_0^b$$

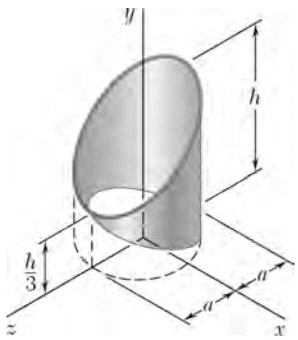
$$= \frac{8ah}{3b^2}\left[\frac{b}{2}(b)^2 - \frac{1}{3}(b)^3\right]$$

$$= \frac{4}{9}abh$$

PROBLEM 5.134 (Continued)

and
$$\int \bar{y}_{EL}\,dV = \int_0^b\int_0^a \frac{1}{2}\left[\frac{16h}{a^2b^2}(ax-x^2)(bz-z^2)\right]\left[\frac{16h}{a^2b^2}(ax-x^2)(bz-z^2)dx\,dz\right]$$
$$= \frac{128h^2}{a^4b^4}\int_0^b\int_0^a (a^2x^2-2ax^3+x^4)(b^2z^2-2bz^3+z^4)dx\,dz$$
$$= \frac{128h^2}{a^2b^4}\int_0^b (b^2z^2-2bz^3+z^4)\left[\frac{a^2}{3}x^3-\frac{a}{2}x^4+\frac{1}{5}x^5\right]_0^a dz$$
$$= \frac{128h^2}{a^4b^4}\left[\frac{a^2}{3}(a)^3-\frac{a}{2}(a)^4+\frac{1}{5}(a)^5\right]\left[\frac{b^2}{3}Z^3-\frac{b}{Z}Z^4+\frac{1}{5}Z^5\right]_0^b$$
$$= \frac{64ah^2}{15b^4}\left[\frac{b^3}{3}(b)^3-\frac{b}{2}(b)^4+\frac{1}{5}(b)^5\right] = \frac{32}{225}abh^2$$

Now
$$\bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}\left(\frac{4}{9}abh\right) = \frac{32}{225}abh^2$$
or $\bar{y} = \dfrac{8}{25}h$ ◀



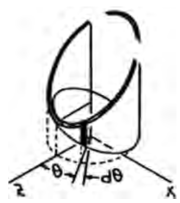
PROBLEM 5.135

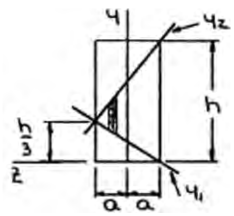
Locate the centroid of the section shown, which was cut from a thin circular pipe by two oblique planes.

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

Assume that the pipe has a uniform wall thickness t and choose as the element of volume A vertical strip of width $ad\theta$ and height $(y_2 - y_1)$. Then





$$dV = (y_2 - y_1)tad\theta, \quad \bar{y}_{EL} = \frac{1}{2}(y_1 + \bar{y}_2)\,\bar{z}_{EL} = z$$

Now
$$y_1 = \frac{\frac{h}{3}}{2a}z + \frac{h}{6} \qquad y_2 = -\frac{\frac{2h}{3}}{2a}z + \frac{2}{3}h$$

$$= \frac{h}{6a}(z+a) \qquad = \frac{h}{3a}(-z+2a)$$

and
$$z = a\cos\theta$$

Then
$$(y_2 - y_1) = \frac{h}{3a}(-a\cos\theta + 2a) - \frac{h}{6a}(a\cos\theta + a)$$

$$= \frac{h}{2}(1-\cos\theta)$$

and
$$(y_1 + y_2) = \frac{h}{6a}(a\cos\theta + a) + \frac{h}{3a}(-a\cos\theta + 2a)$$

$$= \frac{h}{6}(5-\cos\theta)$$

$$dV = \frac{aht}{2}(1-\cos\theta)d\theta \quad \bar{y}_{EL} = \frac{h}{12}(5-\cos\theta), \quad \bar{z}_{EL} = a\cos\theta$$

PROBLEM 5.135 (Continued)

Then $$V = 2\int_0^{\pi} \frac{aht}{2}(1-\cos\theta)d\theta = aht[\theta - \sin\theta]_0^{\pi}$$

$$= \pi aht$$

and $$\int \bar{y}_{EL} dV = 2\int_0^{\pi} \frac{h}{12}(5-\cos\theta)\left[\frac{aht}{2}(1-\cos\theta)d\theta\right]$$

$$= \frac{ah^2t}{12}\int_0^{\pi}(5-6\cos\theta+\cos^2\theta)d\theta$$

$$= \frac{ah^2t}{12}\left[5\theta - 6\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= \frac{11}{24}\pi ah^2t$$

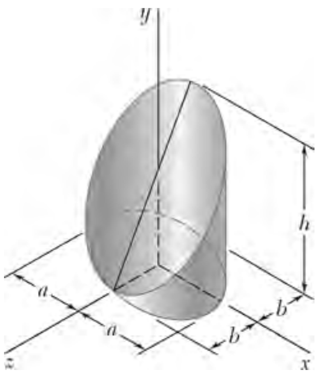
$$\int \bar{z}_{EL} dV = 2\int_0^{\pi} a\cos\theta\left[\frac{aht}{2}(1-\cos\theta)d\theta\right]$$

$$= a^2ht\left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= -\frac{1}{2}\pi a^2ht$$

Now $$\bar{y}V = \int \bar{y}_{EL} dV: \bar{y}(\pi aht) = \frac{11}{24}\pi ah^2t$$ or $\bar{y} = \frac{11}{24}h$ ◀

and $$\bar{z}V = \int \bar{y}_{EL} dV: \ \bar{z}(\pi aht) = -\frac{1}{2}\pi a^2ht$$ or $\bar{z} = -\frac{1}{2}a$ ◀

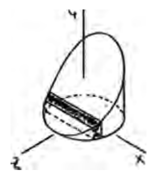


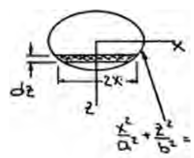
PROBLEM 5.136*

Locate the centroid of the section shown, which was cut from an elliptical cylinder by an oblique plane.

SOLUTION

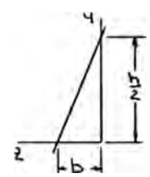
First note that symmetry implies $x = 0$ ◀





Choose as the element of volume a vertical slice of width *zx*, thickness *dz*, and height *y*. Then

$$dV = 2xydz, \quad \bar{y}_{EL} = \frac{1}{24}, \quad \bar{z}_{EL} = z$$



Now $$x = \frac{a}{b}\sqrt{b^2 - z^2}$$

and $$y = -\frac{h/2}{b}z + \frac{h}{2} = \frac{h}{2b}(b - z)$$

Then $$V = \int_{-b}^{b}\left(2\frac{a}{b}\sqrt{b^2 - z^2}\right)\left[\frac{h}{2b}(b - z)\right]dz$$

Let $$z = b\sin\theta \quad dz = b\cos\theta d\theta$$

Then $$V = \frac{ah}{b^2}\int_{\pi/2}^{\pi/2}(b\cos\theta)[b(1 - \sin\theta)]b\cos\theta\, d\theta$$

$$= abh\int_{\pi/2}^{\pi/2}(\cos^2\theta - \sin\theta\cos^2\theta)\,d\theta$$

$$= abh\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2}$$

$$V = \frac{1}{2}\pi abh$$

and $$\int \bar{y}_{EL}dV = \int_{-b}^{b}\left[\frac{1}{2}\times\frac{h}{2b}(b-z)\right]\left\{\left(2\frac{a}{b}\sqrt{b^2-z^2}\right)\left[\frac{h}{2b}(b-z)\right]dz\right\}$$

$$= \frac{1}{4}\frac{ah^2}{b^3}\int_{-b}^{b}(b-z)^2\sqrt{b^2-z^2}\,dz$$

Let $$z = b\sin\theta \quad dz = b\cos\theta d\theta$$

Then $$\int \bar{y}_{EL}dV = \frac{1}{4}\frac{ah^2}{b^3}\int_{-\pi/2}^{\pi/2}[b(1-\sin\theta)]^2(b\cos\theta)\times(b\cos\theta\,d\theta)$$

$$= \frac{1}{4}abh^2\int_{-\pi/2}^{\pi/2}(\cos^2\theta-2\sin\theta\cos^2\theta+\sin^2\theta\cos^2\theta)\,d\theta$$

Now $$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \quad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

So that $$\sin^2\theta\cos^2\theta = \frac{1}{4}(1-\cos^2 2\theta)$$

Then $$\int \bar{y}_{EL}dV = \frac{1}{4}abh^2\int_{-\pi/2}^{\pi/2}\left[\cos^2\theta-2\sin\theta\cos^2\theta+\frac{1}{4}(1-\cos^2 2\theta)\right]d\theta$$

$$= \frac{1}{4}abh^2\left[\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)+\frac{1}{3}\cos^3\theta+\frac{1}{4}\theta-\frac{1}{4}\left(\frac{\theta}{2}+\frac{\sin 4\theta}{8}\right)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{5}{32}\pi abh^2$$

Also $$\int \bar{z}_{EL}dV = \int_{-b}^{b}\left\{2\frac{a}{b}\sqrt{a^2-z^2}\left[\frac{h}{2b}(b-z)\right]dz\right\}$$

$$= \frac{ah}{b^2}\int_{-b}^{b}z(b-z)\sqrt{b^2-z^2}\,dz$$

Let $$z = b\sin\theta \quad dz = b\cos\theta\,d\theta$$

Then $$\int \bar{z}_{EL}dV = \frac{ah}{b^2}\int_{-\pi/2}^{\pi/2}(b\sin\theta)[b(1-\sin\theta)](b\cos\theta)\times(b\cos\theta\,d\theta)$$

$$= ab^2h\int_{-\pi/2}^{\pi/2}(\sin\theta\cos^2\theta-\sin^2\theta\cos^2\theta)d\theta$$

Using $$\sin^2\theta\cos^2\theta = \frac{1}{4}(1-\cos^2 2\theta) \text{ from above}$$

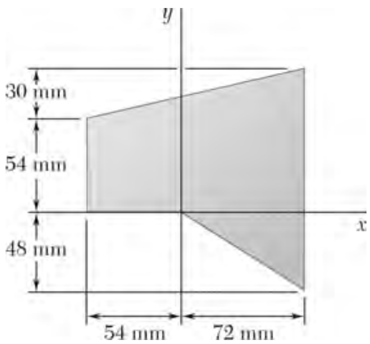
$$\int z_{EL}dV = ab^2h\int_{-\pi/2}^{\pi/2}\left[\sin\theta\cos^2\theta-\frac{1}{4}(1-\cos^2 2\theta)\right]d\theta$$

PROBLEM 5.136* (Continued)

$$= ab^2h\left[-\frac{1}{3}\cos^3\theta - \frac{1}{4}\theta + \frac{1}{4}\left(\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right)\right]_{-\pi/2}^{\pi/2} = -\frac{1}{8}\pi ab^2h$$

Now $\quad \bar{y}V = \int \bar{y}_{EL}dV: \quad \bar{y}\left(\frac{1}{2}\pi\, abh\right) = \frac{5}{32}\pi\, abh^2$ $\quad$ or $\quad \bar{y} = \frac{5}{16}h$ ◀

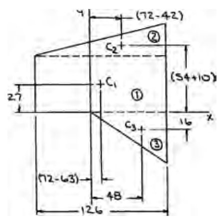
and $\quad \bar{z}V = \int \bar{z}_{EL}dV: \quad \bar{z}\left(\frac{1}{2}\pi\, abh\right) = -\frac{1}{8}\pi\, ab^2h$ $\quad$ or $\quad \bar{z} = -\frac{1}{4}b$ ◀



PROBLEM 5.137

Locate the centroid of the plane area shown.

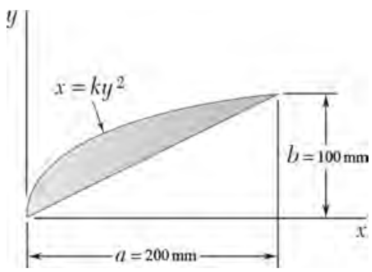
SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$126 \times 54 = 6804$	9	27	61236	183708
2	$\frac{1}{2} \times 126 \times 30 = 1890$	30	64	56700	120960
3	$\frac{1}{2} \times 72 \times 48 = 1728$	48	−16	82944	−27648
Σ	10422			200880	277020

Then
$$\bar{X}\Sigma A = \Sigma \bar{x} A$$
$$\bar{X}(10422 \text{ m}^2) = 200880 \text{ mm}^2 \qquad \text{or} \quad \bar{X} = 19.27 \text{ mm} \blacktriangleleft$$

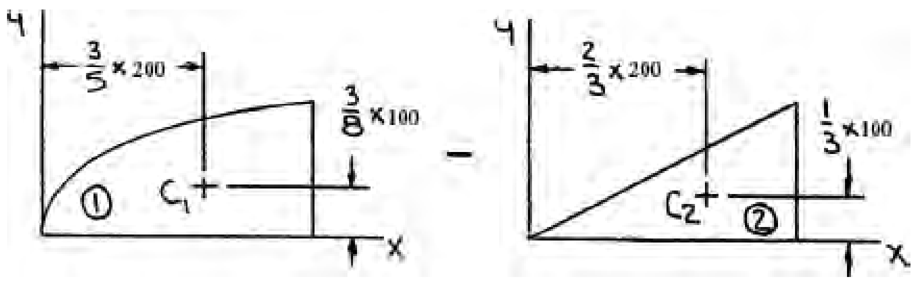
and
$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$
$$\bar{Y}(10422 \text{ m}^2) = 270020 \text{ mm}^3 \qquad \text{or} \quad \bar{Y} = 26.6 \text{ mm} \blacktriangleleft$$



PROBLEM 5.138

Locate the centroid of the plane area shown.

SOLUTION



Dimensions in in.

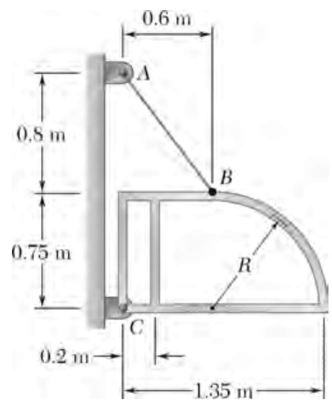
	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{2}{3}(100)(200) = 13333.33$	120	37.5	1.6 10^6	0.5 10^6
2	$-\frac{1}{2}(100)(200) = -10000$	133.333	33.333	–1.3333 10^6	–0.3333 10^6
Σ	3333.33			0.26667 10^6	0.16667 10^6

Then $$\bar{X}\Sigma A = \Sigma \bar{x}A$$

$$\bar{X}(3333.33\text{ mm}^2) = 0.26667\times 10^6\text{ mm}^3$$ or $\bar{X} = 80$ mm ◀

and $$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(3333.33\text{ mm}^2) = 0.166667\times 10^6\text{ mm}^3$$ or $\bar{Y} = 50$ mm ◀

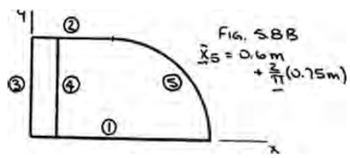


PROBLEM 5.139

The frame for a sign is fabricated from thin, flat steel bar stock of mass per unit length 4.73 kg/m. The frame is supported by a pin at *C* and by a cable *AB*. Determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

SOLUTION

First note that because the frame is fabricated from uniform bar stock, its center of gravity will coincide with the centroid of the corresponding line.

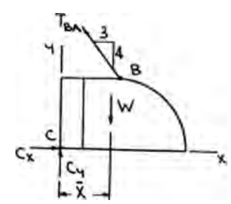


	L, m	$\bar{x}$, m	$\bar{x}L$, m^2
1	1.35	0.675	0.91125
2	0.6	0.3	0.18
3	0.75	0	0
4	0.75	0.2	0.15
5	$\frac{\pi}{2}(0.75) = 1.17810$	1.07746	1.26936
Σ	4.62810		2.5106

Then
$$\bar{X}\Sigma L = \Sigma \bar{x} L$$
$$\bar{X}(4.62810) = 2.5106$$

or
$$\bar{X} = 0.54247 \text{ m}$$

The free-body diagram of the frame is then



where
$$W = (m'\Sigma L)g$$
$$= 4.73 \text{ kg/m} \times 4.62810 \text{ m} \times 9.81 \text{ m/s}^2$$
$$= 214.75 \text{ N}$$

PROBLEM 5.139 (Continued)

Equilibrium then requires

(*a*) $$\Sigma M_C = 0: \quad (1.55\text{ m})\left(\frac{3}{5}T_{BA}\right) - (0.54247\text{ m})(214.75\text{ N}) = 0$$

or $T_{BA} = 125.264\text{ N}$ or $T_{BA} = 125.3\text{ N}$ ◀

(*b*) $$\Sigma F_x = 0: \quad C_x - \frac{3}{5}(125.264\text{ N}) = 0$$

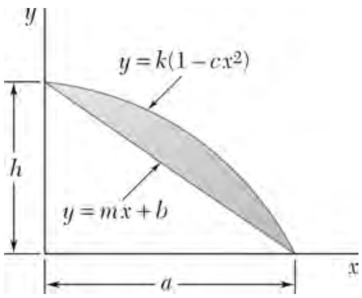
or $\mathbf{C}_x = 75.158\text{ N}$

$$\Sigma F_y = 0: \quad C_y + \frac{4}{5}(125.264\text{ N}) - (214.75\text{ N}) = 0$$

or $\mathbf{C}_y = 114.539\text{ N}\uparrow$

Then $\mathbf{C} = 137.0\text{ N} \measuredangle 56.7°$ ◀

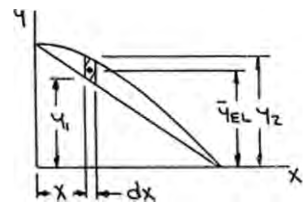
PROBLEM 5.140



Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION

By observation $$y_1 = -\frac{h}{a}x + h = h\left(1 - \frac{x}{a}\right)$$



For y_2: at $$x = 0, \quad y = h: \quad h = k(1-0) \quad \text{or} \quad k = h$$

at $$x = a, \quad y = 0: \quad 0 = h(1 - ca^2) \quad \text{or} \quad C = \frac{1}{a^2}$$

Then $$y_2 = h\left(1 - \frac{x^2}{a^2}\right)$$

Now $$dA = (y_2 - y_1)dx$$

$$= h\left[\left(1 - \frac{x^2}{a^2}\right) - \left(1 - \frac{x}{a}\right)\right]dx$$

$$= h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)dx$$

$$\overline{x}_{EL} = x$$

$$\overline{y}_{EL} = \frac{1}{2}(y_1 - y_2)$$

$$= \frac{h}{2}\left[\left(1 - \frac{x}{a}\right) + \left(1 - \frac{x^2}{a^2}\right)\right]$$

$$= \frac{h}{2}\left(2 - \frac{x}{a} - \frac{x^2}{a^2}\right)$$

Then $$A = \int dA = \int_0^a h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)dx = h\left[\frac{x^2}{2a} - \frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{1}{6}ah$$

PROBLEM 5.140 (Continued)

and $$\int \bar{x}_{EL}\,dA = \int_0^a x\left[h\left(\frac{x}{a}-\frac{x^2}{a^2}\right)dx\right] = h\left[\left(\frac{x^3}{3a}-\frac{x^4}{4a^2}\right)\right]_0^a$$

$$= \frac{1}{12}a^2h$$

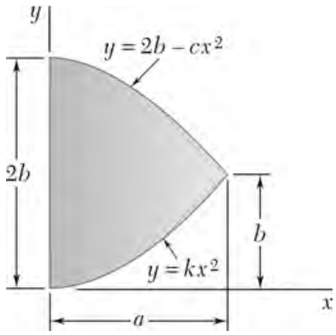
$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{h}{2}\left(2-\frac{x}{a}-\frac{x^2}{a^2}\right)\left[h\left(\frac{x}{a}-\frac{x^2}{a^2}\right)dx\right]$$

$$= \frac{h^2}{2}\int_0^a \left(2\frac{x}{a}-3\frac{x^2}{a^2}+\frac{x^4}{a^4}\right)dx$$

$$= \frac{h^2}{2}\left[\frac{x^2}{a}-\frac{x^3}{a^2}+\frac{x^5}{5a^4}\right]_0^a = \frac{1}{10}ah^2$$

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad x\left(\frac{1}{6}ah\right) = \frac{1}{12}a^2h \qquad \bar{x} = \frac{1}{2}a \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad y\left(\frac{1}{6}ah\right) = \frac{1}{10}a^2h \qquad \bar{y} = \frac{3}{5}h \blacktriangleleft$$



PROBLEM 5.141

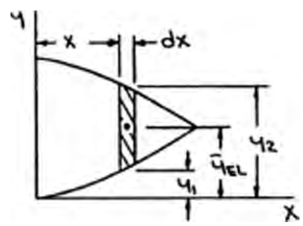
Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION

First note that symmetry implies $\bar{y} = b$ ◀

at $x = a, \quad y = b$

$$y_1: \quad b = ka^2 \quad \text{or} \quad k = \frac{b}{a^2}$$



Then $$y_1 = \frac{b}{a^2}x^2$$

$$y_2: \quad b = 2b - ca^2$$

or $$c = \frac{b}{a^2}$$

Then $$y_2 = b\left(2 - \frac{x^2}{a^2}\right)$$

Now $$dA = (y_2 - y_1)dx_2 = \left[b\left(2 - \frac{x^2}{a^2}\right) - \frac{b}{a^2}x^2\right]dx$$

$$= 2b\left(1 - \frac{x^2}{a^2}\right)dx$$

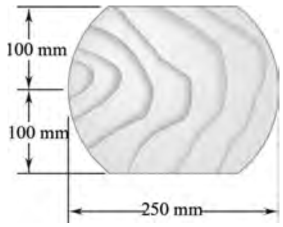
and $$\bar{x}_{EL} = x$$

Then $$A = \int dA \int_0^a 2b\left(1 - \frac{x^2}{a^2}\right)dx = 2b\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{4}{3}ab$$

and $$\int \bar{x}_{EL}dA = \int_0^a x\left[2b\left(1 - \frac{x^2}{a^2}\right)dx\right] = 2b\left[\frac{x^2}{2} - \frac{x^4}{4a^2}\right]_0^a = \frac{1}{2}a^2b$$

$$\bar{x}A = \int \bar{x}_{EL}dA \quad \bar{x}\left(\frac{4}{3}ab\right) = \frac{1}{2}a^2b$$

$\bar{x} = \frac{3}{8}a$ ◀



PROBLEM 5.142

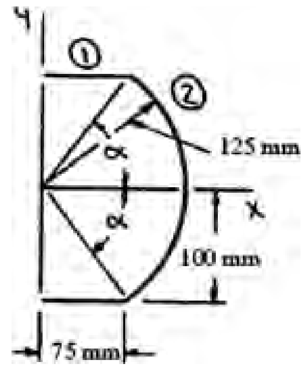
Knowing that two equal caps have been removed from a 250-mm-diameter wooden sphere, determine the total surface area of the remaining portion.

SOLUTION

The surface area can be generated by rotating the line shown about the y axis. Applying the first theorem of Pappus-Guldinus, we have

$$A = 2\pi \bar{X} L = 2\pi \Sigma \bar{x} L$$

$$= 2\pi(2\bar{x}_1 L_1 + \bar{x}_2 L_2)$$



Now $\tan\alpha = \dfrac{4}{3}$

or $\alpha = 53.130°$

Then $$\bar{x}_2 = \frac{125 \text{ mm} \times \sin 53.130°}{53.130° \times \frac{\pi}{180°}}$$

$$= 107.84 \text{ mm}$$

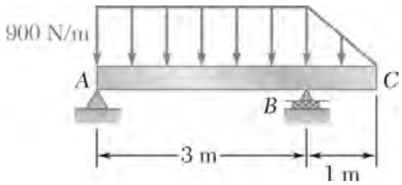
and $$L_2 = 2\left(53.130° \times \frac{\pi}{180°}\right)(125 \text{ mm})$$

$$= 231.8225 \text{ mm}$$

$$A = 2\pi\left[2\left(\frac{75}{2}\text{ mm}\right)(75 \text{ mm}) + (107.84 \text{ mm})(231.8225 \text{ mm})\right]$$

$$= 192420.9 \text{ mm}^2$$

or $A = 192.4 \times 10^3 \text{ mm}^2$ ◀



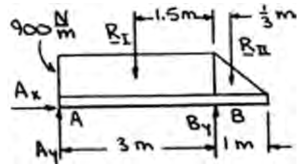
PROBLEM 5.143

Determine the reactions at the beam supports for the given loading.

SOLUTION

$$R_{\mathrm{I}} = (3\text{ m})(900\text{ N/m}) = 2700\text{ N}$$

$$R_{\mathrm{II}} = \frac{1}{2}(1\text{ m})(900\text{ N/m}) = 450\text{ N}$$



Now $\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$

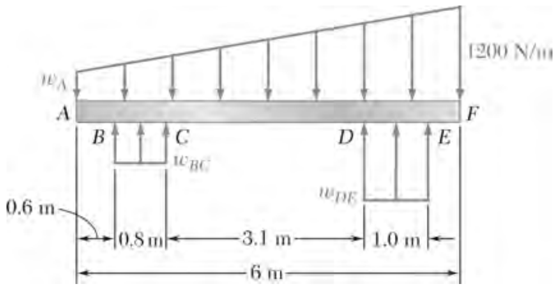
$$\circlearrowleft + \Sigma M_B = 0: \quad -(3\text{ m})A_y + (1.5\text{ m})(2700\text{ N}) - \left(\frac{1}{3}\text{ m}\right)(450\text{ N}) = 0$$

or $A_y = 1300\text{ N}$ $\quad \mathbf{A} = 1300\text{ N}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad 1300\text{ N} - 2700\text{ N} + B_y - 450\text{ N} = 0$$

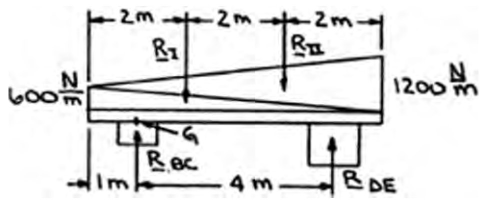
or $B_y = 1850\text{ N}$ $\quad \mathbf{B} = 1850\text{ N}\uparrow$ ◀

PROBLEM 5.144



A beam is subjected to a linearly distributed downward load and rests on two wide supports *BC* and *DE*, which exert uniformly distributed upward loads as shown. Determine the values of w_{BC} and w_{DE} corresponding to equilibrium when $w_A = 600$ N/m.

SOLUTION



We have $$R_{\mathrm{I}} = \frac{1}{2}(6\text{ m})(600\text{ N/m}) = 1800\text{ N}$$

$$R_{\mathrm{II}} = \frac{1}{2}(6\text{ m})(1200\text{ N/m}) = 3600\text{ N}$$

$$R_{BC} = (0.8\text{ m})(W_{BC}\text{ N/m}) = (0.8W_{BC})\text{N}$$

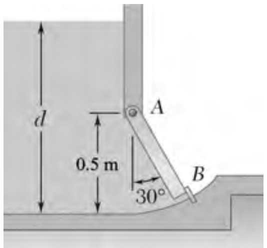
$$R_{DE} = (1.0\text{ m})(W_{DE}\text{ N/m}) = (W_{DE})\text{ N}$$

Then $$+\circlearrowleft \Sigma M_G = 0: \quad -(1\text{ m})(1800\text{ N}) - (3\text{ m})(3600\text{ N}) + (4\text{ m})(W_{DE}\text{ N}) = 0$$

or $$W_{DE} = 3150\text{ N/m} \blacktriangleleft$$

and $$+\uparrow \Sigma F_y = 0: \quad (0.8W_{BC})\text{ N} - 1800\text{ N} - 3600\text{ N} + 3150\text{ N} = 0$$

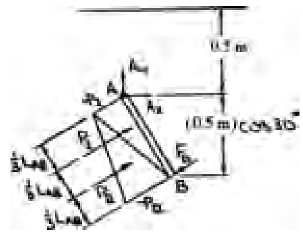
or $$W_{BC} = 2812.5\text{ N/m} \qquad W_{BC} = 2810\text{ N/m} \blacktriangleleft$$



PROBLEM 5.145

The square gate *AB* is held in the position shown by hinges along its top edge *A* and by a shear pin at *B*. For a depth of water $d = 1\,\text{m}$, determine the force exerted on the gate by the shear pin.

SOLUTION



First consider the force of the water on the gate. We have

$$P = \frac{1}{2}Ap$$

$$= \frac{1}{2}A(\delta gh)$$

Then

$$P_{\text{I}} = \frac{1}{2}(0.5\,\text{m})^2(1000 \times 9.81\ \text{N/m}^3)(0.5\ \text{m})$$

$$= 613.125\,\text{N}$$

$$P_{\text{II}} = \frac{1}{2}(0.5\text{m})^2(1000 \times 9.81\ \text{N/m}^3) \times (0.5 + 0.5\cos 30°)\ \text{m}$$

$$= 1144.107\,\text{N}$$

Now

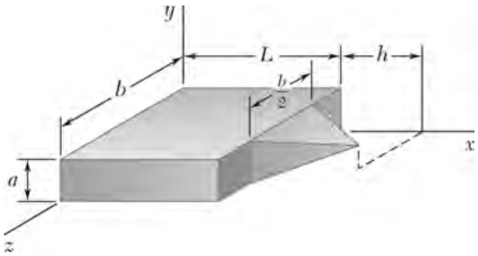
$$\Sigma M_A = 0: \quad \left(\frac{1}{3}L_{AB}\right)P_{\text{I}} + \left(\frac{2}{3}L_{AB}\right)P_{\text{II}} - L_{AB}F_B = 0$$

or

$$\frac{1}{3}(613.125\ \text{N}) + \frac{2}{3}(1144.107\ \text{N}) - F_B = 0$$

or

$$F_B = 967.113\ \text{N} \qquad \mathbf{F}_B = 967\ \text{N} \measuredangle\ 30.0° \blacktriangleleft$$



PROBLEM 5.146

Consider the composite body shown. Determine (*a*) the value of $\bar{x}$ when $h = L/2$, (*b*) the ratio *h/L* for which $\bar{x} = L$.

SOLUTION

	V	$\bar{x}$	$\bar{x}V$
Rectangular prism	Lab	$\frac{1}{2}L$	$\frac{1}{2}L^2ab$
Pyramid	$\frac{1}{3}a\left(\frac{b}{2}\right)h$	$L+\frac{1}{4}h$	$\frac{1}{6}abh\left(L+\frac{1}{4}h\right)$

Then $$\Sigma V = ab\left(L+\frac{1}{6}h\right)$$

$$\Sigma \bar{x}V = \frac{1}{6}ab\left[3L^2 + h\left(L+\frac{1}{4}h\right)\right]$$

Now $$\bar{X}\,\Sigma V = \Sigma \bar{x}V$$

so that $$\bar{X}\left[ab\left(L+\frac{1}{6}h\right)\right] = \frac{1}{6}ab\left(3L^2 + hL + \frac{1}{4}h^2\right)$$

or $$\bar{X}\left(1+\frac{1}{6}\frac{h}{L}\right) = \frac{1}{6}L\left(3+\frac{h}{L}+\frac{1}{4}\frac{h^2}{L^2}\right) \qquad (1)$$

(*a*) $\bar{X} = ?$ when $h = \frac{1}{2}L$

Substituting $\frac{h}{L} = \frac{1}{2}$ into Eq. (1)

$$\bar{X}\left[1+\frac{1}{6}\left(\frac{1}{2}\right)\right] = \frac{1}{6}L\left[3+\left(\frac{1}{2}\right)+\frac{1}{4}\left(\frac{1}{2}\right)^2\right]$$

or $$\bar{X} = \frac{57}{104}L$$ $\bar{X} = 0.548L$ ◀

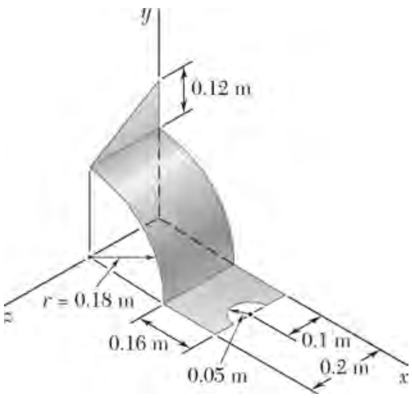
PROBLEM 5.146 (Continued)

(*b*) $\dfrac{h}{L} = ?$ when $\bar{X} = L$

Substituting into Eq. (1) $$L\left(1 + \frac{1}{6}\frac{h}{L}\right) = \frac{1}{6}L\left(3 + \frac{h}{L} + \frac{1}{4}\frac{h^2}{L^2}\right)$$

or $$1 + \frac{1}{6}\frac{h}{L} = \frac{1}{2} + \frac{1}{6}\frac{h}{L} + \frac{1}{24}\frac{h^2}{L^2}$$

or $$\frac{h^2}{L^2} = 12$$ $$\frac{h}{L} = 2\sqrt{3} \blacktriangleleft$$



PROBLEM 5.147

Locate the center of gravity of the sheet-metal form shown.

SOLUTION

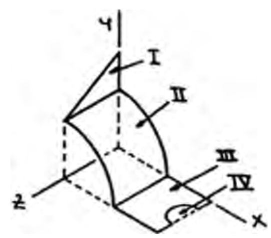
First assume that the sheet metal is homogeneous so that the center of gravity of the form will coincide with the centroid of the corresponding area.

$$\bar{y}_{\mathrm{I}} = 0.18 + \frac{1}{3}(0.12) = 0.22 \text{ m}$$

$$\bar{z}_{\mathrm{I}} = \frac{1}{3}(0.2 \text{ m})$$

$$\bar{x}_{\mathrm{II}} = \bar{y}_{\mathrm{II}} = \frac{2 \times 0.18}{\pi} = \frac{0.36}{\pi} \text{ m}$$

$$\bar{x}_{\mathrm{IV}} = 0.34 - \frac{4 \times 0.05}{3\pi}$$
$$= 0.31878 \text{ m}$$



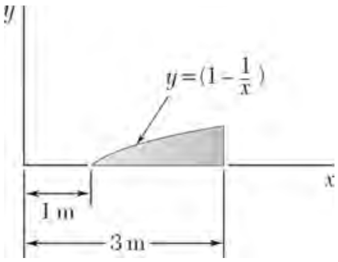
	A, m^2	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}A$, m^3	$\bar{y}A$, m^3	$\bar{z}A$, m^3
I	$\frac{1}{2}(0.2)(0.12) = 0.012$	0	0.22	$\frac{0.2}{3}$	0	0.00264	0.0008
II	$\frac{\pi}{2}(0.18)(0.2) = 0.018\pi$	$\frac{0.36}{\pi}$	$\frac{0.36}{\pi}$	0.1	0.00648	0.00648	0.005655
III	$(0.16)(0.2) = 0.032$	0.26	0	0.1	0.00832	0	0.0032
IV	$-\frac{\pi}{2}(0.05)^2 = -0.00125\pi$	0.31878	0	0.1	–0.001258	0	–0.000393
Σ	0.096622				0.013542	0.00912	0.009262

PROBLEM 5.147 (Continued)

We have $\bar{X}\Sigma V = \Sigma \bar{x}V: \quad \bar{X}(0.096622 \text{ m}^2) = 0.013542 \text{ m}^3$ or $\bar{X} = 0.1402 \text{ m}$ ◀

$\bar{Y}\Sigma V = \Sigma \bar{y}V: \quad \bar{Y}(0.096622 \text{ m}^2) = 0.00912 \text{ m}^3$ or $\bar{Y} = 0.0944 \text{ m}$ ◀

$\bar{Z}\Sigma V = \Sigma \bar{z}V: \quad \bar{Z}(0.096622 \text{ m}^2) = 0.009262 \text{ m}^3$ or $\bar{Z} = 0.0959 \text{ m}$ ◀



PROBLEM 5.148

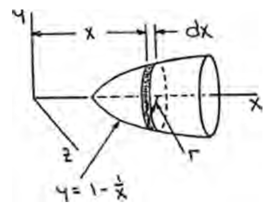
Locate the centroid of the volume obtained by rotating the shaded area about the x axis.

SOLUTION

First, note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx.



Then $$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

Now $r = 1 - \dfrac{1}{x}$ so that $$dV = \pi\left(1 - \frac{1}{x}\right)^2 dx$$

$$= \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

Then $$V = \int_1^3 \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx = \pi\left[x - 2\ln x - \frac{1}{x}\right]_1^3$$

$$= \pi\left[\left(3 - 2\ln 3 - \frac{1}{3}\right) - \left(1 - 2\ln 1 - \frac{1}{1}\right)\right]$$

$$= (0.46944\pi)\ \text{m}^3$$

and $$\int \bar{x}_{EL} dV = \int_1^3 x\left[\pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx\right] = \pi\left[\frac{x^2}{2} - 2x + \ln x\right]_1^3$$

$$= \pi\left\{\left[\frac{3^2}{2} - 2(3) + \ln 3\right] - \left[\frac{1^3}{2} - 2(1) + \ln 1\right]\right\}$$

$$= (1.09861\pi)\ \text{m}$$

Now $$\bar{x}V = \int \bar{x}_{EL} dV \quad : \bar{x}(0.46944\pi\ \text{m}^3) = 1.09861\pi\ \text{m}^4$$

or $\bar{x} = 2.34$ m ◀