Unit 6: Number Theory and its Applications in Cryptography **Text Book: Chapter 4**

Divisibility and Modular Arithmetic

If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that b = ac, or equivalently, if $\frac{b}{a}$ is an integer. When a divides b we say that a is a factor or divisor of b, and that b is a multiple of a. The notation $a \mid b$ denotes that a divides b. We write $a \nmid b$ when a does not divide b.

- Let n and d be positive integers then no. of positive integers not exceeding n, that are divisible by d are $\left[\frac{n}{d}\right]$.
- Q1. How many numbers from 1 to 1500 are divisible by 9?

$$\left\lfloor \frac{1500}{9} \right\rfloor = \left\lfloor 166.7 \right\rfloor$$

Theorem 1:

Let a, b, and c be integers, where $a \neq 0$. Then

- (i) if $a \mid b$ and $a \mid c$, then $a \mid (b+c)$;
- (ii) if $a \mid b$, then $a \mid bc$ for all integers c;
- (iii) if $a \mid b$ and $b \mid c$, then $a \mid c$.

$$a|b\Rightarrow b=k_1a$$
 $a|c\Rightarrow c=k_2a$
 $b+c=(k_1+k_2)a$

Corollary 1:

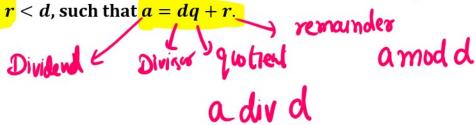
Corollary 1:

If a, b, and c are integers, where $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ whenever m and n are integers.

The Division Algorithm

Theorem 2: Let a be an integer and d a positive integer, then there are unique

integers q and r, with $0 \le r < d$, such that a = dq + r.



Q2. What are the quotients and remainders when

(i) 777 is divided by 21?

$$9 = 777 \text{ div } al = 37$$
 $r = 777 \text{ mod } al = 0$

(ii) -111 is divided by 11?

$$q = -||| \text{div } || = -||$$
 $|| x = -||| \text{mod } || = |0|$

11 is divided by 11?

$$Q = -||| \text{div} || = -|| -||| = ||(-|||) + ||0||$$
 $||-||| = ||(-|||) + ||0||$
 $||-||| = ||(-|||) + ||0||$

Q3. Find the value of

(i) 1,234,567 div 1001
$$= |333|$$

(ii)
$$-100 \ mod \ 101 =$$

$$-|00 = |0|(-|)+|$$
 $|00 = |0|(0)+|00$

Q4. What time does a 12-hour clock read

(i) 80 hours after it reads 11:00?

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(ii) 40 hours before it reads 12:00?

Modular Arithmetic

If a and b are integers and m is a positive integer, then a is *congruent to b modulo* m if m divides a - b. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to $a \equiv b \pmod{m}$. We say that $a \equiv b \pmod{m}$ is a **congruence** and that $a \equiv b \pmod{m}$. If $a \equiv b \pmod{m}$ are not congruent modulo $a \equiv b \pmod{m}$, we write $a \not\equiv b \pmod{m}$.

$$m \mid (a-b)$$
 $\alpha = b \mod m = b$

Theorem 3:

Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \mod m = b \mod m$.

Theorem 4:

Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that a = b + km.

Theorem 5:

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m}$$
 and $ac \equiv bd \pmod{m}$.

Corollary 2:

Let *m* be a positive integer and let *a* and *b* be integers. Then

$$(a+b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$

and

$$ab \mod m = ((a \mod m)(b \mod m)) \mod m.$$

$$15 \mod 3 = 0$$
(8+7)

$$8 \text{mod} 3 + 7 \text{mod} 3$$

 $2 + 1 = 3 \text{mod} 3 = 0$

To be noted:

• $a \equiv b \mod m \Rightarrow a^c \equiv b^c \mod m$

ma-b $m/(a^2-b^2)$

- $a \equiv b \mod m$ and $c \equiv d \mod m \Rightarrow a^c \mod m \equiv b^d \mod m$
- $ac \equiv bc \mod m \Rightarrow a \mod m \equiv b \mod m$

Theorem 6: Let m be a positive integer and let a, b and c be integers. If $ac \equiv$ bc mod m

and
$$gcd(c, m) = 1$$
, then $a \equiv b \mod m$.

Q5.

Suppose that a and b are integers, $a \equiv 4 \pmod{13}$, and

Suppose that a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \le c \le 12$ such that

f) $c \equiv a^3 - b^3 \pmod{13}$.

a)
$$c \equiv 9a \pmod{13}$$
.
b) $c \equiv 11b \pmod{13}$.
c) $c \equiv a + b \pmod{13}$.
d) $c \equiv 2a + 3b \pmod{13}$.
e) $c \equiv a^2 + b^2 \pmod{13}$.
(C) $0 + b \equiv 13 \pmod{13}$
 $0 + b \equiv 0 \pmod{13}$, $0 = 0$

(a)
$$a = 4 \pmod{3}$$

 $qa = 36 \pmod{3}$, $qa = 6 \pmod{3}$, $c = 10$

(b)
$$b = 9 \pmod{3}$$
, $||b| = 99 \pmod{3}$, $||b| = 8 \pmod{3}$
 $c = 8$