Z - test

Test of significance for Difference of Means:

Let $\bar{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1 and variance σ_1^2 and $\bar{x}_2$ be the mean of an independent random sample of size n_2 from another population with mean μ_2 and variance σ_2^2. Then since sample sizes are large,

$$\bar{x}_1 \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right) \text{ and } \bar{x}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

Also $\bar{x}_1 - \bar{x}_2$, being the difference of two independent normal variates is also a normal variate. Then Z, (standard normal variate) corresponding to $\bar{x}_1 - \bar{x}_2$ is given by

$$Z = \frac{(\overline{x}_1 - \overline{x}_2) - E(\overline{x}_1 - \overline{x}_2)}{S.E(\overline{x}_1 - \overline{x}_2)}$$

S.E stands for standard error

$$E(\overline{x}_1 - \overline{x}_2) = E(\overline{x}_1) - E(\overline{x}_2) = \mu_1 - \mu_2 = 0$$

$$V(\overline{x}_1 - \overline{x}_2) = V(\overline{x}_1) + V(\overline{x}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Thus under the null hypothesis. $H_0 : \mu_1 = \mu_2$
the test statistics becomes (for large samples)

$$Z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Q) The means of two single large samples of 1000, and 2000 members are 67.5 inches and 68.0 inches respectively. Can the samples be regarded as drawn from the same population of standard deviation 2.5 inches? Test at 5% level of significance.

Sol :- Given, $n_1 = 1000$, $n_2 = 2000$

$\bar{x}_1 = 67.5$ inches, $\bar{x}_2 = 68.0$ inches

Null hypothesis $H_0 : \mu_1 = \mu_2$ and $\sigma = 2.5$ inches, i.e. the samples have been drawn from the same population of standard deviation 2.5 inches.

Alternative hypothesis : $H_1 : \mu_1 \neq \mu_2$ (Two tailed)

Test statistics Under H_0, the test statistics is

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{67.5 - 68.0}{\sqrt{(2.5)^2\left(\frac{1}{1000} + \frac{1}{2000}\right)}}$$

$$= \frac{-0.5}{2.5 \times \sqrt{\frac{1}{1000} + \frac{1}{2000}}} = \frac{-0.5}{2.5 \times 0.0387} = -5.1$$

$|Z| > 3$

The calculated Z is highly significant and we reject the null hypothesis and conclude that samples are certainly not from the same population with standard deviation 2.5 inches.

Remark ① In the above problem

$\sigma_1^2 = \sigma_2^2 = \sigma^2$. i.e. if the samples have been drawn from the populations with common S.D σ then under $H_0 : \mu_1 = \mu_2$

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

② If $\sigma_1^2 \neq \sigma_2^2$ and σ_1 and σ_2 are unknown, then they are estimated from sample values. This result in some error, which is practically immaterial if sample are large.

These estimates for large samples are given by

$\hat{\sigma}_1 = S_1^2 \approx s_1^2$ and $\hat{\sigma}_2^2 = S_2^2 = s_2^2$ (for samples are large)

$$Z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Ex(2) In a survey of buying habits, 400 women shoppers are chosen at random in super market 'A' located in a certain section of the city. Their average weekly food expenditure is Rs 250 with a standard deviation of Rs 40. For 400 women shoppers chosen at random in super market 'B' in another section of the city, the average weekly food expenditure is Rs 220 with a standard deviation of Rs 55. Test at 1% level of significance whether the average weekly food expenditure of two population shoppers are equal.

Sol: $n_1 = 400$, $\overline{x_1} = 250$, $s_1 = 40$

$n_2 = 400$, $\overline{x_2} = 220$, $s_2 = 55$

Null hypothesis $H_0: \mu_1 = \mu_2$ i.e., the average weekly food expenditures of the two populations of shoppers are equal.

Alternative hypothesis $H_1: \mu_1 \neq \mu_2$

(Two tailed)

$$z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Since σ_1^2 and σ_2^2 are not known, we can take

$\sigma_1^2 = s_1^2$ and $\sigma_2^2 = s_2^2$

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{250 - 220}{\sqrt{\frac{(40)^2}{400} + \frac{(55)^2}{400}}}$$

$$= \frac{30 \times 20}{\sqrt{40^2 + 55^2}} = \underline{8.82}$$

Calculated $|Z| > 2.58$

$\therefore$ Null hypothesis is rejected

So average ~~p~~ expenditure of two populations of shoppers in market A and market B differ significantly.

(3) The average hourly wages of a sample of 150 workers in plant 'A' was Rs 2.56 with a standard deviation of Rs 1.08. The average hourly wage of a sample of 200 workers in plant 'B' was Rs 2.87 with a standard deviation of Rs 1.28. Can an applicant safely assume that hourly wages paid by plant 'B' are higher than those paid by plant 'A'?

Sol :- $n_1 = 150$, $\bar{x}_1 = 2.56$, $s_1 = 1.08$

$n_2 = 200$, $\bar{x}_2 = 2.87$, $s_2 = 1.28$

Null hypothesis : $H_0 : \mu_1 = \mu_2$, there is no significant difference between the mean wages of workers in plant A and plant B.

Alternative hypothesis H_1 ; $\mu_1 < \mu_2$

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{2.56 - 2.87}{\sqrt{\frac{(1.08)^2}{150} + \frac{(1.28)^2}{200}}}$$

$$= \frac{-0.31}{\sqrt{0.016}} = -2.46$$

Here, calculated $Z < -1.645$

So null hypothesis rejected (Critical value of Z for left tailed at 5% level of sig)

we conclude that average hourly wage of plant B is certainly higher

(1) t-test for single mean:

Suppose we want to test:

(i) If a random sample x_i $(i = 1, 2, \dots n)$ of size n has been drawn from a normal population with specified mean, say μ_0, or

(ii) If the sample mean differs significantly from the hypothetical value μ_0 of the population mean.

Under the null hypothesis, H_0:

(i) The sample has been drawn from the population with mean μ_0 or

(ii) There is no significant difference between the sample mean $\bar{x}$ and the population mean μ_0.

the statistic

$$t = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $S^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2$

follows student's t-distribution with $(n-1)$ degree of freedom.

Assumptions for Student's t-test

The following assumptions are made in the Student's t-test

(i) The parent population from which the sample is drawn is normal.

(ii) The sample observations are independent, i.e. the sample is random

(iii) The population standard deviation σ is unknown.

A machinist is making engine parts with axle diameters of 0.700 inch. A random sample of 10 parts shows a mean diameter of 0.742 inch with a standard deviation of 0.040 inch. Compute the statistic you would use to test whether the work is meeting the specification. Also state how would you proceed.

Solution. Here we are given :

$$\mu = 0{\cdot}700 \text{ inche}, \quad \bar{x} = 0{\cdot}742 \text{ inche}, \quad s = 0{\cdot}040 \text{ inche} \quad \text{and} \quad n = 10$$

Null Hypothesis, $H_0 : \mu = 0{\cdot}700$, *i.e., the product is conforming to specifications.*

Alternative Hypothesis, $H_1 : \mu \neq 0{\cdot}700$

Test Statistic. Under H_0, the test statistic is : $t = \dfrac{\bar{x} - \mu}{\sqrt{S^2/n}} = \dfrac{\bar{x} - \mu}{\sqrt{s^2/(n-1)}} \sim t_{(n-1)}$

$$\therefore \qquad t = \frac{\sqrt{9}\,(0{\cdot}742 - 0{\cdot}700)}{0{\cdot}040} = 3{\cdot}15$$

How to proceed further. Here the test statistic 't' follows Student's t-distribution with $10 - 1 = 9$ *d.f.* We will now compare this calculated value with the tabulated value of t for 9 *d.f.* and at certain level of significance, say 5%. Let this tabulated value be denoted by t_0.

(*i*) If calculated 't', *viz.*, $3{\cdot}15 > t_0$, we say that the value of t is significant. This implies that $\bar{x}$ differs significantly from μ and H_0 is rejected at this level of significance and we conclude that the product is not meeting the specifications.

(*ii*) If calculated $t < t_0$, we say that the value of t is not significant, *i.e.*, there is no significant difference between $\bar{x}$ and μ. In other words, the deviation $(\bar{x} - \mu)$ is just due to fluctuations of sampling and null hypothesis H_0 may be retained at 5% level of significance, *i.e.*, we may take the product conforming to specifications.

Example 16·6. *The mean weekly sales of soap bars in departmental stores was 146·3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153·7 and showed a standard deviation of 17·2. Was the advertising campaign successful ?*

Solution. We are given : $n = 22, \bar{x} = 153{\cdot}7, s = 17{\cdot}2$.

Null Hypothesis. The advertising campaign is not successful, *i.e.,* $H_0 : \mu = 146{\cdot}3$

Alternative Hypothesis, $H_1 : \mu > 146{\cdot}3$ (Right-tail).

Test Statistic. Under H_0, the test statistic is : $t = \dfrac{\bar{x} - \mu}{\sqrt{s^2/(n-1)}} \sim t_{22-1} = t_{21}$

$$\therefore \qquad t = \frac{153{\cdot}7 - 146{\cdot}3}{\sqrt{(17{\cdot}2)^2/21}} = \frac{7{\cdot}4 \times \sqrt{21}}{17{\cdot}2} = 9{\cdot}03$$

Conclusion. Tabulated value of t for 21 *d.f.* at 5% level of significance for *single-tailed test* is 1·72. Since calculated value is much greater than the tabulated value, it is

highly significant. Hence we reject the null hypothesis and conclude that the advertising campaign was definitely successful in promoting sales.

Example 16·7. *A random sample of 10 boys had the following I.Q.'s : 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do these data support the assumption of a population mean I.Q. of 100 ? Find a reasonable range in which most of the mean I.Q. values of samples of 10 boys lie.*

Solution. Null hypothesis, H_0 : *The data are consistent with the assum*

Solution. Null hypothesis, H_0 : *The data are consistent with the assumption of a mean I.Q. of 100 in the population, i.e.,* $\mu = 100$.

Alternative hypothesis, H_1 : $\mu \neq 100$.

Test Statistic. Under H_0, the test statistic is : $t = \frac{(\bar{x} - \mu)}{\sqrt{S^2/n}} \sim t_{(n-1)}$,

where $\bar{x}$ and S^2 are to be computed from the sample values of I.Q.'s.

TABLE 16·1 : CALCULATIONS FOR SAMPLE MEAN AND S.D.

x	$(x - \bar{x})$	$(x - \bar{x})^2$
70	−27·2	739·84
120	22·8	519·84
110	12·8	163·84
101	3·8	14·44
88	−9·2	84·64
83	−14·2	201·64
95	−2·2	4·84
98	0·8	0·64
107	9·8	96·04
100	2·8	7·84
Total 972		1833·60

Here $n = 10$, $\bar{x} = \frac{972}{10} = 97 \cdot 2$ and $S^2 = \frac{1833 \cdot 60}{9} = 203 \cdot 73$

$$\therefore \qquad |t| = \frac{|97 \cdot 2 - 100|}{\sqrt{203 \cdot 73 / 10}} = \frac{2 \cdot 8}{\sqrt{20 \cdot 37}} = \frac{2 \cdot 8}{4 \cdot 514} = 0 \cdot 62$$

Tabulated $t_{0 \cdot 05}$ for $(10 - 1)$, *i.e.*, 9 *d.f.* for two-tailed test is 2·262.

Conclusion. Since calculated t is less than tabulated $t_{0 \cdot 05}$ for 9 *d.f.*, H_0 may be accepted at 5% level of significance and we may conclude that the data are consistent with the assumption of mean I.Q. of 100 in the population.

The 95% confidence limits within which the mean I.Q. values of samples of 10 boys will lie are given by :

$$\bar{x} \pm t_{0 \cdot 05} \, S / \sqrt{n} = 97 \cdot 2 \pm 2 \cdot 262 \times 4 \cdot 514 = 97 \cdot 2 \pm 10 \cdot 21 = 107 \cdot 41 \text{ and } 86 \cdot 99$$

Hence the required 95% confidence interval is [86·99, 107·41].

Remark. Aliter for

Example 16·8. *The heights of 10 males of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable to believe that the average height is greater than 64 inches ? Test at 5% significance level assuming that for 9 degrees of freedom* $P(t > 1{\cdot}83) = 0{\cdot}05$.

Solution. *Null Hypothesis*, $H_0 : \mu = 64$ inches

Alternative Hypothesis, $H_1 : \mu > 64$ inches

TABLE 16·2 : CALCULATIONS FOR SAMPLE MEAN AND S.D.

x	70	67	62	68	61	68	70	64	64	66	Total 660
$x-\bar{x}$	4	1	-4	2	-5	2	4	-2	-2	0	0
$(x-\bar{x})^2$	16	1	16	4	25	4	16	4	4	0	90

$$\bar{x} = \frac{\Sigma x}{n} = \frac{660}{10} = 66; \quad S^2 = \frac{1}{n-1}\Sigma(x-\bar{x})^2 = \frac{90}{9} = 10$$

Test Statistic. Under H_0, the test statistic is :

$$t = \frac{\bar{x}-\mu}{\sqrt{S^2/n}} = \frac{66-64}{\sqrt{10/10}} = 2,$$

which follows Student's t-distribution with $10-1=9$ *d.f.*

Tabulated value of t for 9 *d.f.* at 5% level of significance for single (right) tail-test is 1·833. (This is the value $t_{0\cdot10}$ for 9 *d.f.* in the two-tailed tables given at the end of the chapter.)

Conclusion. Since calculated value of t is greater than the tabulated value, it is significant. Hence H_0 is rejected at 5% level of significance and we conclude that the average height is greater than 60 inches.

16·2·7. Critical Values of t. The critical (or significant) values of t at level of significance α and *d.f.* v for *two-tailed test* are given by the equation :

$$P[\,|t| > t_v(\alpha)] = \alpha \quad \ldots(16\cdot5)$$

$$\Rightarrow \quad P[\,|t| \le t_v(\alpha)] = 1 - \alpha \quad \ldots(16\cdot5a)$$

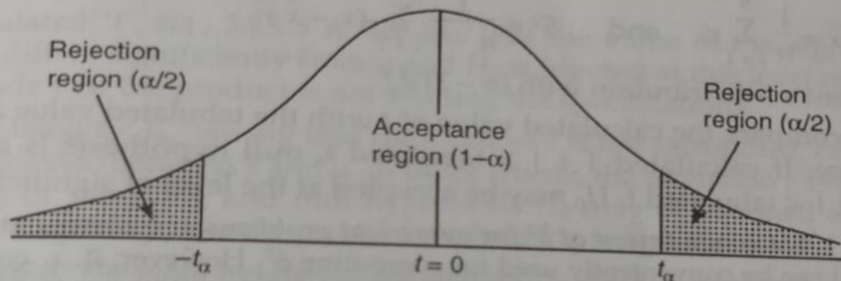

Fig. 16·2 : Critical values of t-distribution

The values $t_v(\alpha)$ have been tabulated in Fisher and Yates' Tables, for different values of α and v and are given in *Table I* at the end of the chapter.

Since t-distribution is symmetric about $t = 0$, we get from (16·5)

$$P(t > t_v(\alpha)] + P[t < -t_v(\alpha)] = \alpha \quad \Rightarrow \quad 2P[t > t_v(\alpha)] = \alpha$$

$$\Rightarrow \quad P[t > t_v(\alpha)] = \alpha/2 \qquad \therefore \; P[t > t_v(2\alpha)] = \alpha \quad \ldots(16\cdot5b)$$

$t_v(2\alpha)$ (from the Tables at the end of the chapter) gives the significant value of t for a single-tail test [Right-tail or Left-tail-since the distribution is symmetrical], at level of significance α and v *d.f.*

Hence the significant values of t at level of significance 'α' for a single-tailed test can be obtained from those of two-tailed test by looking the values at level of significance 2α.

For example,

$t_8(0\cdot05)$ for single-tail test $= t_8(0\cdot10)$ for two-tail test $= 1\cdot86$

$t_{15}(0\cdot01)$ for single-tail test $= t_{15}(0\cdot02)$ for two-tail test $= 2\cdot60$.

TABLE I.

SIGNIFICANT VALUES $t_v(\alpha)$ of t-Distribution
(TWO-TAIL AREAS)
$P[\,|\,t\,|\, > t_v(\alpha)] = \alpha$

d.f.	*Probability (Level of Significance)*					
(v)	0·50	0·10	0·05	0·02	0·01	0·001
1	1·00	6·31	12·71	31·82	63·66	636·62
2	0·82	2·92	4·30	6·97	6·93	31·60
3	0·77	2·35	3·18	4·54	5·84	12·94
4	0·74	2·13	2·78	3·75	4·60	8·61
5	0·73	2·02	2·57	3·37	4·03	6·86
6	0·72	1·94	2·45	3·14	3·71	5·96
7	0·71	1·90	2·37	3·00	3·50	5·41
8	0·71	1·86	2·31	2·90	3·36	5·04
9	0·70	1·83	2·26	2·82	3·25	4·78
10	0·70	1·81	2·23	2·76	3·17	4·59
11	0·70	1·80	2·20	2·72	3·11	4·44
12	0·70	1·78	2·18	2·68	3·06	4·32
13	0·69	1·77	2·16	2·65	3·01	4·22
14	0·69	1·76	2·15	2·62	2·98	4·14
15	0·69	1·75	2·13	2·60	2·95	4·07
16	0·69	1·75	2·12	2·58	2·92	4·02
17	0·69	1·74	2·11	2·57	2·90	3·97
18	0·69	1·73	2·10	2·55	2.86	3.88
19	0.69	1.73	2.09	2.54	2·85	3·85
20	0·69	1·73	2·09	2·53	2·83	3·83
21	0·69	1·72	2·08	2·52	2·82	3·79
22	0·69	1·72	2·07	2·51	2·81	3·77
23	0·69	1·71	2·07	2·50	2·80	3·75
24	0·69	1·71	2·06	5·49	2·79	3·73
25	0·68	1·71	2·06	2·49	2·78	3·71
26	0·68	1·71	2·06	2·48	2·77	3·69
27	0·68	1·70	2·05	2·47	2·76	3·67
28	0·68	1·70	2·05	2·47	2·76	3·66
29	0·68	1·70	2·05	2·46	2·75	3·65
30	0·68	1·70	2·04	2·46	2·58	3·29
∞	0·67	1·65	1·96	2·33		