Q Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

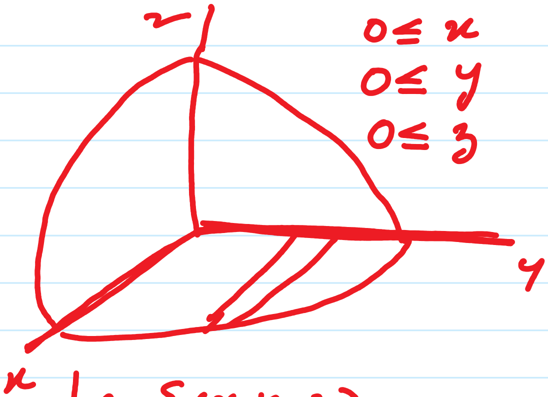
Sol

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$z^2=c^2\left[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right]$$

$z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$



$0\le x$
$0\le y$
$0\le z$

$R=\{(x,y,z):$
$0\le z\le c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$
$0\le y\le b\sqrt{1-\frac{x^2}{a^2}}$
$0\le x\le a$ $\}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left[1-\frac{x^2}{a^2}\right]$$

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$\therefore$ $\frac{x^2}{a^2}=1$

or $x=\pm a$

Required volume $= 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\Big|z\Big|_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dy\,dx$$

$$=8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= 8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{b}\sqrt{\underline{b^2\left(1-\frac{x^2}{a^2}\right)}-y^2}\,dy\,dx$$

let $f = b\sqrt{1-\frac{x^2}{a^2}}$

$$= \frac{8c}{b}\int_0^a\int_0^f \sqrt{f^2-y^2}\,dy\,dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{f^2-y^2}}{2}+\frac{f^2}{2}\sin^{-1}\frac{y}{f}\right]_0^f dx$$

$$= \frac{8c}{b}\int_0^a \left\{\left(0+\frac{f^2}{2}\sin^{-1}1\right)-(0-0)\right\}dx$$

$$= \frac{8c}{2b}\times\frac{\pi}{2}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= \frac{2c\pi}{b}\times b^2\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\left|x-\frac{x^3}{3a^2}\right|_0^a = 2\pi bc\left[\left(a-\frac{a^3}{3a^2}\right)-0\right]$$

$$= 2\pi bc\left[\frac{2a}{3}\right]$$

$$\boxed{\text{Volume} = \frac{4\pi}{3}abc}$$

CTM $\boxed{\text{Volume of } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{4\pi}{3}abc}$

$$\frac{x^2}{1^2}+\frac{y^2}{2^2}+\frac{z^2}{3^2}=1 \qquad \frac{4\pi}{\cancel{3}}\times1\times2\times\cancel{3}=8\pi$$

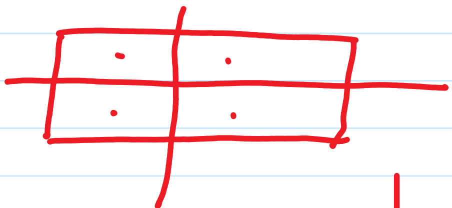
$$x^2+y^2+z^2 \qquad \left(\frac{4\pi}{\ }\right.$$

$$\frac{x^2}{1^2}+\frac{y^2}{1^2}+\frac{z^2}{1^2}=1 \quad , \quad \boxed{\frac{4\pi}{3}}$$

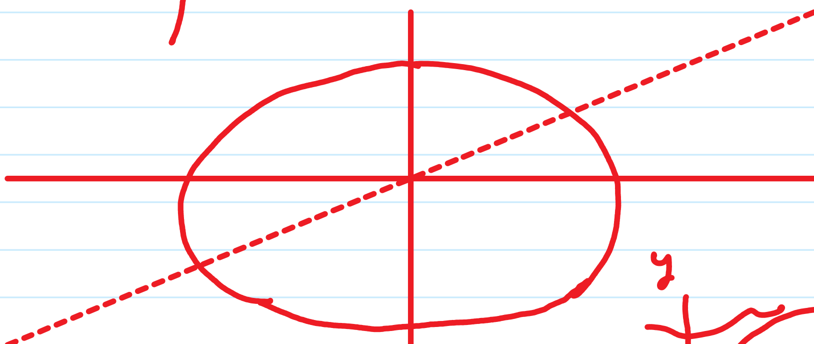
$$\# \quad \boxed{x^2+y^2+z^2=1} \longrightarrow \frac{4}{3}\pi r^3 \rightarrow \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$$

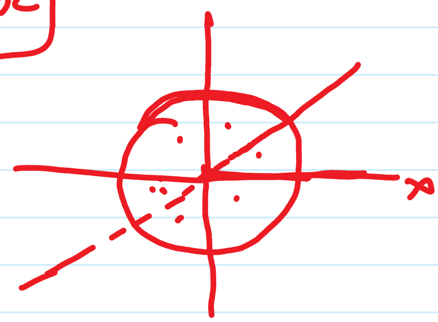
↓ sphere

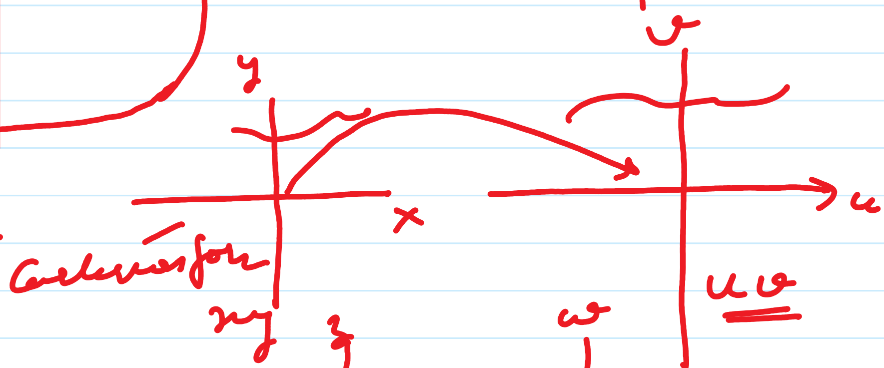
$$\# \quad \boxed{x^2+y^2+z^2=a^2} \longrightarrow \boxed{\frac{4}{3}\pi a^3}$$



$$\boxed{\frac{4}{3}\pi abc}$$

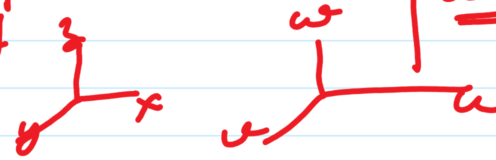




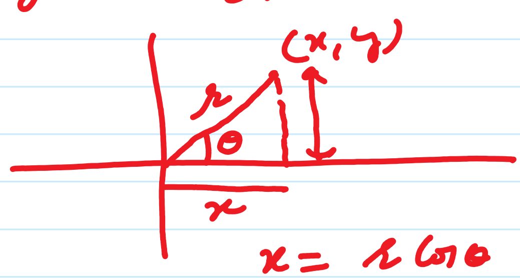


Change of variable

① Cartesian form to polar form



① $x = r\cos\theta$ & $y = r\sin\theta$



$x = r\cos\theta$
$y = r\sin\theta$

① **Double variable**

let $x = \phi(u,v)$ & $y = \psi(u,v)$

CT then $\iint f(x,y)\,\boxed{dx\,dy} = \iint f\underbrace{(\phi(u,v), \psi(u,v))}\,\underline{|J|\,du\,dv}$

where $J = \partial(x,y)$ fo

$$\iint_{R_{xy}} \qquad\qquad \iint_{R'_{uv}}$$

where $J = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$

Triple Integral $x = \phi_1(u,v,w)$, $y = \phi_2(u,v,w)$ & $z = \phi_3(u,v,w)$

$$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint_{R'_{uvw}} f(\phi_1(u,v,w), \phi_2(u,v,w), \phi_3(u,v,w))\,|J|\,du\,dv\,dw$$

where $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$

$$\boxed{\iint dx\,dy = \iint r\,dr\,d\theta}$$

Q

$$\text{Area} = \iint_R dx\,dy = \iint_{R_{r\theta}} |J|\,dr\,d\theta$$

Put $x = r\cos\theta$ & $y = r\sin\theta$

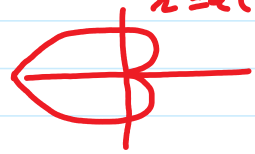
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r[1] = r$$

$\therefore J = r$

$$\therefore \text{Area} = \iint_{R_{xy}} dx\,dy = \iint_{R_{r\theta}} r\,dr\,d\theta$$



→ I Put $x = r\cos\theta$, $y = r\sin\theta$, $\boxed{J = r}$

(II) To change rectangular coordinates (x,y,z) to cylindrical coordinates (ρ, ϕ, z)

$x = \rho\cos\phi$ & $y = \rho\sin\phi$, $z = z$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \partial z/\partial \rho & \partial z/\partial \phi & \partial z/\partial z \end{vmatrix} = \rho$$

III To change rectangular coordinates (x,y,z) to spherical coordinates (r,θ,ϕ).

$x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \partial y/\partial \phi \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \underline{r^2\sin\theta}$$

$$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint f(\qquad)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

Q Evaluate $\iint_R (x+y)^2\,dx\,dy$, where R is the llgm in the xy-plane with vertices $(1,0), (3,1), (2,2)$ & $(0,1)$ using the transformation $u = x+y$ & $v = x-2y$

Sol

$u = x+y$

$v = x-2y$

$\underline{-\quad - \quad +}$

$u - v = 3y$ or $\boxed{y = \frac{u-v}{3}}$

$u = x + \frac{(u-v)}{3}$

or $x = u - \frac{(u-v)}{3}$

$= \frac{3u-u+v}{3} = \frac{2u+v}{3}$

$\boxed{dx\,dy = |J|\,du\,dv}$

$\therefore x = \frac{2u+v}{3}$ & $y = \frac{(u-v)}{3}$

$$\therefore\ x = \frac{2u+v}{3} \ \ \& \ \ y = \frac{(u-v)}{3}$$

$$\boxed{dx\,dy = |J|\,du\,dv}$$

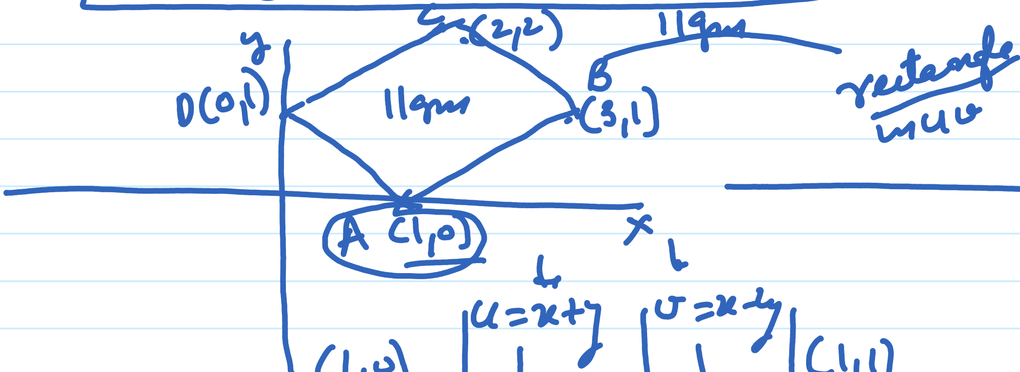
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -1/3 \end{vmatrix}$$

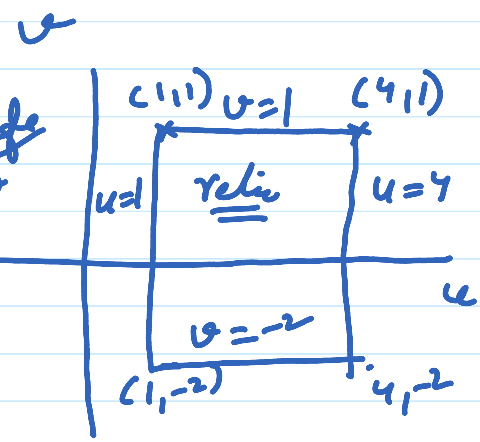
$$= -\frac{2}{9} - \frac{1}{9} = -3/9 = -1/3$$

$$\boxed{J = -1/9}$$

$$u = x+y \ \ \& \ \ v = x-2y$$

$$\boxed{x = \frac{1}{3}(2u+v) \ \ \& \ \ y = \frac{1}{3}(u,v)}$$





	$u=x+y$	$v=x-2y$	
(1,0)	1	1	(1,1)
(3,1)	4	1	(4,1)
(2,2)	4	-2	(4,-2)
(0,1)	1	-2	(1,-2)

rectangle

$$R_{uv} = \{(u,v) : 1 \le u \le 4,\ -2 \le v \le 1\}$$

$$I = \iint_R (x+y)^2\,dx\,dy = \int_{-2}^{1}\int_{1}^{4} u^2\,\underbrace{|J|}\,du\,dv$$

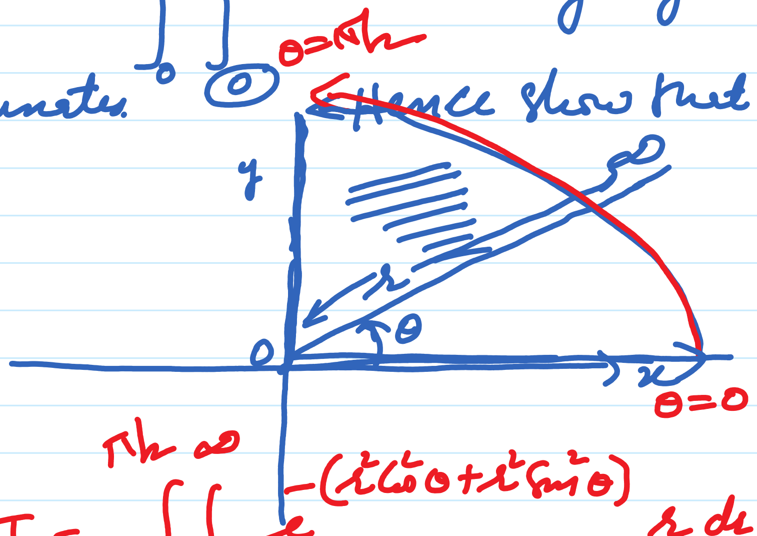
$$= \int_{-2}^{1}\int_{1}^{4} u^2 \left(+\frac{1}{3}\right) du\,dv$$

$$= \frac{1}{3}\int_{-2}^{1} \left|\frac{u^3}{3}\right|_{1}^{4} dv$$

$= 21$ Ans

Q Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$

Sol



Put $x = r\cos\theta$
$y = r\sin\theta$
$J = r$

$$I = \int_0^{\pi/2}\int_0^\infty e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$I = \int_0^\infty\int_0^\infty e^{-x^2}.e^{-y^2}\,dx\,dy$$

$$= \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy$$

$r^2 = t$

$2r\,dr = dt \Rightarrow r\,dr = \frac{1}{2}dt$

$$\int e^{-r^2} r\,dr = \frac{1}{2}\int e^{-t}dt$$

$$= \frac{1}{2}\frac{e^{-t}}{-1}$$

$$= -\frac{e^{-t}}{2} = -\frac{e^{-r^2}}{2}$$

$$= -\int_0^{\pi/2} \left| \frac{e^{-r^2}}{2} \right|_0^\infty d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left\{e^{-\infty} - e^{0}\right\} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}[0-1]\,d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}\left|\theta\right|_0^{\pi/2} = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

\# $\therefore \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}$

or $\int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \frac{\pi}{4}$

$$\text{or} \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \frac{\pi}{4}$$

$$\text{or} \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-x^2}dx = \frac{\pi}{4}$$

$$\text{or} \left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4} \quad \text{or} \quad \boxed{\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}} \quad \underline{\underline{\text{Ans}}}$$

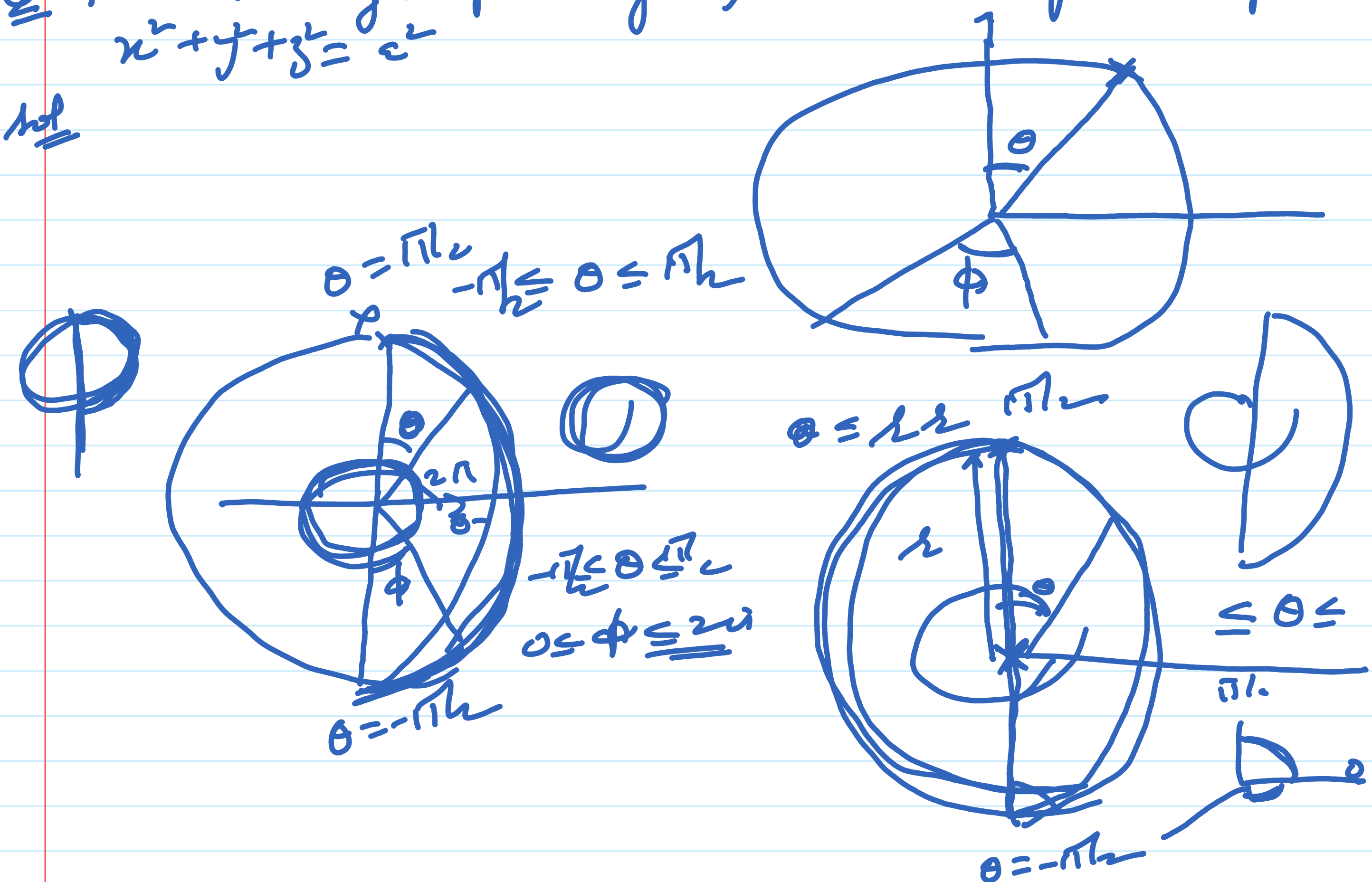
#

Q

$$\boxed{\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}}$$

Q Find ~~the~~ by triple integral, the volume of the sphere $x^2+y^2+z^2=a^2$

Sol



Put $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta,$

Put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$$J = r^2\sin\theta \qquad \iiint dx\,dy\,dz$$

$$\text{volume} = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \int_0^{a} r^2\,dr = \frac{4}{3}\pi a^3 \#$$

Q Find the volume of portion of the sphere $x^2+y^2+z^2=a^2$ lying under the cylinder $x^2+y^2=ay$

$x^2+y^2=ay$

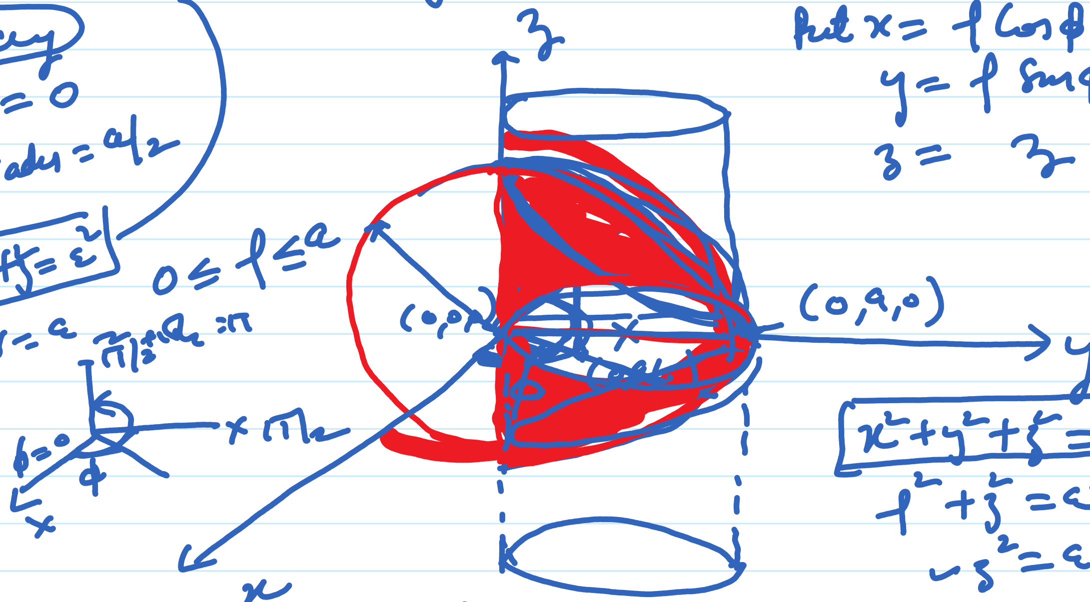
$x^2+y^2-ay=0$

$(0,\frac{a}{2})$, radius $= a/2$

$\boxed{x^2+y^2+z^2=a^2}$

$(0,0,0)$, $r=a$

$0 \le r \le a$



Put $x = r\cos\phi$

$y = r\sin\phi$

$z = z$

$\boxed{x^2+y^2+z^2=a^2}$

$r^2+z^2=a^2$

$\Rightarrow z^2 = a^2-r^2$

$\Rightarrow z = \pm\sqrt{a^2-r^2}$

$0 \le z \le \sqrt{a^2-r^2}$

~~$0 \le r \le$~~

$0 \le \phi \le \pi$

$$\therefore \text{Required volume} = 2\int_0^{\pi}\int_0^{a\sin\phi}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\phi$$

$$= \boxed{\frac{2a^3}{9}(3\pi-4)} \quad \text{Exercise}$$

$x^2+y^2=ay$

$r^2\cos^2\phi + r^2\sin^2\phi = a\,r\sin\phi$

or $\rho^2 - a\rho\sin\phi = 0$

$\rho(\rho - a\sin\phi) = 0$

$\rho = 0$ & $\underline{\rho = a\sin\phi}$