FUZZY LOGIC

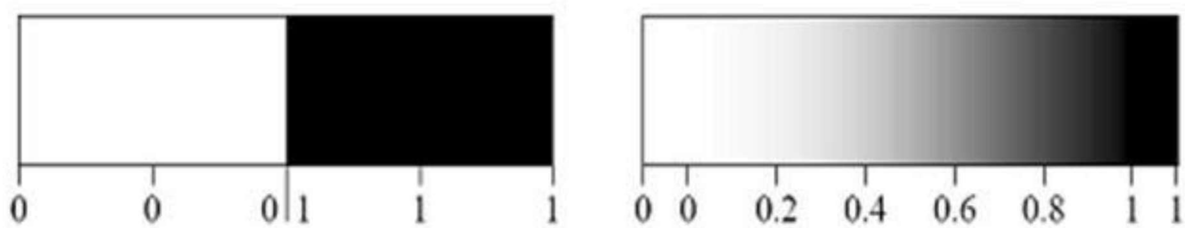
- Fuzzy logic is **the logic** underlying **approximate**, rather than exact, **modes of reasoning**.

- It is an extension of multivalued logic: **Everything**, including truth, **is a matter of degree**.

- It contains as special cases **not only** the classical two-value logic and multivalue logic systems, **but also** probabilistic logic.

- A proposition ***p*** has a **truth value**
  - 0 or 1 in two-value system,
  - element of a set T in multivalue system,
  - **Range over the fuzzy subsets of T** in fuzzy logic.

- Boolean logic uses sharp distinctions.
- Fuzzy logic reflects how people think.
- Fuzzy logic is a set of mathematical principles for



(*a*) Boolean Logic. (*b*) Multi-valued Logic.

- Fuzzy logic is a set of mathematical principles for knowledge representation and reasoning based on degrees of membership.

TYPES AND MODELING OF UNCERTAINTY

Stochastic Uncertainty:

- The probability of hitting the target is 0.8

Lexical Uncertainty:

- "Tall Men", "Hot Days", or "Stable Currencies"
- We will probably have a successful business year.
- The experience of expert A shows that B is Likely to Occur. However, expert C is convinced This Is Not True.

Example: One finds in desert two bottles of fluids with the following labels:

- ✓ bottle 1: there is a probability of 5% that this bottle is poisoned.
- ✓ bottle 2: this bottle contains a liquid which belongs to the set of drinkable water with membership function value of 0.95.

FUZZY vs PROBABILITY

- Fuzzy ≠ Probability
- Probability deals with uncertainty an likelihood
- Fuzzy logic deals with ambiguity an vagueness

- Let L=set of all liquids
  - £ be the subset ={all drinkable liquids}
- Suppose you had been in desert (you must drink!) and you come up with two bottles marked C and A.
- **Bottle C is labeled $\mu_{£}(C)=0.95$ and bottle A is labeled $Pr[A \in £]=0.95$**

• C could contain swamp water, but would not contain any poison. Membership of 0.95 means that the contents of C are fairly similar to perfectly drinkable water.

• The probability that A is drinkable is 0.95, means that over a long run of experiments, the context of A are expected to be drinkable in about 95% of the trials. In other cases it may contain poison.

NEED OF FUZZY LOGIC

- Based on intuition and judgment.
- No need for a mathematical model.
- Provides a smooth transition between members and nonmembers.
- Relatively simple, fast and adaptive.
- Less sensitive to system fluctuations.
- Can implement design objectives, difficult to express mathematically, in linguistic or descriptive rules.

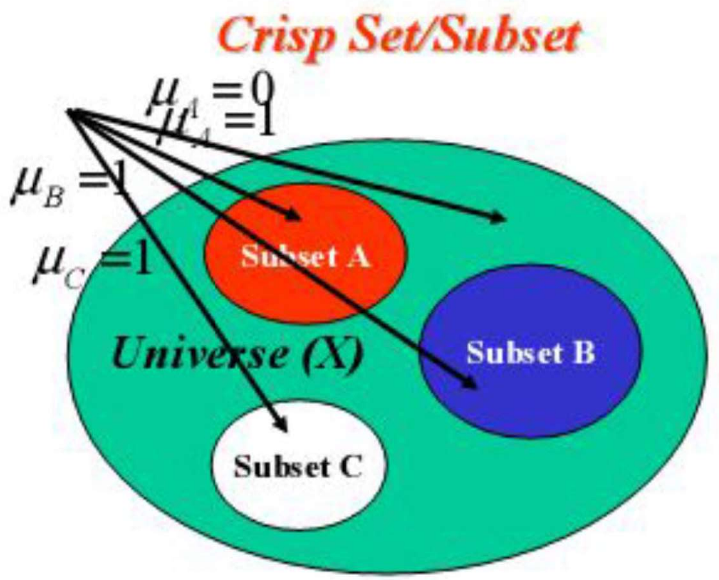
CLASSICAL SETS (CRISP SETS)

Conventional or crisp sets are Binary. An element either belongs to the set or does not.

{True, False}

{1, 0}

CRISP SETS



Crisp Set/Subset
$\mu_A = 0$
$\mu_A = 1$
$\mu_B = 1$
$\mu_C = 1$
Subset A
Universe (X)
Subset B
Subset C

FUZZY SETS

Rules of thumb frequently stated in "fuzzy" linguistic terms.

John is *tall.*

If someone is *tall* **and** *well-built*
then his basketball skill is good.

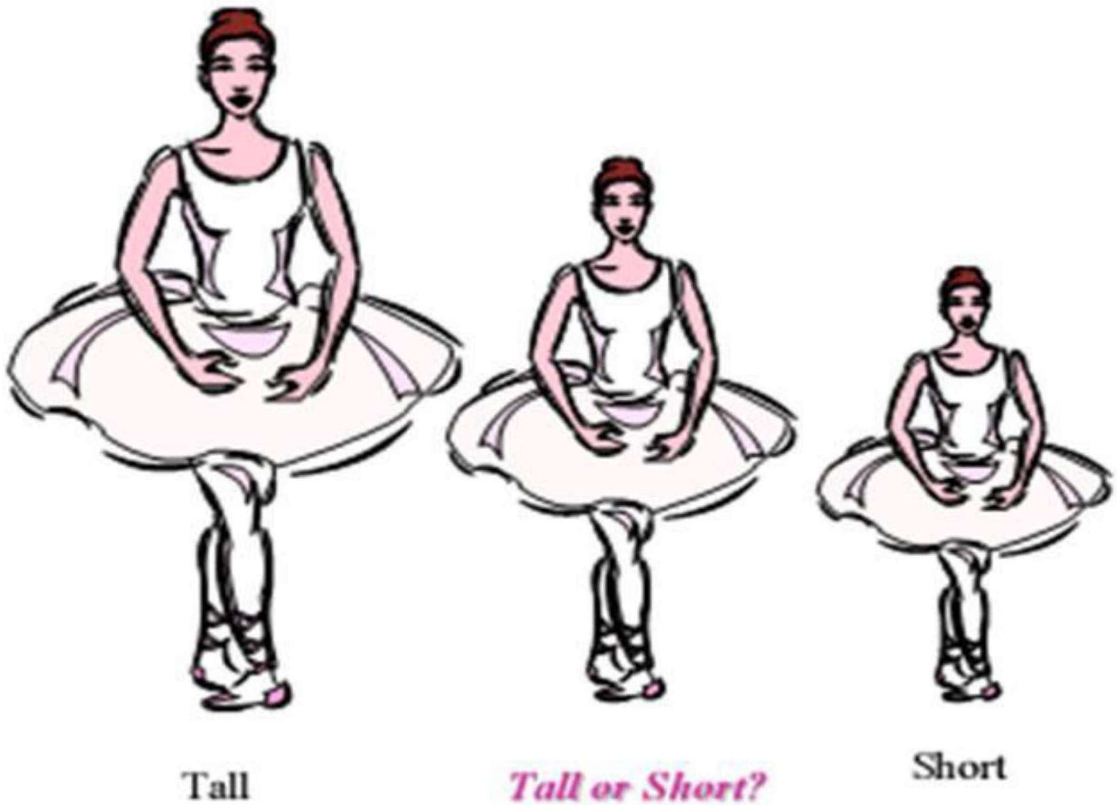
<u>Fuzzy Sets</u>

$0 \leq \mu S(x) \leq 1$ ------ $\mu S(x)$ (or $\mu(S, x)$) is the **degree** of membership of x in set S

$\mu S(x) = 0$ x is not at all in S

$\mu S(x) = 1$ x is fully in S.

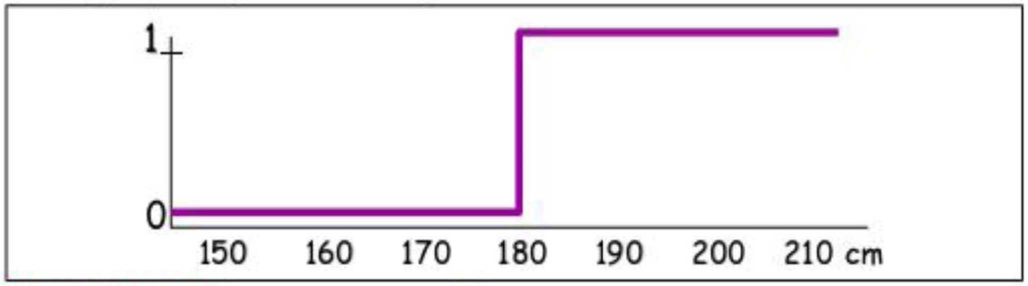
If $\mu S(x) = 0$ or 1, then the set S is **crisp.**


Tall
Tall or Short?
Short

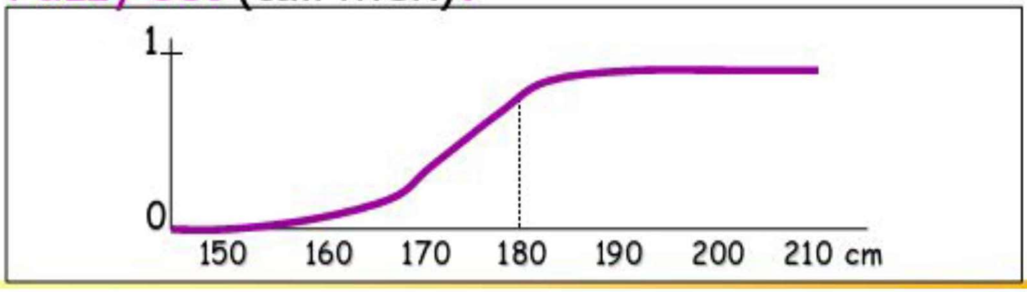
Fuzzy set

- Is a function f: domain → [0,1]

Crisp set (tall men):



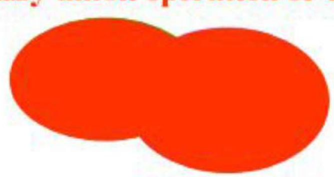
Fuzzy set (tall men):



OPERATIONS ON FUZZY SETS

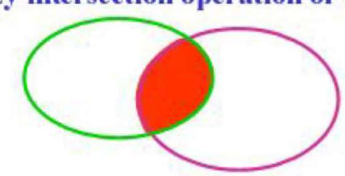
- Union: $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$
- Intersection: $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$
- Complement: $\mu_{\neg A}(x) = 1 - \mu_A(x)$

Fuzzy union operation or fuzzy *OR*



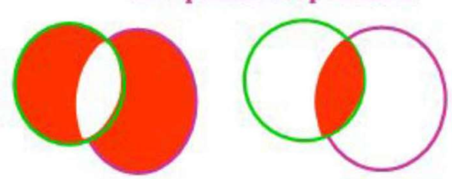
$$\mu_{A+B}(x) = \max[\mu_A(x), \mu_B(x)]$$

Fuzzy intersection operation or fuzzy *AND*



$$\mu_{A\cdot B}(x) = \min[\mu_A(x), \mu_B(x)]$$

Complement operation

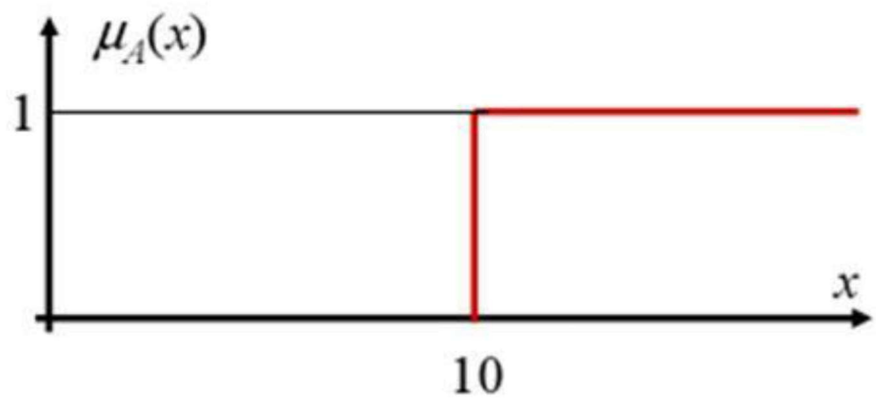


$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

CRISP MEMBERSHIP FUCNTIONS

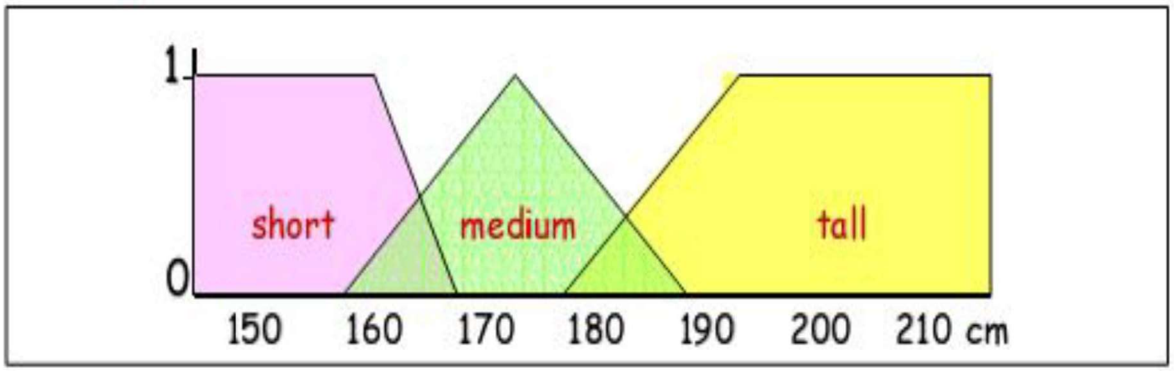
- Crisp membership functions (µ) are either one or zero.
- Consider the example: Numbers greater than 10. The membership curve for the set A is given by

$$A = \{x \mid x > 10\}$$



REPRESENTING A DOMAIN IN FUZZY LOGIC

Fuzzy sets (men's height):



FUZZY MEMBERSHIP FUCNTIONS

- Categorization of element x into a set A
  described through a membership function $\mu_A(x)$
- Formally, given a fuzzy set A of universe X
  $\mu_A(x): X \rightarrow [0,1]$, where
  - $\mu_A(x) = 1$ if x is totally in A
  - $\mu_A(x) = 0$ if x is totally not in A
  - $0 < \mu_A(x) < 1$ if x is partially in A

  $\mu_{Tall}(200) = 1$
  $\mu_{Tall}(160) = 0$
  $0 < \mu_{Tall}(180) < 1$
- (Discrete) Fuzzy set A is represented as:
  $A = \{\mu_A(x_1)/x_1, \mu_A(x_2)/x_2, \ldots, \mu_A(x_n)/x_n\}$

  Tall = {0/160, 0.2/170, 0.8/180, 1/190}

FUZZINESS vs PROBABILITY

- When first exposed to fuzzy logic, humans associate membership functions with density functions.
- This is not so, since:
  - Probability density is an abstraction from empirical frequency.

    ⇒ an aggregate property.
    - how often events occur in different ways.
    - ways that are quite crisp and mutually exclusive after occurrence.
  - Fuzzy relations, by contrast, are properties of single events that are always there, and not different from occurrence to occurrence.

LINGUISTIC VARIABLE

- A linguistic variable associates words or sentences with a measure of belief functions, also called membership function.
- The set of values that it can take is called term set.
- Each value in the set is a fuzzy variable defined over a base variable.
- The base variable defines the Universe of discourse for all the fuzzy variables in the term set.

A linguistic variable is a quintuple [X, $T(X)$, U, G, M] where

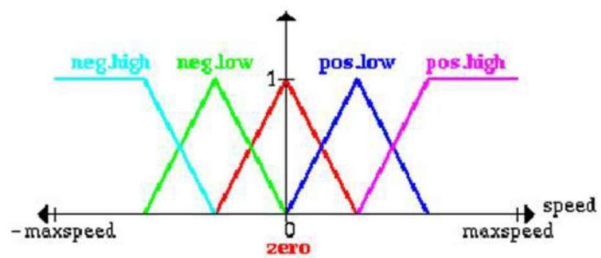
- X is the name of the variable,
- $T(X)$ is the term set, i.e. the set of names of linguistic values of X,
- U is the universe of discourse,
- G is the grammar to generate the names and
- M is a set of semantic rules for associating each X with its meaning.

LINGUISTIC VARIABLE

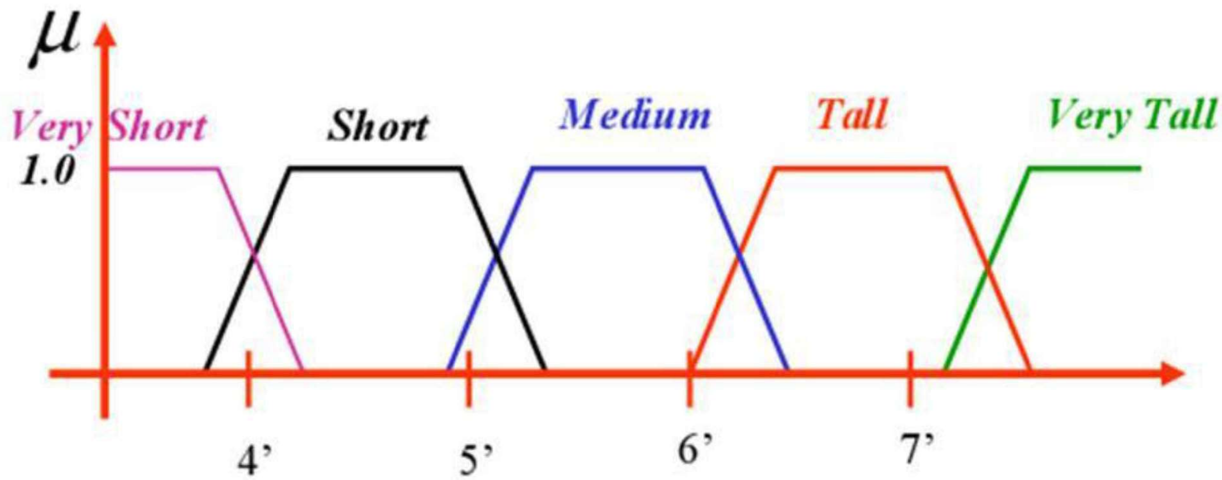
- Let x be a linguistic variable with the label “speed”.
- Terms of x, which are fuzzy sets, could be “positive low”, “negative high” from the term set T:

T = {PostiveHigh, PositiveLow, NegativeLow, NegativeHigh, Zero}

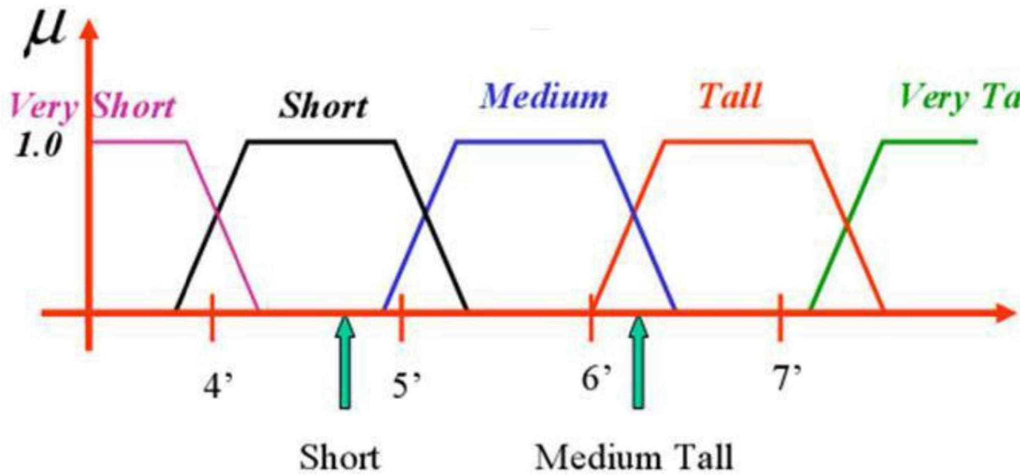
- Each term is a fuzzy variable defined on the base variable which might be the scale of all relevant velocities.



MEMBERSHIP FUCNTIONS



$$\mu = [\mu_{vs}, \mu_{s}, \mu_{m}, \mu_{t}, \mu_{vt}]$$



$$\mu = [0,\ 1,\ 0,\ 0,\ 0]$$

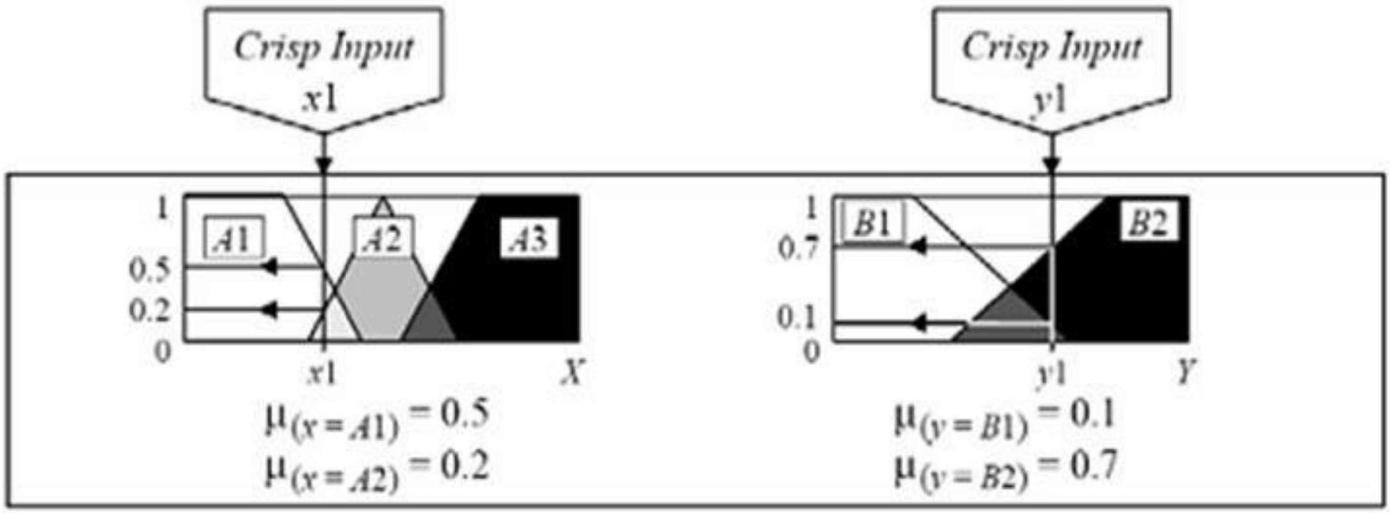
$$\mu = [0,\ 0,\ 0.5,\ 0.5,\ 0]$$

FUZZIFICATION

- Fuzzifier converts a crisp input into a fuzzy variable.
- Definition of the membership functions must
  - reflects the designer's knowledge
  - provides smooth transition between member and nonmembers of a fuzzy set
  - simple to calculate
- Typical shapes of the membership function are Gaussian, trapezoidal and triangular.

- Use crisp inputs from the user.

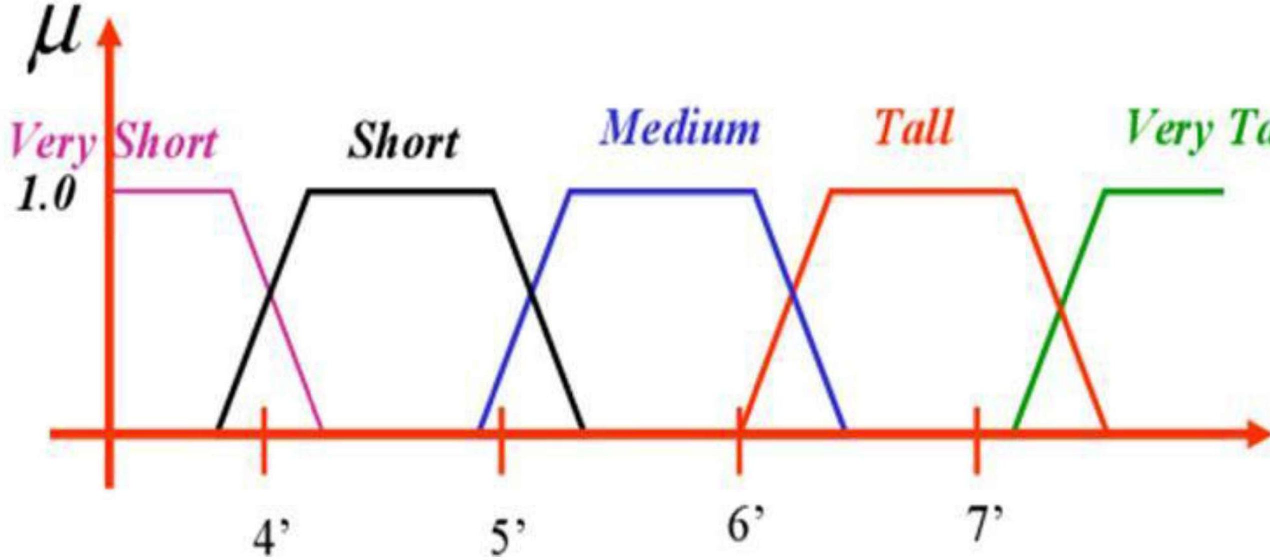
- Determine membership values for all the relevant classes (i.e., in right Universe of Discourse).



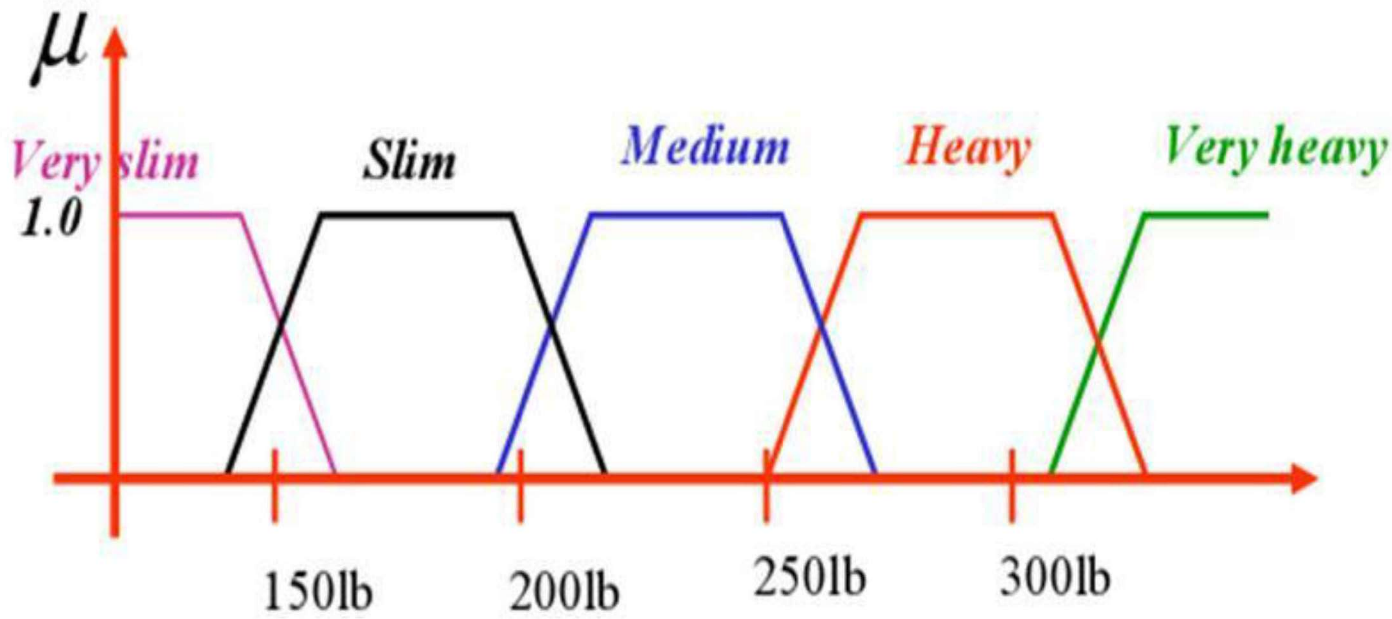
EXAMPLE - FUZZIFICATION

- Assume we want to evaluate the health of a person based on his height and weight.
- The input variables are the crisp numbers of the person's height and weight.
- Fuzzification is a process by which the numbers are changes into linguistic words

FUZZIFICATION OF HEIGHT



FUZZIFICATION OF WEIGHT



FUZZY RULES AND REASONING

The degree of an element in a fuzzy set corresponds to the truth value of a proposition in fuzzy logic systems.

FUZZY RULES

A fuzzy rule is defined as the conditional statement of the form

If x is A
THEN y is B

where x and y are linguistic variables and A and B are linguistic values determined by fuzzy sets on the universes of discourse X and Y.

- The decision-making process is based on rules with sentence conjunctives **AND, OR** and **ALSO**.

- Each rule corresponds to a fuzzy relation.

- Rules belong to a **rule base.**

- Example: If (Distance x to second car is **SMALL) OR** (Distance y to obstacle is **CLOSE**) **AND** (speed v is **HIGH**) **THEN** (perform **LARGE** correction to steering angle θ) **ALSO** (make **MEDIUM** reduction in speed v).

- Three antecedents (or premises) in this example give rise to two outputs (consequences).

FUZZY RULE FORMATION

IF height is tall
THEN weight is heavy.

Here the fuzzy classes height and weight have a given range (i.e., the universe of discourse).

range (height) = [140, 220]
range (weight) = [50, 250]

FORMATION OF FUZZY RULES

Three general forms are adopted for forming fuzzy rules. They are:

- Assignment statements,
- Conditional statements,
- Unconditional statements.

Assignment Statements

y = small
Orange color = orange
$a = s$
Paul is not tall and not very short
Climate = autumn
Outside temperature = normal

Unconditional Statements

Goto sum.
Stop.
Divide by a.
Turn the pressure low.

Conditional Statements

IF y is very cool THEN stop.
IF A is high THEN B is low ELSE B is not low.
IF temperature is high THEN climate is hot.

DECOMPOSITION OF FUZZY RULES

A compound rule is a collection of several simple rules combined together.

- Multiple conjunctive antecedent,
- Multiple disjunctive antecedent,
- Conditional statements (with ELSE and UNLESS).

DECOMPOSITION OF FUZZY RULES

Multiple Conjunctive Antecedants

IF x is $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ THEN y is B_m.
Assume a new fuzzy subset $\underset{\sim}{A}_m$ defined as

$$\underset{\sim}{A}_m = \underset{\sim}{A}_1 \cap \underset{\sim}{A}_2 \cap \cdots \cap \underset{\sim}{A}_n$$

and expressed by means of membership function

$$\mu_{\underset{\sim}{A}_m}(x) = \min[\mu_{\underset{\sim}{A}_1}(x), \mu_{\underset{\sim}{A}_2}(x), \ldots \mu_{\underset{\sim}{A}_n}(x)].$$

Multiple disjunctive antecedent

IF x is $\underset{\sim}{A}_1$ OR x is $\underset{\sim}{A}_2, \ldots$ OR x is $\underset{\sim}{A}_n$ THEN y is $\underset{\sim}{B}_m$.
This can be written as
IF x is $\underset{\sim}{A}_n$ THEN y is $\underset{\sim}{B}_m$,
where the fuzzy set $\underset{\sim}{A}_m$ is defined as

$$\underset{\sim}{A}_m = \underset{\sim}{A}_1 \cup \underset{\sim}{A}_2 \cup \underset{\sim}{A}_3 \cup \cdots \cup \underset{\sim}{A}_n$$

Conditional Statements (With Else and Unless)

IF $\underset{\sim}{A}_1$ (THEN $\underset{\sim}{B}_1$) UNLESS $\underset{\sim}{A}_2$
can be decomposed as
IF $\underset{\sim}{A}_1$ THEN $\underset{\sim}{B}_1$
OR
IF $\underset{\sim}{A}_2$ THEN NOT $\underset{\sim}{B}_1$
IF $\underset{\sim}{A}_1$ THEN ($\underset{\sim}{B}_1$) ELSE IF $\underset{\sim}{A}_2$ THEN ($\underset{\sim}{B}_2$)
can be decomposed into the form
IF $\underset{\sim}{A}_1$ THEN $\underset{\sim}{B}_1$
OR
IF NOT $\underset{\sim}{A}_1$ AND IF $\underset{\sim}{A}_2$ THEN $\underset{\sim}{B}_2$

AGGREGATION OF FUZZY RULES

Aggregation of rules is the process of obtaining the overall consequents from the individual consequents provided by each rule.

- Conjunctive system of rules.
- Disjunctive system of rules.

FUZZY RULE - EXAMPLE

Rule 1: If height is short then weight is light.

Rule 2: If height is medium then weight is medium.

Rule 3: If height is tall then weight is heavy.

Problem: Given

(a) membership functions for short, medium-height, tall, light, medium-weight and heavy;
(b) The three fuzzy rules;
(c) the fact that John's height is 6'1"

estimate John's weight.

Solution:

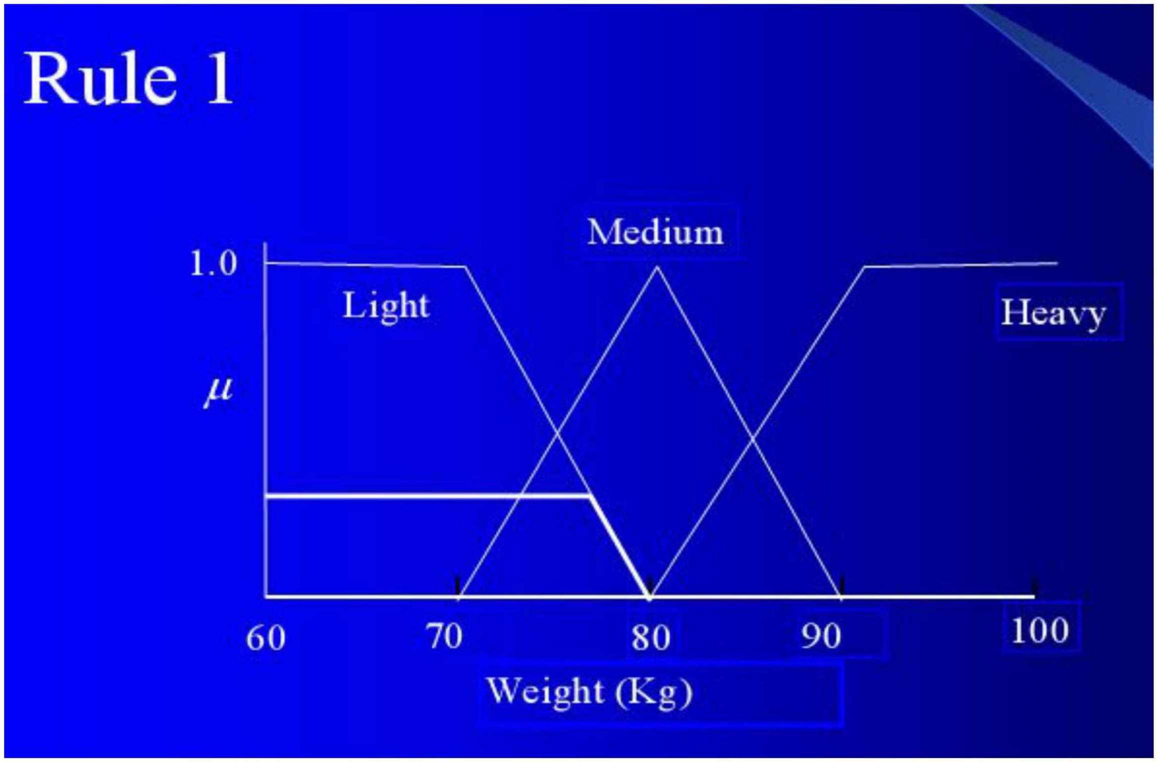
(1) From John's height we know that

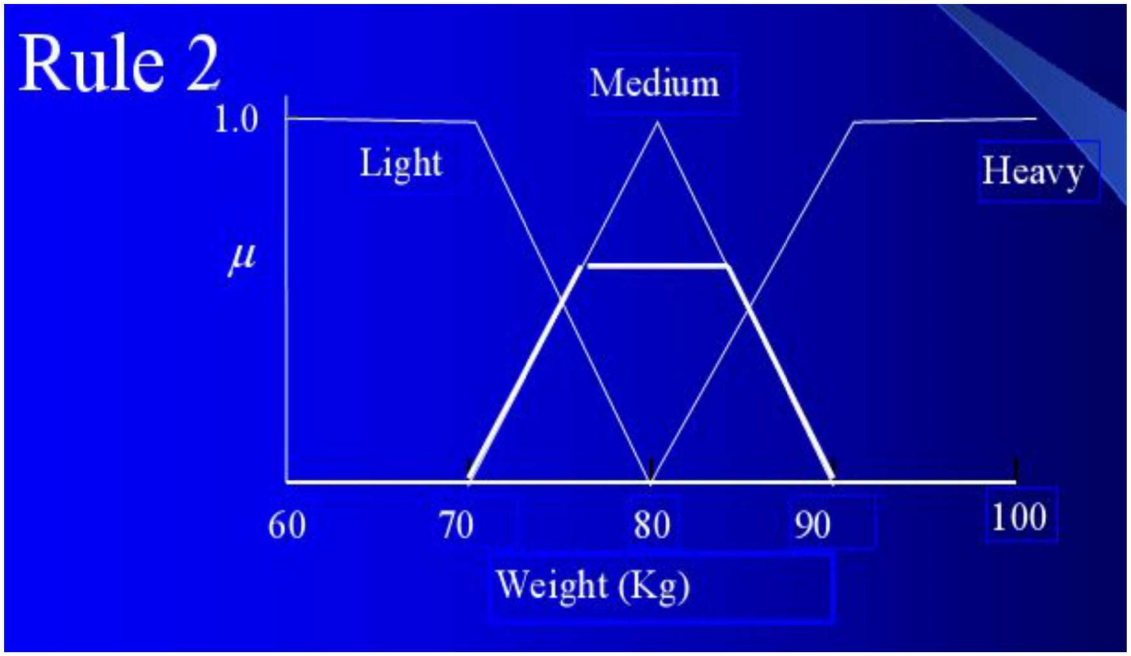
John is short (degree 0.3)

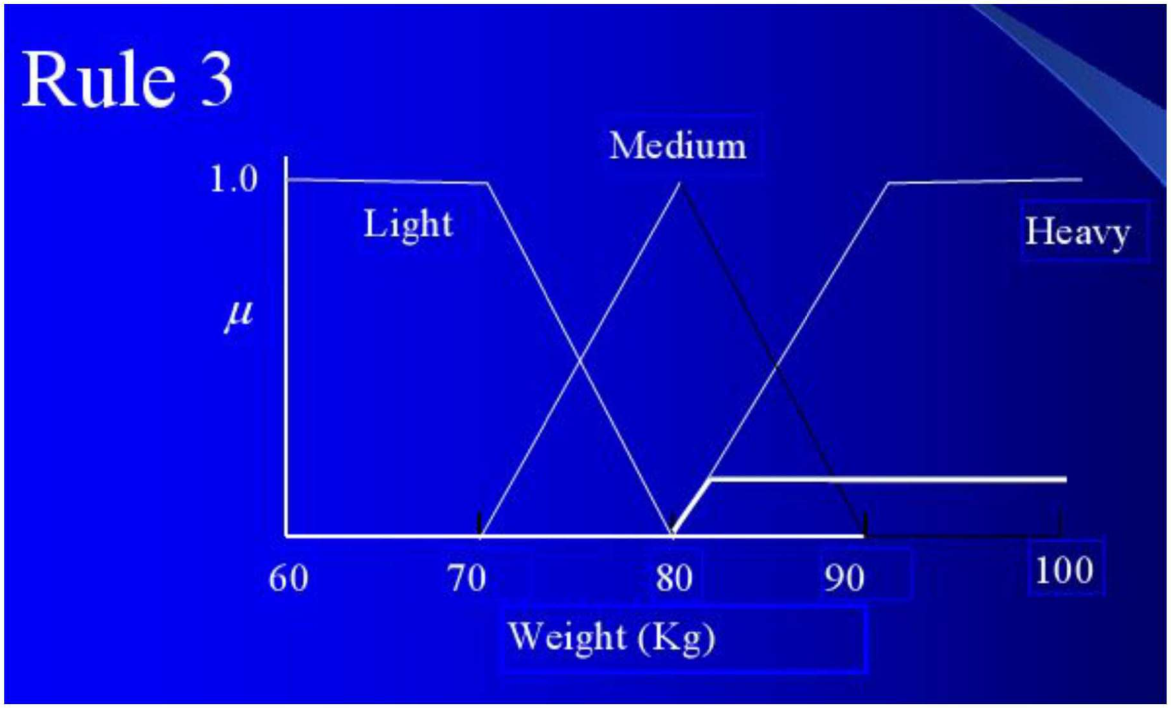
John is of medium height (degree 0.6).

John is tall (degree 0.2).

(2) Each rule produces a fuzzy set as output by truncating the consequent membership function at the value of the antecedent membership.

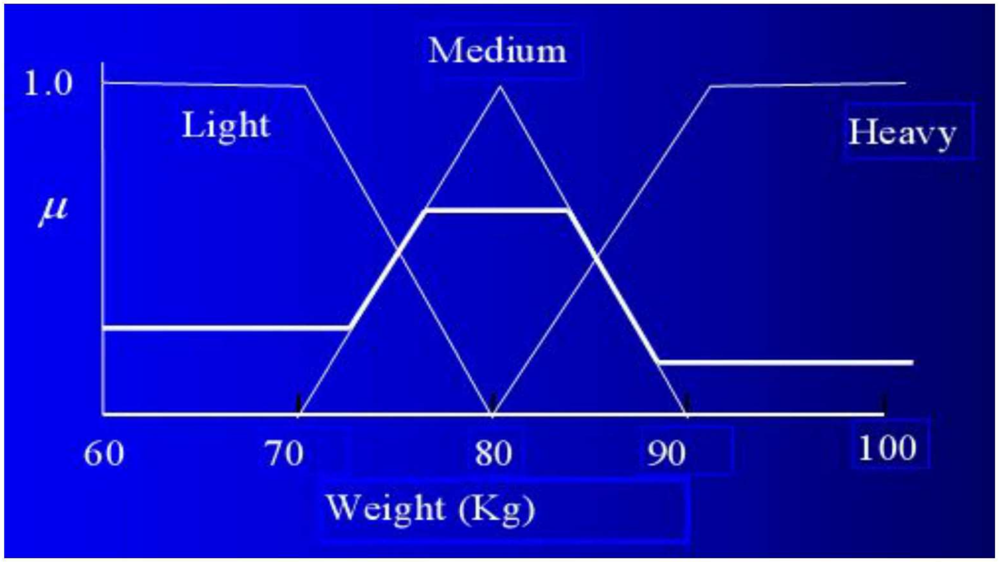

Rule 1
Medium
1.0
Light
Heavy
μ
60
70
80
90
100
Weight (Kg)


Rule 2
1.0
Medium
Light
Heavy
μ
60
70
80
90
100
Weight (Kg)


Rule 3
Medium
1.0
Light
Heavy
μ
60
70
80
90
100
Weight (Kg)

- The cumulative fuzzy output is obtained by OR-ing the output from each rule.

- Cumulative fuzzy output (weight at 6’1”).

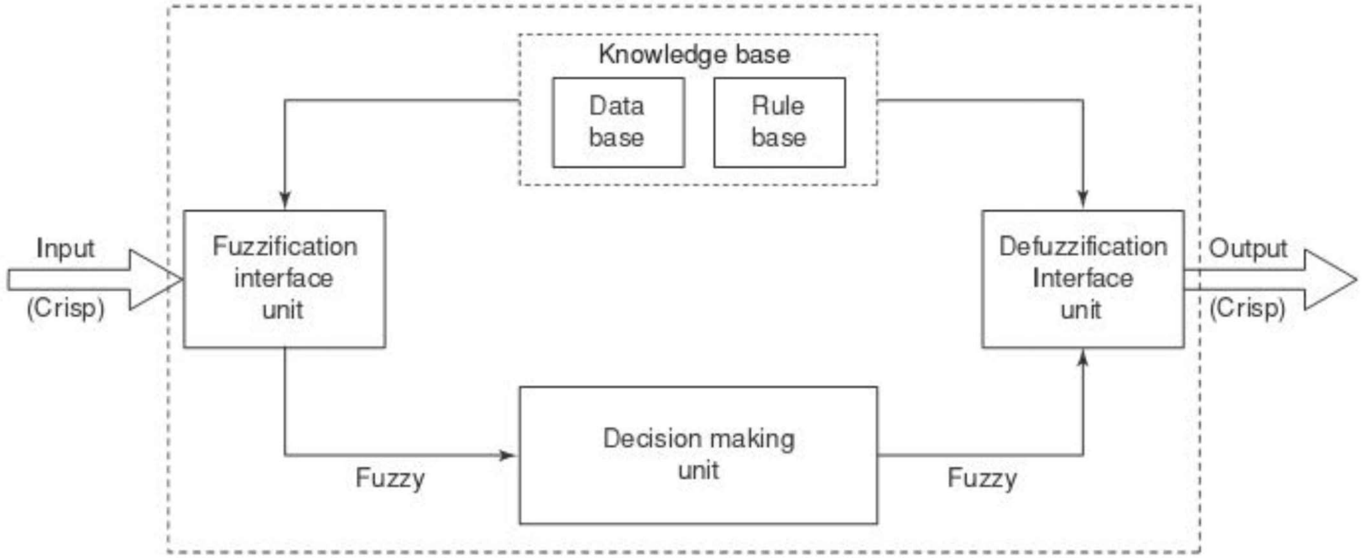


1. De-fuzzify to obtain a numerical estimate of the output.
2. Choose the middle of the range where the truth value is maximum.
3. John's weight = 80 Kg.

FUZZY INFERENCE SYSTEMS (FIS)

- Fuzzy rule based systems, fuzzy models, and fuzzy expert systems are also known as fuzzy inference systems.
- The key unit of a fuzzy logic system is FIS.
- The primary work of this system is decision-making.
- FIS uses "IF...THEN" rules along with connectors "OR" or "AND" for making necessary decision rules.
- The input to FIS may be fuzzy or crisp, but the output from FIS is always a fuzzy set.
- When FIS is used as a controller, it is necessary to have crisp output.
- Hence, there should be a defuzzification unit for converting fuzzy variables into crisp variables along FIS.

BLOCK DIAGRAM OF FIS



TYPES OF FIS

There are two types of Fuzzy Inference Systems:

- Mamdani FIS(1975)
- Sugeno FIS(1985)

MAMDANI FUZZY INFERENCE SYSTEMS (FIS)

- Fuzzify input variables:
  - Determine membership values.
- Evaluate rules:
  - Based on membership values of (composite) antecedents.
- Aggregate rule outputs:
  - Unify all membership values for the output from all rules.
- Defuzzify the output:
  - COG: Center of gravity (approx. by summation).

SUGENO FUZZY INFERENCE SYSTEMS (FIS)

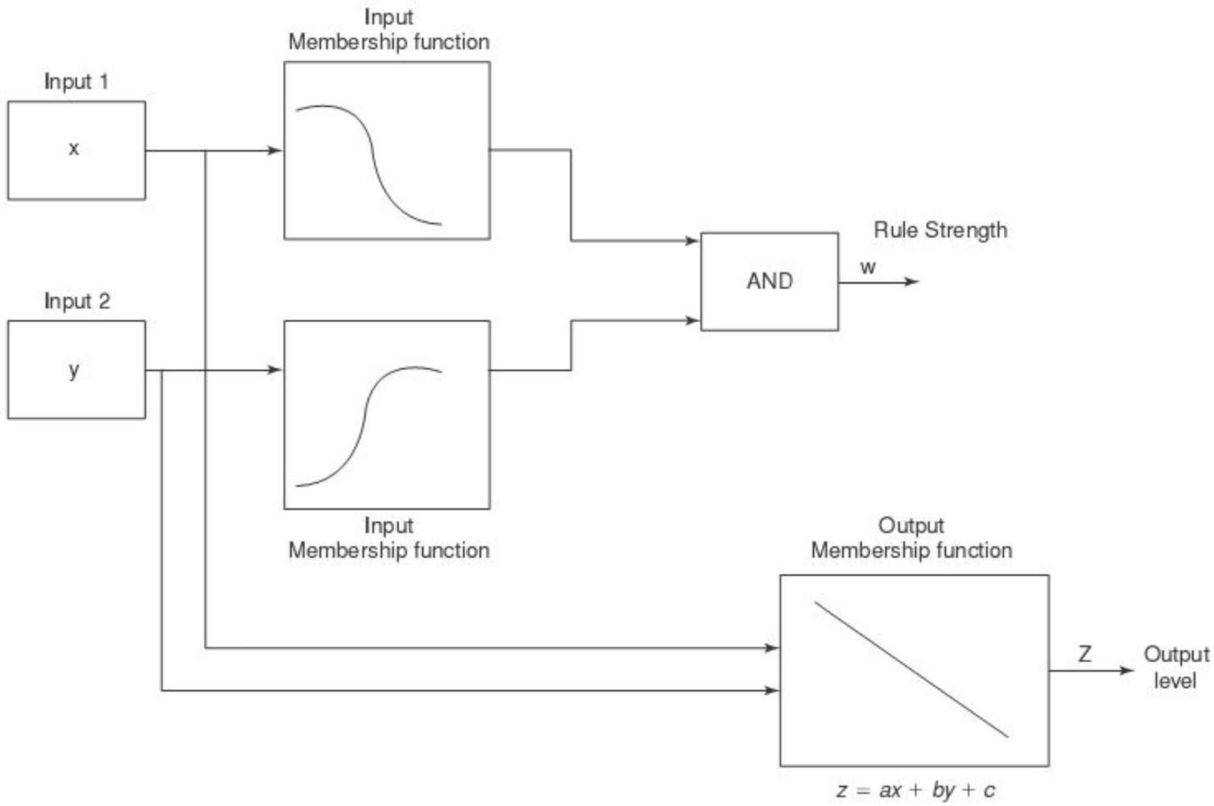
The main steps of the fuzzy inference process namely,

1. fuzzifying the inputs and

2. applying the fuzzy operator are exactly the same as in MAMDANI FIS.

The main difference between Mamdani's and Sugeno's methods is that Sugeno output membership functions are either linear or constant.

SUGENO FIS

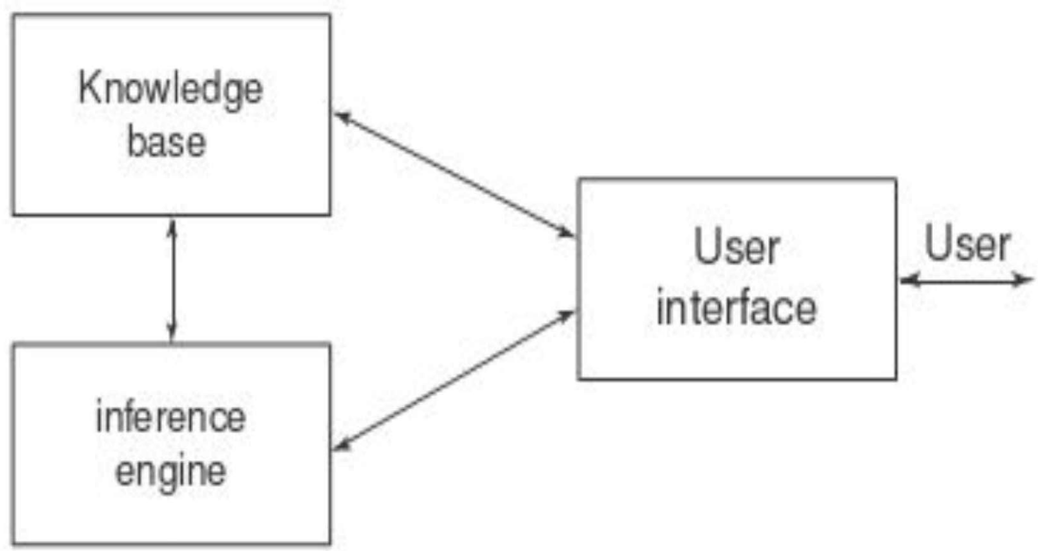

Input
Membership function
Input 1
x
Rule Strength
w
AND
Input 2
y
Input
Membership function
Output
Membership function
Z
Output
level
z = ax + by + c

FUZZY EXPERT SYSTEMS

An expert system contains three major blocks:

- Knowledge base that contains the knowledge specific to the domain of application.
- Inference engine that uses the knowledge in the knowledge base for performing suitable reasoning for user's queries.
- User interface that provides a smooth communication between the user and the system.

BLOCK DIAGRAM OF FUZZY EXPERT SYSTEMS



Examples of Fuzzy Expert System include Z-II, MILORD.