T-test for difference of Mean

Suppose we want to test if two independent samples x_i $(i = 1, 2 \ldots n_1)$ and y_j, $(j = 1, 2, \ldots n_2)$ of size n_1 and n_2 have been drawn from two normal populations with means μ_x and μ_y respectively.

Under the null hypothesis (H_0) that the samples have been drawn from the normal populations with means μ_x and μ_y under the assumption that the population variance are equal, ie

$$\sigma_x^2 = \sigma_y^2 = \sigma^2 \text{ (say)}$$

the statistic

$$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where $S^2 = \dfrac{1}{n_1 + n_2 - 2}\left[\sum_i (x_i - \bar{x})^2 + \sum_j (y_j - \bar{y})^2\right]$

is an unbiased estimate of the common population variance σ^2, follows, Student's t-dist with $(n_1 + n_2 - 2)$ d.f.

<u>Example</u> Below are given the gain in weight (in kgs) of pigs fed on two diets A and B.

Gain in weight

Diet A | 25 32 30 34 24 14 32 24 30 31 35 25

Diet B | 44 34 22 10 47 31 40 30 32 35 18 21 35 29 22

Test, if two diets differ significantly as regards to their effect on increase in weight.

Null hypothesis $H_0: \mu_x = \mu_y$, ie there is no significant difference between the mean increase in weight due to ~~two~~ diets

Alternate hypo: $H_1: \mu_x \neq \mu_y$

Diet A			Diet B		
x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
25	-3	9	44	14	196
32	4	16	34	4	16
30	2	4	22	-8	64
34	6	36	10	-20	400
24	-4	16	47	17	289
14	-14	196	31	1	1
32	4	16	40	10	100
24	-4	16	30	0	0
30	2	4	32	2	4
31	3	9	35	5	25
35	7	49	18	-12	144
25	-3	9	21	-9	81
$\Sigma x = 336$	$\Sigma(x-\bar{x}) = 0$	$\Sigma(x-\bar{x})^2 = 380$	35	5	25
			29	-1	1
			22	-8	64
			$\Sigma y = 450$	$\Sigma(y-\bar{y}) = 0$	$\Sigma(y-\bar{y})^2 = 1410$

$\bar{x} = \frac{336}{12} = 28, \quad \bar{y} = \frac{450}{15} = 30$

$$S^2 = \frac{1}{n_1+n_2-2}\left[\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2\right]$$

$$= \frac{1}{12+15-2}\left[380 + 1410\right] = 71.6$$

Under the null hypothesis

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{S^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{28-30}{\sqrt{71.6\left(\frac{1}{12}+\frac{1}{15}\right)}}$$

$$= \frac{-2}{\sqrt{10.74}} = -0.609$$

Tabulated $t_{0.05}$ for $(12+15-2)=25$ deg of freedom $= 2.06$

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Conclusion: $|t| = 0.609 <$ tabulated t

Ho may be accepted at 5% level of significance and we conclude that the two diets do not differ significantly as regards to their increase in weight.

Observation : What have you noted ?

In single mean t-test, one sample was given and based on that sample, we were testing whether the sample can be considered to have come from a normal population with a hypothetical mean.

But in t-test for difference of mean, we have two samples of different sizes and we are testing whether they have been drawn from two different normal populations.

Ex ② Samples of two types of electric bulbs were tested for length of life and following data were obtained.

	Type I	Type II
Sample No	$n_1 = 8$	$n_2 = 7$
Sample means	$\bar{x}_1 = 1234$ hrs	$\bar{x}_2 = 1036$ hrs
Sample S.D.'s	$s_1 = 36$ hrs	$s_2 = 40$ hrs

Is the difference in means sufficient to warrant that type I is superior to type II regarding the length of life?

Null hypothesis $H_0 : \mu_x = \mu_y$, the two types I and II of electric bulbs are identical.

Alternative hypothesis : $H_1 : \mu_x > \mu_y$, ie type I is superior to type II

Test statistics

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$s^2 = \frac{1}{n_1+n_2-2}\left[\sum (x_1-\bar{x}_1)^2 + \sum (y_1-\bar{y}_1^2)\right]$$

$$= \frac{1}{n_1+n_2-2}\left[n_1 s_1^2 + n_2 s_2^2\right]$$

$$= \frac{1}{13}\left[8 \times 36^2 + 7 \times 40^2\right]$$

$$= 1659.08$$

$$t = \frac{1234 - 1036}{\sqrt{1659.08\left(\frac{1}{8}+\frac{1}{7}\right)}} = \frac{198}{\sqrt{1659.08 \times 0.2679}}$$

$$= 9.39$$

Tabulated t for 13 deg of freedom at 5% level for right (single) tailed test is 1.77

$$9.39 > 1.77$$

<u>Conclusion</u> : Null hypothesis is rejected as calculated t is much greater than tabulated t value.

Hence type I is definitely superior.

Paired t-test for difference of Mean

Consider the case when (i) the sample sizes are equal i.e $n_1 = n_2 = n$ (say), and (ii) the two samples are not independent but sample observations are paired together, i.e, the pair of observations (x_i, y_i) $(i = 1, 2, \ldots n)$ corresponds to the same (ith) sample unit. The problem is to test if the sample means differ significantly or not.

For example, suppose we want to test the efficacy of a particular drug, say, for inducing sleep. Let x_i and y_i $(i = 1, 2, \ldots n)$ be the readings, in hours of sleep, on the ith individual before and after the drug is given respectively. Here instead of applying the difference of mean test discussed in previous class, we apply paired t-test given below

Here we consider the increments $d_i = x_i - y_i$
$(i = 1, 2 - - n)$

Under the null hypothesis, H_0 that increments are due to fluctuations of sampling, i.e., the drug is not responsible for these increments

the statistic: $d = \dfrac{\bar{d}}{\frac{S}{\sqrt{n}}}$

where $\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (d_i - \bar{d})^2$

follows student's t-distribution with $(n-1)$ degree of freedom.

Ex (1) A certain stimulus administered to each of the 12 patients resulted in the following increase of Blood pressure:
5, 2, 8, −1, 3, 0, −2, 1, 5, 0, 4 and 6
Can we conclude that the stimulus, in general, be accompanied by an increase in blood pressure?

Null hypothesis $H_0: \mu_x = \mu_y$ i.e., there is no significant difference in the blood pressure readings of the patients before and after the drug. In other words the given increments are just by chance and not due to stimulus.

Alternative hypothesis $H_1: \mu_x < \mu_y$, i.e the stimulus results in an increase in blood pressure.

Test statistic $t = \dfrac{\bar{d}}{S/\sqrt{n}}$

$$\bar{d} = \frac{1}{n}\sum d \quad \text{and} \quad s^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{(\sum d)^2}{n}\right]$$

d	5	2	8	-1	3	0	-2	1	5	0	4	6
d^2	25	4	64	1	9	0	4	1	25	0	16	36

$$\sum d = 31 \quad \text{and} \quad \sum d^2 = 185$$

$$\bar{d} = \frac{31}{12} = 2.58$$

$$s^2 = \frac{1}{12-1}\left[\sum d^2 - \frac{(\sum d)^2}{n}\right]$$

$$= \frac{1}{11}\left[185 - \frac{(31)^2}{12}\right] = 9.5382$$

$$t = \frac{\bar{d}}{s/\sqrt{n}} = \frac{2.58}{\frac{\sqrt{9.5382}}{\sqrt{12}}} = \frac{2.58 \times \sqrt{12}}{\sqrt{9.5382}}$$

$$= 2.89$$

Tabulated $t_{0.05}$ for 11 d.f $= 1.80$

Calulated $t > $ Tabulated t

Null hypothesis is rejected
Hence we conclude that, the stimulus, will in general be accompanied by an increase in blood pressure.

Ex (2) In a certain experiment to compare two types of animal foods A and B the following result of increase in weights were observed in animals:

Animal number		1	2	3	4	5	6	7	8	Total
Increase weight in pound	Food A	49	53	51	52	47	50	52	53	407
	Food B	52	55	52	53	50	54	54	53	423

(i) Assuming that the two samples of animals are ~~independent~~ independent, can we conclude that food B is better than food A?

(ii) Also examine the case when the same set of eight animals were used in both the fields.

Sol: Null hypothesis H_0: If the increase in weights due to food A and B are denoted by X and Y respectively, then $H_0: \mu_X = \mu_Y$, i.e. there is no significant difference in increase in weight due to diets A and B.

Alternative hypothesis $H_1: \mu_X < \mu_Y$

(1) If two samples are assumed to be independent we will apply t-test for difference of means to test H_0

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Food A			Food B		
X	$d=X-50$	d^2	Y	$D=Y-52$	D^2
49	-1	1	52	0	0
53	3	9	55	3	9
51	1	1	52	0	0
52	2	4	53	1	1
47	-3	9	50	-2	4
50	0	0	54	2	4
52	2	4	54	2	4
53	3	9	53	1	1
Total	7	37	Total	7	23

$\bar{x} = 50 + \frac{7}{8}$
$= 50.875$

$\bar{y} = 52 + \frac{7}{8}$
$= 52.875$

and $\sum(x-\bar{x})^2 = \sum d^2 - \frac{(\sum d^2)}{n_1} = 37 - \frac{49}{8}$
$= 30.875$

and $\sum(y-\bar{y})^2 = \sum D^2 - \frac{(\sum D)^2}{n_2}$
$= 23 - \frac{49}{8} = 16.875$

$S^2 = \frac{1}{n_1+n_2-2}\left[\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2\right]$
$= \frac{1}{14}[30.875 + 16.875] = 3.41$

Tabulated $t_{0.05}$ for $(8+8-2) = 14$ d.f $= 1.76$

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{50.875 - 52.875}{\sqrt{3.41\left(\frac{1}{8} + \frac{1}{8}\right)}} = -2.17$$

The critical region for left tail test is $t < -1.76$

Since calculated t is less than -1.76, H_0 is rejected at 5% level of significance.

So we conclude that food B is superior.

(11) If the same set of animals is used in both the cases, then reading X and Y are not independent but they are paired together and we apply the paired t-test for H_0

$$H_0: \mu_X = \mu_Y$$

$$t = \frac{\bar{d}}{S/\sqrt{n}}$$

X	49	53	51	52	47	50	52	53	Total
Y	52	55	52	53	50	54	54	53	
$d = X - Y$	-3	-2	-1	-1	-3	-4	-2	0	-16
d^2	9	4	1	1	9	16	4	0	44

$$\bar{d} = \frac{\Sigma d}{n} = \frac{-16}{8} = -2$$

$$S^2 = \frac{1}{n-1}\left[\Sigma d^2 - \frac{(\Sigma d)^2}{n}\right]$$

$$= \frac{1}{7}\left(44 - \frac{256}{8}\right) = 1.714$$

$$t = \frac{-2}{\frac{\sqrt{1.714}}{\sqrt{8}}} = \frac{-2 \times \sqrt{8}}{\sqrt{1.714}} = -4.32$$

Tabulated $t_{0.05}$ for $(8-1)=7$ d.f for one tail test is 1.90

Calculated $|t| > 1.90$

Null hypothesis is rejected

So we conclude that food B is superior.