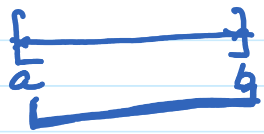
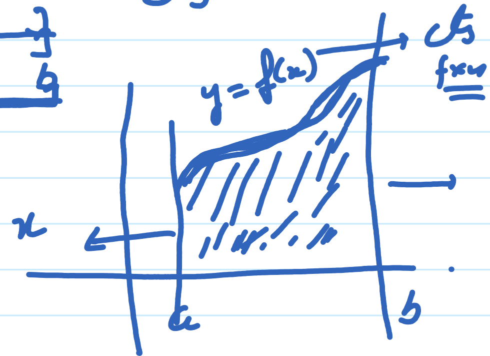
Multiple Integrals

$\iint f(x,y)\,dx\,dy$

① $\int_a^b f(x)\,dx$ where $a \le x \le b$,

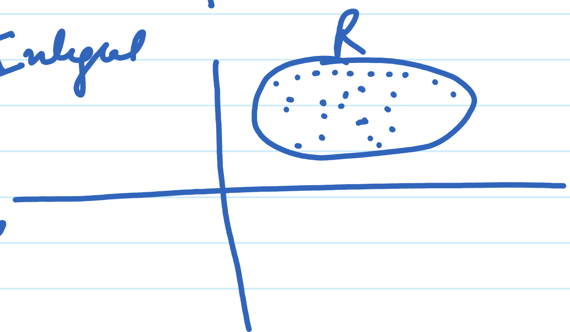




continuous on the region R

① $\iint_R f(x,y)\,dx\,dy \rightarrow$ Double Integral

R → is the region of integration



② $\iiint_R f(x,y,z)\,dx\,dy\,dz \rightarrow$ Triple Integral

③ $\iiint \cdots \int_R f(x_1, x_2, \ldots x_n)\,dx_1\,dx_2\,dx_3 \ldots dx_n$ → Multiple integrals

Evaluation of Double Integral

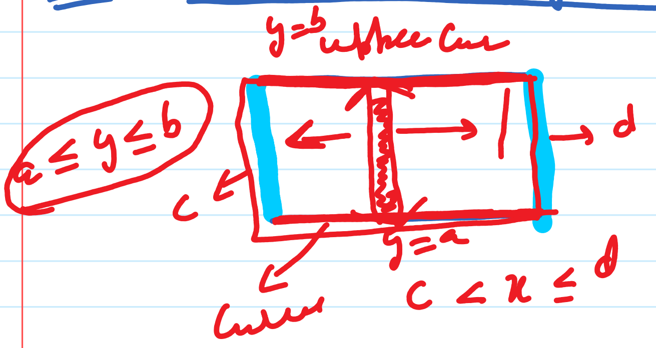
$$I = \iint_R f(x,y)\,dx\,dy = \int\int_R \left[f(x,y)\,dx\right] dy$$

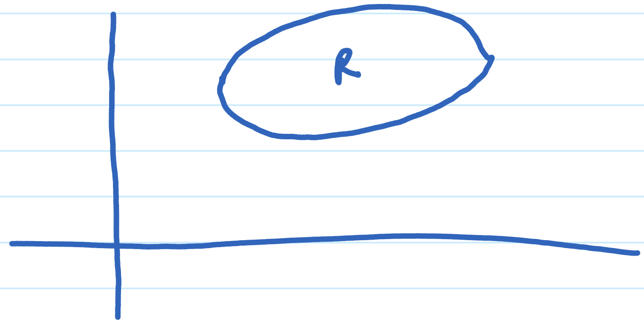
1st, we integrate w.r.t x, then w.r.t 'y'

$$I = \iint_R f(x,y)\,dy\,dx$$

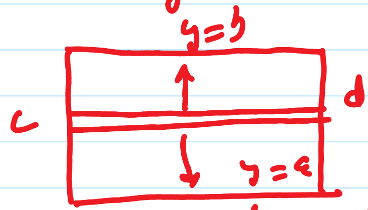
1st w.r.t 'y' then w.r.t 'x'

Case R be the region of integration

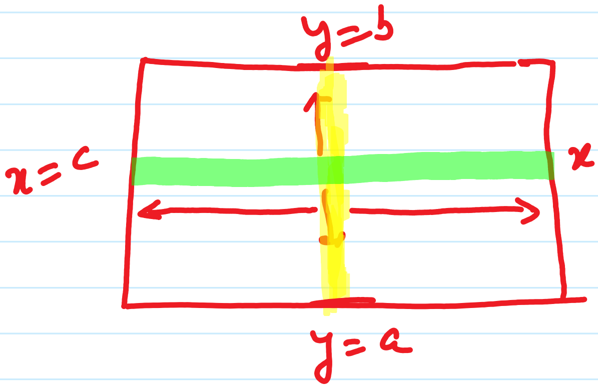




$R = \{(x,y).\ a \le y \le b, c \le x \le d\}$



$R = \{(x,y)\ \ c \le x \le d, a \le y \le b\}$



$a \le y \le b$

$c \le x \le d$

$c \le x \le d, \quad a \le y \le b$

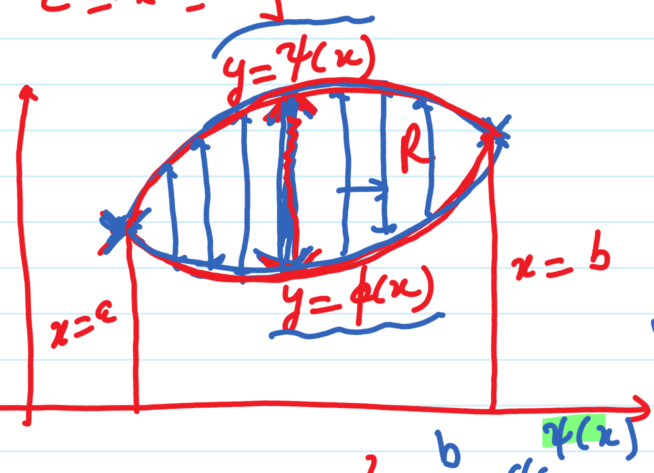
$R\{(x,y): a \le y \le b, \quad c \le x \le d\}$

Case I

$f(x,y)$

$R = \{(x,y): \phi(x) \le y \le \psi(x)$

$a \le x \le b\}$



Vertical strip

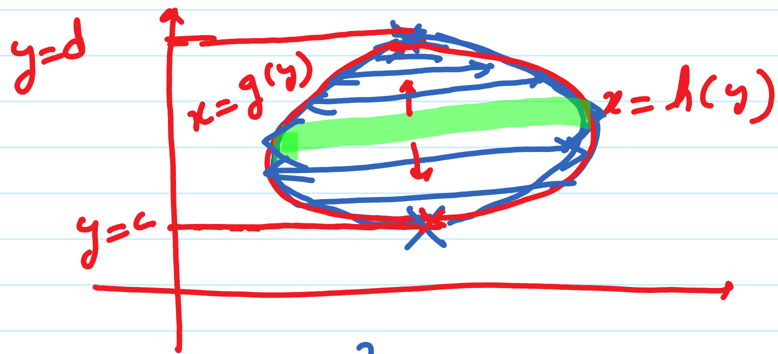
b

$R = \{(x,y): \phi(x) \le y \le \psi(x), \quad a \le x \le b\}$

$$\int_a^b \left(\int_{\phi(x)}^{\psi(x)} f(x,y)\, dy \right) dx$$

$R = \{ (x,y) : \phi(x) \le y \le f(x),\ a \le x \le b \}$ $\qquad \displaystyle\int_a \left(\int_{\phi(x)} f(x,y)\, dy \right) dx$

Case II



$R = \{ (x,y) : g(y) \le x \le h(y);\ c \le y \le d \}$ → Horizontal

$$\int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\, dx \right) dy$$

(1) $$\iint_R [f(x,y) \pm g(x,y)]\, dx\, dy = \iint_R f(x,y)\, dx\, dy \pm \iint_R g(x,y)\, dx\, dy$$

(2) $$\iint_R K f(x,y)\, dx\, dy = K \iint_R f(x,y)\, dx\, dy$$

(3) when $f(x,y)$ is integrable, then $|f(x,y)|$ is also integrable

& $$\left| \iint_R f(x,y)\, dx\, dy \right| \le \iint_R |f(x,y)|\, dx\, dy .$$

$\boxed{\begin{array}{l} f(x,y) = -1 \\ 1 \le x \le 2 \\ 2 \le y \le 3 \end{array}}$

(4) If $0 < f(x,y) \le g(x,y)$ for all $(x,y) \in R$

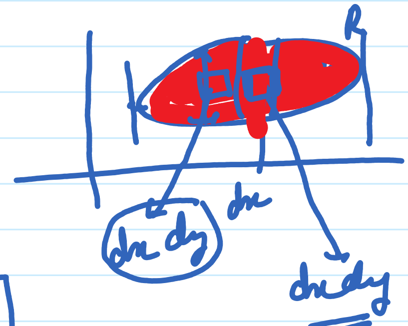
$$\iint f(x,y)\, dx\, dy \le \iint g(x,y)\, dx\, dy$$

$$\iint_R f(x,y)\,dx\,dy \le \iint_R g(x,y)\,dx\,dy$$

(5) If $f(x,y) \geq 0$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)\,dx\,dy \geq 0$$

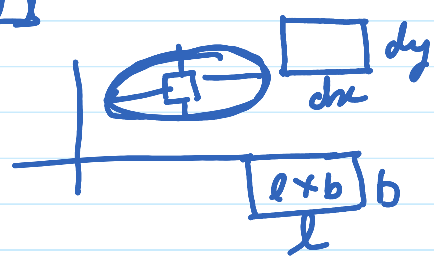
Applications of Double integral



(1) $$I = \iint_R \underline{f(x,y)}\,dx\,dy$$

(1) If $f(x,y) = 1$, then $I = \iint_R 1.\boxed{dx\,dy}$

Area of Region $\underline{\underline{R}}$



(2) If $\boxed{z = f(x,y)}$, then

$$\iint_R z\,dx\,dy \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

gives the volume of the region beneath the surface $z = f(x,y)$ & above the $x-y$ plane.

For example if $z = \sqrt{a^2 - x^2 - y^2}$ & $R:\ x^2 + y^2 \le a^2$,

then $$V = \iint_R \sqrt{a^2 - x^2 - y^2}\,dx\,dy$$

gives the volume of the hemisphere $x^2 + y^2 + z^2 = a^2$,

$z \geq 0$

$z^2 = a^2 - x^2 - y^2$

$z = \sqrt{a^2 - x^2 - y^2}$

If $z = f(x,y)$
$= \sqrt{a^2-x^2-y^2}$

$$I = \iint_R \sqrt{a^2-x^2-y^2}\, dx\,dy$$

Q Evaluate $\displaystyle\int_0^5 \int_0^{x^2} x(x^2+y^2)\, \boxed{dx\,dy}$

Sol $R = \{(x,y) : 0 \le y \le x^2, \quad 0 \le x \le 5\}$

Here $\displaystyle I = \int_0^5 \int_0^{x^2} x(x^2+y^2)\, dy\, dx$

$$= \int_0^5 \int_0^{x^2} (x^3 + xy^2)\, dy\, dx = \int_0^5 \left[x^3 y + \frac{xy^3}{3} \right]_0^{x^2} dx$$

$$= \int_0^5 \left\{ \left(x^5 + \frac{x^7}{3} \right) - 0 \right\} dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3} \right) dx$$

$$= \left| \frac{x^6}{6} + \frac{x^8}{24} \right|_0^5$$

$$= \frac{5^6}{6} + \frac{5^8}{24}$$

$$= 5^6 \left[\frac{1}{6} + \frac{5^2}{24} \right]$$ Ans

imp

Q Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ & the curve $x^2 = 4ay$.

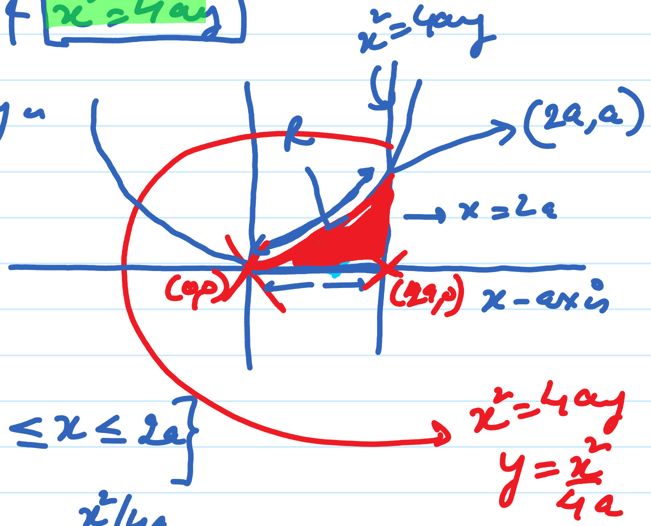
Sol The region is bdd by x-axis, $x = 2a$ & $\boxed{x^2 = 4ay}$

Point of intersection of $x = 2a$ & $x^2 = 4ay$:

$(2a)^2 = 4ay$

or $4ay - 4a^2 = 0$

$4a(y - a) = 0 \Rightarrow \boxed{y = a}$



Here $R = \left\{(x,y) : 0 \le y \le \frac{x^2}{4a},\ 0 \le x \le 2a\right\}$

$$\therefore I = \int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx = \int_0^{2a} \left|\frac{xy^2}{2}\right|_0^{x^2/4a} dx$$

$$= \int_0^{2a}\left(\frac{x}{2}\cdot\frac{x^4}{16a^2} - 0\right)dx = \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

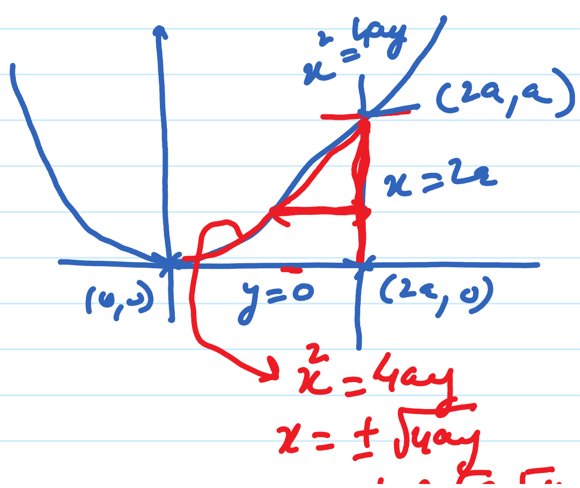
$$= \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a}$$

$$= \frac{a^4}{3}$$

Or

$R = \{(x,y) : 2\sqrt{a}\sqrt{y} \le x \le 2a,\ 0 \le y \le a\}$

$$\therefore I = \int_0^{a}\int_{2\sqrt{a}\sqrt{y}}^{2a} xy\,dx\,dy$$



$\int_0^{a} \int^{2a}$

$x = \pm\sqrt{4ay}$
$= \pm 2\sqrt{a}\sqrt{y}$

In the 1st quad,
$x = 2\sqrt{a}\sqrt{y}$

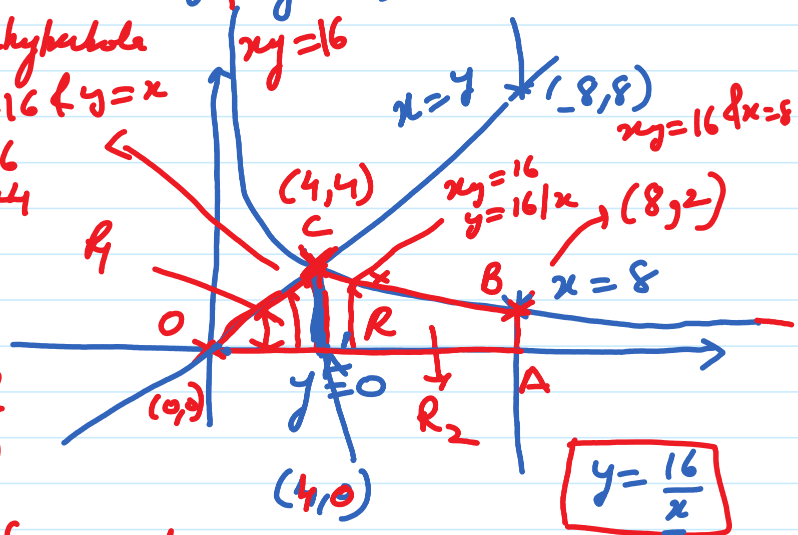
$$= \int_0^a y \left| \frac{x^2}{2} \right|_{2\sqrt{a}\sqrt{y}}^{2a} dy$$

$$= \int_0^a y \left[\frac{4a^2}{2} - \frac{4ay}{2} \right] dy = 2\int_0^a (ay - y^2)\, dy$$

$$= \frac{a^4}{3} \text{ Ans}$$

Q Evaluate $\iint_R x^2\, dx\, dy$, where R is the region in the first quadrant bounded by the lines $x=y$, $y=0$, $x=8$ & the curve $xy=16$. → rectangular hyperbola

Sol



$xy=16$ & $y=x$
$x^2=16$
$x=\pm 4$

$R = R_1 + R_2$

$R_1 = \{(x,y) : 0 \le y \le x,\ 0 \le x \le 4\}$

$R_2 = \{(x,y) : 0 \le y \le \frac{16}{x},\ 4 \le x \le 8\}$

$\boxed{y = \frac{16}{x}}$

$$\iint_R x^2\, dy\, dx = \iint_{R_1} x^2\, dy\, dx + \iint_{R_2} x^2\, dy\, dx$$

$$= \int_0^4 \int_0^x x^2\, dy\, dx + \int_4^8 \int_0^{16/x} x^2\, dy\, dx$$

$$= 448$$

(a) 440
(b) 442
(c) 444
(d) 448 ✓

Change of order of Integration

Q. Change the order of integration in the integral

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$$

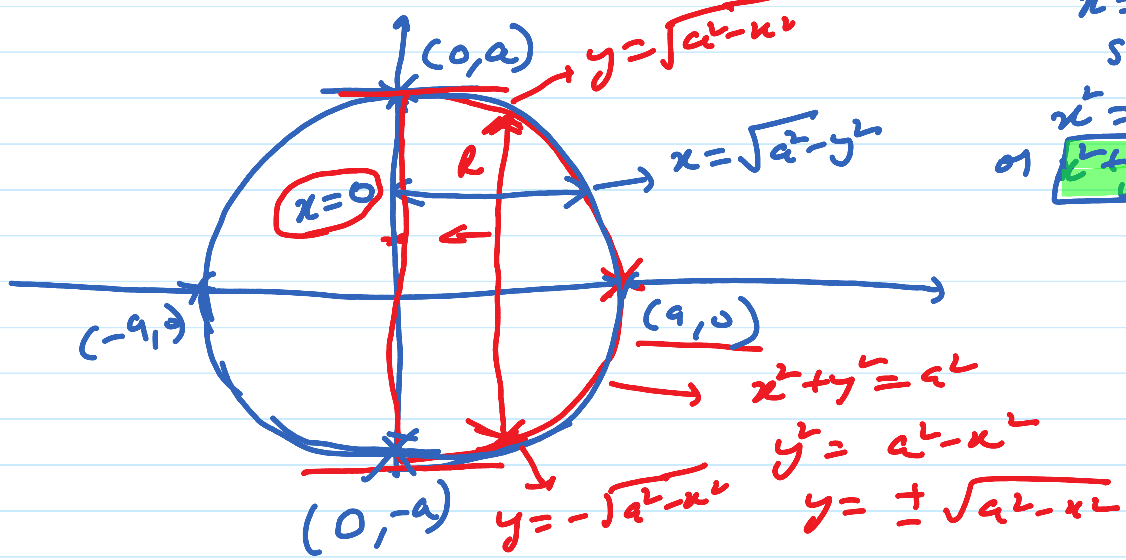
Sol. $R = \{(x,y) : 0 \le x \le \sqrt{a^2-y^2},\ -a \le y \le a\}$

$x = \sqrt{a^2-y^2}$

S.B.S

$x^2 = a^2-y^2$

or $\boxed{x^2+y^2=a^2}$

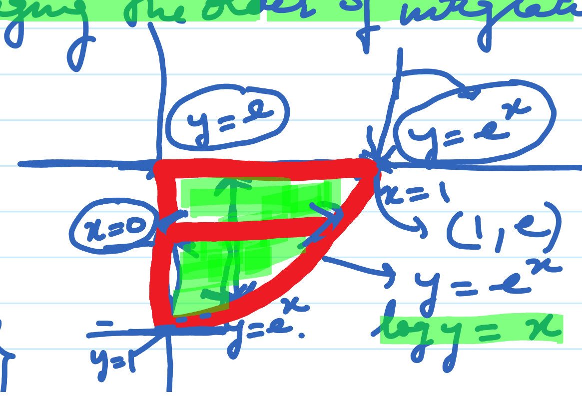


$R_1 = \{(x,y) : -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2},\ 0 \le x \le a\}$

$$\therefore I = \int_{0}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$

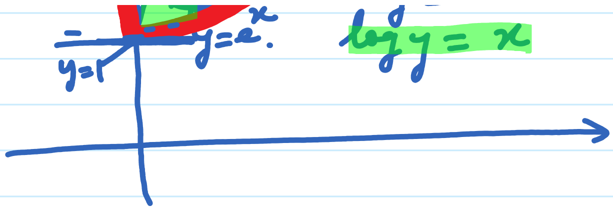
Q. Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$ by changing the order of integration

Sol. $$I = \int_0^1 \int_{e^x}^{e} \frac{1}{\log y}\,dy\,dx$$



$R = \{(x,y) : e^x \le y \le e,\ 0 \le x \le 1\}$

$$R = \left\{ (x,y) \cdot e^x \le y \le e, \quad 0 \le x \le 1 \right\}$$



$\log y = x$

By changing the order of integration

$$R = \left\{ (x,y) : 0 \le x \le \log y ; \quad 1 \le y \le e \right\}$$

$$\therefore I = \int_1^e \int_0^{\log y} \frac{1}{\log y}\, dx\, dy = \int_1^e \frac{1}{\log y} \left| x \right|_0^{\log y} dy$$

$$= \int_1^e \frac{1}{\log y} \cdot \log y \, dy$$

$$= \int_1^e dy = \left| y \right|_1^e = (e-1)$$

Q Change the order of integration & hence evaluate

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dx\, dy$$

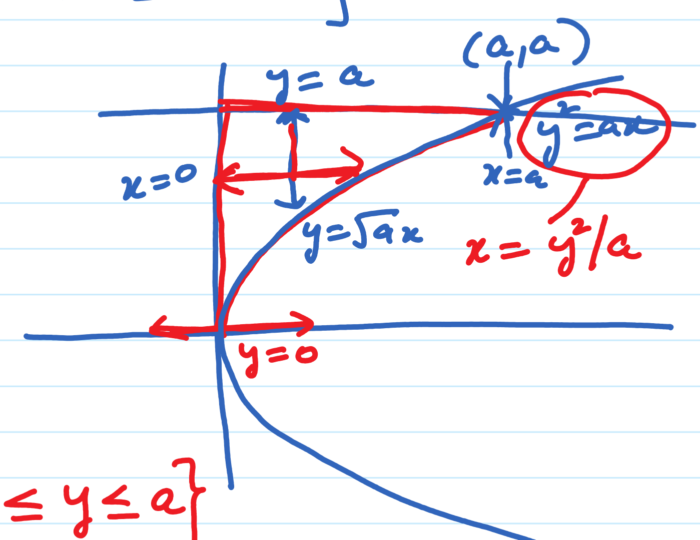
Sol $R = \left\{ (x,y) : \sqrt{ax} \le y \le a, \quad 0 \le x \le a \right\}$

Here $y = \sqrt{ax}$ | $y = a$

$y^2 = ax$

$a^2 = ax$

or $\boxed{x = a}$



$$R = \left\{ (x,y) \cdot 0 \le x \le \frac{y^2}{a}, \quad 0 \le y \le a \right\}$$

$a \quad y^2/a$

$$I = \int_0^a \int_0^{y^2/a} \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dx\, dy$$

$$= \int_0^a \frac{y^2}{a} \int_0^{y^2/a} \frac{1}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\, dx\, dy$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a}$$

$$= \int_0^a \frac{y^2}{a} \left[\sin^{-1}\frac{xa}{y^2}\right]_0^{y^2/a} dy$$

$$= \frac{1}{a}\int_0^a y^2 \left[\sin^{-1}\frac{\cancel{a}}{\cancel{y^2}} \times \frac{\cancel{y^2}}{\cancel{a}} - \sin^{-1} 0\right] dy$$

$$= \frac{1}{a}\int_0^a y^2 \left[\sin^{-1} 1 - 0\right] dy$$

$$= \frac{1}{a}\int_0^a y^2 \left[\pi/2\right] dy = \frac{\pi}{2a}\int_0^a y^2\, dy$$

$$= \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{2\cancel{a}} \times \frac{a^{\cancel{3}\,2}}{3} = \frac{\pi a^2}{6} \quad \underline{\text{Ans}}$$

imp

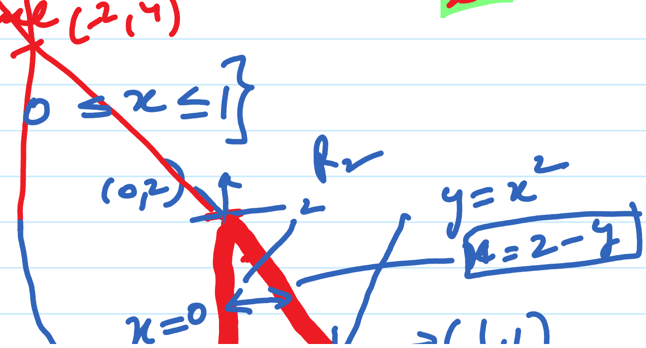
Q Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$

and hence evaluate the same.

Sol $R = \{(x,y) : x^2 \le y \le 2 - x,\ 0 \le x \le 1\}$

Here $\boxed{y = x^2}$ $\boxed{y = 2 - x}$

$\boxed{x + y = 2}$



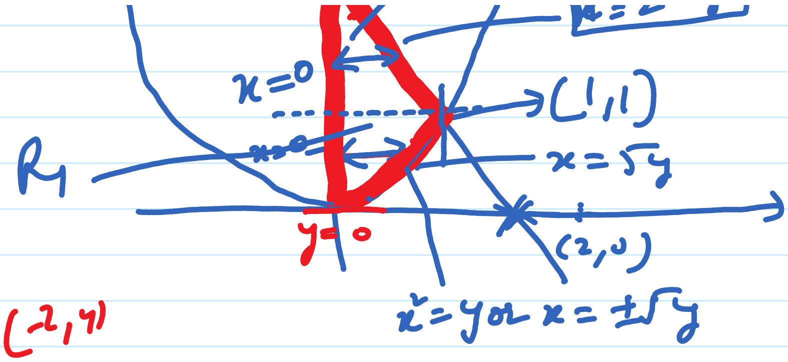
$\boxed{x+y=2}$

$2-x=x^2$

or $x^2+x-2=0$

$x=1,-2$

$y=2-1=1$, $y=2+2=4$ $(-2,4)$



$$R = R_1 + R_2$$

$$R_1=\{(x,y)\cdot\ 0\le x\le\sqrt{y},\quad 0\le y\le 1\}$$

$$R_2=\{(x,y):\ 0\le x\le 2-y,\quad 1\le y\le 2\}$$

$$\iint_R xy\,dx\,dy = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy$$

$$=\int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$=\frac{1}{6}+\frac{5}{24} = 3/8$$

A 3/8 ✓

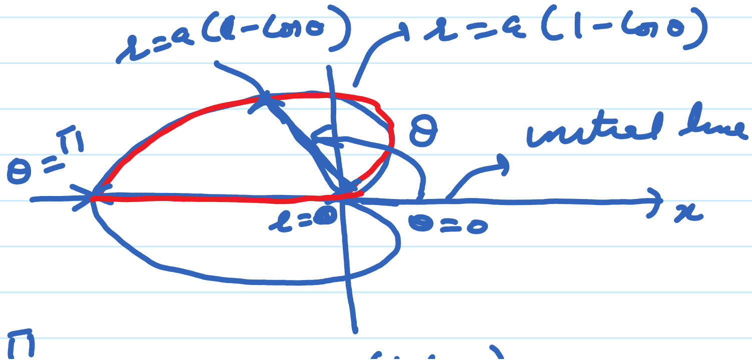
B 3/10

C 2/5

D 1/5

Double integral in polar form

Q evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r=a(1-\cos\theta)$ above the initial line.



Sol

$$R=\{(r,\theta):\ 0\le r\le a(1-\cos\theta),\quad 0\le\theta\le\pi\}$$

$a(1-\cos\theta)$

$$\therefore I = \int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta\, dr\, d\theta = \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta\left[a^2(1-\cos\theta)^2 - 0\right] d\theta$$

$$\int f(x)^n f'(x)\,dx$$

$$\frac{d}{d\theta}(1-\cos\theta) = \sin\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta(1-\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} (1-\cos\theta)^2 \sin\theta\, d\theta$$

$$= \frac{a^2}{2}\left|\frac{(1-\cos\theta)^3}{3}\right|_0^{\pi}$$

$$= \frac{a^2}{2\times 3}\left[(1-\cos\pi)^3 - (1-\cos 0)^3\right]$$

$$= \frac{a^2}{6}\left[2^3 - 0\right] = \frac{a^2}{6_3} \times 8^4 = \frac{4a^2}{3} \text{ Ans}$$

Q. Calculate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ & $r = 4\sin\theta$

Sol. $x = r\cos\theta$ & $y = r\sin\theta$

$x^2+y^2 = r^2$ | $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$

or $r = \sqrt{x^2+y^2}$ | $\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$

$r = 2\sin\theta$

$\sqrt{x^2+y^2} = \frac{2y}{\sqrt{x^2+y^2}}$

or $x^2+y^2 = 2y$ or $\boxed{x^2+y^2-2y=0}$

$r = 4\sin\theta$

$\sqrt{x^2+y^2} = \frac{4y}{\sqrt{x^2+y^2}}$

$x^2+y^2 = 4y$

or $x^2+y^2 = 2y$ or $\boxed{x^2+y^2-2y=0}$ $\sqrt{x^2+y^2}$

$\boxed{x^2+y^2+2ux+2vy+d=0}$ centre $(-u,-v)$, $r=\sqrt{u^2+v^2-d}$

centre $(0,1)$

radius $=\sqrt{0+1-0}$

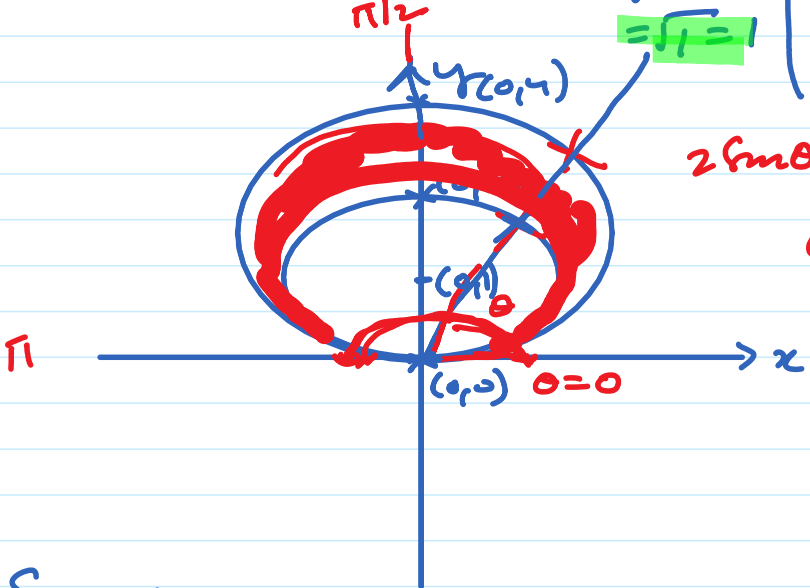
$=\sqrt{1}=1$

$x^2+y^2 = 4y$ $\sqrt{x^2+y^2}$

or $x^2+y^2-4y=0$

Centre $(0,2)$

radius $=2$



$2\sin\theta \le r \le 4\sin\theta$

$0\le\theta\le\pi$

$R=\{(r,\theta):\ 2\sin\theta\le r\le 4\sin\theta,\quad 0\le\theta\le\pi\}$

$$\therefore I = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta = \int_0^{\pi}\left|\frac{r^4}{4}\right|_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\left(4^4\sin^4\theta - 2^4\sin^4\theta\right)d\theta$$

$$= 60\int_0^{\pi}\sin^4\theta\,d\theta$$

$\sin^4(-\theta) = \sin^4\theta$

$$= 60\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

even

$$=120\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= \cancel{120}^{\,\cancel{30}\,15}\ \frac{3\cdot 1}{\cancel{4}\cdot\cancel{2}}\times\frac{\pi}{2}$$

$$= 45\pi = \boxed{22.5\,\pi}$$

$$\int_0^{\pi/2}\sin^n\theta\,d\theta = \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots}$$

$$\int_0 = \frac{(n-1)\ldots}{n(n-2)\ldots}$$

$$= 10\,\frac{3\cdot 1}{\cancel{4}\cdot\cancel{2}} \times \frac{\pi}{2}$$

$$= \frac{45}{2}\pi = \boxed{22.5\,\pi}$$

CTM

$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \times \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{1}{2} \times \frac{\pi}{2}$$

$$I = \iint_R f(x,y)\,dx\,dy$$

if $f(x,y) = 1$

$$I = \iint_R dx\,dy = \text{Area enclosed in the region } R$$

Q Show that the area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4ay$ is $\frac{16}{3}a^2$

Sol The given parabolas are

$$y^2 = 4ax \quad \text{—} \ (1)$$

$$x^2 = 4ay \quad \text{—} \ (2)$$

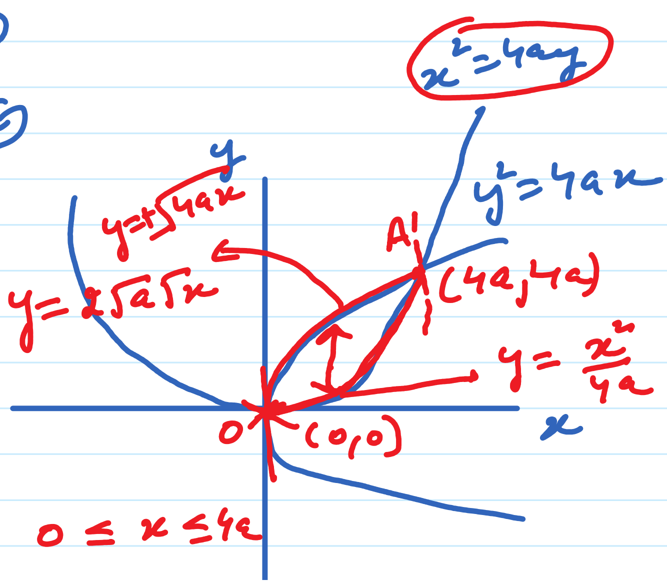
from (1) $x = y^2/4a$, sub in (2)

$$\frac{y^4}{16a^2} = 4ay$$

or $y^4 = 64a^3 y$

or $y(y^3 - 64a^3) = 0$

$y = 0$ & $y = 4a$



$0 \le x \le 4a$

$y=0$ & $y=4a$ $\quad$ $0 \le x \le 4a$

$x=0$ & $x=4a$

$(0,0)$ & $(4a,4a)$

$R = \left\{(x,y) : \frac{x^2}{4a} \le y \le 2\sqrt{a}\sqrt{x} \text{ \& } 0 \le x \le 4a\right\}$

Required area $= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{a}\sqrt{x}} dy\, dx$

$= \int_0^{4a} \left(y \Big|_{x^2/4a}^{2\sqrt{a}\sqrt{x}} \right) dx = \int_0^{4a} \left[2\sqrt{a}\, x^{1/2} - \frac{x^2}{4a}\right] dx$

$= \left| 2\sqrt{a}\, \frac{x^{3/2}}{3/2} - \frac{x^3}{12a} \right|_0^{4a}$

$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} - 0$

$= \frac{16a^2}{3}$

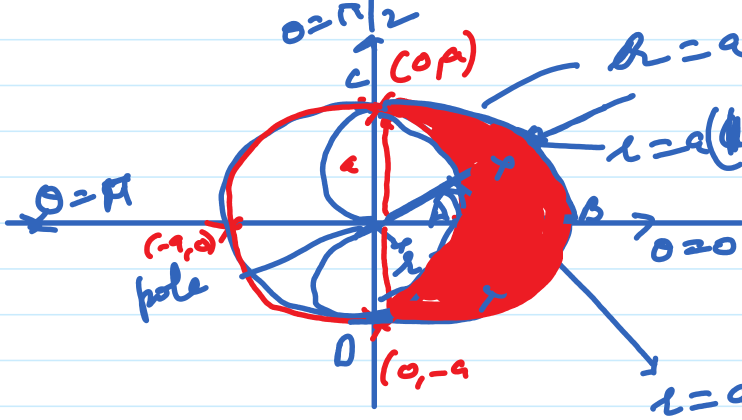
Q Find the area lying inside the cardiod $r = a(1+\cos\theta)$ and outside the circle $r=a$.

Sol $r=a$

$\sqrt{x^2+y^2}=a$

S.B.S

$\boxed{x^2+y^2=a^2}$



when $\theta=\pi/2$

$r=a(1+\cos\pi/2)$

$=a(1+0)$

$=a$

$\iint dx\,dy$

$0 \,|\, (0,$ $\quad r = a$

$$\iint dx\,dy$$

$$\iint_R \boxed{r\,dr\,d\theta}$$

Required area $= 2(\text{area } ABCA)$

$$= 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta$$

$$= 2 \times \frac{1}{2}\int_0^{\pi/2} \left[a^2(1+\cos\theta)^2 - a^2\right] d\theta$$

$$= a^2 \int_0^{\pi/2} \left[\cancel{1} + \cos^2\theta + 2\cos\theta - \cancel{1}\right] d\theta$$

$$= a^2 \int_0^{\pi/2} \left(\cos^2\theta + 2\cos\theta\right) d\theta$$

$$= a^2 \left[\int_0^{\pi/2} \cos^2\theta\, d\theta + 2\int_0^{\pi/2} \cos\theta\, d\theta\right]$$

$$= a^2 \left[\frac{1}{2} \times \frac{\pi}{2} + 2\left(\sin\theta\Big|_0^{\pi/2}\right)\right]$$

$$= a^2 \left(\frac{\pi}{4} + 2\left[\sin\frac{\pi}{2} - \sin 0\right]\right)$$

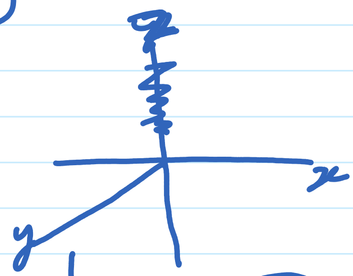
$$= a^2\left[\frac{\pi}{4} + 2\right] = \frac{a^2}{4}(\pi + 8)$$

$\pi/2$

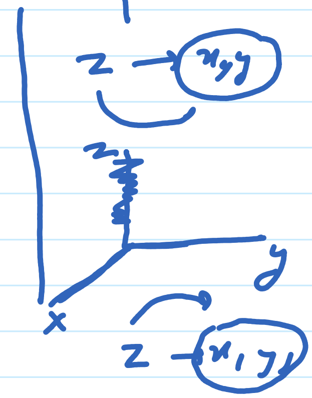
"n ——————— "

$$\int_0^{\pi/2} \sin^n\theta \cos^m\theta \, d\theta = \frac{(n-1) \text{———} \times (m-1)\text{-----}}{(n+m)(m+m\text{---})} \times \frac{\pi}{2}$$

Triple Integral



$$I = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$



Q Evaluate $\displaystyle\int_{-1}^{1} \int_{0}^{2} \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

dx
dz

Sol $\displaystyle I = \int_{-1}^{1} \int_{0}^{2} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

$$= \int_{-1}^{1} \int_{0}^{2} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^{1} \int_{0}^{z} \left[\left\{ x(x+z) + \frac{(x+z)^2}{2} + (x+z)z \right\} - \left\{ x(x-z) + \frac{(x-z)^2}{2} + (x-z)z \right\} \right] dx\, dz$$

$$= \int_{-1}^{1} \int_{0}^{z} \left[\cancel{x^2} + xz + \frac{x^2+z^2+2xz}{2} + \cancel{xz} + z^2 - \cancel{x^2} + \cancel{xz} - \frac{x^2 - z^2 + 2xz}{2} - \cancel{xz} + z^2 \right] dx\, dz$$

$$\int_{-1}^{1}\int_{0}^{z} \qquad - \cancel{xz} + z^2 \Big] \, dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \Big[4xz + 2z^2 \Big] \, dx\, dz$$

$$= \int_{-1}^{1} \left| \frac{4x^2 z}{2} + 2z^2 x \right|_0^z dz$$

$$= \int_{-1}^{1} (2z^3 + 2z^3) \, dz = 4\int_{-1}^{1} z^3 dz = 0 \quad \underline{\text{Ans}}$$

Q Evaluate $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$

$$\begin{array}{l} x \mid y \\ x+y+z = 1 \\ z = 1-x-y \\ x+y = ? \\ y = ?-x \end{array}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left| \frac{z^2}{2} \right|_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy \left\{ (1-x^2-y^2) \right\} dy\, dx$$

$$= \frac{1}{2} \int_0^1 x \int_0^{\sqrt{1-x^2}} \left\{ (1-x^2)y - y^3 \right\} dy\, dx$$

$$= \frac{1}{2}\int_0^1 x \left| \frac{(1-x^2)y^2}{2} - \frac{y^4}{4} \right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}\int_0^1 x \left| 2(1-x^2)y^2 - y^4 \right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}\int_0^1 x \Big[2(1-x^2)(1-x^2) - (1-x^2)^2 \Big\} dx$$

$$= \frac{1}{8}\int_0^1 x\left\{(1-x^2)^2\right\}dx = \frac{1}{8}\int_1^1 x(1+x^4-2x^2)\,dx$$

(A) 1/12
(B) 1/48 ✓
(C) 1/36
(D) 1/9

Volume of Solids

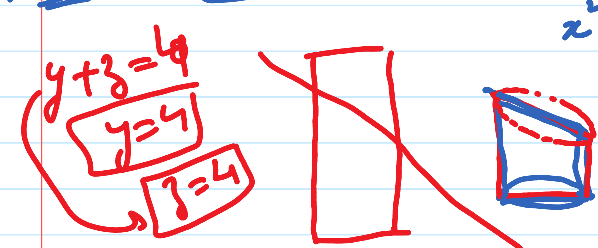
$$\text{Volume} = \iint_R z\,\boxed{dx\,dy} \quad \text{[Ctry]} \qquad \Big|\; dx\,dy = r\,dr\,d\theta$$

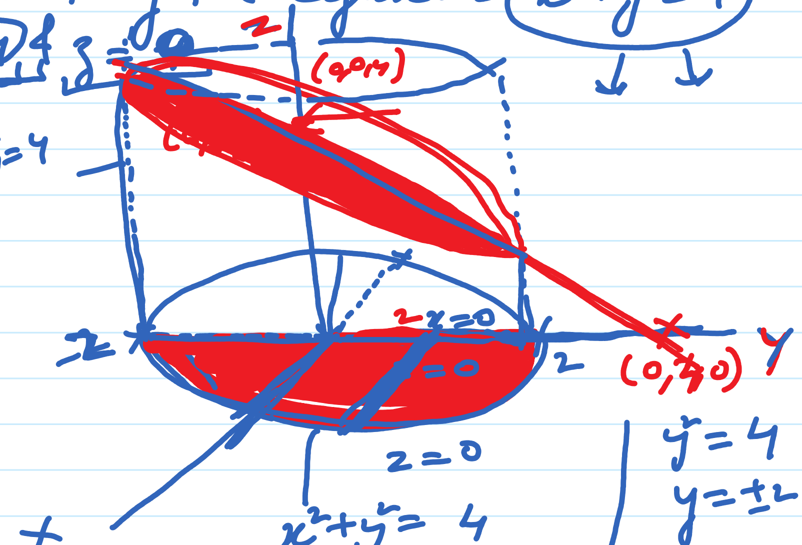
$$\text{or} \quad \iint_R z\,r\,dr\,d\theta \quad \text{[Ctry]}$$

Q Find the volume bounded by the cylinder $x^2+y^2=4$ and the plane $y+z=4$ { $z = 0$

$\boxed{z = 4-y}$

Sol





$z=0$

$x^2+y^2 = 4$ | $y^2=4$, $y=\pm 2$

$x = 4-y^2$

$x = \pm\sqrt{2^2-y^2}$

$\boxed{0 \le x \le \sqrt{2^2-y^2}}$

$$\text{Required volume} = 2\int^{2}\int^{\sqrt{2^2-y^2}} z\,dx\,dy$$

$$\boxed{0 \le x \le \sqrt{2^2-y^2}}$$

Required volume $= 2\int_{-2}^{2}\int_{0}^{\sqrt{2^2-y^2}} z \, dx\, dy$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{2^2-y^2}} (4-y)\, dx\, dy$$

$$= 2\int_{-2}^{2} (4-y) \left| x \right|_{0}^{\sqrt{2^2-y^2}} dy$$

$$= 2\int_{-2}^{2} (4-y)\left(\sqrt{2^2-y^2}\right) dy$$

$$= 2\int_{-2}^{2} 4\sqrt{2^2-y^2}\, dy - 2\int_{-2}^{2} y\sqrt{2^2-y^2}\, dy$$

$$= 16\int_{0}^{2}\sqrt{2^2-y^2}\, dy - 0$$

$$= 16\int_{0}^{2}\sqrt{2^2-y^2}\, dy$$

$$= 16\left[\frac{y\sqrt{2^2-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{0}^{2}$$

$$= 16\left[\left(0 + 2\sin^{-1}1\right) - (0)\right]$$

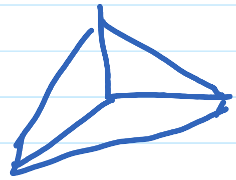
$$= 16 \times \cancel{2} \times \frac{\pi}{\cancel{2}} = \boxed{16\pi} \text{ Ans}$$

<u>Volume by using triple Integral</u>

Q. Calculate the volume of the solid bounded by the planes $x=0, y=0, x+y+z=a$ & $z=0$

Sol $R = \{ (x,y,z) : 0 \le x \le a,\ 0 \le y \le a-x,\ 0 \le z \le a-x-y$

$$\boxed{\begin{array}{l} x+y+z = a \\ z = a-x-y \\ x+y+\cancel{z} = a \\ y = a-x \\ x = a \end{array}}$$



$$\text{volume} = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\,dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left| z \right|_0^{a-x-y} dy\,dx$$

$$= \int_0^a \int_0^{a-x} (a-x-y)\,dy\,dx$$

$$= \int_0^a \left[(a-x)y - \frac{y^2}{2} \right|_0^{a-x} dx$$

$$= \int_0^a \left[(a-x)(a-x) - \frac{(a-x)^2}{2} \right] dx$$

$$= \frac{1}{2} \int_0^a (a-x)^2\,dx \quad = \frac{1}{2} \left| \frac{(a-x)^3}{3(-1)} \right|_0^a$$

$$= -\frac{1}{6}\left[0 - a^3 \right] = a^3/6$$

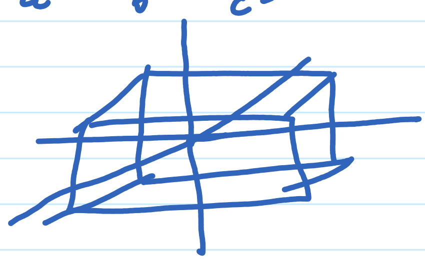
$\frac{4}{3}\pi (2)^3 =$

$\frac{4}{3}\pi abc$

$x^2+y^2+z^2 = 2^2$

Q2 Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol
$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



$$z = \pm\sqrt{c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$$

$$= \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$0 \le z \le c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$