Indeterminate Forms

Consider the ratio $\frac{f(x)}{g(x)}$ of two functions $f(x)$ and $g(x)$.

If at any point $x = a, f(a) = 0, g(a) = 0.$

Then the ratio $\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$

It is called an Indeterminate Form.

If $f(a) = \pm\infty, g(a) = \pm\infty$, Then the ratio $\frac{f(x)}{g(x)}$ takes the form $\frac{\infty}{\infty}$

It is another Indeterminate Form.

Other Indeterminate Forms

The other indeterminate forms are:

$0 \times \infty, 0^0, \infty^0, 1^\infty$ and $\infty - \infty$

In each of these cases, we can reduce the ratio function to the from $\frac{0}{0}$ or $\frac{\infty}{\infty}$

For the indeterminate forms $0^0, \infty^0, 1^\infty$, we take logarithm of the given function and then take the limits.

Important Note

The functions of the form $0^\infty, \infty \times \infty, \infty + \infty, \infty^\infty, \infty^{-\infty}$ are not considered as indeterminate forms.

Note that: $0^\infty = 0$

$\infty \times \infty = \infty$

$\infty + \infty = \infty$

$\infty^\infty = \infty$

$\infty^{-\infty} = 0$

L' Hospital's Rule

L' Hospital rule can be used only when the ratio is of indeterminate from, that is either is of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Suppose that the real valued functions $f(x)$ and $g(x)$ are differentiable in some open interval containing the point $x = a, f(a) = 0, g(a) = 0.$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}, g'(a) \neq 0.$$

Suppose now that $f'(a) = 0 = g'(a)$.

Then we repeat the application of L' Hospital rule on $\frac{f'(x)}{g'(x)}$ and obtain:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}$$

provided the limit exists.
This application of L' Hospital rule can be continued as long as the indeterminate form is obtained

When both $f(a) = \pm\infty, g(a) = \pm\infty$, we get another indeterminate form.

In this case also, L' Hospital rule can be applied. We write

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\left[\frac{1}{g(x)}\right]}{\left[\frac{1}{f(x)}\right]}$$ which is of $\frac{0}{0}$ form.

$\left[\frac{\infty}{\infty}\right]$

L' Hospital rule can also be applied to find the limits $x \to \pm\infty$.

Q Evaluate $\displaystyle \underset{x \to 0}{Lt} \frac{xe^x - \log(1+x)}{x^2}$ $\left\{\text{form } \frac{0}{0}\right]$

$$= \underset{x \to 0}{Lt} \frac{xe^x + e^x - \frac{1}{1+x}}{2x} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{xe^x + e^x + e^x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{0+1+1+1}{2} = 3/2$$

Q $\displaystyle \underset{x \to 1}{Lt} \frac{x^x - x}{x - 1 - \log x}$ $\left[\frac{0}{0}\right]$

$$= \underset{x \to 1}{Lt} \frac{x^x(1+\log x) - 1}{1 - \frac{1}{x}} \quad \left[\frac{0}{0}\right]$$

$y = x^x$

$= x^x(1 + \log x)$

$\log y = x \log x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x$

$\frac{dy}{dx} = x^x(1 + \log x)$

$$= \underset{x \to 1}{Lt} \frac{x^x\left(\frac{1}{x}\right) + x^x(1+\log x)^2}{+\frac{1}{x^2}}$$

$$= \frac{1 + 1(1+0)}{1} = 2 \text{ Ans}$$

1

Q. Evaluate $\lim_{x \to 0} \frac{\log x}{\cot x}$ $\left[\frac{\infty}{\infty}\right]$

$$= \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x}$$

$$= -\lim_{x \to 0} \frac{1}{x} \cdot \sin^2 x \quad \left[\frac{0}{0}\right]$$

$$= -\lim_{x \to 0} \frac{2 \sin x \cos x}{1}$$

$$= 0 \text{ Ans}$$

$\boxed{Q}$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} (1+x)^{1/x} = e}$$

Q. Evaluate $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$

$$= \lim_{x \to 0} \frac{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]\left[x - \frac{x^3}{3!} + \dots\right] - x - x^2}{x^2 + x\left[-x - \frac{x^2}{2} - \frac{1}{3}x^3 - \dots\right]}$$

$$= \lim_{x \to 0} \frac{\left[\left(\cancel{x} + \cancel{x^2} + \frac{x^3}{2!} + \dots\right) - \left(\frac{x^3}{3!} + \frac{x^4}{3!} + \dots\right)\right] - \cancel{x} - \cancel{x^2}}{\cancel{x^2} - \cancel{x^2} - \frac{x^3}{2} - \frac{x^4}{3} - \dots}$$

$$\frac{1}{2} - \frac{1}{6} = \frac{3-1}{6}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + \dots}{-\frac{1}{2}x^3 + \dots} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{3} \times 3x^2 + \dots}{-\frac{1}{2} 3x^2 + \dots} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{2x + \dots}{-3x + \dots} \quad \text{or } \left[\frac{0}{0}\right]$$

$= -\frac{2}{3}$ Ans

14/10/20

③ Form reducible to $\frac{0}{0}$ form

I Form $0 \times \infty$ $\quad \lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = \infty$

then $\lim_{x\to a} f(x)\,g(x) \longrightarrow 0\times\infty$

To evaluate $\quad \lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)}$ $\left[\frac{0}{0} \text{ form}\right]$ ✓

or $= \lim_{x\to a} \frac{g(x)}{1/f(x)}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$ ✓

① Evaluate $\lim_{x\to 0} \tan x \log x$ $\quad [0\times\infty \text{ form}]$

$$= \lim_{x\to 0} \frac{\log x}{1/\tan x}$$

$$= \lim_{x\to 0} \frac{\log x}{\cot x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}^2 x}$$

$$= -\lim_{x\to 0} \frac{\sin^2 x}{x} \quad \left[\frac{0}{0}\right] \longrightarrow \lim_{x\to 0} \frac{\sin x}{x} \sin x$$

✓ $1\times 0 = 0$ ✓

$$= -\lim_{x\to 0} \frac{2\sin x\cos x}{1}$$

$= 0$ Ans ✓

II Form $\infty - \infty$, if $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} \phi(x) = \infty$ then

$\lim_{x\to a} [f(x) - \phi(x)] \longrightarrow \infty - \infty$

$$\lim_{x\to a} [f(x) - \phi(x)] = \lim_{x\to a} \frac{\left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right]}{1/\phi(x)f(x)} \quad \left[\frac{0}{0}\right]$$

$$\lim_{x\to a}[f(x)-\phi(x)] = \lim_{x\to a} \frac{[\phi(x) - f(x)]}{1/\phi(x) f(x)} \qquad \left[\frac{0}{0}\right]$$

Q Evaluate $\lim_{x\to 0}\left[\frac{1}{\sin x} - \frac{1}{x}\right]$ $\qquad [\infty - \infty]$

$$= \lim_{x\to 0} \frac{\left[\frac{1}{1/x} - \frac{1}{1/\sin x}\right]}{1/\frac{1}{x}\frac{1}{\sin x}}$$

$$= \lim_{x\to 0} \left[\frac{x - \sin x}{x \sin x}\right] \qquad \left[\frac{0}{0}\right] \text{form}$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{x(\cos x) + 1.\sin x} \qquad \left[\frac{0}{0}\right] \text{form}$$

$$= \lim_{x\to 0} \frac{+\sin x}{x \times -\sin x + \cos x + \cos x}$$

$$= \frac{0}{0+1+1} = 0 \text{ Ans}$$

III Form $0^0, 1^\infty, \infty^0$

If $y = \lim_{x\to a} [f(x)]^{\phi(x)}$

$\# \log y = \lim_{x\to a} \phi(x) \log f(x) \qquad [0\times\infty]$

Q Evaluate $y = \lim_{x\to \pi/2} (\sin x)^{\tan x} \qquad [1]^\infty$

Sol $\log y = \lim_{x\to\pi/2} \tan x \log \sin x \qquad [0\times\infty]$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{1/\tan x}$$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{\cot x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\pi/2} \frac{1/\sin x \times \cos x}{-\operatorname{cosec}^2 x}$$

$$= -\lim_{x\to\pi/2} \frac{\sin^2 x \cos x}{\sin x}$$

$$= -\lim_{x\to\pi/2} \sin x \cos x$$

$$= -0 = 0$$

$$\therefore \log y = 0$$

or $\boxed{y = e^0 = 1}$

(11) $y = \lim_{x\to 0} \left[\frac{a^x + b^x + c^x}{3}\right]^{1/x}$ $[1^\infty]$

$$\therefore \log y = \lim_{x\to 0} \frac{1}{x} \log\left[\frac{a^x+b^x+c^x}{3}\right]$$

$$= \lim_{x\to 0} \frac{\log(a^x+b^x+c^x) - \log 3}{x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{(a^x+b^x+c^x)}(a^x \log a + b^x \log b + c^x \log c) - 0}{1}$$

$$= \frac{1}{3}[\log a + \log b + \log c]$$

$$= \frac{1}{3} \log abc$$

$$\log y = \log(abc)^{1/3}$$

or $\boxed{y = (abc)^{1/3}}$

Q $y = \lim_{x\to 0} \left[\frac{\tan x}{x}\right]^{1/x^2}$

$$= \lim_{x\to 0} \left[\frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots}{x}\right]^{1/x^2}$$

$$= \lim_{x\to 0}\left[\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots}{x}\right]^{1/x^2}$$

$$= \lim_{x\to 0}\left[1+\frac{1}{3}x^2+\frac{2}{15}x^4+\cdots\right]^{1/x^2}$$

$$= \lim_{x\to 0}\left[1+x^2\left(\frac{1}{3}+\frac{2}{15}x^2+\cdots\right)\right]^{1/x^2}$$

$$= \lim_{x\to 0}\left[1+x^2 t\right]^{1/x^2} \qquad \left[\text{where } t=\frac{1}{3}+\frac{2}{15}x^2+\cdots\right.$$

$$= \lim_{x\to 0}\left[(1+x^2t)^{\frac{1}{tx^2}}\right]^{t} \qquad \lim_{x\to 0}(1+x)^{1/x}=e$$

$$= \lim_{x\to 0}\left[e\right]^{t}$$

$$= \lim_{x\to 0} e^{\left[\frac{1}{3}+\frac{2}{15}x^2+\cdots\right]}$$

$$= e^{1/3} \quad \text{Ans}$$