CSE322

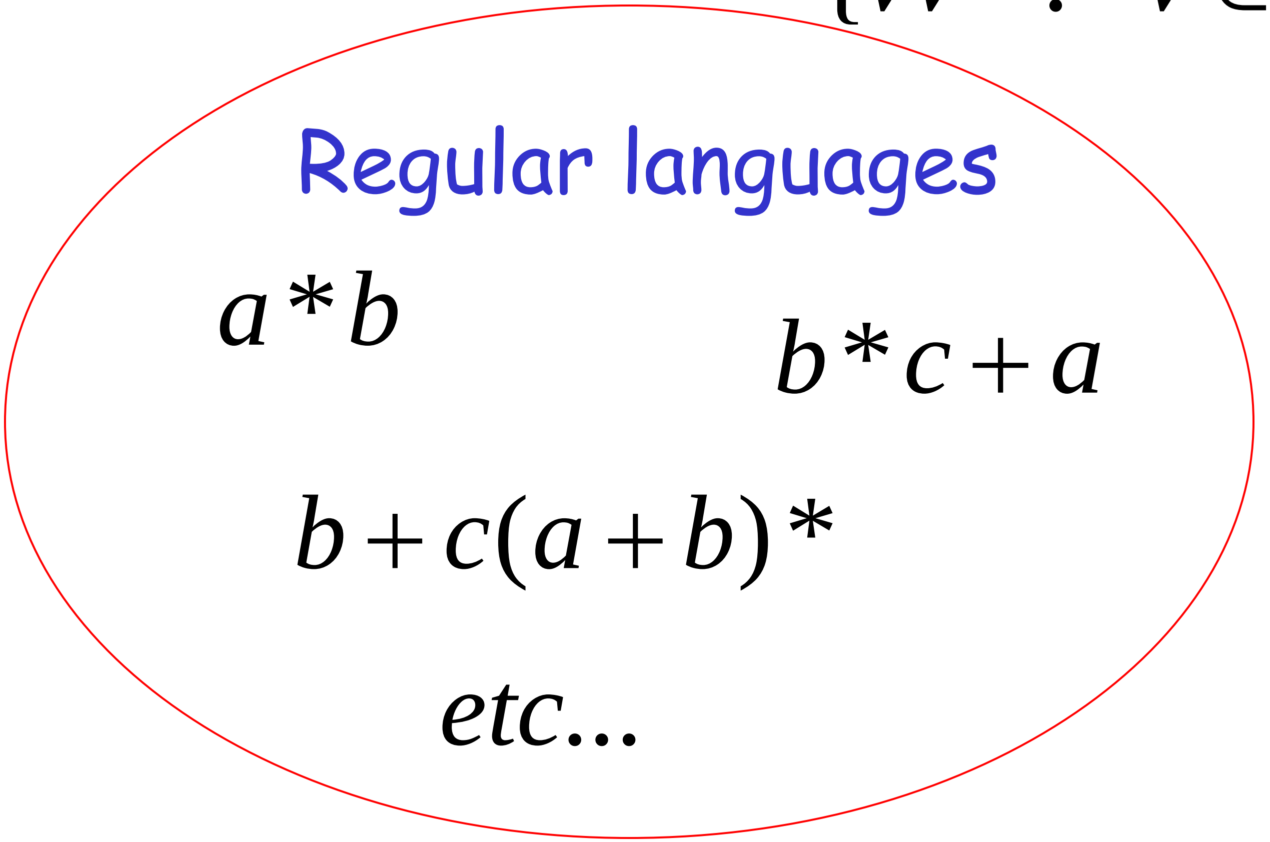
PUMPING LEMMA FOR REGULAR SETS AND ITS APPLICATIONS

Lecture #11

Non-regular languages

$$\{a^n b^n : \quad n \geq 0\}$$

$$\{vv^R : \quad v \in \{a,b\}^*\}$$



Regular languages

$$a^*b$$

$$b^*c + a$$

$$b + c(a+b)^*$$

$$etc...$$

How can we prove that a language L is not regular?

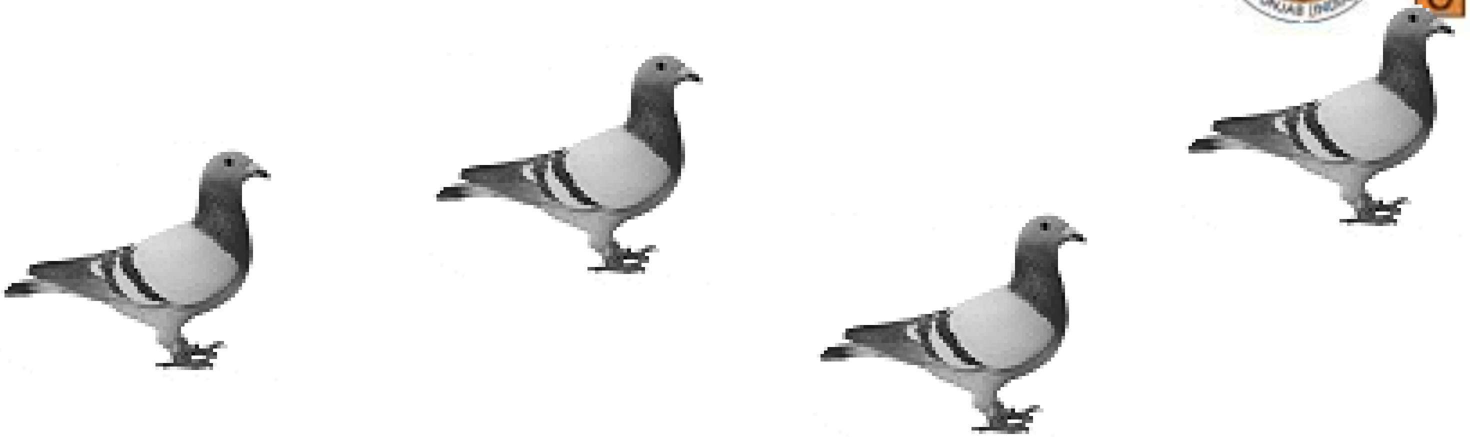
Prove that there is no DFA that accepts L

Problem: this is not easy to prove

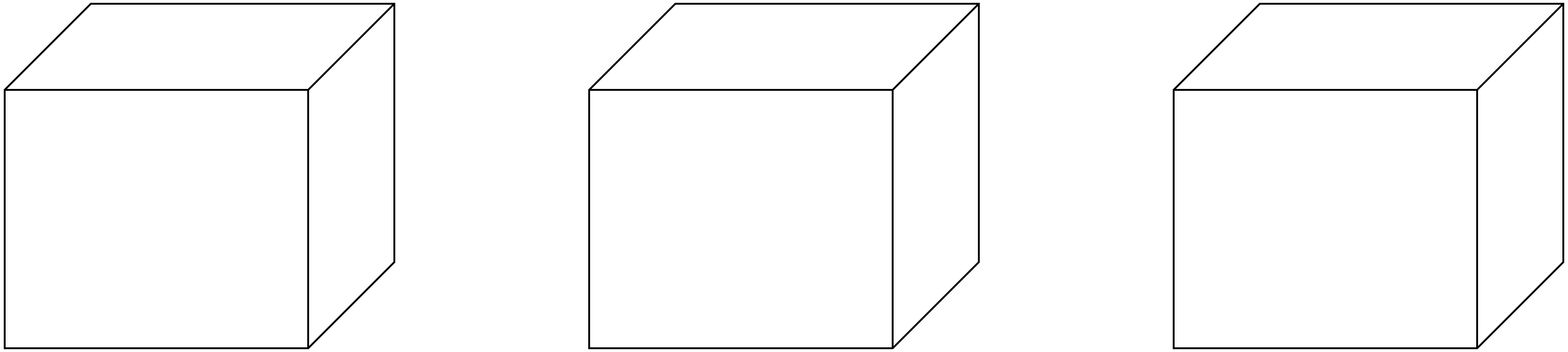
Solution: the Pumping Lemma !!!

The Pigeonhole Principle

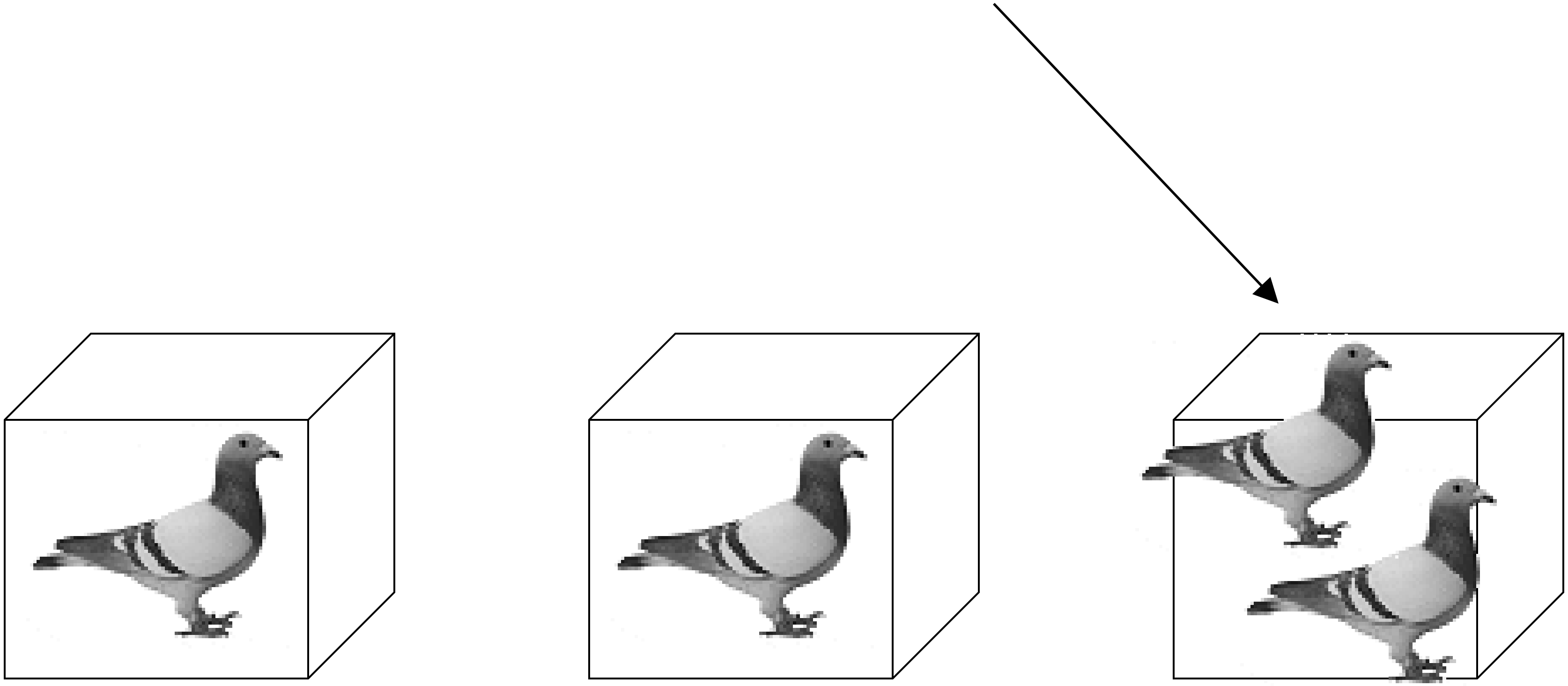
4 pigeons



3 pigeonholes

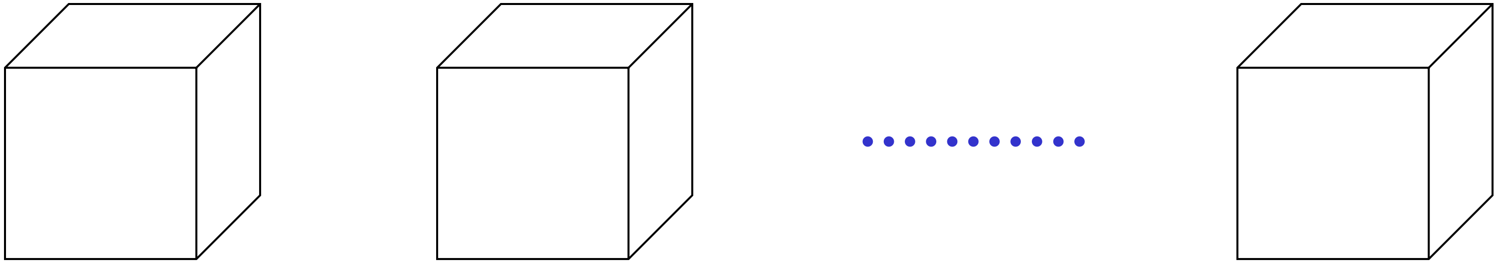


A pigeonhole must
contain at least two pigeons



n pigeons

m pigeonholes $n > m$



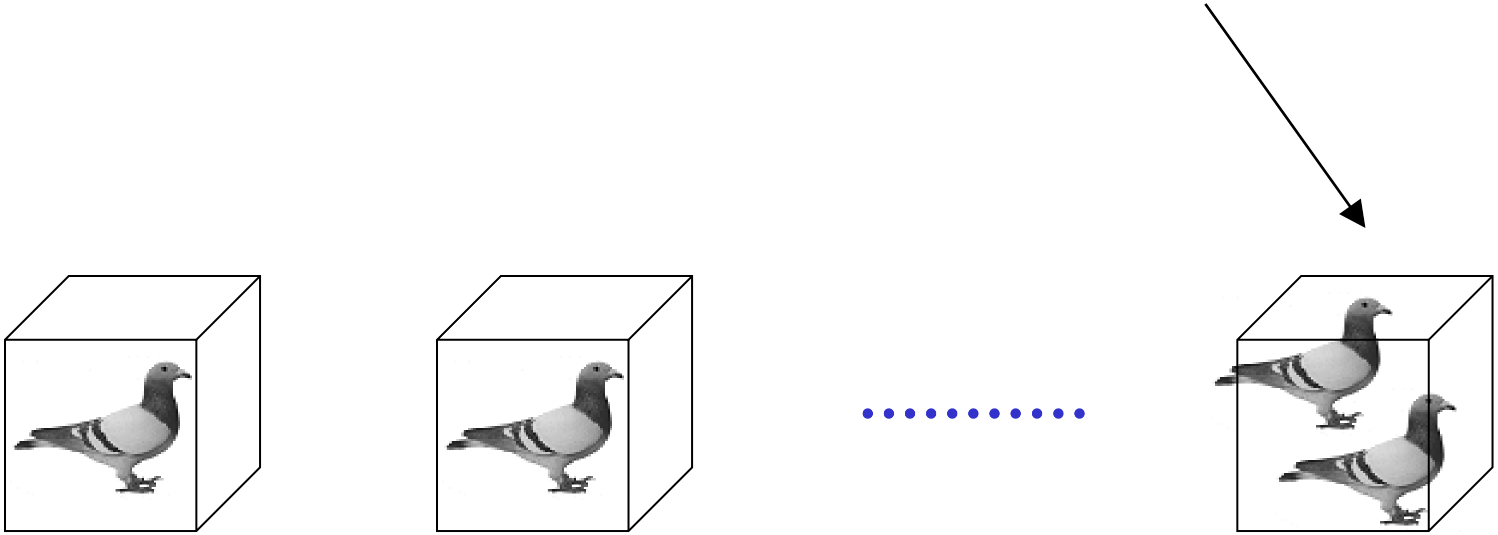
The Pigeonhole Principle

n pigeons

m pigeonholes

$$n > m$$

There is a pigeonhole with at least 2 pigeons

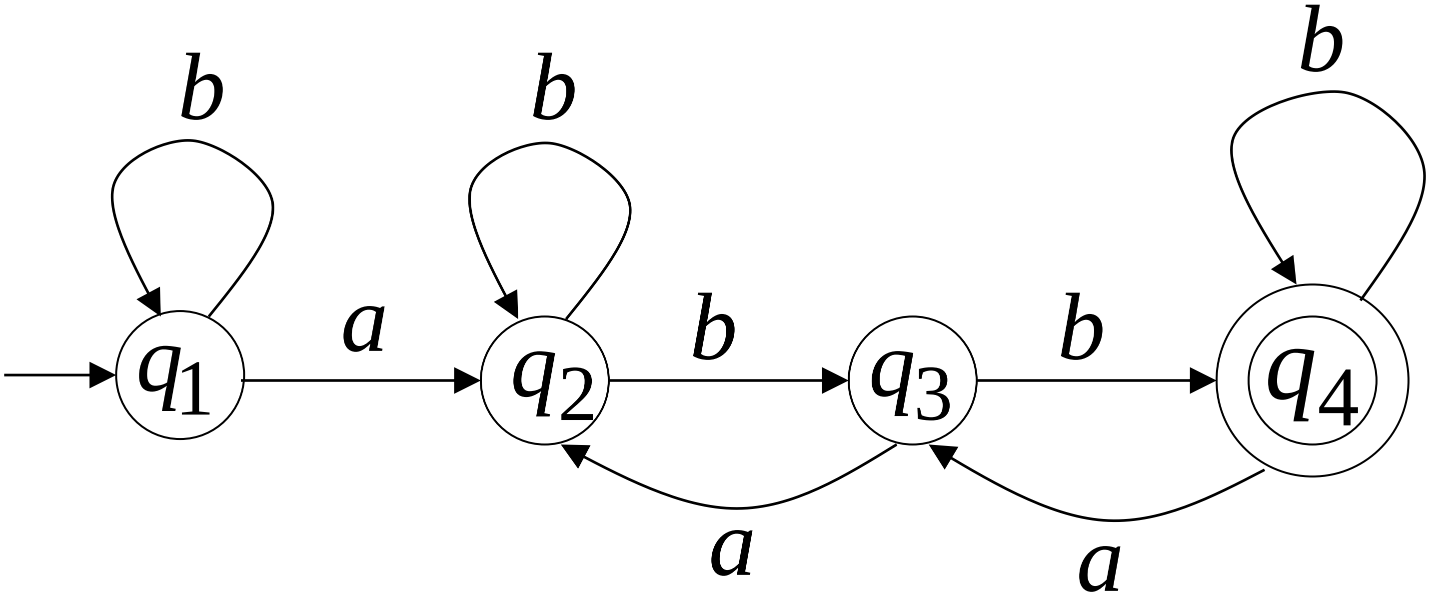


The Pigeonhole Principle

and

DFAs

DFA with 4 states

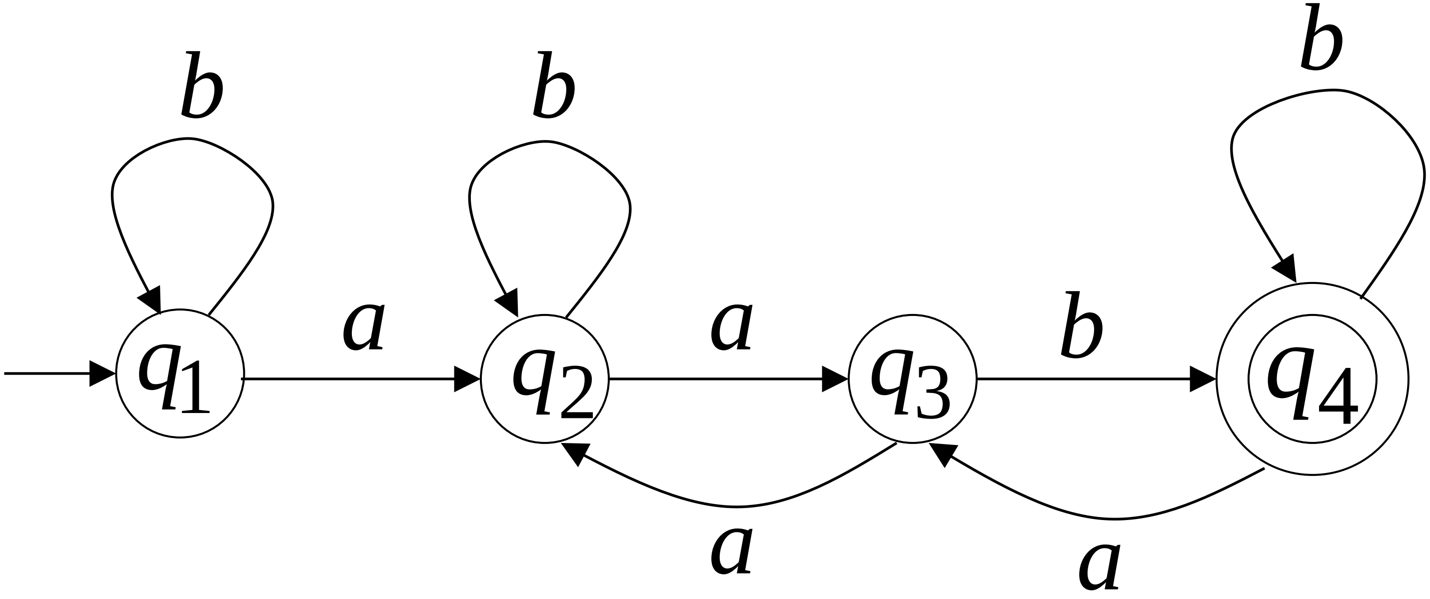


In walks of strings: a

aa

aab

no state is repeated



In walks of strings: $aabb$

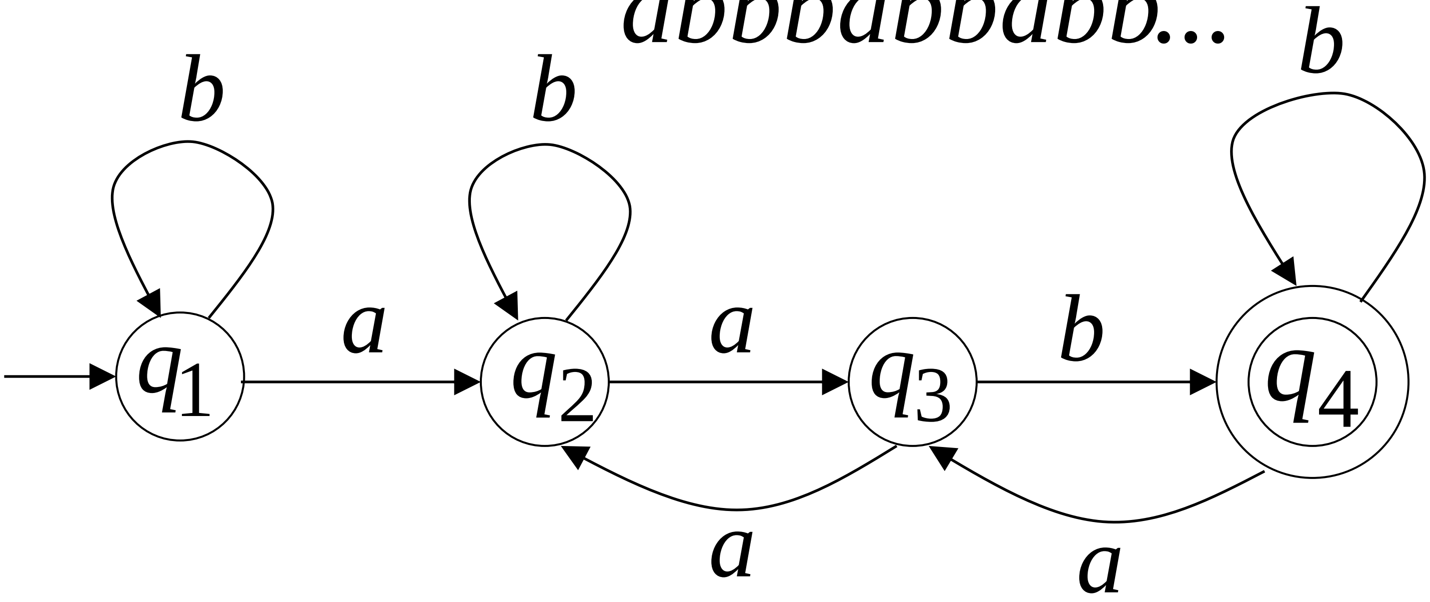
$bbaa$

$abbabb$

$abbbabbabb...$

a state

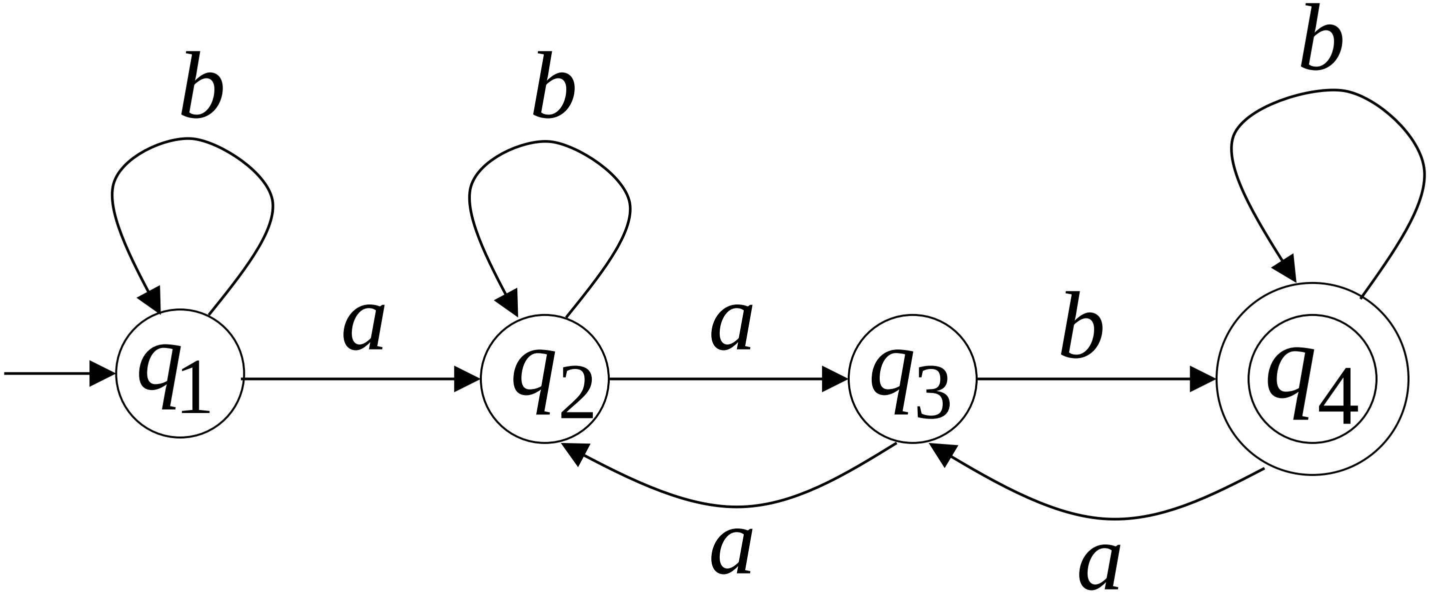
is repeated



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

Thus, a state must be repeated

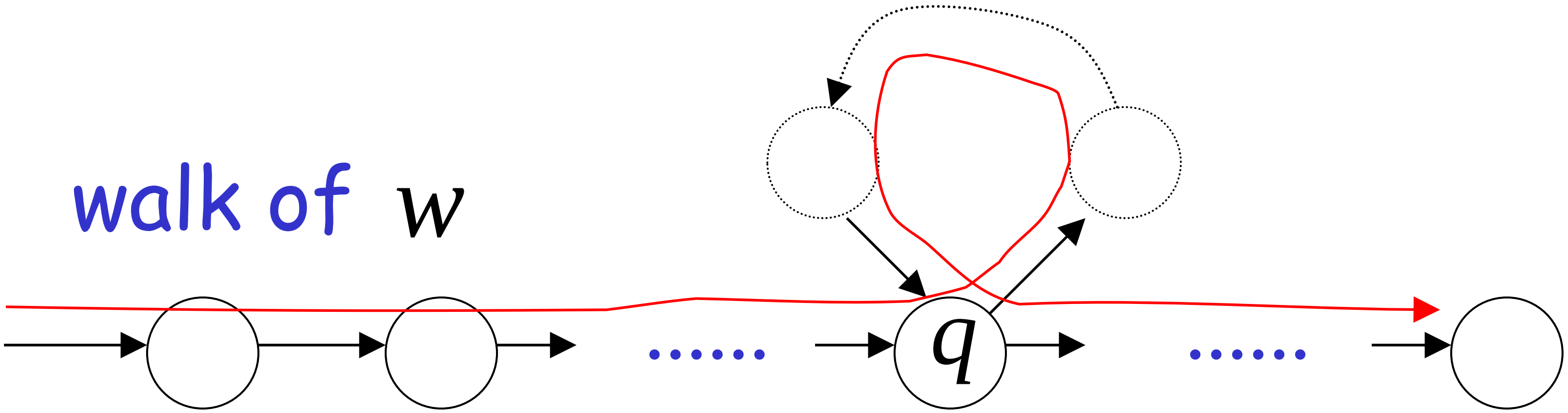


In general, for any DFA:

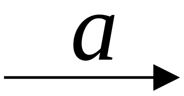
String w has length $\geq$ number of states

$\Downarrow$

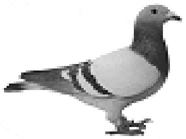
A state q must be repeated in the walk of w

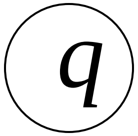


In other words for a string w:

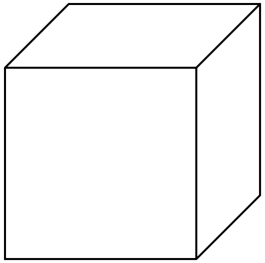


transitions are pigeons

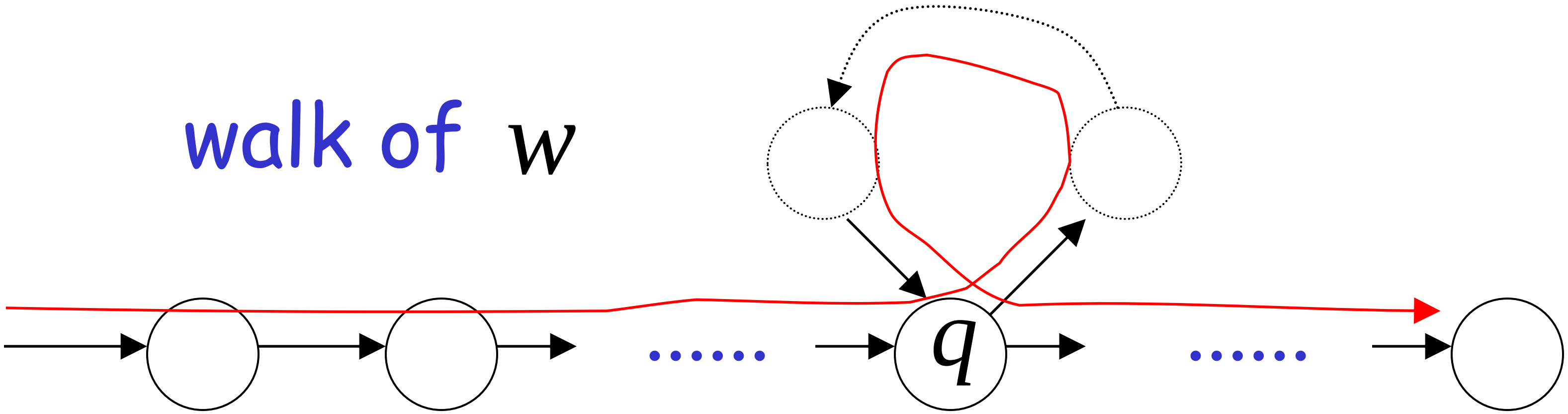




states are pigeonholes



walk of w

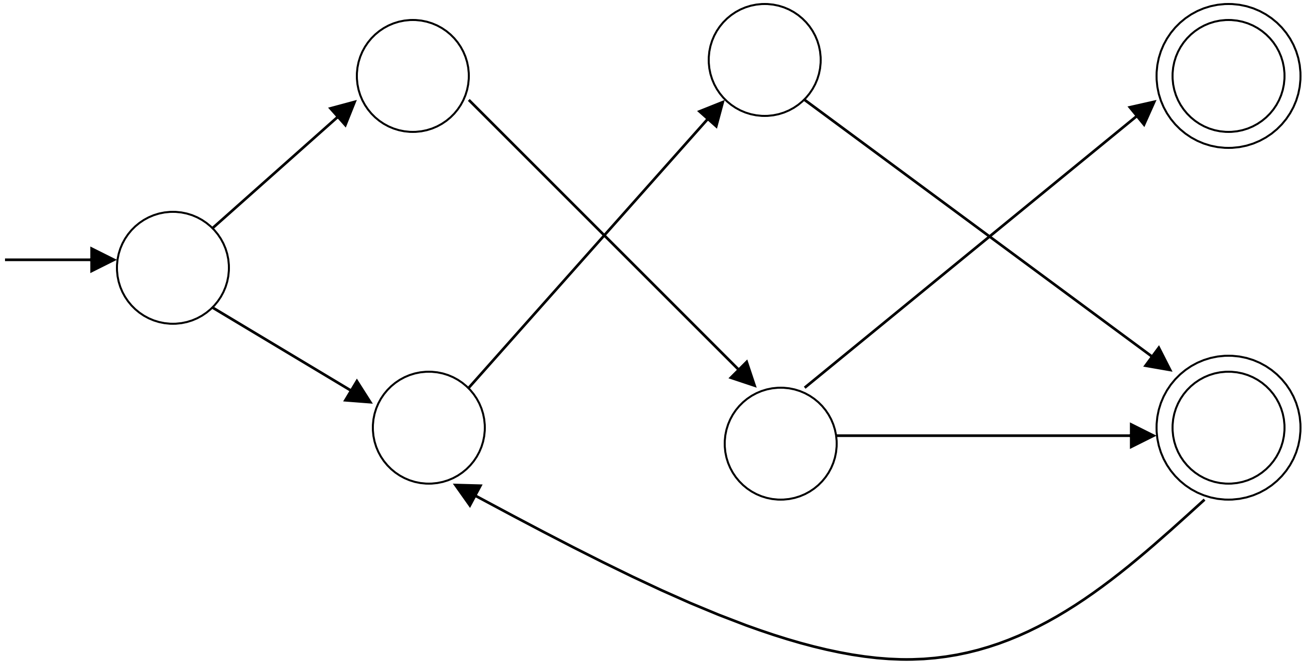


Repeated state

The Pumping Lemma

Take an infinite regular language L

There exists a DFA that accepts L



m
states

Take string w with $w \in L$

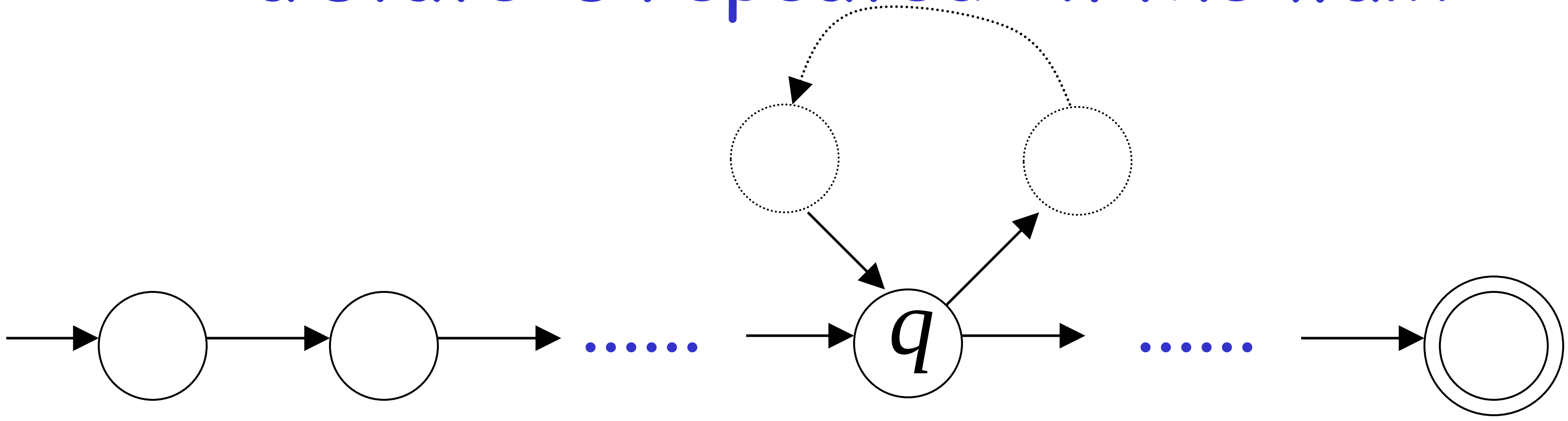
There is a walk with label w:

walk w

If string w has length $|w| \geq m$ (number of states of DFA)

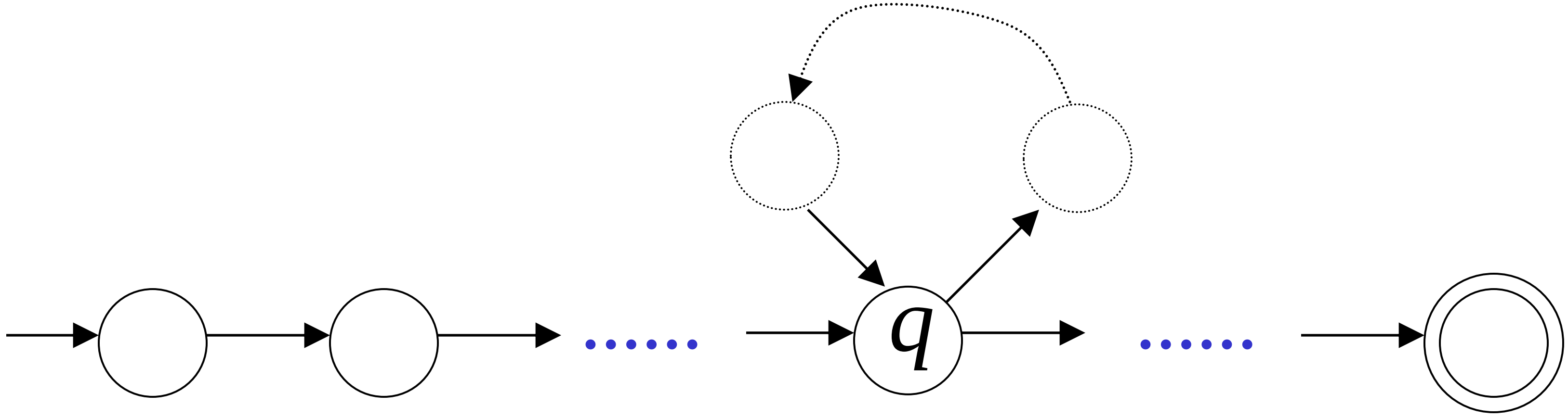
then, from the pigeonhole principle:

a state is repeated in the walk w



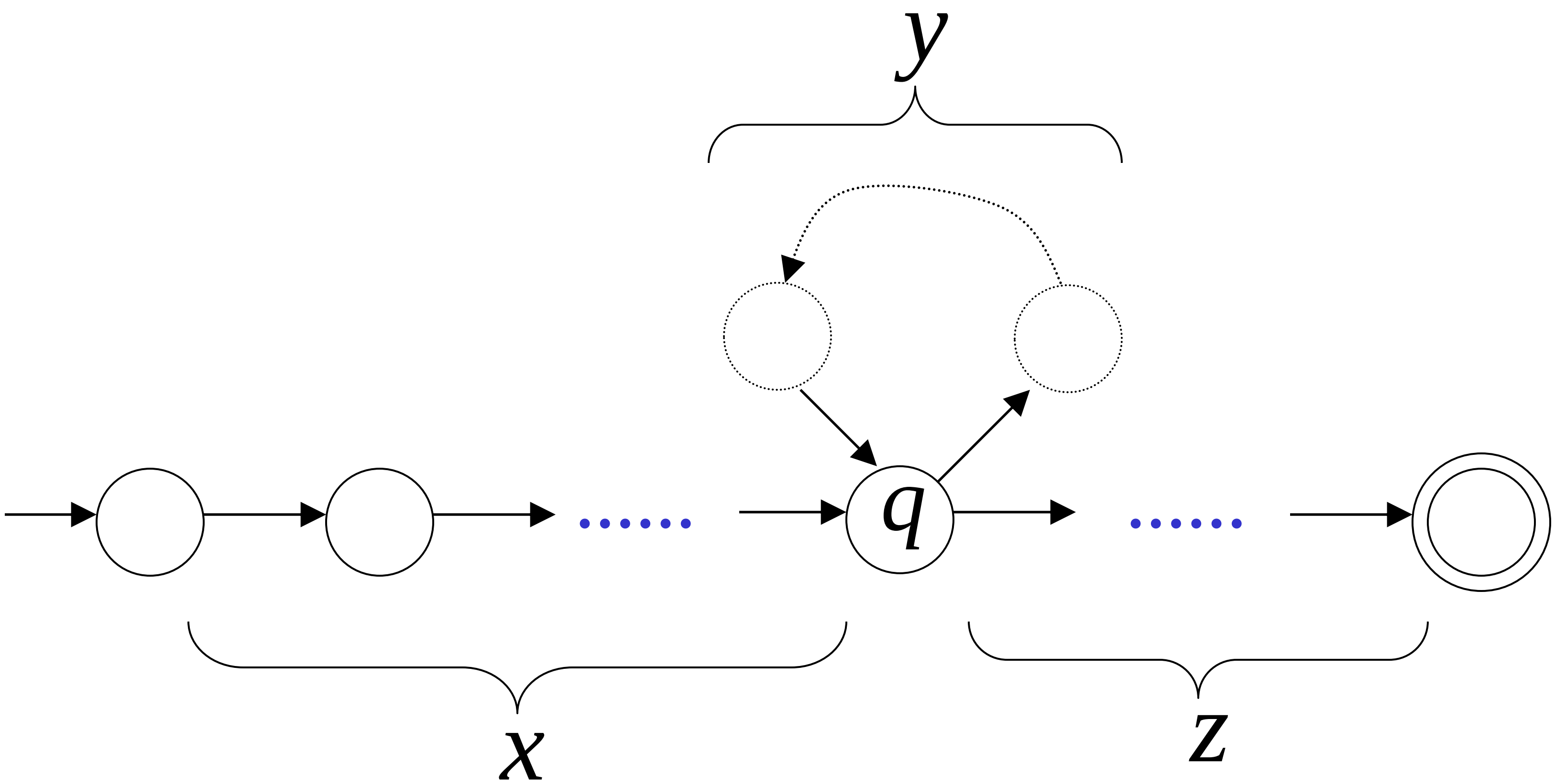
walk w

Let q be the first state repeated in the walk of w



walk w

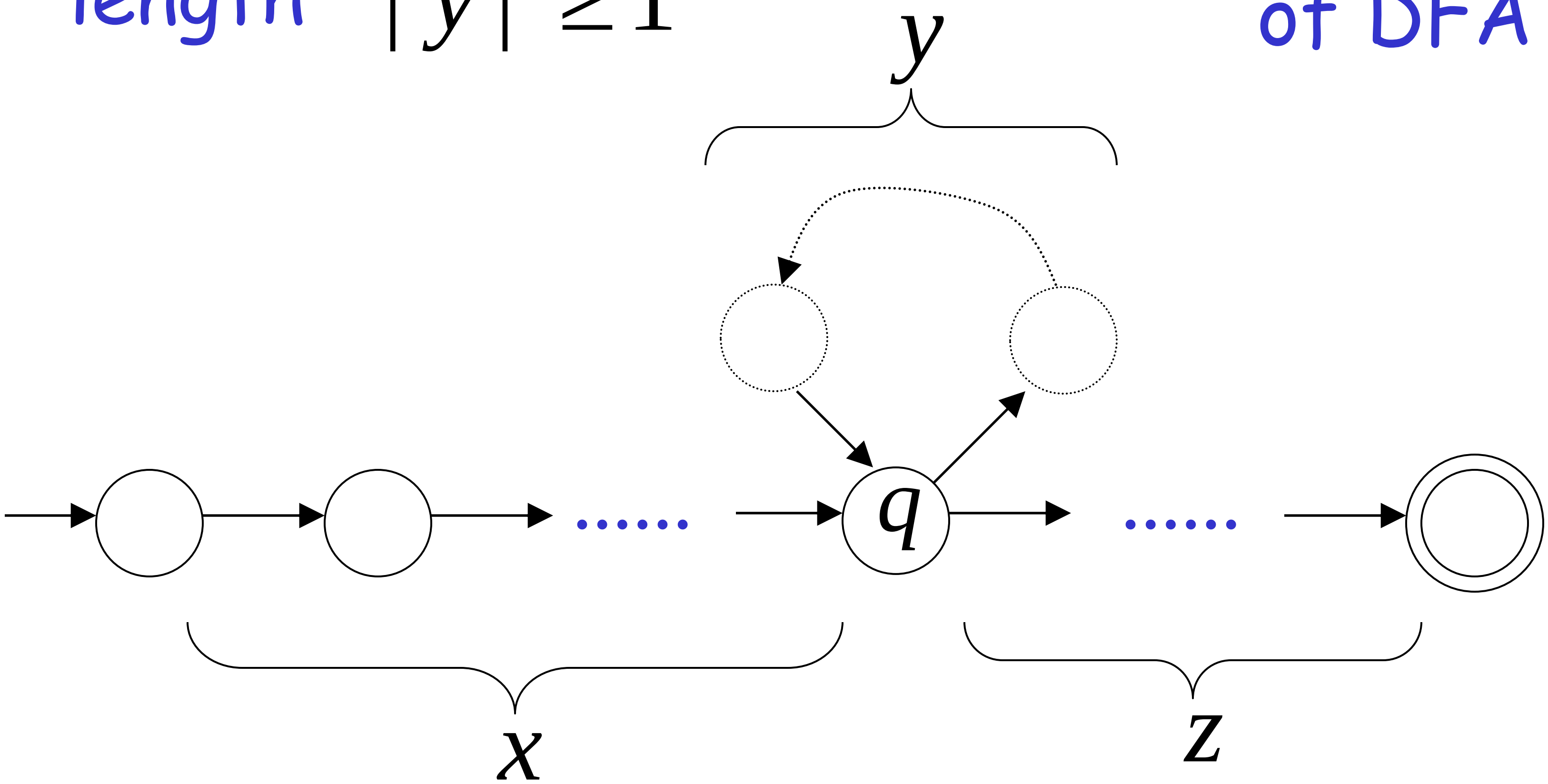
Write $w = x\ y\ z$



Observations:

length $|x\,y| \leq m$ number of states of DFA

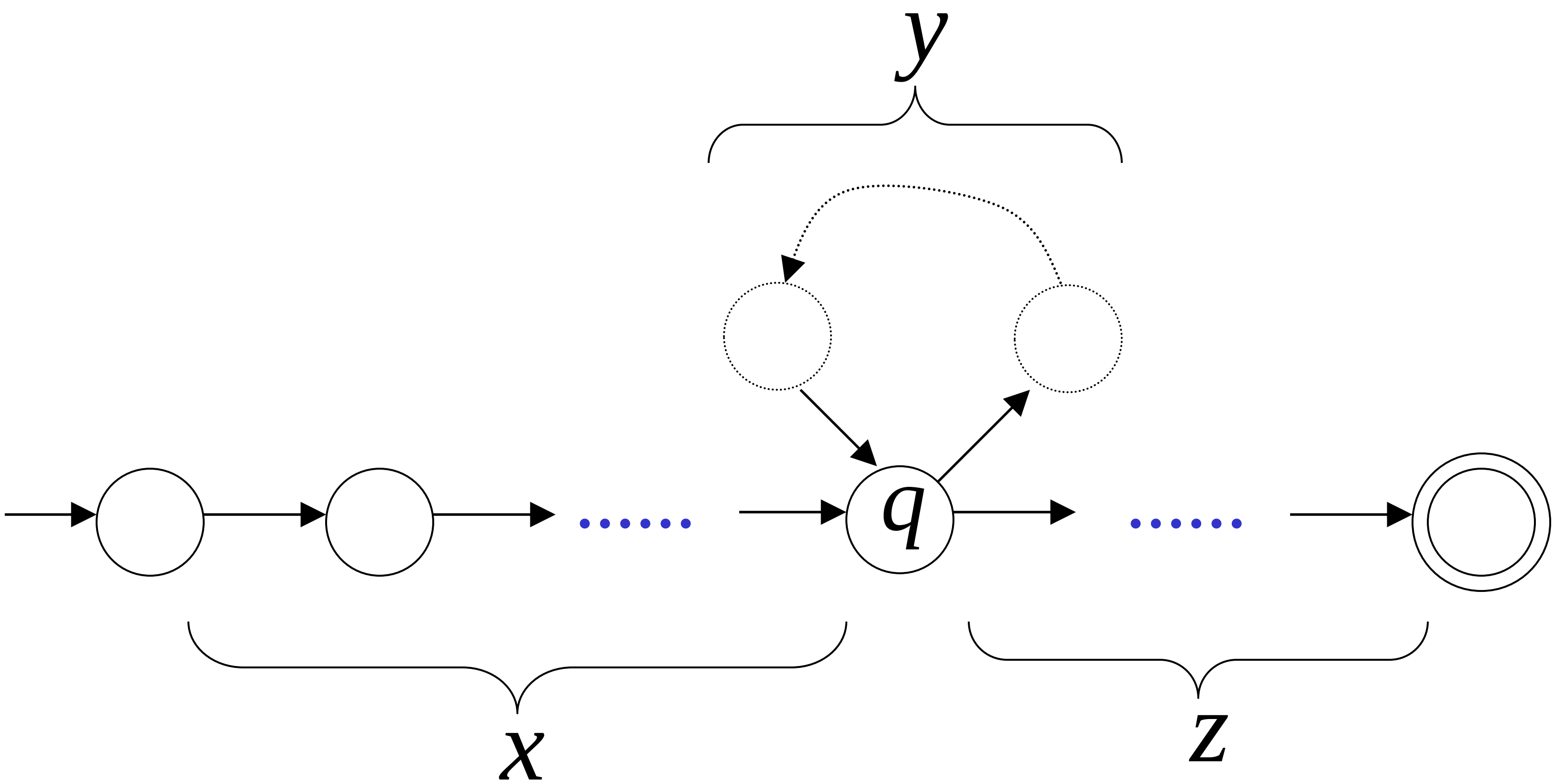
length $|y| \geq 1$



Observation:

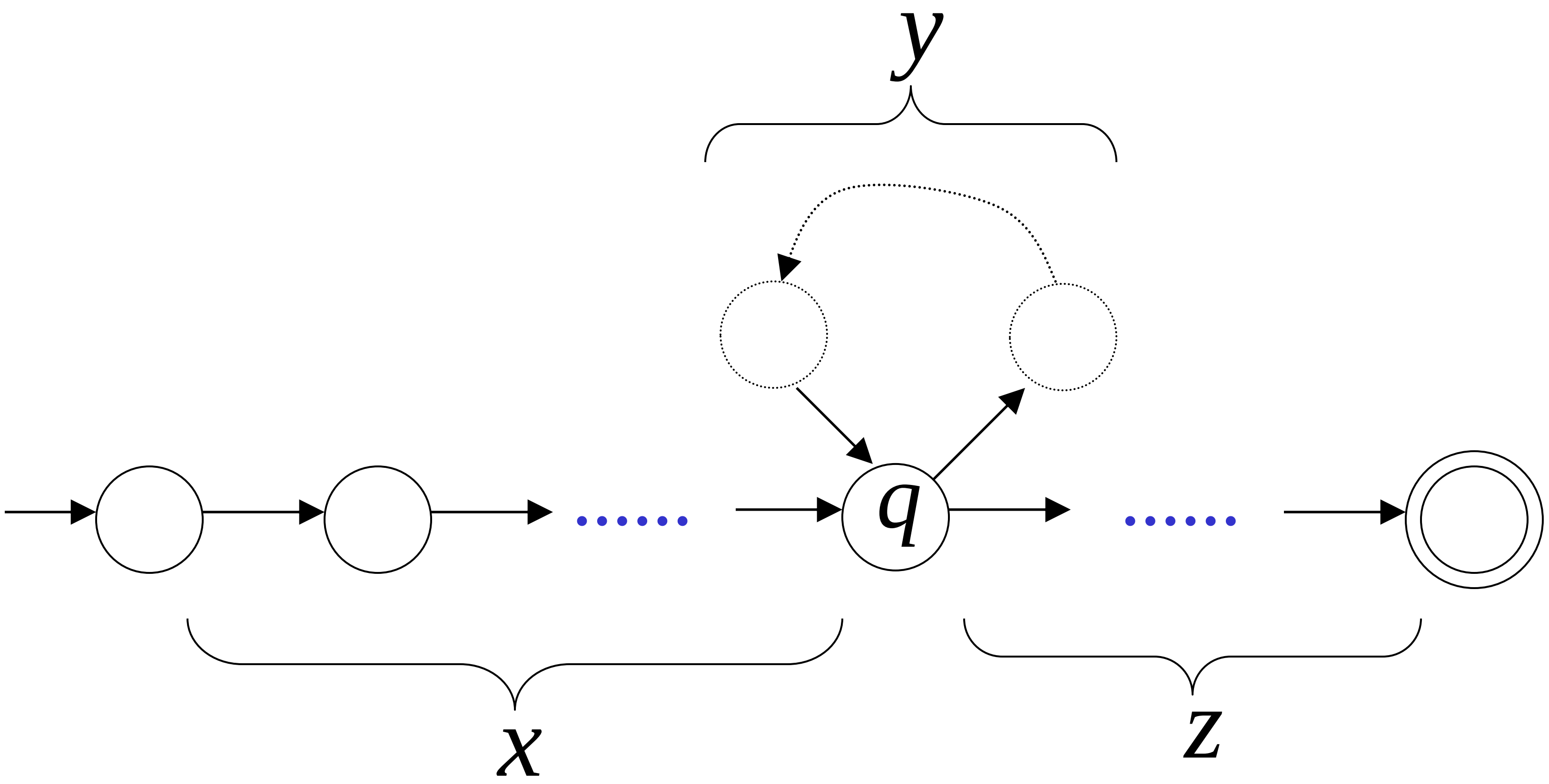
The string $x\ z$

is accepted



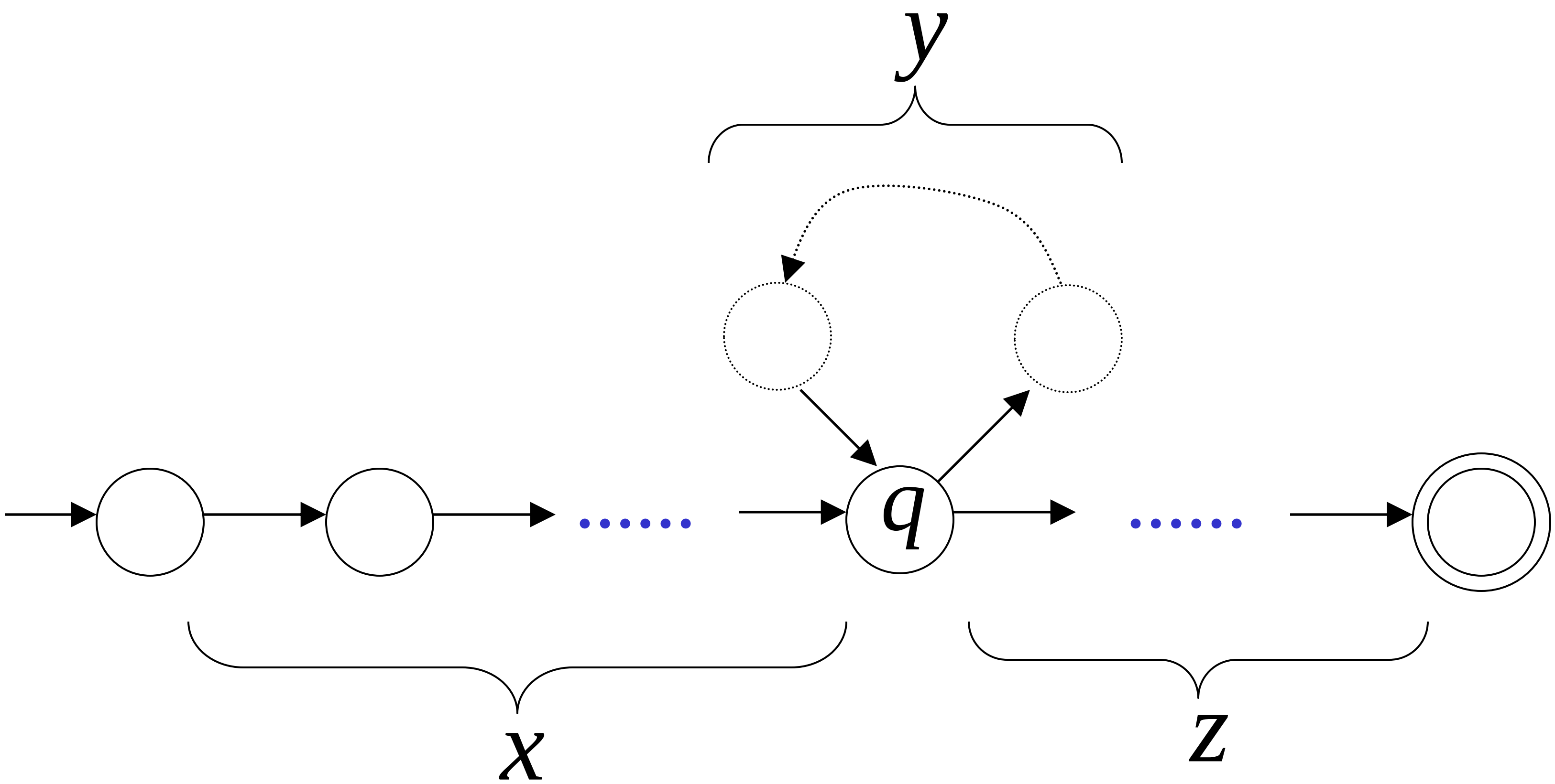
Observation:

The string is accepted $x\ y\ y\ z$



Observation:

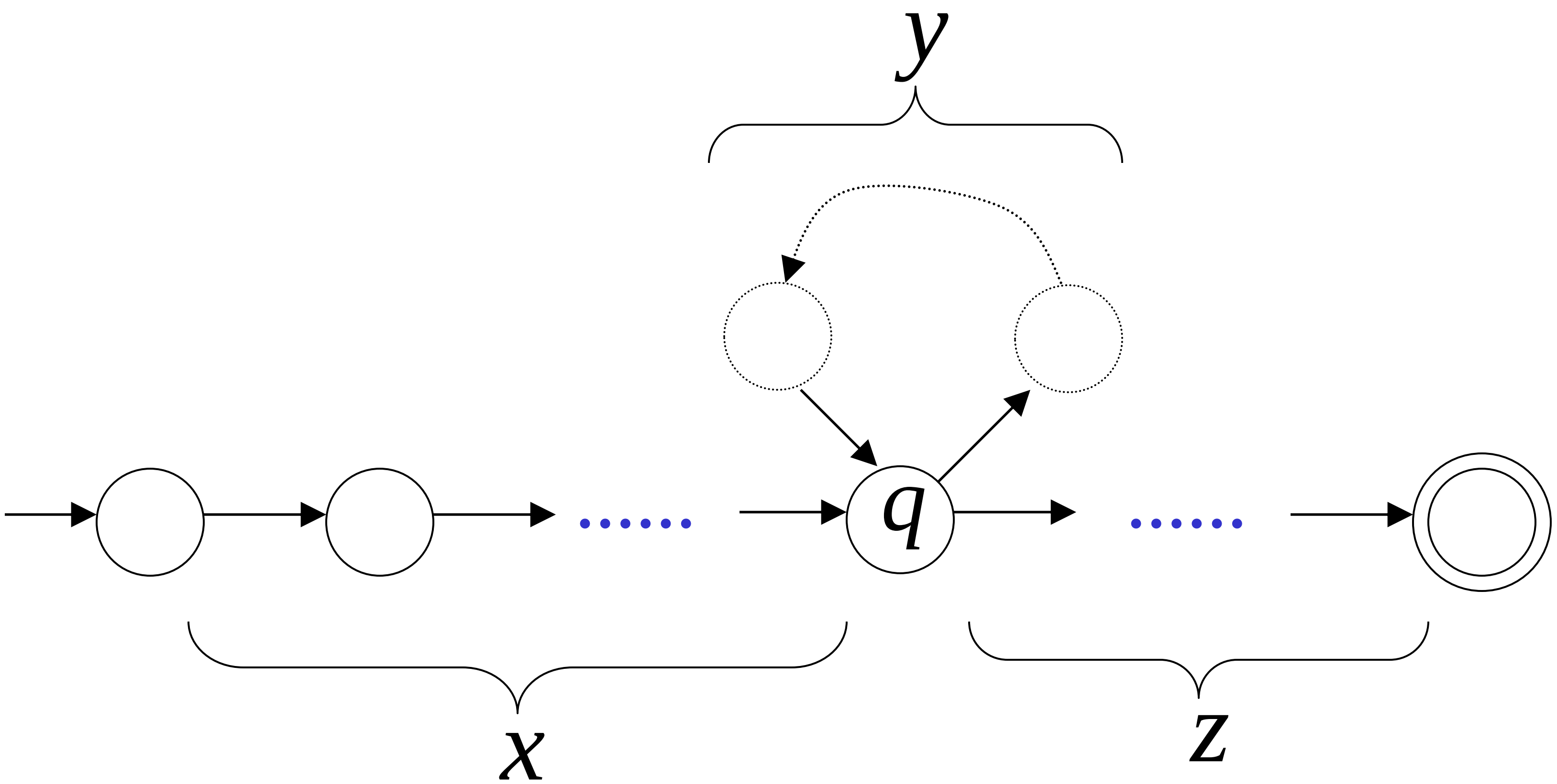
The string is accepted $x\,y\,y\,y\,z$



In General:

The string $x\, y^i\, z$

is accepted $i = 0, 1, 2, \ldots$

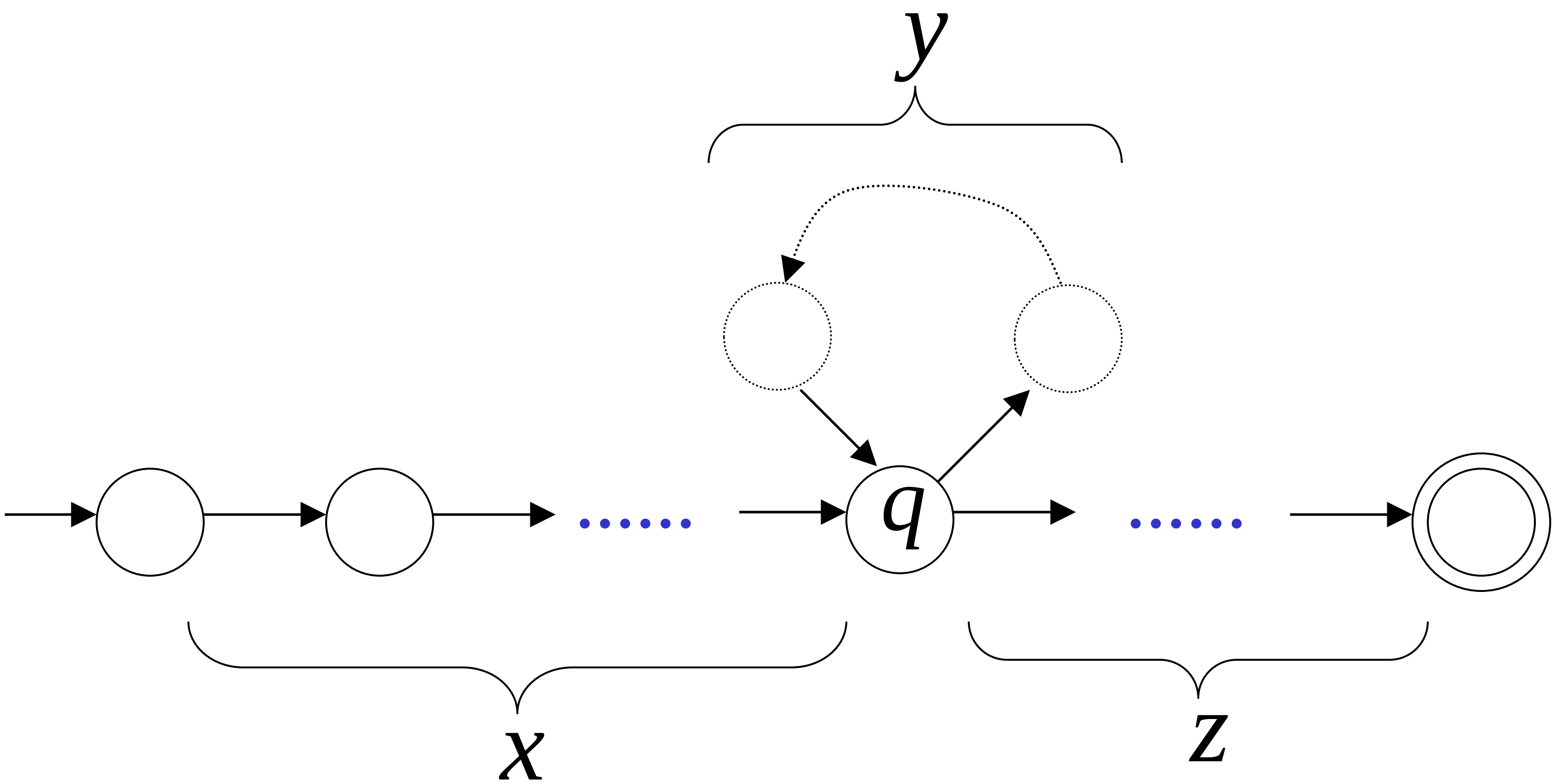


In General:

$x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Language accepted by the DFA



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Applications

of

the Pumping Lemma

Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$L = \{a^n b^n : n \geq 0\}$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^m b^m$

Write: $a^m b^m = x\,y\,z$

From the Pumping Lemma

it must be that length $|x\,y| \leq m, \quad |y| \geq 1$

$$xyz = a^m b^m \;=\; \mathrlap{\overbrace{}^{m}\overbrace{}^{m}}\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots ab\ldots b}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\mathrlap{\underbrace{a...ab...b}_{z}}a...a}^{m+k}\overbrace{b...b}^{m} \in L$$

Thus: $a^{m+k}b^m \in L$

$$a^{m+k}b^m \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^n : n \geq 0\}$

⇓

$$a^{m+k}b^m \notin L$$

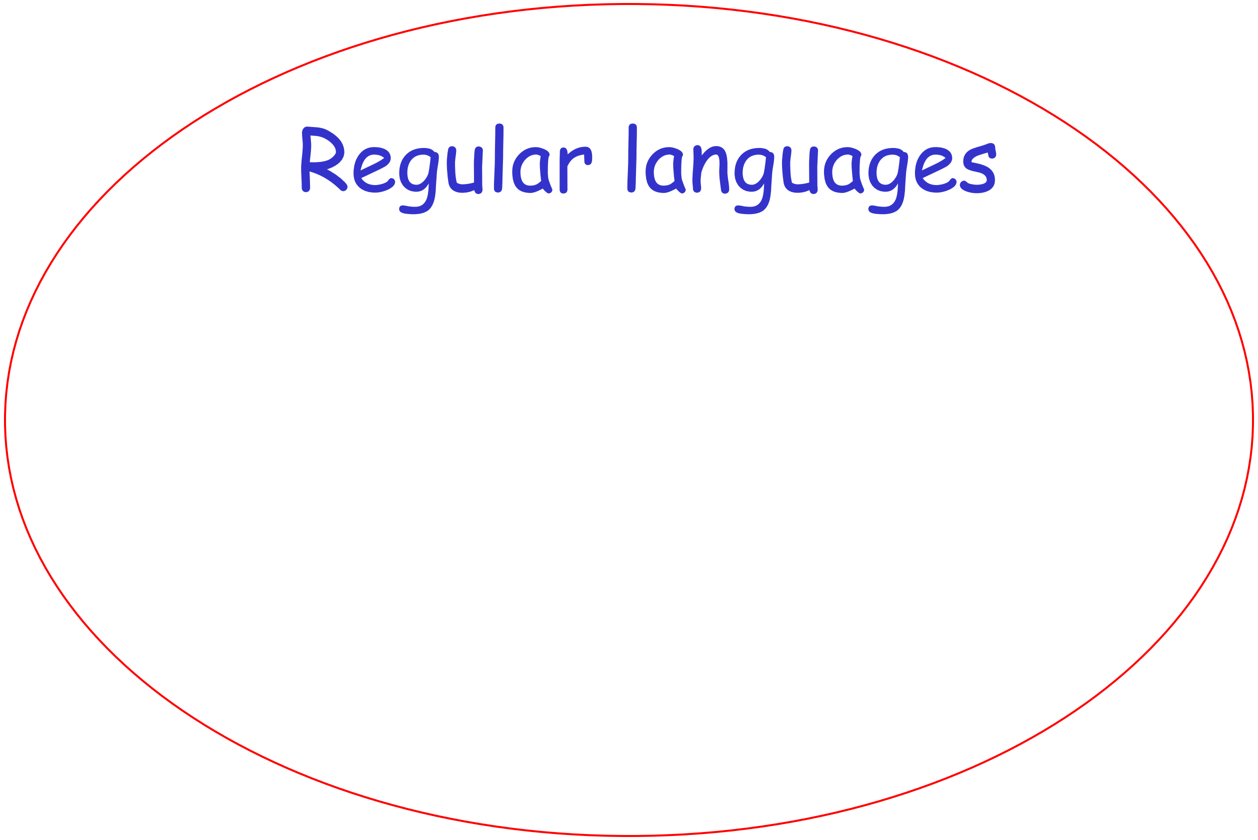
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$\{a^n b^n : \quad n \geq 0\}$$



Regular languages

More Applications

of

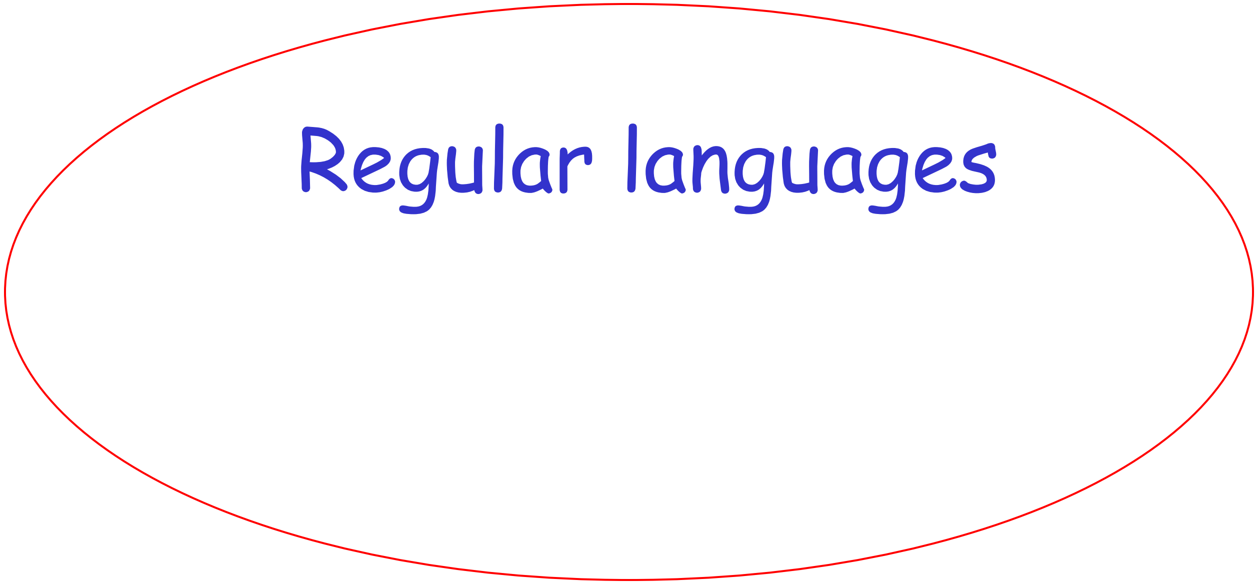
the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L$ $i = 0, 1, 2, \ldots$

Non-regular languages

$$L = \{vv^R : v \in \Sigma^*\}$$


Regular languages

Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

Write $a^m b^m b^m a^m = x\,y\,z$

From the Pumping Lemma

it must be that length $|\,x\,y\,| \leq m, \quad |\,y\,| \geq 1$

$$xyz = \rlap{$\overbrace{}^{m}$}\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{\ldots a\overbrace{b\ldots b}^{m}\overbrace{b\ldots b}^{m}\overbrace{a\ldots a}^{m}}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^i\ z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^2\ z \in L$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{...ab...bb...ba...a}_{z} \in L$$

$$\overbrace{a...aa...aa...a...a}^{m+k}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}$$

Thus: $a^{m+k} b^m b^m a^m \ \in L$

$$a^{m+k}b^m b^m a^m \quad \in L \quad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k}b^m b^m a^m \quad \notin L$$

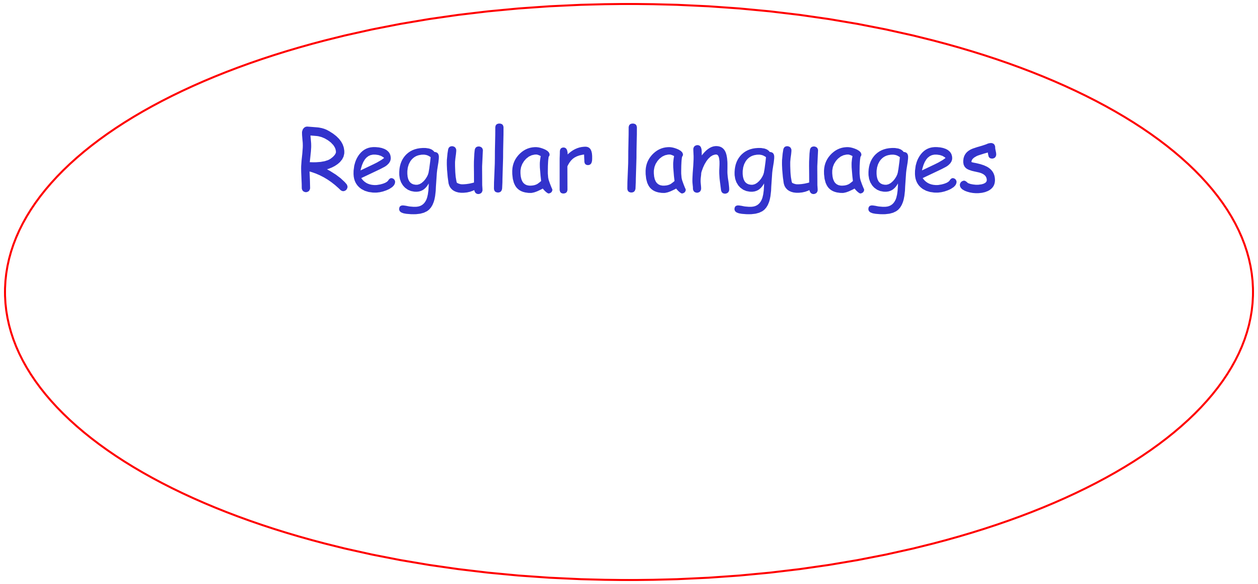
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$



Regular languages

Theorem: The language

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} :\ n, l \geq 0\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$L=\{a^n b^l c^{n+l}:\ n,l\geq 0\}$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w|\geq m$

We pick $w=a^m b^m c^{2m}$

Write $a^m b^m c^{2m} = x\,y\,z$

From the Pumping Lemma

it must be that length $|x\,y| \leq m, \quad |y| \geq 1$

$$xyz = \rlap{$\overbrace{}^{m}\overbrace{}^{m}\overbrace{}^{2m}$}\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots ab\ldots bc\ldots cc\ldots c}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^i\ z \ \in\ L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^0\ z\ = xz\ \in\ L$

$x\ y\ z = a^m b^m c^{2m} \quad y = a^k, \quad k \geq 1$

From the Pumping Lemma: $\quad xz \in L$

$$xz = \underbrace{a...a}_{x}\underbrace{\overbrace{a...a}^{m-k}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}}_{z} \in L$$

Thus: $\quad a^{m-k} b^m c^{2m} \quad \in L$

$$a^{m-k}b^{m}c^{2m} \quad \in L \qquad\qquad k \geq 1$$

BUT: $$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

$$\Downarrow$$

$$a^{m-k}b^{m}c^{2m} \quad \notin L$$

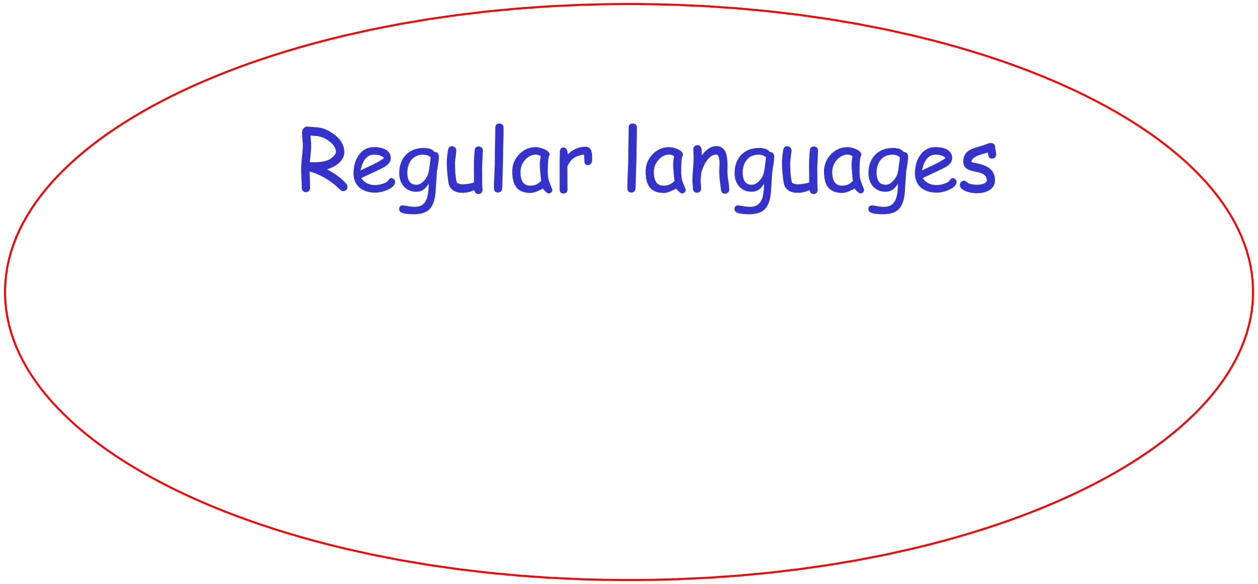
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^{n!} : \; n \geq 0\}$$



Regular languages

Theorem: The language

$$L = \{a^{n!} : \; n \geq 0\}$$ is not regular

$$n! \;=\; 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : \; n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$L = \{a^{n!} : n \geq 0\}$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^{m!}$

Write $a^{m!} = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\,y| \le m, \quad |y| \ge 1$

$$xyz = a^{m!} = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...aa...aa...a}_{z}$$

$$\overbrace{a...aa...aa...a}^{m}\overbrace{a...aa...a}^{m!-m}$$

Thus: $y = a^k, \quad 1 \le k \le m$

$$x\,y\,z = a^{m!} \qquad y = a^{k}, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $x\,y^{i}\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^{2}\,z \in L$

$$x\,y\,z = a^{m!} \qquad y = a^{k}, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\,y^{2}\,z \in L$

$$xy^2z = \mathrlap{\overbrace{}^{m+k}}\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...a\overbrace{a...aa...a}^{m!-m}}_{z} \in L$$

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \quad \in L \qquad 1 \leq k \leq m$$

Since:

$$L = \{a^{n!}: \ n \geq 0\}$$

$$\Downarrow$$

There must exist p such that:

$$m!+k = p!$$

However:

for $m > 1$

$$
\begin{aligned}
m! + k &\le m! + m \\
&\le m! + m! \\
&< m!m + m! \\
&= m!(m+1) \\
&= (m+1)!
\end{aligned}
$$

$$\Downarrow$$

$$m! + k < (m+1)!$$

$$\Downarrow$$

$m! + k \ne p!$ for any p

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $L = \{a^{n!} : \; n \geq 0\}$

$$\Downarrow$$

$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language