Conditional Probability

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Let A and B are any two events, $B \neq \phi$, then $P(A/B)$ denotes the conditional probability of occurrence of event A when B has already occurs.

Example:- (1) Let a bag contain 2 red balls and 3 black balls. One ball is drawn from the bag and this ball is not replaced in the bag. Then a second ball is drawn from the bag.

Let A: The event of occurence of a red ball on the first draw

B: The event of occurence of a black ball on the first draw

C: The event of occurence of a black ball in second draw

$$\begin{bmatrix} 2\ R \\ 3\ B \end{bmatrix}$$

$$P(C/A) = \frac{3}{4}$$

$$P(C/B) = \frac{2}{4} = \frac{1}{2}$$

(ii) Throw a die, $S = \{1, 2, 3, 4, 5, 6\}$

A: The event of occurence of a number greater than 4 $= \{5, 6\}$

B: The event of occurence of an odd number $= \{1, 3, 5\}$

$$P(A/B) = \frac{1}{3}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B/A) = \frac{P(A \cap B)}{P(A)}$$

If A and B are independent events, then probability of occurence of event A is not affected by occurence or not occurence of event B.

$$P(A/B) = P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A) \Rightarrow \boxed{P(A \cap B) = P(A)\,P(B)} \text{---} *$$

Remark : For three independent events A, B and C

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

In general, If $A_1, A_2, \ldots A_n$ are independent events then $P(A_1 \cap A_2 \cap \cdots \cap A_n)$

$$= P(A_1)\,P(A_2) \cdots P(A_n)$$

Some results:

(1) $P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B})$

(2) The events A and ϕ are independent

(3) The events A and S are independent.

Proof: (1) $P(A \cup B) = 1 - P(\overline{\overline{A} \cup \overline{B}})$

$= 1 - P(\overline{A} \cap \overline{B})$

$= 1 - P(\overline{A})P(\overline{B})$

(2) $P(A \cap \phi) = P(\phi) = 0$

$P(A)P(\phi) = P(A) \cdot 0 = 0$

So $P(A \cap \phi) = P(A)P(\phi)$

(3) $A \cap S = A$

$P(A \cap S) = P(A) = P(A) \cdot 1 = P(A) \cdot P(S)$

(4) If A and B are independent events, then

(I) A and $\overline{B}$ are independent events

(II) $\overline{A}$ and B are independent events

(III) $\overline{A}$ and $\overline{B}$ are independent events

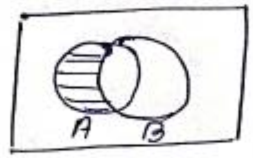
Proof:- (I) $P(A\cap\overline{B})$

$= P(A) - P(A\cap B)$

$= P(A) - P(A)P(B)$

$= P(A)(1-P(B))$

$= P(A)\,P(\overline{B})$, So A and $\overline{B}$ are independent



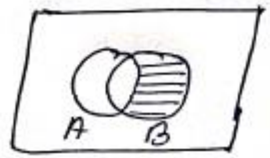
(II) $P(\overline{A}\cap B)$

$= P(B) - P(A\cap B)$

$= P(B) - P(A)P(B)$

$= P(B)(1-P(A))$

$= P(\overline{A})\,P(B)$, So $\overline{A}$ and B are independent.



(III) $P(\overline{A}\cap\overline{B}) = P(\overline{A\cup B}) = 1 - P(A\cup B)$

$= 1 - [P(A) + P(B) - P(A\cap B)]$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= (1-P(A)) - P(B)(1-P(A))$

$= (1-P(A))(1-P(B)) = P(\overline{A})P(\overline{B})$

(5) If A and B are two events such that $B \neq \phi$, then

$$P(A/B) + P(\overline{A}/B) = 1$$

Proof : $P(A/B) + P(\overline{A}/B)$

$$= \frac{P(A \cap B)}{P(B)} + \frac{P(\overline{A} \cap B)}{P(B)} = \frac{P(A \cap B) + P(\overline{A} \cap B)}{P(B)}$$

$$= \frac{P(B)}{P(B)} = 1$$

(1) For two events A and B, $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cup B) = 0.8$. Find the conditional probability $P(A/B)$ and $P(B/A)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad 0.8 = 0.5 + 0.6 - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = 1.1 - 0.8 = 0.3$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.6} = \frac{1}{2}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{0.3}{0.5} = \frac{3}{5}$$

(2) A dice is thrown twice and the sum of the numbers appearing on them is noted to be 8. What is the condetional probability that the number 5 has appeared atleast once.

Define the events

A: The number 5 appears atleast once

B: The sum of number appearing is 8.

$$A = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$$
$$(1,5),(2,5),(3,5),(4,5),(6,5)\}$$

$$B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

$$A \cap B = \{(3,5),(5,3)\}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{2/36}{5/36} = 2/5$$

(3) Events E and F are given to be independent. If it is given that $P(E) = 0.4$ and $P(E \cup F) = 0.55$ Find $P(F)$

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{1}{15}$ (d) none

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$\Rightarrow 0.55 = 0.4 + P(F) - P(E)P(F)$$

$$\Rightarrow 0.15 = P(F)(1 - P(E)) = P(F) \times 0.6$$

$$\therefore P(F) = \frac{0.15}{0.6} = \frac{1}{4}$$

A problem in a question paper is given to three students in a class to be solved. The probabilities of their solving the problem are 0.5, 0.7 and 0.8 respectively. Find the probability that the problem will be solved.

P(Problem will be solved)

$= 1 -$ P(Problem will not be solved)

$= 1 - P(\overline{A \cup B \cup C}) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$

$= 1 - P(\overline{A})P(\overline{B})P(\overline{C}) = 1 - (1-0.5)(1-0.7)(1-0.8)$

$= 1 - 0.5 \times 0.3 \times 0.2 = 1 - 0.030 = 0.97$

A pair of dice is thrown together till a sum of 4 or 8 obtained. Determine the probability that the sum 4 appears before 8.

A: Sum of 4 is obtained : $A = \{(1,3),(2,2),(3,1)\}$

B: Sum of 8 is obtained : $B = \{(2,6),(3,5),(4,4)$

C: Sum other than 4 or 8 obtained | $(5,3),(6,2)\}$

$P(A) = \frac{3}{36}$, $P(B) = \frac{5}{36}$, $P(C) = \frac{28}{36}$

$$P(4 \text{ appears before } 8) = A + CA + CCA + CCCA + \cdots$$

$$= \frac{3}{36} + \frac{28}{36}\cdot\frac{3}{36} + \left(\frac{28}{36}\right)^2\cdot\frac{3}{36} + \left(\frac{28}{36}\right)^3\frac{3}{36} + \cdots$$

$$= \frac{3/36}{1 - 28/36} = \frac{3/36}{8/36} = \frac{3}{8}$$