Fourier Series

$(0,2)$ $\quad 2l = 2-0 = 2$ $\quad \boxed{l=1}$

$(\alpha, \alpha+2\pi)$ $\quad 2l = 2\pi$ $\quad \boxed{l=\pi}$

length of interval = 2l

$f(x) \to \underline{(0,2l)}$

$f(x) \to$ Cosine ✓, Sines ✓

If $f(x)$ is a fxn defined on $\underline{(\alpha, \alpha+2l)}$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \Big] \to \text{Fourier series}$$

where

$$a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

} Euler's Formulae

(1) If $l=\pi$

then $f(x)$ is defined on $(\alpha, \alpha+2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)\,dx, \quad a_n = \dfrac{1}{\pi}\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$$

(ii) if $\alpha = -l$ $\quad f(x) \longrightarrow (\alpha, \alpha+2l) \longrightarrow (-l, l)$

If $f(x) \longrightarrow (-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$, $\quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

(iii) $\boxed{\alpha = -l}$ — $\boxed{f(x) \longrightarrow (-l, l)}$

(i) If $f(x)$ is even $\quad \int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$

(ii) If $f(x)$ is odd $\quad \int_{-l}^{l} f(x)\,dx = 0$

(i) If $f(x)$ is even then $\quad f(x)\cos\frac{n\pi x}{l} \rightarrow$ even even $\rightarrow$ even

$f(x)\sin\frac{n\pi x}{l} \rightarrow$ even odd $\rightarrow$ odd

(ii) If $f(x)$ is odd then $\quad f(x)\cos\frac{n\pi x}{l} \rightarrow$ odd $\times$ even $\rightarrow$ odd

$f(x)\sin\frac{n\pi x}{l} \rightarrow$ odd $\times$ odd $\rightarrow$ even

(i) $f(x)$ — $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum^{\infty} a_n \cos\frac{n\pi x}{l} + \sum^{\infty} b_n \sin\frac{n\pi x}{l}$$

(I) $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

(i) $f(x)$ is even

then $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_0^l f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0$

$f(x) \longrightarrow (-l, l)$ & if $f(x)$ is even then $\boxed{b_n = 0}$

(ii) $f(x)$ is odd $\rightarrow$ $f(x)$ — $(-l, l)$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = 0$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Half Range Fourier Series

$\boxed{\text{length } (2l)}$

$f(x) \longrightarrow (0, l)$

H.R FS

Half Range Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

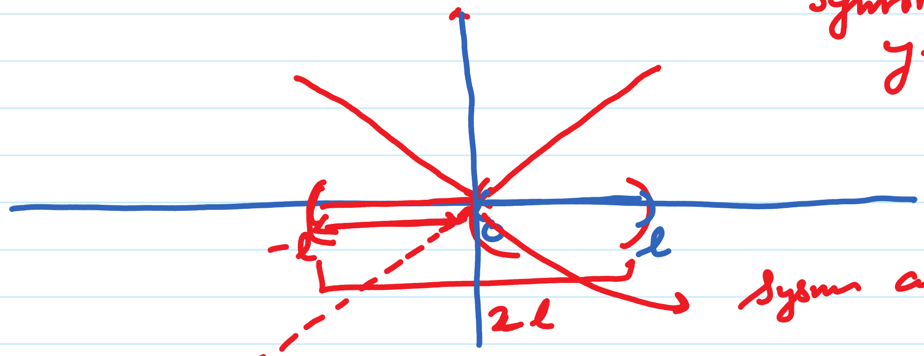
$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Half Range Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

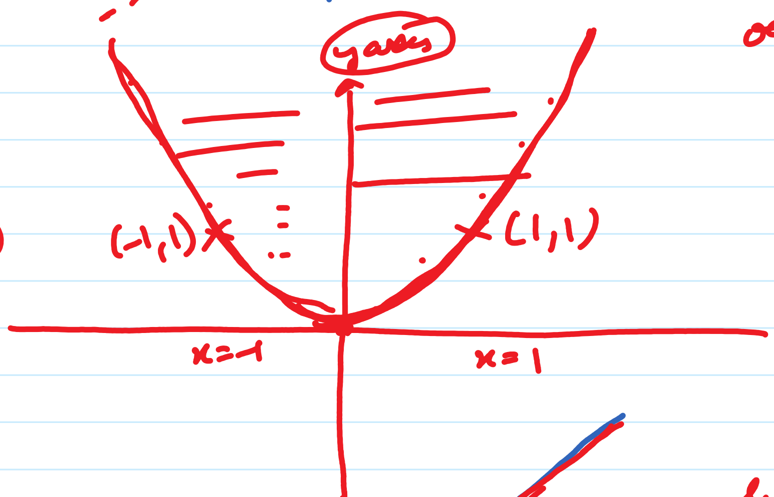
Symmetrical about y-axis → even fn



Symm about origin → odd fn

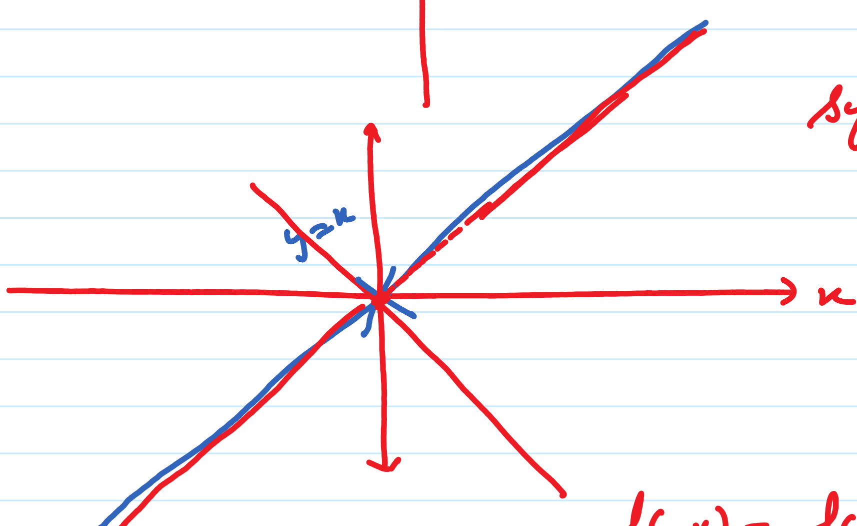
$f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$



$f(x) = x$

Symmetrical origin $\boxed{\text{odd fn}}$



$f(-x) = f(x) \rightarrow$ even

$= -f(x) \rightarrow$ odd

$f(-x) = f(x) \rightarrow$ even
$= -f(x) \rightarrow$ odd

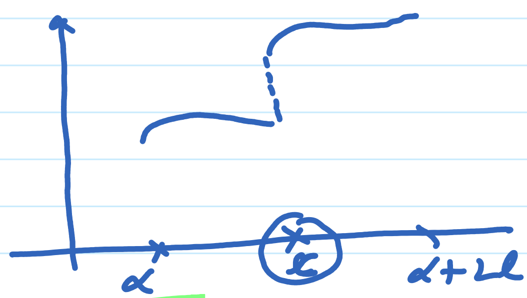
$f(x) \rightarrow (\alpha, \alpha+2l)$

1. $f(x)$ is periodic, single valued & finite
2. $f(x)$ has a finite number of discontinuities in any one period
3. $f(x)$ has at the most a finite number of maxima & minima.

Dirichlet's Conditions

Functions having point of discontinuity

$$f(x) = \begin{cases} \phi(x), & \alpha < x < c \\ \psi(x), & c < x < \alpha+2l \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{l}\left[\int_{\alpha}^{\alpha+2l} f(x)\,dx\right] = \frac{1}{l}\left[\int_{\alpha}^{c} f(x)\,dx + \int_{c}^{\alpha+2l} f(x)\,dx\right]$$

$$a_n = \frac{1}{l}\left[\int_{\alpha}^{c} f(x)\cos\frac{n\pi x}{l}\,dx + \int_{c}^{\alpha+2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$b_n = \frac{1}{l}\left[\int_{\alpha}^{c} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{c}^{\alpha+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

At $x = c$

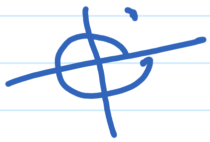
Here $f(x) = \dfrac{[f(c-0)+f(c+0)]}{2}$

① $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = \left| \frac{\sin nx}{n} \right|_{\alpha}^{\alpha+2\pi} = \frac{1}{n}\left[\sin n(\alpha+2\pi) - \sin n\alpha\right]$

$= \frac{1}{n}\left[\sin(2n\pi + n\alpha) - \sin n\alpha\right]$

$= \frac{1}{n}\left[\sin n\alpha - \sin n\alpha\right]$

$= 0$ ✓



② $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = 0$

③ $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} 2\cos mx \cos nx \, dx$ $\boxed{m \neq n}$

$\displaystyle = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} \left[\cos(m+n)x + \cos(m-n)x\right] dx$

$\displaystyle = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right|_{\alpha}^{\alpha+2\pi}$

$\displaystyle = \frac{1}{2}\left[\left\{\frac{\sin(m+n)(\alpha+2\pi)}{m+n} + \frac{\sin(m-n)(\alpha+2\pi)}{m-n}\right\} - \left\{\frac{\sin(m+n)\alpha}{m+n} + \frac{\sin(m-n)\alpha}{m-n}\right\}\right]$

$\displaystyle = \frac{1}{2}\left[\left\{\frac{\sin(m+n)\alpha}{m+n} + \frac{\sin(m-n)\alpha}{m-n}\right\} - \left\{\frac{\sin(m+n)\alpha}{m+n} + \frac{\sin(m-n)\alpha}{m-n}\right\}\right]$

$= 0$

$\alpha+2\pi$ $\qquad$ $\alpha+2\pi$ $\qquad$ $\cos 2A = 2\cos^2 A - 1$

(iv) $$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} 2\cos^2 nx \, dx$$

$\cos 2A = 2\cos^2 A - 1$

$$= \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} (1+\cos 2nx)\, dx$$

$$= \frac{1}{2}\left| x + \frac{\sin 2nx}{2n} \right|_{\alpha}^{\alpha+2\pi}$$

$$= \frac{1}{2}\left[\left\{\alpha+2\pi+\frac{1}{2n}\sin 2n(\alpha+2\pi)\right\} - \left\{\alpha + \frac{1}{2n}\sin 2n\alpha\right\}\right]$$

$$= \frac{1}{2}\left[\cancel{\alpha}+2\pi+\frac{1}{2n}\sin 2n\alpha - \cancel{\alpha} - \frac{1}{2n}\sin 2n\alpha\right]$$

$$= \frac{1}{2}\times 2\pi = \pi$$

$\sin(2n\alpha + \underline{4n\pi})$
$= \sin 2n\alpha$

(v) $$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n. \end{cases}$$

(5) $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ 0 & \text{if } m = n \end{cases}$$

(6) $$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f(x) \text{ is even}$$
$$= 0 \quad \text{if } f(x) \text{ is odd}$$

$\frac{n\pi x}{l}$

Q. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

$2l = 2\pi$

Q Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

$$\boxed{2l = 2\pi \quad \boxed{l = \pi}}$$

Sol: $e^{-x} = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left|\frac{e^{-x}}{-1}\right|_0^{2\pi}$$

$$= -\frac{1}{\pi}\left[e^{-2\pi} - e^{0}\right]$$

$$= \frac{1}{\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{1^2+n^2}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$= \frac{1}{\pi(1+n^2)}\left|e^{-x}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[\left\{e^{-2\pi}(-\cos 2n\pi + n\sin 2n\pi)\right\} - \left\{e^{0}(-\cos 0 + n\sin 0)\right\}\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-1+0) - (-1+0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-e^{-2\pi} + 1\right]$$

$$\cos n\pi = (-1)^n$$
$$\sin n\pi = 0$$

$$= \frac{(1-e^{-2\pi})}{\pi} \cdot \frac{1}{n^2+1}$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$

$$\int e^{ax}\sin bx\,dx$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{1+n^2}\left[-\sin nx - n\cos nx\right]\right|_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[\left\{e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi)\right\} - \left\{e^{0}(-\sin 0 - n\cos 0)\right\}\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(0-n) + n\right] = \frac{n}{\pi(1+n^2)}\left[1-e^{-2\pi}\right]$$

$$= \frac{(1-e^{-2\pi})}{\pi}\cdot\frac{n}{n^2+1}$$

$\therefore$ from (1)

$$e^{-x} = \frac{1}{2}\left[\frac{1}{\pi}(1-e^{-2\pi})\right] + \sum_{n=1}^{\infty}\frac{(1-e^{-2\pi})}{\pi}\cdot\frac{1}{n^2+1}\cos nx$$

$$+ \sum_{n=1}^{\infty}\frac{(1-e^{-2\pi})}{\pi}\cdot\frac{n}{n^2+1}\sin nx$$

$$= \frac{1}{\pi}(1-e^{-2\pi})\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx + \sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx\right]$$

$$= \frac{(1-e^{-2\pi})}{\pi}\left[\frac{1}{2} + \left\{\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \cdots\right\} + \left\{\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \cdots\right\}\right]$$

Q/ $f(x) = e^{-x}$, $0 < x < 2\pi$

$\frac{1}{\pi}\int^{2\pi} e^{-x}\cos nx\,dx$

Q $f(x) = e^{-x}, \quad 0 < x < 2\pi$

$a_4 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos 4x\,dx$ $\qquad a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$

$a_3 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos 3x\,dx$ $\qquad a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$

$b_1 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin x\,dx$

Q Find a Fourier Series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$

Sol $x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= 0 - \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= -\frac{2}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi} = -\frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$= -2\pi^2/3$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\underline{x\cos nx\,dx} - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$$= 0 - \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= -\frac{2}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\left\{0 + \frac{2\pi\cos n\pi}{n^2} - 0\right\} - \left\{0 + 0 - 0\right\}\right]$$

$$= -\frac{2}{\cancel{\pi}}\left[\frac{2\cancel{\pi}(-1)^n}{n^2}\right. = -\frac{4(-1)^n}{n^2} = \frac{4(-1)^{n+1}}{n^2}$$

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx - 0$$

$$= \frac{2}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi\cos n\pi}{n} - 0\right\} - \left\{0 + 0\right\}\right] = -\frac{2}{\cancel{\pi}} \times \cancel{\pi}\frac{(-1)^n}{n}$$

$$= -\frac{2}{n}(-1)^n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore x - x^2 = \frac{1}{2} \times -\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

$$x - x^2 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\Big] \#$$

$$\left[\ 1\cos x - 1\cos 2x + 1\cos 3x - 1\cos 4x + \cdots\right]$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \cdots\right]$$

$$+ 2\left[\underbrace{\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots}\right]$$

At $\boxed{x=0}$

$$0-0 = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] + 2\left[0 - 0 \cdots\right]$$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots}$$

$-\pi$ —— π

$$\Rightarrow \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

or $\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$

Q Expand $f(x) = x\sin x$ as a Fourier series in the interval $0 < x < 2\pi$.

Sol $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\Big| x(-\cos x) - (1)(-\sin x)\Big|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(-2\pi) - 0\right] = -2$$

$$\boxed{a_0 = -2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$\frac{1}{\cdot}\int^{2\pi} x\,(2\cos nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - (1)\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

if $n \neq 1$

$$= \frac{1}{2\pi}\left[\left\{2\pi\left[-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right] - 0\right\} - \{0\}\right]$$

$$= \left[-\frac{1}{n+1} + \frac{1}{n-1}\right] = \left[\frac{-n+1+n+1}{n^2-1}\right]$$

$$\boxed{a_n = \frac{2}{n^2-1}}\quad , \text{ if } n \neq 1$$

For $\underline{n=1}$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\cos x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{\frac{-2\pi\times 1}{2} - 0\right\} - \{0\}\right]$$

$$= \frac{1}{2\pi}\times -\frac{2\pi}{2} = -1/2$$

$$\therefore \boxed{a_1 = -1/2}\ ,\ \boxed{a_n = \frac{2}{n^2-1},\quad n \neq 1}$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$

$= 0 \quad$ if $n \neq 1$

For $n=1$, $\quad b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx$

$= \pi$

Q. Find the Fourier series expansion for $f(x)$, if

$$f(x) = \begin{cases} -\pi \ ; & -\pi < x < 0 \\ x \ ; & 0 < x < \pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \pi^2/8$

$2l = 2\pi$

$\boxed{l = \pi}$

Sol $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{(1)}$

$$f(-x) = \begin{cases} -\pi \ ; & -\pi < -x < 0 \\ -x \ ; & 0 < -x < \pi \end{cases}$$

$$= \begin{cases} -\pi \ ; & \pi > x > 0 \\ -x \ ; & -\pi < x < 0 \end{cases}$$

$$= \begin{cases} -x \ ; & -\pi < x < 0 \\ -\pi \ ; & 0 < x < \pi \end{cases}$$

$\neq -f(x), f(x)$

where $a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right\}$

$= \frac{1}{\pi}\left\{-\int_{-\pi}^{0} \pi \, dx + \int_0^{\pi} x \, dx\right\}$

$$= \frac{1}{\pi}\left[-\pi \left| x \right|_{-\pi}^{0} + \left| \frac{x^2}{2} \right|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-\pi(0+\pi) + \frac{\pi^2}{2} - 0 \right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2} \right] = \frac{1}{\pi} \times -\frac{\pi^2}{2} = -\pi/2$$

$$\boxed{a_0 = -\pi/2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\, dx + \int_{0}^{\pi} f(x)\cos nx\, dx \right]$$

$$= \frac{1}{\pi}\left[-\pi \int_{-\pi}^{0} \cos nx\, dx + \int_{0}^{\pi} x\cos nx\, dx \right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left\{ x\left(\frac{\sin nx}{n} \right) - (1)\left(-\frac{\cos nx}{n^2} \right) \right\}_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-\pi[0+0] + \left\{ \left(0 + \frac{\cos n\pi}{n^2} \right) - \left(0 + \frac{1}{n^2} \right) \right\} \right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi - 1}{n^2} \right] = \frac{1}{\pi n^2}\left((-1)^n - 1 \right)$$

$$a_n = \frac{1}{\pi n^2}\left((-1)^n - 1 \right)$$

$$a_1 = -\frac{2}{\pi \cdot 1^2}, \quad a_2 = 0, \quad a_3 = -\frac{2}{\pi \cdot 3^2}, \quad a_4 = 0, \; \ldots$$

Also $b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_{0}^{\pi} f(x)\sin nx\, dx \right]$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \ldots \, dx + \int_{0}^{\pi} \ldots \right]$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{-\cos nx}{n}\right|_{-\pi}^{0} + \left| x\left(\frac{-\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{1}{n} - \frac{\cos n\pi}{n}\right] + \left\{\left(\frac{-\pi\cos n\pi}{n} + 0\right) - (0+0)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{1}{\cancel{\pi}} \times \frac{\cancel{\pi}}{n}\left[1 - 2\cos n\pi\right]$$

$$\boxed{\cos n\pi = (-1)^n}$$

$$\boxed{b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]}$$

$b_1 = 1$

$= 2$

$= 3$

$= 5$

$$b_1 = \frac{1}{1}\left[1 - 2(-1)^1\right] = 1[1+2] = 3$$

$$\boxed{b_1 = 3}, \quad b_2 = \frac{1}{2}\left[1 - 2(-1)^2\right] = \frac{1}{2}[1-2]$$

$$= -1/2$$

$$b_3 = \frac{1}{3}\left[1 - 2(-1)\right] = 1$$

$$b_4 = \frac{1}{4}[1-2] = -1/4 \text{ etc.}$$

Substituting these values in equation (1), we get

$$f(x) = -\frac{\pi}{2} \times \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}\left((-1)^n - 1\right)\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - 2(-1)^n\right)\sin nx$$

$$f(x) = -\frac{\pi}{2}\times\frac{1}{2} + \sum_{n=1} \frac{1}{\pi n^2}[(-1)^n - 1]\cos nx + \sum_{n=1} \frac{1}{n}[\ldots$$

$$= -\frac{\pi}{4} + \frac{1}{\pi}\left[\sum_{n=1}^{\infty} \frac{1}{n^2}[(-1)^n-1]\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}[1-2(-1)^n]\sin nx\right]$$

$$= -\frac{\pi}{4} + \frac{1}{\pi}\left[\frac{-2}{1^2}\cdot\cos x + 0 + \frac{-2}{3^2}\cos 3x + 0 + \frac{-2}{5^2}\cos 5x + \cdots\right]$$
$$+ \left[3\sin x - \frac{1}{2}\sin 2x + \sin 3x - \frac{1}{4}\sin 4x + \cdots\right]$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$
$$+ \left[3\sin x - \frac{1}{2}\sin 2x + \sin 3x - \frac{1}{4}\sin 4x + \cdots\right]$$

At $x = 0$

$$\underline{f(0)} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] + 0 \quad \text{— (2)}$$

$\sin 0 = 0$
$\cos 0 = 1$

$$f(x) = \begin{cases} f(x) & -\pi < x < c \\ f(x); & c < x \ldots \end{cases}$$

$$f(c) = \frac{f(c^-) + f(c^+)}{2}$$

$\boxed{\text{disconti}}$

At $x = 0$

$$f(\underline{0}) = \frac{f(0-) + f(0^+)}{2}$$

$$= \frac{(-\pi) + 0}{2}$$

$$\boxed{f(0) = -\pi/2}$$

$\therefore$ from (2)

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

or $$-\frac{\pi}{2} + \frac{\pi}{4} = \frac{-2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

or $$-\frac{\pi}{4} \times -\frac{\pi}{2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \quad -\frac{\pi}{4} \times -\frac{\pi}{2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \quad \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}}$$

Q. Find the Fourier series expansion of $f(x) = 2x - x^2$ in $\boxed{(0,3)}$ & hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots - \infty = \pi^2/12$$

Sol $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$

$2l = 3 - 0 = 3$

$\boxed{l = 3/2}$

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_0^3 (2x - x^2)\,dx =$$

$$= \frac{2}{3}\left| \frac{2x^2}{2} - \frac{x^3}{3} \right|_0^3$$

$$= \frac{2}{3}\left[(9 - 9) - 0\right] = 0$$

$$a_n = \frac{1}{l}\int_0^3 f(x) \cos\frac{n\pi x}{l}\,dx = \frac{2}{3}\int_0^3 (2x - x^2) \cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left| (2x - x^2)\left[\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3}\right] - (2 - 2x)\left[\frac{-\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right] + (-2)\left[\frac{-\sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right] \right|_0^3$$

$$= \frac{2}{3}\left[\left\{0 - \frac{(-4)(-1)}{(2n\pi/3)^2} + 0\right\} - \left\{0 - \frac{2(-1)}{(2n\pi/3)^2}\right\} \right]$$

$$= \frac{2}{3}\left[\left\{0 - \frac{(-4)(-1)}{(2n\pi/3)^2} + 0\right\} - \left\{0 - \frac{2(-1)}{(2n\pi/3)^2}\right\}\right]$$

$$= \frac{2}{3}\left[\frac{-4\cdot 9}{4n^2\pi^2} - \frac{2\times 9}{4n^2\pi^2}\right] = \frac{\cancel{2}}{\cancel{3}} \times \frac{-\cancel{6}\times 9}{\cancel{4}n^2\pi^2} = \frac{-9}{n^2\pi^2}$$

$$\boxed{a_n = -\frac{9}{n^2\pi^2}}$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= 3/n\pi$$

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

or $$2x - x^2 = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$$

At $x = 3/2$

$$2\cdot\frac{3}{2} - \left(\frac{3}{2}\right)^2 = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi + 0$$

or $$3 - \frac{9}{4} = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

$$\frac{3}{4}\times\frac{\pi^2}{9_3} = -\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

or $$\boxed{1 - \ldots + \ldots - \ldots + \cdots = \pi^2}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Q. Find a Fourier Series to represent x^2 in the interval $(-l, l)$

Sol. $f(x) = x^2$ is an even function $\therefore \boxed{b_n = 0}$

hence Fourier series reduces to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\therefore \quad a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left|\frac{x^3}{3}\right|_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{3} - 0\right] = \frac{2l^2}{3}$$

& $a_n = \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left[x^2 \left[\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right] - (2x)\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right] + (2)\left[\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 + 2l \times \frac{l^2}{n^2\pi^2}\cos n\pi - 0\right\} - \left\{0\right\}\right]$$

$$= \frac{2}{\cancel{l}}\left[\frac{2l^{\cancel{3}\,2}}{n^2\pi^2}\cos n\pi\right] = \frac{4l^2}{n^2\pi^2}\cos n\pi$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n$$

$$\therefore f(x) = \frac{2l^2}{3\times 2} + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{\ } + \frac{4l^2}{\ } \sum^{\infty} \frac{(-1)^n}{\ } \cos\frac{n\pi x}{\ }$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[-\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} - \frac{1}{3^2}\cos\frac{3\pi x}{l} - \text{---}\right]$$

$$f(x) = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} - \cdots\right]$$

Q obtain Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1+\frac{2x}{\pi}\,; & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}\,; & 0 \le x \le \pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{---} = \frac{\pi^2}{8}$

Sol

$$f(-x) = \begin{cases} 1-\frac{2x}{\pi}\,; & \underline{-\pi \le -x \le 0} \\ 1+\frac{2x}{\pi}\,; & \underline{0 \le -x \le \pi} \end{cases}$$

$$= \begin{cases} 1-\frac{2x}{\pi}\,; & \pi \geq x \geq 0 \text{ or } 0 \le x \le \pi \\ 1+\frac{2x}{\pi}\,; & 0 \geq x \geq -\pi \text{ or } \boxed{-\pi \le x \le 0} \end{cases}$$

$$= \begin{cases} 1+\frac{2x}{\pi}\,; & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}\,; & 0 \le x \le \pi \end{cases}$$

$$\boxed{f(-x) = f(x)}$$

$\therefore f(x)$ is an even function

$1 - \frac{2\pi}{\pi} = -1$

$a_0\ y = f(x) = \begin{cases} 1+\frac{2x}{\pi}\,; & -\pi \le x \le 0 \end{cases}$

$1 - \frac{2\pi}{\pi} = -1$

Q. $y = f(x) = \begin{cases} 1 + \frac{2x}{\pi} ; & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} , & 0 \le x \le \pi \end{cases}$

$1 - \frac{2\pi}{\pi} =$

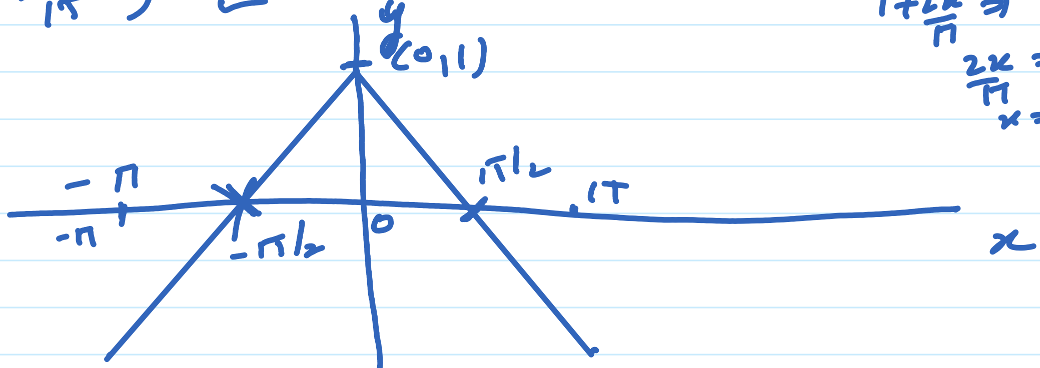
$1 + \frac{2x}{\pi} \Rightarrow$

$\frac{2x}{\pi} = -1$

$x = -\pi/2$

$1 - \frac{2x}{\pi} = 0$

$x = \pi/2$



$f(x)$ is symmetrical about y-axis $\therefore$ $f(x)$ is even fun

$\therefore \boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\cancel{\pi} x}{\cancel{\pi}}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{\cancel{2}x^2}{\pi \times \cancel{2}} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^{\cancel{2}}}{\cancel{\pi}} \right] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{ 0 - \frac{2}{\pi}\frac{\cos n\pi}{n^2} \right\} - \left\{ 0 - \frac{2}{\pi}\cdot\frac{1}{n^2} \right\} \right]$$

$$= \frac{2}{\pi}\left[-\frac{2}{\pi}\frac{\cos n\pi}{n^2} + \frac{2}{\pi n^2} \right] = \frac{2}{\pi} \times \frac{2}{\pi n^2}\left[1 - \cos n\pi\right]$$

$$= \frac{2}{\pi}\left\{ -\frac{2}{\pi}\frac{\cos n\pi}{n^2} + \frac{2}{\pi n^2} \right] = \frac{2}{\pi}\times\frac{2}{\pi n^2}\left[1-\cos n\pi\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$a_1 = \frac{8}{1^2\pi^2},\ a_2 = 0,\ a_3 = \frac{8}{3^2\pi^2},\ a_4 = 0,\ a_5 = \frac{8}{5^2\pi^2}, \cdots\cdots$$

$$\therefore f(x) = \frac{8}{1^2\pi^2}\cos x + 0 + \frac{8}{3^2\pi^2}\cos 3x + 0 + \frac{8}{5^2\pi^2}\cos 5x + \text{———}$$

$$\boxed{f(0) = \frac{f(0-)+f(0+)}{2}}$$

$$= \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\cdots\right]$$

Q Express $f(x) = x$ as a half range sine series in $0<x<2$. $(0<x<l)$

Sol $\boxed{l=2}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= \left[\left\{\frac{2\times 2}{n\pi}(-\cos n\pi) - 0\right\} - \left\{0\right\}\right]$$

$$= -\frac{4}{n\pi}\cos n\pi = -\frac{4}{n\pi}(-1)^n = \frac{4}{n\pi}(-1)^{n+1}$$

$n\pi$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{2}$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left[\frac{1}{1} \sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + \cdots \right]$$

Q Express $f(x) = x$ a a half range Cosine Series in $0 < x < 2$

$0 < x < l$

Sol $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$a_0 = \frac{2}{l} \int_0^2 x \, dx = \frac{x^2}{2}\Big|_0^2 = \frac{1}{2}\left[2^2 - 0\right] = 2$$

$$a_n = \frac{2}{2} \int_0^2 x \cos \frac{n\pi x}{2} \, dx$$

$$= \left| x \left[\frac{\sin \frac{n\pi x}{2}}{n\pi/2} \right] - (1)\left(\frac{-\cos \frac{n\pi x}{2}}{n^2\pi^2/2^2} \right) \right|_0^2$$

$$= \left\{ 0 + \frac{2^2}{n^2\pi^2} \cos n\pi \right\} - \left\{ 0 + \frac{2^2}{n^2\pi^2} \cdot 1 \right\}$$

$$= \frac{4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore f(x) = \frac{2}{2} + \sum^{\infty} \frac{4}{n^2\pi^2}\left\{(-1)^n - 1\right\} \cos \frac{n\pi x}{2}$$

$$\therefore f(x) = \frac{2}{2} + \sum_{n=1} \frac{4}{n^2\pi^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{2}$$

$$= 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{2}$$

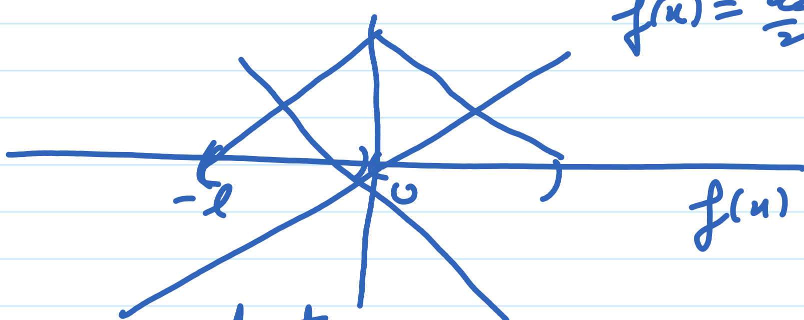
$$= 1 + \frac{4}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{2} + 0 - \frac{2}{3^2}\cos\frac{3\pi x}{2} + 0 - \frac{2}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

<u>Half Range</u> $(0, l)$

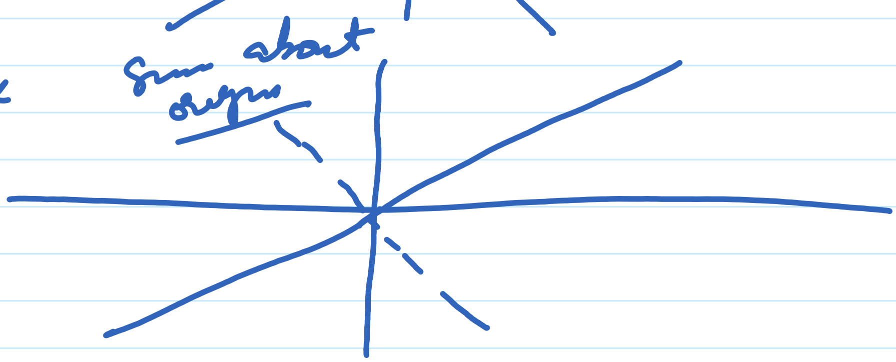
$\boxed{\begin{matrix}(-l, l)\\(0, 2l)\end{matrix}}$

$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$



$f(x) = \sum b_n \cos\frac{n\pi x}{l}$

$f(x) = x$



Complex Form of Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}, \text{ where } c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

Q. Find the Complex form of the Fourier series of

Q. Find the Complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.

$\boxed{f(x) \quad -l < x \le l}$ $\boxed{l = 1}$

Sol $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n\pi x}{1}}$ — ⊗

where $c_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-i n\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+n\pi i)x}$

$\boxed{\sinh\theta = \frac{e^{\theta} + e^{-\theta}}{2}}$

$= \frac{1}{2}\left| \frac{e^{-(1+n\pi i)x}}{-(1+n\pi i)} \right|_{-1}^{1}$

$= \frac{1}{2}\left[\frac{e^{-(1+n\pi i)}}{-(1+n\pi i)} + \frac{e^{-(1+n\pi i)(-1)}}{(1+n\pi i)} \right]$

$= \frac{1}{2}\left[-\frac{e^{-(1+n\pi i)}}{1+n\pi i} + \frac{e^{(1+n\pi i)}}{1+n\pi i} \right]$

or $\frac{1}{2}\left[\frac{e^{(1+n\pi i)} - e^{-(1+n\pi i)}}{1+n\pi i} \right]$

$= \frac{1}{2}\left[\frac{e^{1} . e^{n\pi i} - e^{-1} e^{-n\pi i}}{1+n\pi i} \right]$

$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$

$= \frac{1}{2}\left[\frac{e[\cos n\pi + i\sin n\pi] - e^{-1}[\cos n\pi - i\sin n\pi]}{1+n\pi i} \right]$

$\sin n\pi = 0$

$= \frac{1}{2}\left[\frac{e(-1)^n - e^{-1}((-1)^n)}{1+n\pi i} \right]$

-1

$= \frac{1}{2} \Big[\quad \frac{\quad}{1+n\pi i} \Big]$

$= \frac{1}{2} (-1)^n \frac{[e - e^{-1}]}{1+n\pi i} \times \frac{1-n\pi i}{1-n\pi i}$

$= (-1)^n \frac{[e-e^{-1}]}{2} \frac{(1-n\pi i)}{\underbrace{1-n^2\pi^2 i^2}}$

$= \frac{(-1)^n (1-n\pi i)}{1+n^2\pi^2} \times \frac{(e-e^{-1})}{2}$ ✓

$$\boxed{c_n = \frac{(-1)^n (1-n\pi i)\sinh 1}{1+n^2\pi^2}}$$

$\boxed{\text{sinh cos nx}}$

-1
✓ 0
1
2

$\cos n\pi = (-1)^n$
$\sin \pi = 0$
$\sin 2\pi = 0$
$\sin -\pi = 0$

$(a-b)(a+b) = a^2 - b^2$
$i^2 = -1$

$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$

$\sinh 1 = \frac{e^1 - e^{-1}}{2}$

Q If $f(x) = |\cos x|$, expand $f(x)$ as Fourier Series in the interval $(-\pi, \pi)$

Sol $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi} + \sum_{n=1}^{\infty} b_n \sin n\pi x$ $\quad b_n = 0$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} |\cos x|\, dx$

$= \frac{2}{\pi} \left[\int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx \right]$

$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx \right]$

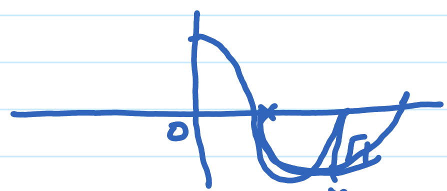
$= \frac{2}{\pi} \left[|\sin x|_0^{\pi/2} - |\sin x|_{\pi/2}^{\pi} \right]$

$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$ if $f(x)$ is even

$= 0$ if fn is odd

$\cos x = 0$

$\boxed{x = \pi/2}$



$0 \le x \le \pi/2$, $|\cos x| = +\cos x$

$\pi/2 \le x \le \pi$, $\cos x$ is $-$ve

$|\cos x| = -\cos x$

$$= \frac{2}{\pi}\left[\left|\sin x\right|_0^{\pi/2} - \left|\sin x\right|_{\pi/2}^{\pi} \right]$$

$\pi/2 \le x \le \pi$, $\cos x$ is $-$ve

$|\cos x| = -\cos x$

$$= \frac{2}{\pi}\left[\left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin\pi - \sin\pi/2\right) \right]$$

$$= \frac{2}{\pi}\left[1 - 0 - 0 + 1 \right] = 4/\pi$$

$$\therefore \boxed{a_0 = 4/\pi}$$

$$\& \quad a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} |\cos x| \cos nx \, dx + \int_{\pi/2}^{\pi} |\cos x| \cos nx \, dx \right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx \, dx - \int_{\pi/2}^{\pi} \cos x \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos nx \cos x \, dx - \int_{\pi/2}^{\pi} 2\cos nx \cos x \, dx \right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \left[\cos(n+1)x + \cos(n-1)x\right] dx - \int_{\pi/2}^{\pi} \left[\cos(n+1)x + \cos(n-1)x\right] dx \right]$$

$$= \frac{1}{\pi}\left[\left| \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right|_0^{\pi/2} - \left| \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right|_{\pi/2}^{\pi} \right]$$

if $n \neq 1$

$$= \frac{1}{\pi}\left[\left\{ \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right\} - \left\{ 0 \right\} \right.$$

$$\left. \left\{ \sin(n+1)\pi + \sin(n-1)\pi \right\} - \left\{ \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right\} \right]$$

$$-\left[\left\{\frac{\sin(n+1)\pi}{n+1} + \frac{\sin(n-1)\pi}{n-1}\right\} - \left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi} \times \cos\frac{n\pi}{2}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right] = -\frac{4\cos n\pi/2}{\pi(n^2-1)}, n \neq 1$$

$$\therefore \boxed{a_n = -\frac{4}{\pi}\frac{\cos n\pi/2}{(n^2-1)}, \quad n \neq 1}$$

$\sin(n+1)\pi/2$
$= \sin\left(\frac{n\pi}{2} + \pi/2\right)$
$= \sin\left(\frac{\pi}{2} + \frac{n\pi}{2}\right)$
$= \cos n\pi/2$

$\sin(n-1)\pi/2$
$= \sin\left(\frac{n\pi}{2} - \pi/2\right)$
$= \sin -\left(\frac{\pi}{2} - n\pi/2\right)$
$= -\sin(\pi/2 - n\pi/2)$
$= -\cos\frac{n\pi}{2}$

Also $a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x \, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} |\cos x| \cos x \, dx + \int_{\pi/2}^{\pi} |\cos x| \cos x \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos^2 x \, dx - \int_{\pi/2}^{\pi} 2\cos^2 x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int^{\pi/2} (1+\cos 2x)\, dx - \int^{\pi} (1+\cos 2x)\, dx\right]$$

$\cos 2A = 2\cos^2 A - 1$
$2\cos^2 A = 1 + \cos 2A$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} (1+\cos 2x)\, dx - \int_{\pi/2}^{\pi} (1+\cos 2x)\, dx \right]$$

$2\cos^2 A = 1 + \cos 2A$

$$\boxed{a_1 = 0}$$

$A = 1$

$B = \pi$

$C = 0$

$D = -\pi$

$$\therefore |\cos x| = \frac{1}{2}\times\frac{4}{\pi} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{-4\cos n\pi/2}{\pi(n^2-1)} \cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{\cos n\pi/2 \cos nx}{n^2-1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{(-1)\cos 2x}{3} + 0 + \frac{1.\cos 4x}{15} - - - \right]$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + - - - - \right]$$

$\cos n\pi/2$

$\cos \pi = -1$

$\cos \frac{3\pi}{2}$

$= \cos(\pi + \pi/2)$

$= -\cos \pi/2 = 0$

$n=4$

$\cos 4\frac{\pi}{2} = \cos 2\pi = 1$

$\cos 5\pi/2 = 0$

$$\therefore \boxed{|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + - - -\right]}$$

Q. Obtain Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$.

Sol

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x\, dx$

$$= \frac{2}{\pi}\left[x(-\cos x) - (1)(-\sin x) \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(-\pi\cos\pi + \sin\pi) - 0 \right] = \frac{2}{\pi}\left[-\pi \times -1\right] = 2$$

$$= \frac{2}{\pi}\left[\left(-\pi\cos\pi + \sin\pi\right) - 0\right] = \frac{2}{\pi}\left[-\pi \times -1\right] = 2$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \left[2\cos nx \sin x\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left[-\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]\right]_0^{\pi}$$

$n \neq 1$

$$= \frac{1}{\pi}\left[\left\{\pi\left(-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right) - 0\right\} - \left\{0\right\}\right]$$

$$= \frac{1}{\cancel{\pi}} \times \cancel{\pi}\left[\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right], \; n \neq 1$$

$$\boxed{a_n = \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, \; n \neq 1}$$

Now $a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \int_0^{\pi} x \left(2\sin x \cos x\right) dx$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos 2x}{2}\right\} - (1)\left\{-\frac{\sin 2x}{4}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\ldots}{2}\right\} - (1)\left\{-\frac{\ldots}{4}\right\}\right]_0$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\pi \cos 2\pi}{2} - 0\right\} - \left\{0\right\}\right] = \frac{1}{\pi}\times -\frac{\pi}{2}(1)^2 = \frac{-1\cancel{\pi}/2}{\cancel{\pi}} = -1/2$$

$$\boxed{a_1 = -1/2}$$

$$\therefore\ x \sin x = \frac{2}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\}\cos nx$$