CHAPTER 13

Linear Differential Equations

13.1 DEFINITIONS

Linear differential equations *are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = X,$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = X$$

where $k_1, k_2, \ldots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

13.2 (1) THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = 0 \qquad \ldots(1)$$

then $c_1 y_1 + c_2 y_2$ $(= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1).

$$\therefore \qquad \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \ldots + k_n y_1 = 0 \qquad \ldots(2)$$

and

$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + k_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \ldots + k_n y_2 = 0 \qquad \ldots(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + k_1 \frac{d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \ldots + k_n (c_1 y_1 + c_2 y_2)$$

$$= c_1\left(\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1}y_1}{dx^{n-1}} + \dots + k_n y_1\right) + c_2\left(\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1}y_2}{dx^{n-1}} + \dots + k_n y_2\right)$$

$$= c_1(0) + c_2(0) = 0 \qquad \text{[By (2) and (3)]}$$

i.e.,
$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1}u}{dx^{n-1}} + \dots + k_n y = 0 \qquad \dots(4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \dots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \dots + c_ny_n$ $(= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + k_n y = X \qquad \dots(5)$$

then
$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1}v}{dx^{n-1}} + \dots + k_n v = X \qquad \dots(6)$$

Adding (4) and (6), we have $\dfrac{d^n(u+v)}{dx^n} + k_1 \dfrac{d^{n-1}(u+v)}{dx^{n-1}} + \dots + k_n(u+v) = X$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)** *and the part v is called the* **particular integral (P.I.)** *of (5).*

$\therefore$ the complete solution (C.S.) of (5) is **y = C.F. + P.I.**

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e. a particular solution of (5).

13.3 OPERATOR D

Denoting $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$\dfrac{dy}{dx} = Dy$, $\dfrac{d^2y}{dx^2} = D^2y$, $\dfrac{d^3x}{dx^3} = D^3y$ etc., the equation (5) above can be written in the *symbolic form* $(D^n + k_1D^{n-1} + \dots + k_n)y = X$, *i.e.*, $f(D)y = X$,
where $f(D) = D^n + k_1D^{n-1} + \dots + k_n$, *i.e.*, a polynomial in D.

Thus *the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., f (D) can be factorised by ordinary rules of algebra and the factors may be taken in any order.* For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)\,y = (D+3)(D-1)y \text{ or } (D-1)(D+3)\,y.$$

13.4 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation $\dfrac{d^n y}{dx^n} + k_1 \dfrac{d^{n-1}y}{dx^{n-1}} + k_2 \dfrac{d^{n-2}y}{dx^{n-2}} + \dots + k_n y = 0$ $\qquad \dots(1)$

where k's are constants.

The equation (1) in *symbolic form is*

$$(D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n)y = 0 \qquad \dots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \dots, m_n$ be its roots.

Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2)\dots(D - m_n)y = 0 \qquad \dots(3)$$

Now (3) will be satisfied by the solution of $(D - m_n)y = 0$, *i.e.*, by $\dfrac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ its solution is $y e^{-m_n x} = c_n$, *i.e.*, $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1) y = 0$, $(D - m_2)y = 0$ etc. *i.e.*, by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$ ...(4)

Case II. *If two roots are equal (i.e., $m_1 = m_2$)*, then (4) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$y = C e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \qquad [\because \; c_1 + c_2 = \text{one arbitrary constant } C]$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows :

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1) z = 0$ or $\dfrac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$. $\therefore$ its solution is $ze^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $\qquad (D - m_1)y = z = c_1 e^{m_1 x}$ or $\dfrac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ ...(5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$ye^{-m_1 x} = \int c_1 e^{m_1 x} \, dx + c_2 = c_1 x + c_2 \text{ or } y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (*i.e.*, $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary, i.e.*, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

[$\because$ by *Euler's Theorem*, $e^{i\theta} = \cos \theta + i \sin \theta$]

$$= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Case IV. *If two points of imaginary roots be equal i.e.*, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] + \ldots + c_n e^{m_n x}.$$

Example 13.1. *Solve* $\dfrac{d^2 x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0$, *given* $x(0) = 0$, $\dfrac{dx}{dt}(0) = 15$. *(V.T.U., 2010)*

Solution. Given equation in *symbolic form is* $(D^2 + 5D + 6)\, x = 0$.

Its A.E. is $D^2 + 5D + 6 = 0$, *i.e.*, $(D + 2)(D + 3) = 0$ whence $D = -2, -3$.

$\therefore$ $\qquad$ C.S. is $x = c_1 e^{-2t} + c_2 e^{-3t}$ and $\dfrac{dx}{dt} = -2ae^{-2t} - 3c_2 e^{-3t}$

When $t = 0$, $x = 0$. $\quad \therefore \quad 0 = c_1 + c_2$ (*i*)

When $t = 0$, $dx/dt = 15$ $\quad \therefore \quad 15 = -2c_1 - 3c_2$ ...(*ii*)

Solving (*i*) and (*ii*), $c_1 = 15, c_2 = -15$.
Hence the required solution is $x = 15\,(e^{-2t} - e^{-3t})$.

Example 13.2. *Solve* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution. Given equation in *symbolic form is* $(D^2 + 6D + 9) = 0$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.*, $(D + 3)^2 = 0$ whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2 t)\, e^{-3t}$.

Example 13.3. *Solve* $(D^3 + D^2 + 4D + 4) = 0.$

Solution. Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$ *i.e.*, $(D^2 + 4)(D + 1) = 0$ $\therefore$ $D = -1, \pm 2i$.

Hence the C.S. is $y = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e., $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$.

Example 13.4. *Solve (i)* $(D^4 - 4D + 4)\, y = 0$ *(Bhopal, 2008)*
(ii) $(D^2 + 1)^3\, y = 0$ *where* $D \equiv d/dx$.

Solution. (*i*) The A.E. equation is $D^4 - 4D^2 + 4 = 0$ or $(D^2 - 2)^2 = 0$

$\therefore$ $D^2 = 2, 2$ *i.e.*, $D = \pm\sqrt{2}, \pm\sqrt{2}$.

Hence the C.S. is $((c_1 + c_2 x)\, e^{\sqrt{2}x} + (c_3 + c_4 x)\, e^{-\sqrt{2}x})$ [Roots being repeated]

(*ii*) The A.E. equation is $(D^2 + 1)^3 = 0$

$\therefore$ $D = \pm i, \pm i, \pm i$.

Hence the C.S. is $y = e^{\alpha x}\,[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$

i.e., $y = (c_1 + c_2 + c_3 x^2)\cos x + (c_4 + c_5\, x + c_6 x^2)\sin x$.

Example 13.5. *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0.$

Solution. Given equation in *symbolic form is* $(D^4 + 4)\, x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$ or $(D^2 + 2)^2 - (2D)^2 = 0$

or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

whence $D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$ *i.e.*, $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{t}(c_3 \cos t + c_4 \sin t)$.

PROBLEMS 13.1

Solve :

1. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 13x = 0, x(0), \dfrac{dx(0)}{dt} = 2.$ (V.T.U., 2008)
2. $y'' - 2y' + 10y = 0, y(0) = 4, y'(0) = 1.$
3. $4y''' + 4y'' + y' = 0.$
4. $\dfrac{d^3y}{dx^3} + y = 0.$ (V.T.U., 2000 S)
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$
6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0.$ (J.N.T.U., 2005)
7. $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0.$ (V.T.U., 2008)
8. $(D^2 + 1)^2 (D - 1)y = 0.$
9. If $\dfrac{d^4x}{dt^4} = m^4 x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$.

13.5 INVERSE OPERATOR

(1) Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e.,
$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ *satisfies the equation* $f(D)y = X$ *and is, therefore, its particular integral.*

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2)
$$\mathbf{\frac{1}{D}X = \int X dx}$$

Let
$$\frac{1}{D}X = y \qquad ...(i)$$

Operating by D,
$$D\frac{1}{D}X = Dy \quad i.e., \ X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x, $y = \int X\,dx$, no constant being added as (i) does not contain any constant.

Thus
$$\frac{1}{D}X = \int X\,dx.$$

(3)
$$\mathbf{\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.}$$

Let
$$\frac{1}{D-a}X = y \qquad ...(ii)$$

Operating by $D - a$, $(D-a)\cdot\frac{1}{D-a}X = (D-a)y.$

or
$$X = \frac{dy}{dx} - ay, \ i.e., \ \frac{dy}{dx} - ay = X$$ which is a Leibnitz's linear equation.

$\therefore$ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$ no constant being added as (ii) doesn't contain any constant.

Thus
$$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

13.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + ... + k_n y = X$

which is *symbolic form of* $(D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n)y = X.$

$$\therefore \qquad \text{P.I.} = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n}X.$$

Case I. When $\mathbf{X = e^{ax}}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$D^ne^{ax} = a^ne^{ax}$$

$$\therefore \quad (D^n + k_1D^{n-1} + ... + k_n)e^{ax} = (a^n + k_1a^{n-1} + ... + k_n)e^{ax}, \quad i.e., \quad f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$, $\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$ or $e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$

$\therefore$ dividing by $f(a)$,

$$\frac{\mathbf{1}}{\mathbf{f(D)}} \mathbf{e^{ax}} = \frac{\mathbf{1}}{\mathbf{f(a)}} \mathbf{e^{ax}} \text{ provided } f(a) \neq 0 \qquad ...(1)$$

If f(a) = 0, *the above rule fails and we proceed further.*

Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + ... + k_n = 0$.

$\therefore$ $D - a$ is a factor of $f(D)$. Suppose $f(D) = (D - a)\,\phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax} \qquad \text{[By (1)]}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \int e^{ax} \cdot e^{-ax} dx \qquad \text{[By §13.5 (3)]}$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = x \frac{1}{\phi(a)} e^{ax} \quad i.e., \quad \frac{\mathbf{1}}{\mathbf{f(D)}} \mathbf{e^{ax}} = \mathbf{x} \frac{\mathbf{1}}{\mathbf{f'(a)}} \mathbf{e^{ax}} \qquad ...(2)$$

$$\left[\begin{array}{l} \because f'(D) = (D-a)\phi'(D) + 1 \cdot \phi(D) \\ \therefore f'(a) = 0 \times \phi'(a) + \phi(a) \end{array}\right.$$

If $f'(a) = 0$, then applying (2) again, we get $\frac{\mathbf{1}}{\mathbf{f(D)}} \mathbf{e^{ax}} = \mathbf{x^2} \frac{\mathbf{1}}{\mathbf{f''(a)}} \mathbf{e^{ax}}$, *provided* $f''(a) \neq 0$ ...(3)

and so on.

Example 13.6. *Find the P.I. of* $(D^2 + 5D + 6)y = e^x$.

Solution. P.I. $= \frac{1}{D^2 + 5D + 6} e^x$ [Put $D = 1$] $= \frac{1}{1^2 + 5.1 + 6} e^x = \frac{e^x}{12}$.

Example 13.7. *Find the P.I. of* $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x$.

Solution. P.I. $= \frac{1}{(D+2)(D-1)^2} [e^{-2x} + 2 \sinh x] = \frac{1}{(D+2)(D-1)^2} [e^{-2x} + e^x - e^{-x}]$

Let us evaluate each of these terms separately.

$$\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2} \cdot \left[\frac{1}{(D-1)^2} e^{-2x}\right]$$

$$= \frac{1}{D+2} \cdot \frac{1}{(-2-1)^2} e^{-2x} = \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{1}{9} \cdot x \cdot \frac{1}{1} e^{-2x} = \frac{x}{9} e^{-2x} \qquad \left[\because \frac{d}{dD}(D+2) = 1\right]$$

$$\frac{1}{(D+2)(D-1)^2} e^x = \frac{1}{1+2} \cdot \frac{1}{(D-1)^2} e^x = \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^x = \frac{x^2}{6} e^x \qquad \left[\because \frac{d^2}{dD^2}(D-1)^2 = 2\right]$$

and $$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{e^{-x}}{4}$$

Hence, P.I. $= \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$.

Case II. When X = sin (ax + b) or cos (ax + b).

Since $D \sin(ax + b) = a \cos(ax + b)$

$D^2 \sin(ax + b) = -a^2 \sin(ax + b)$

$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

i.e., $$D^2 \sin(ax+b) = (-a^2)\sin(ax+b)$$

$$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

In general $$(D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b)$$

$\therefore$ $$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

Operating on both sides $1/f(D^2)$,

$$\frac{1}{f(D^2)}\cdot f(D^2)\ \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

or $$\sin(ax+b) = f(-a^2)\,\frac{1}{f(D^2)}\sin(ax+b)$$

$\therefore$ Dividing by $f(-a^2)$ $\quad$ $\mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)}$ *provided* $f(-a^2) \neq 0$ ...(4)

If $\mathbf{f(-a^2) = 0}$, *the above rule fails and we proceed further.*

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ [Euler's theorem]

$$\therefore \quad \frac{1}{f(D^2)}\sin(ax+b) = \text{I.P. of } \frac{1}{f(D^2)}e^{i(ax+b)} \qquad [\text{Since } f(-a^2) = 0 \ \therefore \text{ by (2)}]$$

$$= \text{I.P. of } x\frac{1}{f'(D^2)}e^{i(ax+b)} \qquad \text{where } D^2 = -a^2$$

$\therefore$ $\mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = x\,\dfrac{1}{f'(-a^2)}\sin(ax+b)}$ *provided* $f'(-a^2) \neq 0$ ...(5)

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cdot\sin(ax+b) = x^2\,\dfrac{1}{f''(-a^2)}\sin(ax+b)}$, *provided* $f''(-a^2) \neq 0$, and so on.

Similarly, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = \dfrac{1}{f(-a^2)}\cos(ax+b)}$, *provided* $f(-a^2) \neq 0$

If $f(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x\cdot\dfrac{1}{f'(-a^2)}\cos(ax+b)}$, *provided* $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\,\dfrac{1}{f''(-a^2)}\cos(ax+b)}$, *provided* $f''(-a^2) \neq 0$ and so on.

Example 13.8. *Find the P.I. of $(D^3 + 1)y = \cos(2x - 1)$.*

Solution. P.I. $= \dfrac{1}{D^3+1}\cos(2x-1)$ [Put $D^2 = -2^2 = -4$]

$$= \frac{1}{D(-4)+1}\cos(2x-1) \qquad [\text{Multiply and divide by } 1 + 4D]$$

$$= \frac{(1+4D)}{(1-4D)(1+4D)}\cos(2x-1) = (1+4D)\cdot\frac{1}{1-16D^2}\cos(2x-1) \qquad [\text{Put } D^2 = -2^2 = -4]$$

$$= (1+4D)\frac{1}{1-16(-4)}\cos(2x-1) = \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)].$$

Example 13.9. *Find the P.I. of $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$.*

Solution. Given equation in *symbolic form* is $(D^3 + 4D)y = \sin 2x$

$\therefore$ $$\text{P.I.} = \frac{1}{D(D^2+4)}\sin 2x$$ [$\because D^2 + 4 = 0$ for $D^2 = -2^2$, $\therefore$ Apply (5) 477]

$$= x\frac{1}{3D^2+4}\sin 2x$$ $\left[\because \frac{d}{dD}[D^3+4D] = 3D^2+4\right]$

[Put $D^2 = -2^2 = -4$]

$$= x\frac{1}{3(-4)+4}\sin 2x = -\frac{x}{8}\sin 2x.$$

Case III. When $X = x^m$.

Here $$\text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m + 1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 13.10. *Find the P.I. of* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4.$

Solution. Given equation in *symbolic form* is $(D^2 + D)y = x^2 + 2x + 4$.

$\therefore$ $$\text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1 - D + D^2 - \ldots)(x^2+2x+4) = \frac{1}{D}[x^2 + 2x + 4 - (2x+2) + 2]$$

$$= \int (x^2+4)dx = \frac{x^3}{3} + 4x.$$

Case IV. When $X = e^{ax}V$, *V being a function of x.*

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u + e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D+a)^2u$$

and in general, $$D^n(e^{ax}u) = e^{ax}(D+a)^n u$$

$\therefore$ $$f(D)(e^{ax}u) = e^{ax}f(D+a)u$$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

$$e^{ax}u = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

Now put $f(D+a)u = V$, *i.e.*, $u = \dfrac{1}{f(D+a)}V$, so that $e^{ax}\dfrac{1}{f(D+a)}V = \dfrac{1}{f(D)}(e^{ax}V)$

i.e., $$\mathbf{\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.} \quad \ldots(6)$$

Example 13.11. *Find P.I. of $(D^2 - 2D + 4)y = e^x \cos x$.*

Solution. $$\text{P.I.} = \frac{1}{D^2-2D+4}e^x\cos x$$ [Replace D by $D + 1$]

$$= e^x\frac{1}{(D+1)^2 - 2(D+1)+4}\cos x = e^x\frac{1}{D^2+3}\cos x$$ [Put $D^2 = -1^2 = -1$]

$$= e^x\frac{1}{-1+3}\cos x = \frac{1}{2}e^x\cos x.$$

Case V. When X is any other function of x.

Here $$\text{P.I.} = \frac{1}{f(D)}X.$$

If $f(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}$$

$$\therefore \quad \text{P.I.} = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right]X$$

$$= A_1 \frac{1}{D - m_1}X + A_2 \frac{1}{D - m_2}X + \ldots + A_n \frac{1}{D - m_n}X$$

$$= A_1 . e^{m_1 x} \int X e^{-m_1 x} dx + A_2 . e^{m_2 x} \int X e^{-m_2 x} dx + \ldots + A_n . e^{m_n x} \int X e^{-m_n x} dx \qquad \text{[By §13.5 ...(3)]}$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

13.7 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

of which the *symbolic form* is

$$(D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n)y = X.$$

Step I. To find the complementary function

(i) *Write the A.E.*

i.e., $D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n = 0$ and solve it for D.

(ii) Write the *C.F.* as follows :

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \ldots$ (real and different roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
2. $m_1, m_1, m_3 \ldots$ (two real and equal roots)	$(c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots$
3. $m_1, m_1, m_1, m_4 \ldots$ (three real and equal roots)	$(c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3 \ldots$ (a pair of imaginary roots)	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \ldots$ (2 pairs of equal imaginary roots)	$e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + \ldots$

Step II. To find the particular integral

From symbolic form P.I. $= \dfrac{1}{D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n} X = \dfrac{1}{f(D)}$ or $\dfrac{1}{\phi(D^2)}X$

(i) *When* $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)}e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x\frac{1}{f'(D)}e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)}e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $f'(D)$ = diff. coeff. of $f(D)$ w.r.t. D

$f''(D)$ = diff. coeff. of $f'(D)$ w.r.t. D, etc.

(*ii*) *When* $X = \sin(ax+b)$ or $\cos(ax+b)$.

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) \neq 0]$$

$$= x \frac{1}{\phi'(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)\text{], put } D^2 = -a^2 \qquad [\phi'(-a^2) \neq 0, \phi''(-a^2) \neq 0]$$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. D, etc.

(*iii*) *When* $X = x^m$, m being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

(*iv*) *When* $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\frac{1}{f(D+a)} V$ as in (*i*), (*ii*), and (*iii*).

(*v*) *When X is any function of x.*

$$\text{P.I.} = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx.$$

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example 13.12. *Solve* $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1-e^x)^2$.

Solution. Given equation in *symbolic form* is $(D^2 + D + 1)y = (1 - e^x)^2$

(*i*) *To find C.F.*

Its A.E. is $D^2 + D + 1 = 0$, $\therefore$ $D = \frac{1}{2}(-1 + \sqrt{3}i)$

Thus C.F. $= e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2+D+1}(1 - 2e^x + e^{2x}) = \frac{1}{D^2+D+1}(e^{0x} - 2e^x + e^{2x})$$

$$= \frac{1}{0^2+0+1}e^{0x} - 2 \cdot \frac{1}{1^2+1+1}e^x + \frac{1}{2^2+2+1}e^{2x} = 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$$

(*iii*) Hence the C.S. is $y = e^{-x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) + 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$.

Example 13.13. *Solve $y'' + 4y' + 4y = 3\sin x + 4\cos x$, $y(0) = 1$ and $y'(0) = 0$.* (J.N.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 + 4D + 4)y = 3\sin x + 4\cos x$

(*i*) *To find C.F.*

Its A.E. is $(D + 2)^2 = 0$ where $D = -2, -2$ $\therefore$ C.F. $= (c_1 + c_2x)e^{-2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4}(3\sin x + 4\cos x) = \frac{1}{-1 + 4D + 4}(3\sin x + 4\cos x)$$

$$= \frac{4D - 3}{16D^2 - 9}(3\sin x + 4\cos x) = \frac{(4D - 3)}{-16 - 9}(3\sin x + 4\cos x)$$

$$= \frac{-1}{25}\{3(4\cos x - 3\sin x) + 4(-4\sin x - 3\cos x)\} = \sin x$$

(*iii*) C.S. is $y = (c_1 + c_2)\, e^{-2x} + \sin x$

When $x = 0, y = 1,$ $\therefore$ $1 = c_1$

Also $y' = c_2e^{-2x} + (c_1 + c_2x)(-2)e^{-2x} + \cos x$.

When $x = 0, y' = 0,$ $\therefore$ $0 = c_2 - 2c_1 + 1$, *i.e.*, $c_2 = 1$.

Hence the required solution is $y = (1 + x)\, e^{-2x} + \sin x$.

Example 13.14. *Solve $(D - 2)^2 = 8(e^{2x} + \sin 2x + x^2)$.*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D - 2)^2 = 0$, $\therefore$ $D = 2, 2$.

Thus C.F. $= (c_1 + c_2x)\, e^{2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2\right]$$

Now $\dfrac{1}{(D-2)^2}e^{2x} = x^2\dfrac{1}{2(1)}e^{2x}$ [$\because$ by putting $D = 2$, $(D - 2)^2 = 0$, $2(D - 2) = 0$]

$$= \frac{x^2e^{2x}}{2}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2 - 4D + 4}\sin 2x = \frac{1}{(-2^2) - 4D + 4}\sin 2x$$

$$= -\frac{1}{4}\int \sin 2x\, dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

and $$\frac{1}{(D-2)^2}x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2}.x^2 = \frac{1}{4}\left[1 + (-2)\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2 + \ldots\right]x^2$$

$$= \frac{1}{4}\left(1 + D + \frac{3D^2}{4} + \ldots\right)x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Thus P.I. $= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

(*iii*) Hence the C.S. is $y = (c_1 + c_2x)\, e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Example 13.15. *Find the complete solution of $y'' - 2y' + 2y = x + e^x\cos x$.* (U.P.T.U., 2002)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 2)y = x + e^x\cos x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 2 = 0$ $\therefore$ $D = \dfrac{2 \pm \sqrt{(4-8)}}{2} = 1 \pm i$.

Thus C.F. $= e^x(c_1\cos x + c_2\sin x)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2}\left[1 - \left(D - \frac{D^2}{2}\right)\right]^{-1}(x) + e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}(\cos x)$$

$$= \frac{1}{2}\left(1 + D - \frac{D^2}{2}\right)x + e^x \frac{1}{D^2 + 1}\cos x \qquad \text{[Case of failure]}$$

$$= \frac{1}{2}(x + 1 - 0) + e^x \cdot x\frac{1}{2D}\cos x = \frac{1}{2}(x+1) + \frac{xe^x}{2}\int \cos x\, dx = \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

(*iii*) Hence the C.S. is $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$.

Example 13.16. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x.$

(V.T.U., 2008 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 3D + 2 = 0$ or $(D-2)(D-1) = 0$ whence $D = 1, 2$.

Thus C.F. $= c_1e^x + c_2e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(xe^{3x} + \sin 2x) = \frac{1}{D^2 - 3D + 2}(e^{3x} \cdot x) + \frac{1}{D^2 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2 - 3(D+3) + 2}(x) + \frac{1}{-4 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{D^2 + 3D + 2}(x) - \frac{3D - 2}{9D^2 - 4}(\sin 2x) = \frac{e^{3x}}{2} \cdot \left[1 + \left\{\frac{3D + D^2}{2}\right\}\right]^{-1} x - \frac{(3D-2)}{9(-4) - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{3D}{2} \ldots\right)x + \frac{1}{40}(6\cos 2x - 2\sin 2x) = \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{2x} + e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Example 13.17. *Solve* $\frac{d^2y}{dx^2} - 4y = x \sinh x.$ *(Madras, 2000 S)*

Solution. Given equation in *symbolic form* is $(D^2 - 4)y = x \sinh x$.

(*i*) *To find C.F.*

Its A.E. is $D^2 - 4 = 0$, whence $D = \pm 2$.

Thus C.F. $= c_1e^{2x} + c_2e^{-2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 4}x \sinh x = \frac{1}{D^2 - 4}x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2 - 4}e^x \cdot x - \frac{1}{D^2 - 4}e^{-x} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4}x - e^{-x}\frac{1}{(D-1)^2 - 4}x\right] = \frac{1}{2}\left[e^x \frac{1}{D^2 + 2D - 3}x - e^{-x}\frac{1}{D^2 - 2D - 3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}.x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}\cdot x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}+...\right)x - e^{-x}\left(1-\frac{2D}{3}+...\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right)-e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right)-\frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

(*iii*) Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Example 13.18. *Solve* $(D^2 - 1)y = x \sin 3x + \cos x$.

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 1 = 0$, whence $D = \pm 1$. $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2-1}(x\sin 3x + \cos x) = \frac{1}{D^2-1}x\,(\text{I.P. of } e^{3ix}) + \frac{1}{D^2-1}\cos x$$

$$= \text{I.P. of } \frac{1}{D^2-1}e^{3ix}.x + \frac{1}{(-1)^2-1}\cos x = \text{I.P. of}\left[e^{3ix}\frac{1}{(D+3i)^2-1}x\right] - \frac{\cos x}{2}$$

[Replacing D by $D + 3i$]

$$= \text{I.P. of}\left[e^{3ix}\frac{1}{D^2+6iD-10}x\right]-\frac{\cos x}{2}$$

$$= \text{I.P. of}\left[e^{3ix}.\frac{1}{-10}\left(1-\frac{3iD}{5}-\frac{D^2}{10}\right)^{-1}x\right]-\frac{\cos x}{2}$$ [Expand by Binomial theorem]

$$= \text{I.P. of}\left[e^{3ix}.\frac{1}{-10}\left(1+\frac{3iD}{5}+......\right)x\right]-\frac{\cos x}{2} = \text{I.P. of}\left[-\frac{e^{3ix}}{10}\left(x+\frac{3i}{5}\right)\right]-\frac{\cos x}{2}$$

$$= \text{I.P. of}\left[\frac{-1}{10}(\cos 3x + i\sin 3x)\left(x+\frac{3i}{5}\right)\right]-\frac{\cos x}{2}$$

$$= -\frac{1}{10}\text{ I.P. of}\left[\left(x\cos 3x - \frac{3\sin 3x}{5}\right)+i\left(x\sin 3x+\frac{3}{5}\cos 3x\right)\right]-\frac{\cos x}{2}$$

$$= -\frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right)-\frac{\cos x}{2}.$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x + 25\cos x)$.

Example 13.19. *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$. *(S.V.T.U., 2007 ; J.N.T.U., 2006 ; U.P.T.U., 2005)*

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 1)\,y = xe^x \sin x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 1 = 0$, *i.e.*, $(D - 1)^2 = 0$

$\therefore$ $D = 1, 1$. Thus C.F. $= (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x \cdot x \sin x = e^x \cdot \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x \, dx \qquad \text{[Integrate by parts]}$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1 \cdot (-\cos x)\, dx\right] = e^x \int [-x \cos x + \sin x]\, dx$$

$$= e^x\left[-\left\{x \sin x - \int 1 \cdot \sin x \, dx\right\} - \cos x\right] = e^x[-x \sin x - \cos x - \cos x]$$

$$= -e^x(x \sin x + 2 \cos x).$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x)e^x - e^x(x \sin x + 2 \cos x)$.

Example 13.20. *Solve* $(D^4 + 2D^2 + 1)y = x^2 \cos x$. (*Nagarjuna, 2008 ; Rajasthan, 2005*)

Solution. (*i*) *To find C.F.*

Its A.E. is $(D^2 + 1)^2 = 0$ whose roots are $D = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{(D^2+1)^2} x^2 (\text{Re.P. of } e^{ix})$$

$$= \text{Re.P. of } \left\{\frac{1}{(D^2+1)^2} e^{ix} \cdot x^2\right\} = \text{Re.P. of } \left\{e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2\right\}$$

$$= \text{Re.P. of } \left\{e^{ix} \frac{1}{(D^2+2iD)^2} x^2\right\} = \text{Re.P. of } \left[e^{ix}\left\{-\frac{1}{4D^2}\left(1 - \frac{i}{2}D\right)^{-2} x^2\right\}\right]$$

$$= \text{Re.P. of } \left[-\frac{1}{4} e^{ix} \cdot \frac{1}{D^2}\left\{1 + 2\frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \ldots\right\} x^2\right]$$

$$= \text{Re.P. of } \left\{-\frac{1}{4} e^{ix} \cdot \frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right\} = \text{Re.P. of } \left\{-\frac{1}{4} e^{ix} \cdot \frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)\right\}$$

$$= -\frac{1}{4} \text{Re.P. of } \left\{e^{ix}\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)\right\} = \frac{-1}{48} \text{ Re.P. of } \{(\cos x + i \sin x)(x^4 + 4ix^3 - 9x^2)\}$$

$$= -\frac{1}{48}[(x^4 - 9x^2)\cos x - 4x^3 \sin x]$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + \frac{1}{48}\,[4x^3 \sin x - x^2(x^2 - 9)\cos x\}$.

Example 13.21. *Solve* $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$. (*J.N.T.U., 2006 ; U.P.T.U., 2004*)

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 4D + 4 = 0$ *i.e.*, $(D-2)^2 = 0$. $\therefore$ $D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2 e^{2x} \sin 2x) = 8e^{2x} \frac{1}{(D+2-2)^2}(x^2 \sin 2x)$$

$$= 8e^{2x} \frac{1}{D^2}(x^2 \sin 2x) = 8e^{2x} \cdot \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2}\,dx\right\}$$

$$= 8e^{2x}\int\left\{-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right\}dx$$

$$= 8e^{2x}\left[\left\{\frac{-x^2}{2}\frac{\sin 2x}{2} - \int(-x)\frac{\sin 2x}{2}dx\right\} + \left\{\int\frac{x}{2}\sin 2x\,dx\right\} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(\frac{-x^2}{4} + \frac{1}{8}\right)\sin 2x + \int x\sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x\left(\frac{-\cos 2x}{2}\right) - \int 1 \cdot \left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

(*iii*) Hence the C.S. is $y = e^{2x}[c_1 + c_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$.

Example 13.22. *Solve* $\dfrac{d^2y}{dx^2} + a^2y = \sec ax.$

Solution. Given equation in *symbolic form* is $(D^2 + a^2)y = \sec ax$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm ia$.

Thus C.F. $= c_1 \cos ax + c_2 \sin ax$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax \qquad \text{[Resolving into partial fractions]}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$

Now $\quad \dfrac{1}{D - ia}\sec ax = e^{iax}\displaystyle\int \sec ax \cdot e^{-iax}\,dx \qquad \left[\because \dfrac{1}{D - a}X = e^{ax}\displaystyle\int Xe^{-ax}\,dx\right]$

$$= e^{iax}\int\frac{\cos ax - i\sin ax}{\cos ax}dx = e^{iax}\int(1 - i\tan ax)dx = e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Changing i to $-i$, we have

$$\frac{1}{D + ia}\sec ax = e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}$$

Thus $\quad \text{P.I.} = \dfrac{1}{2ia}\left[e^{iax}\left\{x + \dfrac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \dfrac{i}{a}\log\cos ax\right\}\right]$

$$= \frac{x}{a}\frac{e^{iax} - e^{-iax}}{2i} + \frac{1}{a^2}\log\cos ax \cdot \frac{e^{iax} + e^{-iax}}{2} = \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax \cdot \cos ax.$$

(*iii*) Hence the C.S. is

$$y = c_1\cos ax + c_2\sin ax + (1/a)x\sin ax + (1/a^2)\cos ax\log\cos ax.$$

PROBLEMS 13.2

Solve :

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$ (V.T.U., 2005)

2. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$. Also find y when $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$.

3. $\frac{d^2x}{dt^2} + n^2x = k\cos(nt + \alpha)$.

4. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$.

5. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$. (Bhopal, 2002 S)

6. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$. (Madras, 2000)

7. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$. (V.T.U., 2004)

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$. (Delhi, 2002)

9. $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$. (Nagarjuna, 2008)

10. $\frac{d^2y}{dx^2} - y = e^x + x^2e^x$. (Nagpur, 2009)

11. $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$. (Mumbai, 2006)

12. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$. (V.T.U., 2006)

13. $(D^2 + 1)^2 y = x^4 + 2\sin x \cos 3x$. (Madras, 2006)

14. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$. (Bhopal, 2008)

15. $(D^4 + D^2 + 1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$. (Rajasthan, 2006)

16. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x \cos x$. (V.T.U., 2010)

17. $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$. (Raipur, 2005 ; Anna, 2002 S)

18. $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$.

19. $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.

20. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$. (P.T.U., 2003)

21. $\frac{d^2y}{dx^2} + 16y = x\sin 3x$. (V.T.U., 2010 S)

22. $(D^2 + 2D + 1)y = x\cos x$. (Rajasthan, 2006)

23. $(D^2 - 1)y = x\sin x + (1 + x^2)e^x$.

24. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. (S.V.T.U., 2009)

25. $(D^2 + a^2)y = \tan ax$. (V.T.U., 2005)

13.8 TWO OTHER METHODS OF FINDING P.I.

I. Method of variation of parameters. *This method is quite general and applies to equations of the form*

$$y'' + py' + qy = X \quad ...(1)$$

where p, q, and X are functions of x. It gives P.I. $= -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$ ...(2)

where y_1 *and* y_2 *are the solutions of* $y'' + py' + qy = 0$ ...(3)

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ *is called the Wronskian* of* y_1, y_2.

Proof. Let the C.F. of (1) be $y = c_1y_1 + c_2y_2$

Replacing c_1, c_2 (regarded as *parameters*) by unknown functions $u(x)$ and $v(x)$, let the P.I. be

$$y = uy_1 + vy_2 \quad ...(4)$$

Differentiating (4) w.r.t. x, we get $y' = uy'_1 + vy'_2 + u'y_1 + v'y_2$

*Named after the Polish mathematician and philosopher *Hoene Wronsky* (1778–1853).

$$= uy_1' + vy_2' \qquad ...(5)$$

on assuming that $u'y_1 + v'y_2 = 0$ ...(6)

Differentiate (4) and substitute in (1). Then noting that y_1 and y_2, satisfy (3), we obtain

$$u'y_1' + v'y_2' = X \qquad ...(7)$$

Solving (6) and (7), we get

$$u' = -\frac{y_2 X}{W}, v' = \frac{y_1 X}{W} \qquad \text{where } W = y_1y_2' - y_2y_1'$$

Integrating $u = -\int \frac{y_2 X}{W}\, dx$, $v = \int \frac{y_1 X}{W}\, dx$. Substituting these in (4), we get (2).

Example 13.23. *Using the method of variation of parameters, solve*

$$\frac{d^2y}{dx^2} + 4y = \tan 2x. \qquad \textit{(V.T.U., 2008 ; Bhopal, 2007 ; S.V.T.U., 2006 S)}$$

Solution. Given equation in *symbolic form* is $(D^2 + 4)y = \tan 2x$.

(i) To find C.F.

Its A.E. is $D^2 + 4 = 0$, $\therefore$ $D = \pm 2i$

Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

(ii) To find P.I.

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Thus,

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2}\, dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2}\, dx$$

$$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x)dx + \frac{1}{2}\sin 2x \int \sin 2x\, dx$$

$$= -\frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x) - \sin 2x] - \frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$.

Example 13.24. *Solve, by the method of variation of parameters, $d^2y/dx^2 - y = 2/(1 + e^x)$.*

(V.T.U., 2005 ; Hissar, 2005)

Solution. Given equation is $D^2 - 1 = 2/(1 + e^x)$

A.E. is $D^2 - 1 = 0$, $D = \pm 1$, $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

Here $y_1 = e^x$, $y_2 = e^{-x}$ and $X = 2/(1 + e^x)$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^xe^{-x} - e^xe^{-x} = -2.$$

Thus

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx = -e^x \int \frac{e^{-x}}{-2}\cdot\frac{2}{1+e^x}dx + e^{-x}\int \frac{e^x}{-2}\cdot\frac{2}{1+e^x}dx$$

$$= e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx - e^{-x}\log(1+e^x) = e^x\left[e^{-x} - \int \frac{e^{-x}}{e^{-x}+1}dx\right] - e^{-x}\log(1+e^x)$$

$$= e^x[-e^{-x} + \log(e^{-x}+1)] - e^{-x}\log(1+e^x) = -1 + e^x\log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Hence C.S. is $y = c_1e^x + c_2e^{-x} - 1 + e^x\log(e^{-x}+1) - e^{-x}\log(e^x+1)$.

Example 13.25. *Solve by the method of variation of parameters* $y'' - 6y' + 9y = e^{3x}/x^2$.

(*Nagpur, 2009 ; S.V.T.U., 2009*)

Solution. Given equation is $(D^2 - 6D + 9)y = e^{3x}/x^2$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.* $(D-3)^2 = 0$ $\therefore$ C.F. $= (c_1 + c_2x)e^{3x}$

Here $y_1 = e^{3x}$, $y_2 = xe^{3x}$ and $X = e^{3x}/x^2$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx = -e^{3x} \int \frac{xe^{3x}}{e^{6x}} \frac{e^{3x}}{x^2}\,dx + xe^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2}\,dx$$

$$= -e^{3x} \int \frac{dx}{x} + xe^{3x} \int x^{-2}\,dx = -e^{3x}(\log x + 1)$$

Hence C.S. is $\quad y = (c_1 + c_2x)e^{3x} - e^{3x}(\log x + 1).$

Example 13.26. *Solve, by the method of variation of parameters,* $y'' - 2y' + y = e^x \log x$.

(*V.T.U., 2006 ; Kurukshetra, 2005 ; Madras, 2003*)

Solution. Given equation in symbolic form is $(D^2 - 2D + 1)\,y = e^x \log x$

(*i*) *To find C.F.*

Its A.E. is $(D-1)^2 = 0$, $\therefore$ $D = 1, 1$

Thus C.F. is $y = (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

Here $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)\,e^x \end{vmatrix} = e^{2x}$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx$$

$$= -e^x \int \frac{xe^x \,.\, e^x \log x}{e^{2x}}\,dx + xe^x \int \frac{e^x \,.\, e^x \log x}{e^{2x}}\,dx = -e^x \int x \log x\,dx + xe^x \int \log x\,dx$$

$$= -e^x \left(\frac{x^2}{2}\log x - \int \frac{1}{x} \cdot \frac{x^2}{2}\,dx\right) + x\,.\,e^x\left(x \log x - \int \frac{1}{x}\,.\,x\,dx\right)$$

$$= -e^x \left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + x\ e^x(x \log x - x) = \frac{1}{4}\,x^2e^x(2\log x - 3)$$

Hence C.S. is $y = (c_1 + c_2x)\,e^x + \dfrac{1}{4}\,x^2e^x(2\log x - 3)$.

II. Method of undetermined coefficients

To find the P.I. of $f(D)\,y = X$, we assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of X. Thus when (*i*) $X = 2e^{3x}$, trial solution $= ae^{3x}$.

(*ii*) $X = 3 \sin 2x$, trial solution $= a_1 \sin 2x + a_2 \cos 2x$

(*iii*) $X = 2x^3$, trial solution $= a_1x^3 + a_2x^2 + a_3x + a_4$

However when $X = \tan x$ or $\sec x$, this method fails, since the number of terms obtained by differentiating $X = \tan x$ or $\sec x$ is infinite.

The above method holds so long as no term in the trial solution appears in the C.F. If any term of the trial solution appears in the C.F., we multiply this trial solution by the lowest positive integral power of x which is large enough so that none of the terms which are then present, appear in the C.F.

Example 13.27. *Solve* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$. *(V.T.U., 2008)*

Solution. Here C.F. $= e^{-x}(c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x)$

Assume P.I. as $y = a_1x^2 + a_2x + a_3 + a_4e^{-x}$

$\therefore$ $Dy = 2a_1x + a_2 - a_4e^{-x}$ and $D^2y = 2a_1 + a_4e^{-x}$

Substituting these in the given equation, we get

$$4a_1x^2 + (4a_1 + 4a_2)x + (2a_1 + 2a_2 + 4a_3) + 3a_4e^{-x} = 2x^2 + 3e^{-x}$$

Equating corresponding coefficients on both sides, we get

$$4a_1 = 2,\ 4a_1 + 4a_2 = 0,\ 2a_1 + 2a_2 + 4a_3 = 0,\ 3a_4 = 3$$

Then $a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_3 = 0, a_4 = 1$. Thus P.I. $= \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$

$\therefore$ C.S. is $y = e^{-x}(c_1 \cos \sqrt{3x} + c_2 \sin \sqrt{3x}) + \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$.

Example 13.28. *Solve* $(D^2 + 1)y = \sin x$.

Solution. Here C.F. $= c_1 \cos x + c_2 \sin x$

We would normally assume a trial solution as $a_1 \cos x + a_2 \sin x$.

However, since these terms appear in the C.F., we multiply by x and assume the trial P.I. as

$$y = x(a_1 \cos x + a_2 \sin x)$$

$\therefore$ $Dy = (a_1 + a_2x)\cos x + (a_2 - a_1x)\sin x$ and $D^2y = (2a_2 - a_1x)\cos x - (2a_1 + a_2x)\sin x$

Substituting these in the given equation, we get $2a_1 \cos x - 2a_2 \sin x = \sin x$

Equating corresponding coefficients,

$$2a_1 = 0, \quad -2a_2 = 1 \quad \text{so that } a_1 = 0, a_2 = -\frac{1}{2}. \quad \text{Thus P.I.} = -\frac{1}{2}x \sin x$$

$\therefore$ C.S. is $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x \sin x$.

Example 13.29. *Solve by the method of undetermined coefficients,*

$$\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x.$$

Solution. Its A.E. is $D^2 - 1 = 0$, $\therefore$ $D = \pm 1$.

Thus C.F. $= c_1e^x + c_2e^{-x}$

Assume P.I. as $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - e^{2x}(c_3 \cos 3x + c_4 \sin 3x)$

$$\therefore \quad \frac{dy}{dx} = e^{3x}\{(3c_1 + 2c_2)\cos 2x + (3c_2 - 2c_1)\sin 2x\} - e^{2x}\{(2c_3 + 3c_4)\cos 3x + (2c_4 - 3c_3)\sin 3x\}$$

and

$$\frac{d^2y}{dx^2} = e^{3x}\{(5c_1 + 12c_2)\cos 2x + (5c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 5c_3)\cos 3x - (5c_4 + 12c_3)\sin 3x\}$$

Substituting these in the given equation, we get

$$e^{3x}\{(4c_1 + 12c_2)\cos 2x + (4c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 6c_3)\cos 3x - (6c_4 + 12c_3)\sin 3x\}$$
$$= e^{3x}\cos 2x - e^{2x}\sin 3x$$

Equating corresponding coefficients,

$$4c_1 + 12c_2 = 1,\ 4c_2 - 12c_1 = 0\ ;\ 12c_4 - 6c_3 = 0,\ 6c_4 + 12c_3 = -1$$

whence $c_1 = 1/40, c_2 = 3/40, c_3 = -1/15, c_4 = -1/30$

Thus P.I. $= \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x) + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)$

Hence C.S. is $y = c_1e^x + c_2e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)$.

PROBLEMS 13.3

Solve by the method of variation of parameters :

1. $\dfrac{d^2y}{dx^2} + a^2y = \text{cosec}\, ax.$

2. $\dfrac{d^2y}{dx^2} + y = \sec x.$ *(Bhopal, 2007)*

3. $\dfrac{d^2y}{dx^2} + y = \tan x.$ *(P.T.U., 2005 ; Raipur, 2004)*

4. $\dfrac{d^2y}{dx^2} + y = x \sin x.$ *(S.V.T.U., 2007 ; J.N.T.U., 2005)*

5. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x/]x.$ *(V.T.U., 2006)*

6. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \dfrac{1}{1+e^{-x}}.$ *(V.T.U., 2010 S ; U.P.T.U., 2005)*

7. $y'' - 2y' + 2y = e^x \tan x.$ *(V.T.U., 2010)*

8. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = e^x \sin x.$ *(U.P.T.U., 2003)*

9. $\dfrac{d^2y}{dx^2} + y = \dfrac{1}{1+\sin x}.$ *(V.T.U., 2004)*

Solve by the method of undetermined coefficients :

10. $(D^2 - 3D + 2)\, y = x^2 + e^x.$ *(V.T.U., 2003 S)*

11. $\dfrac{d^2y}{dx^2} + y = 2\cos x.$ *(V.T.U., 2000 S)*

12. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{3x} + \sin x.$ *(V.T.U., 2008)*

13. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x + \sin x.$ *(V.T.U., 2010)*

14. $(D^2 - 2D + 3)\, y = x^3 + \cos x.$

15. $(D^2 - 2D)\, y = e^x \sin x.$ *(V.T.U., 2006)*

13.9 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now we shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

I. Cauchy's homogeneous linear equation*. An equation of the form

$$x^n \frac{d^ny}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_{n-1} x \frac{dy}{dx} + k_n y = X \qquad \ldots(1)$$

where X is a function of x, is called *Cauchy's homogeneous linear equation.*

Such equations can be reduced to linear differential equations with constant coefficients, by putting

$$x = e^t \quad \text{or} \quad t = \log x. \text{ Then if } D = \frac{d}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}, \quad i.e., \quad x\frac{dy}{dx} = Dy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

i.e., $$x^2 \frac{d^2y}{dx^2} = D(D-1)\,y. \text{ Similarly, } x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)\,y \text{ and so on.}$$

After making these substitutions in (1), there results a linear equation with constant coefficients, which can be solved as before.

Example 13.30. *Solve* $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x.$ *(V.T.U., 2010)*

Solution. This is a Cauchy's homogeneous linear.

*See footnote p. 144.

Put $x = e^t$, *i.e.*, $t = \log x$, so that $x \dfrac{dy}{dx} = Dy$, $x^2 \dfrac{d^2y}{dx^2} = D(D-1)y$ where $D = \dfrac{d}{dt}$

Then the given equation becomes $[D(D-1) - D + 1]\, y = t$ or $(D-1)^2 y = t$ ...(*i*)

which is a linear equation with constant coefficients.

Its A.E. is $(D-1)^2 = 0$ whence $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 t)e^t$ and P.I. $= \dfrac{1}{(D-1)^2}\, t = (1-D)^{-2}\, t = (1 + 2D + 3D^2 + \ldots)\, t = t + 2.$

Hence the solution of (*i*) is $y = (c_1 + c_2 t)e^t + t + 2$ or, putting $t = \log x$ and $e^t = x$, we get

$y = (c_1 + c_2 \log x)\, x + \log x + 2$ as the required solution of (*i*).

Example 13.31. *Solve* $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. (*P.T.U., 2003*)

Solution. Put $x = e^t$ *i.e.*, $t = \log x$ so that $x\, dy/dx = Dy$, $x^2 \dfrac{d^2y}{dx^2} = D\,(D-1)\,y$

Then the given equation becomes

$$[D\,(D-1) + 3D + 1]\, y = \frac{1}{(1-e^t)^2} \quad \text{or} \quad (D^2 + 2D + 1)\, y = \frac{1}{(1-e^t)^2}$$

Its A.E. is $D^2 + 2D + 1 = 0$ or $(D+1)^2 = 0$ *i.e.*, $D = -1, -1$.

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x)\, e^{-t} = (c_1 + c_2 \log x)\, \frac{1}{x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\, \frac{1}{(1-e^t)^2} = \frac{1}{D+1}\, u, \text{ where } u = \frac{1}{D+1} \cdot \frac{1}{(1-e^t)^2} \text{ i.e. } \frac{du}{dt} + u = (1-e^t)^{-2}$$

which is Leibnitz's linear equation having I.F. $= e^t$

$$\therefore \text{ its solution is} \qquad ue^t = \int \frac{e^t}{(1-e^t)^2}\, dt = \frac{1}{1-e^t} \quad \text{or} \quad u = \frac{e^{-t}}{1-e^t}$$

$$\therefore \qquad \text{P.I.} = \frac{1}{D+1}\left(\frac{e^{-t}}{1-e^t}\right) = e^{-t} \int \frac{1}{1-e^t}\, dt = \frac{1}{x} \int \frac{dx}{x(1-x)}$$

$$= \frac{1}{x} \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \frac{1}{x}\, [\log x - \log(1-x)] = \frac{1}{x} \log \frac{x}{x-1}$$

$$\text{Hence the solution is} \qquad y = \left\{c_1 + c_2 \log x + \log \frac{x}{x-1}\right\} \frac{1}{x}.$$

Example 13.32. *Solve* $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = \log x \sin(\log x)$.

(*Kurukshetra, 2006 ; Madras, 2006 ; Kerala, 2005*)

Solution. Putting $x = e^t$ *i.e.* $t = \log x$, the given equation becomes

$[D\,(D-1) + D + 1]\, y = t \sin t$ *i.e.* $(D^2 + 1)\, y = t \sin t$ ...(*i*)

Its A.E. is $D^2 + 1 = 0$ *i.e.* $D = \pm i$.

$\therefore$ C.F. $= c_1 \cos t + c_2 \sin t$

and $\quad P.I. = \dfrac{1}{D^2+1}\, t \sin t = \dfrac{1}{D^2+1}\, t\, (\text{I.P. of } e^{it})$

$$= \text{I.P. of } e^{it}\, \frac{1}{(D+i)^2 + 1}\, t = \text{I.P. of } e^{it} \cdot \frac{1}{D^2 + 2iD}\, t$$

$$= \text{I.P. of } e^{it} \frac{1}{2iD(1 + D/2i)} t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 - \frac{iD}{2}\right)^{-1} t$$

$$= \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 + \frac{iD}{2} + \ldots\right) t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(t + \frac{i}{2}\right)$$

$$= \text{I.P. of } \frac{e^{it}}{2i} \int \left(t + \frac{i}{2}\right) dt = \text{I.P. of } \frac{e^{it}}{2i}\left(\frac{t^2}{2} + \frac{it}{2}\right)$$

$$= \text{I.P. of } e^{it}\left(-\frac{i}{4}t^2 + \frac{t}{4}\right) = \text{I.P. of } (\cos t + i \sin t)\left(-\frac{it^2}{4} + \frac{t}{4}\right) = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

Hence the C.S. of (*i*) is $y = c_1 \cos t + c_2 \sin t - \frac{t^2}{4} \cos t + \frac{t}{4} \sin t$

or $\quad y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{4}(\log x)^2 \cos(\log x) + \frac{1}{4}\log(\log x)\sin(\log x)$

which is the required solution.

Example 13.33. *Solve* $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \log x \frac{\sin(\log x) + 1}{x}$. *(I.S.M., 2001)*

Solution. Put $x = e^t$, *i.e.*, $t = \log x$ so that $x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$\{D(D-1) - 3D + 1\}y = t\frac{\sin t + 1}{e^t} \quad \text{or} \quad (D^2 - 4D + 1)y = e^{-t} t (\sin t + 1)$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 4D + 1 = 0$ whence $D = 2 \pm \sqrt{3}$

$\therefore$ $\quad$ C.F. $= c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} = e^{2t}(c_1 e^{\sqrt{3t}} + c_2 e^{-\sqrt{3t}})$

and $\quad$ P.I. $= \frac{1}{D^2 - 4D + 1} e^{-t} t (\sin t + 1) = e^{-t} \frac{1}{(D-1)^2 - 4(D-1) + 1} t (\sin t + 1)$

$$= e^{-t}\left\{\frac{1}{D^2 - 6D + 6}t + \frac{1}{D^2 - 6D + 6} t \sin t\right\}$$

$$\frac{1}{D^2 - 6D + 6} t = \frac{1}{6}\left(1 - \frac{6D - D^2}{6}\right)^{-1} t = \frac{1}{6}(1 + D)t = \frac{1}{6}(t + 1)$$

$$\frac{1}{D^2 - 6D + 6} t \sin t = \text{I.P. of } \frac{1}{D^2 - 6D + 6} e^{it} \cdot t$$

$$= \text{I.P. of } e^{it}\frac{1}{(D+i)^2 - 6(D+i) + 6} t = \text{I.P. of } e^{it}\frac{1}{D^2 + (2i - 6)D + (5 - 6i)} t$$

$$= \text{I.P. of } \frac{e^{it}}{5 - 6i}\left\{1 + \frac{(2i - 6)D + D^2}{5 - 6i}\right\}^{-1} t = \text{I.P. } \frac{e^{it}}{5 - 6i}\left(1 - \frac{2i - 6}{5 - 6i} D\right) t$$

$$= \text{I.P. of } \frac{(5 + 6i)}{61}(\cos t + i \sin t)\left(t - \frac{2i - 6}{5 - 6i}\right)$$

$$= \text{I.P. of } \frac{1}{61}\{(5\cos t - 6\sin t) + i(5\sin t + 6\cos t)\}\left(t + \frac{42 + 26i}{61}\right)$$

$$= \frac{26}{3721}(5\cos t - 6\sin t) + \frac{1}{61}(5\sin t + \cos t)\left(t + \frac{42}{61}\right)$$

$$= \frac{t}{61}(5 \sin t + \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)$$

$\therefore$ $$\text{P.I.} = e^{-t}\left[\frac{1}{6}(t+1) + \frac{1}{61}(5 \sin t + 6 \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)\right]$$

Hence $$y = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}) + e^{-t}\left[\frac{1}{6}(t+1) + \frac{t}{61}(5 \sin t + 6 \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)\right]$$

or $$y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5 \sin(\log x) + 6 \cos(\log x)\} + \frac{2}{3721}\{27 \sin(\log x) + 191 \cos(\log x)\}\right].$$

Example 13.34. *Solve* $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^{x^x}$. *(Kurukshetra, 2005 ; U.P.T.U., 2005)*

Solution. Putting $x = e^t$, *i.e.*, $t = \log x$, the given equation becomes

$[D(D-1) + 4D + 2]\, y = e^{e^t}$ *i.e.*, $(D^2 + 3D + 2)\, y = e^{e^t}$

Its A.E. is $D^2 + 3D + 2 = 0$ whence $D = -1, -2$.

$\therefore$ C.F. $= c_1 e^{-t} + c_2 e^{-2t} = c_1 x^{-1} + c_2 x^{-2}$

and $$\text{P.I.} = \frac{1}{(D^2 + 3D + 2)} e^{e^t} = \frac{1}{(D+1)(D+2)} e^{e^t} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^t}.$$

Now $$\frac{1}{D+1} e^{e^t} = \frac{1}{D+1} e^{-t} \cdot e^t e^{e^t} = e^{-t} \frac{1}{(D-1)+1} e^t e^{e^t}$$

$$= e^{-t} \frac{1}{D} e^t e^{e^t} = e^{-t} \int e^{e^t} d(e^t) = x^{-1} \int e^x dx = x^{-1} e^x$$

$$\frac{1}{D+2} e^{e^t} = \frac{1}{D+2} e^{-2t} \cdot e^{2t} e^{e^t} = e^{-2t} \frac{1}{(D-2)+2} e^{2t} e^{e^t}$$

$$= e^{-2t} \frac{1}{D} e^{e^t} e^{2t} = e^{-2t} \int e^{e^t} e^t \, d(e^t)$$

$$= x^{-2} \int e^x \, x \, dx \qquad [\because \; e^t = x]$$

$$= x^{-2}(xe^x - e^x) \qquad \text{[Integrating by parts]}$$

$\therefore$ $$\text{P.I.} = x^{-1} e^x - x^{-2}(xe^x - e^x) = x^{-2} e^x$$

Hence the required solution is $y = c_1 x^{-1} + x^{-2}(c_2 + e^x)$.

II. Legendre's linear equation*. An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = X \qquad \dots(2)$$

where k's are constants and X is a function of x, is called *Legendre's linear equation.*

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax + b = e^t$, *i.e.*, $t = \log(ax + b)$.

Then, if $$D = \frac{d}{dt}, \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a}{ax+b} \cdot \frac{dy}{dt} \text{ i.e. } (ax+b)\frac{dy}{dx} = a\, Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

* A French mathematician *Adrien Marie Legender* (1752 – 1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

i.e., $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)\,y$. Similarly, $(ax+b)^3 \dfrac{d^3y}{dx^3}\; a^3D(D-1)(D-2)\,y$ and so on.

After making these replacements in (2), there results a linear equation with constant coefficients.

Example 13.35. *Solve* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$. ...(*i*)

(*V.T.U., 2009 ; J.N.T.U., 2005 ; Kerala, 2005*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $1+x=e^t$, *i.e.*, $t=\log(1+x)$, so that $(1+x)\dfrac{dy}{dx} = Dy$

and $$(1+x)^2\frac{d^2y}{dx^2} = D(D-1)\,y, \text{ where } D = \frac{d}{dt}$$

Then (*i*) becomes $D(D-1)\,y + Dy + y = 2\sin t$

or $$(D^2+1)\,y = 2\sin t \qquad ...(ii)$$

This is a linear equation with constant co-efficients

Its A.E. is $D^2+1=0$, whence $D=\pm i$ $\therefore$ C.F. $= c_1\cos t + c_2\sin t$

and $$\text{P.I.} = 2\frac{1}{D^2+1}\sin t = 2t\,.\,\frac{1}{2D}\sin t$$

$$= t\int \sin t\,dt = -t\cos t \qquad [\because \text{ on replacing } D^2 \text{ by } -1^2,\ D^2+1=0]$$

Hence the solution of (*ii*) is $y = c_1\cos t + c_2\sin t - t\cos t$ and on replacing t by $\log(1+x)$, we get $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$ as the required solution.

Example 13.36. *Solve* $(2x-1)^2\dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} - 2y = 8x^2 - 2x + 3$. (*V.T.U., 2006*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $2x-1=e^t$ *i.e.*, $t=\log(2x-1)$ so that $(2x-1)\dfrac{dy}{dx} = 2Dy$

and $$(2x-1)^2\frac{d^2y}{dx^2} = 4D(D-1)\,y, \text{ where } D=\frac{d}{dt}.$$

Then the given equation becomes

$$4D(D-1)\,y + 2Dy - 2y = 8\left(\frac{1+e^t}{2}\right)^2 - 2\left(\frac{1+e^t}{2}\right) + 3$$

or $$2D^2y - Dy - y = e^{2t} + \frac{3}{2}e^t + 2 \qquad ...(i)$$

This is a linear equation with constant coefficients.

Its A.E. is $2D^2 - D - 1 = 0$ whence $D = 1, -1/2$.

$\therefore$ C.F. $= c_1e^t + c_2e^{-t/2}$

and $$\text{P.I.} = \frac{1}{2D^2-D-1}\left(e^{2t} + \frac{3}{2}e^t + 2\right) = \frac{1}{2.4-2-1}e^{2t} + \frac{3}{2}\frac{t}{4D-1}e^t + 2\,.\,\frac{1}{2.0^2-0-1}e^{0t}$$

$[\because$ on putting $t=1$, $2D^2-D-1=0$

$$= \frac{1}{5}e^{2t} + \frac{3t}{2}\cdot\frac{1}{4-1}e^t - 2 = \frac{1}{5}e^{2t} + \frac{t}{2}e^t - 2$$

Hence the solution of (*i*) is

$$y = c_1e^t + c_2e^{-t/2} + \frac{1}{5}e^{2t} + \frac{1}{2}te^t - 2 \text{ and on replacing } t \text{ by } \log(2x-1),$$

$$y = c_1 (2x - 1) + c_2 (2x - 1)^{-1/2} + \frac{1}{5} (2x - 1)^2 + \frac{1}{2} (2x - 1) \log (2x - 1) - 2.$$

which is the required solution.

PROBLEMS 13.4

Solve :

1. $x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = x^2.$

2. $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} - 4y = x^4.$

3. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = (1 + x^2).$ *(S.V.T.U., 2007)*

4. $x \dfrac{d^2y}{dx^2} - \dfrac{2y}{x} = x + \dfrac{1}{x^2}.$ *(V.T.U., 2005 S)*

5. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \dfrac{d^2u}{dr^2} + r \dfrac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

Solve :

6. $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = \log x.$ *(Bhopal, 2009)*

7. $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = x + \log x$ *(Bhopal, 2008)*

8. $x^2y'' + xy' + y = 2\cos^2 (\log x).$ *(V.T.U., 2011)*

9. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10 \left(x + \dfrac{1}{x} \right)$ *(S.V.T.U., 2006 ; P.T.U., 2003)*

10. $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1 - x)^2}.$ *(P.T.U., 2003)*

11. $x^2 \dfrac{d^2y}{dx^2} + 5x \dfrac{dy}{dx} + 4y = x \log x.$ *(U.P.T.U., 2004)*

12. $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 12y = x^3 \log x.$ *(Bhopal, 2008)*

13. $(2x + 3)^2 \dfrac{d^2y}{dx^2} - (2x + 3) \dfrac{dy}{dx} - 12y = 6x.$ *(V.T.U., 2007 ; Kerala, 2005 ; Anna, 2002 S)*

14. $(x - 1)^3 \dfrac{d^3y}{dx^3} + 2(x - 1)^2 \dfrac{d^2y}{dx^2} - 4 (x - 1) \dfrac{dy}{dx} + 4y = 4 \log (x - 1).$ *(Nagpur, 2009)*

15. $(1 + x)^2 \dfrac{d^2y}{dx^2} + (1 + x) \dfrac{dy}{dx} + y = \sin [2 \log (1 + x)]$ *(P.T.U., 2006 ; V.T.U., 2004)*

16. $(3x + 2)^2 \dfrac{d^2y}{dx^2} + 5 (3x + 2) \dfrac{dy}{dx} - 3y = x^2 + x + 1.$ *(Mumbai, 2006)*

13.10 (1) LINEAR DEPENDENCE OF SOLUTIONS

Consider the initial value problem consisting of the homogeneous linear equation

$$y'' + py' + qy = 0 \qquad \text{...(1)}$$

with variable coefficients $p (x)$ and $q (x)$ and two initial conditions $y (x_0) = k_0, y' (x_0) = k_1$ ...(2)

Let its general solution be $y = c_1 y_1 + c_2 y_2$ ...(3)

which is made up of two linearly dependent solutions y_1 and y_2.*

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is any fixed point on I, then the above initial value problem has a **unique solution** *$y(x)$ on the interval I.*

* As in §2.12, y_1, y_2 are said to be *linearly dependent* in an interval I, if and only if there exist numbers λ_1, λ_2 not both zero such that $\lambda_1 y_1 + \lambda_2 y_2 = 0$ for all x in I.

If no such numbers other than zero exist, then y_1, y_2 are said to be *linearly independent*.

(2) Theorem. *If $p(x)$ and $q(x)$ are continuous on an open interval I, then the solutions y_1 and y_2 of (1) are linearly dependent in I if and only if the Wronskian† $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ for some x_0 on I.* If there is an $x = x_1$ in I at which $W(y_1, y_2) \neq 0$, then y_1, y_2 *are linearly independent on I.*

Proof. If y_1, y_2 are linearly dependent solutions of (1) then there exist two constants c_1, c_2 not both zero, such that

$$c_1 y_1 + c_2 y_2 = 0 \quad \text{...(4)}$$

Differentiating w.r.t. x, $c_1 y_1' + c_2 y_2' = 0$ ...(5)

Eliminating c_1, c_2 from (4) and (5), we get

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

Conversely, suppose $W(y_1, y_2) = 0$ for some $x = x_0$ on I and show that y_1, y_2 are linearly dependent.

Consider the equation

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \end{aligned}\right\} \quad \text{...(6)}$$

which, on eliminating c_1, c_2, give $W(y_1, y_2) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = 0$

Hence the system has a solution in which c_1, c_2 are not both zero.

Now introduce the function $\bar{y}(x) = c_1 y_1(x) + c_2 y_2(x)$

Then $y(x)$ is a solution of (1) on I. By (6), this solution satisfies the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$. Also since $p(x)$ and $q(x)$ are continuous on I, this solution must be unique. But $\mathbf{y = 0}$ is obviously another solution of (1) satisfying the given initial conditions. Hence $\bar{\mathbf{y}} = y$ *i.e.*, $c_1 y_1 + c_2 y_2 = 0$ in I. Now since c_1, c_2 are not both zero, it implies that y_1 and y_2 are linearly dependent on I.

Example 13.37. *Show that the two functions sin 2x, cos 2x are independent solutions of $y'' + 4y = 0$.*

Solution. Substituting $y_1 = \sin 2x$ (or $y_2 = \cos 2x$) in the given equation we find that y_1, y_2 are its solutions.

Also $$W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$$

for any value of x. Hence the solutions y_1, y_2 are linearly independent.

PROBLEMS 13.5

Solve :

1. Show that e^{-x}, xe^{-x} are independent solutions of $y'' + 2y' + y = 0$ in any interval.
2. Show that $e^x \cos x$, $e^x \sin x$ are independent solutions of the equation $xy'' - 2y' = 0$.
3. If y_1, y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$, show that the *Wronskian* can be expressed as $W(y_1, y_2) = ce^{-\int p dx}$

13.11 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Quite often we come across linear differential equations in which there are two or more dependent variables and a single independent variable. Such equations are known as *simultaneous linear equations.* Here we shall deal with systems of linear equations with constant coefficients only. Such a system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the dependent variables is obtained in a similar manner.

Example 13.38. *Solve the simultaneous equations :*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

being given $x = y = 0$ when $t = 0$. *(S.V.T.U., 2009 ; Kurukshetra, 2005)*

† See footnote on p. 486.

Solution. Taking $d/dt = D$, the given equations become

$$(D+5)x - 2y = t \qquad ...(i)$$

$$2x + (D+1)y = 0 \qquad ...(ii)$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (i) by 2 and operating on (ii) by $D+5$ and then subtracting, we get

$$[-4-(D+5)(D+1)]\,y = 2t \text{ or } (D^2+6D+9)\,y = -2t$$

Its auxiliary equation is $D^2 + 6D + 9 = 0$, *i.e.*, $(D+3)^2 = 0$

whence $\quad D = -3, -3 \quad \therefore \quad \text{C.F.} = (c_1 + c_2 t)\,e^{-3t}$

and $\quad \text{P.I.} = \dfrac{1}{(D+3)^2}(-2t) = -\dfrac{2}{9}\left(1+\dfrac{D}{3}\right)^{-2} t = -\dfrac{2}{9}\left(1-\dfrac{2D}{3}+...\right) t = -\dfrac{2t}{9}+\dfrac{4}{27}$

Hence $\quad y = (c_1 + c_2 t)\,e^{-3t} - \dfrac{2t}{9} + \dfrac{4}{27} \qquad ...(iii)$

Now to find x, either eliminate y from (i) and (ii) and solve the resulting equation or substitute the value of y in (ii). Here, it is more convenient to adopt the latter method.

From (iii), $\quad Dy = c_2 e^{-3t} + (c_1 + c_2 t)(-3)\,e^{-3t} - \dfrac{2}{9}$

$\therefore$ Substituting for y and Dy in (ii), we get

$$x = -\frac{1}{2}[Dy + y] = \left[\left(c_1 - \frac{1}{2}c_2\right) + c_2 t\right]e^{-3t} + \frac{t}{9} + \frac{1}{27} \qquad ...(iv)$$

Hence (iii) and (iv) constitute the solutions of the given equations.

Since $x = y = 0$ when $t = 0$, the equations (iii) and (iv) give

$$0 = c_1 + \frac{4}{27} \text{ and } c_1 - \frac{1}{2}c_2 + \frac{1}{27} = 0 \text{ whence } c_1 = -\frac{4}{27},\ c_2 = -\frac{2}{9}.$$

Hence the desired solutions are

$$x = -\frac{1}{27}(1+6t)\,e^{-3t} + \frac{1}{27}(1+3t),\ y = -\frac{2}{27}(2+3t)\,e^{-3t} + \frac{2}{27}(2-3t).$$

Example 13.39. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 2y + \sin t = 0,\ \dfrac{dy}{dt} - 2x - \cos t = 0$ *given that* $x = 0$ *and* $y = 1$ *when* $t = 0$.

Solution. Given equations are

$$Dx + 2y = -\sin t \quad ...(i); \qquad -2x + Dy = \cos t \qquad ...(ii)$$

Eliminating x by multiplying (i) by 2 and (ii) by D and then adding, we get

$$4y + D^2y = -2\sin t - \sin t \text{ or } (D^2+4)y = -3\sin t$$

Its A.E. is $D = \pm 2i \quad \therefore \quad \text{C.F.} = c_1 \cos 2t + c_2 \sin 2t$

$$\text{P.I.} = -3\,\frac{1}{D^2+4}\sin t = -3\,\frac{1}{-1+4}\sin t = -\sin t$$

$\therefore \quad y = c_1 \cos 2t + c_2 \sin 2t - \sin t \qquad ...(iii)$

and $\quad dy/dt = -2\sin 2t + 2c_2 \cos 2t - \cos t \qquad ...(iv)$

Substituting (iii) in (ii), we get

$$2x = Dy - \cos t = -2c_1 \sin 2t + 2c_2 \cos 2t - 2\cos t$$

or $\quad x = -c_1 \sin 2t + c_2 \cos 2t + -\cos t \qquad ...(v)$

When $t = 0, x = 0, y = 1$, (iii) and (v) give $1 = c_1,\ 0 = c_2 - 1$

Hence $x = \cos 2t - \sin 2t - \cos t,\ y = \cos 2t + \sin 2t - \sin t$.

Example 13.40. *Solve the simultaneous equations*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \qquad \textit{(U.P.T.U., 2001)}$$

Solution. Given equations are

$$Dx + (D-2)y = 2\cos t - 7\sin t \qquad ...(i)$$
$$(D+2)x - Dy = 4\cos t - 3\sin t \qquad ...(ii)$$

Eliminate y by operating on (i) by D and (ii) by $(D-2)$ and then adding, we get

$$D^2x + (D-2)(D+2)x = -2\sin t - 7\cos t + 4(-\sin t - 2\cos t) - 3(\cos t - 2\sin t)$$

or $$2(D^2-2)x = -18\cos t \text{ or } (D^2-2)x = -9\cos t$$

Its A.E. is $$D^2 - 2 = 0 \text{ or } D = \pm\sqrt{2}, \quad \therefore \text{ C.F.} = c_1 e^{\sqrt{st}} + c_2 e^{-\sqrt{2t}}$$

$$\text{P.I.} = (-9)\frac{1}{D^2-2}\cos t = \frac{-9\cos t}{-1-2} = 3\cos t.$$

Hence $x = c_1 e^{\sqrt{2t}} + c_2 e^{-\sqrt{2t}}\ 3\cos t.$

Now substituting this value of x in (ii), we get

$$Dy = (D+2)(c_1 e^{\sqrt{2t}} + c_2 e^{-\sqrt{2t}} + 3\cos t) - 4\cos t + 3\sin t$$
$$= c_1\sqrt{2}e^{\sqrt{2t}} + 2c_1 e^{\sqrt{2t}} + c_2(-\sqrt{2}e^{-2t}) + 2c_2\, e^{-\sqrt{2t}} - 3\sin t + 6\cos t - 4\cos t + 3\sin t$$
$$= (2+\sqrt{2})c_1 e^{\sqrt{2t}} + (2-\sqrt{2})c_2 e^{-\sqrt{2t}} + 2\cos t$$

Hence $$y = (\sqrt{2}+1)c_1 e^{\sqrt{2t}} - (\sqrt{2}-1)c_2 e^{-\sqrt{2t}} + 2\sin t + c_3.$$

Example 13.41. *The small oscillations of a certain system with two degrees of freedom are given by the equations*

$$D^2x + 3x - 2y = 0$$
$$D^2x + D^2y - 3x + 5y = 0$$

where $D = d/dt$. If $x = 0, y = 0, Dx = 3, Dy = 2$ when $t = 0$, find x and y when $t = 1/2$.

Solution. Given equations are $(D^2+3)x - 2y = 0 \qquad ...(i)$

$$(D^2-3)x + (D^2+5)y = 0 \qquad ...(ii)$$

To eliminate x, operate these equations by D^2-3 and D^2+3 respectively and subtract (i) from (ii). Then

$$[(D^2+3)(D^2+5) + 2(D^2-3)]\,y = 0 \quad \text{or} \quad (D^4 + 10D^2 + 9)y = 0$$

Its auxiliary equation is $D^4 + 10D^2 + 9 = 0$ whence $D = \pm i, \pm 3i$

Thus $$y = c_1\cos t + c_2\sin t + c_3\cos 3t + c_4\sin 3t \qquad ...(iii)$$

To find x, we eliminate y from (i) and (ii).

$\therefore$ operating (i) by D^2+5 and multiplying (ii) by 2 and adding, we get

$$(D^4 + 10D^2 + 9)x = 0. \text{ Thus } x = k_1\cos t + k_2\sin t + k_3\cos 3t + k_4\sin 3t \qquad ...(iv)$$

To find the relations between the constants in (iii) and (iv), substitute these values of x and y either of the given equations, say (i). This gives

$$2(k_1-c_1)\cos t + 2(k_2-c_2)\sin t - 2(3k_3+c_3)\cos 3t - 2(3k_4+c_4)\sin 3t = 0$$

which must hold for all values of t.

$\therefore$ Equating to zero the coefficients of $\cos t$, $\sin t$, $\cos 3t$ and $\sin 3t$, we get

$$k_1 = c_1,\ k_2 = c_2,\ k_3 = -c_3/3,\ k_4 = -c_4/3$$

Thus $$x = c_1\cos t + c_2\sin t - \frac{1}{3}(c_3\cos 3t + c_4\sin 3t) \qquad ...(v)$$

Hence (iii) and (iv) constitute the solutions of (i) and (ii).

Since $x = y = 0$, when $t = 0$; $\therefore$ (iii) and (v) give

$$0 = c_1 + c_3 \text{ and } c_1 - \frac{1}{3}c_3 = 0 \text{ i.e. } c_1 = c_3 = 0$$

Thus (iii) and (v) reduce to

and $$\left.\begin{aligned} y &= c_2\sin t + c_4\sin 3t \\ x &= c_2\sin t - \frac{c_4}{3}\sin 3t \end{aligned}\right\} \qquad ...(vi)$$

$\therefore$ $Dx = c_2 \cos t - c_4 \cos 3t$ and $Dy = c_2 \cos t + 3c_4 \cos 3t$.

Since $Dx = 3$ and $Dy = 2$ when $t = 0$

$\therefore$ $3 = c_2 - c_4$ and $2 = c_2 + 3c_4$, whence $c_2 = 11/4$, $c_4 = -\frac{1}{4}$.

Hence equation (*vi*) becomes $x = \frac{1}{4}\left(11 \sin t + \frac{1}{3} \sin 3t\right), y = \frac{1}{4}(11 \sin t - \sin 3t)$ ...(*vii*)

$\therefore$ when $t = 1/2$, $x = \frac{1}{4}\left[11 \sin (0.5) + \frac{1}{3} \sin (1.5)\right] = \frac{1}{4}\left[[11 (0.4794 + \frac{1}{3}(0.9975)\right] = 1.4015$

and $y = \frac{1}{4}[11 \sin (0.5) - \sin (1.5)] = 1.069$.

Example 13.42. *Solve the simultaneous equations:* $\frac{dx}{dt} = 2y, \frac{dy}{dt} = 2z, \frac{dz}{dt} = 2x$.

(*S.V.T.U., 2006 S ; U.P.T.U., 2004*)

Solution. Differentiating first equation w.r.t. t, $\frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 2(2z)$

Again differentiating w.r.t. t, $\frac{d^3x}{dt^2} = 4\frac{dz}{dt} = 4(2x)$ ...(*i*)

or $(D^3 - 8)x = 0$

Its A.E. is $D^3 - 8 = 0$ or $(D - 2)(D^2 + 2D + 4) = 0$

or $D = 2, -1 \pm i\sqrt{3}$

$\therefore$ the solution of (*i*) is $x = c_1 e^{2t} + e^{-t}(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t})$ ...(*ii*)

From the first equation, we have $y = \frac{1}{2}\frac{dx}{dt}$

$$\therefore \quad y = \frac{1}{2}[2c_1 e^{2t} + (-1)e^{-t}(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t}) + e^t(-\sqrt{3}\, c_2 \sin \sqrt{3t} + \sqrt{3}\, c_3 \cos \sqrt{3t})]$$

$$\text{or} \quad y = c_1 e^{2t} + \frac{1}{2}e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin \sqrt{3t}\right\} \quad ...(iii)$$

From the second equation, we have $z = \frac{1}{2}\frac{dy}{dt}$

$$\therefore \quad z = \frac{1}{2}2c_1 e^{2t} + \frac{1}{4}\Big[(-1)e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin \sqrt{3t}\right\}$$
$$+ e^{-t}\left\{\sqrt{3}(c_2 - \sqrt{3}c_3)\sin \sqrt{3t} - \sqrt{3}(c_3 + \sqrt{3}c_2)\cos \sqrt{3t}\right\}\Big]$$

$$= c_1 e^{2t} + \frac{1}{4}e^{-t}\left\{(-2c_2 - 2\sqrt{3}c_3)\cos \sqrt{3t} - (2\sqrt{3}\, c_2 - 2c_3)\sin \sqrt{3t}\right\}$$

$$\text{or} \quad z = c_1 e^{2t} - \frac{1}{2}e^{-t}\left\{(\sqrt{3}\, c_2 - c_3)\sin \sqrt{3t} + (c_2 + \sqrt{3}c_3)\cos \sqrt{3t}\right\} \quad ...(iv)$$

Hence the equations (*ii*), (*iii*) and (*iv*) taken together give the required solution.

PROBLEMS 13.6

Solve the following simultaneous equations :

1. $\frac{dx}{dt} = 5x + y, \frac{dy}{dt} = y - 4x$.
2. $\frac{dx}{dt} + y = \sin t, \frac{dy}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$.

(*Bhopal, 2009 ; J.N.T.U., 2006 ; Kerala, 2005*)

3. $\frac{dx}{dt} + 2x + 3y = 0, 3x + \frac{dy}{dt} + 2y = 2e^{2t}$. (*Delhi, 2002*)
4. $\frac{dx}{dt} - 7x + y = 0, \frac{dy}{dt} - 2x - 5y = 0$.
5. $\frac{dx}{dt} + 2y = e^t, \frac{dy}{dt} - 2x = e^{-t}$. (*Bhopal, 2002 S*)
6. $\frac{dx}{dt} + 2x - 3y = t; \frac{dy}{dt} - 3x + 2y = e^{2t}$. (*Nagpur, 2009*)
7. $(D-1)x + Dy = 2t + 1, (2D+1)x + 2Dy = t$.
8. $(D+1)x + (2D+1)y = e^t, (D-1)x + (D+1)y = 1$.
9. $Dx + Dy + 3x = \sin t, Dx + y - x = \cos t$. (*U.P.T.U., 2003*)
10. $t\frac{dx}{dt} + y = 0, t\frac{dy}{dt} + x = 0$ given $x(1) = 1, y(-1) = 0$.
11. $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t, \frac{dx}{dt} + y - x = \cos t$.
12. $\frac{d^2x}{dt^2} - 3x - 4y = 0, \frac{d^2y}{dt^2} + x + y = 0$. (*U.P.T.U., 2005*)
13. $\frac{d^2x}{dt^2} + y = \sin t, \frac{d^2y}{dt^2} + x = \cos t$. (*U.P.T.U., 2004*)
14. A mechanical system with two degrees of freedom satisfies the equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4; 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$$

Obtain expression for x and y in terms of t, given $x, y, dx/dt, dy/dt$ all vanish at $t = 0$.

13.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 13.7

Fill up the blanks or choose the correct answer in the following problems :

1. The complementary function of $(D^4 - a^4)y = 0$ is ....
2. P.I. of the differential equation $(D^2 + D + 1)y = \sin 2x$ is ....
3. P.I. of $y'' - 3y' + 2y = 12$ is ....
4. The Wronskian of x and e^x is ....
5. The C.F. of $y'' - 2y' + y = xe^x \sin x$ is
   (*a*) $C_1e^x + C_2e^{-x}$ (*b*) $(C_1x + C_2)e^x$ (*c*) $(C_1 + C_2x)e^{-x}$ (*d*) None of these. (*V.T.U., 2010*)
6. The general solution of the differential equation $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is ....
7. The particular integral of $(D^2 + a^2)y = \sin ax$ is
   (*a*) $-\frac{x}{2a}\cos ax$ (*b*) $\frac{x}{2a}\cos ax$ (*c*) $-\frac{ax}{2}\cos ax$ (*d*) $\frac{ax}{2}\cos ax$.
8. The solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$, is ....
9. The solution of the differential equation $y'' + y = 0$ satisfying the conditions $y(0) = 1$ and $y(\pi/2) = 2$, is ....
10. $e^{-x}(c_1 \cos\sqrt{3x} + c_2 \sin\sqrt{3x}) + c_3e^{2x}$ is the general solution of
    (*a*) $d^3y/dx^3 + 4y = 0$ (*b*) $d^3y/dx^3 - 8y = 0$
    (*c*) $d^3y/dx^3 + 8y = 0$ (*d*) $d^3y/dx^3 - 2d^2y/dx^2 + dy/dx - 2 = 0$.
11. The solution of the differential equation $(D^2 + 1)^2 y = 0$ is ....
12. The particular integral of $d^2y/dx^2 + y = \cos h\, 3x$ is ....
13. The solution of $x^2y'' + xy' = 0$ is ....
14. The general solution of $(D^2 - 2)^2, y = 0$ is.......
15. P.I. of $(D+1)^2 y = xe^{-x}$ is ....
16. If $f(D) = D^2 - 2, \frac{1}{f(D)}e^{2x} =$ ....
17. If $f(D) = D^2 + 5, \frac{1}{f(D)}\sin 2x =$ ....
18. The particular integral of $(D+1)^2 y = e^{-x}$ is ....
19. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is ....

20. P.I. of $(D^2 + 4)y = \cos 2x$ is

(a) $\frac{1}{2} \sin 2x$ (b) $\frac{1}{2} x \sin 2x$ (c) $\frac{1}{4} x \sin 2x$ (d) $\frac{1}{2} x \cos 2x$. (Bhopal, 2008)

21. By the method of undetermined coefficients y_p of $y'' + 3y' + 2y = 12x^2$ is of the form

(a) $a + bx + cx^2$ (b) $a + bx$ (c) $ax + bx^2 + cx^3$ (d) None of these. (V.T.U., 2010)

22. In the equation $\frac{dx}{dt} + y = \sin t + 1$, $\frac{dy}{dt} + x = \cos t$ if $y = \sin t + 1 + e^{-t}$, then $x =$ ....

23. $(x^2D^2 + xD + 7)y = 2/x$ converted to a linear differential equation with constant coefficients is ....

24. P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ is

(a) $\frac{x^2}{3} + 4x$ (b) $\frac{x^3}{3} + 4$ (c) $\frac{x^3}{3} + 4x$ (d) $\frac{x^3}{3} + 4x^2$.

25. The solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ is given by

(a) $y = C_1e^{x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (b) $y = C_1e^{-x} + C_2e^{-2x} + \frac{1}{2}e^{3x}$

(c) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (d) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{-3x}$.

26. The particular integral of the differential equation $(D^3 - D)y = e^x + e^{-x}$, $D = \frac{d}{dx}$ is

(a) $\frac{1}{2}(e^x + e^{-x})$ (b) $\frac{1}{2}x(e^x + e^{-x})$ (c) $\frac{1}{2} x^2 (e^x + e^{-x})$ (d) $\frac{1}{2} x^2 (e^x - e^{-x})$.

27. The complementary function of the differential equation $x^2y'' - xy' + y = \log x$ is ....

28. The homogeneous linear differential equation whose auxiliary equation has roots $1, -1$ is ......

29. The particular integral of $(D^2 - 6D + 9)y = \log 2$ is .... (V.T.U., 2011)

30. To transform $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}$ into a linear differential equation with constant coefficients, put $x =$ ....

31. The particular integral of $(D^2 - 4)y = \sin 3x$ is

(a) 1/4 (b) – 1/13 (c) 1/5 (d) None of these. (V.T.U., 2010)

32. The solution of $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$ is ....

33. The differential equation whose auxiliary equation has the roots $0, -1, -1$ is ....

34. Complementary function of $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 2x \log x$ is

(a) $(C_1 + C_2 x)e^x$ (b) $(C_1 + C_2 \log x)x$ (d) $(C_1 + C_2 x) \log x$ (d) $(C_1 + C_2 \log x)e^x$. (Bhopal, 2008)

35. The general solution of $(D^2 - D - 2)x = 0$ is $x = c_1e^t + c_2e^{-2t}$ (True or False)

36. $\frac{1}{f(D)}(x^2e^{ax}) = \frac{1}{f(D+a)}(e^{ax}x^2)$. (True or False)