12·9·2. Test of Significance for Difference of Proportions. Suppose we want to compare two distinct populations with respect to the prevalence of a certain attribute, say A, among their members. Let X_1, X_2 be the number of persons possessing the given attribute A in random samples of sizes n_1 and n_2 from the two populations respectively. Then sample proportions are given by

$$p_1 = X_1/n_1 \quad \text{and} \quad p_2 = X_2/n_2$$

If P_1 and P_2 are the population proportions, then

$$E(p_1) = P_1, \; E(p_2) = P_2 \qquad \text{[c.f. Equation (12·4a)]}$$

and

$$V(p_1) = \frac{P_1 Q_1}{n_1} \quad \text{and} \quad V(p_2) = \frac{P_2 Q_2}{n_2}$$

Since for large samples, p_1 and p_2 are asymptotically normally distributed, $(p_1 - p_2)$ is also normally dstributed. Then the standard variable corresponding to the difference $(p_1 - p_2)$ is given by

$$Z = \frac{(p_1 - p_2) - E(p_1 - p_2)}{\sqrt{V(p_1 - p_2)}} \sim N(0, 1)$$

Under the *null hypothesis* $H_0 : P_1 = P_2$, *i.e.*, there is no significant difference between the sample proportions, we have

$$E(p_1 - p_2) = E(p_1) - E(p_2) = P_1 - P_2 = 0 \qquad \text{(Under } H_0\text{)}$$

Also $\quad V(p_1 - p_2) = V(p_1) + V(p_2)$,

the covariance term $\text{Cov}(p_1, p_2)$ vanishes, since sample proportions are independent.

$$\therefore \qquad V(p_1 - p_2) = \frac{P_1 Q_1}{n_1} + \frac{P_2 Q_2}{n_2} = PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right),$$

since under $H_0 : P_1 = P_2 = P$, (say), and $Q_1 = Q_2 = Q$.

Hence under $H_0 : P_1 = P_2$, the test statistic for the difference of proportions becomes

$$Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1) \qquad \ldots(12{\cdot}5)$$

In general, we do not have any information as to the proportion of A's in the populations from which the samples have been taken. Under $H_0 : P_1 = P_2 = P$, (say), an unbiased estimate of the population proportion P, based on both the samples is given by

$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2} \qquad \ldots(12{\cdot}5a)$$

The estimate is unbiased, since

$$E(\hat{P}) = \frac{1}{n_1 + n_2} E\left[n_1 p_1 + n_2 p_2\right] = \frac{1}{n_1 + n_2}\left[n_1 E(p_1) + n_2 E(p_2)\right]$$

$$= \frac{1}{n_1 + n_2}\left[n_1 P_1 + n_2 P_2\right] = P \qquad [\because P_1 = P_2 = P, \text{ under } H_0]$$

Thus (12·5) along with (12·5a) gives the required test statistic.

Example 12·6. *Random samples of 400 men and 600 women were asked whether they would like to have a flyover near their residence. 200 men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and women in favour of the proposal, are same against that they are not, at 5% level.*

Solution. *Null Hypothesis* $H_0 : P_1 = P_2 = P$, (say), *i.e.*, there is no significant difference between the opinion of men and women as far as proposal of flyover is concerned.

Alternative Hypothesis, $H_1 : P_1 \neq P_2$ (two-tailed).

We are given :

$n_1 = 400$, X_1 = Number of men favouring the proposal = 200

$n_2 = 600$, X_2 = Number of women favouring the proposal = 325

$\therefore$ p_1 = Proportion of men favouring the proposal in the sample

$$= \frac{X_1}{n_1} = \frac{200}{400} = 0{\cdot}5$$

p_2 = Proportion of women favouring the proposal in the sample

$$= \frac{X_2}{n_2} = \frac{325}{600} = 0{\cdot}541$$

Test Statistic. Since samples are large, the test statistic under the Null Hypothesis, H_0 is :

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

where $\hat{P} = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \dfrac{X_1 + X_2}{n_1 + n_2} = \dfrac{200 + 325}{400 + 600} = \dfrac{525}{1000} = 0{\cdot}525$

$\Rightarrow \quad \hat{Q} = 1 - \hat{P} = 1 - 0{\cdot}525 = 0{\cdot}475$

$$\therefore \quad Z = \frac{0{\cdot}500 - 0{\cdot}541}{\sqrt{0{\cdot}525 \times 0{\cdot}475 \times \left(\frac{1}{400} + \frac{1}{600}\right)}}$$

$$= \frac{-0{\cdot}041}{\sqrt{0{\cdot}525 \times 0{\cdot}475 \times (10/2{,}400)}}$$

$$= \frac{-0{\cdot}041}{\sqrt{0{\cdot}001039}} = \frac{-0{\cdot}041}{0{\cdot}0323} = -1{\cdot}269$$

Conclusion. Since $|Z| = 1{\cdot}269$ which is less than $1{\cdot}96$, it is not significant at 5% level of significance. Hence H_0 may be accepted at 5% level of significance and we may conclude that men and women do not differ significantly as regards proposal of flyover is concerned.

Example 12·7. *A company has the head office at Calcutta and a branch at Bombay. The personnel director wanted to know if the workers at the two places would like the introduction of a new plan of work and a survey was conducted for this purpose. Out of a sample of 500 workers at Calcutta, 62% favoured the new plan. At Bombay out of a sample of 400 workers, 41% were against the new plan. Is there any significant difference between the two groups in their attitude towards the new plan at 5% level ?*

Solution. In the usual notations, we are given :

$$n_1 = 500,\ p_1 = 0{\cdot}62 \text{ and } n_2 = 400,\ p_2 = 1 - 0{\cdot}41 = 0{\cdot}59$$

Null hypothesis, $H_0 : P_1 = P_2$, *i.e.*, there is no significant difference between the two groups in their attitude towards the new plan.

Alternative hypothesis, $H_1 : P_1 \neq P_2$ (Two-tailed).

Test Statistic. Under H_0, the test statistic for large samples is :

$$Z = \frac{p_1 - p_2}{\text{S.E. } (p_1 - p_2)} = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

where $$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{500 \times 0{\cdot}62 + 400 \times 0{\cdot}59}{500 + 400} = 0{\cdot}607$$

and $$\hat{Q} = 1 - \hat{P} = 0{\cdot}393$$

$\therefore$ $$Z = \frac{0{\cdot}62 - 0{\cdot}59}{\sqrt{0{\cdot}607 \times 0{\cdot}393 \times \left(\frac{1}{500} + \frac{1}{400}\right)}}$$

$$= \frac{0{\cdot}03}{\sqrt{0{\cdot}00107}} = \frac{0{\cdot}03}{0{\cdot}0327} = 0{\cdot}917.$$

Critical region. At 5% level of significance, the critical value of Z for a two-tailed test is 1·96. Thus the critical region consists of all values of $Z \geq 1{\cdot}96$ or $Z \leq -1{\cdot}96$.

Conclusion. Since the calculated value of $|Z| = 0{\cdot}917$ is less than the critical value of Z (1·96), it is not significant at 5% level of significance. Hence the data do not provide us any evidence against the null hypothesis which may be accepted, and we conclude that there is no significant difference between the two groups in their attitude towards the new plan.

Example 12·8. *Before an increase in excise duty on tea, 800 persons out of a sample of 1,000 persons were found to be tea drinkers. After an increase in duty, 800 people were tea drinkers in a sample of 1,200 people. Using standard error of proportion, state whether there is a significant decrease in the consumption of tea after the increase in excise duty ?*

Solution. In the usual notations, we have $n_1 = 1{,}000$; $n_2 = 1{,}200$

p_1 = Sample proportion of tea drinkers before increase in excise duty

$$= \frac{800}{1000} = 0{\cdot}80$$

p_2 = Sample proportion of tea drinkers after increase in excise duty

$$= \frac{800}{1200} = 0{\cdot}67$$

Null Hypothesis, $H_0 : P_1 = P_2$, *i.e.,* there is no significant difference in the consumption of tea before and after the increase in excise duty.

Alternative Hypothesis, $H_1 : P_1 > P_2$ (Right-tailed alternative).

Test Statistic. Under the null hypothesis, the test statistic is

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1) \qquad \text{(Since samples are large)}$$

where

$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{800 + 800}{1000 + 1200} = \frac{16}{22}, \text{ and } \hat{Q} = 1 - \hat{P} = \frac{6}{22}$$

$$\therefore \quad Z = \frac{0{\cdot}80 - 0{\cdot}67}{\sqrt{\frac{16}{22} \times \frac{6}{22} \times \left(\frac{1}{1000} + \frac{1}{1200}\right)}}$$

$$= \frac{0{\cdot}13}{\sqrt{\frac{16}{22} \times \frac{6}{22} \times \frac{11}{6000}}} = \frac{0{\cdot}13}{0{\cdot}019} = 6{\cdot}842$$

Conclusion. Since Z s much greater than 1·645 as well as 2·33 (since test is one-tailed), it is highly significant at both 5% and 1% levels of significance. Hence,
we reject the null hypothesis H_0 and conclude that there is a significant decease in the consumption of tea after increase in the excise duty.

Example 12·10. *On the basis of their total scores, 200 candidates of a civil service examination are divided into two groups, the upper 30 per cent and the remaining 70 per cent. Consider the first question of this examination. Among the first group, 40 had the correct answer, whereas among the second group, 80 had the correct answer. On the basis of these results, can one conclude that the first question is no good at discriminating ability of the type being examined here?*

Solution. Here, we have

n = Total number of candidates = 200

n_1 = The number of candidates in the upper 30% group

$= \frac{30}{100} \times 200 = 60$

n_2 = The number of candidates in the remaining 70% group

$= \frac{70}{100} \times 200 = 140$

X_1 = The number of candidates, with correct answer in the first group = 40

X_2 = The number of candidates, with correct answer in the second group = 80

$$\therefore \quad p_1 = \frac{X_1}{n_1} = \frac{40}{60} = 0{\cdot}6666 \text{ and } p_2 = \frac{X_2}{n_2} = \frac{80}{140} = 0{\cdot}5714$$

Null Hypothesis, H_0 : There is no significant difference in the sample proportions, *i.e.,* $P_1 = P_2$, *i.e.,* the first question is no good at dicriminating the ability of the type being examined here.

Alternative Hypothesis, $H_1 : P_1 \neq P_2$.

Test Statistic. Under H_0 the test statistic is :

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{P}\hat{Q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1) \quad \text{(since samples are large).}$$

where

$$\hat{P} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{40 + 80}{60 + 140} = 0{\cdot}6, \quad \hat{Q} = 1 - \hat{P} = 0{\cdot}4$$

$$\therefore \qquad Z = \frac{0{\cdot}6666 - 0{\cdot}5714}{\sqrt{0{\cdot}6 \times 0{\cdot}4 \left(\frac{1}{60} + \frac{1}{140}\right)}} = \frac{0{\cdot}0953}{0{\cdot}0756} = 1{\cdot}258$$

Conclusion. Since $|Z| < 1{\cdot}96$, the data are consistent with the null hypothesis at 5% level of significance. Hence we conclude that the first question is not good enough to distinguish between the ability of the two groups of candidates.

12·13. Test of Significance for Single Mean. We have proved that if x_i, $(i = 1, 2, \ldots, n)$ is a random sample of size n from a normal population with mean μ and variance σ^2, then the sample mean is distributed normally with mean μ and variance σ^2/n, *i.e.*, $\bar{x} \sim N(\mu, \sigma^2/n)$. However, this result holds, *i.e.*, $\bar{x} \sim N(\mu, \sigma^2/n)$, even in random sampling from non-normal population provided the sample size n is large [*c.f.* Central Limit Theorem, § 8·10].

Thus for large samples, the *standard normal variate* corresponding to $\bar{x}$ is :

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Under the *null hypothesis*, H_0 that the sample has been drawn from a population with mean μ and variance σ^2, *i.e.*, there is no significant difference between the sample mean ($\bar{x}$) and population mean (μ), the test statistic (for large samples), is :

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \qquad \ldots(12\cdot9a)$$

Remarks 1. If the population s.d. σ is unknown then we use its estimate provided by the sample variance given by [See (12·8*b*)]:

$$\hat{\sigma}^2 = s^2 \quad \Rightarrow \quad \hat{\sigma} = s \text{ (for large samples).}$$

2. Confidence limits for μ. 95% confidence interval for μ is given by :

$$|Z| \leq 1{\cdot}96, \text{ i.e., } \left| \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right| \leq 1{\cdot}96$$

$$\Rightarrow \qquad \bar{x} - 1{\cdot}96\sigma/\sqrt{n} \leq \mu \leq \bar{x} + 1{\cdot}96\sigma/\sqrt{n} \qquad \ldots(12{\cdot}10)$$

and $\bar{x} \pm 1{\cdot}96\sigma/\sqrt{n}$ are known as 95% confidence limits for μ. Similarly, 99% confidence limits for μ are $\bar{x} \pm 2{\cdot}58\sigma/\sqrt{n}$ and 98% confidence limits for μ are $\bar{x} \pm 2{\cdot}33\sigma/\sqrt{n}$.

However, in sampling from a finite population of size N, the corresponding 95% and 99% confidence limits for μ are respectively

$$\bar{x} \pm 1{\cdot}96 \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} \text{ and } \bar{x} \pm 2{\cdot}58 \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} \ldots(12{\cdot}10a)$$

3. The confidence limits for any parameter (P, μ, etc.) are also known as its *fiducial limits.*

Example 12·15. *A sample of 900 members has a mean 3·4 cms,. and s.d. 2·61 cms. Is the sample from a large population of mean 3·25 cms. and s.d. 2·61 cms. ?*

If the population is normal and its mean is unknown, find the 95% and 98% fiducial limits of true mean.

Solution. *Null hypothesis,* (H_0) : The sample has been drawn from the population with mean $\mu = 3{\cdot}25$ cms., and S.D. $\sigma = 2{\cdot}61$ cms.

Alternative Hypothesis, $H_1 : \mu \neq 3{\cdot}25$ (Two-tailed).

Test Statistic. Under H_0, the test statistic is :

Here, we are given

$$\bar{x} = 3{\cdot}4 \text{ cms.}, \; n = 900 \text{ cms.}, \; \mu = 3{\cdot}25 \text{ cms. and } \sigma = 2{\cdot}61 \text{ cms.}$$

$$Z = \frac{3{\cdot}40 - 3{\cdot}25}{2{\cdot}61/\sqrt{900}} = \frac{0{\cdot}15 \times 30}{2{\cdot}61} = 1{\cdot}73$$

Here,

Since $|Z| < 1{\cdot}96$, we conclude that the data don't provide us any evidence against the null hypothesis (H_0) which may, therefore, be accepted at 5% level of significance.

95% *fiducial limits* for the population mean μ are :

Since

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$$\bar{x} \pm 1{\cdot}96\, \sigma/\sqrt{n} \;\Rightarrow\; 3{\cdot}40 \pm 1{\cdot}96 \times 2{\cdot}61/\sqrt{900}$$

$$\Rightarrow \quad 3{\cdot}40 \pm 0{\cdot}1705, \; \textit{i.e.,} \; 3{\cdot}5705 \text{ and } 3{\cdot}2295$$

98% *fiducial limits* for μ are given by :

$$\bar{x} \pm 2{\cdot}33 \frac{\sigma}{\sqrt{n}}, \; \textit{i.e.,} \; 3{\cdot}40 \pm 2{\cdot}33 \times \frac{2{\cdot}61}{30}$$

$$\Rightarrow \quad 3{\cdot}40 \pm 0{\cdot}2027 \; \textit{i.e.,} \; 3{\cdot}6027 \text{ and } 3{\cdot}1973$$

$\Rightarrow$

98% j

Remark. 2·33 is the value z_1 of Z from standard normal probability integrals, such that $P(|Z| > z_1) = 0{\cdot}98 \;\Rightarrow\; P(Z > z_1) = 0{\cdot}49$.

$$\bar{x} \pm 2{\cdot}33 \frac{\sigma}{\sqrt{n}}, \; \textit{i.e.,} \; 3{\cdot}40 \pm 2{\cdot}33 \times \frac{2{\cdot}61}{30}$$

$$\Rightarrow \quad 3{\cdot}40 \pm 0{\cdot}2027 \; \textit{i.e.,} \; 3{\cdot}6027 \text{ and } 3{\cdot}1973$$

Remark. 2·33 is the value z_1 of Z from standard normal probability integrals, such that $P(|Z| > z_1) = 0{\cdot}98 \;\Rightarrow\; P(Z > z_1) = 0{\cdot}49$.

Ex (2) A sample of 100 students is taken from a large population. The mean height of the student in this sample is 160 cm. Can it be reasonably regarded that, in the population, the mean height is 165 cm. and the S.D is 10 cm?

Null hypothesis : H_0: The sample has been drawn from population with mean height 165 cm, i.e $\mu = 165$ cms

Alternative hypothesis

$H_1 : \mu \neq 165$ cms (Two tailed)

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \cancel{165}\ \frac{160-165}{\frac{10}{\sqrt{100}}} = -5$$

$Z = -5$

Now $|Z| = 5 > 1.96$

Thus the null hypothesis is rejected at 5% level of significance.

So it is not correct to assume that the sample has been drawn from a population with mean height 165 cms.

Ex (3) The mean breaking strength of the cables supplied by a manufacturer is 1800 with a S.D of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. In order to test this claim, a sample of 50 cables is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1% level of significance.

Sol: $\bar{x} = 1850, n = 50, \mu = 1800, \sigma = 100$

Null hypothesis $H_0 : \bar{x} = \mu$

Alternate hypothesis $H_1 : \bar{x} > \mu$ (Right tailed)

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{1850 - 1800}{100/\sqrt{50}} = 3.54$$

Here $|Z| > 2.33$

Null hypothesis is rejected and alternate hypothesis is accepted

So we conclude that the data suggest that there is increase in breaking strength