Lecture 12

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Linear Recurrence relations with Constant Coefficients

A linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = R(n)$$ Difference Eqn

where $c_0, c_1, c_2, \ldots, c_k$ are real numbers and $c_0, c_k \neq 0$

Linear $\rightarrow$ $a_n, a_{n-1}, \ldots a_{n-k}$ $\rightarrow$ degree 1 a_n^2 ✗ $a_n a_{n-1}$ ✗

Constant Coeff $\rightarrow$ $C_0, C_1, \ldots C_k$ $\rightarrow$ real no.

Degree = Highest subscript − Lowest subscript
$= n-(n-k) = k$

Homogeneous $\rightarrow$ $R(n)=0$ No term without sequence.

Non-homo $\rightarrow$ $R(n) \neq 0$

Classify the following R.R. as Linear/Non-linear, Homo/Non-homo, Constant/Variable. If it is linear, find degree.

① $a_n = a_{n-1} + 2a_{n-3}$ Linear, Homogeneous, Constant Coefficient

$a_n - a_{n-1} - 2a_{n-3} = 0$

Degree $= n-(n-3) = 3$

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② $a_n = \frac{a_{n-1}}{n}$ Homogeneous, Linear, Degree $= 1$
Variable Coefficient

③ $1 \cdot a_n = 1 \cdot a_{n-1}^2$ Homogeneous, Non-linear, Constant Coeff.

④ $1 \cdot a_n^1 = 1 \cdot a_{n-1}^1 + 1 \cdot a_{n-2}^1 + n+3$

Linear, Non-homogeneous, Constant Coefficient
Degree $= n-(n-2) = 2$

$a_n \quad a_{n-1}^2 \quad e^{a_n}$, n^2, e^n, $(n-1)^3$

⑤ $a_n^1 = 2a_{n-1}^1 + n a_{n-4}^1 + 3$

(A) ~~Non~~-hom Vari Linear dg=4

(B) Non-homo Vari Linear

(C) Non homo Const Linear

(D) Non-homo Vari Non-linear

$a_n - 2a_{n-1} - 0a_{n-2} - 0a_{n-3} - na_{n-4} = 3$

Linear Homogeneous Recurrence relations with Constant Coefficients

Linear Recurrence relations with constant coefficients with constant coefficients

Homogeneous　　　　Non-homogeneous.

Shift Operator: E

$$E(a_n) = a_{n+1}, \quad E^2(a_n) = a_{n+2}$$

$$E(a_{n-1}) = a_n, \quad E^k(a_{n-k}) = a_n$$

Forward Difference Operator:

Δ

$$\Delta(a_n) = a_{n+1} - a_n$$

$$\Delta(a_n) = E(a_n) - a_n$$

$$\Delta(a_n) = (E-1)a_n$$

$$\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta$$

Linear Homogeneous R.R. with constant coefficient

$$C_0 a_n + C_1 a_{n-1} + \dots + C_k a_{n-k} = 0$$

$C_0, C_1, \dots, C_k \to$ real no., $\quad C_0, C_k \neq 0$

Degree $= k$

Degree = k

$$\alpha_0 a_{n+k} + \alpha_1 a_{n+k-1} + \dots + \alpha_k a_n = 0$$

Rules to solve

1. Make least index as n.
2. Represent R.R. in the form of shift operator E.
3. Find auxilliary roots.
4. Case I: If roots are real and distinct.

$$r_1, r_2, \dots, r_k$$

$$a_n = \beta_1 (r_1)^n + \beta_2 (r_2)^n + \dots + \beta_k (r_k)^n$$

Case II. If roots are real and repeated

$$r_1, r_2, r_2, r_3, r_3, r_3, \dots r_{k-6}$$

$$a_n = \beta_1 r_1^n + (\beta_2 + \beta_3 n) r_2^n + (\beta_4 + \beta_5 n + \beta_6 n^2) r_3^n + \dots \beta_k (r_{k-6})^n$$

Case III If roots are complex.

$$a \pm ib =$$

$$a_n = R^n (\beta \cos n\theta + \beta_2 \sin n\theta)$$

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$$R = \sqrt{a^2+b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Q10. Solve $y_{n+2} + y_{n+1} - 6y_n = 0, \;\; y_0 = -1, y_1 = 8$

Degree $= (n+2) - n = 2$

$$E^2(y_n) + E(y_n) - 6y_n = 0 \qquad (E^2+E-6)y_n = 0$$

$a=1 \quad b=1 \quad c=-6$

$$m^2 + m - 6 = 0, \quad m = -3, 2$$

$$m^2 + 3m - 2m - 6 = 0 \qquad (m+3)(m-2) = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y_n = C_1(-3)^n + C_2(2)^n$$

$$y_0 = -1 \xrightarrow{\text{Put } n=0} C_1 + C_2 = -1$$

$$y_1 = 8 \xrightarrow{\text{Put } n=1} -3C_1 + 2C_2 = 8$$

$$C_1 = -2, \quad C_2 = 1$$

$$y_n = -2(-3)^n + (2)^n$$

Try solving the recurrence relation of Tower of Hanoi problem using its initial conditions.

Q13. Solve $y_{n+3} - 2y_{n+2} - 5y_{n+1} + 6y_n = 0$

Q14. Solve $y_{n+3} + 3y_{n+2} + 3y_{n+1} + y_n = 0, \quad y_0 = 1, y_1 = -2, y_2 = -1$

Q11. Solve $y_n - 6y_{n-1} + 9y_{n-2} = 0, \; y_0 = 1, y_1 = 6$

Q12. Solve $y_n + 2y_{n-1} + 4y_{n-2} = 0$