Exercise 15.4

In problems **1** to **8**, evaluate the line integral $\int_C f(x, y)\, ds$ or $\int_C f(x, y, z)\, ds$.

1. $f(x, y, z) = 2x + 3y$, C is given by $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 3$.
2. $f(x, y, z) = xy^2 z$, C is the line segment from $(1, 2, 2)$ to $(2, 3, 5)$.
3. $f(x, y) = x^2 - y^2$, C is the closed curve $x = 3 \cos t$, $y = 3 \sin t$, $0 \le t \le 2\pi$.
4. $f(x, y, z) = 3x + 2z$, C is the parabola $y = z^2$, $x = 1$ for $0 \le z \le 4$.
5. $f(x, y, z) = x + 2y + z$, C is given by $x = 5 \cos t$, $y = 3$, $z = 5 \sin t$, $0 \le t \le \pi$.
6. $f(x, y, z) = y + z$, C is given by $x = t^2$, $y = t$, $z = 2$, $0 \le t \le 3$.
7. $f(x, y, z) = x + y^2 + yz$, C is the curve $y = 2x$, $z = 2$ from $(1, 2, 2)$ to $(3, 6, 2)$.
8. $f(x, y, z) = x^2 + y + z^2$, C is the curve $2x = y = z$ for $2 \le x \le 4$.

In problems **9** to **15**, evaluate the line integral $\int_C \mathbf{v} \cdot d\mathbf{r}$.

9. $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, C is the line segment from $(1, 2, 2)$ to $(3, 6, 6)$.
10. $\mathbf{v} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$, C is the circle $x^2 + y^2 = 9$, $z = 0$ going around once in the anti-clockwise direction.
11. $\mathbf{v} = x\mathbf{i} + (\sin y)\mathbf{j} + \mathbf{k}$, C is given by $x = t^2$, $y = t$, $z = 2t$, $0 \le t \le 1$.
12. $\mathbf{v} = x^2 y\,\mathbf{i} - xy^2\mathbf{j}$, $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 3$.
13. $\mathbf{v} = e^x\mathbf{i} + xe^{xy}\mathbf{j} + \mathbf{k}$, $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 2$.
14. $\mathbf{v} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 2\pi$.
15. $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j} + xyz^2\,\mathbf{k}$, C is the circle $x^2 + y^2 - 2y = 2$, $z = 1$ going around once in the anti-clockwise direction.

In problems **16** to **18**, evaluate the line integrals $\int_C f(x, y)\, dx$ and $\int_C f(x, y)\, dy$

16. $f(x, y) = xy$, $x = 3 \cos t$, $y = 3 \sin t$, $0 \le t \le \pi/4$.
17. $f(x, y) = x^2 + 2x^2 y + 3y^2$, $x = t$, $y = 2t^2$, $0 \le t \le 2$
18. $f(x, y) = x + y$, $x = 2 \cos t$, $y = 3 \sin t$, $0 \le t \le \pi/2$.

In problems **19** to **21**, evaluate the line integrals $\int_C f(x, y, z)dx$, $\int_C f(x, y, z)dy$ and $\int_C f(x, y, z)\,dz$.

19. $f(x, y, z) = xyz$, $x = 2\cos t$, $y = 2\sin t$, $z = t$, $0 \le t \le \pi/2$.

20. $f(x, y, z) = x + y + z$, $x = t$, $y = t$, $z = t^2$, $1 \le t \le 2$.

21. $f(x, y, z) = x^2 - y - z$, $x = t^2$, $y = t$, $z = 2t$, $0 \le t \le 1$.

In problems **22** to **27**, evaluate the line integrals $\int_C f(x, y)\,dx + g(x, y)dy$.

22. $f(x, y) = y$, $g(x, y) = x$, C is $y = 2x^2$ from (0, 0) to (2, 8).

23. $f(x, y) = xy$, $g(x, y) = x + 2y$, C is (i) $y = x - 1$, (ii) $x = y^2 + 1$ from (1, 0) to (2, 1).

24. $f(x, y) = y$, $g(x, y) = x^2$, C is the line segment from (1, 1) to (3, 5).

25. $f(x, y) = 2xy^2$, $g(x, y) = 2x + 3y$, C is the parabola $x = y^2$ from (1, – 1) to (4, 2).

26. $f(x, y) = y$, $g(x, y) = -2x$, C is given by $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le \pi$.

27. $f(x, y) = (x + y)/(x^2 + y^2)$, $g(x, y) = -(x - y)/(x^2 + y^2)$, C is the circle $x^2 + y^2 = 9$ going around once in the anti-clockwise direction.

In problems **28** to **30**, evaluate the line integrals $\int_C f dx + g\,dy + h dz$.

28. $f = 2x + y$, $g = y^2$, $h = (x + z)$, C is $y^2 = 2x$, $z = x$, $0 \le x \le 2$.

29. $f = z$, $g = x$, $h = y$, C consists of line segments from (0, 0, 0) to (1, 2, 3) and then from (1, 2, 3) to (3, 4, 5).

30. $f = x/(yz)$, $g = y/(xz)$, $h = z/(xy)$, C is given by $x = \cos t$, $y = \sin t$, $z = \cos t$, $\pi/6 \le t \le \pi/3$.

31. Let C be a smooth directed plane curve. Let **T** and **n** denote the unit tangent and unit normal fields respectively to C, satisfying $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. If $\mathbf{u} = f\,\mathbf{i} + g\,\mathbf{j}$ and $\mathbf{v} = -g\,\mathbf{i} + f\mathbf{j}$, show that $\int_C \mathbf{u} \cdot \mathbf{n}\,ds = \int_C \mathbf{v} \cdot \mathbf{T}\,ds$.

In problems **32** to **36**, find the work done by the force **F** in moving a particle from a point P to the point Q.

32. $\mathbf{F} = x^2\,\mathbf{i} + yz\,\mathbf{j} + z\,\mathbf{k}$, C is the line segment from (1, 2, 2) to (3, 4, 2).

33. $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$, C is the curve with parametric representation $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $1 \le t \le 2$.

34. $\mathbf{F} = (2x + y)\,\mathbf{i} + (4y - x)\,\mathbf{j}$, C is taken once round the triangle with vertices at (2, 2), (4, 2) and (4, 4) taken counter clockwise.

35. $\mathbf{F} = 16y\,\mathbf{i} + (3x^2 + 2)\,\mathbf{j}$, C is the right half of the ellipse $x^2 + a^2y^2 = a^2$ from (0, 1) to (0, –1).

36. $\mathbf{F} = (2x - y - z)\,\mathbf{i} + (x + y - z^2)\,\mathbf{j} + (3x - 2y + 4z)\,\mathbf{k}$, C is taken once round the circle $x^2 + y^2 = 16$ in the x-y plane, in the anti-clockwise direction.

37. Find the circulation of $\mathbf{F} = e^x[(\sin y)\,\mathbf{i} + (\cos y)\,\mathbf{j}]$ around the curve C, where C is the rectangle with vertices (0, 0), (2, 0), (2, $\pi/2$) and (0, $\pi/2$), taken in that order.

Show that in the problems **38** to **45**, the line integrals are independent of path of integration Evaluate the integrals.

38. $$\int_P^Q 2xy^2\,dx + (2x^2y + 1)\,dy,\quad P: (-1, 2),\ Q: (2, 3).$$

39. $$\int_P^Q (1 - \sin x \sin y)dx + (1 + \cos x \cos y)\,dy,\quad P: (\pi/4, \pi/4),\ Q: (\pi/2, 0).$$

40. $\displaystyle\int_P^Q \frac{-x\,dy + y\,dx}{x^2}$, $P:(1, 1)$, $Q:(3, 4)$ on any path not crossing the y-axis.

41. $\displaystyle\int_P^Q (y^3 + 2xy^2)dx + (3xy^2 + 2x^2y + 1)\,dy$, $P:(-2, 1)$, $Q:(1, 2)$,

42. $\displaystyle\int_P^Q e^{yz}\,[\sin y\,dx + (xz \sin y + x \cos y)dy + xy \sin y\,dz]$, $P:(1, \pi/4, 2)$, $Q:(2, \pi/2, 4)$.

43. $\displaystyle\int_P^Q (3x^2 + 2xyz)\,dx + (1 + x^2z)\,dy + x^2y\,dz$, $P:(1, 1, 1)$, $Q:(-2, -3, -4)$.

44. $\displaystyle\int_P^Q (2xz + y)\,dx + (x + z)\,dy + (x^2 + y)dz$, $P:(-1, 2, 3)$, $Q:(2, 2, 4)$.

45. $\displaystyle\int_P^Q yz \cos(x + y - z)\,(dx + dy - dz) + \sin(x + y - z)\,(z\,dy + y\,dz)$, $P:(\pi/2, 1, 1)$, $Q:(\pi, 2, 1)$.

In problems **46** to **50**, find whether the given vector field **F** is conservative (gradient field). If it is, find the potential function.

46. $\mathbf{F} = \cosh(x + y)\,(\mathbf{i} + \mathbf{j})$.

47. $\mathbf{F} = (2x + ye^{xy})\,\mathbf{i} + (2y + xe^{xy})\,\mathbf{j}$.

48. $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

49. $\mathbf{F} = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\mathbf{k}$.

50. $\mathbf{F} = (y^3 - 3x^2z)\,\mathbf{i} + 3xy^2\mathbf{j} - x^3\mathbf{k}$.

51. Let C be a positively oriented simple closed path enclosing a simply connected region R. Use Green's theorem to show that

$$\text{area of the region} = R = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

In problems **52** to **56**, verify Green's theorem by evaluating both integrals.

52. $\displaystyle\oint_C (x + y)dx + x^2dy$, C is the triangle with vertices at $(0, 0)$, $(2, 0)$ and $(2, 4)$, taken in that order.

53. $\displaystyle\oint_C (xy^2 - 2xy)dx + (x^2y + 3)\,dy$, C is the rectangle with vertices at $(-1, 0)$, $(1, 0)$, $(1, 1)$ and $(-1, 1)$, taken in that order.

54. $\displaystyle\oint_C x^3dy - y^3dx$, C is the circle $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$.

55. $\displaystyle\oint_C (xy^2 + 2xy)dx + x^2y\,dy$, C is the boundary of the region enclosing $y^2 = 4x$, $x = 3$.

56. $\displaystyle\oint_C (x - y)\,dx + 3xy\,dy$, C is the boundary of the region enclosing $x^2 = 4y$ and $y^2 = 4x$.

In problems **57** to **65**, evaluate the line integral using the Green's theorem. The orientation of C is anti-clockwise unless otherwise stated.

57. $\displaystyle\oint_C 3x^2y\,dx - 2xy^2\,dy$, C is the boundary of the region $x^2 + y^2 \le 16$, $x \ge 0$, $y \ge 0$.

58. $\oint_C y^2 dx + x^2 dy$, C is the circle $(x + 1)^2 + (y - 2)^2 = 16$.

59. $\oint_C (x^2 + y^2) dx + (5x^2 - 3y) dy$, C is the boundary of the region enclosing $x^2 = 4y$, $y = 4$.

60. $\oint_C xy^2 dx + 5x^3 dy$, C is the rectangle with vertices at $(-1, 0)$, $(2, 0)$, $(2, 2)$ and $(-1, 2)$.

61. $\oint_C (x^2 - y^3) dx + (x^3 + y^2) dy$, C is the ellipse $x^2 + 4y^2 = 64$.

62. $\oint_C (y^3 - xy) dx + (xy + 3xy^2) dy$, C is the boundary of the region in the first quadrant enclosed by $x = 0$, $y = 1 - x^2$, and $y = x^2$.

63. $\oint_C e^x (\sin y\, dx + \cos y\, dy)$, C is the ellipse $4(x + 1)^2 + 9(y - 3)^2 = 36$.

64. $\oint_C xy\, dx + x^3 dy$, C is the boundary of the region enclosed by $y = 0$, $x^2 + y^2 = 4$, $y \ge 0$.

65. $\oint_C x^2 y\, dx + y^2 dy$, C is the boundary of the region enclosed by $y^2 = x$ and $y = x$.

66. Evaluate $\int_C x dy - y\, dx$, where C is the line segment joining the points (a_1, b_1) and (a_2, b_2). Using this result and the result in problem **51**, show that the area of a polygon with vertices at (a_1, b_1), (a_2, b_2), $\ldots$, (a_n, b_n) taken in the anti-clockwise direction is

Area $= [(a_1 b_2 - a_2 b_1) + (a_2 b_3 - a_3 b_2) + \ldots + (a_{n-1} b_n - a_n b_{n-1}) + (a_n b_1 - a_1 b_n)]/2$.

Use the result of problem **51** to find the area bounded by the given closed curves in problems **67** to **69**.

67. Ellipse : $x = a \cos \theta$, $y = b \sin \theta$, $0 \le \theta \le 2\pi$, a and b are positive constants.

68. Circle : $x = a \cos \theta$, $y = a \sin \theta$, $0 \le \theta \le 2\pi$, a is a positive constant.

69. Hypocycloid : $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $a > 0$, $0 \le \theta \le 2\pi$.

In problems **70** to **72** evaluate $\int_C (\partial u/\partial n) ds$ using the result given in example 15.38.

70. $u = x^2 + y^2$, C is the boundary of the region $x^2 + y^2 \le 9$, $x \ge 0$, $y \ge 0$.

71. $u = x^3 - xy^2$, C is the rectangle with vertices at $(1, 1)$, $(3, 1)$, $(3, 4)$ and $(1, 4)$.

72. $u = xy^3 + x^2y^2$, C is the circle $x^2 + y^2 = 4$.

In problems **73** and **74**, use Green's Theorem to evaluate the double integral in terms of a line integral (choose appropriate functions f or g or f and g).

73. $\iint_R y^2 dx\, dy$, R is the region enclosed by the ellipse $x^2 + 4y^2 = 16$.

74. $\iint_R (y - x)\, dx\, dy$, R is the region enclosed by the circle $x^2 + y^2 = 4$.