Q Find the eigen values & eigen vector of a matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0 \quad \text{—— (1)}$$

let $\lambda = -2$

$$(-2)^3 - 7(-2)^2 + 36 = 0$$

$$-8 - 28 + 36 = 0$$

$$0 = 0$$

$\therefore$ $\lambda = -2$ is a root of equation (1)

The remaining roots are given by

$$\begin{array}{c|cccc|c} -2 & 1 & -7 & 0 & 36 & \\ & \downarrow & -2 & 18 & -36 & \\ \hline & 1 & -9 & 18 & & 0 \end{array}$$

0 → Remainder

quotient $= \lambda^2 - 9\lambda + 18$

The remaining roots are given by

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda^2 - 6\lambda - 3\lambda + 18 = 0$$

$$\lambda(\lambda - 6) - 3(\lambda - 6) = 0$$

$$(\lambda - 3)(\lambda - 6) = 0$$

$$\lambda = 3 \ \& \ \lambda = 6$$

The eigen values are $\lambda = -2, 3 \ \& \ 6$

The eigen values are $\lambda = -2, 3 \& 6$

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = -2$.

Consider $(A + 2I)X = 0$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = \lambda x$

$(A - \lambda I)X = 0$

$|A - \lambda I| = 0$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & -20 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad — (2)$$

✓ Number of eigen vectors $= \underline{3 - 2 = 1}$

Here from (2) $x + 7y + z = 0$

$-20y = 0 \Rightarrow \boxed{y = 0}$

$\therefore x + 0 + z = 0$

$\boxed{x = -z}$

$z = -x$

$X = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$

$\therefore (-1, 0, 1)$ is the eigen vector corresponding to eigen value $\boxed{\lambda = -2}$

$\therefore$ $(-1, 0, 1)$ is the eigen vector corresponding to eigen value $\boxed{\lambda = -2}$

$X \rightarrow \lambda$

$\alpha X \rightarrow \lambda$

$\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ ✓

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to $\lambda = 3$

consider $(A - 3I)X = 0$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nr of eigen vectors $= 3 - 2 = 1$

$\therefore$ $x + 2y + z = 0$

$5y + 5z = 0 \Rightarrow \boxed{y = -z}$

$\therefore$ $x - 2z + z = 0$

or $\boxed{x = z}$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} z$$

$(1, -1, 1)$ is the eigen vector for $\lambda = 3$

(1, -1, 1) is the eigen vector of $\lambda = 3$

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector for $\lambda = 6$

Do solve yourself

$\boxed{(1, 2, 1)}$

Exceptional Case

✓ repeated eigen value

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nr of eigen vectors $3 - 1 = 2$

$$x + 2y + 3z = 0$$

✓ or $x = -2y - 3z$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y - 3z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore$ (-2, 1, 0) & (-3, 0, 1) will be eigen vectors

$$\boxed{|A - \lambda I| = 0}$$

① $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

trace of A = Sum of elements on the main diagonal

Characteristic eqn

$$\lambda^2 - (\text{trace } A) + \det A = 0$$

$$\boxed{\lambda^2 - 7\lambda + 6 = 0}$$

② $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Imp

Characteristic eqn

$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{Sum of minors of}$

$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{Sum of minors of elements on the main diagonal})\lambda - \det A = 0$

$$\therefore \quad \lambda^3 - 7\lambda^2 + (4 + (-8) + 4)\lambda - (-36) = 0$$

$$\boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$

Properties of Eigen Value

① Any square matrix A and its transpose A' have the same eigen values

$$(A - \lambda I)' = A' - \lambda I'$$

$$\therefore |(A - \lambda I)'| = |A' - \lambda I|$$

$$|(A - \lambda I)| = |A' - \lambda I| \qquad [\because |B| = |B'|]$$

$$\therefore \quad |A - \lambda I| = 0 \text{ iff } |A' - \lambda I| = 0$$

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \quad \& \quad A' = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$\lambda = 1, 6$

$\lambda^2 - 7\lambda + 6 = 0$

$\lambda = 1, 6$

② αA has the eigen value $\alpha\lambda$ & the corresponding eigen vector is X

If $\lambda \longrightarrow A \longrightarrow X$

$$AX = \lambda X$$

$$(\alpha A)X = (\alpha\lambda)X$$

$$\Rightarrow \alpha\lambda \longrightarrow \alpha A$$

$\lambda \longrightarrow A \longrightarrow X$

$AX = \lambda X$

$\alpha A X = \alpha \lambda X$

$\alpha \ (\alpha A)X = (\alpha\lambda)X$

Ex. $\lambda = 1, 6$ are $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$\lambda = 2 \ \& \ 12 \longleftarrow 2A = \begin{bmatrix} 10 & 8 \\ 2 & 4 \end{bmatrix}$$

③ The eigen values of the triangular matrix are just the diagonal elements

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\lambda = 1, 2 \,\&\, 1}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

ie $\begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\left[(2-\lambda)(1-\lambda)\right] = 0$$

$$\Rightarrow \lambda = 1, 2, 1$$

(4) If X is the eigen vector for λ

then $AX = \lambda X$

Multiply b.s by scalar α

$$\alpha A X = \alpha \lambda X$$

$$(\alpha A) X = (\alpha \lambda) X$$

$\Rightarrow \alpha\lambda$ is the eigen value of αA whose eigen vector is X

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then $\lambda = \underline{1, 6}$

Find $2A$ & its eigen values

$$\alpha A = 2\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 2 & 4 \end{bmatrix}, \quad \lambda = \underline{2, 12}$$

$\therefore$

(5) $$\underline{A}X = \lambda X$$

Premultiply b.s by A

$$A(AX) = A(\lambda X)$$

or $A^2 X = \lambda(AX)$

$= \lambda(\lambda X)$

or $A^2 X = \lambda^2 X$

Similarly $A(A^2 X) = A(\lambda^2 X)$

$A^3 X = \lambda^2 (AX)$

$= \lambda^2 (\lambda X)$

$= \lambda^3 X$

or $A^3 X = \lambda^3 X$

or generalisation

$A^m X = \lambda^m X$, where m is any +ve integer

∴ λ^m is the eigen value of A^m & the corresponding eigen vector is X, m is any +ve integer

(6)

$$AX = \lambda X$$

$$AX - KIX = \lambda X - KX$$

or $(A - KI)X = (\lambda - k)X$

⇒

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $\lambda = 1, 6$

$A - 2I = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$, $\lambda = 1-2, 6-2$

$= -1, 4$

$\lambda^2 - 3\lambda - 4 = 0$

$\lambda^2 - 4\lambda + \lambda - 4 = 0$

$\lambda(\lambda - 4) + 1(\lambda - 4) = 0$

$\lambda = -1, 4$

(6) $A - KI$ has the eigen values $\lambda - k$, for any scalar k, and the corresponding eigen vector X.

(7) $AX = \lambda X$,

Premultiply b.s by A^{-1}

$$A^{-1}AX = A^{-1}(\lambda X)$$

$$IX = \lambda(A^{-1}X)$$

or $$A^{-1}X = \frac{1}{\lambda}X$$

or $$A^{-1}X = \lambda^{-1}X$$

A^{-1} (if exist) has eigenvalues $1/\lambda$ & the corresponding eigenvector is X

<u>Ex</u> $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $\lambda = 1, 6$, $|A| = +6 \neq 0$

. eigen values of A^{-1}, $\lambda = 1, \frac{1}{6}$

(8) $(A-KI)^{-1}$ has the eigenvalue $\frac{1}{\lambda - K}$, where K is scalar, & X is corresponding eigen vector

<u>imp</u>

(9) The eigen values of idempotent matrix are either zero or unity

<u>Sol</u> let A be an idempotent matrix i.e $A^2 = A$

let λ be an <u>eigen value of A</u>, then there exists vector X s.t

$$AX = \lambda X \quad \text{—— (1)}$$

Premultiply b.s by A

$$A(AX) = A(\lambda X)$$

$$\text{or } A^2X = \lambda(AX)$$

$$= \lambda(\lambda X)$$

$$= \lambda^2 X$$

i.e $$A^2X = \lambda^2 X$$

But $A^2 = A$

$$\therefore AX = \lambda^2 X \quad \text{—— (2)}$$

from (1) & (2)

$$\lambda^2 X = \lambda X$$

$$\text{or } (\lambda^2 - \lambda)X = 0$$

$$\text{or } \lambda^2 - \lambda = 0$$

$$\Rightarrow \lambda(\lambda-1)=0$$

$$\boxed{\lambda = 0 \text{ or } \lambda = 1}$$

Imp

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation

Ex If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, verify C-H Theorem.

Characteristic equation is $\boxed{|A-\lambda I|=0}$

$$\lambda^2 - 7\lambda + 6 = 0$$

Put $\lambda = A$

$$A^2 - 7A + 6I = 0 \longrightarrow (1)$$

Now $A^2 = AA$

$$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}$$

Sub these values in (1)

$$\begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence CH-Theorem is verified

$$A^2 - 7A + 6I = 0$$

or $6I = -A^2 + 7A$

Multiply b.s by A^{-1}

$$\boxed{\begin{aligned} 6A^{-1}I &= A^{-1}(-A^2+7A) \\ &= -A^{-1} + 7A^{-1}A \end{aligned}}$$
$$= -A + 7I$$

$$6A^{-1} = -A + 7I$$

or $$6A^{-1} = -\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} & \\ -1 & 5 \end{bmatrix}$$

or $\boxed{A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}}$

Imp

Spectrum of A $\rightarrow$ The set of the eigenvalues of A

Spectral radius of A $\rightarrow$ The largest eigen value in magnitude is called the spectral radius of A

eg $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, $\lambda = -2, 3 \,\&\, 6$

The spectrum $= \{-2, 3, 6\}$.

spectral radius $= \underline{6}$

Q verify CH-Theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ & Compute A^{-1}.

✓ Also find the matrix represented by

✓ $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.