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Q Find the eigen values & eigen vector of a matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0 \quad \text{— (1)}$$

let $\lambda = -2$

$$(-2)^3 - 7(-2)^2 + 36 = 0$$

$$-8 - 28 + 36 = 0$$

$$0 = 0$$

$\therefore$ $\lambda = -2$ is a root of equation (1)

The remaining roots are given by

$$\begin{array}{c|ccc|c} -2 & 1 & -7 & 0 & 36 \\ & \downarrow & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & 0 \end{array} \rightarrow \text{Remainder}$$

quotient $= \lambda^2 - 9\lambda + 18$

The remaining roots are given by

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda^2 - 6\lambda - 3\lambda + 18 = 0$$

$$\lambda(\lambda - 6) - 3(\lambda - 6) = 0$$

$$(\lambda - 3)(\lambda - 6) = 0$$

$$\lambda = 3 \text{ \& } \lambda = 6$$

The eigen values are $\lambda = -2, 3 \text{ \& } 6$

The eigen values are $\lambda = -2, 3 \& 6$

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = -2$.

Consider $(A + 2I)X = 0$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX = \lambda X$

$(A - \lambda I)X = 0$

$|A - \lambda I| = 0$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & -20 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (2)}$$

✓ Number of eigen vectors $= 3 - 2 = 1$

Here from (2) $x + 7y + z = 0$

$-20y = 0 \Rightarrow \boxed{y = 0}$

$\therefore x + 0 + z = 0$

$\boxed{x = -z}$

$z = -x$

$X = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore (-1, 0, 1)$ is the eigen vector corresponding to eigen value $\boxed{\lambda = -2}$

$\therefore$ $(-1, 0, 1)$ is the eigen vector corresponding to eigen value $\boxed{\lambda = -2}$

$X \to \lambda$

$\underline{\alpha X} \to \lambda$

$\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ ✓

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to $\lambda = 3$

consider $(A - 3I)X = 0$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nr of eigen vectors $= 3 - 2 = 1$

$\therefore$ $x + 2y + z = 0$

$5y + 5z = 0 \Rightarrow \boxed{y = -z}$

$\therefore$ $x - 2z + z = 0$

or $\boxed{x = z}$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} z$$

$(1, -1, 1)$ is the eigen vector for $\lambda = 3$

$(1,-1,1)$ is the eigen vector of $\lambda = 3$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector for $\lambda = 6$

Do solve yourself

$\boxed{(1,2,1)}$

Exceptional Case

✓ repeated eigen value

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nr of eigen vectors $3 - 1 = 2$

$$x + 2y + 3z = 0$$

✓ or $x = -2y - 3z$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y - 3z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore$ $(-2,1,0)$ & $(-3,0,1)$ will be eigen vectors

$\boxed{|A - \lambda I| = 0}$

① $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ trace of A = Sum of elements on the main diagonal

Characteristic eqn

$$\lambda^2 - (\text{trace } A) + \det A = 0$$

$$\boxed{\lambda^2 - 7\lambda + 6 = 0}$$

② $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Imp

Characteristic eqn

$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{Sum of minors of}$

$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{Sum of minors of elements on the main diagonal})\lambda - \det A = 0$

$\therefore \lambda^3 - 7\lambda^2 + (4+(-8)+4)\lambda - (-36) = 0$

$$\boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$

Properties of Eigen Value

① Any square matrix A and its transpose A' have the same eigen values

$$(A - \lambda I)' = A' - \lambda I'$$

$$\therefore |(A - \lambda I)'| = |A' - \lambda I|$$

$$|(A - \lambda I)| = |A' - \lambda I| \qquad [\because |B| = |B'|]$$

$$\therefore |A - \lambda I| = 0 \text{ iff } |A' - \lambda I| = 0$$

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \ \& \ A' = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$\lambda = 1, 6$

$\lambda^2 - 7\lambda + 6 = 0$

$\lambda = 1, 6$

② αA has the eigen value $\alpha\lambda$ & the corresponding eigen vector is X

$$\boxed{\begin{gathered} \lambda \to A \to X \\ AX = \lambda X \end{gathered}}$$

$\alpha AX = \alpha\lambda X$

or $(\alpha A)X = (\alpha\lambda)X$

If $\lambda \longrightarrow A \longrightarrow X$

$$AX = \lambda X$$

$$(\alpha A)X = (\alpha\lambda)X$$

$$\Rightarrow \alpha\lambda \longrightarrow \underline{\alpha A}$$

Ex $\quad \lambda = 1, 6 \quad$ are $\quad A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$\lambda = 2 \ \& \ 12 \longleftarrow 2A = \begin{bmatrix} 10 & 8 \\ 2 & 4 \end{bmatrix}$$

③ The eigen values of the triangular matrix are just the diagonal elements

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\lambda = 1, 2 \,\&\, 1}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

ie $\begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\left[(2-\lambda)(1-\lambda)\right] = 0$$

$$\Rightarrow \lambda = 1, 2, 1$$

(4) If X is the eigen vector for λ

then $AX = \lambda X$

Multiply b.s by scalar α

$$\alpha A X = \alpha \lambda X$$

$$a \; (\alpha A) X = (\alpha \lambda) X$$

$\Rightarrow \alpha\lambda$ is the eigen value of αA whose eigen vector is X

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then $\lambda = \underline{1, 6}$

Find $2A$ & its eigen values

$\alpha A = 2\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 2 & 4 \end{bmatrix}$, $\lambda = \underline{2, 12}$

$\therefore$

(5) $$\underline{A}X = \lambda X$$

Premultiply b.s by A

$$A(AX) = A(\lambda X)$$

or $$A^2 X = \lambda(AX)$$
$$= \lambda(\lambda X)$$

or $$A^2 X = \lambda^2 X$$

Similarly $$A(A^2 X) = A(\lambda^2 X)$$
$$A^3 X = \lambda^2 (AX)$$
$$= \lambda^2 (\lambda X)$$
$$= \lambda^3 X$$

or $$A^3 X = \lambda^3 X$$

on generalisation
$$A^m X = \lambda^m X, \text{ where } m \text{ is any +ve integer}$$

∴ λ^m is the eigen value of A^m & the corresponding eigen vector is X, m is any +ve integer

(6)

$$AX = \lambda X$$

$$AX - KIX = \lambda X - KX$$

or $$(A - KI)X = (\lambda - k)X$$

⇒

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $\lambda = 1, 6$

$A - 2I = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$, $\lambda = 1-2, 6-2$
$= -1, 4$

$$\lambda^2 - 3\lambda - 4 = 0$$
$$\lambda^2 - 4\lambda + \lambda - 4 = 0$$
$$\lambda(\lambda - 4) + 1(\lambda - 4) = 0$$
$$\lambda = -1, 4$$

(6) $A - KI$ has the eigen values $\lambda - k$, for any scalar k, and the corresponding eigen vector X.

(7) $$AX = \lambda X,$$

Premultiply b.s by A^{-1}

$$A^{-1}AX = A^{-1}(\lambda X)$$

$$IX = \lambda(A^{-1}X)$$

or $$A^{-1}X = \frac{1}{\lambda}X$$

or $$A^{-1}X = \lambda^{-1}X$$

A^{-1} (if exist) has eigenvalues $1/\lambda$ & the corresponding eigenvector is X

Ex $\quad A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $\lambda = 1, 6$, $|A| = +6 \neq 0$

. eigen values of A^{-1}, $\lambda = 1, \frac{1}{6}$

(8) $(A-KI)^{-1}$ has the eigenvalue $\frac{1}{\lambda - K}$, where K is scalar, & X is corresponding eigen vector

imp

(9) The eigen values of idempotent matrix are either zero or unity

Sol $\quad$ let A be an idempotent matrix i.e $A^2 = A$

let λ be an eigenvalue of A, then there exists vector X s.t

$$AX = \lambda X \quad \text{—} (1)$$

Pre multiply b.s by A

$$A(AX) = A(\lambda X)$$

$$\text{or } A^2X = \lambda(AX)$$

$$= \lambda(\lambda X)$$

$$= \lambda^2 X$$

i.e $$A^2X = \lambda^2 X$$

But $A^2 = A$

$$\therefore AX = \lambda^2 X \quad \text{—} (2)$$

from (1) & (2)

$$\lambda^2 X = \lambda X$$

$$\text{or } (\lambda^2 - \lambda)X = 0$$

$$\text{or } \lambda^2 - \lambda = 0$$

$$\Rightarrow \lambda(\lambda-1)=0$$

$$\boxed{\lambda=0 \text{ or } \lambda=1}$$

Imp

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation

Ex If $A=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$, verify C-H Theorem.

Characteristic equation is $\boxed{|A-\lambda I|=0}$

$$\lambda^2-7\lambda+6=0$$

Put $\lambda=A$

$$A^2-7A+6I=0 \rightarrow (1)$$

Now $A^2 = AA$

$$=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}29 & 28\\ 7 & 8\end{bmatrix}$$

Sub these values in (1)

$$\begin{bmatrix}29 & 28\\ 7 & 8\end{bmatrix}-7\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}+6\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

or $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Hence CH-Theorem is verified

$$A^2-7A+6I=0$$

or $6I=-A^2+7A$

Multiply b.s by A^{-1}

$$\boxed{\begin{aligned}6A^{-1}I &= A^{-1}(-A^2+7A)\\ &= -A^{-1}A^2+7A^{-1}A\end{aligned}}$$

$$=-A+7I$$

$$6A^{-1}=-A+7I$$

or $6A^{-1}=-\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}2 & -4\\ -1 & 5\end{bmatrix}$$

$$= \begin{bmatrix} & \\ -1 & 5 \end{bmatrix}$$

or $\boxed{A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}}$

imp

Spectrum of A → The set of the eigenvalues of A

Spectral radius of A → The largest eigen value in magnitude is called the spectral radius of A

eg $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, $\lambda = -2, 3$ & 6

The spectrum $= \{-2, 3, 6\}$

spectral radius $= \underline{6}$

Q verify CH-Theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ &

Compute A^{-1}.

✓ Also find the matrix represented by

✓ $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Solution → characteristic eq^n of A is given

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

Put $\lambda = A$

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{—} (1)$$

or $3I = A^3 - 5A^2 + 7A$

Multiply b.s by A^{-1}

$$3A^{-1} = A^2 - 5A + 7I$$

or $\boxed{A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I \right]}$ * solve it

$$3A = A^2 - 5A + 7I$$

$$\text{or } \boxed{A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I\right]}$$ * solve it

Consider

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A[A^3 - 5A^2 + 7A - 3I]$$

$$+ A^2 + A + I$$

using eq (1)

$$= A^5 \times 0 + A \times 0 + A^2 + A + I$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Q If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then show that

$A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. Hence, find A^{50}.

Sol Characteristic eqn of A is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Using CH-Theorem

$$A^3 - A^2 - A + I = 0 \quad \text{--- (1)}$$

from (1)

$$A^3 - A^2 = A - I$$

Pre multiplying b.s by A, we get

$$A^4 - A^3 = A^2 - A$$

$$A^5 - A^4 = A^3 - A^2$$

$$- - - - - - - -$$

$$x + y = 2$$
$$2x + y = 3$$

$$A^5 - A^4 = A^3 - A^2$$

- - - - - - - - - - -

$$A^{n-1} - A^{n-2} = A^{n-3} - A^{n-4}$$

$$A^{n} - A^{n-1} = A^{n-2} - A^{n-3}$$

$x+y=2$		
$2x+y=3$		
$3x+2y=$ …		

$x+y=1$
$2x-y=3$
$3x+4y=$ …

On adding vertically

$$A^n - A^2 = A^{n-2} - I$$

or $$\boxed{A^n = A^{n-2} + A^2 - I}$$

Here $$A^n = A^{n-2} + (A^2 - I) \quad — (2) \qquad n \geq 3$$

Change n to $n-2$ in (2)

$$A^{n-2} = A^{n-4} + A^2 - I$$

Sub the value of A^{n-2} in (2)

$$A^n = A^{n-4} + (A^2 - I) + (A^2 - I)$$

$$\boxed{A^n = A^{n-4} + 2(A^2 - I)} \quad — (3)$$

Changing n to $n-4$ in (2), we get

$$A^{n-4} = A^{n-6} + A^2 - I$$

Sub it in eq (3)

$$A^n = A^{n-6} + (A^2 - I) + 2(A^2 - I)$$

$$A^n = A^{n-6} + 3(A^2 - I)$$

On generalizing

$$A^n = A^{n-(n-2)} + \frac{(n-2)}{2}(A^2 - I)$$

or $$A^n = A^2 + \frac{(n-2)}{2}(A^2 - I)$$

$$= A^2 + \frac{1}{2}(nA^2 - nI - 2A^2 + 2I)$$

$$= \cancel{A^2} + \frac{n}{2}A^2 - \frac{n}{2}I - \cancel{A^2} + I$$

$$= \cancel{A^2} + \frac{n}{2} A^2 - \frac{n}{2} I - \cancel{A^2} + I$$

$$\boxed{A^n = \frac{n}{2} A^2 - \frac{1}{2}(n-2) I}$$

Put $n = 50$, we get

$$A^{50} = 25 A^2 - \frac{1}{2}(48) I$$

$$= 25 A^2 - 24 I$$