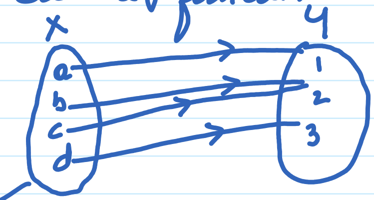
Monday, September 14, 2020 9:43 AM

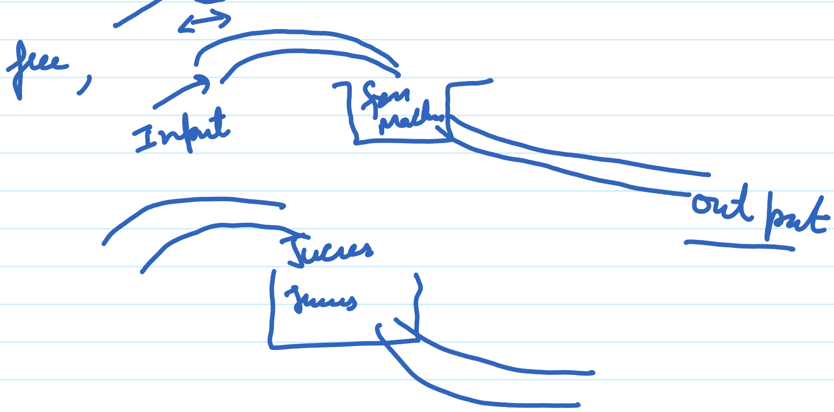
Function →

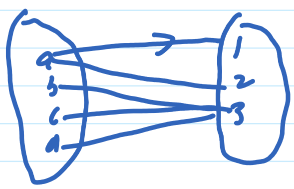
let X & Y be two non-empty sets

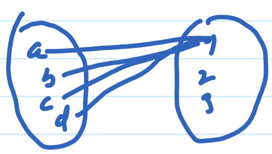
? Define $f : X \to Y$

A rule which associates every element of a set X to a unique element of set Y is called a function

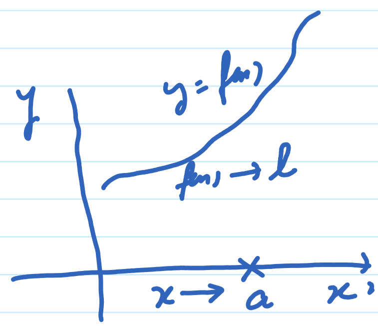


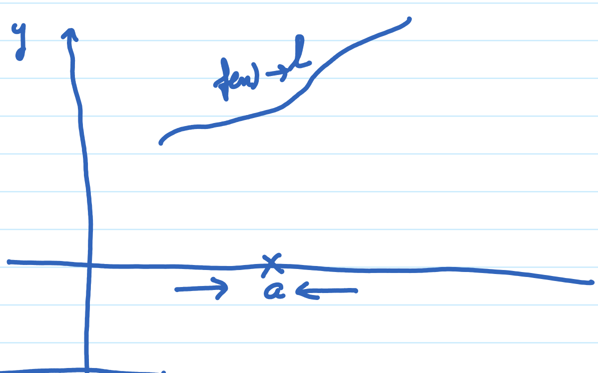






limit of function →



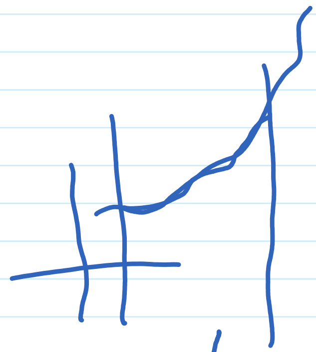


$$\boxed{\lim_{x \to a} f(x) = l}$$

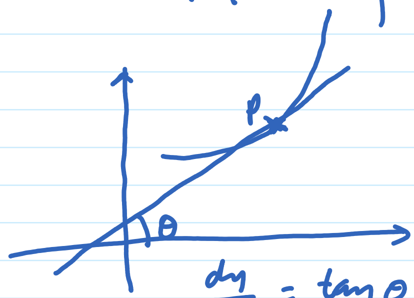
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$$

Continuous function →

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$



Derivatives of a function at a point



$\frac{dy}{dx} = \tan\theta$

Let f be a function with domain D.
The derivative of f at the point x of its domain is defined as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ provided}$$

the limit exists and is finite.
The derivative (or differential coeff) is denoted by $f'(x)$ or $\frac{d}{dx} f(x)$.

The derivative of f at $c \in D$, c being an interior point of D, is also defined as $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, provided this limit exists & is finite

$$\boxed{f(x)} \quad f'(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

① **First step rule**

$$y = f(x) \quad \text{--- ①}$$

Let δx be the increment in x & corresponding to this δy be the increment in y, we get

$$y + \delta y = f(x + \delta x) \quad \text{--- ②}$$

Subtract ① from ②, we get

$$\delta y = f(x + \delta x) - f(x) \quad \text{--- ③}$$

Divide both sides by $\delta x \neq 0$, we get

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Taking limit as $\delta x \to 0$, on both sides, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \boxed{\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}}$$

or $$\boxed{\frac{dy}{dx} = f'(x)}$$

Q. Find the derivative of x^n w.r.t x

$\boxed{\frac{d}{dx} x^n = n x^{n-1}}$

Sol. let $y = x^n$ — (1)

let $\delta x \to x$, $\delta y \to y$

$$y + \delta y = (x+\delta x)^n \quad \text{— (2)}$$

subtract (1) from (2), we get

$$\delta y = (x+\delta x)^n - x^n$$

Divide both sides by $\delta x \neq 0$

$$\frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}$$

Taking limit as $\delta x \to 0$, on both side, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{(x+\delta x) - x}$$

$$\frac{dy}{dx} = n x^{n-1}$$

$\boxed{\lim_{x \to a} \frac{x^n - a^n}{x - a} = n x^{n-1}}$

or $$\boxed{\frac{d x^n}{dx} = n x^{n-1}}$$

Q. Find the derivative of $(ax+b)^n$ by using five step rule.

Sol. $y = (ax+b)^n$

$$\frac{dy}{dx} = n(ax+b)^{n-1} a$$

Q. Find the derivative of $f(x) = x^2$ w.r.t x, by using definition

Q Find the derivative of $f(x) = x^2$ w.r.t x, by using definition

Sol

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + h^2 + 2xh - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+2x)}{\cancel{h}}$$

$$= 2x.$$

$$\therefore \boxed{f'(x) = 2x}$$

Q Find the derivative of the following functions using Ist principle.

$\lim_{x\to a} \frac{f(x)}{g(x)}$

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

$\lim_{x\to a} \frac{f'(x)}{g'(x)}$

Sol (1)

$$y = x^3 - 27 \quad \text{—(1)}$$

$$y + \delta y = (x+\delta x)^3 - 27 \quad \text{—(2)}$$

Sub (1) from (2), we get

$$\delta y = \left((x+\delta x)^3 - 27\right) - (x^3 - 27)$$

$$= (x+\delta x)^3 - x^3$$

$$\frac{\delta y}{\delta x} = \frac{\cancel{x^3} + (\delta x)^3 + 3x^2\delta x + 3x(\delta x)^2 - \cancel{x^3}}{\delta x}$$

$$= \frac{(\cancel{\delta x})\left[(\delta x)^2 + 3x^2 + 3x\,\delta x\right]}{\cancel{\delta x}}$$

$$\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} (\delta x)^2 + 3x^2 + 3x\,\delta x$$

$$= 0 + 3x^2 + 0$$

or $\boxed{\frac{dy}{dx} = 3x^2}$ #

ab-initio
or
five step rule
or
first method

ab- initio

Ex

Q $y = (x-1)(x-2)$ (2) $\frac{1}{x}$ (3) $\frac{x+1}{}$

as - ... or

Ex
Q. $y = (x-1)(x-2)$ ② $\frac{1}{x^2}$ ③ $\frac{x+1}{x-1}$ first prin.

Q Prove that differential Coefficient of a Constant is zero

Sol
let $y = c$ — ①

let δy be the increment in y & corresponding to this δx be the increment in x

$\therefore \; y + \delta y = c$ — ②

Sub ① from ② $\delta y = 0$

Divide by $\delta x \neq 0$ $\frac{\delta y}{\delta x} = 0$

Taking limit as $\delta x \to 0$, we get

$$\underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x} = 0$$

$$\therefore \boxed{\frac{dy}{dx} = 0}$$

Q If $y = u + v$ then $\frac{dy}{dx} = \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

Q If u & v are two functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Q $$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \quad v \neq 0$$

Problem Differentiate $x^4 + 7x^3 + 8x^2 + 3x + 2$ w.r.t x

Sol let $y = x^4 + 7x^3 + 8x^2 + 3x + 2$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 + 7x^3 + 8x^2 + 3x + 2)$$

$$= \frac{d}{dx}x^4 + 7\frac{d}{dx}x^3 + 8\frac{dx^2}{dx} + 3\frac{dx}{dx} + \frac{d}{dx}2$$

$$\boxed{\frac{dx^n}{dx} = nx^{n-1}}$$

$$\boxed{\frac{dy}{dx} = 4x^3 + 21x^2 + 16x + 3}$$

Q If $y = (x-1)(x-2)$ then find $\frac{dy}{dx}$

Sol ① $y = x^2 - 3x + 2$

Sol (i) $y = x^2 - 3x + 2$

$\therefore \frac{dy}{dx} = 2x - 3$

Or $\frac{dy}{dx} = (x-1)\frac{d}{dx}(x-2) + (x-2)\frac{d}{dx}(x-1)$

$= (x-1) + (x-2)$

$= 2x - 3$

Q. If $y = \frac{x^2+3x+2}{x^{1/2}}$

$= \frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{2}{x^{1/2}}$

$= x^{3/2} + 3x^{1/2} + 2x^{-1/2}$

$\therefore \frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-\frac{1}{2}} + 2 \times -\frac{1}{2}x^{-3/2}$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} - x^{-3/2}$

Q. If $y = (x+1)^3(x+2)^5(2x-7)^4$

$\frac{d}{dx}uvw = uv\frac{dw}{dx} + uw\frac{dv}{dx} + wv\frac{du}{dx}$

$\frac{dy}{dx} = (x+1)^3(x+2)^5\frac{d}{dx}(2x-7)^4$

$+ (x+1)^3(2x-7)^4\frac{d}{dx}(x+2)^5$

$+ (x+2)^5(2x-7)^4\frac{d}{dx}(x+1)^3$

$= (x+1)^3(x+2)^5\,4(2x-7)^3 \times 2 + (x+1)^3(2x-7)^4 \times 5(x+2)^4$

$+ (x+2)^5(2x-7)^4 \times 3(x+1)^2$

$= (x+1)^2(x+2)^4(2x-7)^3\left[8(x+1)(x+2) + 5(x+1)(2x-7) + 3(x+2)(2x-7)\right]$

Q. If $y = \frac{x^2-7x}{x-5}$, find $\frac{dy}{dx}$

Sol $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2-7x}{x-5}\right]$

$= \frac{(x-5)\frac{d}{dx}(x^2-7x) - (x^2-7x)\frac{d}{dx}(x-5)}{\quad}$

$$(x-5)^2$$

$$= \frac{(x-5)(2x-7) - (x^2-7x).1}{(x-5)^2}$$

$$= \frac{2x^2 - \cancel{7x} - 10x + 35 - x^2 + \cancel{7x}}{(x-5)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2-10x+35}{(x-5)^2}}$$

Q If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ then show that $\underline{2x\frac{dy}{dx} + y} = 2\sqrt{x}$ — (1)

Sol $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$

Sub in L.H.S of eqn (1)

$$2x\left[\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right] + \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$= x^{1/2} - \cancel{x^{-1/2}} + \sqrt{x} + \cancel{\frac{1}{\sqrt{x}}}$$

$$= 2\sqrt{x} = R.H.S$$

Q
$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

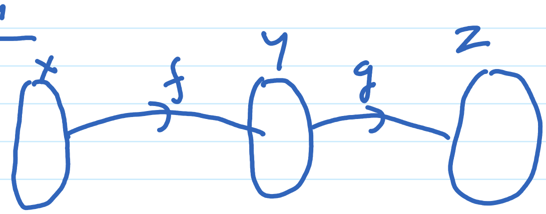
$$\frac{d}{dx}\tan x = \sec^2 x$$

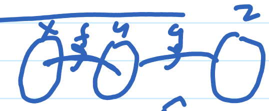
$$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

Composite function





$g(f(x)) = ?$

$g \circ f : X \to Z$

$g(f(x)) = (g \circ f)(x)$

$$g(f(x)) = (g \circ f)(x)$$

$g \circ f : X \to Z$

Q Find $\frac{dy}{dx}$, if $y = \left[\frac{2x-1}{2x+1}\right]^2$

Sol let $u = \frac{2x-1}{2x+1}$

$y = u^2$, $u = \frac{2x-1}{2x+1}$ $\qquad y \to u \to x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Here $\frac{dy}{du} = 2u$,

and $\frac{du}{dx} = \frac{d}{dx}\left[\frac{2x-1}{2x+1}\right]$

$$= \frac{(2x+1)(2) - (2x-1)(2)}{(2x+1)^2}$$

$$= \frac{4x+2-4x+2}{(2x+1)^2} = \frac{4}{(2x+1)^2}$$

$$\therefore \frac{dy}{dx} = 2\left(\frac{2x-1}{2x+1}\right) \times \frac{4}{(2x+1)^2} = \frac{8(2x-1)}{(2x+1)^3} \quad \#$$

Q $\frac{d}{dx} a^x = a^x \log a$

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$

$\frac{d}{dx} \log_e x = \frac{1}{x}$

Q If $y = \sqrt{15x^2 - x + 1}$ then $\frac{dy}{dx} = ?$

$\frac{dy}{dx} = \frac{1}{2}(15x^2-x+1)^{-1/2} \frac{d}{dx}(15x^2-x+1)$

$y = (15x^2-x+1)^{1/2}$

$\frac{dy}{dx} = \frac{d}{dx}(15x^2-x+1)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(15x^2-x+1)^{-1/2}\frac{d}{dx}(15x^2-x+1)$$

$$= \frac{1}{2\sqrt{15x^2-x+1}}\left[30x-1\right]$$

$$\frac{dy}{dx} = \frac{d}{dx}(15x^2-x+1)^{1/2}$$

$$= \frac{1}{2}(15x^2-x+1)^{-1/2} \times \frac{d}{dx}(15x^2-x+1)$$

$$= \frac{1}{2}(15x^2-x+1) \times (30x-1)$$

or

let $u = 15x^2-x+1$

$y = u^{1/2}$, $u = $

$y \to u \to x$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Q

Q If $y = \sqrt{\frac{1-x}{1+x}}$, prove that $(1-x^2)\frac{dy}{dx} + y = 0$

Sol

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \times \frac{d}{dx}\left[\frac{1-x}{1+x}\right]$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}} \times \left[\frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}} \times \left[\frac{-1-\cancel{x}-1+\cancel{x}}{(1+x)^2}\right]$$

$$= \frac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$$

Sub the values in LHS $= (1-x^2)\frac{dy}{dx} + y$

$$= (1-x)(1+x) \times -\frac{1}{(1-x)^{1/2}(1+x)^{3/2}} + \sqrt{\frac{1-x}{1+x}}$$

$$= -\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-x}{1+x}} = 0$$

Q $y = e^{x(1+\log x)}$

$x(1+\log x)$

Q $y = e^{x(1+\log x)}$

$$\frac{dy}{dx} = e^{x(1+\log x)} \times \frac{d}{dx}\, \underline{x(1+\log x)}$$

$$= e^{x(1+\log x)} \times \left[x \times \frac{1}{x} + 1 + \log x \right]$$

$$= e^{x(1+\log x)} (2 + \log x)$$

Q If $y = \log((x+2)(x^3 - x))$ | $\log uv = \log u + \log v$

$$y = \log(x+2) + \log(\underline{x^3 - x})$$

$$\frac{dy}{dx} = \frac{1}{x+2} \cdot 1 + \frac{1}{x^3 - x} \times (3x^2 - 1)$$

$$\frac{dy}{dx} = \frac{1}{x+2} + \frac{(3x^2 - 1)}{x^3 - x}$$

Q If $y = \log\left[\frac{x^2 + x + 1}{x^2 - x + 1}\right]$ | $\log\left[\frac{u}{v}\right] = \log u - \log v$

$$y = \log(x^2 + x + 1) - \log(x^2 - x + 1)$$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2 + x + 1} \times (2x+1) - \frac{1}{x^2 - x + 1} \times (2x - 1)$$

Q If $y = \log(x + \sqrt{a^2 + x^2})$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{a^2 + x^2}} \times \frac{d}{dx}\left[x + \sqrt{a^2 + x^2}\right]$$

<u>Parametric equation</u> →

When two variables x & y are functions of third variable (say t), then these equations are called Parametric equations

$$\begin{array}{l} x \searrow \\ \quad\quad t \\ y \nearrow \end{array}$$

let $\underline{x = \phi(t)}$ & $y = \psi(t)$ are called parameter equations

Here $\frac{dx}{dt} = \phi'(t)$ & $\frac{dy}{dt} = \psi'(t)$

then $\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\psi'(t)}{\phi'(t)}}$

Q If $x = at^2$ —(1), $y = 2at$ —(2), find $\frac{dy}{dx}$

Sol Diff (1) & (2) w.r.t 't', we get

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t} \text{ Ans}$$

Q If $x = a\left[\frac{1+t^2}{1-t^2}\right]$ —(1) & $y = \frac{2t}{1-t^2}$ —(2), find $\frac{dy}{dx}$

Sol Diff (1) & (2) w r t 't', we get

$$\frac{dx}{dt} = a\left[\frac{(1-t^2)\times 2t - (1+t^2)\times -2t}{(1-t^2)^2}\right]$$

$$= a\left[\frac{2t - 2t^3 + 2t + 2t^3}{(1-t^2)^2}\right]$$

$$= \frac{4at}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \qquad \text{Solve it}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{2(1+t^2)}{\cancel{(1-t^2)^2}}}{\frac{4at}{\cancel{(1-t^2)^2}}} = \frac{1+t^2}{2at}$$

Q. Diff $\log(x e^x)$ w.r.t $x \log x$

Sol. let $y = \log(x e^x)$

$= \log x + \log e^x$

$y = \log x + x$

$\frac{dy}{dx} = \frac{1}{x} + 1$

$= \frac{1+x}{x}$

$u = x \log x$

$\frac{du}{dx} = x \cdot \frac{1}{x} + \log x$

$= 1 + \log x$

$\frac{dy}{du}$

$\therefore \frac{dy}{du} = \frac{\frac{1+x}{x}}{1+\log x} = \frac{1+x}{x(1+\log x)}$

Q. Derivatives of Implicit functions.

① $x - y - \pi = 0 \Rightarrow y = x - \pi$ or $x = y + \pi$ explicit functions

② $x + \sin xy - y = 0$

$y \to x$ $x \to y$ Not possible

Implicit function

Q. Find $\frac{dy}{dx}$ if $x^6 + y^6 + 6x^2y^2 = 16$

Sol. we have

$x^6 + y^6 + 6x^2y^2 = 16$

Diff it w.r.t 'x', we get

$6x^5 + 6y^5\frac{dy}{dx} + 6\left[x^2 2y\frac{dy}{dx} + y^2 \cdot 2x\right] = 0$

$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

or $6x^5 + 6y^5\frac{dy}{dx} + 12x^2y\frac{dy}{dx} + 12y^2x = 0$

or $\left[6y^5 + 12x^2y\right]\frac{dy}{dx} = -6x^5 - 12y^2x$

or $\left[6y^5 + 12x^2 y\right]\frac{dy}{dx} = -6x^5 - 12y^2 x$

or $\frac{dy}{dx} = -\frac{(6x^5 + 12y^2 x)}{6y^5 + 12x^2 y}$

$= \frac{-x(x^4 + 2y^2)}{y(y^4 + 2x^2)}$ Ans

Q If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} + e^{y-x} = 0$

Sol Here $e^x + e^y = e^{x+y}$ — (1)

Divide b.s by e^{x+y}, we get

$\frac{e^x}{e^{x+y}} + \frac{e^y}{e^{x+y}} = 1$

or $e^{-y} + e^{-x} = 1$ — (2)

Diff eq^n (2) w.r.t 'x', we get

$e^{-y}\left(-\frac{dy}{dx}\right) + e^{-x} \times -1 = 0$

$\frac{d}{dx}e^{-y} = e^{(-y)}\frac{d}{dx}(-y)$

or $-\left[\frac{1}{e^y}\cdot\frac{dy}{dx} + \frac{1}{e^x}\right] = 0$

or $\frac{1}{e^y}\frac{dy}{dx} + \frac{1}{e^x} = 0$

Multiply throughout by $e^y \neq 0$, we get

$\frac{dy}{dx} + \frac{e^y}{e^x} = 0$

or $\frac{dy}{dx} + e^{y-x} = 0$

logarithmic Differentiation

let $y = u^v$ — (1) where u & v are functions of x only,

Take log on both sides, we get

$\log y = \log u^v$

$\log y = \underbrace{v}\,\underbrace{\log u}$ — (2)

Diff (2) w.r.t 'x', we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = v\cdot\frac{d}{dx}\log u + \log u\cdot\frac{dv}{dx}$$

$$= v\cdot\frac{1}{u}\frac{du}{dx} + \log u\cdot\frac{dv}{dx}$$

or $$\frac{dy}{dx} = y\left[\frac{v}{u}\cdot\frac{du}{dx} + \log u\cdot\frac{dv}{dx}\right]$$

Q Differentiate x^x w.r.t 'x'

Sol let $y = x^x$

$$\log y = x\log x$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x$$

or $$\frac{dy}{dx} = y[1+\log x]$$

$$\boxed{\frac{dy}{dx} = x^x(1+\log x)}$$

Q Diff $(x^x)^x$ w.r.t 'x'

$$\boxed{(x^x)^2 = x^{2x}}$$

$$(x^x)^x = x^{x\cdot x} = x^{x^2}$$

Sol let $y = (x^x)^x = x^{x^2}$

Taking log on b.s

$$\log y = \log x^{x^2}$$

$$= x^2\log x$$

Diff b.s with respect to 'x', we get

$$\frac{1}{y}\frac{dy}{dx} = x^2\times\frac{1}{x} + \log x\, 2x$$

$$= x + 2x\log x$$

or $$\frac{dy}{dx} = y[x + 2x\log x]$$

$$\# \quad \frac{dy}{dx} = x^{x^2}[x + 2x\log x]$$

Q If $x^y = e^{x-y}$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{[1+\log x]^2}$ or $\dfrac{\log x}{[\log(xe)]^2}$

Sol Here $x^y = e^{x-y}$

Taking log on both sides, we get

$$\log x^y = \log e^{(x-y)}$$

$$y \log x = (x-y)\log e$$

$$y \log x = x - y$$

or $$y\log x + y = x$$

$$y(1+\log x) = x$$

$$y = \frac{x}{1+\log x}$$

Diff b.s w.r.t 'x', we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x \times \frac{1}{x}}{(1+\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

or $$= \frac{\log x}{(\log e + \log x)^2} = \frac{\log x}{(\log(ex))^2}$$ Ans

$x = \phi(t)$

$x = t^2$

$y = \psi(t)$

$= 2t+1$

Q Find $\frac{dy}{dx}$, when $y = 3\tan x + 5\log_a x + \sqrt{x} - 3e^x + \frac{1}{x}$ —(1)

Sol Diff (1) with respect to 'x', we get

$$\frac{dy}{dx} = 3\sec^2 x + 5\frac{1}{x}\log_a e + \frac{1}{2}x^{-1/2} - 3e^x - \frac{1}{x^2}$$

Q Diff $(x^2+7x+2)(e^x - \sin x)$ w.r.t x

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e \quad \text{If } a = e}$$

$\frac{d}{dx} \log_e e = \frac{1}{x} \log_e e$
$= \frac{1}{x}$

Q Differentiate $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$ w.r.t x

Sol let $y = \left(\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right)^{1/2}$

Taking log on both sides, we get

$$\log y = \log\left[\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right]^{1/2}$$

$$= \frac{1}{2}\log\left[\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right]$$

$$= \frac{1}{2}\left[\log(x-3)(x^2+4) - \log(3x^2+4x+5)\right]$$

$$\log y = \frac{1}{2}\left[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)\right] \quad -(1)$$

Diff (1) w.r.t 'x', we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3} + \frac{1}{x^2+4} \times 2x - \frac{1}{3x^2+4x+5} \times (6x+4)\right]$$

or $$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{(6x+4)}{3x^2+4x+5}\right]$$

$$= \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{(6x+4)}{3x^2+4x+5}\right] \text{Ans}$$

#

Q Diff $x^{\log x} + (\log x)^x$ w.r.t x

Sol $y = x^{\log x} + (\log x)^x$

$y = u + v$, $-(1)$ where $u = x^{\log x}$, $v = (\log x)^x$

$\log(u+v) \neq \log u + \log v$

Diff (1) $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad -(2)$

$$-(1), \quad u = x^{\log x}, \quad v = (\log x)^x$$

Diff (1) $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ — (2)

$u = x^{\log x}$

Taking log on both sides

$\log u = \log x^{\log x}$

$= \log x \log x$

$= (\log x)^2$

Diff b.s w r t 'x', we get

$\dfrac{1}{u}\dfrac{du}{dx} = 2(\log x)^1 \times \dfrac{1}{x}$

or $\dfrac{du}{dx} = 2u \dfrac{\log x}{x}$

$= \dfrac{2x^{\log x}\log x}{x}$

$v = (\log x)^x$

$\log v = \log(\log x)^x$

$\log v = x \log(\log x)$

Diff b.s. w.r.t 'x'

$\dfrac{1}{v}\dfrac{dv}{dx} = x\dfrac{d}{dx}\log(\log x) + \log(\log x)\dfrac{d}{dx}x$

$= x \dfrac{1}{\log x} \times \dfrac{1}{x} + \log(\log x)$

$= \dfrac{1}{\log x} + \log(\log x)$

or $\dfrac{dv}{dx} = v\left[\dfrac{1}{\log x} + \log(\log x)\right]$

$= (\log x)^x\left[\dfrac{1}{\log x} + \log(\log x)\right]$

$\therefore$ from (2) $\dfrac{dy}{dx} = \dfrac{2x^{\log x}\log x}{x} + (\log x)^x\left[\dfrac{1}{\log x} + \log(\log x)\right]$

Q Diff $\cos x^3 . \sin^2(x^5)$ w.r.t 'x'

Sol let $y = \cos x^3 . \sin^2(x^5)$

$\dfrac{dy}{dx} = \cos x^3 \dfrac{d}{dx}\sin^2(x^5) + \sin^2(x^5)\dfrac{d}{dx}\cos x^3$

$= \cos x^3 \; 2\sin(x^5)\dfrac{d}{dx}(\sin(x^5)) + \sin^2(x^5) \times -\sin(x^3) \times \dfrac{d}{dx}x^3$

$= \cos x^3 \; 2\sin(x^5) \; \cos(x^5) \times \dfrac{d}{dx}x^5 - \sin^2(x^5)\sin x^3 \times 3x^2$

$= 10\cos x^3 \sin x^5 \cos x^5 \quad x^4 - 3\sin^2(x^5)\sin x^3 \; x^2$

Q If $y = \sin\left[\sqrt{\sin\sqrt{x}}\right]$ — (1), where $x > 0$, find $\dfrac{dy}{dx}$

Sol

$$\boxed{\frac{dy}{dx} = \frac{\cos\left[\sqrt{\sin\sqrt{x}}\right]\cos\sqrt{x}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}}$$

Diff ① w.r.t 'x', we get

$$\frac{dy}{dx} = \cos\left[\sqrt{\sin\sqrt{x}}\right] \times \frac{d}{dx}(\sin\sqrt{x})^{1/2}$$

$$= \cos\left[\sqrt{\sin\sqrt{x}}\right] \times \frac{1}{2\sqrt{\sin\sqrt{x}}} \times \frac{d}{dx}\sin\sqrt{x}$$

$$= \cos\left[\sqrt{\sin\sqrt{x}}\right] \times \frac{1}{2\sqrt{\sin\sqrt{x}}} \times \cos\sqrt{x} \times \frac{1}{2\sqrt{x}} \times 1$$

$$= \frac{\cos\left[\sqrt{\sin\sqrt{x}}\right]\cos\left[\sqrt{x}\right]}{4\sqrt{x}\sqrt{\sin\sqrt{x}}} \quad \text{Ans}$$

Q If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \cdots \infty}}}$, Prove that $(2y-1)\frac{dy}{dx} = -\sin x$.

$y = \sqrt{x + \sqrt{x + \cdots \infty}}$

Sol for ① $y = \sqrt{\cos x + y}$

S.B.S.

$$y^2 = \cos x + y$$

$$2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

or $2y\frac{dy}{dx} - \frac{dy}{dx} = -\sin x$

or $\boxed{(2y-1)\frac{dy}{dx} = -\sin x}$ Ans

Q If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cdots \infty}}}}$, Prove that

$$\boxed{\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}}$$

Sol $y = \dfrac{\sin x}{1 + \cos x} = \dfrac{(1+y)\sin x}{(1+y) + \cos x}$

$$\text{or} \quad y = \frac{\frac{1+y}{(1+y)}\sin x}{1+y+\cos x}$$

$$y(1+y+\cos x) = (1+y)\sin x$$

$$\text{or} \quad y+y^2+y\cos x = (1+y)\sin x$$

exercise

Derivatives of Inverse Trigonometric functions

① $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, where $-1 < x < 1$ $\quad \left[\begin{array}{l} 1-x^2>0 \\ 1>x^2 \\ \text{or} \end{array} \right.$

② $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$, where $-1 < x < 1$

③ $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$, where $-\infty < x < \infty$

④ $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$, where $-\infty < x < \infty$

⑤ $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$, where $x>1$ or $x<-1$

⑥ $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$, where $x>1$ or $x<-1$

$x^2-1>0$
or $x^2>1$
$|x|>1$
$x>1$ or $x<-1$

Q Diff $\sin^{-1}(e^x)$ w.r.t x

Sol

$$y = \sin^{-1}(e^x)$$

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(e^x)$$

$$= \frac{1}{\sqrt{1-e^{2x}}} \times \frac{d}{dx}e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

Q Diff $y = \cos^{-1}\left[\frac{3\cos x - 4\sin x}{5}\right]$ w.r.t 'x',

Sol let $y = \cos^{-1}\left[\frac{3}{5}\cos x - \frac{4}{5}\sin x\right]$ —①

Put $\frac{3}{5} = r\cos\theta$ & $\frac{4}{5} = r\sin\theta$

Sq & adding $\frac{9}{25}+\frac{16}{25} = r^2(\cos^2\theta+\sin^2\theta)$

$1 = r^2$ or $\boxed{r=1}$

or dividing $\frac{r\sin\theta}{r\cos\theta} = \frac{4/5}{3/5}$

or $\tan\theta = \frac{4}{3}$ or $\theta = \tan^{-1}\left[\frac{4}{3}\right]$

$\therefore y = \cos^{-1}\left[\underline{r\cos\theta\cos x - r\sin\theta\sin x}\right]$

$= \cos^{-1}\left[\cos(x+\theta)\right]$ $\because r=1$

$y = x+\theta$

$y = x + \tan^{-1}\left[\frac{4}{3}\right]$

$\therefore \frac{dy}{dx} = 1+0 = 1$

or $\boxed{\frac{dy}{dx}=1}$ Ans

Some Important Substitutions

C.T.M

1. a^2+x^2, put $x = a\tan\theta$
2. $\sqrt{a^2-x^2}$, put $x = a\sin\theta$ or $x = a\cos\theta$
3. $\sqrt{x^2-a^2}$, put $x = a\sec\theta$
4. both $1+x^2$ & $1-x^2$, put $\underline{x^2=\cos\theta}$ or $\underline{x=\tan\theta}$

Q Diff $\cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$, $0<x<1$ w.r.t x

Sol let $y = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$

Put $x = \tan\theta$

$y = \cos^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$= \cos^{-1}\cos 2\theta$

$= 2\theta$

$y = 2\tan^{-1}x$

$x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$$= 2\theta$$

$$y = 2\tan^{-1}x$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

Q Diff $\tan^{-1}\left[\frac{3x - x^3}{1-3x^2}\right]$ w.r.t $\tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$

Sol let $y = \tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right]$

Put $x = \tan\theta$ ✓

$$\therefore y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right]$$

$$= \tan^{-1}\tan 3\theta$$

$$y = 3\theta$$
$$= 3\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+x^2}$$

$$u = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

Put $x = \sin\theta$

$$\therefore u = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]$$

$$= \tan^{-1}\tan\theta$$

$$u = \theta = \sin^{-1}x$$

$$\therefore u = \sin^{-1}x$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{du} = \frac{dy/dx}{du/dx} = \frac{3/1+x^2}{\frac{1}{\sqrt{1-x^2}}} = \frac{3\sqrt{1-x^2}}{1+x^2} \text{ Ans}$$

Derivatives of higher order

$$y = f(x)$$

Diff $\frac{dy}{dx} = f'(x)$ → order → '1'

Diff it again $\frac{d}{dx}\left[\frac{dy}{dx}\right] = f''(x)$

or

$$\frac{d^2y}{dx^2} = f''(x) \longrightarrow \text{order } 2$$

Again Diff $\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = f'''(x)$

$$\frac{d^3y}{dx^3} = f'''(x) \quad \text{— order } 3$$

— — — — . .

dx^3

$-----\ldots$

$$\frac{d^n y}{dx^n} = f^{(n)}(x) \quad \rightarrow \text{order } n$$

Q Find $\frac{d^3y}{dx^3}$ when $y = \frac{\log x}{x}$

Sol $y = \frac{\log x}{x}$ —(1)

Diff it w.r.t 'x')

$$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2}.$$

$$\frac{dy}{dx} = \frac{1 - \log x}{x^2} \quad \text{—(2)}$$

Again diff it w.r.t 'x', we get

$$\frac{d^2y}{dx^2} = \frac{-3x + 2x \log x}{x^4} \quad \text{Do it}$$

$$= \frac{-3 + 2\log x}{x^3} \quad \text{—(3)}$$

Diff it again

$$\frac{d^3y}{dx^3} = \frac{11 - 6\log x}{x^4} \quad \text{Ans}$$