EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\, \boldsymbol{dx} + \mathbf{N}(x, y)\, \boldsymbol{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y\ \text{cons.})} \mathbf{M\, dx} + \int (\textbf{terms of N not containing x})\, \mathbf{dy} = \mathbf{c}$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Solve $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$

Solve $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\} dx + (x + \log x - x\sin y)\,dy = 0.$

Solve $(1 + 2xy\cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)\,dy = 0.$

Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

Solution. Here $\quad M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\, e^{xy^2} - 3y^2$

$\therefore$
$$\frac{\partial M}{\partial y} = 2y\, e^{xy^2} + y^2\, e^{xy^2} \cdot 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y\ \text{const.})} (y^2 e^{y^2 x} + 4x^3) dx + \int (-3y^2)\, dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Solution. Here $M = y(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$\therefore$
$$\frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\,\text{const})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int_{(y\,\text{const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x) y + x \cos y = c.$$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

$$\int_{(y \text{ const})} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 . 2x dx - \int 2x\, dx \right] = c$$

$$x + y \sin x^2 - yx^2 = c.$$

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x.$

$$\therefore \qquad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int_{(y\text{ const.})} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

Solve the following equations :

1. $(x^2 - ay)\,dx = (ax - y^2)\,dy.$
2. $(x^2 + y^2 - a^2)\,xdx + (x^2 - y^2 - b^2)\,ydy = 0$
3. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$
4. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$
5. $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$
6. $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$
7. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$
8. $\dfrac{2x}{y^3}\,dx + \dfrac{y^2 - 3x^2}{y^4}\,dy = 0$
9. $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$
10. $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$
11. $(2xy + y - \tan y)\,dx + x^2 - x \tan^2 y + \sec^2 y)\,dy = 0.$