CSE 322

Pumping Lemma for Context Free Grammar

Lecture #24

Background Information for the Pumping Lemma for Context-Free Languages

Definition: Let G = (V, T, P, S) be a CFL. If every production in P is of the form

$$A \rightarrow BC$$

or

$$A \rightarrow a$$

where A, B and C are all in V and a is in T, then G is in Chomsky Normal Form (CNF).

Example: (not quite!)

$S \rightarrow AB \mid BA \mid aSb$
$A \rightarrow a$
$B \rightarrow b$

Theorem: Let L be a CFL. Then L – {ε} is a CFL.

Theorem: Let L be a CFL not containing {ε}. Then there exists a CNF grammar G such that L = L(G).

Definition: Let T be a tree. Then the <u>height</u> of T, denoted h(T), is defined as follows:

- If T consists of a single vertex then h(T) = 0
- If T consists of a root r and subtrees T_1, T_2, … T_k, then $h(T) = \max_i\{h(T_i)\} + 1$

Lemma: Let G be a CFG in CNF. In addition, let w be a string of terminals where A=>*w and w has a derivation tree T. If T has height $h(T)\geq 1$, then $|w| \leq 2^{h(T)-1}$.

Proof: By induction on h(T) (exercise).

Corollary: Let G be a CFG in CNF, and let w be a string in L(G). If $|w| \geq 2^k$, where $k \geq 0$, then any derivation tree for w using G has height at least k+1.

Proof: Follows from the lemma.

Pumping Lemma for Context-Free Languages

Lemma:

Let G = (V, T, P, S) be a CFG in CNF, and let $n = 2^{|V|}$. If z is a string in L(G) and $|z| \geq n$, then there exist strings u, v, w, x and y in T* such that z=uvwxy and:

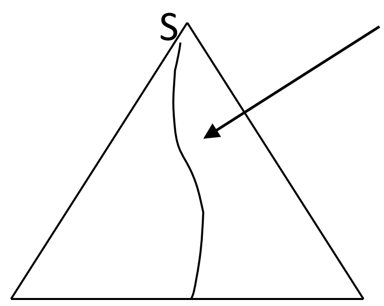
- $|vx| \geq 1$ (i.e., $|v| + |x| \geq 1$, or, non-null)
- $|vwx| \leq n$ (the loop in generating this substring)
- $uv^i wx^i y$ is in L(G), for all $i \geq 0$
- *Note: u could be of any length, so, vwx is not a prefix*
  - *unlike that (uv of uvw) in RL pumping lemma*

- **Proof:**

  Since $|z| \geq n = 2^k$, where $k = |V|$, it follows from the corollary that any derivation tree for z has height at least k+1.

  By definition such a tree contains a path of length at least k+1.

  Consider the longest such path in the tree:



t

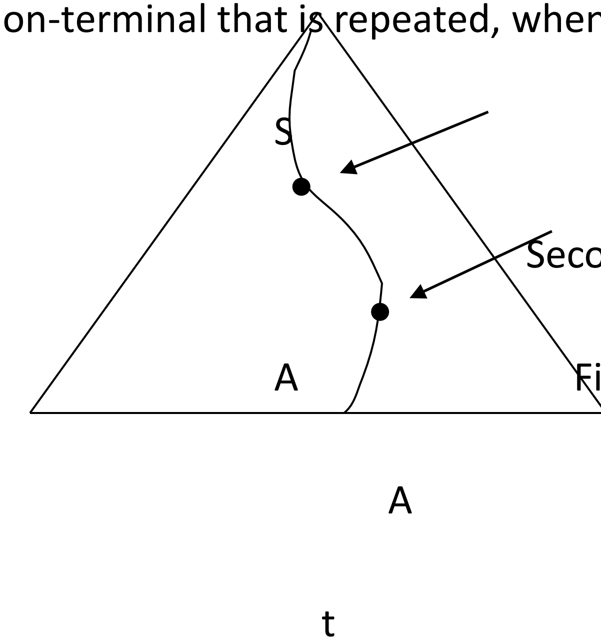
yield of T is z

Such a path has:

- Length $\geq$ k+1 (i.e., number of edges in the path is $\geq$ k+1)
- At least k+2 nodes
- 1 terminal
- At least k+1 non-terminals

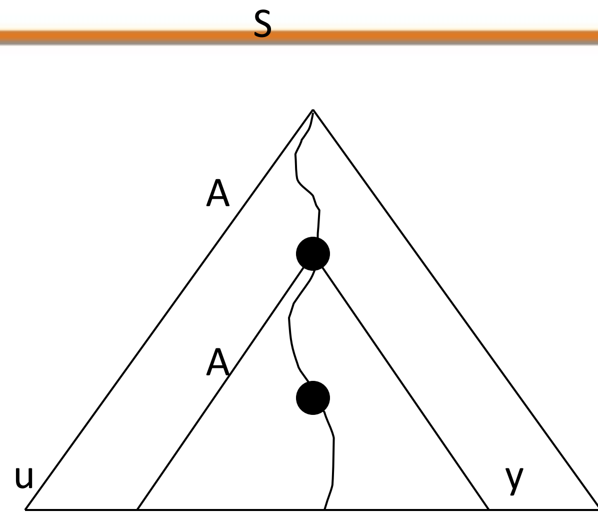
- Since there are only k non-terminals in the grammar, and since k+1 appear on this long path, it follows that some non-terminal (and perhaps many) appears at least twice on this path.

- Consider the first non-terminal that is repeated, when traversing the path from the leaf to the root.

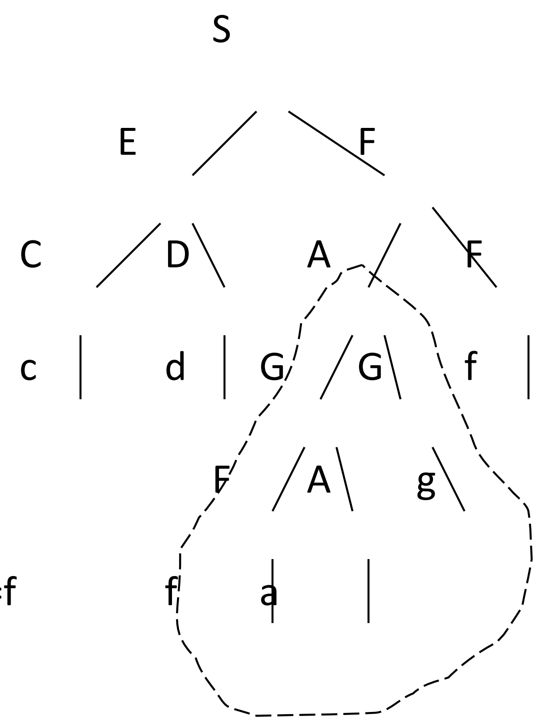


This path, and the non-terminal A will be used to break up the string z.

Generic Description:

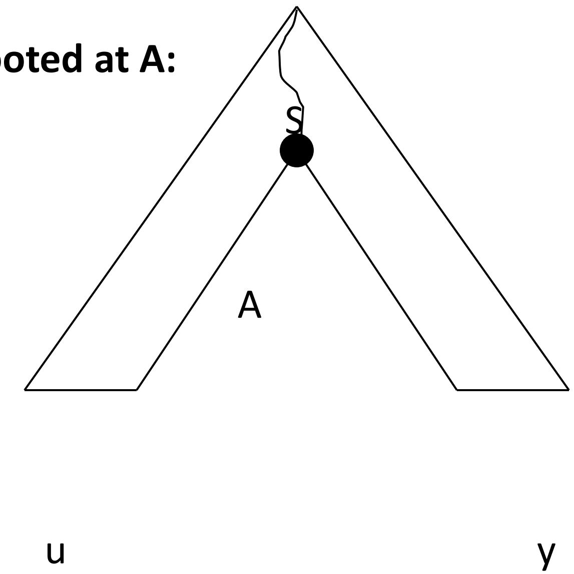


Example:



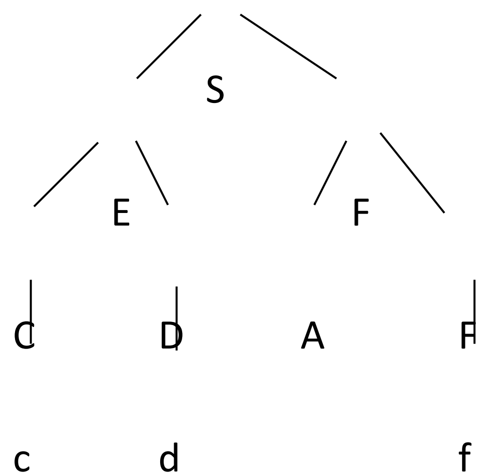
In this case u = cd and y =f

Cut out the subtree rooted at A:



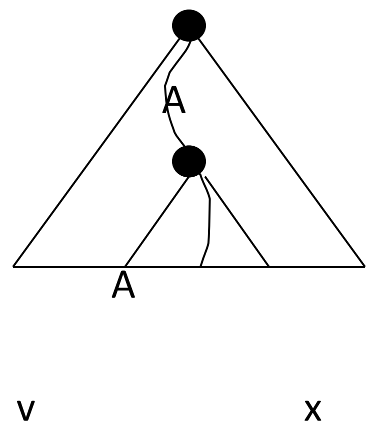
S =>* uAy (1)

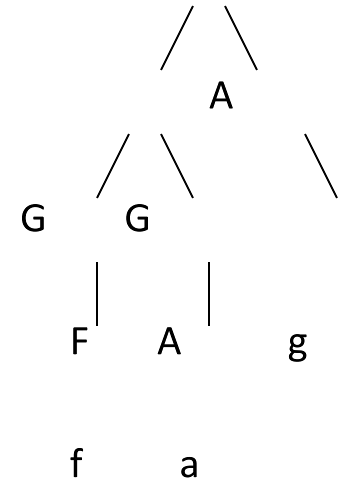
Example:



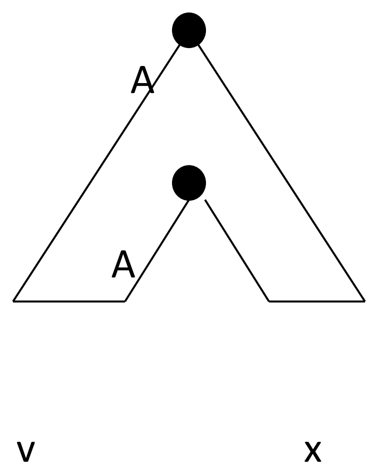
S =>* cdAf

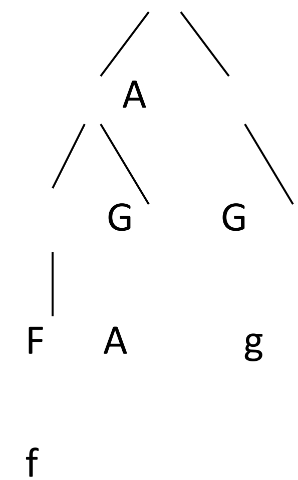
Consider the subtree rooted at A:





Cut out the subtree rooted at the first occurrence of A:

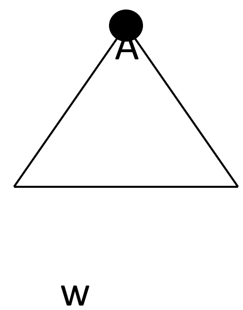


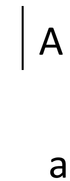


A =>* vAx (2)

A =>* fAg

Consider the smallest subtree rooted at A:





$A \Rightarrow^* w$ (3) $A \Rightarrow^* a$

Collectively (1), (2) and (3) give us:

$S \Rightarrow^* uAy$	(1)
$\Rightarrow^* uvAxy$	(2)
$\Rightarrow^* uvwxy$	(3)
$\Rightarrow^* z$	since $z=uvwxy$

In addition, (2) also tells us:

$$
\begin{aligned}
S &\Rightarrow^* uAy && (1)\\
&\Rightarrow^* uvAxy && (2)\\
&\Rightarrow^* uv^2Ax^2y && (2)\\
&\Rightarrow^* uv^2wx^2y && (3)
\end{aligned}
$$

More generally:

$$S \Rightarrow^* uv^iwx^iy \qquad \text{for all } i>=1$$

And also:

$$
\begin{aligned}
S &\Rightarrow^* uAy && (1)\\
&\Rightarrow^* uwy && (3)
\end{aligned}
$$

Hence:

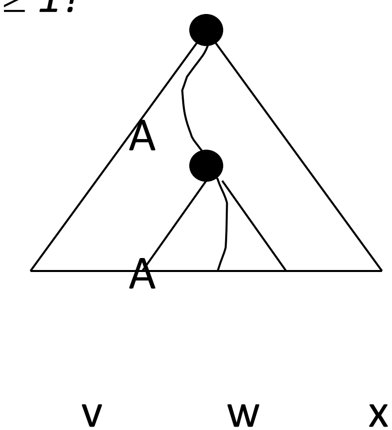
$$S \Rightarrow^* uv^iwx^iy \qquad \text{for all } i>=0$$

Consider the statement of the Pumping Lemma:

–*What is n?*

n = 2^k, where k is the number of non-terminals in the grammar.

–*Why is $|v| + |x| \geq 1$?*



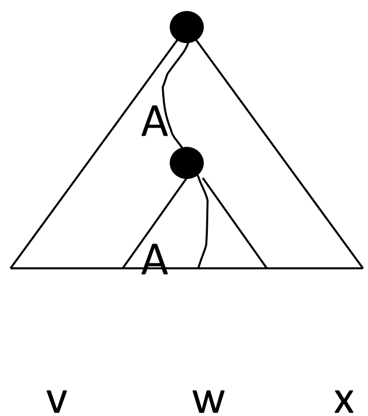
Since the height of this subtree is $\geq$ 2, the first production is A->V_1V_2. Since no non-terminal derives the empty string (in CNF), either V_1 or V_2 must derive a non-empty v or x. More specifically, if w is generated by V_1, then x contains at least one symbol, and if w is generated by V_2, then v contains at least one symbol.

–Why is $|vwx| \leq n$?

Observations:

- The repeated variable was the first repeated variable on the path from the bottom, and therefore (by the pigeon-hole principle) the path from the leaf to the second occurrence of the non-terminal has length at most k+1.

- Since the path was the largest in the entire tree, this path is the longest in the subtree rooted at the second occurrence of the non-terminal. Therefore the subtree has height $\leq k+1$. From the lemma, the yield of the subtree has length $\leq 2^k = n$.



Closure Properties for Context-Free Languages

- **Theorem:** The CFLs are closed with respect to the union, concatenation and Kleene star operations.

- **Proof:** (details left as an exercise) Let L_1 and L_2 be CFLs. By definition there exist CFGs G_1 and G_2 such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$.

  - For union, show how to construct a grammar G_3 such that $L(G_3) = L(G_1) \cup L(G_2)$.
  - For concatenation, show how to construct a grammar G_3 such that $L(G_3) = L(G_1)L(G_2)$.
  - For Kleene star, show how to construct a grammar G_3 such that $L(G_3) = L(G_1)^*$. ￼

- **Theorem:** The CFLs are not closed with respect to intersection.

- **Proof:** (counter example) Let

$$L_1 = \{a^i b^i c^j \mid i,j \geq 1\}$$

and

$$L_2 = \{a^i b^j c^j \mid i,j \geq 1\}$$

Note that both of the above languages are CFLs. If the CFLs were closed with respect to intersection then

$$L_1 \cap L_2$$

would have to be a CFL. But this is equal to:

$$\{a^i b^i c^i \mid i \geq 0\}$$

which is not a CFL. ⍰

$$\overline{L_1 \cup L_2} = \overline{L_1} \cap \overline{L_2}$$

Lemma: Let L_1 and L_2 be subsets of Σ^*. Then .

Theorem: The CFLs are not closed with respect to complementation.

Proof: (by contradiction) Suppose that the CFLs were closed with respect to complementation, and let L_1 and L_2 be CFLs. Then:

$\overline{L_1}$

$\overline{L_2}$ would be a CFL

$\overline{L_1} \cup \overline{L_2}$ would be a CFL

$\overline{\overline{L_1} \cup \overline{L_2}}$ would be a CFL

would be a CFL

$$\overline{\overline{L_1} \cup \overline{L_2}} = \overline{\overline{L_1}} \cap \overline{\overline{L_2}} = L_1 \cap L_2$$

But by the lemma:

Theorem: Let L be a CFL and let R be a regular language. Then $L \cap R$ is a CFL.

Proof: (exercise – sort of) ☐

Question: Is $L \cap R$ regular?

Answer: Not always. Let L = {$a^i b^i$ | i >= 0} and R = {$a^i b^j$ | i,j >= 0}, then $L \cap R = L$ which is not regular.