of homogeneous linear equations. In this section, we shall discuss methods for finding the general solution of a non-homogeneous linear equation (see Eq. (5.1)) of the form

$$L[y] = a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \ldots + a_{n-1}(x)y' + a_n(x)y = r(x),\ a_0(x) \neq 0, \tag{5.46}$$

when the general solution of the corresponding homogeneous linear equation $L[y] = 0$ is known. We present the following theorem.

Theorem 5.5 If $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ is a basis and $c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ is the general solution of the corresponding homogeneous linear equation $L[y] = 0$ and if $y_p(x)$ is any particular solution (a solution not containing any arbitrary constants) of the non-homogeneous equation (5.46), then the general solution of equation (5.46) is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) + y_p(x). \tag{5.47}$$

Proof Since $y_p(x)$ is a particular solution, we have

$$L[y_p(x)] = a_0 y_p^{(n)} + a_1 y_p^{(n-1)} + \ldots + a_{n-1}y_p' + a_n y_p = r(x). \tag{5.48}$$

Subtracting Eq. (5.48) from (5.46), we obtain

$$a_0(y^{(n)} - y_p^{(n)}) + a_1(y^{(n-1)} - y_p^{(n-1)}) + \ldots + a_{n-1}(y' - y_p') + a_n(y - y_p) = 0. \tag{5.49}$$

Denote $y - y_p = z$. Then, from Eq. (5.49) we obtain

$$a_0 z^{(n)} + a_1 z^{(n-1)} + \ldots + a_{n-1}z' + a_n z = 0. \tag{5.50}$$

But, this equation is the corresponding homogeneous equation of Eq. (5.46), whose basis is $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. Hence, the general solution of Eq. (5.50) is given by

$$z = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x).$$

Replacing $z = y - y_p$, and taking y_p to the right hand side, we obtain

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x). \tag{5.51}$$

Since, this solution contains n arbitrary constants, it is the general solution of the Eq. (5.46).

From the above theorem, we conclude that the solution of a non-homogeneous equation consists of the sum of the following two parts.

(i) The general solution of the corresponding homogeneous equation. This solution is called the *complementary function* and is denoted by $y_c(x)$.

(ii) A particular solution of the non-homogeneous equation. This solution is also called a *particular integral* of the non-homogeneous equation and is denoted by $y_p(x)$.

The general solution of the non-homogeneous equation is then written as

$$y(x) = y_c(x) + y_p(x).$$

Now, suppose that the right hand side $r(x)$ is the sum of a number of functions

$$r(x) = r_1(x) + r_2(x) + \ldots + r_m(x). \tag{5.52}$$

Let $y_{p_i}(x), i = 1, 2, \ldots, m$ be any particular solutions, not containing any arbitrary constants, of the equations

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = r_i(x),\ i = 1, 2, \ldots, m. \tag{5.53}$$

Then, $y_{p_1} + y_{p_2} + \ldots + y_{p_m}$ is the particular integral of the equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = r_1(x) + r_2(x) + \ldots + r_m(x) = r(x)$$

and hence of the given non-homogeneous linear equation. This can be proved by summing Eq. (5.53) over i. In other words, if the right hand side of Eq. (5.46) consists of sum of a number of functions, then particular integrals of the Eq. (5.53) can be obtained with respect to each of the functions and the particular integral of Eq. (5.46) is then given by the sum of these particular integrals.

The methods for finding $y_c(x)$ have been discussed in the previous section. In the remaining part of this section, we shall derive methods for finding the particular integral $y_p(x)$ of the non-homogeneous equation.

5.4.1 Method of Variation of Parameters

Consider the second order non-homogeneous linear equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), \quad a_0(x) \neq 0. \tag{5.54}$$

We shall discuss a general method of solution, called the method of *variation of parameters*, which can always be used to find a particular integral whenever the complementary function of the equation is known. Consider first, the solution of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0. \tag{5.55}$$

Using the methods given in the previous section, we can find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the equation (5.55). The complementary function is given by

$$y_c(x) = Ay_1(x) + By_2(x) \tag{5.56}$$

where A and B are arbitrary constants. The idea behind the method of variation of parameters is to vary the parameters A and B. That is, we assume A and B to be functions of x and determine $A(x)$, $B(x)$ such that

$$y(x) = A(x)y_1(x) + B(x)y_2(x) \tag{5.57}$$

is the general solution of Eq. (5.54). Now, $y(x)$ contains two functions $A(x)$ and $B(x)$ which are to be determined. Therefore, we need two equations to determine them. One equation is obtained by substituting $y(x)$ from Eq. (5.57) in Eq. (5.54). The determination of the second equation is at our disposal. This equation is chosen such that the determination of $A(x)$ and $B(x)$ is simple. Differentiating Eq. (5.57), we obtain

$$y'(x) = A'y_1 + Ay_1' + By_2' + B'y_2 = (A'y_1 + B'y_2) + (Ay_1' + By_2'). \tag{5.58}$$

If we differentiate this equation again, then the equation would contain the second derivatives A'' and B'' of the unknown functions. In order that these derivatives are not used, we set in Eq. (5.58)

$$A'y_1 + B'y_2 = 0. \tag{5.59}$$

which gives us the second equation to determine $A(x)$ and $B(x)$. Now, differentiating $y'(x) = Ay_1' + By_2'$, we obtain

$$y''(x) = Ay_1'' + A'y_1' + By_2'' + B'y_2'. \tag{5.60}$$

Substituting the expressions for $y(x)$, $y'(x)$ and $y''(x)$ in Eq. (5.54), we obtain

$$a_0(x)[Ay_1'' + A'y_1' + By_2'' + B'y_2'] + a_1(x)[Ay_1' + By_2'] + a_2(x)[Ay_1 + By_2] = r(x)$$

or

$$a_0(x)[A'y_1' + B'y_2'] + A[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1]$$

$$+ B[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] = r(x).$$

Since, $y_1(x)$ and $y_2(x)$ are the solutions of the homogeneous equation (5.55), we obtain

$$a_0(x)[A'y_1' + B'y_2'] = r(x), \text{ or } A'y_1' + B'y_2' = \frac{r(x)}{a_0(x)} = g(x). \tag{5.61}$$

Since $a_0(x) \neq 0$ on the given interval I, $g(x)$ is continuous on I. Solving the equations

$$A'y_1 + B'y_2 = 0$$

$$A'y_1' + B'y_2' = g(x),$$

we obtain

$$A' = -\frac{g(x)y_2}{y_1y_2' - y_2y_1'}, \quad B' = \frac{g(x)y_1}{y_1y_2' - y_2y_1'}. \tag{5.62}$$

We note that the Wronskian $W(y_1, y_2)$ is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \neq 0$$

since y_1, y_2 are the linearly independent solutions of the homogeneous equation. Hence, we can write Eqs. (5.62) as

$$A' = -\frac{g(x)y_2}{W(x)}, \text{ and } B' = \frac{g(x)y_1}{W(x)}. \tag{5.63}$$

Integrating, we obtain

$$A(x) = -\int \frac{g(x)y_2(x)}{W(x)}\,dx + c_1 \text{ and } B(x) = \int \frac{g(x)y_1(x)}{W(x)}\,dx + c_2. \tag{5.64}$$

Substituting in Eq. (5.57), we obtain the general solution which contains two arbitrary constants. If we do not add the arbitrary constants while carrying out integrations of Eqs. (5.63), then we obtain the particular solution as $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$, which does not contain any arbitrary constants. The general solution is then given by $y(x) = y_c(x) + y_p(x)$.

The method is applicable both for constant coefficient and variable coefficient problems. The method can also be easily extended to equations of any order. At each differentiation step, we set the part containing the derivatives of the unknown functions to zero, until we arrive at the final substitution step. For example, consider the third order equation

$$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = r(x), \quad a_0(x) \neq 0. \tag{5.65}$$

The complementary function is

$$y(x) = Ay_1(x) + By_2(x) + Cy_3(x)$$

where y_1, y_2, y_3 are the linearly independent solutions of the corresponding homogeneous equation and A, B, C are arbitrary constants. We assume the solution as

$$y(x) = A(x)y_1(x) + B(x)y_2(x) + C(x)y_3(x). \tag{5.66}$$

Following the procedure discussed earlier, we obtain the required equations for determining $A(x)$, $B(x)$ and $C(x)$ as

$$A'(x)\,y_1 + B'(x)y_2 + C'(x)y_3 = 0$$

$$A'(x)y_1' + B'(x)y_2' + C'(x)y_3' = 0$$

and
$$A'(x)y_1'' + B'(x)y_2'' + C'(x)y_3'' = \frac{r(x)}{a_0(x)} = g(x). \tag{5.67}$$

The determinant of the coefficient matrix is the Wronskian $W(y_1, y_2, y_3) \neq 0$. We determine $A(x)$, $B(x)$, $C(x)$ and substitute in Eq. (5.66) to obtain the general solution.

Example 5.32 Find the general solution of the equation $y'' + 3y' + 2y = 2e^x$, using the method of variation of parameters.

Solution The corresponding homogeneous equation is $y'' + 3y' + 2y = 0$. The characteristic equation is $m^2 + 3m + 2 = 0$ and its roots are $m = -1, -2$. Hence, the complementary function is

$$y_c(x) = Ay_1(x) + By_2(x) = A\,e^{-x} + Be^{-2x}$$

where $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$ are two linearly independent solutions of the homogeneous equation. Assume the general solution as

$$y(x) = A(x)e^{-x} + B(x)e^{-2x}$$

We have $g(x) = r(x)/a_0(x) = 2e^x$.

The Wronskian of $y_1(x)$, $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}.$$

Using Eq. (5.64), we obtain the solutions for $A(x)$ and $B(x)$ as

$$A(x) = -\int \frac{g(x)y_2(x)}{W}\,dx + c_1 = -\int \frac{2e^x e^{-2x}}{-e^{-3x}}\,dx + c_1 = e^{2x} + c_1$$

$$B(x) = \int \frac{g(x)y_1(x)}{W}\,dx + c_2 = \int \frac{2e^x e^{-x}}{-e^{-3x}}\,dx + c_2 = -\frac{2}{3}e^{3x} + c_2.$$

The general solution is

$$y(x) = A(x)e^{-x} + B(x)e^{-2x}$$

$$= (e^{2x} + c_1)e^{-x} + \left(-\frac{2}{3}e^{3x} + c_2\right)e^{-2x} = c_1e^{-x} + c_2e^{-2x} + \frac{1}{3}e^x.$$

Example 5.33 Find the general solution of the equation $y'' + 16y = 32 \sec 2x$, using the method of variation of parameters.

Solution The characteristic equation of the corresponding homogeneous equation is $m^2 + 16 = 0$. The characteristic roots are $m = \pm 4i$. The complementary function is given by

$$y_c(x) = Ay_1(x) + By_2(x) = A\cos 4x + B\sin 4x$$

where $y_1(x) = \cos 4x$ and $y_2(x) = \sin 4x$ are two linearly independent solutions of the homogeneous equation. By the method of the variation of parameters, we write the general solution as

$$y(x) = A(x) \cos 4x + B(x) \sin 4x.$$

We have $g(x) = r(x)/a_0(x) = 32 \sec 2x$. The Wronskian of y_1, y_2 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4 \sin 4x & 4 \cos 4x \end{vmatrix} = 4.$$

Therefore, from Eq. (5.64), we obtain

$$A(x) = -\int \frac{g(x) y_2(x)}{W} dx + c_1 = -\frac{1}{4} \int 32 \sec 2x \sin 4x \, dx + c_1$$

$$= -16 \int \sin 2x \, dx + c_1 = 8 \cos 2x + c_1 .$$

$$B(x) = \int \frac{g(x) y_1(x)}{W} dx + c_2 = \frac{1}{4} \int 32 \sec 2x \cos 4x \, dx + c_2$$

$$= 8 \int \frac{2 \cos^2 2x - 1}{\cos 2x} dx + c_2 = 8 \int (2 \cos 2x - \sec 2x) \, dx + c_2$$

$$= 8 \sin 2x - 4 \ln | \sec 2x + \tan 2x | + c_2.$$

The general solution is

$$y(x) = A(x) \cos 4x + B(x) \sin 4x$$

$$= c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x \cos 4x + 8 \sin 2x \sin 4x$$

$$- 4 \sin 4x \ln | \sec 2x + \tan 2x |$$

$$= c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x - 4 \sin 4x \ln | \sec 2x + \tan 2x |.$$

Example 5.34 Find the general solution of the equation $y''' - 6y'' + 11y' - 6y = e^{-x}$.

Solution The characteristic equation of the corresponding homogeneous equation is $m^3 - 6m^2 + 11m - 6 = 0$ and its roots are $m = 1, 2, 3$. The complementary function is given by

$$y_c(x) = A e^x + B e^{2x} + C e^{3x}.$$

By the method of variation of parameters, we assume the solution as

$$y(x) = A(x) e^x + B(x) e^{2x} + C(x) e^{3x}.$$

We have
$$g(x) = r(x)/a_0(x) = e^{-x}.$$

From Eqs. (5.67), the equations for determining $A(x)$, $B(x)$ and $C(x)$ are

$$A' e^x + B' e^{2x} + C' e^{3x} = 0$$

$$A' e^x + 2B' e^{2x} + 3C' e^{3x} = 0$$

$$A' e^x + 4B' e^{2x} + 9C' e^{3x} = e^{-x}.$$

The Wronskian of $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ is given by

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x}.$$

By the Cramer's rule, we obtain

$$WA' = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^{-x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{4x}, \text{ or } A' = \frac{e^{4x}}{2e^{6x}} = \frac{1}{2} e^{-2x}.$$

Integrating, we get $A = -\frac{1}{4} e^{-2x} + c_1$.

Similarly, we have

$$WB' = \begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^{-x} & 9e^{3x} \end{vmatrix} = -2e^{3x}, \text{ or } B' = -\frac{2e^{3x}}{2e^{6x}} = -e^{-3x}.$$

$$WC' = \begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^{-x} \end{vmatrix} = e^{2x}, \text{ or } C' = \frac{e^{2x}}{2e^{6x}} = \frac{1}{2} e^{-4x}.$$

Integrating, we obtain $B(x) = \frac{1}{3} e^{-3x} + c_2$ and $C(x) = -\frac{1}{8} e^{-4x} + c_3$. The general solution is

$$y(x) = A(x)e^x + B(x)e^{2x} + C(x)e^{3x}$$

$$= \left(-\frac{1}{4} e^{-2x} + c_1\right)e^x + \left(\frac{1}{3} e^{-3x} + c_2\right)e^{2x} + \left(-\frac{1}{8} e^{-4x} + c_3\right)e^{3x}$$

$$= c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{24} e^{-x}.$$

Example 5.35 It is given that $y_1 = x$ and $y_2 = 1/x$ are two linearly independent solutions of the associated homogeneous equation of $x^2y'' + xy' - y = x$, $x \neq 0$. Find a particular integral and the general solution of the equation.

Solution By the method of variation of parameters, we write

$$y(x) = A(x)x + B(x)\left(\frac{1}{x}\right).$$

The Wronskian of $y_1(x) = x$ and $y_2(x) = 1/x$ is given by

$$W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{2}{x}, \quad x \neq 0.$$

We have $g(x) = r(x)/a_0(x) = 1/x$. Using Eq. (5.64), we obtain

$$A(x) = -\int \frac{g(x)y_2(x)}{W}\, dx = -\int \frac{1}{x^2}\left(-\frac{x}{2}\right) dx = \frac{1}{2}\ln|x| + c_1.$$

$$B(x) = \int \frac{g(x)y_1(x)}{W}\, dx = \int \frac{1}{x}\left(-\frac{x^2}{2}\right) dx = -\frac{1}{4}x^2 + c_2.$$

The particular integral is

$$y_p(x) = A(x)x + B(x)\left(\frac{1}{x}\right) = \frac{x}{2}\ln|x| - \frac{x}{4}.$$

The general solution is

$$y(x) = y_c(x) + y_p(x) = c_1 x + \frac{1}{x}c_2 + \frac{x}{2}\ln|x| - \frac{x}{4}$$

or

$$y(x) = c_1^{*}x + \frac{1}{x}c_2 + \frac{x}{2}\ln|x|, \quad \text{where} \quad c_1^{*} = c_1 - \frac{1}{4}.$$

Exercise 5.4

Find the general solution of the following differential equations, using the method of variation of parameters.

1. $y'' - 2y' - 3y = e^x$.
2. $y'' - 4y' + 4y = e^{-2x}$.
3. $y'' + 4y = \cos x$.
4. $y'' + y = \sec x$.
5. $y'' + y = \operatorname{cosec} x$.
6. $y'' + y = \tan x$.
7. $y'' - 4y' + 3y = e^x$.
8. $y'' + 4y = \sec 2x$.
9. $y'' + 4y = \cos 2x$.
10. $y'' + 4y' + 4y = e^{-2x}\sin x$.
11. $y'' + 6y' + 9y = e^{-3x}/x$.
12. $y'' + 2y' + 2y = e^{-x}\cos x$.

In the following problems, using the method of variation of parameters and the given linearly independent solutions, find a particular integral and the general solution.

13. $x^2y'' + xy' - y = x^3$, $y_1 = x$, $y_2 = 1/x$.
14. $x^2y'' + xy' - 4y = x^2\ln|x|$, $y_1 = x^2$, $y_2 = 1/x^2$.
15. $x^2y'' - xy' + y = 1/x^4$, $y_1 = x$, $y_2 = x\ln|x|$.
16. $x^2y'' - 2xy' + 2y = x^3 + x$, $y_1 = x$, $y_2 = x^2$.
17. $y'' + 4y' + 8y = 16\,e^{-2x}\operatorname{cosec}^2 2x$, $y_1 = e^{-2x}\cos 2x$, $y_2 = e^{-2x}\sin 2x$.
18. $y''' + 4y' = \sec 2x$, $y_1 = 1$, $y_2 = \cos 2x$, $y_3 = \sin 2x$.
19. $y''' - 6y'' + 12y' - 8y = e^{2x}/x$, $y_1 = e^{2x}$, $y_2 = xe^{2x}$, $y_3 = x^2e^{2x}$.
20. Show that the general solution of the equation $y'' + k^2y = g(x)$, where $k \neq 0$ and $g(x)$ is continuous on I, can always be written as

$$y(x) = A\cos kx + B\sin kx + \frac{1}{k}\int_0^x \sin k(x-t)g(t)\,dt.$$

5.4.2 Method of Undetermined Coefficients

In the previous section, we have discussed the method of variation of parameters for finding the solution of the differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = r(x)$$

where $a_0, a_1, \ldots, a_n$ are constants. In the cases when the right hand side $r(x)$ is of a special form containing exponentials, polynomials, cosine and sine functions, sums or products of these functions, then the particular integral can be easily obtained by the method of undetermined coefficients. The basic idea behind this approach is as follows.

If $r(x)$ is of exponential form e^{mx}, then its derivatives also contain exponentials e^{mx} only, that is, if $r(x) = pe^{mx}$, p constant, then we can choose the particular integral as $y_p(x) = ce^{mx}$, c constant and determine c by substituting $y_p(x)$ in the given equation and comparing both sides of the equation. That is, the equation is identically satisfied.

If $r(x)$ is a cosine or a sine function, cos mx or sin mx, then their derivatives contain the terms cos mx and sin mx. In other words, if $r(x) = p \cos mx$ or $p \sin mx$, p constant, then we can choose the particular integral as $y_p(x) = c_1 \cos mx + c_2 \sin mx$. The constants c_1, c_2 are determined by substituting $y_p(x)$ in the given equation and comparing both sides of the equation.

If $r(x)$ is of the form x^m, then its derivatives contain the terms $x^m, x^{m-1}, \ldots, x, 1$. Hence, when $r(x) = px^m$, p constant then we can choose the particular integral as

$$y_p(x) = c_0 x^m + c_1 x^{m-1} + \ldots + c_{m-1}x + c_m \tag{5.68}$$

where $c_0, c_1, \ldots, c_m$ are constants.

If $r(x)$ is of the forms $e^{ax} \cos bx$ or $e^{ax} \sin bx$ then their derivatives contain the terms $e^{ax} \cos bx$ and $e^{ax} \sin bx$. Hence, when $r(x) = e^{ax} \cos bx$ or $e^{ax} \sin bx$, then we can choose the particular integral as

$$y_p(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx). \tag{5.69}$$

However, if any term in the choice of the particular integral is also a solution of the corresponding homogeneous equation, that is, a term in the complementary function, then we multiply this term by x or by x^m (if the term in the complementary function corresponds to a multiple root of multiplicity m). If $r(x)$ is the sum of a number of functions, then the contribution with respect to each of the terms is included in the choice of the particular integral.

Example 5.36 Using the method of undetermined coefficients find the general solution of the differential equation $y'' + y = 32x^3$.

Solution The characteristic equation of the homogeneous equation is $m^2 + 1 = 0$ and its roots are $m = \pm i$. The complementary function is $y_c(x) = A \cos x + B \sin x$.
Since $r(x) = 32x^3$, we choose the particular integral as

$$y_p(x) = c_1x^3 + c_2x^2 + c_3x + c_4.$$

Substituting in the given equation, we get

$$(6c_1x + 2c_2) + (c_1x^3 + c_2x^2 + c_3x + c_4) = 32x^3.$$

Comparing the coefficients of various powers of x, we get

$$c_1 = 32,\ c_2 = 0,\ 6c_1 + c_3 = 0,\ 2c_2 + c_4 = 0.$$

The solution of the system is $c_1 = 32$, $c_2 = 0$, $c_3 = -192$, $c_4 = 0$. Therefore, $y_p(x) = 32x^3 - 192x$. The general solution is

$$y(x) = A \cos x + B \sin x + 32x(x^2 - 6).$$

Example 5.37 Find the general solution of the differential equation $y'' - 2y' - 3y = 6e^{-x} - 8e^x$.

Solution The characteristic equation of the homogeneous equation is $m^2 - 2m - 3 = 0$ and its roots are $m = -1, 3$.

The complementary function is $y_c(x) = Ae^{-x} + Be^{3x}$.

We note that e^{-x} appears both as a term in $y_c(x)$ (due to the simple root $m = -1$) and the right hand side $r(x)$. The term e^x appears only in $r(x)$. Hence, we choose the particular integral as

$$y_p(x) = c_1 x e^{-x} + c_2 e^x.$$

We have $\quad y_p'(x) = c_1(1 - x)e^{-x} + c_2 e^x, \; y_p''(x) = -c_1(2 - x)e^{-x} + c_2 e^x.$

Substituting in the given equation, we get

$$c_1[-(2 - x) - 2(1 - x) - 3x]e^{-x} + c_2[1 - 2 - 3]e^x = 6x^{-x} - 8e^x$$

or

$$-4c_1 e^{-x} - 4c_2 e^x = 6e^{-x} - 8e^x.$$

Comparing the coefficients of e^{-x} and e^x, we get $c_1 = -3/2$, $c_2 = 2$. The general solution is

$$y(x) = Ae^{-x} + Be^{3x} - \frac{3}{2}xe^{-x} + 2e^x.$$

Example 5.38 Find the general solution of the equation $y'' + 9y = \cos 3x$.

Solution The characteristic equation of the homogeneous equation is $m^2 + 9 = 0$ and its roots are $m = \pm 3i$. The complementary function is

$$y_c(x) = A \cos 3x + B \sin 3x.$$

We note that $\cos 3x$ appears as a term in $y_c(x)$ and the right hand side $r(x)$. Hence, we choose the particular integral as

$$y_p(x) = x(c_1 \cos 3x + c_2 \sin 3x).$$

We have $\quad y_p'(x) = c_1 \cos 3x + c_2 \sin 3x + 3x(-c_1 \sin 3x + c_2 \cos 3x)$

$$y_p''(x) = 6(-c_1 \sin 3x + c_2 \cos 3x) + 9x(-c_1 \cos 3x - c_2 \sin 3x).$$

Substituting in the given equation, we get

$$y_p'' + 9y_p = \sin 3x\,[-6c_1 - 9xc_2 + 9xc_2] + \cos 3x\,[6c_2 - 9xc_1 + 9xc_1] = \cos 3x$$

or

$$-6c_1 \sin 3x + 6c_2 \cos 3x = \cos 3x.$$

Comparing both sides, we get $c_1 = 0$ and $c_2 = 1/6$. The particular integral is $y_p(x) = (x \sin 3x)/6$. The general solution is

$$y(x) = A \cos 3x + B \sin 3x + \frac{1}{6}x \sin 3x.$$

Example 5.39 Find the general solution of the equation $y'' + 4y' + 4y = 12e^{-2x}$.

Solution The characteristic equation of the homogeneous equation is $m^2 + 4m + 4 = (m + 2)^2 = 0$ and its roots are $m = -2, -2$.

The complementary function is $y_c(x) = (Ax + B)e^{-2x}$.

We note that e^{-2x} and xe^{-2x} are present in the complementary function (due to the double root

$m = -2$) and e^{-2x} is also a term on the right hand side $r(x)$. Therefore, we choose the particular integral as

$$y_p(x) = c_1 x^2 e^{-2x}.$$

We have $\quad y_p'(x) = c_1[2x - 2x^2]e^{-2x}, \; y_p''(x) = c_1[2 - 8x + 4x^2]e^{-2x}.$

Substituting in the given equation, we get

$$y_p'' + 4y_p' + 4y_p = c_1[(2 - 8x + 4x^2) + 4(2x - 2x^2) + 4x^2]e^{-2x} = 12e^{-2x}$$

or

$$2c_1 e^{-2x} = 12e^{-2x}.$$

Comparing both sides, we get $c_1 = 6$. Therefore, the particular integral is $y_p(x) = 6x^2e^{-2x}$. The general solution is

$$y(x) = (Ax + B)e^{-2x} + 6x^2e^{-2x} = (Ax + B + 6x^2)e^{-2x}.$$

Example 5.40 Find the general solution of the equation $y'' - 4y' + 13y = 12e^{2x} \sin 3x$.

Solution The characteristic equation of the homogeneous equation is $m^2 - 4m + 13 = 0$. The roots of this equation are

$$m = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i.$$

The complementary function is $y_c(x) = e^{2x}(A \cos 3x + B \sin 3x)$.

We note that $e^{2x} \sin 3x$ appears both in the complementary function and the right hand side $r(x)$. Therefore, we choose

$$y_p(x) = xe^{2x}(c_1 \cos 3x + c_2 \sin 3x).$$

We have

$$y_p'(x) = (1 + 2x)e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + 3xe^{2x}(-c_1 \sin 3x + c_2 \cos 3x)$$

$$y_p''(x) = (4 + 4x)e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$
$$+ 6(1 + 2x)e^{2x}(-c_1 \sin 3x + c_2 \cos 3x) + 9xe^{2x}(-c_1 \cos 3x - c_2 \sin 3x).$$

Substituting in the given equation, we get

$$y_p'' - 4y_p' + 13y_p = e^{2x} \cos 3x\, [c_1(4 + 4x) + 6c_2(1 + 2x) - 9c_1x - 4c_1(1 + 2x)$$
$$- 12xc_2 + 13c_1x] + e^{2x} \sin 3x\, [c_2(4 + 4x) - 6c_1(1 + 2x) - 9c_2x$$
$$- 4c_2(1 + 2x) + 12c_1x + 13xc_2] = 12e^{2x} \sin 3x$$

or

$$6c_2 e^{2x} \cos 3x - 6c_1 e^{2x} \sin 3x = 12 e^{2x} \sin 3x.$$

Comparing both sides, we get $c_1 = -2$ and $c_2 = 0$. Therefore, the particular integral is $y_p(x) = -2xe^{2x} \cos 3x$. The general solution is

$$y(x) = e^{2x}\,[A \cos 3x + B \sin 3x - 2x \cos 3x].$$

Example 5.41 Find the general solution of the differential equation $y''' - 2y'' - 5y' + 6y = 18e^x$.

Solution The characteristic equation of the homogeneous equation is

$$m^3 - 2m^2 - 5m + 6 = (m-1)(m+2)(m-3) = 0, \text{ or } m = 1, -2, 3.$$

The complementary function is $y_c(x) = Ae^x + Be^{-2x} + Ce^{3x}$.

Choose the particular integral as $y_p(x) = c_1 x e^x$.

We have $\quad y_p' = c_1(1+x)e^x,\ y_p'' = c_1(2+x)e^x,\ y_p''' = c_1(3+x)e^x.$

Substituting in the given equation, we get

$$y_p''' - 2y_p'' - 5y_p' + 6y_p = c_1 e^x[(3+x) - 2(2+x) - 5(1+x) + 6x]$$
$$= -6c_1 e^x = 18e^x.$$

Comparing both sides, we get $c_1 = -3$. Hence, the particular integral is $y_p = -3xe^x$. The general solution is

$$y(x) = Ae^x + Be^{-2x} + Ce^{3x} - 3xe^x.$$

Example 5.42 Find the general solution of the differential equation

$$y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}.$$

Solution The characteristic equation of the homogeneous equation is

$$m^3 - 6m^2 + 12m - 8 = (m-2)^3 = 0, \text{ or } m = 2, 2, 2.$$

The complementary function is $y_c(x) = (Ax^2 + Bx + C)e^{2x}$. Note that $m = 2$ is a triple root and e^{2x} is contained in a term in $r(x)$. Therefore, we choose the particular integral as

$$y_p(x) = c_1 x^3 e^{2x} + c_2 e^{-x}.$$

We have $\quad y_p' = c_1(3x^2 + 2x^3)e^{2x} - c_2 e^{-x},\ y_p'' = c_1(6x + 12x^2 + 4x^3)e^{2x} + c_2 e^{-x},$

$$y_p''' = c_1(6 + 36x + 36x^2 + 8x^3)e^{2x} - c_2 e^{-x}.$$

Substituting in the given equation, we get

$$y_p''' - 6y_p'' + 12y_p' - 8y_p = c_1 e^{2x}[(6 + 36x + 36x^2 + 8x^3) - 6(6x + 12x^2 + 4x^3)$$
$$+ 12(3x^2 + 2x^3) - 8x^3] + c_2 e^{-x}[-1 - 6 - 12 - 8]$$
$$= 6c_1 e^{2x} - 27c_2 e^{-x} = 12e^{2x} + 27e^{-x}.$$

Comparing both sides, we get $c_1 = 2$ and $c_2 = -1$. Therefore, the particular integral is $y_p(x) = 2x^3 e^{2x} - e^{-x}$. The general solution is

$$y(x) = (Ax^2 + Bx + C)e^{2x} + 2x^3 e^{2x} - e^{-x}.$$

Exercise 5.5

Find the general solution of the following differential equations by the method of undetermined coefficients.

1. $y'' - 3y' - 10y = 1 + x^2$.
2. $2y'' - y' - 3y = x^3 + x + 1$.

3. $4y'' - y = e^x + e^{3x}$.
4. $3y'' + 2y' - y = e^{-2x} + x$.
5. $y'' + 6y' + 8y = e^{-3x} + e^x$.
6. $y'' + 4y' + 3y = 6e^{-x}$.
7. $2y'' + 3y' - 2y = 5e^{-2x} + e^x$.
8. $y'' - y' - 6y = 5e^{-2x} + 10e^{3x}$.
9. $3y'' + 5y' - 2y = 14e^{x/3}$.
10. $y'' + 3y' + 2y = \cos x + \sin x$.
11. $y'' + y' - 6y = 39 \cos 3x$.
12. $y'' + 4y' - 5y = 34 \cos 2x - 2 \sin 2x$.
13. $y'' + 25y = 50 \cos 5x + 30 \sin 5x$.
14. $y'' + 16y = 16 \sin 4x$.
15. $y'' - 4y' + 4y = 8e^{2x} + e^{3x}$.
16. $4y'' - 4y' + y = 6e^{x/2}$.
17. $y'' + 6y' + 9y = 26e^{-3x} + 5e^{2x}$.
18. $y'' + y = e^x \sin x$.
19. $y'' + 2y' + 10y = e^{-x} \sin 3x$.
20. $y'' - 4y' + 5y = 16e^{2x} \cos x$.
21. $y'' - 6y' + 13y = 6e^{3x} \sin x \cos x$.
22. $y'' + 4y' + 4y = 6e^{-2x} \cos^2 x$.
23. $y'' + 3y' + 2y = 12e^{-x} \sin^3 x$.
24. $y'' - 4y' + 3y = 4 \cosh 3x$.
25. $y''' + 4y'' - y' - 4y = 18e^{-x}$.
26. $y''' + 3y'' - 4y = 12e^{-2x} + 9e^x$.
27. $y''' - 9y'' + 27y' - 27y = 36e^{3x}$.
28. $y''' - y'' + y' - y = 6 \cos 2x$.
29. $y''' - 2y'' + 4y' - 8y = 8(x^2 + \cos 2x)$.
30. $y^{iv} - 256y = 128 \cos 4x$.
31. $y^{iv} - y = x^4 + 1$.
32. $y^{iv} + 3y''' + 3y'' + y' = 2x + 4$.
33. $y^{iv} - 3y'' - 4y = 60e^{2x}$.
34. $y^{iv} + 6y''' + 12y'' + 8y' = 60e^{-2x}$.
35. $y^{iv} - 16y'' = 8x + 16$.

5.4.3 Solution of Euler-Cauchy Equation

In the previous sections, we have discussed methods for finding the solution of the constant coefficient differential equations. Closed form solutions do not exist, in general, for the variable coefficient linear equations. However, for the *Euler-Cauchy equation*

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x y' + a_n y = r(x), \quad x \neq 0 \tag{5.70}$$

where $a_0, a_1, \dots, a_n$ are constants, closed form solutions can be obtained by using one of the following two procedures.

We shall illustrate these procedures using the second order equation

$$a_0 x^2 y'' + a_1 x y' + a_2 y = r(x), \quad a_0 \neq 0, x \neq 0. \tag{5.71}$$

Consider first, the corresponding homogeneous equation

$$a_0 x^2 y'' + a_1 x y' + a_2 y = 0. \tag{5.72}$$

We attempt to find a solution of the form $y = x^m$. We have $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Substituting in Eq. (5.72), we get

$$[a_0 m(m-1) + a_1 m + a_2] x^m = 0. \tag{5.73}$$

Cancelling x^m, we get

$$a_0 m(m-1) + a_1 m + a_2 = a_0 m^2 + (a_1 - a_0)m + a_2 = 0 \tag{5.74}$$

which is called the *auxiliary* equation corresponding to the Eq. (5.72). Equation (5.74) has two roots $m = m_1, m_2$, which may be real and distinct, real and equal or complex conjugates. In these cases, we obtain the following solutions.

Real and distinct roots

If the roots m_1 and m_2 are real and distinct, then the two linearly independent solutions are

$$y_1(x) = x^{m_1} \quad \text{and} \quad y_2(x) = x^{m_2}. \tag{5.75}$$

The general solution is given by

$$y(x) = Ax^{m_1} + Bx^{m_2} \tag{5.76}$$

where A, B are arbitrary constants.

Example 5.43 Find the solution of the differential equation $x^2y'' + 2xy' - 2y = 0$.

Solution Here, $a_0 = 1$, $a_1 = 2$ and $a_2 = -2$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = m^2 + m - 2 = 0, \quad \text{or} \quad (m+2)(m-1) = 0.$$

The roots of this equation are $m = 1, -2$. Hence, the two linearly independent solutions are

$$y_1(x) = x, \quad \text{and} \quad y_2(x) = x^{-2}.$$

The general solution is $y(x) = Ax + (B/x^2)$.

Example 5.44 Find the solution of the differential equation $2x^2y'' + xy' - 6y = 0$.

Solution Here, $a_0 = 2$, $a_1 = 1$, and $a_2 = -6$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = 2m^2 - m - 6 = 0, \quad \text{or} \quad (m-2)(2m+3) = 0.$$

The roots of this equation are $m = 2, -3/2$. The two linearly independent solutions are

$$y_1(x) = x^2, \quad \text{and} \quad y_2(x) = x^{-3/2}.$$

The general solution is $y(x) = Ax^2 + \dfrac{B}{x\sqrt{x}}$.

Real and equal roots

Let the roots of the auxiliary equation be real and equal, that is, $m = m_1$ is a double root. Then $m_1 = (a_0 - a_1)/(2a_0)$. Since, the discriminant of Eq. (5.74) vanishes in this case, we can also write $m_1^2 = a_2/a_0$ (product of roots). Then, $y_1(x) = x^{m_1}$ is one of the linearly independent solutions. The second linearly independent solution can now be obtained by the method of reduction of order (see section 5.3.3). Write $y_2(x) = u(x)y_1(x)$. We have

$$y_2' = uy_1' + u'y_1, \; y_2'' = uy_1'' + 2u_1'y_1' + u''y_1.$$

Substituting in Eq. (5.72) and simplifying, we get

$$a_0x^2(uy_1'' + 2u'y_1' + u''y_1) + a_1x(uy_1' + u'y_1) + a_2uy_1 = 0$$

or
$$a_0y_1x^2u'' + xu'(2a_0xy_1' + a_1y_1) + u(a_0x^2y_1'' + a_1xy_1' + a_2y_1) = 0. \tag{5.77}$$

Since $y_1(x)$ is a solution of Eq. (5.72), the third term in Eq. (5.77) vanishes. Further, since $y_1(x) = x^{m_1}$ where $m_1 = (a_0 - a_1)/(2a_0)$, we obtain

$$2a_0xy_1' + a_1y_1 = (2a_0m_1 + a_1)x^{m_1} = a_0x^{m_1} = a_0y_1.$$

Therefore, Eq. (5.77) simplifies to

$$a_0y_1x^2u'' + a_0xu'y_1 = (xu'' + u')a_0xy_1 = 0$$

Since $x \neq 0$, $y_1 \neq 0$, $a_0 \neq 0$, we get $xu'' + u' = 0$. Separating the variables, we get

$$\frac{u''}{u'} = -\frac{1}{x}$$

Integrating, we get for $x > 0$

$$\ln|u'| = -\ln x, \quad \text{or} \quad u' = \frac{1}{x}.$$

Integrating again, we get $u = \ln x$.

Therefore, $y_2 = uy_1 = y_1 \ln x$. Since $y_2/y_1 = \ln x$, is not a constant, the two solutions y_1, y_2 are linearly independent. The general solution in this case is

$$y(x) = Ay_1 + By_2 = (A + B\ln x)y_1 = (A + B\ln x)x^{m_1} \tag{5.78}$$

where $m_1 = (a_0 - a_1)/(2a_0)$.

Example 5.45 Find the solution of the differential equation $4x^2y'' + y = 0$.

Solution Here, $a_0 = 4$, $a_1 = 0$, $a_2 = 1$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = 4m^2 - 4m + 1 = 0, \quad \text{or} \quad (2m-1)^2 = 0.$$

The equation has the double root $m = 1/2$. The general solution is (from Eq. (5.78))

$$y(x) = (A + B\ln x)\,x^{1/2}, \quad x > 0.$$

Complex roots

Let the roots of the auxiliary equation (5.74) be a complex conjugate pair, $m = p \pm iq$. Then the solutions are given by

$$x^m = x^{p\pm iq} = x^p x^{\pm iq} = x^p(e^{\ln x})^{\pm iq}$$

$$= x^p e^{\pm iq\ln x} = x^p[\cos(q\ln x) \pm i\sin(q\ln x)], \quad x > 0.$$

Therefore, we can take the two linearly independent solutions as

$$y_1(x) = x^p\cos(q\ln x), \quad \text{and} \quad y_2(x) = x^p\sin(q\ln x). \tag{5.79}$$

Example 5.46 Find the general solution of the equation $4x^2y'' + 8xy' + 17y = 0$.

Solution Here, $a_0 = 4$, $a_1 = 8$ and $a_2 = 17$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = 4m^2 + 4m + 17 = 0.$$

The roots of this equation are $m = \dfrac{-4 \pm \sqrt{16 - 272}}{8} = \dfrac{-4 \pm 16i}{8} = -\dfrac{1}{2} \pm 2i = p \pm iq.$

The general solution is (from Eq. (5.79))

$$y(x) = Ax^{-1/2}\cos(2\ln x) + Bx^{-1/2}\sin(2\ln x).$$

The method considered here is easily applicable for the homogeneous equations. However, for non-homogeneous equations, finding a particular solution is difficult.

We now discuss a method which can be applied for the solution of general Euler-Cauchy equation given by Eq. (5.71). For $x > 0$, we change the independent variable to

$$x = e^t, \quad \text{or} \quad t = \ln x, \quad x > 0. \tag{5.80}$$

The case $x < 0$ can also be considered by writing the transformation as

$$|x| = e^t, \quad \text{or} \quad t = \ln |x|. \tag{5.81}$$

This transformation always reduces the Euler-Cauchy equation into a linear equation with constant coefficients. The solution of this equation can tnen be obtained using the methods discussed in the previous sections. Finally, the solution of the given equation, in terms of the original variable x, is obtained by replacing t by $\ln x$.

When $x = e^t$, $t = \ln x$, we have

$$\frac{d}{dx} = \frac{d}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{d}{dt}, \quad \text{or} \quad x\frac{d}{dx} = \frac{d}{dt} \tag{5.82}$$

$$\frac{d^2}{dx^2} = -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{d}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x^2}\frac{d^2}{dt^2}$$

or
$$x^2\frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt} = \frac{d}{dt}\left(\frac{d}{dt} - 1\right). \tag{5.83}$$

In operator notation, set $D = d/dx$, $D^2 = d^2/dx^2$, $\theta = d/dt$, $\theta^2 = d^2/dt^2$ etc. Then, Eqs. (5.82), (5.83) can be written as

$$xD = \theta, \quad x^2D^2 = \theta^2 - \theta = \theta(\theta - 1) \tag{5.84}$$

or
$$xDy = \theta y, \quad x^2D^2y = \theta(\theta - 1)y. \tag{5.85}$$

By induction, we can prove that

$$x^n D^n y = \theta(\theta - 1)\ldots[\theta - (n - 1)]y. \tag{5.86}$$

Substituting in the non-homogeneous second order linear equation (5.71), we obtain the reduced equation as

$$a_0\,\theta(\theta - 1)\,y + a_1\,\theta y + a_2\,y = a_0\,\theta^2 y + (a_1 - a_0)\,\theta y + a_2 y = r(e^t). \tag{5.87}$$

This is a linear equation with constant coefficients. The methods described in the previous sections can be applied to find its solution.

Example 5.47 Find the general solution of the equation $2x^2y'' + 3xy' - 3y = x^3$.

Solution Using the transformation $x = e^t$, we get (using Eqs. (5.82) and 5.83))

$$2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 3\frac{dy}{dt} - 3y = e^{3t}, \quad \text{or} \quad 2\frac{d^2y}{dt} + \frac{dy}{dt} - 3y = e^{3t}. \tag{5.88}$$

This is a linear, constant coefficient equation. Substituting, $y = e^{mt}$, the characteristic equation of the corresponding homogeneous equation is obtained as

$$2m^2 + m - 3 = 0, \quad \text{or} \quad (m - 1)(2m + 3) = 0, \quad \text{or} \quad m = 1, -3/2.$$

The complementary function is $y_c(t) = Ae^t + Be^{-3t/2}$.

Let the particular integral be written as $y_p = ce^{3t}$. Substituting in Eq. (5.88), we obtain

$$(18 + 3 - 3)ce^{3t} = e^{3t}, \quad \text{or} \quad c = 1/18.$$

The particular integral is $y_p = e^{3t}/18$.

The general solution is $y(t) = Ae^{t} + Be^{-3t/2} + \frac{1}{18}e^{3t}$.

Substituting $e^t = x$, we get the general solution as

$$y(x) = Ax + \frac{B}{x\sqrt{x}} + \frac{x^3}{18}.$$

Example 5.48 Find the general solution of the equation $x^2y'' + 5xy' + 3y = \ln x, \quad x > 0.$

Solution Using the transformation $x = e^t$, we obtain

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 5\frac{dy}{dt} + 3y = \ln(e^t), \quad \text{or} \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = t. \tag{5.89}$$

The characteristic equation of the corresponding homogeneous equation is

$$m^2 + 4m + 3 = 0, \quad \text{or} \quad (m + 1)(m + 3) = 0, \quad \text{or} \quad m = -1, -3.$$

The complementary function is $y_c(t) = Ae^{-t} + Be^{-3t}$.

Let the particular integral be written as $y_p = c_1t + c_2$. Substituting in Eq. (5.89), we get

$$4c_1 + 3(c_1t + c_2) = t.$$

Comparing the coefficients of t and the constant terms on both sides, we obtain $3c_1 = 1$ and $4c_1 + 3c_2 = 0$. The solution is $c_1 = 1/3$, $c_2 = -4/9$.

The particular integral is $y_p = \frac{t}{3} - \frac{4}{9}$.

The general solution of the given equation is

$$y(t) = Ae^{-t} + Be^{-3t} + \frac{t}{3} - \frac{4}{9}.$$

Substituting $e^t = x$, we get the general solution as

$$y(x) = \frac{A}{x} + \frac{B}{x^3} + \frac{1}{3}\ln x - \frac{4}{9}.$$

Example 5.49 Find the general solution of the equation $x^2y'' - 5xy' + 13y = 30x^2$.

Solution Using the transformation $x = e^t$, we obtain

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 5\frac{dy}{dt} + 13y = 30e^{2t}, \quad \text{or} \quad \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 13y = 30e^{2t}. \tag{5.90}$$

The characteristic equation of the corresponding homogeneous equation is

$$m^2 - 6m + 13 = 0, \quad \text{or} \quad m = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

The complementary function is $y_c(t) = e^{3t}(A \cos 2t + B \sin 2t)$.

Let the particular integral be written as $y_p = ce^{2t}$. Substituting in equation (5.90), we obtain

$$(4 - 12 + 13)ce^{2t} = 30e^{2t}, \quad \text{or} \quad c = 6.$$

The particular integral is $y_p = 6e^{2t}$.
The general solution is $y(t) = e^{3t}(A \cos 2t + B \sin 2t) + 6e^{2t}$.
Substituting $e^t = x$, we get

$$y(x) = x^3[A \cos (2 \ln x) + B \sin (2 \ln x)] + 6x^2.$$

Example 5.50 Find the general solution of the equation

$$x^3y''' + 5x^2y'' + 5xy' + y = x^2 + \ln x, \quad x > 0.$$

Solution Using the transformation $x = e^t$, we get (in operator notation)

$$[\theta(\theta-1)(\theta-2) + 5\theta(\theta-1) + 5\theta + 1]\, y = e^{2t} + t$$

or
$$[\theta^3 - 3\theta^2 + 2\theta + 5\theta^2 - 5\theta + 5\theta + 1]\, y = e^{2t} + t$$

or
$$[\theta^3 + 2\theta^2 + 2\theta + 1]\, y = e^{2t} + t \tag{5.91}$$

where $\theta = d/dt$.

The characteristic equation of the corresponding homogeneous equation is

$$m^3 + 2m^2 + 2m + 1 = 0, \quad \text{or} \quad (m+1)(m^2 + m + 1) = 0.$$

Its roots are $m = -1, \dfrac{-1 \pm i\sqrt{3}}{2}$.
The complementary function is

$$y_c(t) = Ae^{-t} + [B \cos (\sqrt{3}t/2) + C \sin (\sqrt{3}t/2)]e^{-t/2}.$$

Let the particular integral be written as $y_p = c_1e^{2t} + c_2t + c_3$.

Then, $$y_p' = 2c_1e^{2t} + c_2,\ y_p'' = 4c_1e^{2t},\ y_p''' = 8c_1e^{2t}.$$

Substituting in Eq. (5.91), we obtain

$$(8 + 8 + 4 + 1)c_1e^{2t} + c_2t + 2c_2 + c_3 = e^{2t} + t, \quad \text{or} \quad 21c_1e^{2t} + c_2t + 2c_2 + c_3 = e^{2t} + t,$$

Comparing both sides, we get $c_1 = 1/21$, $c_2 = 1$, $c_3 = -2$.
The particular integral is $y_p = \dfrac{1}{21}e^{2t} + t - 2$.
The general solution is

$$y(t) = Ae^{-t} + [B \cos (\sqrt{3}t/2) + C \sin (\sqrt{3}t/2)]e^{-t/2} + \frac{1}{21}e^{2t} + t - 2.$$

Substituting $e^t = x$, we get

$$y(x) = \frac{A}{x} + \frac{1}{\sqrt{x}}[B \cos (\sqrt{3} \ln x/2) + C \sin (\sqrt{3} \ln x/2)] + \left(\frac{x^2}{21} + \ln x - 2\right).$$

Example 5.51 Find the general solution of the equation

$$x^3y''' - 3xy' + 3y = 16x + 9x^2 \ln x, \quad x > 0.$$

Solution Using the transformation $x = e^t$, we get (in operator notation)

$$[\theta(\theta-1)(\theta-2) - 3\theta + 3]y = 16e^t + 9te^{2t}$$

or
$$(\theta^3 - 3\theta^2 - \theta + 3)y = 16e^t + 9te^{2t} \tag{5.92}$$

where $\theta = d/dt$. The characteristic equation of the corresponding homogeneous equation is

$$m^3 - 3m^2 - m + 3 = 0, \quad \text{or} \quad (m-1)(m+1)(m-3) = 0, \quad \text{or} \quad m = \pm 1, 3.$$

The complementary function is given by $y_c(t) = Ae^t + Be^{-t} + Ce^{3t}$. Note that e^t, which is one of the linearly independent solutions, also appears as a term on the right hand side of Eq. (5.92). Hence, by the method of undetermined parameters, we write the particular solution as

$$y_p = (c_1 t + c_2)e^{2t} + c_3 t\, e^t$$

We have

$$y_p' = (c_1 + 2c_1 t + 2c_2)e^{2t} + (1+t)c_3 e^t$$

$$y_p'' = (4c_1 + 4c_1 t + 4c_2)e^{2t} + (2+t)c_3 e^t$$

$$y_p''' = (12c_1 + 8c_1 t + 8c_2)e^{2t} + (3+t)c_3 e^t.$$

Substituting in Eq. (5.92), we obtain

$$[(12c_1 + 8c_1 t + 8c_2) - 3(4c_1 + 4c_1 t + 4c_2) - (c_1 + 2c_1 t + 2c_2) + 3(c_1 t + c_2)]e^{2t}$$

$$+ [(3+t)c_3 - 3(2+t)c_3 - (1+t)c_3 + 3c_3 t]e^t = 16e^t + 9te^{2t}$$

or

$$-(c_1 + 3c_1 t + 3c_2)e^{2t} - 4c_3 e^t = 16e^t + 9te^{2t}.$$

Comparing both sides, we obtain $c_1 + 3c_2 = 0$, $-3c_1 = 9$, $-4c_3 = 16$. The solution is $c_1 = -3$, $c_2 = 1$ and $c_3 = -4$.

The particular integral is $y_p(t) = (1 - 3t)e^{2t} - 4t\,e^t$.

The general solution is $y(t) = Ae^t + Be^{-t} + Ce^{3t} + (1-3t)e^{2t} - 4t\,e^t$.

Substituting $x = e^t$, we obtain the general solution as

$$y(x) = Ax + \frac{B}{x} + Cx^3 + (1 - 3\ln x)x^2 - 4x\ln x.$$

Exercise 5.6

Find the general solution of the following homogeneous differential equations (Assume $x > 0$ in Problems 1 to 20).

1. $x^2y'' + xy' - 4y = 0$.
2. $x^2y'' + 4xy' + 2y = 0$. $-1, -2$
3. $x^2y'' + xy' - y = 0$.
4. $9x^2y'' + 15xy' + y = 0$.
5. $4x^2y'' + 16xy' + 9y = 0$.
6. $2x^2y'' + 2xy' + y = 0$.
7. $x^2y'' + 3xy' + y = 0$.
8. $x^2y'' - xy' + 5y = 0$.
9. $x^2y'' + 3xy' + 10y = 0$.
10. $9x^2y'' + 3xy' + 10y = 0$.
11. $x^3y''' + 2x^2y'' = 0$.
12. $x^3y''' + xy' - y = 0$.
13. $x^3y''' + 4x^2y'' + 2xy' - 2y = 0$.
14. $x^3y''' + 9x^2y'' + 18xy' + 6y = 0$.
15. $x^3y''' - 2xy' + 4y = 0$.
16. $x^3y''' + 3x^2y'' + 14xy' + 34y = 0$.
17. $x^4y^{iv} + 3x^3y''' = 0$.
18. $x^4y^{iv} + 6x^3y''' + 4x^2y'' - 2xy' - 4y = 0$.
19. $4x^4y^{iv} + 24x^3y''' + 43x^2y'' + 19xy' - 4y = 0$.
20. $x^4y^{iv} + 6x^3y''' + 5x^2y'' - xy' + y = 0$.

Find the general solution of the following differential equations (Assume $x > 0$ in Problems 21 to 40).

21. $x^2y'' - 2y = 2x + 6$.
22. $x^2y'' - 3xy' + 3y = 2 + 3\ln x$.

23. $x^2y'' + 2xy' - 2y = 6x - 14$.

24. $x^2y'' + 2xy' - 6y = 15x^2$.

25. $x^2y'' + 2xy' = \cos(\ln x)$.

26. $x^2y'' + 5xy' - 5y = 24x \ln x$.

27. $4x^2y'' + y = 25 \sin(\ln x)$.

28. $x^2y'' - 3xy' + 4y = x^3$.

29. $4x^2y'' + 16xy' + 9y = 19\cos(\ln x) + 22\sin(\ln x)$.

30. $x^2y'' + 2xy' - 2y = x\sin(\ln x)$.

31. $x^2y'' - 2xy' - 4y = 6x^2 + 4\ln x$.

32. $x^3y''' + 8x^2y'' + 5xy' - 5y = 42x^2$.

33. $x^3y''' + 6x^2y'' - 12y = 12/x^2$.

34. $x^3y''' - 3x^2y'' + 7xy' - 8y = 3x^3 + 8x$.

35. $4x^3y''' + 12x^2y'' + xy' + y = 50\sin(\ln x)$.

36. $(3x+1)^2y'' + (3x+1)y' + y = 6x$.

37. $(x+2)^3y''' + (x+2)^2y'' + (x+2)y' - y = 24x^2$.

38. $x^4y^{iv} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = 10/x^3$.

39. $4x^4y^{iv} + 16x^3y''' - x^2y'' + 9xy' - 9y = 14x^2 + 1$.

40. $x^4y^{iv} + 6x^3y''' + 12x^2y'' + 6xy' + 4y = 2/x^2$.

Find the solutions of the following differential equations, which satisfy the given conditions.

41. $2x^2y'' + 3xy' - y = x$, $y(1) = 1$, $y(4) = 41/16$.

42. $4x^2y'' + y = \ln x$, $x > 0$, $y(1) = 0$, $y(e) = 5$.

43. $x^2y'' - 3xy' + 3y = 5x^2 - x$, $y(1) = 1$, $y'(1) = 3/2$.

44. $x^2y'' - xy' + 2y = 6$, $y(1) = 1$, $y'(1) = 2$.

45. $x^2y'' + 3xy' + 10y = 9x^2$, $y(1) = 5/2$, $y'(1) = 8$.

5.5 Operator Methods for Finding Particular Integrals

In section 5.3.1, we have introduced the differential operator D, where $D = d/dx$. For example, we can write

$$L(y) = a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (aD^2 + bD + c)y = F(D)y.$$

Since D is a differential operator, its inverse D^{-1} defines the integral operator, such that $D^{-1}Df(x) = f(x)$.

In this section, we develop symbolic short cut methods for finding a particular integral of a linear non-homogeneous equation with constant coefficients.

Consider the linear non-homogeneous equation with constant coefficients

$$L(y) = a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\frac{dy}{dx} + a_ny = r(x) \tag{5.93}$$

or

$$L(y) = (a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n)y = F(D)y = r(x) \tag{5.94}$$

where $F(D) = a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n$, and $a_0, a_1, \ldots, a_n$ are constants. From Eq. (5.94), we write the particular integral as

$$y_p(x) = [F(D)]^{-1}r(x). \tag{5.95}$$

In the following, we develop methods for finding $[F(D)]^{-1}r(x)$ for particular cases of $r(x)$.

5.5.1 Case $r(x) = e^{\alpha x}$.

When $y = e^{\alpha x}$, we have

$$F(D)y = (a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n)e^{\alpha x}$$
$$= (a_0 \alpha^n + a_1 \alpha^{n-1} + \ldots + a_{n-1}\alpha + a_n)e^{\alpha x} = F(\alpha)e^{\alpha x}.$$

Case $F(\alpha) \neq 0$

We may now symbolically write this equation as

$$y = [F(D)]^{-1}F(\alpha)e^{\alpha x} = F(\alpha)[F(D)]^{-1}e^{\alpha x}$$

since $F(\alpha)$ is a constant. We can further write

$$\frac{1}{F(\alpha)}y = [F(D)]^{-1}e^{\alpha x}, \quad \text{or} \quad [F(D)]^{-1}e^{\alpha x} = \frac{1}{F(\alpha)}e^{\alpha x}.$$

Hence, if $r(x) = e^{\alpha x}$, we obtain

$$y_p(x) = [F(D)]^{-1}e^{\alpha x} = \frac{1}{F(\alpha)}e^{\alpha x}, \quad F(\alpha) \neq 0. \tag{5.96}$$

We can verify that this result is true. Operating with $F(D)$ on both sides, we get

$$F(D)y_p(x) = F(D)\cdot\frac{1}{F(\alpha)}e^{\alpha x} = \frac{1}{F(\alpha)}F(D)e^{\alpha x}$$
$$= \frac{1}{F(\alpha)}F(\alpha)e^{\alpha x} = e^{\alpha x}.$$

Example 5.52 Find the general solution of the differential equation $y'' - 2y' - 3y = 3e^{2x}$.

Solution In operator notation, the given equation is $(D^2 - 2D - 3)y = 3e^{2x}$. The characteristic equation of the corresponding homogeneous equation is

$$(m^2 - 2m - 3) = (m - 3)(m + 1) = 0. \text{ Its roots are } m = -1, 3.$$

The complementary function is given by $y_c(x) = Ae^{-x} + Be^{3x}$.

We have $F(D) = D^2 - 2D - 3$. The particular integral is

$$y_p(x) = [F(D)]^{-1}r(x) = [D^2 - 2D - 3]^{-1}(3e^{2x}) = \frac{3e^{2x}}{F(2)} = \frac{3}{-3}e^{2x} = -e^{2x}.$$

The general solution is

$$y(x) = y_c(x) + y_p(x) = Ae^{-x} + Be^{3x} - e^{2x}.$$

Example 5.53 Find the general solution of the equation $y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$.

Solution The given equation in operator notation is

$$F(D)y = (D^3 - 2D^2 - 5D + 6)y = 4e^{-x} - e^{2x}, \quad \text{where} \quad F(D) = D^3 - 2D^2 - 5D + 6.$$

The characteristic equation of the corresponding homogeneous equation is

$$m^3 - 2m^2 - 5m + 6 = 0, \quad \text{or} \quad (m - 1)(m + 2)(m - 3) = 0.$$

The roots of this equation are $m = 1, -2, 3$. The complementary function is

$$y_c(x) = Ae^{x} + Be^{-2x} + Ce^{3x}.$$

The particular integral is

$$y_p(x) = [F(D)]^{-1}(4e^{-x} - e^{2x})$$
$$= [F(D)]^{-1}(4e^{-x}) - [F(D)]^{-1}e^{2x}$$
$$= \frac{4}{F(-1)} e^{-x} - \frac{1}{F(2)} e^{2x} = \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$

The general solution is

$$y(x) = y_c(x) + y_p(x) = Ae^{x} + Be^{-2x} + Ce^{3x} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$

Before we discuss the case when $F(\alpha) = 0$, let us derive the following result

$$F(D)[g(x)e^{\alpha x}] = e^{\alpha x} F(D + \alpha) g(x). \tag{5.97}$$

By Leibniz theorem, we have

$$D^n[e^{\alpha x}g(x)] = (D^n e^{\alpha x})g + {}^nC_1(D^{n-1}e^{\alpha x})(Dg) + \ldots + e^{\alpha x}(D^n g)$$
$$= \alpha^n e^{\alpha x} g + {}^nC_1 \alpha^{n-1} e^{\alpha x} Dg + {}^nC_2 \alpha^{n-2} e^{\alpha x}(D^2 g) + \ldots + e^{\alpha x}(D^n g)$$
$$= e^{\alpha x}[D^n g + {}^nC_1(D^{n-1}g)\alpha + {}^nC_2(D^{n-2}g)\alpha^2 + \ldots + \alpha^n g]$$

(by reversing the order of terms and using the result ${}^nC_r = {}^nC_{n-r}$)

$$= e^{\alpha x}[D^n + {}^nC_1 \alpha D^{n-1} + {}^nC_2 \alpha^2 D^{n-2} + \ldots + \alpha^n]g$$
$$= e^{\alpha x}[D + \alpha]^n g.$$

Substituting the expressions for $n = 1, 2, \ldots$ on the left hand side of Eq. (5.97), we obtain

$$F(D)[e^{\alpha x}g(x)] = [a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n](e^{\alpha x} g)$$
$$= a_0 D^n(e^{\alpha x} g) + a_1 D^{n-1}(e^{\alpha x} g) + \ldots + a_n(e^{\alpha x} g)$$
$$= a_0 e^{\alpha x}(D + \alpha)^n g + a_1 e^{\alpha x}(D + \alpha)^{n-1} g + \ldots + a_n e^{\alpha x} g$$
$$= e^{\alpha x}[a_0 (D + \alpha)^n + a_1(D + \alpha)^{n-1} + \ldots + a_n]g$$
$$= e^{\alpha x} F(D + \alpha) g.$$

Case $F(\alpha) = 0$.

Let us now consider the case $F(\alpha) = 0$. From the theory of polynomial equations, $(D - \alpha)$ is a factor of $F(D)$. If $F'(\alpha)$ also vanishes, then $(D - \alpha)^2$ is a factor of $F(D)$. If $F(\alpha) = 0 = F'(\alpha) = \ldots = F^{(r-1)}(\alpha)$, $F^r(\alpha) \neq 0$, then $(D - \alpha)^r$ is a factor of $F(D)$. Then, we can write

$$F(D) = (D - \alpha)^r G(D), \quad G(\alpha) \neq 0. \tag{5.98}$$

Let us now write the particular integral of $F(D)y = e^{\alpha x}$ as

$$y_p(x) = [F(D)]^{-1}e^{\alpha x} = [(D - \alpha)^r G(D)]^{-1} e^{\alpha x} = [(D - \alpha)^r]^{-1}[G(D)]^{-1}e^{\alpha x}$$
$$= [(D - \alpha)^r]^{-1}[G(\alpha)]^{-1}e^{\alpha x} \qquad \text{(since } G(\alpha) \neq 0 \text{ and using Eq. (5.96))}$$
$$= \frac{1}{G(\alpha)}[(D - \alpha)^r]^{-1}[e^{\alpha x} \cdot 1] = \frac{1}{G(\alpha)} e^{\alpha x}[(D + \alpha - \alpha)^r]^{-1}[1]$$

(using Eq. (5.97))

$$= \frac{1}{G(\alpha)} e^{\alpha x}[D^r]^{-1}[1] = \frac{1}{G(\alpha)} e^{\alpha x}[D^{-r}][1]$$

$$= \frac{1}{G(\alpha)} e^{\alpha x} \frac{x^r}{r!} = \frac{x^r}{r!} \frac{e^{\alpha x}}{G(\alpha)} \tag{5.99}$$

since D^{-r} represents integration r times.

Therefore, when $F(\alpha) = 0$, $F'(\alpha) \neq 0$, we have $F(D) = (D - \alpha)G(D)$ and the particular integral of $F(D)y = e^{\alpha x}$ is given by

$$y_p(x) = \frac{x}{1!} \frac{e^{\alpha x}}{G(\alpha)}. \tag{5.100}$$

Generalization to the case $r(x) = e^{\alpha x}h(x)$.

Irrespective of whether $F(\alpha)$ vanishes or does not vanish, the above result can be extended to the case $r(x) = e^{\alpha x}h(x)$. We have the particular integral in this case as

$$y_p(x) = [F(D)]^{-1}[e^{\alpha x} h(x)] = e^{\alpha x}[F(D + \alpha)]^{-1}h(x) \tag{5.101}$$

using Eq. (5.97). Now, $[F(D + \alpha)]^{-1}h(x)$ can be evaluated when $h(x)$ is of some particular forms.

Example 5.54 Find the general solution of the equation $y'' + y' - 6y = 5e^{-3x}$.

Solution The equation in operator notation is $(D^2 + D - 6)y = 5e^{-3x}$, where

$$F(D) = D^2 + D - 6 = (D + 3)(D - 2).$$

The characteristic equation of the corresponding homogeneous equation $(D^2 + D - 6)y = 0$ is

$$m^2 + m - 6 = 0, \quad \text{or} \quad (m + 3)(m - 2) = 0, \quad \text{or} \quad m = 2, -3.$$

The complementary function is $y_c(x) = Ae^{2x} + Be^{-3x}$.

Now, $F(m) = m^2 + m - 6$, $F(-3) = 0$ and $F'(-3) = -5 \neq 0$. Therefore,

$$y_p(x) = [(D + 3)(D - 2)]^{-1}(5e^{-3x}) = 5(D + 3)^{-1}[(D - 2)^{-1}e^{-3x}]$$

$$= 5(D + 3)^{-1}(-5)^{-1}e^{-3x} = -(D + 3)^{-1}[e^{-3x} \cdot 1] \quad \text{(using Eq. (5.96))}$$

$$= -e^{-3x}(D - 3 + 3)^{-1} \cdot 1 = -e^{-3x} D^{-1}(1) = -x e^{-3x}. \quad \text{(using Eq. (5.99))}$$

We might have also used the formula (5.100) directly where $G(D) = D - 2$. The general solution is

$$y(x) = y_c(x) + y_p(x) = Ae^{2x} + Be^{-3x} - x e^{-3x}.$$

Example 5.55 Find the general solution of the equation $4y'' - 4y' + y = e^{x/2}$.

Solution The characteristic equation of the corresponding homogeneous equation is

$$4m^2 - 4m + 1 = 0, \quad \text{or} \quad (2m - 1)^2 = 0. \text{ Its roots are } m = 1/2, 1/2.$$

The complementary function is $y_c(x) = (A + Bx)e^{x/2}$. We have

$$F(D) = 4D^2 - 4D + 1 = (2D - 1)^2, \text{ where } F(1/2) = 0, \text{ and } F'(1/2) = 0.$$

The particular integral is

$$y_p(x) = (2D - 1)^{-2}(e^{x/2} \cdot 1) = e^{x/2}\left[2\left(D + \frac{1}{2}\right) - 1\right]^{-2}(1)$$

$$= \frac{1}{4} e^{x/2} D^{-2}(1) = \frac{x^2}{8} e^{x/2}.$$

The general solution is $y(x) = (A + Bx)e^{x/2} + (x^2 e^{x/2})/8$.

Example 5.56 Find the general solution of the equation $9y''' + 3y'' - 5y' + y = 42e^x + 64e^{x/3}$

Solution The characteristic equation of the corresponding homogeneous equation $9y''' + 3y'' - 5y' + y = 0$ is

$$9m^3 + 3m^2 - 5m + 1 = 0, \quad \text{or} \quad (m + 1)(3m - 1)^2 = 0.$$

The roots of this equation are $m = -1, 1/3, 1/3$. The complementary function is

$$y_c(x) = Ae^{-x} + (Bx + C)e^{x/3}$$

We have $F(D) = 9D^3 + 3D^2 - 5D + 1 = (D + 1)(3D - 1)^2$ and $F(1/3) = 0$, $F'(1/3) = 0$.

The particular integral is

$$y_p(x) = [(D + 1)(3D - 1)^2]^{-1}(42e^x + 64e^{x/3})$$

$$= [(D + 1)(3D - 1)^2]^{-1}(42e^x) + [(D + 1)(3D - 1)^2]^{-1}(64e^{x/3}).$$

Since $F(1) \neq 0$ and $F(1/3) = 0$, we obtain

$$y_p(x) = [(1 + 1)(3 - 1)^2]^{-1}(42e^x) + (3D - 1)^{-2}[(D + 1)^{-1}(64e^{x/3})]$$

$$= \frac{21}{4}e^x + (3D - 1)^{-2}\left[\frac{64}{(4/3)}e^{x/3}\right]$$

$$= \frac{21}{4}e^x + 48e^{x/3}\left[3\left(D + \frac{1}{3}\right) - 1\right]^{-2}(1)$$

$$= \frac{21}{4}e^x + \frac{48}{9}e^{x/3}D^{-2}(1) = \frac{21}{4}e^x + \frac{8}{3}x^2e^{x/3}.$$

The general solution is

$$y(x) = Ae^{-x} + (Bx + C)e^{x/3} + \frac{21}{4}e^x + \frac{8}{3}x^2e^{x/3}.$$

Example 5.57 Find the general solution of the equation $16y'' + 8y' + y = 48xe^{-x/4}$.

Solution The characteristic equation of the corresponding homogeneous equation is

$16m^2 + 8m + 1 = 0$, or $(4m + 1)^2 = 0$. Its roots are $m = -1/4, -1/4$.

The complementary function is $y_c(x) = (Ax + B)e^{-x/4}$.

We have $F(D) = 16D^2 + 8D + 1 = (4D + 1)^2$ where $F(-1/4) = 0$, and $F'(-1/4) = 0$.

The particular integral is

$$y_p(x) = (4D + 1)^{-2}(48xe^{-x/4}) = 48e^{-x/4}\left[4\left(D - \frac{1}{4}\right) + 1\right]^{-2} x$$

$$= 48e^{-x/4}(4D)^{-2}(x) = 3e^{-x/4}D^{-2}(x) = \frac{1}{2}x^3e^{-x/4},$$

The general solution is $y(x) = (Ax + B)e^{-x/4} + \frac{1}{2}x^3e^{-x/4}$.

5.5.2 Case $r(x) = \cos \alpha x$ or $\sin \alpha x$.

Consider first, the case when $F(D)$ contains even powers of D.
When $f(x) = \cos \alpha x$, we have

$$D^2 f = -\alpha^2 \cos \alpha x,\ D^4 f = (-\alpha^2)^2 \cos \alpha x,\ D^6 f = (-\alpha^2)^3 \cos \alpha x, \ldots$$

Let
$$F(D^2)y = [a_0(D^2)^n + a_1(D^2)^{n-1} + a_2(D^2)^{n-2} + \ldots + a_n]y.$$

Now, let $y = \cos \alpha x$, then

$$F(D^2) \cos \alpha x = [a_0(D^2)^n + a_1(D^2)^{n-1} + \ldots + a_n] \cos \alpha x$$
$$= [a_0(-\alpha^2)^n + a_1(-\alpha^2)^{n-1} + \ldots + a_n] \cos \alpha x = F(-\alpha^2) \cos \alpha x \quad (5.102)$$

Case $F(-\alpha^2) \neq 0$

From Eq. (5.102), we symbolically write

$$\cos \alpha x = [F(D^2)]^{-1}[F(-\alpha^2) \cos \alpha x] = F(-\alpha^2)[F(D^2)]^{-1} \cos \alpha x$$

or

$$[F(D^2)]^{-1} \cos \alpha x = \frac{\cos \alpha x}{F(-\alpha^2)}.$$

Therefore, the particular integral of the equation $F(D^2)y = \cos \alpha x$ is given by

$$y_p(x) = [F(D^2)]^{-1} \cos \alpha x = \frac{\cos \alpha x}{F(-\alpha^2)},\quad F(-\alpha^2) \neq 0. \quad (5.103)$$

It is easy to show that similar formula holds when $r(x) = \sin \alpha x$. That is, if $F(D^2)y = \sin \alpha x$, then

$$y_p(x) = [F(D^2)]^{-1} \sin \alpha x = \frac{\sin \alpha x}{F(-\alpha^2)}. \quad (5.104)$$

When odd powers of D also exist in $F(D)$, we can follow the same procedure to obtain $y_p(x)$. Let $F(D) = F_1(D^2) + F_2(D)$, where $F_2(D)$ contains odd powers of D. Then

$$[F_1(D^2) + F_2(D)] \cos \alpha x = [F_1(-\alpha^2) + F_2(D)] \cos \alpha x$$

Since $F(D)$ has constant coefficients, we obtain

$$[F_1(D^2) + F_2(D)]^{-1} \cos \alpha x = [F_1(-\alpha^2) + F_2(D)]^{-1} \cos \alpha x, \quad (5.105)$$

We now simplify $F_1(-\alpha^2) + F_2(D)$ and multiply it by $F_3(D)[F_3(D)]^{-1}$, where $F_3(D)$ contains odd powers of D, such that $[F_1(-\alpha^2) + F_2(D)]F_3(D)$ contains only even powers of D. Formula (5.105) is applied and the procedure is repeated to obtain $y_p(x)$. We illustrate this technique through examples.

Example 5.58 Find the general solution of the equation $y'' + 4y = 6 \cos x$.

Solution It is easy to verify that the complementary function is

$$y_c(x) = A \cos 2x + B \sin 2x.$$

We have $F(D^2) = D^2 + 4$ and $r(x) = 6 \cos x$, that is $\alpha = 1$. Since $F(-\alpha^2) = -\alpha^2 + 4$ and $F(-1) = -1 + 4 = 3 \neq 0$, we have

$$y_p(x) = [(D^2 + 4)]^{-1}(6 \cos x) = \frac{6 \cos x}{F(-\alpha^2)} = \frac{6 \cos x}{F(-1)} = 2 \cos x.$$

The general solution is $y(x) = A \cos 2x + B \sin 2x + 2 \cos x$.

Example 5.59 Find the general solution of the equation $2y'' + y' - y = 16 \cos 2x$.

Solution The characteristic equation of the corresponding homogeneous equation is

$$2m^2 + m - 1 = 0, \quad \text{or} \quad (m + 1)(2m - 1) = 0.$$

Its roots are $m = -1, 1/2$. The complementary function is

$$y_c(x) = Ae^{-x} + Be^{x/2}.$$

We have $F(D) = 2D^2 + D - 1$, $r(x) = 16 \cos 2x$. Therefore, $\alpha = 2$. Using Eq. (5.105), we get

$$y_p(x) = [(2D^2 + D - 1)]^{-1}(16 \cos 2x) = 16[2(-4) + D - 1]^{-1} \cos 2x$$

$$= 16(D - 9)]^{-1} \cos 2x = 16(D + 9)[(D + 9)(D - 9)]^{-1} \cos 2x$$

$$= 16(D + 9)[(D^2 - 81)]^{-1} \cos 2x = -\frac{16}{85}(D + 9) \cos 2x$$

$$= -\frac{16}{85}(9 \cos 2x - 2 \sin 2x)$$

The general solution is

$$y(x) = Ae^{-x} + Be^{x/2} - \frac{16}{85}(9 \cos 2x - 2 \sin 2x).$$

Example 5.60 Find the general solution of the equation $y'' - 5y' + 4y = 65 \sin 2x$.

Solution The characteristic equation of the corresponding homogeneous equation is

$m^2 - 5m + 4 = 0$, or $(m - 1)(m - 4) = 0$. Its roots are $m = 1, 4$.

The complementary function is $y_c(x) = Ae^x + Be^{4x}$.

We have $F(D) = D^2 - 5D + 4$, $r(x) = 65 \sin 2x$. Therefore $\alpha = 2$. Using Eq. (5.105) we get the particular integral as

$$y_p(x) = (D^2 - 5D + 4)^{-1}(65 \sin 2x) = 65[-4 - 5D + 4]^{-1}(\sin 2x)$$

$$= -\frac{65}{5} D^{-1}(\sin 2x) = \frac{13}{2} \cos 2x.$$

since integral of $\sin 2x$ is $(-\cos 2x)/2$. The general solution is

$$y(x) = Ae^x + Be^{4x} + \frac{13}{2} \cos 2x.$$

Example 5.61 Find the general solution of the equation $y''' - y'' + 4y' - 4y = \sin 3x$.

Solution The characteristic equation of the homogeneous equation is

$m^3 - m^2 + 4m - 4 = 0$, or $(m - 1)(m^2 + 4) = 0$. Its roots are $m = 1, \pm 2i$.

The complementary function is $y_c(x) = Ae^x + B \cos 2x + C \sin 2x$.

We have $F(D) = D^3 - D^2 + 4D - 4 = (D - 1)(D^2 + 4)$, $r(x) = \sin 3x$, $\alpha = 3$.

Using Eq. (5.105) we get the particular integral as

$$y_p(x) = [(D - 1)(D^2 + 4)]^{-1}(\sin 3x) = [(D - 1)(-9 + 4)]^{-1} \sin 3x$$

$$= -\frac{1}{5}(D+1)(D+1)^{-1}(D-1)^{-1}\sin 3x = -\frac{1}{5}(D+1)(D^2-1)^{-1}\sin 3x$$

$$= -\frac{1}{5}(D+1)(-9-1)^{-1}\sin 3x = \frac{1}{50}(D+1)\sin 3x = \frac{1}{50}(\sin 3x + 3\cos 3x).$$

The general solution is

$$y(x) = Ae^x + B\cos 2x + C\sin 2x + (3\cos 3x + \sin 3x)/50.$$

Remark 3

Note that the above results hold good when $r(x)$ is also of the form $\cos(\alpha x + a)$ or $\sin(\alpha x + b)$.

Case $F(-\alpha^2) = 0$.

When $F(-\alpha^2) = 0$, we write $\cos\alpha x = \text{Re}(e^{i\alpha x})$ and $\sin\alpha x = \text{Im}(e^{i\alpha x})$ and apply the formula (5.97). We shall illustrate this technique through the following examples.

Example 5.62 Find the general solution of the equation $y'' + y = 6\sin x$.

Solution The complementary function is $y_c(x) = A\cos x + B\sin x$.

We have $F(D^2) = D^2 + 1$, $r(x) = 6\sin x$. Therefore $\alpha = 1$ and $F(-\alpha^2) = F(-1) = 0$.

We write the particular integral as

$$y_p(x) = (D^2+1)^{-1}(6\sin x) = \text{Im}(D^2+1)^{-1}(6e^{ix})$$

$$= 6\,\text{Im}\{e^{ix}[(D+i)^2+1]^{-1}(1)\} = 6\,\text{Im}\{e^{ix}[D^2+2iD]^{-1}(1)\}$$

$$= 6\,\text{Im}\{e^{ix}D^{-1}[(D+2i)^{-1}](1)\} = 6\,\text{Im}\{e^{ix}D^{-1}(0+2i)^{-1}(1)\} \quad (\because 1 = e^{0x})$$

$$= 3\,\text{Im}\left\{\frac{1}{i}e^{ix}x\right\} = 3x\,\text{Im}\{-i(\cos x + i\sin x)\} = -3x\cos x.$$

The general solution is $y(x) = A\cos x + B\sin x - 3x\cos x$.

Example 5.63 Find the general solution of the equation $y'' - 4y' + 13y = 18e^{2x}\sin 3x$.

Solution The characteristic equation of the homogeneous equation is

$$m^2 - 4m + 13 = 0. \text{ Its roots are } m = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i.$$

The complementary function is $y_c(x) = e^{2x}(A\cos 3x + B\sin 3x)$.

We have $F(D) = D^2 - 4D + 13$ and $r(x) = 18e^{2x}\sin 3x$.

We write the particular integral as

$$y_p(x) = 18[D^2 - 4D + 13]^{-1}(e^{2x}\sin 3x)$$

$$= 18e^{2x}[(D+2)^2 - 4(D+2) + 13]^{-1}(\sin 3x) \qquad \text{(using Eq. (5.97))}$$

$$= 18e^{2x}[D^2+9]^{-1}(\sin 3x) = 18e^{2x}\{\text{Im}(D^2+9)^{-1}(e^{3ix})\}$$

$$= 18e^{2x}\{\text{Im}\, e^{3ix}[(D+3i)^2+9]^{-1}(1)\}$$

$$= 18e^{2x}\{\text{Im}\, e^{3ix}[D^2+6iD]^{-1}(1)\}$$

$$= 18e^{2x}\{\text{Im } e^{3ix}D^{-1}(D + 6i)^{-1}(1)\}$$

$$= 18e^{2x}\{\text{Im } e^{3ix}D^{-1}(0 + 6i)^{-1}(1)\} = 18e^{2x}\left\{\text{Im } \frac{1}{6i}xe^{3ix}\right\}$$

$$= 3xe^{2x}\text{ Im }\{-i(\cos 3x + i\sin 3x)\} = -3xe^{2x}\cos 3x.$$

The general solution is

$$y(x) = e^{2x}(A\cos 3x + B\sin 3x) - 3xe^{2x}\cos 3x.$$

5.5.3 Case $r(x) = x^{\alpha}$, $\alpha > 0$ and integer.

The particular integral of $F(D)y = x^{\alpha}$, is

$$y_p(x) = [F(D)]^{-1}x^{\alpha}$$

Symbolically, we expand the operator $[F(D)]^{-1}$ as an infinite series in ascending powers of D and operate on x^{α}.

Example 5.64 Find the general solution of the equation $y'' + 16y = 64x^2$.

Solution The complementary function is $y_c(x) = A\cos 4x + B\sin 4x$.
The particular integral is

$$y_p(x) = (D^2 + 16)^{-1}(64x^2) = \frac{64}{16}\left[1 + \frac{D^2}{16}\right]^{-1}(x^2)$$

$$= 4\left[1 - \frac{D^2}{16} + \frac{D^4}{256} - \ldots\right]x^2 = 4\left[x^2 - \frac{1}{8}\right]$$

The general solution is $y(x) = A\cos 4x + B\sin 4x + 4x^2 - (1/2)$.

Example 5.65 Find the general solution of the equation $y'' + 4y' + 3y = x\sin 2x$.

Solution The characteristic equation of the corresponding homogeneous equation is

$$m^2 + 4m + 3 = 0, \quad \text{or} \quad (m + 1)(m + 3) = 0. \text{ Its roots are } m = -1, -3.$$

The complementary function is $y_c(x) = Ae^{-x} + Be^{-3x}$.

The particular integral is

$$y_p(x) = [D^2 + 4D + 3]^{-1}(\text{Im } xe^{2ix}) = \text{Im }\{e^{2ix}[(D + 2i)^2 + 4(D + 2i) + 3]^{-1}(x)\}$$

$$= \text{Im }\{e^{2ix}[D^2 + 4(1 + i)D + (8i - 1)]^{-1}(x)\}$$

$$= \text{Im}\left\{\frac{e^{2ix}}{8i - 1}\left[1 + \frac{4(1 + i)D}{8i - 1} + \frac{D^2}{8i - 1}\right]^{-1}(x)\right\}$$

$$= \text{Im}\left\{\frac{e^{2ix}}{8i - 1}\left[1 - \frac{4(1 + i)D}{8i - 1} + \ldots\right](x)\right\}$$

$$= \text{Im}\left\{\frac{(8i + 1)}{(-65)}e^{2ix}\left[x - \frac{4(8i + 1)(1 + i)}{(-65)}\right]\right\}$$

$$= \text{Im}\left\{-\frac{1}{65}(8i+1)(\cos 2x + i\sin 2x)\left[x + \frac{4}{65}(9i-7)\right]\right\}$$

$$= \text{Im}\left\{-\frac{1}{65}[(\cos 2x - 8\sin 2x) + i(\sin 2x + 8\cos 2x)]\left[\left(x - \frac{28}{65}\right) + \frac{36}{65}i\right]\right\}$$

$$= -\frac{1}{4225}[65x(8\cos 2x + \sin 2x) - 28(8\cos 2x + \sin 2x) + 36(\cos 2x - 8\sin 2x)]$$

$$= -\frac{1}{4225}[65x(8\cos 2x + \sin 2x) - 188\cos 2x - 316\sin 2x].$$

The general solution is

$$y(x) = Ae^{-x} + Be^{-3x} - \frac{1}{4225}[65x(8\cos 2x + \sin 2x) - 188\cos 2x - 316\sin 2x].$$

Example 5.66 Find the general solution of the equation $y^{iv} + 3y'' = 108x^2$.

Solution The characteristic equation of the homogeneous equation is

$m^4 + 3m^2 = 0$, or $m^2(m^2 + 3) = 0$. Its roots are $m = 0, 0, \pm\sqrt{3}i$.

The complementary function is $y_c(x) = A + Bx + (C\cos\sqrt{3}x + D\sin\sqrt{3}x)$.
We have $F(D) = D^4 + 3D^2 = D^2(D^2 + 3)$. The particular integral is given by

$$y_P(x) = 108[D^2(D^2+3)]^{-1}(x^2) = 108[D^{-2}]\frac{1}{3}\left[1 + \frac{D^2}{3}\right]^{-1}(x^2)$$

$$= 36[D^{-2}]\left[1 - \frac{D^2}{3} + \frac{D^4}{9} - \ldots\right](x^2) = 36D^{-2}\left[x^2 - \frac{2}{3}\right]$$

$$= 36\left[\frac{x^4}{12} - \frac{x^2}{3}\right] = 3x^4 - 12x^2.$$

The general solution is $y(x) = A + Bx + (C\cos\sqrt{3}x + D\sin\sqrt{3}x) + 3x^4 - 12x^2$.

Exercise 5.7

Find the general solution of the following differential equations.

1. $(D^2 + 5D + 4)y = 18e^{2x}$.
2. $(D^2 - 1)y = 8e^{3x}$.
3. $(D^2 - 3D - 4)y = e^x + 6e^{5x}$.
4. $(D^2 + D + 2)y = e^{x/2}$.
5. $(D^2 + 3D + 3)y = 7e^x$.
6. $(D^2 - 2D + 1)y = 5e^{4x} + 4e^{2x}$.
7. $(9D^2 - 6D + 1)y = 4e^{-x}$.
8. $(D^2 - 6D + 9)y = 14e^{3x}$.
9. $(D^2 + D - 6)y = e^{2x}$.
10. $(2D^2 - 3D - 2)y = xe^{-x/2}$.
11. $(D^2 - 1)y = 6xe^x$.
12. $(4D^2 + 9D + 2)y = xe^{-2x}$.
13. $(9D^2 + 6D + 1)y = e^{-x/3}$.
14. $(2D^2 + 7D - 4)y = xe^{-4x}$.
15. $(D^3 + 2D^2 - 5D - 6)y = 4e^x$.
16. $(2D^3 + 3D^2 - 3D - 2)y = 10e^{2x}$.
17. $(D^3 - 2D^2 - D + 2)y = e^{3x}$.
18. $(D^3 - 6D^2 + 12D - 8)y = 18e^{2x}$.

19. $(2D^3 - 3D^2 + 1)y = 16e^x$.
20. $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$.
21. $(D^2 + 16)y = \cos 2x$.
22. $(2D^2 - 5D + 3)y = \sin x$.
23. $(3D^2 - 7D + 2)y = \sin x + \cos x$.
24. $(2D^2 - 7D + 3)y = \sin 2x$.
25. $(D^2 + D + 1)y = 16 \cos x$.
26. $(8D^2 - 12D + 5)y = 16 \sin x$.
27. $(D^2 + 9)y = \sin 3x$.
28. $(D^2 + 3)y = \cos \sqrt{3}x$.
29. $(D^2 + 2D + 5)y = e^{-x} \cos 2x$.
30. $(D^2 - 4D + 5)y = 24e^{2x} \sin x$.
31. $(D^2 - 6D + 13)y = 28e^{3x} \sin 2x$.
32. $(D^2 - 2D + 10)y = 16e^x \cos 3x + 24e^x \sin 3x$.
33. $(D^3 - 3D^2 + D - 3)y = 6 \cos x$.
34. $(D^3 - D^2 + 9D - 9)y = 30 \cos 3x$.
35. $(D^3 - 4D^2 + 9D - 10)y = 24e^x \sin 2x$.
36. $(4D^3 - 12D^2 + 13D - 10)y = 16e^{x/2} \cos x$.
37. $(D^4 + 5D^2 + 4)y = 16 \sin x + 64 \cos 2x$.
38. $(D^2 + 25)y = 9x^3 + 4x^2$.
39. $(D^2 + 6D + 9)y = 4x^2 - 1$.
40. $(D^2 - 2D - 3)y = 2x^2 + 6x$.
41. $(D^2 - 5D + 6)y = x \cos 2x$.
42. $(D^2 + D - 2)y = x^2 \sin x$.
43. $(D^2 - D - 6)y = xe^{-2x}$.
44. $(D^2 + 7D + 12)y = e^x \sin 2x$.
45. $(D^2 + 4D + 3)y = e^{2x} \cos x$.
46. $(D^2 + 3D + 4)y = e^x \cos (\sqrt{7}x/2)$.
47. $(D^2 + 3D + 2)y = xe^x \sin x$.
48. $(D^2 + 9)y = xe^{2x} \cos x$.
49. $(4D^2 + 8D + 3)y = xe^{-x/2} \cos x$.
50. $(D^4 + 3D^2 + 2)y = 16x^2 \cos x$.
51. If $(2D - 1)y = e^{3x}$, then prove that $(D - 3)(2D - 1)y = 0$. Find the general solution of the second equation and substituting in the first equation obtain the general solution of the first order equation.
52. If $F(D)y = (D - m)y = r(x)$, then show that the particular integral can be written as

$$y_p(x) = e^{mx} \int e^{-mx} r(x)\, dx.$$

53. Show that $y = \frac{1}{n} \int_a^x r(t) \sin n(x - t)\, dt$ is the solution of the equation $y'' + n^2 y = r(x)$.
54. If u is a function of x, then show that

$$F(D)xu = xF(D)u + F'(D)u$$

where $F(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_n$, and a_i are constants.

55. Let a given differential equation be of the form $F(D)y = r(x) = xu(x)$. Then, using the result in problem 54 prove that the particular integral $y(x)$ can be written as

$$y(x) = [F(D)]^{-1} xu(x) = x[F(D)]^{-1} u(x) - [F'(D)\{F(D)\}^{-2}]u(x).$$

56. The particular integral of the equation $F(D)y = e^{mx}$ is

$$y_p(x) = \frac{x}{1!} \frac{e^{mx}}{G(m)}, \quad \text{where} \quad F(D) = (D - m)G(D),\ G(m) \neq 0,$$

$$y_p(x) = \frac{x^2}{2!} \frac{e^{mx}}{G(m)}, \quad \text{where} \quad F(D) = (D - m)^2 G(D),\ G(m) \neq 0, \text{ etc.}$$

Show that these particular integrals can be written as

$$[F(D)]^{-1} e^{mx} = x\left[\frac{1}{F'(m)}\right] e^{mx}, \quad F(m) = 0,\ F'(m) \neq 0$$

$$[F(D)]^{-1} e^{mx} = x^2\left[\frac{1}{F''(m)}\right] e^{mx}, \quad F(m) = 0,\ F'(m) = 0,\ F''(m) \neq 0, \text{ etc.}$$

Use these formulas to evaluate the particular integral in Problems 8, 9, 13 and 19 of this exercise.

57. If $F(D)$ can be factorised into n distinct factors $F(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, then show that the particular integral of $F(D)y = r(x)$, can be written as

$$y_p(x) = A_1 e^{m_1 x} \int e^{-m_1 x} r(x)\,dx + A_2 e^{m_2 x} \int e^{-m_2 x} r(x)\,dx + \ldots + A_n e^{m_n x} \int e^{-m_n x} r(x)\,dx$$

Use this formula to evaluate the particular integral in problem 40 of this exercise.

58. The forced oscillations of a mechanical system with periodic input are governed by the non-homogeneous equation

$$m\ddot{y} + c\dot{y} + ky = F_0 \cos \omega t,$$

where $m > 0$, $c > 0$ and $k > 0$. Obtain its general solution when (i) $c \neq 0$ (forced damped oscillations), (ii) $c = 0$ (forced undamped oscillations).

5.6 Simultaneous Linear Equations

In the previous sections, we have discussed the solution of a single linear differential equation, in which y is the dependent variable and x is the independent variable. In this section, we consider the solution of a system of two linear first order equations in two dependent variables y_1 and y_2 and one independent variable t. We shall restrict ourselves to the solution of constant coefficient equations. For example, the equations

$$\text{(i)}\ 6\frac{dy_1}{dt} + 5\frac{dy_2}{dt} + 3y_1 + y_2 = 0, \qquad \frac{dy_2}{dt} - 5y_1 + 3y_2 = e^t$$

$$\text{(ii)}\ 3\frac{dy_1}{dt} + 2y_1 + y_2 = e^{-t}, \qquad \frac{dy_1}{dt} + \frac{dy_2}{dt} - 2y_1 + 3y_2 = t$$

are two systems of linear, constant coefficient first order equations. These two systems can respectively be written in operator form as

$$\text{(i)}\ (6D + 3)y_1 + (5D + 1)y_2 = 0, \quad \text{and} \quad \text{(ii)}\ (3D + 2)y_1 + y_2 = e^{-t},$$

$$-5y_1 + (D + 3)y_2 = e^t, \qquad (D - 2)y_1 + (D + 3)y_2 = t$$

where $D = d/dt$.

The solution of such systems can be obtained by eliminating one of the variables and solving the resulting linear, second order equation for the second variable. Sometimes, elimination of one of the variables may also produce a first order equation for the second variable.

We illustrate the method of obtaining the solution through the following examples.