Chapter 3

Matrices and Eigenvalue Problems

3.1 Introduction

In modern mathematics, matrix theory occupies an important place and has applications in almost all branches of engineering and physical sciences. Matrices of order $m \times n$ form a vector space and they define linear transformations which map vector spaces consisting of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ into another vector space consisting of vectors in $\mathbb{R}^n$ or $\mathbb{C}^m$ under a given set of rules of vector addition and scalar multiplication. A matrix does not denote a number and no value can be assigned to it. The usual rules of arithmetic operations do not hold for matrices. The rules defining the operations on matrices are usually called its algebra. In this chapter, we shall discuss the matrix algebra and its use in solving linear system of algebraic equations $\mathbf{Ax} = \mathbf{b}$ and solving the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$.

3.2 Matrices

An $m \times n$ matrix is an arrangement of mn objects (not necessarily distinct) in m *rows* and n *columns* in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{3.1}$$

We say that the matrix is of *order* $m \times n$ (m by n). The objects $a_{11}, a_{12}, \ldots, a_{mn}$ are called the *elements* of the matrix. Each element of the matrix can be a real or a complex number or a function of one or more variables or any other object. The element a_{ij} which is common to the ith row and the jth column is called its *general element*. The matrices are usually denoted by boldface uppercase letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ etc. When the order of the matrix is understood, we can simply write $\mathbf{A} = (a_{ij})$. If all the elements of a matrix are real, it is called a *real matrix*, whereas if one or more elements of a matrix are complex it is called a *complex matrix*. We define the following particular types of matrices.

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Row Vector A matrix of order $1 \times n$, that is, it has one row and n columns is called a *row vector* or a row matrix of order n and is written as

$$[a_{11}\ a_{12}\ \dots\ a_{1n}], \text{ or } [a_1\ a_2\ \dots\ a_n]$$

in which a_{1j} (or a_j) is the jth element.

Column vector A matrix of order $m \times 1$, that is, it has m rows and one column is called a *column vector* or a *column matrix* of order m and is written as

$$\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}, \text{ or } \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in which b_{j1} (or b_j) is the jth element.

The number of elements in a row/column vector is called its *order*. The vectors are usually denoted by boldface lower case letters **a**, **b**, **c**, ... etc. If a vector has n elements and all its elements are real numbers, then it is called an *ordered n-tuple* in $\mathbb{R}^n$, whereas if one or more elements are complex numbers, then it is called an ordered *n-tuple* in $\mathbb{C}^n$.

Rectangular matrix A matrix **A** of order $m \times n$, $m \neq n$ is called a *rectangular matrix*.

Square matrices A matrix **A** of order $m \times n$ in which $m = n$, that is number of rows is equal to the number of columns is called a square matrix of order n. The elements a_{ii}, that is the elements $a_{11}, a_{22}, \dots, a_{nn}$ are called the *diagonal elements* and the line on which these elements lie is called the *principal diagonal* or the *main diagonal* of the matrix. The elements a_{ij}, when $i \neq j$ are called the *off-diagonal elements*. The sum of the diagonal elements of a square matrix is called the *trace* of the matrix.

Null matrix A matrix **A** of order $m \times n$ in which all the elements are zero is called a *null matrix* or a *zero matrix* and is denoted by **0**.

Diagonal matrix A square matrix **A** in which all the off-diagonal elements a_{ij}, $i \neq j$ are zero is called a diagonal matrix. For example

$$\mathbf{A} = \begin{bmatrix} a_{11} & & & \mathbf{0} \\ & a_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & a_{nn} \end{bmatrix} \text{ is a } \textit{diagonal matrix} \text{ of order } n.$$

A diagonal matrix is denoted by **D**. It is also written as diag $[a_{11}\ a_{22}\ \dots\ a_{nn}]$.

If all the elements of a diagonal matrix of order n are equal, that is $a_{ii} = \alpha$ for all i, then the matrix is called a *scalar matrix* of order n.

If all the elements of a diagonal matrix of order n are 1, then the matrix

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$$\mathbf{A} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$ is called an *unit matrix* or an *identity matrix* of order n.

An identity matrix is denoted by $\mathbf{I}$.

Equal matrices Two matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{p\times q}$ are said to be equal, when

(i) they are of the same order, that is $m = p$, $n = q$ and

(ii) their corresponding elements are equal, that is $a_{ij} = b_{ij}$ for all i, j.

Submatrix A matrix obtained by omitting some rows and or columns from a given matrix $\mathbf{A}$ is called a *submartix* of $\mathbf{A}$. As a convention, the given matrix $\mathbf{A}$ is also taken as the submatrix of $\mathbf{A}$.

3.2.1 Matrix Algebra

The basic operations allowed on matrices are

(i) multiplication of a matrix by a scalar,

(ii) addition/subtraction of two matrices,

(iii) multiplication of two matrices.

Note that there is no concept of dividing a matrix by a matrix. Therefore, the operation $\mathbf{A}/\mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are matrices is not defined.

Multiplication of a matrix by a scalar

Let α be a scalar (real or complex) and $\mathbf{A} = (a_{ij})$ be a given matrix of order $m \times n$. Then

$$\mathbf{B} = \alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij}) \quad \text{for all } i \text{ and } j. \tag{3.2}$$

The order of the new matrix $\mathbf{B}$ is same as that of the matrix $\mathbf{A}$.

Addition/subtraction of two matrices

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two matrices of the same order. Then

$$\mathbf{C} = (c_{ij}) = \mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}), \quad \text{for all } i \text{ and } j \tag{3.3a}$$

and

$$\mathbf{D} = (d_{ij}) = \mathbf{A} - \mathbf{B} = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}), \quad \text{for all } i \text{ and } j. \tag{3.3b}$$

The order of the new matrix $\mathbf{C}$ or $\mathbf{D}$ is the same as that of the matrices $\mathbf{A}$ and $\mathbf{B}$. Matrices of the same order are said to be *conformable* for addition/subtraction.

If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$ are p matrices which are conformable for addition and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are any scalars, then

$$\mathbf{C} = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \ldots + \alpha_p\mathbf{A}_p \tag{3.4}$$

is called a linear combination of the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_p$. The order of the matrix $\mathbf{C}$ is same as that of $\mathbf{A}_i$, $i = 1, 2, \ldots, p$.

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Properties of the matrix addition and scalar multiplication

Let **A**, **B**, **C** be the matrices which are conformable for addition and α, β be scalars. Then

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. (commutative law)
2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (associative law).
3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$ (**0** is the null matrix of the same order as **A**).
4. $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.
5. $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$.
6. $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$.
7. $\alpha(\beta\mathbf{A}) = \alpha\beta\mathbf{A}$.
8. $1 \times \mathbf{A} = \mathbf{A}$ and $0 \times \mathbf{A} = \mathbf{0}$.

Multiplication of two matrices

The product **AB** of two matrices **A** and **B** is defined only when the number of columns in **A** is equal to the number of rows in **B**. Such matrices are said to be *conformable* for multiplication. Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{B} = (b_{ij})$ be an $n \times p$ matrix. Then the product matrix

$$\mathbf{C} = (c_{ij}) = \mathbf{AB} = \underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & & & & \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}}$$

is a matrix of order $m \times p$. The general element of the product matrix **C** is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}. \tag{3.5}$$

In the product **AB**, **B** is said to be pre-multiplied by **A** or **A** is said to be post-multiplied by **B**. If **A** is a row matrix of order $1 \times n$ and **B** is a column matrix of order $n \times 1$, then **AB** is a matrix of order 1×1, that is a single element, and **BA** is a matrix of order $n \times n$.

Remark 1

(a) It is possible that for two given matrices **A** and **B**, the product matrix **AB** is defined but the product matrix **BA** may not be defined. For example, if **A** is a 2×3 matrix and **B** is a 3×4 matrix, then the product matrix **AB** is defined and is a matrix of order 2×4, whereas the product matrix **BA** is not defined.

(b) If both the product matrices **AB** and **BA** are defined, then both the matrices **AB** and **BA** are square matrices. In general $\mathbf{AB} \neq \mathbf{BA}$. Thus, the matrix product is not commutative. If $\mathbf{AB} = \mathbf{BA}$, then the matrices **A** and **B** are said to *commute* with each other.

(c) If $\mathbf{AB} = \mathbf{0}$, then it does not always imply that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. For example, let

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$$\mathbf{A} = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

then
$$\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 0 & 0 \\ ax+by & 0 \end{bmatrix} \neq \mathbf{AB}.$$

(d) If $\mathbf{AB} = \mathbf{AC}$, it does not always imply that $\mathbf{B} = \mathbf{C}$.

(e) Define $\mathbf{A}^k = \mathbf{A} \times \mathbf{A} \ldots \times \mathbf{A}$ (k times). Then, a matrix $\mathbf{A}$ such that $\mathbf{A}^k = \mathbf{0}$ for some positive integer k is said to be *nilpotent*. The smallest value of k for which $\mathbf{A}^k = \mathbf{0}$ is called the *index of nilpotency* of the matrix $\mathbf{A}$.

(f) If $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A}$ is called an *idempotent matrix*.

Properties of matrix multiplication

1. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively, then
$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \qquad \text{(associative law)}$$
is a matrix of order $m \times q$.

2. If $\mathbf{A}$ is a matrix of order $m \times n$ and $\mathbf{B}, \mathbf{C}$ are matrices of order $n \times p$, then
$$\mathbf{A}\,(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \qquad \text{(left distributive law).}$$

3. If $\mathbf{A}, \mathbf{B}$ are matrices of order $m \times n$ and $\mathbf{C}$ is a matrix of order $n \times p$, then
$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \qquad \text{(right distributive law).}$$

4. If $\mathbf{A}$ is a matrix of order $m \times n$ and $\mathbf{B}$ is a matrix of order $n \times p$, then
$$\alpha(\mathbf{AB}) = \mathbf{A}(\alpha\mathbf{B}) = (\alpha\mathbf{A})\mathbf{B}$$
for any scalar α.

3.2.2 Some Special Matrices

We now define some special matrices.

Transpose of a matrix The matrix obtained by interchanging the corresponding rows and columns of a given matrix $\mathbf{A}$ is called the *transpose matrix* of $\mathbf{A}$ and is denoted by $\mathbf{A}^T$ or $\mathbf{A}'$, that is, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}.$$

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix. Also, both the product matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are defined, and

$$\mathbf{A}^T\mathbf{A} = (n \times m)(m \times n) \text{ is an } n \times n \text{ square matrix}$$

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and $$\mathbf{AA}^T = (m \times n)(n \times m) \text{ is an } m \times m \text{ square matrix.}$$

A column vector **b** can also be written as $[b_1\ b_2\ \dots\ b_n]^T$.

The following results can be easily verified

1. The transpose of a row matrix is a column matrix and the transpose of a column matrix is a row matrix.
2. $(\mathbf{A}^T)^T = \mathbf{A}$.
3. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$, when the matrices **A** and **B** are conformable for addition.
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, when the matrices **A** and **B** are conformable for multiplication.

   If the product $\mathbf{A}_1\ \mathbf{A}_2\ \dots\ \mathbf{A}_p$ is defined, then

$$[\mathbf{A}_1\ \mathbf{A}_2\ \dots\ \mathbf{A}_p]^T = \mathbf{A}_p^T\ \mathbf{A}_{p-1}^T\ \dots\ \mathbf{A}_1^T.$$

Remark 2

The product of a row vector $\mathbf{a}_i = (a_{i1}\ a_{i2}\ \dots\ a_{in})$ of order $1 \times n$ and a column vector $\mathbf{b}_j = (b_{1j}\ b_{2j}\ \dots\ b_{nj})^T$ of order $n \times 1$ is called the *dot product* or the *inner product* of the vectors $\mathbf{a}_i$ and $\mathbf{b}_j$, that is

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

which is a scalar. In terms of the inner products, the product matrix **C** in Eq. (3.5) can be written as

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_p \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{b}_p \\ \cdots & \cdots & \cdots & \\ \mathbf{a}_m \cdot \mathbf{b}_1 & \mathbf{a}_m \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_m \cdot \mathbf{b}_p \end{bmatrix}. \tag{3.6}$$

Symmetric and skew-symmetric matrices A real square matrix $\mathbf{A} = (a_{ij})$ is said to be

symmetric, if $a_{ij} = a_{ji}$ for all i and j, that is $\mathbf{A} = \mathbf{A}^T$

skew-symmetric, if $a_{ij} = -a_{ji}$ for all i and j, that is $\mathbf{A} = -\mathbf{A}^T$.

Remark 3

(a) In a skew-symmetric matrix $\mathbf{A} = (a_{ij})$, all its diagonal elements are zero.

(b) The matrix which is both symmetric and skew-symmetric must be a null matrix.

(c) For any real square matrix **A**, the matrix $\mathbf{A} + \mathbf{A}^T$ is always symmetric and the matrix $\mathbf{A} - \mathbf{A}^T$ is always skew-symmetric. Therefore, a real square matrix **A** can be written as the sum of a symmetric matrix and a skew-symmetric matrix. That is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).$$

Triangular matrices A square matrix $\mathbf{A} = (a_{ij})$ is called a *lower triangular matrix* if $a_{ij} = 0$, whenever $i < j$, that is all elements above the principal diagonal are zero and an *upper triangular matrix* if $a_{ij} = 0$, whenever $i > j$, that is all the elements below the principal diagonal are zero.

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Conjugate matrix Let $\mathbf{A} = (a_{ij})$ be a complex matrix. Let $\bar{a}_{ij}$ denote the complex conjugate of a_{ij}. Then, the matrix $\bar{\mathbf{A}} = (\bar{a}_{ij})$ is called the *conjugate matrix* of A.

Hermitian and skew-Hermitian matrices A complex matrix $\mathbf{A}$ is called an *Hermitian matrix* if $\bar{\mathbf{A}} = \mathbf{A}^T$ or $\mathbf{A} = (\bar{\mathbf{A}})^T$ and a *skew-Hermitian* matrix if $\bar{\mathbf{A}} = -\mathbf{A}^T$ or $\mathbf{A} = -(\bar{\mathbf{A}})^T$. Sometimes, a Hermitian matrix is denoted by $\mathbf{A}^H$ or $\mathbf{A}^*$.

Remark 4

(a) If $\mathbf{A}$ is a real matrix, then an Hermitian matrix is same as a symmetric matrix and a skew-Hermitian matrix is same as a skew-symmetric matrix.

(b) In an Hermitian matrix, all the diagonal elements are real (let $a_{jj} = x_j + iy_j$; then $a_{jj} = \bar{a}_{jj}$ gives $x_j + iy_j = x_j - iy_j$ or $y_j = 0$ for all j).

(c) In a skew-Hermitian matrix, all the diagonal elements are either 0 or pure imaginary (let $a_{jj} = x_j + iy_j$; then $a_{jj} = -\bar{a}_{jj}$ gives $x_j + iy_j = -(x_j - iy_j)$ or $x_j = 0$ for all j).

(d) For any complex square matrix $\mathbf{A}$, the matrix $\mathbf{A} + \bar{\mathbf{A}}^T$ is always an Hermitian matrix and the matrix $\mathbf{A} - \bar{\mathbf{A}}^T$ is always a skew-Hermitian matrix. Therefore, a complex square matrix $\mathbf{A}$ can be written as the sum of an Hermitian matrix and a skew-Hermitian matrix, that is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \bar{\mathbf{A}}^T) + \frac{1}{2}(\mathbf{A} - \bar{\mathbf{A}}^T).$$

Example 3.1 Let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric matrices of the same order. Show that the matrix $\mathbf{AB}$ is symmetric if and only if $\mathbf{AB} = \mathbf{BA}$, that is the matrices $\mathbf{A}$ and $\mathbf{B}$ commute.

Solution Since the matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetric, we have

$$\mathbf{A}^T = \mathbf{A} \quad \text{and} \quad \mathbf{B}^T = \mathbf{B}.$$

Let $\mathbf{AB}$ be symmetric. Then

$$(\mathbf{AB})^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{B}^T\mathbf{A}^T = \mathbf{AB}, \quad \text{or} \quad \mathbf{BA} = \mathbf{AB}.$$

Now, let $\mathbf{AB} = \mathbf{BA}$. Taking transpose on both sides, we get

$$(\mathbf{AB})^T = (\mathbf{BA})^T = \mathbf{A}^T\mathbf{B}^T = \mathbf{AB}.$$

Hence, the result.

3.2.3 Determinants

With every square matrix $\mathbf{A}$ of order n, we associate a determinant of order n which is denoted by $det\,(\mathbf{A})$ or $|\mathbf{A}|$. The determinant has a value and this value is real if the matrix $\mathbf{A}$ is real and may be real or complex, if the matrix is complex. A determinant of order n is defined as

$$det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

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$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij} \tag{3.7}$$

where M_{ij} and A_{ij} are the *minors* and *cofactors* of a_{ij} respectively.

We give now some important properties of determinants.

1. If all the elements of a row (or column) are zero then the value of the determinant is zero.
2. $|\mathbf{A}| = |\mathbf{A}^T|$.
3. If any two rows (or columns) are interchanged, then the value of the determinant is multiplied by (-1).
4. If the corresponding elements of two rows (or columns) are proportional to each other, then the value of the determinant is zero.
5. If each element of a row (or column) is multiplied by a scalar α then the value of the determinant is multiplied by the scalar α. Therefore, if β is a factor of each element of a row (or column), then this factor β can be taken out of the determinant.

   Note that when we multiply a matrix by a scalar α, then every element of the matrix is multiplied by α. Therefore, $|\alpha\mathbf{A}| = \alpha^n |\mathbf{A}|$ where $\mathbf{A}$ is a matrix of order n.
6. If a non-zero constant multiple of the elements of some row (or column) is added to the corresponding elements of some other row (or column), then the value of the determinant remains unchanged.
7. $|\mathbf{A} + \mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$, in general.

Remark 5

When the elements of the jth row are multiplied by a non-zero constant k and added to the corresponding elements of the ith row, we denote this operation as $R_i \leftarrow R_i + kR_j$, where R_i is the ith row of $|\mathbf{A}|$. The elements of the jth row remain unchanged whereas the elements of the ith row get changed. This operation is called an *elementary row operation*. Similarly, the operation $C_i \leftarrow C_i + kC_j$, where C_i is the ith column of $|\mathbf{A}|$, is called the *elementary column operation*. Therefore, under elementary row (or column) operations, the value of a determinant is unchanged.

Product of two determinants

If $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of the same order, then

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|.$$

Since $|\mathbf{A}| = |\mathbf{A}^T|$, we can multiply two determinants in any one of the following ways

(i) row by row, (ii) column by column,

(iii) row by column, (iv) column by row.

The value of the determinant is same in each case.

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Rank of a matrix

The *rank* of a matrix **A**, denoted by r or $r(\mathbf{A})$ is the order of the largest non-zero minor of $|\mathbf{A}|$. Therefore, the rank of a matrix is the largest value of r, for which there exists at least one $r \times r$ submartix of **A** whose determinant is not zero. Thus, for an $m \times n$ matrix $r \leq \min(m, n)$. For a square matrix **A** of order n, the rank $r = n$ if $|\mathbf{A}| \neq 0$, otherwise $r < n$. The rank of a null matrix is zero and if the rank of matrix is 0, then it must be a null matrix.

Example 3.2 Find all values of μ for which rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

is equal to 3.

Solution Since the matrix **A** is of order 4, $r(\mathbf{A}) \leq 4$. Now, $r(\mathbf{A}) = 3$, if $|\mathbf{A}| = 0$ and there is at least one submatrix of order 3 whose determinant is not zero. Expanding the determinant through the elements of first row, we get

$$|\mathbf{A}| = \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix} = \mu\left[\mu(\mu - 6) + 11\right] - 6$$

$$= \mu^3 - 6\mu^2 + 11\mu - 6 = (\mu - 1)(\mu - 2)(\mu - 3).$$

Setting $|\mathbf{A}| = 0$, we obtain $\mu = 1, 2, 3$. For $\mu = 1, 2, 3$, the determinant of the leading third order submatrix

$$|\mathbf{A}_1| = \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 0 & 0 & \mu \end{vmatrix} = \mu^3 \neq 0.$$

Hence, $r(\mathbf{A}) = 3$, when $\mu = 1$ or 2 or 3. For other values of μ, $r(\mathbf{A}) = 4$.

3.2.4 Inverse of a Square Matrix

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. Then, **A** is called a

(i) *singular matrix* if $|\mathbf{A}| = 0$,

(ii) *non-singular matrix* if $|\mathbf{A}| \neq 0$.

In other words, a square matrix of order n is singular if its rank $r(\mathbf{A}) < n$ and non-singular if its rank $r(\mathbf{A}) = n$. A square non-singular matrix **A** of order n is said to be *invertible*, if there exists a non-singular square matrix **B** of order n such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{3.8}$$

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where **I** is an identity matrix of order n. The matrix **B** is called the *inverse matrix* of **A** and we write $\mathbf{B} = \mathbf{A}^{-1}$ or $\mathbf{A} = \mathbf{B}^{-1}$. Hence, we say that $\mathbf{A}^{-1}$ is the inverse of the matrix **A**, if

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}. \tag{3.9}$$

The inverse, $\mathbf{A}^{-1}$ of the matrix **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\, adj(\mathbf{A}) \tag{3.10}$$

where $adj(\mathbf{A})$ = adjoint matrix of **A**
= transpose of the matrix of cofactors of **A**.

Remark 6

(a) $$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

We have

$$(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}.$$

Pre-multiplying both sides first by $\mathbf{A}^{-1}$ and then by $\mathbf{B}^{-1}$ we obtain

$$\mathbf{B}^{-1}\mathbf{A}^{-1}(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\,\mathbf{B}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \text{ or } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

In general, we have $(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_p)^{-1} = \mathbf{A}_p^{-1}\mathbf{A}_{p-1}^{-1} \ldots \mathbf{A}_1^{-1}$.

(b) If **A** and **B** are non-singular matrices, then **AB** is also a non-singular matrix.

(c) If $\mathbf{AB} = \mathbf{0}$ and **A** is a non-singular matrix, then **B** must be null matrix, since $\mathbf{AB} = \mathbf{0}$ can be pre-multiplied by $\mathbf{A}^{-1}$. If **B** is non-singular matrix, then **A** must be a null matrix, since $\mathbf{AB} = \mathbf{0}$ can be post-multiplied by $\mathbf{B}^{-1}$.

(d) If $\mathbf{AB} = \mathbf{AC}$ and **A** is a non-singular matrix, then $\mathbf{B} = \mathbf{C}$ (see Remark 1(d)).

(e) $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$, in general.

Properties of inverse martices

1. If $\mathbf{A}^{-1}$exists, then it is unique.
2. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
3. $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$. (From $(\mathbf{AA}^{-1})^T = \mathbf{I}^T = \mathbf{I}$, we get $(\mathbf{A}^{-1})^T\mathbf{A}^T = \mathbf{I}$. Hence, the result).
4. Let $\mathbf{D} = \text{diag}\,(d_{11}, d_{22}, \ldots, d_{nn})$, $d_{ii} \neq 0$. Then, $\boldsymbol{D}^{-1} = \text{diag}\,(1/d_{11}, 1/d_{22}, \ldots 1/d_{nn})$.
5. The inverse of a non-singular upper or lower triangular matrix is respectively an upper or a lower triangular matrix.
6. The inverse of a non-singular symmetric matrix is a symmetric matrix.
7. $(\mathbf{A}^{-1})^n = \mathbf{A}^{-n}$ for any positive integer n.

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Example 3.3 Show that the matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ where $\mathbf{I}$ is an identity matrix of order 3. Hence, find the matrix (i) $\mathbf{A}^{-1}$ and (ii) $\mathbf{A}^{-2}$.

Solution We have

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}.$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix}.$$

Substituting in $\mathbf{B} = \mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I}$, we get

$$\mathbf{B} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix} - \begin{bmatrix} 24 & -6 & -30 \\ 90 & 6 & -30 \\ 30 & 24 & 54 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 55 & 11 & 0 \\ 0 & 11 & 33 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

(i) Premultiplying $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - \mathbf{I} = \mathbf{0}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^3 - 6\mathbf{A}^{-1}\mathbf{A}^2 + 11\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1} = \mathbf{0}$$

or $$\mathbf{A}^{-1} = \mathbf{A}^2 - 6\mathbf{A} + 11\,\mathbf{I}$$

$$= \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 12 & 0 & -6 \\ 30 & 6 & 0 \\ 0 & 6 & 18 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

(ii) $$\mathbf{A}^{-2} = (\mathbf{A}^{-1})^2 = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -11 & 10 \\ -160 & 61 & -55 \\ 55 & -21 & 19 \end{bmatrix}.$$

We can also write

$$\mathbf{A}^{-2} = (\mathbf{A}^{-1})\,(\mathbf{A}^{-1}) = \mathbf{A} - 6\,\mathbf{I} + 11(\mathbf{A}^{-1}).$$

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3.2.5 Solution of $n \times n$ Linear System of Equations

Consider the system of n equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \tag{3.11}$$

In matrix form, we can write the system of equations (3.11) as

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.12}$$

where
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and **A, b, x** are respectively called the *coefficient matrix*, the right hand side column vector and the solution vector. If $\mathbf{b} \neq \mathbf{0}$, that is, at least one of the elements $b_1, b_2, \ldots, b_n$ is not zero, then the system of equations is called *non-homogeneous*. If $\mathbf{b} = \mathbf{0}$, then the system of equations is called *homogeneous*. The system of equations is called *consistent* if it has at least one solution and *inconsistent* if it has no solution.

Non-homogeneous system of equations

The non-homogeneous system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved by the following methods.

Matrix method

Let **A** be non-singular. Pre-multiplying $\mathbf{A}\mathbf{x} = \mathbf{b}$ by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3.13}$$

The system of equations is consistent and has a unique solution. If $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ (trivial solution) is the only solution.

Cramer's rule

Let **A** be non-singular. The Cramer's rule for the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}, \quad i = 1, 2, \ldots, n \tag{3.14}$$

where $|\mathbf{A}_i|$ is the determinant of the matrix $\mathbf{A}_i$ obtained by replacing the ith column of **A** by the right hand side column vector **b**.

We discuss the following cases.

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Case 1 When $|\mathbf{A}| \neq 0$, the system of equations is consistent and the unique solution is obtained by using Eq. (3.14).

Case 2 When $|\mathbf{A}| = 0$ and one or more of $|\mathbf{A}_i|$, $i = 1, 2, \ldots, n$, are not zero, then the system of equations has no solution, that is the system is inconsistent.

Case 3 When $|\mathbf{A}| = 0$ and all $|\mathbf{A}_i| = 0$, $i = 1, 2, \ldots, n$, then the system of equations is consistent and has infinite number of solutions. The system of equations has at least a one-parameter family of solutions.

Homogeneous system of equations

Consider the homogeneous system of equations

$$\mathbf{Ax} = \mathbf{0}. \tag{3.15}$$

Trivial solution $\mathbf{x} = \mathbf{0}$ is always a solution of this system.

If $\mathbf{A}$ is non-singular, then again $\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{0} = \mathbf{0}$ is the solution.

Therefore, a homogeneous system of equations is always consistent. We conclude that non-trivial solutions for $\mathbf{Ax} = \mathbf{0}$ exist if and only if $\mathbf{A}$ is singular. In this case, the homogeneous system of equations has infinite number of solutions.

Example 3.4 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

has a unique solution. Solve this system using (i) matrix method, (ii) Cramer's rule.

Solution We find that

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3) - 2(-1-1) + 1\,(3-1) = 10 \neq 0.$$

Therefore, the coefficient matrix $\mathbf{A}$ is non-singular and the given system of equations has a unique solution. Let $\mathbf{x} = [x, y, z]^T$.

(i) We obtain
$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

Therefore,
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, $x = 2$, $y = -1$ and $z = 1$.

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(ii) We have
$$|\mathbf{A}_1| = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(1+3) - 0 + 2(3-1) = 20.$$

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(0+6) - 2(4-2) + 1(-12-0) = -10.$$

$$|\mathbf{A}_3| = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-0) - 2(-2-4) + 1(0-4) = 10.$$

Therefore,
$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = 2,\ y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = -1,\ z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = 1.$$

Example 3.5 Show that the system of equations

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

has infinite number of solutions. Hence, find the solutions.

Solutions We find that

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 0, \quad |\mathbf{A}_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 0,$$

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 0, \quad |\mathbf{A}_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0.$$

Therefore, the system of equations has infinite number of solutions. Using the first two equations

$$x_1 - x_2 = 3 - 3x_3$$
$$2x_1 + 3x_2 = 2 - x_3$$

and solving, we obtain $x_1 = (11 - 10\,x_3)/5$ and $x_2 = (5x_3 - 4)/5$ where x_3 is arbitrary. This solution satisfies the third equation.

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Example 3.6 Show that the system of equations

$$\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$$

is inconsistent.

Solution We find that

$$|\mathbf{A}| = \begin{vmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{vmatrix} = 0, \quad |\mathbf{A}_1| = \begin{vmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{vmatrix} = 0, \quad |\mathbf{A}_2| = \begin{vmatrix} 4 & 6 & 3 \\ 2 & 7 & 1 \\ 2 & 7 & 2 \end{vmatrix} = 6.$$

Since $|\mathbf{A}| = 0$ and $|\mathbf{A}_2| \neq 0$, the system of equations is inconsistent.

Example 3.7 Solve the homogeneous system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -2 \\ 4 & 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution We find that $|\mathbf{A}| = 0$. Hence, the given system has infinite number of solutions. Solving the first two equations

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3z \\ 2z \end{bmatrix}$$

we obtain $x = 13z$, $y = -8z$ where z is arbitrary. This solution satisfies the third equation.

Exercise 3.1

1. Given the matrices $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, verify that

   (i) $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$, (ii) $|\mathbf{A}+\mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$.

2. If $\mathbf{A}^T = [1, -5, 7]$, $\mathbf{B} = [3, 1, 2]$, verify that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

3. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \mathbf{0}$. Hence, find $\mathbf{A}^{-1}$.

4. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 5\mathbf{A} + 11\mathbf{I} = \mathbf{0}$. Hence, find $\mathbf{A}^{-1}$.

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5. For the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, verify that

(i) $[adj\,(\mathbf{A})]^T = adj\,(\mathbf{A}^T)$, (ii) $[adj\,(\mathbf{A})]^{-1} = adj\,(\mathbf{A}^{-1})$.

6. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that

(i) $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$, (ii) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

7. For the matrices $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 0 & 9 \end{bmatrix}$, verify that

(i) $adj\,(\mathbf{AB}) = adj\,(\mathbf{A})\,adj\,(\mathbf{B})$, (ii) $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

8. For any non-singular matrix $\mathbf{A} = (a_{ij})$ of order n, show that

(i) $|adj\,(\mathbf{A})| = |\mathbf{A}|^{n-1}$, (ii) $adj\,(adj(\mathbf{A})) = |\mathbf{A}|^{n-2}\,\mathbf{A}$.

9. For any non-singular matrix $\mathbf{A}$, show that $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$.

10. For any symmetric matrix $\mathbf{A}$, show that $\mathbf{BAB}^T$ is symmetric, where $\mathbf{B}$ is any matrix for which the product matrix $\mathbf{BAB}^T$ is defined.

11. If $\mathbf{A}$ is a symmetric matrix, prove that $(\mathbf{BA}^{-1})^T(\mathbf{A}^{-1}\mathbf{B}^T)^{-1} = \mathbf{I}$ where $\mathbf{B}$ is any matrix for which the product matrices are defined.

12. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices, then prove that

(i) $\mathbf{A}+\mathbf{B}$ is symmetric, (ii) $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are both symmetric,

(iii) $\mathbf{AB}-\mathbf{BA}$ is skew-symmetric.

13. If $\mathbf{A}$ and $\mathbf{B}$ are non-singular commutative and symmetric matrices, then prove that

(i) $\mathbf{AB}^{-1}$, (ii) $\mathbf{A}^{-1}\mathbf{B}$, (iii) $\mathbf{A}^{-1}\mathbf{B}^{-1}$

are symmetric.

14. Let $\mathbf{A}$ be a non-singular matrix. Show that

(i) if $\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\ldots+\mathbf{A}^n = \mathbf{0}$, then $\mathbf{A}^{-1} = \mathbf{A}^n$,

(ii) if $\mathbf{I}-\mathbf{A}+\mathbf{A}^2-\ldots+(-1)^n\mathbf{A}^n = \mathbf{0}$, then $\mathbf{A}^{-1} = (-1)^{n-1}\mathbf{A}^n$.

15. Let $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{A}$ be non-singular square matrices of order n and $\mathbf{PAQ} = \mathbf{I}$, then show that $\mathbf{A}^{-1} = \mathbf{QP}$.

16. If $\mathbf{I}-\mathbf{A}$ is a non-singular matrix, then show that

$$(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\ldots$$

assuming that the series on the right hand side converges.

17. For any three non-singular matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, each of order n, show that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

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Solve the following system of equations:

18. $x - y + z = 2, \quad 2x + 3y - z = 5, \quad x + y - z = 0.$

19. $x + 2y + 3z = 6, \quad 2x + 4y + z = 7, \quad 3x + 2y + 9z = 14.$

20. $-x + y + 2z = 2, \quad 3x - y + z = 3, \quad -x + 3y + 4z = 6.$

21. $2x - z = 1, \quad 5x + y = 7, \quad y + 3z = 5.$

22. Determine the values of k for which the system of equations

$$x - ky + z = 0, \quad kx + 3y - kz = 0, \quad 3x + y - z = 0$$

has (i) only trivial solution, (ii) non-trivial solution.

23. Find the value of θ for which the system of equations

$$2(\sin\theta)\,x + y - 2z = 0, \quad 3x + 2(\cos 2\theta)y + 3z = 0, \quad 5x + 3y - z = 0$$

has a non-trivial solution.

24. If the system of equations $x + ay + az = 0, \quad bx + y + bz = 0, \quad cx + cy + z = 0$, where a, b, c are non-zero and non-unity, has a non-trivial solution, then show that

$$\frac{a}{1-a}\,\frac{b}{1-b} + \frac{c}{1-c} = -1.$$

25. Find the values of λ and μ for which the system of equations

$$x + 2y + z = 6, \quad x + 4y + 3z = 10, \quad x + 4y + \lambda z = \mu$$

has (i) a unique solution, (ii) infinite number of solutions, (iii) no solution.

Find the rank of the matrix **A**, where **A** is given by

26. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$.

27. $\begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix}$.

28. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$.

29. $\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{bmatrix}$.

30. (a) $\begin{bmatrix} 2 & 1 & 5 & -1 \\ -1 & 2 & 5 & 3 \\ 3 & 2 & 9 & -1 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & c_1 & -b_1 & a_2 \\ -c_1 & 0 & a_1 & b_2 \\ b_1 & -a_1 & 0 & c_2 \\ -a_2 & -b_2 & -c_2 & 0 \end{bmatrix}$, $a_i, b_i, c_i \neq 0, i = 1, 2.$

31. Prove that if **A** is an Hermitian matrix, then i**A** is a skew-Hermitian matrix and if **A** is a skew-Hermitian matrix, then i**A** is a Hermitian matrix.

32. Prove that if **A** is a real matrix and $\mathbf{A}^n \to \mathbf{0}$ as $n \to \infty$, then $\mathbf{I} + \mathbf{A}$ is invertible.

33. Let **A**, **B** be $n \times n$ real matrices Then, show that

(i) Trace $(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha$ Trace $(\mathbf{A}) + \beta$ Trace $(\mathbf{B})$ for any scalars α and β.

(ii) Trace $(\mathbf{AB})$ = Trace $(\mathbf{BA})$, (iii) $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ is never true.

34. If **B**, **C** are $n \times n$ matrices, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, $\mathbf{BC} = \mathbf{CB}$ and $\mathbf{C}^2 = \mathbf{0}$, then show that $\mathbf{A}^{p+1} = \mathbf{B}^p[\mathbf{B} + (p+1)\,\mathbf{C}]$ for any positive integer p.

35. Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n, such that $a_{ij} = d$, $i \neq j$ and $a_{ij} = c$, $i = j$. Then, show that $|\mathbf{A}| = (c - d)^{n-1}[c + (n-1)d]$.

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Identity the following matrices as symmetric, skew-symmetric, Hermitian, skew-Hermitian or none of these;

36. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 4 \\ -3 & -4 & 6 \end{bmatrix}$.

37. $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

38. $\begin{bmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{bmatrix}$.

39. $\begin{bmatrix} 1 & 2+4i & 1-i \\ 2-4i & -5 & 3-5i \\ 1+i & 3+5i & 6 \end{bmatrix}$.

40. $\begin{bmatrix} 1 & 2+4i & 1-i \\ -2+4i & -5 & 3-5i \\ -1-i & 3-5i & 6 \end{bmatrix}$.

41. $\begin{bmatrix} 0 & 2+4i & 1-i \\ -2+4i & 0 & 3-5i \\ -1-i & -3-5i & 0 \end{bmatrix}$.

42. $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$

43. $\begin{bmatrix} 0 & -i & 1+i \\ -i & -2i & 0 \\ -1+i & 0 & i \end{bmatrix}$.

44. $\begin{bmatrix} 1 & -1 & i \\ -1 & 0 & 1-i \\ -i & 1+i & 2 \end{bmatrix}$.

45. $\begin{bmatrix} 1 & 2i & -i \\ -2i & i & 1 \\ i & 1 & 2 \end{bmatrix}$.

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3.4 Solution of General linear System of Equations

In section 3.2.5, we have discussed the matrix method and the Cramer's rule for solving a system of n equations in n unknowns, $\mathbf{A}\mathbf{x} = \mathbf{b}$. We assumed that the coefficient matrix $\mathbf{A}$ is non-singular, that is $|\mathbf{A}| \neq 0$, or the rank of the matrix $\mathbf{A}$ is n. The matrix method requires evaluation of n^2 determinants each of order $(n-1)$, to generate the cofactor matrix, and one determinant of order n, whereas the Cramer's rule requires evaluation of $(n+1)$ determinants each of order n. Since the evaluation of high order determinants is very time consuming, these methods are not used for large values of n, say $n > 4$. In this section, we discuss a method for solving a general system of m equations in n unknowns, given by

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.28}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

are respectively called the *coefficient matrix*, *right hand side column vector* and the *solution vector*. The order of the matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{x}$ are respectively $m \times n$, $m \times 1$ and $n \times 1$.

The matrix

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] \tag{3.29}$$

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is called the *augmented matrix* and has m rows and $(n + 1)$ columns. The augmented matrix describes completely the system of equations. The solution vector of the system of equations (3.28) is an n-tuple $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations. There are three possibilities:

(i) the system has a unique solution,
(ii) the system has no solution,
(iii) the system has infinite number of solutions.

The system of equations is said to be *consistent*, if it has atleast one solution and *inconsistent*, if it has no solution. Using the concepts of ranks and vector spaces, we now obtain the necessary and sufficient conditions for the existence and uniqueness of the solution of the linear system of equations.

3.4.1 Existence and Uniqueness of the Solution

Let V_n be a vector space consisting of n-tuples in $\mathbb{R}^n$(or $\mathbb{C}^n$). The row vectors $R_1, R_2, \ldots, R_m$ of the $m \times n$ matrix **A** are n-tuples which belong to V_n. Let S be the subspace of V_n generated by the rows of **A**. Then, S is called the *row-space* of the matrix **A** and its dimension is called the *row-rank* of **A** and is denoted by $rr(\mathbf{A})$. Therefore,

$$\textit{row-rank of } \mathbf{A} = rr(\mathbf{A}) = dim\ (S). \tag{3.30}$$

Similarly, we define the *column-space* of **A** and the *column-rank* of **A** denoted by $cr(\mathbf{A})$.

Since the row-space of $m \times n$ matrix **A** is generated by m row vectors of **A**, we have dim $(S) \leq m$ and since S is a subspace of V_n, we have $dim\ (S) \leq n$. Therefore, we have

$$rr\ (\mathbf{A}) \leq \min\ (m, n) \quad \text{and} \quad \text{similarly} \quad cr(\mathbf{A}) \leq \min\ (m, n). \tag{3.31}$$

Theorem 3.6 Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix. Then the row-rank and column-rank of **A** are same.

Now, we state an important result which is known as the *fundamental theorem of linear algebra.*

Theorem 3.7 The non-homogeneous system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where **A** is an $m \times n$ matrix, has a solution if and only if the matrix **A** and the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ have the same rank.

In section 3.2.3, we defined the rank of $m \times n$ matrix **A** in terms of the determinants of the submatrices of **A**. An $m \times n$ matrix has rank r if it has at least one square submatrix of order r which is non-singular and all square submatrices of order greater than r are singular. This approach is very time consuming when n is large. Now, we discuss an alternative procedure to obtain the rank of a matrix.

3.4.2 Elementary Row and Column Operation

The following three operations on a matrix **A** are called the *elementary row operations:*

(i) Interchange of any two rows (written as $R_i \sim R_j$).
(ii) Multiplication/division of any row by a non-zero scalar (written as αR_i)
(iii) Adding/subtracting a scalar multiple of any row to another row (written as $R_i \leftarrow R_i + \alpha R_j$, that is α multiples of the elements of the jth row are added to the corresponding elements of the ith row. The elements of the jth row remain unchanged, whereas, the elements of the ith row get changed).

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These operations change the form of **A** but do not change the row-rank of **A** as they do not change the row-space of **A**. A matrix **B** is said to be *row equivalent* to a matrix **A**, if the matrix **B** can be obtained from the matrix **A** by a finite sequence of elementary row operations. Then, we usually write $\mathbf{B} \approx \mathbf{A}$. We observe that

(i) every matrix is row equivalent to itself.

(ii) if **A** is row equivalent to **B**, then **B** is row equivalent to **A**.

(iii) if **A** is row equivalent to **B** and **B** is row equivalent to **C**, then **A** is row equivalent to **C**.

The above operations performed on columns (that is column in place of row) are called *elementary column operations*.

3.4.3 Echelon Form of a Matrix

An $m \times n$ matrix is called a *row echelon* matrix or in *row echelon* form if the number of zeros preceeding the first non-zero entry of a row increases row by row until a row having all zero entries (or no other elimination is possible) is obtained. Therefore, a matrix is in row echelon form if the following are satisfied.

(i) If the ith row contains all zeros, it is true for all subsequent rows.

(ii) If a column contains a non-zero entry of any row, then every subsequent entry in this column is zero, that is, if the ith and $(i + 1)$th rows are both non-zero rows, then the initial non-zero entry of the $(i + 1)$th row appears in a later column than that of the ith row.

(iii) Rows containing all zeros occur only after all non-zero rows.

For example, the following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{A} = (a_{ij})$ be a given $m \times n$ matrix. Assume that $a_{11} \neq 0$. If $a_{11} = 0$, we interchange the first row with some other row to make the element in the (1, 1) position as non-zero. Using elementary row operations, we reduce the matrix **A** to its row echelon form (elements of first column below a_{11} are made zero, then elements in the second column below a_{22} are made zero and so on).

Similarly, we define the column echelon form of a matrix.

Rank of A The number of non-zero rows in the row echelon form of a matrix **A** gives the rank of the matrix **A** (that is, the dimension of the row-space of the matrix **A**) and the set of the non-zero rows in the row echelon form gives the basis of the row-space.

Similar results hold for column echelon matrices.

Remark 13

(i) If **A** is a square matrix, then the row-echelon form is an upper triangular matrix and the column echelon form is a lower triangular matrix.

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(ii) This approach can be used to examine whether a given set of vectors are linearly independent or not. We form the matrix with each vector as its row (or column) and reduce it to the row (column) echelon form. The given vectors are linearly independent, if the row echelon form has no row with all its elements as zeros. The number of non-zero rows is the dimension of the given set of vectors and the set of vectors consisting of the non-zero rows is the basis.

Example 3.25 Reduce the following matrices to row echelon form and find their ranks.

$$\text{(i)} \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}, \qquad \text{(ii)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}.$$

Solution Let the given matrix be denoted by **A**. We have

$$\text{(i)}\ \mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + 2R_1 \end{matrix} \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix} R_3 + 2R_2 \approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is the row echelon form of **A**. Since the number of non-zero rows in the row echelon form is 2, we get rank (**A**) = 2.

$$\text{(ii)}\ \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 8R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} R_3 + R_2 \\ R_4 - 5R_2 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the number of non-zero rows in the echelon form of **A** is 2, we get rank (**A**) = 2.

Example 3.26 Reduce the following matrices to column echelon form and find their ranks.

$$\text{(i)} \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 4 & -1 & 5 \end{bmatrix}, \qquad \text{(ii)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}.$$

Solution Let the given matrix be denoted by **A**. We have

$$\text{(i)}\ \mathbf{A} = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix} \begin{matrix} C_2 - C_1/3 \\ C_3 - 7C_1/3 \end{matrix} \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 5/3 \\ 4 & -7/3 & -7/3 \\ 2 & 1/3 & 1/3 \end{bmatrix} C_3 - C_2 \approx \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 0 \\ 4 & -7/3 & 0 \\ 2 & 1/3 & 0 \end{bmatrix}.$$

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Since the column echelon form of **A** has two non-zero columns, rank (**A**) = 2.

$$\text{(ii)}\ \mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix} \begin{matrix} C_2 - C_1 \\ C_3 + C_1 \\ C_4 - C_1 \end{matrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & -2 \\ 1 & -1 & 2 & 1 \\ 1 & -2 & 4 & 2 \end{bmatrix} \begin{matrix} C_3 + 2C_2 \\ C_4 + C_2 \end{matrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

Since the column echelon form of **A** has 2 non-zero columns, rank (**A**) = 2.

Example 3.27 Examine whether the following set of vectors is linearly independent. Find the dimension and the basis of the given set of vectors.

(i) (1, 2, 3, 4), (2, 0, 1, – 2), (3, 2, 4, 2),

(ii) (1, 1, 0, 1), (1, 1, 1, 1), (–1, 1, 1, 1), (1, 0, 0, 1),

(iii) (2, 3, 6, –3, 4), (4, 2, 12, –3, 6), (4, 10, 12, –9, 10).

Solution Let each given vector represent a row of a matrix **A**. We reduce **A** to row echelon form. If all the rows of the echelon form have some non-zero elements, then the given set of vectors are linearly independent.

$$\text{(i)}\ \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{bmatrix} R_3 - R_2 \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2. The basis can be taken as the set of vectors {(1 2 3 4), (0, –4, –5, –10)}.

$$\text{(ii)}\ \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{matrix} \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_2 \sim R_3 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} R_4 + R_2/2$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix} R_4 - R_3/2 \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since all the rows in the row echelon form of **A** are non-zero, the given set of vectors are linearly independent and the dimension of the given set of vectors is 4. The set of vectors {(1, 1, 0, 1), (0, 2, 1, 2), (0, 0, 1, 0), (0, 0, 0, 1)} or the given set itself forms the basis.

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(iii) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 4 & 2 & 12 & -3 & 6 \\ 4 & 10 & 12 & -9 & 10 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 4 & 0 & -3 & 2 \end{bmatrix} R_3 + R_4 \approx \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Since all the rows in the echelon form of **A** are not non-zero, the given set of vectors are linearly dependent. Since the number of non-zero rows is 2, the dimension of the given set of vectors is 2 and its basis can be taken as the set $\{(2, 3, 6, -3, 4), (0, -4, 0, 3, -2)\}$.

3.4.4 Gauss Elimination Method for Non-homogeneous Systems

Consider a non-homogeneous system of m equations in n unknowns

$$\mathbf{Ax} = \mathbf{b} \tag{3.32}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

We assume that at least one element of **b** is not zero. We write the augmented matrix of order $m \times (n + 1)$ as

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

and reduce it to the row echelon form by using elementary row operations. We need a maximum of $(m - 1)$ stages of eliminations to reduce the given augmented matrix to the equivalent row echelon form. This process may terminate at an earlier stage. We then have an equivalent system of the form

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & b_1 \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2r} & \cdots & \bar{a}_{2n} & \bar{b}_2 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & a^*_{rr} & \cdots & a^*_{rn} & b^*_r \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b^*_{r+1} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & b^*_m \end{array}\right]. \tag{3.33}$$

where $r \le m$ and $a_{11} \ne 0$, $\bar{a}_{22} \ne 0$, ... , $a^*_{rr} \ne 0$ are called *pivots*. We have the following cases:

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(a) Let $r < m$ and one or more of the elements $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are not zero. Then, rank $(\mathbf{A}) \neq$ rank $(\mathbf{A} \mid \mathbf{b})$ and the system of equations has no solution.

(b) Let $m \geq n$ and $r = n$ (the number of columns in $\mathbf{A}$) and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{b}) = n$ and the system of equations has a unique solution. We solve the nth equation for x_n, the $(n-1)$th equation for x_{n-1} and so on. This procedure is called the *back substitution method.*

For example, if we have 10 equations in 5 variables, then the augmented matrix is of order 10×6. When rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{b}) = 5$, the system has a unique solution.

(c) Let $r < n$ and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zeros. In this case, r unknowns, $x_1, x_2, \ldots, x_r$ can be determined in terms of the remaining $(n-r)$ unknowns $x_{r+1}, x_{r+2}, \ldots, x_n$ by solving the rth equation for x_r, $(r-1)$th equation for x_{r-1} and so on. In this case, we obtain an $(n-r)$ parameter family of solutions, that is infinitely many solutions.

Remark 14

(a) We do not, normally use column elementary operations in solving the linear system of equations. When we interchange two columns, the order of the unknowns in the given system of equations is also changed. Keeping track of the order of unknowns is quite difficult.

(b) Gauss elimination method may be written as

$$(\mathbf{A} \mid \mathbf{b}) \xrightarrow[\text{row operations}]{\text{Elementary}} (\mathbf{B} \mid \mathbf{c}).$$

The matrix $\mathbf{B}$ is the row echelon form of the matrix $\mathbf{A}$ and $\mathbf{c}$ is the new right hand side column vector. We obtain the solution vector (if it exists) using the back substitution method.

(c) If $\mathbf{A}$ is a square matrix of order n, then $\mathbf{B}$ is an upper triangular matrix of order n.

(d) Gauss elimination method can be used to solve p systems of the form $\mathbf{Ax} = \mathbf{b}_1$, $\mathbf{Ax} = \mathbf{b}_2, \ldots,$ $\mathbf{Ax} = \mathbf{b}_p$ which have the same coefficient matrix but different right hand side column vectors. We form the augmented matrix as $(\mathbf{A} \mid \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p)$, which has m rows and $(n+p)$ columns. Using the elementary row operations, we obtain the row equivalent system $(\mathbf{B} \mid \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p)$, where $\mathbf{B}$ is the row echelon form of $\mathbf{A}$. Now, we solve the systems $\mathbf{Bx} = \mathbf{c}_1$, $\mathbf{Bx} = \mathbf{c}_2, \ldots, \mathbf{Bx} = \mathbf{c}_p$, using the back substitution method.

Remark 15

(a) If at any stage of elimination, the pivot element becomes zero, then we interchange this row with any other row below it such that we obtain a non-zero pivot element. We normally choose the row such that the pivot element becomes largest in magnitude.

(b) For an $n \times n$ system, we require $(n-1)$ stages of elimination. It is possible to compute the total number of additions, subtractions, multiplications and divisions. This number is called the *operation count* of the method. The operation count of the Gauss elimination method for solving an $n \times n$ system is $n(n^2 + 3n - 1)/3$. For large n, the operation count is approximately $n^3/3$.

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Example 3.28 Solve the following systems of equations (if possible) using Gauss elimination method.

$$\text{(i)}\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}, \qquad \text{(ii)}\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix},$$

$$\text{(iii)}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

Solution We write the augmented matrix and reduce it to row echelon form by applying elementary row operations.

$$\text{(i)}\ (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right] \begin{array}{l} R_2 - R_1/2 \\ R_3 + R_1/2 \end{array} \approx \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right] R_3 + 5R_2/3$$

$$\approx \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 0 & 8/3 & -8/3 \end{array}\right].$$

Using the back substitution method, we obtain the solution as

$$\frac{8}{3}z = -\frac{8}{3}, \quad \text{or} \quad z = -1,$$

$$-\frac{3}{2}y + \frac{5}{2}z = -4, \quad \text{or} \quad y = 1,$$

$$2x + y - z = 4, \quad \text{or} \quad x = 1.$$

Therefore, the system of equations has the unique solution $x = 1$, $y = 1$, $z = -1$.

$$\text{(ii)}\ (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right] \begin{array}{l} R_2 - R_1/2 \\ R_3 - 2R_1 \end{array} \approx \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & -2 & 1 & -3 \end{array}\right] R_3 - 2R_2 \approx \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

We find that rank $(\mathbf{A}) = 2$ and rank $(\mathbf{A} \mid \mathbf{b}) = 3$. Therefore, the system of equations has no solution.

$$\text{(iii)}\ (\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 5R_1 \end{array} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] R_3 - R_2 \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

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The system is consistent and has infinite number of solutions. We find that the last equation is satisfied for all values of x, y, z. From the second equation, we get $3y - 3z = 0$, or $y = z$. From the first equation, we get $x - y + z = 1$, or $x = 1$. Therefore, we obtain the solution $x = 1$, $y = z$ and z is arbitrary.

Example 3.29 Solve the following system of equations using Gauss elimination method.

(i) $4x - 3y - 9z + 6w = 0$
$2x + 3y + 3z + 6w = 6$
$4x - 21y - 39z - 6w = -24,$

(ii) $x + 2y - 2z = 1$
$2x - 3y + z = 0$
$5x + y - 5z = 1$
$3x + 14y - 12z = 5.$

Solution We have

(i) $$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right] \begin{array}{l} R_2 - R_1/2 \\ R_3 - R_1 \end{array} \approx \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & -18 & -30 & -12 & -24 \end{array}\right] R_3 + 4R_2$$

$$= \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is consistent and has infinite number of solutions. Choose w as arbitrary. From the second equation, we obtain

$$\frac{9}{2}y + \frac{15}{2}z = 6 - 3w, \quad \text{or} \quad y = \frac{2}{9}\left(6 - 3w - \frac{15}{2}z\right) = \frac{1}{3}(4 - 5z - 2w).$$

From the first equation, we obtain

$$4x = 3y + 9z - 6w = 4 - 5z - 2w + 9z - 6w = 4 + 4z - 8w$$

or

$$x = 1 + z - 2w.$$

Thus, we obtain a two parameter family of solutions

$$x = 1 + z - 2w \quad \text{and} \quad y = (4 - 5z - 2w)/3$$

where z and w are arbitrary.

(ii) $$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & -3 & 1 & 0 \\ 5 & 1 & -5 & 1 \\ 3 & 14 & -12 & 5 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 5R_1 \\ R_4 - 3R_1 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & -9 & 5 & -4 \\ 0 & 8 & -6 & 2 \end{array}\right] \begin{array}{l} R_3 - 9R_2/7 \\ R_4 + 8R_2/7 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & -2/7 & -2/7 \end{array}\right] R_4 - R_3/5 \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & -10/7 & -10/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

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The last equation is satisfied for all values of x, y, z. From the third equation, we obtain $z = 1$. Back substitution gives $y = 1$, $x = 1$. Hence, the system of equations has a unique solution $x = 1$, $y = 1$ and $z = 1$. Since $R_4 = (24R_1 - 7R_2 + R_3)/5$, the last equation is redundent.

3.4.5 Gauss-Jordan Method

In this method, we perform elementary row transformations on the augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ and reduce it to the form

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{c}]$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{c}$ is the solution vector. This reduction is equivalent to finding the solution as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. The first step is same as in the Gauss elimination method. From second step onwards, we make elements below and above the pivot as zeros, using elementary row transformations. Finally, we divide each row by its pivot to obtain the form $[\mathbf{I} \mid \mathbf{c}]$. Alternately, at every step, the pivot can be made as 1 before elimination. Then, $\mathbf{c}$ is the solution vector.

This method is more expensive (larger operation count) than the Gauss elimination. Hence, we do not normally use the Gauss-Jordan method for finding the solution of a system. However, this method is very useful for finding the inverse $(\mathbf{A}^{-1})$ of a matrix $\mathbf{A}$. We consider the augmented matrix $[\mathbf{A} \mid \mathbf{I}]$ and reduce it to the form

$$[\mathbf{A} \mid \mathbf{I}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{A}^{-1}]$$

using elementary row transformations. If we are solving the system of equations (3.28), then we have $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, and the matrix multiplication on the right hand side gives the solution vector.

Remark 16

If any pivot element at any stage of elimination becomes zero, then we interchange rows as in the Gauss elimination method.

Example 3.30 Using the Gauss-Jordan method, solve the system of equations $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

Solution We perform elementary row transformations on the augmented matrix and reduce it the form $[\mathbf{I} \mid \mathbf{c}]$. We get

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 4 \\ 0 & 2 & 0 & 1 \end{array}\right] R_2/3$$

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$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ R_3 - 2R_2 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 10/3 & -5/3 \end{array}\right] R_3/(10/3)$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \begin{array}{l} R_1 + 2R_3/3 \\ R_2 + 5R_3/3 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right].$$

Hence, the solution vector is

$$\mathbf{x} = [1 \quad 1/2 \quad -1/2]^T.$$

Example 3.31 Using Gauss-Jordan method, find the inverse of the matix $\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Solution We have

$$(\mathbf{A} \mid \mathbf{I}) = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right].$$

The pivot element a_{11} is -1. We make it 1 by multiplying the first row by -1. Therefore,

$$(\mathbf{A} \mid \mathbf{I}) \approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \\ R_3 + R_1 \end{array} \approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] R_2/2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ R_3 - 2R_2 \end{array} \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] (-R_3)/5$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 3/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \begin{array}{l} R_1 - 3R_3/2 \\ R_3 - 7R_3/2 \end{array} \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right].$$

Hence,

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}.$$

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3.4.6 Homogeneous System of Linear Equations

Consider the homogeneous system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{3.34}$$

where $\mathbf{A}$ is an $m \times n$ matrix. The homogenous system is always consistent since $\mathbf{x} = \mathbf{0}$ (trivial solution) is always a solution. In this case, rank $(\mathbf{A})$ = rank $(\mathbf{A} \mid \mathbf{0})$. Therefore, for the homogeneous system to have a non-trivial solution, we require that rank $(\mathbf{A}) < n$. If rank $(\mathbf{A}) = r < n$, we obtain an $(n - r)$ parameter family of solutions which form a vector space of dimension $(n - r)$ as $(n - r)$ parameters can be chosen arbitrarily.

The solution space of the homogeneous system is called the *null space* and its dimension is called the *nullity* of $\mathbf{A}$. Therefore, we obtain the result

$$\text{rank } (\mathbf{A}) + \text{nullity } (\mathbf{A}) = n \text{ (see Theorem 3.5).}$$

Remark 17

(a) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of a linear homogeneous system, then $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is also a solution of the homogenous system for any scalars α, β. This result does not hold for non-homogenous systems.

(b) A homogeneous system of m equations in n unknowns and $m < n$, always possesses a non-trivial solution.

Theorem 3.8 If a non-homogeneous system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has solutions, then all these solutions are of the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$ where $\mathbf{x}_0$ is any fixed solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ is any solution of the corresponding homogeneous system.

Proof Let $\mathbf{x}$ be any solution and $\mathbf{x}_0$ be any fixed solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$. Therefore, we have

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{A}\mathbf{x}_0 = \mathbf{b}.$$

Subtracting, we get

$$\mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}_0 = \mathbf{0}, \quad \text{or} \quad \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}.$$

Thus, the difference $\mathbf{x} - \mathbf{x}_0$ between any solution $\mathbf{x}$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$ and any fixed solution $\mathbf{x}_0$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is a solution of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$, say $\mathbf{x}_h$. Hence, the result.

Remark 18

If the non-homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix $(m \geq n)$ has a unique solution, that is rank $(\mathbf{A}) = n$, then the corresponding homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution, that is $\mathbf{x}_h = \mathbf{0}$.

Example 3.32 Solve the following homogeneous system of equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, where $\mathbf{A}$ is given by

$$\text{(i)} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}, \qquad \text{(ii)} \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \qquad \text{(iii)} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix}.$$

Find the rank $(\mathbf{A})$ and nullity $(\mathbf{A})$.

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Solution We write the augmented matrix $(\mathbf{A} \mid \mathbf{0})$ and reduce it to row echelon form.

(i) $$(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 0 \end{array}\right] \begin{array}{l} R_2 - R_1/2 \\ R_3 - 3R_1/2 \end{array} \approx \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 1/2 & 0 \end{array}\right] R_3 + R_2/3 \approx \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since, rank $(\mathbf{A}) = 2 =$ number of unknowns, the system has only a trivial solution. Hence, nullity $(\mathbf{A}) = 0$.

(ii) $$(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right] R_3 - 3R_2 \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right].$$

Since rank $(\mathbf{A}) = 3 =$ number of unknowns, the homogeneous system has only a trivial solution. Therefore, nullity $(\mathbf{A}) = 0$.

(iii) $$(\mathbf{A} \mid \mathbf{0}) = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 4 & 0 \\ 3 & 2 & -6 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \approx \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -1 & -3 & -2 & 0 \end{array}\right] R_3 + R_2 \approx \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, rank $(\mathbf{A}) = 2$ and the number of unknowns is 4. Hence, we obtain a two parameter family of solutions as $x_2 = -3x_3 - 2x_4$, $x_1 = -x_2 + x_3 - x_4 = 4x_3 + x_4$, where x_3 and x_4 are arbitrary. Therefore, nullity $(\mathbf{A}) = 2$.

Exercise 3.3

Using the elementary row operations, determine the ranks of the following matrices.

1. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ 5 & -5 & 11 \end{bmatrix}$.

3. $\begin{bmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 4 & -13 & -5 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & -1 \\ 8 & 13 & 14 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 7 & 11 & 15 & 19 \\ 9 & 15 & 21 & 27 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}$.

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10. $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$.

Using the elementary column operations, determine the rank of the following matrices.

11. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}$. **12.** $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$. **13.** $\begin{bmatrix} 1 & 2 & 3 & 2 \\ -1 & 1 & 3 & -5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$.

14. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$. **15.** $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$.

Determine whether the following set of vectors is linearly independent. Find also its dimension.

16. $\{(3, 2, 4), (1, 0, 2), (1, -1, -1)\}$.

17. $\{(2, 2, 1), (1, -1, 1), (1,0,1)\}$.

18. $\{(2, 1, 0), (1, -1, 1), (4, 1, 2), (2, -3, 3)\}$.

19. $\{(2, 2, 1), (2, i, -1), (1 + i, -i, 1)\}$.

20. $\{(1, 1, 1), (i, i, i), (1 + i, -1 - i, i)\}$

21. $\{(1, 1, 1, 1), (-1, 1, 1, -1), (1, 0, 1, 1), (1, 1, 0,1)\}$.

22. $\{(1, 2, 3, 1), (2, 1, -1, 1), (4, 5, 5, 3), (5, 4, 1, 3)\}$.

23. $\{(1, 2, 3, 4), (0, 1, -1, 2), (1, 4, 1, 8), (3, 7, 8, 14)\}$.

24. $\{(1, 1, 0, 1), (1, 1, 1, 1), (4, 4, 1, 1), (1, 0, 0, 1)\}$.

25. $\{(2, 2, 0, 2), (4, 1, 4, 1), (3, 0, 4, 0)\}$.

Determine which of the following systems are consistent and find all the solutions for the consistent systems.

26. $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$. **27.** $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. **28.** $\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$.

29. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -9 & 2 \\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$. **30.** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$. **31.** $\begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

32. $\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. **33.** $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. **34.** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$.

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35. $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

Find all the solutions of the following homogeneous systems $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A}$ is given as the following.

36. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}.$ **37.** $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -7 \\ -1 & -2 & 11 \end{bmatrix}.$ **38.** $\begin{bmatrix} 3 & -11 & 5 \\ 4 & 1 & -10 \\ 4 & 9 & -6 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 6 & 12 \end{bmatrix}.$ **40.** $\begin{bmatrix} 2 & -1 & -3 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & -7 & -13 & -1 \\ -1 & 5 & 9 & 1 \end{bmatrix}.$ **41.** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$

42. $\begin{bmatrix} 3 & 1 & 1 & 4 \\ 0 & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{bmatrix}$ **43.** $\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & -2 & -3 \\ 3 & 0 & -5 & -1 \\ 5 & -1 & -7 & -4 \end{bmatrix}.$ **44.** $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}.$

45. $\begin{bmatrix} 1 & 1 & -2 & -1 \\ 2 & 1 & 1 & -2 \\ 3 & 2 & -1 & -3 \\ 4 & 2 & 2 & -4 \end{bmatrix}.$

Using the Gauss-Jordan method, find the inverses of the following matrices.

46. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$ **47.** $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$ **48.** $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$

49. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$ **50.** $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$

3.5 Eigenvalue Problems

Let $\mathbf{A} = (a_{ij})$ be a square matrix of order n. The matrix $\mathbf{A}$ may be singular or non-singular. Consider the homogeneous system of equations

$$\mathbf{Ax} = \lambda\mathbf{x} \quad \text{or} \quad (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{3.35}$$

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where λ is a scalar and **I** is an identity matrix of order n. The homogeneous system of equations (3.35) always has a trivial solution. We need to find values of λ for which the homogeneous system (3.35) has non-trivial solutions. The values of λ for which non-trivial solutions of the homogeneous system (3.35) exist, are called the *eigenvalues* or the *characteristic values* of **A** and the corresponding non-trivial solution vectors **x** are called the *eigenvectors* or the *characteristic vectors* of **A**. If **x** is a non-trivial solution of the homogeneous system (3.35), then α **x**, where α is any constant is also a solution of the homogeneous system. Hence, an eigenvector is unique only upto a constant multiple. The problem of determining the eigenvalues and the corresponding eigenvectors of a square matrix **A** is called an *eigenvalues problem.*

3.5.1 Eigenvalues and Eigenvectors

If the homogeneous system (3.35) has a non-trivial solution, then the rank of the coefficient matrix $(\mathbf{A} - \lambda\mathbf{I})$ is less than n, that is the coefficient matrix must be singular. Therefore,

$$det\ (\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.. \tag{3.36}$$

Expanding the determinant given in Eq. (3.36), we obtain a polynomial of degree n in λ, which is of the form

$$P_n(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = (-1)^n [\lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} - \ldots + (-1)^n c_n] = 0$$

or

$$\lambda^n - c_1\lambda^{n-1} + c_2 \lambda^{n-2} - \ldots + (-1)^n c_n = 0. \tag{3.37}$$

where $c_1, c_2, \ldots, c_n$ can be expressed in terms of the elements a_{ij} of the matrix **A**. This equation is called the *characteristic equation* of the matrix **A**. The polynomial equation $P_n(\lambda) = 0$ has n roots which can be real or complex, simple or repeated. The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the polynomial equation $P_n(\lambda) = 0$ are called the *eigenvalues*. By using the relation between the roots and the coefficients, we can write

$$\begin{aligned} &\lambda_1 + \lambda_2 + \ldots + \lambda_n = c_1 = a_{11} + a_{22} + \ldots + a_{nn} \\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-1}\lambda_n = c_2 \\ &\vdots \\ &\lambda_1\lambda_2 \ldots \lambda_n = c_n. \end{aligned} \tag{3.38}$$

If we set $\lambda = 0$ in Eq. (3.36), then we get

$$|\mathbf{A}| = (-1)^{2n} c_n = c_n = \lambda_1\lambda_2 \ldots \lambda_n.$$

Therefore, we get

sum of eigenvalues = trace (**A**), and product of eigenvalues = $|\mathbf{A}|$.

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The set of the eigenvalues is called the *spectrum* of **A** and the largest eigenvalue in magnitude is called the *spectral radius* of **A** and is denoted by ρ (**A**). If $|\mathbf{A}| = 0$, that is the martix is singular, then from Eq. (3.38), we find that atleast one of the eigenvalues must be zero. Conversely, if one of the eigenvalues is zero, then $|\mathbf{A}| = 0$. Note that if **A** is a diagonal or an upper triangular or a lower triangular matrix, then the diagonal elements of the matrix **A** are the eigenvalues of **A**.

After determining the eigenvalues λ_i's, we solve the homogeneous system $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x} = \mathbf{0}$ for each λ_i, $i = 1, 2, \ldots, n$ to obtain the corresponding eigenvectors.

Properties of eigenvalues and eigenvectors

Let λ be an eigenvalue of **A** and **x** be its corresponding eigenvector. Then, we have the following results.

1. α **A** has eigenvalue $\alpha\lambda$ and the corresponding eigenvector is **x**.

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \alpha\,\mathbf{A}\mathbf{x} = (\alpha\lambda)\mathbf{x}.$$

2. $\mathbf{A}^m$ has eigenvalue λ^m and the corresponding eigenvector is **x** for any positive interger m. Pre-multiplying both sides of $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ by **A**, we get

$$\mathbf{A}\,\mathbf{A}\mathbf{x} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{A}\mathbf{x} = \lambda(\lambda\mathbf{x}) \text{ or } \mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}.$$

Therefore, $\mathbf{A}^2$ has the eigenvalue λ^2 and the corresponding eigenvector is **x**. Pre-multiplying successively m times, we obtain the result.

3. $\mathbf{A} - k\mathbf{I}$ has the eigenvalue $\lambda - k$, for any scalar k and the corresponding eigenvector is **x**.

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{A}\mathbf{x} - k\,\mathbf{I}\mathbf{x} = \lambda\mathbf{x} - k\,\mathbf{x}$$

or

$$(\mathbf{A} - k\mathbf{I})\mathbf{x} = (\lambda - k)\mathbf{x}.$$

4. $\mathbf{A}^{-1}$(if it exists) has the eigenvalue $1/\lambda$ and the corresponding eigenvector is **x**. Pre-multiplying both sides of $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{A}^{-1}\mathbf{x} \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}.$$

5. $(\mathbf{A} - k\mathbf{I})^{-1}$ has the eigenvalue $1/(\lambda - k)$ and the corresponding eigenvector is **x** for any scalar k.
6. **A** and $\mathbf{A}^T$ have the same eigenvalues (since a determinant can be expanded by rows or by columns) but different eigenvectors, (see Example 3.41).
7. For a real matrix **A**, if $\alpha + i\beta$ is an eigenvalue, then its conjugate $\alpha - i\beta$ is also an eigenvalue (since the characteristic equation has real coefficients). When the matrix **A** is complex, this property does not hold.

We now present an important result which gives the relationship of a matrix **A** and its characteristic equation.

Theorem 3.9 (Cayley-Hamilton theorem) Every square matrix **A** satisfies its own characteristic equation

$$\mathbf{A}^n - c_1\mathbf{A}^{n-1} + \ldots + (-1)^{n-1} c_{n-1}\,\mathbf{A} + (-1)^n c_n\,\mathbf{I} = \mathbf{0}. \tag{3.39}$$

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Proof The cofactors of the elements of the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ are polynomials in λ of degree $(n-1)$ or less. Therefore, the elements of the adjoint matrix (transpose of the cofactor matrix) are also polynomials in λ of degree $(n-1)$ or less. Hence, we can express the adjoint matrix as a polynomial in λ whose coefficients $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ are square matrices of order n having elements as functions of the elements of the matrix $\mathbf{A}$. Thus, we can write

$$adj(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n.$$

We also have

$$(\mathbf{A} - \lambda\mathbf{I})\ adj\ (\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}|\ \mathbf{I}.$$

Therefore, we can write for any λ

$$(\mathbf{A} - \lambda\mathbf{I})\ (\mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \ldots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n)$$
$$= \lambda^n\mathbf{I} - c_1\lambda^{n-1}\mathbf{I} + \ldots + (-1)^{n-1}\ c_{n-1}\ \lambda\mathbf{I} + (-1)^n\ c_n\mathbf{I}$$

Comparing the coefficients of various powers of λ, we obtain

$$-\mathbf{B}_1 = \mathbf{I}$$
$$\mathbf{AB}_1 - \mathbf{B}_2 = c_1\mathbf{I}$$
$$\mathbf{AB}_2 - \mathbf{B}_3 = c_2\mathbf{I}$$
$$\ldots$$
$$\mathbf{AB}_{n-1} - \mathbf{B}_n = (-1)^{n-1}\ c_{n-1}\mathbf{I}$$
$$\mathbf{AB}_n = (-1)^n\ c_n\mathbf{I}.$$

Pre-multiplying these equations by $\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots, \mathbf{A}, \mathbf{I}$ respectively and adding, we get

$$\mathbf{A}^n - c_1\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\ \mathbf{A} + (-1)^n c_n\ \mathbf{I} = \mathbf{0}$$

which proves the theorem.

Remark 19

(a) We can use Eq. (3.39) to find $\mathbf{A}^{-1}$ (if it exists) in terms of the powers of the matrix $\mathbf{A}$. Pre-multiplying both sides in Eq. (3.39) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{A}^n - c_1\ \mathbf{A}^{-1}\mathbf{A}^{n-1} + \ldots + (-1)^{n-1}c_{n-1}\mathbf{A}^{-1}\mathbf{A} + (-1)^n c_n\ \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$$

or
$$\mathbf{A}^{-1} = -\frac{(-1)^n}{c_n}\ [\mathbf{A}^{n-1} - c_1\mathbf{A}^{n-2} + \ldots + (-1)^{n-1}\ c_{n-1}\mathbf{I}] \tag{3.40}$$

(b) We can use Eq.(3.39) to obtain $\mathbf{A}^n$ in terms of lower powers of $\mathbf{A}$ as

$$\mathbf{A}^n = c_1\mathbf{A}^{n-1} - c_2\mathbf{A}^{n-2} + \ldots + (-1)^{n-1}c_n\mathbf{I}. \tag{3.41}$$

Example 3.33 Verify Cayley-Hamilton theorem for the martrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

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Also, (i) obtain $\mathbf{A}^{-1}$ and $\mathbf{A}^3$, (ii) find eigenvalues of $\mathbf{A}$, $\mathbf{A}^2$ and verify that eigenvalues of $\mathbf{A}^2$ are squares of those of $\mathbf{A}$, (iii) find the spectral radius of $\mathbf{A}$.

Solution The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\{(1-\lambda)^2 - 4\} - 2\{-(1-\lambda) - 2\}$$

$$= (1-\lambda)(\lambda^2 - 2\lambda - 3) - 2(\lambda - 3) = -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0.$$

Now,

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

We have

$$-\mathbf{A}^3 + 3\mathbf{A}^2 - \mathbf{A} + 3\mathbf{I} = -\begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix} + 3\begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}. \tag{3.42}$$

Hence, $\mathbf{A}$ satisfies the characteristic equation $-\lambda^3 + 3\lambda^2 - \lambda + 3 = 0$.

(i) From Eq. (3.42), we get

$$\mathbf{A}^{-1} = \frac{1}{3}[\mathbf{A}^2 - 3\mathbf{A} + \mathbf{I}] = \frac{1}{3}\left[\begin{pmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 6 & 0 \\ -3 & 3 & 6 \\ 3 & 6 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}.$$

From Eq. (3.42), we get

$$\mathbf{A}^3 = 3\mathbf{A}^2 - \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} -3 & 12 & 12 \\ 0 & 9 & 12 \\ 0 & 18 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}.$$

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(ii) Eigenvalues of **A** are the roots of

$$\lambda^3 - 3\lambda^2 + \lambda - 3 = 0 \text{ or } (\lambda - 3)(\lambda^2 + 1) = 0 \text{ or } \lambda = 3, i, -i.$$

The characteristic equation of $\mathbf{A}^2$ is given by

$$\begin{vmatrix} -1-\lambda & 4 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 6. & 5-\lambda \end{vmatrix} = (-1-\lambda)\,[(3-\lambda)(5-\lambda) - 24] = 0$$

or $\quad (\lambda + 1)(\lambda^2 - 8\lambda - 9) = 0$ or $(\lambda + 1)(\lambda - 9)(\lambda + 1) = 0.$

The eigenvalues of $\mathbf{A}^2$ are 9, – 1, – 1 which are the squares of the eigenvalues of **A**.

(iii) The spectral radius of **A** is given by

$$\rho(\mathbf{A}) = \text{largest eigenvalue in magnitude} = \max_i |\lambda_i| = 3.$$

Example 3.34 If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then show that $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$ for $n \geq 3$. Hence, find $\mathbf{A}^{50}$.

Solution The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0, \quad \text{or} \quad \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Using Cayley-Hamilton theorem, we get

$$\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{0}, \quad \text{or} \quad \mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I}.$$

Pre-multiplying both sides successively by **A**, we obtain

$$\mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I}$$

$$\mathbf{A}^4 - \mathbf{A}^3 = \mathbf{A}^2 - \mathbf{A}$$

$$\ldots$$

$$\mathbf{A}^{n-1} - \mathbf{A}^{n-2} = \mathbf{A}^{n-3} - \mathbf{A}^{n-4}$$

$$\mathbf{A}^n - \mathbf{A}^{n-1} = \mathbf{A}^{n-2} - \mathbf{A}^{n-3}.$$

Adding these equations, we get

$$\mathbf{A}^n - \mathbf{A}^2 = \mathbf{A}^{n-2} - \mathbf{I}, \quad \text{or} \quad \mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}, \quad n \geq 3.$$

Using this equation recursively, we get

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$$\begin{aligned}\mathbf{A}^n &= (\mathbf{A}^{n-4} + \mathbf{A}^2 - \mathbf{I}) + \mathbf{A}^2 - \mathbf{I} = \mathbf{A}^{n-4} + 2(\mathbf{A}^2 - \mathbf{I}) \\ &= (\mathbf{A}^{n-6} + \mathbf{A}^2 - \mathbf{I}) + 2(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^{n-6} + 3(\mathbf{A}^2 - \mathbf{I}) \\ &\cdots \\ &= \mathbf{A}^{n-(n-2)} + \frac{1}{2}(n-2)(\mathbf{A}^2 - \mathbf{I}) = \frac{n}{2}\mathbf{A}^2 - \frac{1}{2}(n-2)\mathbf{I}.\end{aligned}$$

Substituting $n = 50$, we get

$$\mathbf{A}^{50} = 25\mathbf{A}^2 - 24\mathbf{I} = 25\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} - 24\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 25 & 1 & 0\\ 25 & 0 & 1\end{bmatrix}.$$

Example 3. 35 Find the eigenvalues and the corresponding eigenvectors of the following matrices.

(i) $\mathbf{A} = \begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}$, (ii) $\mathbf{A} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$, (iii) $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$.

Solution

(i) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & 4\\ 3 & 2-\lambda\end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 3\lambda - 10 = 0, \quad \text{or} \quad \lambda = -2, 5.$$

Corresponding to the eigenvalue $\lambda = -2$, we have

$$(\mathbf{A} + 2\mathbf{I})\,\mathbf{x} = \begin{pmatrix}3 & 4\\ 3 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \text{ or } 3x_1 + 4x_2 = 0 \text{ or } x_1 = -\frac{4}{3}x_2.$$

Hence, the eigenvector $\mathbf{x}$ is given by

$$\mathbf{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-4x_2/3\\ x_2\end{bmatrix} = x_2\begin{bmatrix}-4/3\\ 1\end{bmatrix}.$$

Since an eigenvector is unique upto a constant multiple, we can take the eigenvector as $[-4, 3]^T$.

Corresponding to the eigenvalue $\lambda = 5$, we have

$$(\mathbf{A} - 5\mathbf{I})\,\mathbf{x} = \begin{pmatrix}-4 & 4\\ 3 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \text{ or } x_1 - x_2 = 0, \text{ or } x_1 = x_2.$$

Therefore, the eigenvalue is given by $\mathbf{x} = (x_1, x_2)^T = x_1(1, 1)^T$ or $(1, 1)^T$.

(ii) The characteristic equation of $\mathbf{A}$ is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & 1\\ -1 & 1-\lambda\end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 2\lambda + 2 = 0, \quad \text{or} \quad \lambda = 1 \pm i.$$

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Corresponding to the eigenvalue $\lambda = 1 + i$, we have

$$[\mathbf{A} - (1+i)\,\mathbf{I}]\,\mathbf{x} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$-ix_1 + x_2 = 0 \text{ and } -x_1 - ix_2 = 0.$$

Both the equations reduce to $-x_1 - ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = -i$. Therefore, the eigenvector is $\mathbf{x} = [-i, 1]^T$.

Corresponding to the eigenvalue $\lambda = 1 - i$, we have

$$[\mathbf{A} - (1-i)\mathbf{I}]\,\mathbf{x} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$ix_1 + x_2 = 0 \text{ and } -x_1 + ix_2 = 0.$$

Both the equations reduce to $-x_1 + ix_2 = 0$. Choosing $x_2 = 1$, we get $x_1 = i$. Therefore, the eigenvector is $x = [i, 1]^T$.

Remark 20

For a real matrix **A**, the eigenvalues and the corresponding eigenvectors can be complex.

(iii) The characteristic equation of **A** is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \text{ or } (1-\lambda)(2-\lambda)(3-\lambda) = 0 \text{ or } \lambda = 1, 2, 3.$$

Corresponding to the eigenvalue $\lambda = 1$, we have

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{cases} x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \end{cases}.$$

We obtain two equations in three unknowns. One of the variables x_1, x_2, x_3 can be chosen arbitrarily. Taking $x_3 = 1$, we obtain the eigenvector as $[-1, -1, 1]^T$.

Corresponding to the eigenvalue $\lambda = 2$, we have

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x_1 = 0$, $x_3 = 0$ and x_2 arbitrary. Taking $x_2 = 1$, we obtain the eigenvector as $[0, 1, 0]^T$.

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Corresponding to the eigenvalue $\lambda = 3$, we have

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} x_1 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Choosing $x_3 = 1$, we obtain the eigenvector as $[\,0, -1, 1]^T$.

Example 3.36 Find the eigenvalues and the corresponding eigenvectors of the following matrices.

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(ii) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(iii) } \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution In each of the above problems, we obtain the characteristic equation as $(1 - \lambda)^3 = 0$. Therefore, the eigenvalues are $\lambda = 1, 1, 1$, a repeated value. Since a 3×3 matrix has 3 eigenvalues, it is important to know, whether the given matrix has 3 linearly independent eigenvectors, or it has lesser number of linearly independent eigenvectors.

Corresponding to the eigenvalue $\lambda = 1$, we obtain the following eigenvectors.

(i)
$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ arbitrary.} \end{cases}$$

Choosing $x_1 = 1$, we obtain the solution as $[1, 0, 0]^T$.

Hence, $\mathbf{A}$ has only one independent eigenvector.

(ii)
$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{cases} x_2 = 0 \\ x_1, x_3 \text{ arbitrary.} \end{cases}$$

Taking $x_1 = 0$, $x_3 = 1$ and $x_1 = 1$, $x_3 = 0$, we obtain two linearly independent solutions

$$\mathbf{x}_1 = [0, 0, 1]^T, \quad \mathbf{x}_2 = [1, 0, 0]^T.$$

In this case $\mathbf{A}$ has two linearly independent eigenvectors.

(iii)
$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system is satisfied for arbitrary values of all the three variables. Hence, we obtain three linearly independent eigenvectors, which can be taken as

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$$\mathbf{x}_1 = [1, 0, 0]^T, \quad \mathbf{x}_2 = [0, 1, 0]^T, \quad \mathbf{x}_3 = [0, 0, 1]^T.$$

We now state some important results regarding the relationship between the eigenvalues of a matrix and the corresponding linearly independent eigenvectors.

1. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. If λ is an eigenvalue of multiplicity m of a square matix $\mathbf{A}$ of order n, then the number of linearly independent eigenvectors associated with λ is given by

$$p = n - r, \quad \text{where} \quad r = \text{rank } (\mathbf{A} - \lambda\mathbf{I}), \; 1 \le p \le m.$$

Remark 21

In Example 3.35, all the eigenvalues are distinct and therefore, the corresponding eigenvectors are linearly indenpendent. In Example 3.36, the eigenvalue $\lambda = 1$ is of multiplicity 3. We find that in

(i) Example 3.36(i), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 2 and we obtain one linearly independent eigenvector.

(ii) Example 3.36(ii), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 1 and we obtain two linearly independent eigenvectors.

(iii) Example 3.36(iii), the rank of the matrix $\mathbf{A} - \mathbf{I}$ is 0 and we obtain three linearly independent eigenvectors.

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Exercise 3.4

Verify the Cayley-Hamilton theorem for the matrix **A**. Find $\mathbf{A}^{-1}$, if it exists, where **A** is as given in Problems 1 to 6.

1. $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$.

2. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & i & i \\ i & 1 & i \\ i & i & 1 \end{bmatrix}$.

Find all the eigenvalues and the corresponding eigenvectors of the matrices given in Problems 7 to 18. Which of the matrices are diagonalizable?

7. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

9. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & 1 & i \\ 1 & 0 & i \\ -i & -i & 1 \end{bmatrix}$.

12. $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$.

13. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

14. $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & -2 & 1 \\ -1 & 1 & -2 & 1 \end{bmatrix}$.

15. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

16. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{bmatrix}$.

17. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

18. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Show that the matrices given in Problems 19 to 24 are diagonalizable. Find the matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

19. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$.

20. $\begin{bmatrix} -3 & -2 & 1 \\ -2 & 0 & 4 \\ -6 & -3 & 5 \end{bmatrix}$.

21. $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$.

22. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

23. $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$.

24. $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

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Find the matrix **A** whose eigenvalues and the corresponding eigenvectors are as given in Problems 25 to 30.

25. Eigenvalues: 2, 2, 4; Eigenvectors: $(-2, 1, 0)^T$, $(-1, 0, 1)^T$, $(1, 0, 1)^T$.

26. Eigenvalues: 1, –1, 2; Eigenvectors: $(1, 1, 0)^T$, $(1, 0, 1)^T$, $(3, 1, 1)^T$.

27. Eigenvalues: 1, 2, 3; Eigenvectors: $(1, 2, 1)^T$, $(2, 3, 4)^T$, $(1, 4, 9)^T$.

28. Eigenvalues: 1, 1, 1; Eigenvectors: $(-1, 1, 1)^T$, $(1, -1, 1)^T$, $(1, 1, -1)^T$.

29. Eigenvalues: 0, –1, 1; Eigenvectors: $(-1, 1, 0)^T$, $(1, 0, -1)^T$, $(1, 1, 1)^T$.

30. Eigenvalues: 0, 0, 3; Eigenvectors: $(1, 2, -1)^T$, $(-2, 1, 0)^T$, $(3, 0, 1)^T$.

31. Let a 4×4 matrix **A** have eigenvalues 1, –1, 2, –2, Find the value of the determinant of the matrix $\mathbf{B} = 2\mathbf{A} + \mathbf{A}^{-1} - \mathbf{I}$.

32. Let a 3×3 matrix **A** have eigenvalues 1, 2, –1. Find the trace of the matrix $\mathbf{B} = \mathbf{A} - \mathbf{A}^{-1} + \mathbf{A}^2$.

33. Show that the matrices **A** and $\mathbf{P}^{-1}\mathbf{AP}$ have the same eigenvalues.

34. Let **A** and **B** be square matrices of the same order. Then, show that **AB** and **BA** have the same eigenvalues but different eigenvectors.

35. Show that the matrices $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{BA}^{-1}$ have the same eigenvlaues but different eigenvectors.

36. An $n \times n$ matrix **A** is *nilpotent* if for some positive integer k, $\mathbf{A}^k = \mathbf{0}$. Show that all the eigenvalues of a nilpotent matrix are zero.

37. If **A** is an $n \times n$ diagonalizable matrix and $\mathbf{A}^2 = \mathbf{A}$, then show that each eigenvalue of **A** is 0 or 1.

38. Show that the matrix $\mathbf{A} = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$, $a \neq b$, is transformed to a diagonal matrix $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, where **P** is of the form $\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\tan 2\theta = \dfrac{2h}{a-b}$.

39. Let **A** be similar to **B**. Then show that (i) $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$, (ii) $\mathbf{A}^m$ is similar to $\mathbf{B}^m$ for any positive integer m, (iii) $|\mathbf{A}| = |\mathbf{B}|$.

40. Let **A** and **B** be symmetric matrices of the same order. Then, show that **AB** is symmetric if and only if $\mathbf{AB} = \mathbf{BA}$.

41. For any square matrix **A**, show that $\mathbf{A}^T\mathbf{A}$ is symmetric.

42. Let **A** be a non-singular matrix. show that $\mathbf{A}^T\mathbf{A}^{-1}$ is symmetric if and only if $\mathbf{A}^2 = (\mathbf{A}^T)^2$.

43. If **A** is a symmetric matrix and $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, then show that **P** is an orthogonal matrix.

44. Show that the product of two orthogonal matrices of the same order is also an orthogonal matrix.

45. Find the conditions that a matrix $\mathbf{A} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal.

46. If **A** is an orthogonal matrix, show that $|\mathbf{A}| = \pm 1$.

47. Prove that the eigenvectors of a symmetric matrix corresponding to distinct eigenvalues are orthogonal.

48. A matrix **A** is called a *normal matrix* if $\mathbf{A}\overline{\mathbf{A}}^T = \overline{\mathbf{A}}^T\mathbf{A}$. Show that the Hermitian, skew-Hermitian and unitary matrices are normal.

49. If a matrix **A** can be diagonalized using an orthogonal matrix, then show that **A** is symmetric.

50. Suppose that a matrix **A** is both unitary and Hermitian. Then, show that $\mathbf{A} = \mathbf{A}^{-1}$.

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