Lecture 17

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Generating Function

The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

$a_n = 1,\quad G(x) = \frac{1}{1-x} \leftarrow 1 + x + x^2 + x^3 + \ldots \quad (1-x)^{-1}$

$a_n = (-1)^n,\quad G(x) = \frac{1}{1+x} \qquad 1 - x + x^2 - x^3 + \quad (1+x)^{-1}$

$G(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots = 1 + (2x) + (2x)^2 + (2x)^3 + \ldots$

$(1-2x)^{-1} \quad a_n = 2^n$

$G(x) = 1 - 2x + 4x^2 - 8x^3 + \ldots = 1 - 2x + (2x)^2 - (2x)^3 + \ldots$

$= (1+2x)^{-1} \quad a_n = 2^n(-1)^n$

$G(x) = 1 + x^2 + x^4 + x^6 + \ldots = 1 + x^2 + (x^2)^2 + \ldots$

$= (1 - 1\cdot x^2)^{-1}$

$a_{2n} \to x^{2n} \cdot 1$

$a_n \quad x^n$

$a_n = \begin{cases} 1 & 2 \mid n \\ 0 & 2 \nmid n \end{cases}$ 2 divides n

$G(x) = 1 - x^3 + x^6 - x^9 \ldots = (1+x^3)^{-1} \to (-1)^n x^{3n}$

$a_n = \begin{cases} (-1)^{n/3} & 3 \mid n \\ 0 & 3 \nmid n \end{cases}$

$\underbrace{(-1)^n x^{3n}}_{a_{3n}}$

$$a_n = \begin{cases} \ldots \\ 0 & 3 \nmid n \end{cases}$$

a_{3n}

$$G(x) = 1 - 2x^2 + 4x^4 - 8x^8 \ldots$$
$$= 1 - (2x^2) + (2x^2)^2 - (2x^2)^3$$
$$= (1 + 2x^2)^{-1}$$

$$a_n = \begin{cases} 2^{n/2}(-1)^{n/2}, & 2 \mid n \\ 0 & 2 \nmid n \end{cases}$$

$$, \quad a_{2n} = 2^n x^{2n} (-1)^n$$

$$a_{2n+1} = 0$$

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$ $= 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$ $\quad {}^nC_k$ $\quad (1-x)^n \to a_k = (-1)^k\, {}^nC_k$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$ $= 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$ $\quad {}^nC_k\, a^k$ $\quad k \leq 4$ $\quad (1-2x)^4, \; {}^4C_k\, 2^k (-1)^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$ $= 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise ${}^nC_{k/r}$ $\quad r \mid k$ 0 $\quad r \nmid k$
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \leq n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = \boxed{1 + 2x + 3x^2 + \cdots}$	$k+1$ $\quad a_k = (k+1)$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ $= 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$ $\frac{1}{(1-x)^n} = (1-x)^{-n}$ $\quad {}^{n+k-1}C_{n-1}$

$(1-x)^{-3}$

$$1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1(1-x^6)}{1-x}$$

$$ {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n = (1+x)^n$$

$n \to$ +ve integer

$n \to +ve$ integer

$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n b^n$

Q28.

For each of these generating functions, provide a closed formula for the sequence it determines.

a) $(x^2+1)^3$
b) $(3x-1)^3$
c) $1/(1-2x^2)$
d) $x^2/(1-x)^3$
e) $x-1+(1/(1-3x))$
f) $(1+x^3)/(1+x)^3$

(a) $(1+x^2)^3$

$a_{2k} \to {}^3C_k x^{2k}$

$$a_k = \begin{cases} {}^3C_{k/2} & 2 \mid k \\ 0 & 2 \nmid k \end{cases}, \quad k \leq 6$$

(b) $(3x-1)^3 =$

${}^3C_k (-1)^k (3x)^k$

$a_k = {}^3C_k (-1)^k (3)^k \quad , k \leq 3$

(c) $\frac{1}{(1-2x^2)} = (1-2x^2)^{-1}$

$(2x^2)^k, \; 2^k x^{2k}$

$\uparrow$

a_{2k}

$$a_k = \begin{cases} 2^{k/2} & 2 \mid k \\ 0 & 2 \nmid k \end{cases}$$

(d) $\frac{x^2}{(1-x)^3} = x^2 (1-x)^{-3} = \boxed{x^2 \sum_{k=0}^{\infty} {}^{3+k-1}C_{3-1} x^k}$

$a_{k+2} \to {}^{k+2}C_2 \, x^{k+2}$

${}^{k+2}C_2$

$a_{k+2} \rightarrow \dots C_2\, x \qquad {}^{k+2}C_2$

$a_k = {}^{k}C_2 \, , \; k \geq 2$

$a_0 = a_1 = 0$

Q27.

Find a closed form for the generating function for each of these sequences. (Assume a general form for the terms of the sequence, using the most obvious choice of such a sequence.)

a) $-1, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, \ldots$

b) $1, 3, 9, 27, 81, 243, 729, \ldots$

c) $0, 0, 3, -3, 3, -3, 3, -3, \ldots$

d) $1, 2, 1, 1, 1, 1, 1, 1, 1, \ldots$

e) $\binom{7}{0}, 2\binom{7}{1}, 2^2\binom{7}{2}, \ldots, 2^7\binom{7}{7}, 0, 0, 0, 0, \ldots$ $(1+2x)^7$

(e) ${}^7C_0 + 2\,{}^7C_1 x + 2^2\,{}^7C_2 x^2 + \ldots 2^7\,{}^7C_7 x^7$

(a) $G(x) = (-1) + (-1)x^1 + (-1)x^2 + \ldots (-1)x^6$

$G(x) = -\left(1 + x + x^2 + \ldots + x^6\right) = -\left(\frac{1(1-x^7)}{1-x}\right)$

(b) $G(x) = 1 + 3x^1 + 9x^2 + 27x^3 + 81x^4 + \ldots$

$G(x) = 1 + (3x)^1 + (3x)^2 + (3x)^3 + \ldots$

$G(x) = (1-3x)^{-1} = \frac{1}{1-3x}$

(c) $G(x) = 3x^2 - 3x^3 + 3x^4 - 3x^5 + \ldots$

$G(x) = 3x^2\left(1 - x + x^2 - x^3 + \ldots\right) = 3x^2(1+x)^{-1}$

$G(x) = \frac{3x^2}{1+x}$

Q29.

Find the coefficient of x^9 in the power series of each of these functions.

a) $(1 + x^3 + x^6 + x^9 + \cdots)^3$

b) $(x^2 + x^3 + x^4 + x^5 + x^6 + \cdots)^3$