11.11 EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\,\boldsymbol{dx} + \mathbf{N}(x, y)\,\boldsymbol{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \qquad ...(1)$$

where u is some function of x any y.

But $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad ...(2)$$

$\therefore$ equating coefficients of dx and dy in (1) and (2), we get $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \qquad \text{(Assumption)}$$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : *i.e.*, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \quad i.e., \; M = \frac{\partial u}{\partial x}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \qquad \left\{\begin{array}{l} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ (given)} \\ \text{and } \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \end{array}\right. \qquad ...(3)$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \qquad ...(4)$$

$\therefore$ $$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\}dy \qquad \text{[By (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\} + f(y)\,dy = du + f(y)\,dy = d\,[u + \int f(y)dy] \qquad ...(5)$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d[u + \int f(y)dy] = 0$

Integrating $\quad u + \int f(y)dy = 0.$

But $\quad u = \int_{y\text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y\text{ cons.})} \mathbf{M\,dx} + \int \textbf{(terms of N not containing x) dy} = \mathbf{c}$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0.$ *(V.T.U., 2006)*

Solution. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\, e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y\, e^{xy^2} + y^2\, e^{xy^2} .\, 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y \text{ const.})} (y^2 e^{y^2 x} + 4x^3) dx + \int (-3y^2)\, dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1+\frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\, dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int_{(y \text{ const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\, y + x \cos y = c.$$

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y \text{ const})} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 .\, 2x dx - \int 2x\, dx\right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Kurukshetra, 2005)*

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y \text{ const.})} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

Example 11.29. *Solve* $(2x^2 + 3y^2 - 7)\,xdx - (3x^2 + 2y^2 - 8)\,ydy = 0.$ *(U.P.T.U., 2005)*

Solution. Given equation can be written as

$$\frac{ydy}{xdx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{ydy + xdx}{ydy - xdx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$$ [By componendo & dividendo]

or $$\frac{xdx + ydy}{x^2 + y^2 - 3} = 5 \cdot \frac{xdx - ydy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c'$$ [Writing $c = \log c'$]

or $$x^2 + y^2 - 3 = c' (x^2 - y^2 - 1)^5$$

which is the required solution.

PROBLEMS 11.7

Solve the following equations :

1. $(x^2 - ay)\,dx = (ax - y^2)\,dy.$
2. $(x^2 + y^2 - a^2)\,xdx + (x^2 - y^2 - b^2)\,ydy = 0$ *(Kurukshetra, 2005)*
3. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$
4. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$
5. $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$
6. $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$ *(V.T.U., 2008)*
7. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$
8. $\dfrac{2x}{y^3}dx + \dfrac{y^2 - 3x^2}{y^4}dy = 0$
9. $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$
10. $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$ *(Marathwada, 2008)*
11. $(2xy + y - \tan y)\,dx + x^2 - x \tan^2 y + \sec^2 y)\,dy = 0.$ *(Nagpur, 2009)*

11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) I.F. found by inspection. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d\,(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \frac{xdy}{xy} - \frac{ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right); \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right).$$

Example 11.30. *Solve* $y\,(2xy + e^x)\,dx = e^x\,dy.$ *(Kurukshetra, 2005)*

Solution. It is easy to note that the terms ye^xdx and e^xdy should be put together.

$\therefore$ $$(ye^xdx - e^xdy) + 2xy^2\,dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\frac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor $(Mx + Ny \neq 0)$.*

Example 11.31. *Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.* *(Osmania, 2003 S)*

Solution. This equation is homogeneous in x and y.

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ the solution is $\int_{(y\text{ const})} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$ or $\frac{x}{y} - 2\log x + 3\log y = c$.

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.
If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

Example 11.32. *Solve $(1 + xy)\,ydx + (1 - xy)\,xdy = 0$.* *(S.V.T.U., 2008)*

Solution. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\qquad M = (1 + xy)\,y, N = (1 - xy)\,x.$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1 + xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$\therefore$ the solution is $\int_{(y\text{ const})} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

or $$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation $Mdx + Ndy = 0$,

(*a*) *if* $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ *be a function of x only* $= f(x)$ *say, then* $e^{\int f(x)dx}$ *is an integrating factor.*

(*b*) *if* $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ *be a function of y only* $= F(y)$ *say, then* $e^{\int F(y)dy}$ *is an integrating factor.*

Example 11.33. *Solve $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$.* *(S.V.T.U., 2009 ; Mumbai, 2007)*

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\frac{y^2}{x^3} - \frac{1}{4^4} e^{1/x^3}\right) dx - \frac{y}{x^2}\, dy = 0$

which is an exact equation.

$\therefore$ the solution is $\int_{(y\text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

or $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx + 0 = c$$

or $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3}\int e^{x^{-3}} (-3x^{-4})\, dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2}\frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

Example 11.34. *Solve* $(xy^3 + y)\, dx + 2\,(x^2y^2 + x + y^4)\, dy = 0.$

Solution. Here $M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)}(4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\, dx + (2x^2y^3 + 2xy + 2y^5)\, dy = 0$, which is an exact equation.

$\therefore$ its solution is $\int_{(y\text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = 0$

or $$\int_{(y\text{ const})} (xy^4 + y^2) dx + \int 2y^5\, dy = c \quad \text{or} \quad \frac{1}{2}x^2y^4 + xy^2 + \frac{1}{3}y^6 = c.$$

Example 11.35. *Solve* $(y \log y)\, dx + (x - \log y)\, dy = 0$ *(U.P.T.U., 2004)*

Solution. Here $M = y \log y$ and $N = x - \log y$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y}(1 - \log y - 1) = -\frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y}(x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \frac{\partial}{\partial y}(\log y) = \frac{\partial}{\partial x}\left(\frac{x - \log y}{y}\right)\right.$

$\therefore$ its solution is $\int_{(y\text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $$\log y \int dx + \int \left(\frac{-\log y}{y}\right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2}(\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b\, (mydx + nxdy) + x^{a'} y^{b'}\, (m'ydx + n'xdy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n},\ \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

Example 11.36. *Solve $y(xy + 2x^2y^3)\,dx + x\,(xy - x^2y^2)\,dy = 0$.* (*Hissar, 2005 ; Kurukshetra, 2005*)

Solution. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^ay^b\,(mydx + nxdy) + x^{a'}y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1$, $m = n = 1$; $a' = b' = 2$, $m' = 2$, $n' = -1$.

$$\therefore \qquad \text{I.F.} = x^hy^k.$$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n},\ \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

i.e. $$\frac{1+h+1}{1} = \frac{1+k+1}{1},\ \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or $$h - k = 0,\ h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\int_{(y\ const)} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

or $$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c \quad \text{or} \quad 2\log x - \log y - 1/xy = c.$$

PROBLEMS 11.8

Solve the following equations :

1. $xdy - ydx + a(x^2 + y^2)\,dx = 0$.
2. $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$. (*U.P.T.U., 2005*)
3. $ydx - xdy + \log x\,dx = 0$.
4. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$.
5. $(x^3y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0$.
6. $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)\,xdy = 0$.
7. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$.
8. $(4xy + 3y^2 - x)\,dx + x\,(x + 2y)dy = 0$ (*Mumbai, 2006*)
9. $x^4\dfrac{dy}{dx} + x^3y + \text{cosec}\,(xy) = 0$.
10. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ (*Mumbai, 2006*)
11. $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0$. (*Kurukshetra, 2006*)
12. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)dy = 0$. (*Rajasthan, 2005*)
13. $2ydx + x(2\log x - y)\,dy = 0$. (*P.T.U., 2005*)

11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \dots + P_n = 0 \qquad \dots(1)$$

where $P_1, P_2, \dots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)]\,\dots\,[p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y),\ p = f_2(x, y),\ \dots,\ p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0,\ \dots,\ F_n\,(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \,.\, F_2(x, y, c) \,\ldots\ldots\, F_n(x, y, c) = 0.$$

Example 11.37. *Solve* $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Solution. Given equation is $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$ where $p = \frac{dy}{dx}$ or $p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0.$

Factorising $(p + y/x)(p - x/y) = 0.$

Thus we have $\quad p + y/x = 0 \quad \ldots(i) \quad$ and $\quad p - x/y = 0 \quad \ldots(ii)$

From (*i*), $\quad \frac{dy}{dx} + \frac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (*ii*), $\quad \frac{dy}{dx} - \frac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 11.38. *Solve* $p^2 + 2py \cot x = y^2$. *(Bhopal, 2008 ; Kerala, 2005)*

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $\quad p + y \cot x = \pm\, y \operatorname{cosec} x$

i.e., $\quad p = y\,(-\cot x + \operatorname{cosec} x) \quad \ldots(i)$

or $\quad p = y\,(-\cot x - \operatorname{cosec} x) \quad \ldots(ii)$

From (*i*), $\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\frac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, $\log y = \log \tan \frac{x}{2} - \log \sin x + \log c = \log \frac{c \tan x/2}{\sin x}$

or $\quad y = \frac{c}{2 \cos x^2/2}$ or $y\,(1 + \cos x) = c \quad \ldots(iii)$

From (*ii*), $\quad \frac{dy}{dx} = -\,y\,(\cot x + \operatorname{cosec} x) \quad$ or $\quad \frac{dy}{y} = -\,(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$

or $\quad y = \frac{c}{2 \sin^2 \frac{x}{2}} \quad$ or $\quad y(1 - \cos x) = c \quad \ldots(iv)$

Thus combining (*iii*) and (*iv*), the required general solution is

$$y\,(1 \pm \cos x) = c.$$

PROBLEMS 11.9

Solve the following equations :

1. $y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0.$
2. $p(p + y) = x\,(x + y).$ *(V.T.U., 2011)*
3. $y = x\,[p + \sqrt{(1 + p^2)}]$.
4. $xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\frac{dy}{dx} + xy = 0.$
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$ *(Madras, 2003)*

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \qquad ...(1)$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $\quad F(x, p, c) = 0. \qquad ...(2)$

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c), y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 11.39. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2) \qquad ...(i)$

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $\quad p + 2x\dfrac{dp}{dx} + \left(p + 2x\dfrac{dp}{dx}\right) \cdot \dfrac{p}{1 + x^2p^4} = 0$ or $\left(p + 2x\dfrac{dp}{dx}\right)\left(1 + \dfrac{p}{1 + x^2p^4}\right) = 0$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\displaystyle\int \frac{dx}{x} + 2\int \frac{dp}{p}$ = a constant

or $\quad \log x + 2\log p = \log c \quad$ or $\quad \log xp^2 = \log c$

whence $\quad xp^2 = c \quad$ or $\quad p = \sqrt{(c/x)} \qquad ...(ii)$

Eliminating p from (i) and (ii), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$

or $\quad y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (i).

Obs. The significance of the factor $1 + p/(1 + x^2p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (i) whereas *we are interested only in finding general solution.*

Caution. Sometimes one is tempted to write (ii) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x}\right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is *incorrect*.

Example 11.40. *Solve* $y = 2px + p^n$. *(Bhopal, 2009)*

Solution. Given equation is $\quad y = 2px + p^n \qquad ...(i)$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

or $$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \qquad ...(ii)$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int \frac{2}{p}dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (*ii*) is

$$x(\text{I.F.}) = \int(-np^{n-2}) \cdot (\text{I.F.})\, dp + c \quad \text{or} \quad xp^2 = -n\int p^n\, dp + c = -\frac{np^{n+1}}{n+1} + c$$

or
$$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad ...(iii)$$

Substituting this value of x in (*i*), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n}p^n$ ...(*iv*)

The equations (*iii*) and (*iv*) taken together, with parameter p, constitute the general solution (*i*).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's* equation, are solvable for y and lead to Leibnitz's equation in dx/dp.

PROBLEMS 11.10

Solve the following equations :

1. $y = x + a\tan^{-1}p$.
2. $y + px = x^4p^2$. (*S.V.T.U., 2007*)
3. $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$.
4. $xp^2 + x = 2yp$.
5. $y = xp^2 + p$.
6. $y = p\sin p + \cos p$.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad ...(1)$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y.

Example 11.41. *Solve* $y = 2px + y^2p^3$. (*Bhopal, 2008*)

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(=\dfrac{1}{p}\right) = \dfrac{1}{2} \cdot \dfrac{p\left(1 - 2y \cdot p^3 - y^2 3p^2\dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or
$$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or
$$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or
$$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $\qquad py = c.$ ...(*i*)

Thus eliminating from the given equation and (*i*), we get $y = 2\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

PROBLEMS 11.11

Solve the following equations :

1. $p^3 - 4xyp + 8y^2 = 0$. (*Kanpur, 1996*)
2. $p^3y + 2px = y$.
3. $x - yp = ap^2$. (*Andhra, 2000*)
4. $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$. (*S.V.T.U., 2008*)

11.14 CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x \dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$

or
$$[x + f'(p)]\frac{dp}{dx} = 0 \qquad \therefore \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \qquad ...(2)$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p by *c*.

Obs. If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. *This is the* **singular solution** of (1) *which gives the envelope of the family of straight lines* (3).

To obtain the singular solution, we proceed as follows :

(*i*) Find the general solution by replacing p by c *i.e.*, (3)

(*ii*) Differentiate this w.r.t. c giving $x + f(c) = 0$. ...(4)

(*iii*) Eliminate c from (3) and (4) which will be the *singular solution*.

Example 11.42. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Solution. Given equation can be written as

$\sin^{-1} p = y - xp$ or $y = px + \sin^{-1} p$ which is the *Clairaut's equation.*

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \qquad ...(ii)$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (*i*), we get

$$y = N(x^2 - 1) + \sin^{-1}\{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.

Example 11.43. *Solve $(px - y)(py + x) = a^2p$.* (*V.T.U., 2011 ; J.N.T.U., 2006*)

Solution. Put $x^2 = u$ and $y^2 = v$ so that $2xdx = du$ and $2ydy = dv$

$$\therefore \quad p = \frac{dy}{dx} = \frac{dv}{y}\Big/\frac{du}{x} = \frac{x}{y}P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy *Alexis Claude Clairaut* (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

Then the given equation becomes $\left(\frac{xP}{y}.x - y\right)\left(\frac{xP}{y}.y + x\right) = a^2\frac{xP}{y}$

or $(uP - v)(P + 1) = a^2P$ or $uP - v = \frac{a^2P}{P+1}$

or $v = uP - a^2P/(P + 1)$, which is Clairaut's form.

$\therefore$ its solution is $v = uc - a^2c/(c + 1)$, *i.e.*, $y^2 = cx^2 - a^2c/(c + 1)$.

PROBLEMS 11.12

1. Find the general and singular solution of the equations :
   (*i*) $xp^2 - yp + a = 0$. (*J.N.T.U., 2006*) (*ii*) $p = \log(px - y)$.
   (*iii*) $y = px + \sqrt{(a^2p^2 + b^2)}$ (*W.B.T.U., 2005*) (*iv*) $\sin px \cos y = \cos px \sin y + p$ (*P.T.U., 2006*)

Solve the following equations :

2. $y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\frac{dy}{dx}$.
3. $(y - px)(p - 1) = p$.
4. $(x-a)\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - y = 0$.
5. $x^2(y - px) = yp^2$.
6. $(px + y)^2 = py^2$.
7. $(px - y)(x + py) = 2p$.

11.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.13

Fill up the blanks or choose the correct answer in the following problems :

1. $y = cx - c^2$, is the general solution of the differential equation
   (*i*) $(y')^2 - xy' + y = 0$ (*ii*) $y'' = 0$ (*iii*) $y' = c$ (*iv*) $(y')^2 + xy' + y = 0$.
2. The differential equation having a basis for its solution as sinh $6x$ and cosh $6x$ is
   (*i*) $y'' + 36y = 0$ (*ii*) $y'' - 36y = 0$ (*iii*) $y'' + 6y = 0$ (*iv*) none of these.
3. The differential equation $(dx/dy)^2 + 5y^{1/3} = x$ is
   (*i*) linear of degree 3 (*ii*) non-linear of order 1 and degree 6
   (*iii*) non-linear of order 1 and degree 2.
4. The differential equation $y dx/dy + 1 = y, y(0) = 1$, has
   (*i*) a unique solution (*ii*) two solutions
   (*iii*) infinite number of solutions (*iv*) no solution
5. Solution of $(x^2 + y^2)\,dy = xy\,dx$ is ....
6. Solution of $(3x - 2y)\,dx = xdy$ is ....
7. Solution of $dy/dx - y = 2xy^2 e^{-x}$ is ....
8. The differential equation $(y^2e^{xy^2} + 6x)\,dx + (2xye^{xy^2} - 4y)\,dy = 0$ is
   (*i*) linear, homogeneous and exact (*ii*) non-linear, homogeneous and exact
   (*iv*) non-linear, non-homogeneous and exact (*iv*) non-linear, non-homogeneous and inexact.
9. Solution of $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2}$ is ....
10. Solution of $dy/dx = \frac{x^3 + y^3}{xy^2}$ is ......
11. The differential equation $(x + x^8 + ay^2)\,dx + (y^8 - y + bxy)\,dy = 0$ is exact if
    (*i*) $b = 2a$ (*ii*) $a = b$ (*iii*) $a \neq 2b$ (*iv*) $a = 1, b = 3$.
12. Solution of $xy\,(1 + xy^2)\,dy = dx$ is ....
13. Solution of $xp^2 - yp + a = 0$ is ....
14. The differential equation $p = \log(px - y)$ has the solution ....
15. Solution of $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ is ....