Q Find the eigen values & eigen vector of a matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0 \quad \text{--- (1)}$$

let $\lambda = -2$

$$(-2)^3 - 7(-2)^2 + 36 = 0$$

$$-8 - 28 + 36 = 0$$

$$0 = 0$$

$\therefore$ $\lambda = -2$ is a root of equation (1)

The remaining roots are given by

-2	1	-7	0	36
	$\downarrow$	-2	18	-36
	1	-9	18	0 → Remainder

quotient = $\lambda^2 - 9\lambda + 18$

The remaining roots are given by

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda^2 - 6\lambda - 3\lambda + 18 = 0$$

$$\lambda(\lambda - 6) - 3(\lambda - 6) = 0$$

$$(\lambda-3)(\lambda-6)=0$$

$$\lambda=3 \text{ \& } \lambda=6$$

The eigen values are $\lambda=-2, 3 \text{ \& } 6$

let $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda=-2$.

Consider $(A+\underline{2}I)X=0$

$$\begin{bmatrix} \underline{3} & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX=\lambda X$

$(A-\lambda I)X=0$

$|A-\lambda I|=0$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-3R_1, \quad R_3 \to R_3-3R_1$

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & -20 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3-R_2$

$$\sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- } (2)$$

✓ Number of eigen vectors $= \underline{3-2=1}$

Here from (2) $x+7y+z=0$

$-20y=0 \Rightarrow \boxed{y=0}$

$\therefore x+0+z=0$

$\boxed{x=-z}$

$z=-x$

$X=\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}x$

$$\boxed{x = -3}$$

$$\therefore\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} z$$

$$X = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x$$

$\therefore$ $(-1, 0, 1)$ is the eigen vector corresponding to eigen value $\boxed{\lambda = -2}$

$X \to \lambda$

$\alpha X \to \lambda$

$$\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \checkmark$$

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector corresponding to $\lambda = 3$

consider $(A - 3I)X = 0$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $\quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

No of eigen vectors $= 3 - 2 = 1$

$\therefore$ $x + 2y + z = 0$

$5y + 5z = 0 \Rightarrow \boxed{y = -z}$

$\therefore \quad x + 2y + z = 0$

$5y + 5z = 0 \Rightarrow \boxed{y = -z}$

$\therefore \quad x - 2z + z = 0$

or $\boxed{x = z}$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} z$$

$(1, -1, 1)$ is the eigen vector for $\lambda = 3$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector for $\lambda = 6$

Do solve yourself

$\boxed{(1, 2, 1)}$

Exceptional Case

✓ repeated eigen value

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nr of eigen vectors $3 - 1 = 2$

$x + 2y + 3z = 0$

✓ or $x = -2y - 3z$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y - 3z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} z$$

$\therefore \quad (-2, 1, 0)$ & $(-3, 0, 1)$ will be eigen vectors

$\boxed{|A - \lambda I| = 0}$

① $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ trace of A = Sum of elements on the main diagonal

Characteristic eqⁿ

$$\lambda^2 - (\text{trace } A) + \det A = 0$$

$$\boxed{\lambda^2 - 7\lambda + 6 = 0}$$

(2) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Imp //

Characteristic eqⁿ

$$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{sum of minors of elements on the main diagonal})\lambda - \det A = 0$$

$$\therefore \lambda^3 - 7\lambda^2 + (4 + (-8) + 4)\lambda - (-36) = 0$$

$$\boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$

Properties of Eigen value

(1) Any square matrix A and its transpose A' have the same eigen values

$$(A - \lambda I)' = A' - \lambda I'$$

$$\therefore |(A - \lambda I)'| = |A' - \lambda I|$$

$$\therefore |(A - \lambda I)| = |A' - \lambda I| \qquad [\because |B| = |B'|]$$

$$\therefore |A - \lambda I| = 0 \text{ iff } |A' - \lambda I| = 0$$

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \quad \& \quad A' = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

$\lambda = 1, 6$

$\lambda^2 - 7\lambda + 6 = 0$

$\lambda = 1, 6$

(2) αA has the eigen value $\alpha\lambda$ & the corresponding eigen vector ... X

(3) αA has the eigen value $\alpha\lambda$ & the corresponding eigen vector is X

If $\lambda \longrightarrow A \longrightarrow X$

$$AX = \lambda X$$

$$(\alpha A)X = (\alpha\lambda)X$$

$$\Rightarrow \alpha\lambda \longrightarrow \underline{\alpha A}$$

$$\boxed{\begin{array}{l} \lambda \to A \to X \\ AX = \lambda X \end{array}}$$

$$\alpha A X = \alpha \lambda X$$

$$\alpha \quad (\alpha A)X = (\alpha\lambda)X$$

beg $\lambda = 1, 6$ are $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$\lambda = 2$ & 12 $\longleftarrow$ $2A = \begin{bmatrix} 10 & 8 \\ 2 & 4 \end{bmatrix}$

Q The eigen values of the triangular matrix are just the diagonal elements

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\lambda = 1, 2 \text{ \& } 1}$$