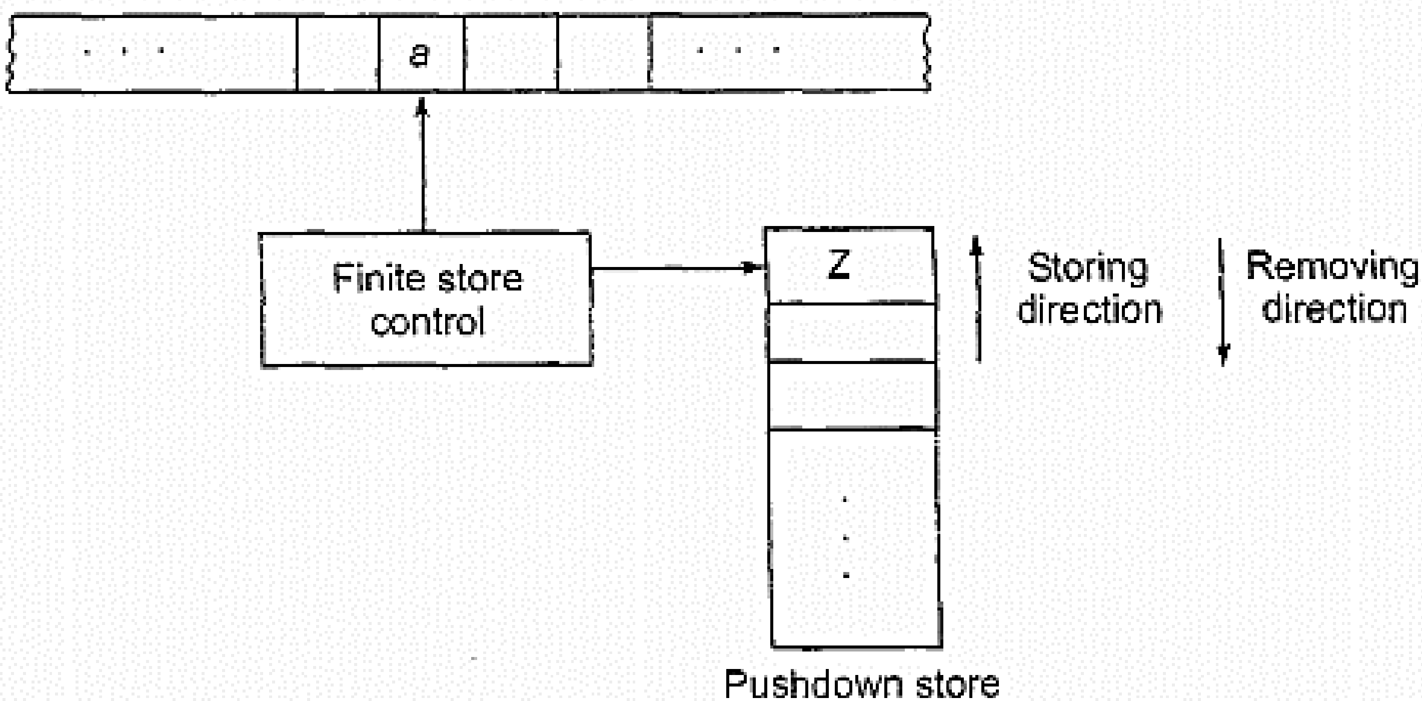
CSE322

Representation of Pushdown Automata
&
Description and Model of Pushdown Automata

Lecture #30

Pushdown Automata

- Adding additional auxiliary memory to Finite Automaton; in form of ‘ Stack’ ; is Pushdown Automaton.
- While removing the elements LIFO (Last In First Out) basis.



Model of a pushdown automaton.

- Has Read only Input Tape
- An input Alphabet
- Finite state control
- Set of final states
- Initial state
- In Addition to this has Stack “Pushdown Store”.
- It is a Read Write Pushdown Store, as element added to PDA or removed from PDA
- PDA is in some state and on reading an input symbol and the topmost symbol in PDA, it moves to a new state and writes(adds) a string of symbol in PDA.

Pushdown Automata

Pushdown automata are for context-free languages what finite automata are for regular languages.

PDAs are *recognizing automata* that have a single stack (= memory):

Last-In First-Out *pushing* and *popping*

Difference: PDAs are inherently nondeterministic.
(They are not practical machines.)

Definition 7.1 A pushdown automaton consists of

(i) a finite nonempty set of states denoted by Q,
(ii) a finite nonempty set of input symbols denoted by Σ,
(iii) a finite nonempty set of pushdown symbols denoted by Γ,
(iv) a special state called the initial state denoted by q_0,
(v) a special pushdown symbol called the *initial symbol* on the pushdown store denoted by Z_0,
(vi) a set of final states, a subset of Q denoted by F, and
(vii) a transition function δ from $Q \times (\Sigma \cup \{\Lambda\}) \times \Gamma$ to the set of finite subsets of $Q \times \Gamma^*$.

Symbolically, a pda is a 7-tuple, namely $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$.

Note: When $\delta(q, a, Z) = \emptyset$ for $(q, a, Z) \in Q \times (\Sigma \cup \{\Lambda\}) \times \Gamma$, we do not mention it.

Let

$$A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$

where

$$Q = \{q_0, q_1, q_f\}, \quad \Sigma = \{a, b\}, \quad \Gamma = \{a, Z_0\}, \quad F = \{q_f\}$$

and δ is given by

$$\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}, \ \delta(q_1, b, a) = \{(q_1, \Lambda)\}$$
$$\delta(q_0, a, a) = \{(q_0, aa)\}, \ \delta(q_1, \Lambda, Z_0) = \{(q_1, \Lambda)\}$$
$$\delta(q_0, b, a) = \{(q_1, \Lambda)\}$$

$$A = (\{q_0, q_1, q_f\}, \{a, b, c\}, \{a, b, Z_0\}, \delta, q_0, Z_0, \{q_f\})$$

is a pda, where δ is defined as

$$\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}, \quad \delta(q_0, b, Z_0) = \{(q_0, bZ_0)\} \tag{7.5}$$

$$\delta(q_0, a, a) = \{(q_0, aa)\}, \quad \delta(q_0, b, a) = \{(q_0, ba)\} \tag{7.6}$$

$$\delta(q_0, a, b) = \{(q_0, ab)\}, \quad \delta(q_0, b, b) = \{(q_0, bb)\} \tag{7.7}$$

$$\delta(q_0, c, a) = \{(q_1, a)\}, \quad \delta(q_0, c, b) = \{(q_1, b)\}, \ \delta(q_0, c, Z_0) = \{(q_1, Z_0)\} \tag{7.8}$$

$$\delta(q_1, a, a) = \delta(q_1, b, b) = \{(q_1, \Lambda)\} \tag{7.9}$$

$$\delta(q_1, \Lambda, Z_0) = \{(q_f, Z_0)\} \tag{7.10}$$

bacab

$$(q_0, bacab, Z_0) \vdash (q_0, acab, bZ_0) \quad \text{by Rule (7.5)}$$

$$\vdash (q_0, cab, abZ_0) \quad \text{by Rule (7.7)}$$

$$\vdash (q_1, ab, abZ_0) \quad \text{by Rule (7.8)}$$

$$\vdash (q_0\ b, bZ_0) \quad \text{by Rule (7.9)}$$

$$\vdash (q_1, \Lambda, Z_0) \quad \text{by Rule (7.10)}$$

$$\vdash (q_f, \Lambda, Z_0) \quad \text{by Rule (7.10)}$$

i.e.

$$(q_0, bacab, Z_0) \overset{*}{\vdash} (q_f, Z_0)$$

7.2 ACCEPTANCE BY pda

A pda has final states like a nondeterministic finite automaton and has also the additional structure, namely PDS. So we can define acceptance of input strings by pda in terms of final states or in terms of PDS.

Definition 7.6 Let $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a pda. The set accepted by pda by final state is defined by

$$T(A) = \{w \in \Sigma^* \mid (q_0, w, Z_0) \overset{*}{\vdash} (q_f, \Lambda, \alpha) \text{ for some } q_f \in F \text{ and } \alpha \in \Gamma^*\}$$

Construct a PDA for

$$\{a^n b^n : \quad n \geq 0\}$$

Construct a pda A accepting $L = \{wcw^T \mid w \in \{a, b\}^*\}$ by final state.

$$\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}, \quad \delta(q_0, b, Z_0) = \{(q_0, bZ_0)\}$$

$$\delta(q_0, a, a) = \{(q_0, aa)\}, \quad \delta(q_0, b, a) = \{(q_0, ba)\}$$

$$\delta(q_0, a, b) = \{(q_0, ab)\}, \quad \delta(q_0, b, b) = \{(q_0, bb)\}$$

$$\delta(q_0, c, a) = \{(q_1, a)\}, \quad \delta(q_0, c, b) = \{(q_1, b)\}, \ \delta(q_0, c, Z_0) = \{(q_1, Z_0)\}$$

$$\delta(q_1, a, a) = \delta(q_1, b, b) = \{(q_1, \Lambda)\}$$

$$\delta(q_1, \Lambda, Z_0) = \{(q_f, Z_0)\}$$

Construct a deterministic pda accepting $L = \{w \in \{a, b\}^* \mid$ the number of a's in w equals the number of b's in $w\}$ by final state.

We define a pda M as follows:

$$M = (\{q_0, q_1\}, \{a, b\}, \{a, b, Z_0\}, \delta, q_0, Z_0, \{q_1\})$$

where δ is defined by

$$\delta(q_0, a, Z_0) = \{(q_1, Z_0)\}$$

$$\delta(q_0, b, Z_0) = \{(q_0, bZ_0)\}$$

$$\delta(q_0, a, b) = \{(q_0, \Lambda)\}$$

$$\delta(q_0, b, b) = \{(q_0, bb)\}$$

$$\delta(q_1, a, Z_0) = \{(q_1, aZ_0)\}$$

$$\delta(q_1, b, Z_0) = \{(q_0, aZ_0)\}$$

$$\delta(q_1, a, a) = \{(q_1, aa)\}$$

$$\delta(q_1, b, a) = \{(q_1, \Lambda)\}$$

Construct a pda M accepting $L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$ by final state.

Theorem 7.4 If $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is a pda, then there exists a context-free grammar G such that $L(G) = N(A)$.

Proof We first give the construction of G and then prove that $N(A) = L(G)$.

Step 1 (Construction of G). We define $G = (V_N, \Sigma, P, S)$, where

$$V_N = \{S\} \cup \{[q, Z, q'] \mid q, q' \in Q, Z \in \Gamma\}$$

i.e. any element of V_N is either the new symbol S acting as the start symbol for G or an ordered triple whose first and third elements are states and the second element is a pushdown symbol.

The productions in P are induced by moves of pda as follows:

R_1: S-productions are given by $S \rightarrow [q_0, Z_0, q]$ for every q in Q.

R_2: Each move erasing a pushdown symbol given by $(q', \Lambda) \in \delta(q, a, Z)$ induces the production $[q, Z, q'] \rightarrow a$.

R_3: Each move not erasing a pushdown symbol given by $(q_1, Z_1Z_2 \ldots Z_m) \in \delta(q, a, Z)$ induces many productions of the form

$$[q, Z, q'] \rightarrow a[q_1, Z_1, q_2][q_2, Z_2, q_3] \ldots [q_m, Z_m, q']$$

where each of the states $q', q_2, \ldots, q_m$ can be any state in Q. Each move yields many productions because of R_3. We apply this construction to an example before proving that $L(G) = N(A)$.

Construct a context-free grammar G which accepts $N(A)$, where

$$A = (\{q_0, q_1\}, \{a, b\}, \{Z_0, Z\}, \delta, q_0, Z_0, \emptyset)$$

and δ is given by

$$\delta(q_0, b, Z_0) = \{(q_0, ZZ_0)\}$$
$$\delta(q_0, \Lambda, Z_0) = \{(q_0, \Lambda)\}$$
$$\delta(q_0, b, Z) = \{(q_0, ZZ)\}$$
$$\delta(q_0, a, Z) = \{(q_1, Z)\}$$
$$\delta(q_1, b, Z) = \{(q_1, \Lambda)\}$$
$$\delta(q_1, a, Z_0) = \{(q_0, Z_0)\}$$

The productions are

$$P_1: S \to [q_0, Z_0, q_0]$$

$$P_2: S \to [q_0, Z_0, q_1]$$

$\delta(q_0, b, Z_0) = \{(q_0, ZZ_0)\}$ yields

$$P_3: [q_0, Z_0, q_0] \to b[q_0, Z, q_0][q_0, Z_0, q_0]$$

$$P_4: [q_0, Z_0, q_0] \to b[q_0, Z, q_1][q_1, Z_0, q_0]$$

$$P_5: [q_0, Z_0, q_1] \to b[q_0, Z, q_0][q_0, Z_0, q_1]$$

$$P_6: [q_0, Z_0, q_1] \to b[q_0, Z, q_1][q_1, Z_0, q_1]$$

$\delta(q_0, \Lambda, Z_0) = \{(q_0, \Lambda)\}$ gives

$$P_7: [q_0, Z_0, q_0] \to \Lambda$$

$\delta(q_0, b, Z) = \{(q_0, ZZ)\}$ gives

$$P_8: [q_0, Z, q_0] \to b[q_0, Z, q_0][q_0, Z, q_0]$$

$$P_9: [q_0, Z, q_0] \to b[q_0, Z, q_1][q_1, Z, q_0]$$

$$P_{10}: [q_0, Z, q_1] \to b[q_0, Z, q_0][q_0, Z, q_1]$$

$$P_{11}: [q_0, Z, q_1] \to b[q_0, Z, q_1][q_1, Z, q_1]$$

$\delta(q_0, a, Z) = \{(q_1, Z)\}$ yields

$$P_{12}: [q_0, Z, q_0] \to a[q_1, Z, q_0]$$

$$P_{13}: [q_0, Z, q_1] \to a[q_1, Z, q_1]$$

$\delta(q_1, b, Z) = \{(q_1, \Lambda)\}$ gives

$$P_{14}: [q_1, Z, q_1] \to b$$

$\delta(q_1, a, Z_0) = \{(q_0, Z_0)\}$ gives

$$P_{15}: [q_1, Z_0, q_0] \to a[q_0, Z_0, q_0]$$

$$P_{16}: [q_1, Z_0, q_1] \to a[q_0, Z_0, q_1]$$

Construct a pda accepting $\{a^n b^m a^n \mid m, n \geq 1\}$ by null store. Construct the corresponding context-free grammar accepting the same set.

$$R_1: \delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$$

$$R_2: \delta(q_0, a, a) = \{(q_0, aa)\}$$

$$R_3: \delta(q_0, b, a) = \{(q_1, a)\}$$

$$R_4: \delta(q_1, b, a) = \{(q_1, a)\}$$

$$R_5: \delta(q_1, a, a) = \{(q_1, \Lambda)\}$$

$$R_6: \delta(q_1, \Lambda, Z_0) = \{(q_1, \Lambda)\}$$

$$P_1: S \to [q_0, Z_0, q_0], \qquad P_2: S \to [q_0, Z_0, q_1]$$

$\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$ induces

$$P_3: [q_0, Z_0, q_0] \to a[q_0, a, q_0][q_0, Z_0, q_0]$$

$$P_4: [q_0, Z_0, q_0] \to a[q_0, a, q_1][q_1, Z_0, q_0]$$

$$P_5: [q_0, Z_0, q_1] \to a[q_0, a, q_0][q_0, Z_0, q_1]$$

$$P_6: [q_0, Z_0, q_1] \to a[q_0, a, q_1][q_1, Z_0, q_1]$$

$\delta(q_0, a, a) = \{(q_0, aa)\}$ yields

$$P_7: [q_0, a, q_0] \to a[q_0, a, q_0][q_0, a, q_0]$$

$$P_8: [q_0, a, q_0] \to a[q_0, a, q_1][q_1, a, q_0]$$

$$P_9: [q_0, a, q_1] \to a[q_0, a, q_0][q_0, a, q_1]$$

$$P_{10}: [q_0, a, q_1] \to a[q_0, a, q_1][q_1, a, q_1]$$

$\delta(q_0, b, a) = (q_1, a)\}$ gives

$$P_{11}: [q_0, a, q_0] \to b[q_1, a, q_0]$$

$$P_{12}: [q_0, a, q_1] \to b[q_1, a, q_1]$$

$\delta(q_1, b, a) = \{(q_1, a)\}$ yields

$$P_{13}: [q_1, a, q_0] \to b[q_1, a, q_0]$$

$$P_{14}: [q_1, a, q_1] \to b[q_1, a, q_1]$$

$\delta(q_1, a, a) = \{(q_1, \Lambda)\}$ gives

$$P_{15}: [q_1, a, q_1] \to a$$

$\delta(q_1, \Lambda, Z_0) = \{(q_1, \Lambda)\}$ yields

$$P_{16}: [q_1, Z_0, q_1] \to \Lambda$$

Step 1 (Construction of A) Let $L = L(G)$, where $G = (V_N, \Sigma, P, S)$ is a context-free grammar. We construct a pda A as

$$A = (\{q\}, \Sigma, V_N \cup \Sigma, \delta, q, S, \emptyset)$$

where δ is defined by the following rules:

$$R_1\text{: } \delta(q, \Lambda, A) = \{(q, \alpha) \mid A \to \alpha \text{ is in } P\}$$

$$R_2\text{: } \delta(q, a, a) = \{(q, \Lambda)\} \quad \text{for every } a \text{ in } \Sigma$$

Construct a pda A equivalent to the following context-free grammar: $S \to 0BB$, $B \to 0S \mid 1S \mid 0$. Test whether 010^4 is in $N(A)$.

Define pda A as follows:

$$A = (\{q\}, \{0, 1\}, \{S, B, 0, 1\}, \delta, q, S, \emptyset)$$

δ is defined by the following rules:

R_1: $\delta(q, \Lambda, S) = \{(q, 0BB)\}$

R_2: $\delta(q, \Lambda, B) = \{(q, 0S), (q, 0S), (q, 0)\}$

1S

R_3: $\delta(q, 0, 0) = \{(q, \Lambda)\}$

R_4: $\delta(q, 1, 1) = \{(q, \Lambda)\}$

$$
\begin{array}{lll}
& (q,\ 010^4,\ S) & \\
\vdash & (q,\ 010^4,\ 0BB) & \text{by Rule } R_1 \\
\vdash & (q,\ 10^4,\ BB) & \text{by Rule } R_3 \\
\vdash & (q,\ 10^4,\ 1SB) & \text{by Rule } R_2 \text{ since } (q,\ 1S) \in \alpha(q,\ \Lambda,\ B) \\
\vdash & (q,\ 0^4,\ SB) & \text{by Rule } R_4 \\
\vdash & (q,\ 0^4,\ 0BBB) & \text{by Rule } R_1 \\
\vdash & (q,\ 0^3,\ BBB) & \text{by Rule } R_3 \\
\overset{*}{\vdash} & (q,\ 0^3,\ 000) & \text{by Rule } R_2 \text{ since } (q,\ 0) \in \alpha(q,\ \Lambda,\ B) \\
\overset{*}{\vdash} & (q,\ \Lambda,\ \Lambda) & \text{by Rule } R_3
\end{array}
$$

Convert the grammar $S \to aSb \mid A$, $A \to bSa \mid S \mid \Lambda$ to a pda that accepts the same language by empty stack.