Problems on Probability

The odds that person X speaks the truth are 3 :2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an-identical point.

Solution. Let us define the events:

$A : X$ speaks the truth, $\qquad B : Y$ speaks the truth

Then $\overline{A}$ and $\overline{B}$ represent the complementary events that X and Y tell a lie respectively. We are given:

$$P(A) = \frac{3}{3+2} = \frac{3}{5} \quad \Rightarrow \quad P(\overline{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

and $\qquad$ $$P(B) = \frac{5}{5+3} = \frac{5}{8} \quad \Rightarrow \quad P(\overline{B}) = 1 - \frac{5}{8} = \frac{3}{8}$$

The event E that X and Y contradict each other on an identical point can happen in the following mutually exclusive ways:

(i) X speaks the truth and Y tells a lie, i.e., the event $A \cap \overline{B}$ happens,

(ii) X tells a lie and Y speaks the truth, i.e., the event $\overline{A} \cap B$ happens.

Hence by addition theorem of probability the required probability is given by:

$$P(E) = P(i) + P(ii) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

$$= P(A) . P(\overline{B}) + P(\overline{A}) P(B),$$

[Since A and B are independent]

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40} = 0.475$$

Hence A and B are likely to contradict each other on an identical point in 47.5% of the cases.

(b) One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third gun respectively. If $P(E_1) = 0.5$, $P(E_2) = 0.6$ and $P(E_3) = 0.8$ and. E_1, E_2, E_3 are independent events, find the probability that (a) exactly one hit is registered, (b) at least two hits are registered.

(a) Exactly one hit can be registered in the following mutually exclusive ways: (i) $E_1 \cap \overline{E_2} \cap \overline{E_3}$ happens, (ii) $\overline{E_1} \cap E_2 \cap \overline{E_3}$ happens, (iii) $\overline{E_1} \cap \overline{E_2} \cap E_3$ happens. Hence by addition probability theorem, the required probability 'p' is given by :

$$p = P(E_1 \cap \overline{E_2} \cap \overline{E_3}) + P(\overline{E_1} \cap E_2 \cap \overline{E_3}) + P(\overline{E_1} \cap \overline{E_2} \cap E_3)$$

$$= P(E_1)\,P(\overline{E_2})\,P(\overline{E_3}) + P(\overline{E_1})\,P(E_2)\,P(\overline{E_3}) + P(\overline{E_1})\,P(\overline{E_2})\,P(E_3)$$

(Since E_1, E_2 and E_3 are independent)

$$= 0.5 \times 0.4 \times 0.2 + 0.5 \times 0.6 \times 0.2 + 0.5 \times 0.4 \times 0.8 = 0.26.$$

(b) At least two hits can be registered in the following mutually exclusive ways: (i) $E_1 \cap E_2 \cap \overline{E_3}$ happens (ii) $E_1 \cap \overline{E_2} \cap E_3$ happens, (iii) $\overline{E_1} \cap E_2 \cap E_3$ happens. (iv) $E_1 \cap E_2 \cap E_3$ happens.

Required probability

$$= P(E_1 \cap E_2 \cap \overline{E_3}) + P(E_1 \cap \overline{E_2} \cap E_3) + P(\overline{E_1} \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= 0.5 \times 0.6 \times 0.2 + 0.5 \times 0.4 \times 0.8 + 0.5 \times 0.6 \times 0.8 + 0.5 \times 0.6 \times 0.8$$

$$= 0.06 + 0.16 + 0.24 + 0.24 = 0.70$$

If A and B are two independent events such
that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{5}$, then

(a) $P(A \cup B) = 3/5$ (b) $P(A/B) = \frac{1}{2}$

(c) $P(A/A \cup B) = \frac{5}{6}$ (d) $P(A \cap B / A' \cup B') = 0$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{5} - \frac{1}{2} \times \frac{1}{5} = \frac{3}{5}$$

$$P(A/B) = \frac{1}{2}$$

$$P(A \mid A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{\frac{1}{2}}{\frac{3}{5}} = \frac{5}{6}$$

$$P(A \cap B \mid A' \cup B') = P(A \cap B \mid (A \cap B)') = 0$$

For two events A and B, et $P(A) = P(A/B) = \frac{1}{4}$
and $P(B/A) = \frac{1}{2}$, then

(a) A and B are independent

(b) A and B are mutually exclusive

(c) $P(A'/B) = \frac{3}{4}$ (d) $P(B'/A') = \frac{1}{2}$

$P(A) = P(A/B) \Rightarrow P(A) = \dfrac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A)P(B)$

$P(A \cap B) = P(A)P(B/A)$ A and B are independent

$\qquad = \dfrac{1}{4} \times \dfrac{1}{2} = \dfrac{1}{8} \neq 0$

$P(A'/B) = P(A') = 1 - P(A) = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$P(B'/A') = P(B') = 1 - P(B) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$\left[\because P(B) = P(B/A) = \dfrac{1}{2} \right]$

If A and B are two indipendent events such that
$P(A' \cap B) = 2/15$ and $P(A \cap B') = 1/6$, then find
P(B)

$$P(A' \cap B) = \frac{2}{15} \implies P(A') \, P(B) = \frac{2}{15}$$

$$\implies (1-a)b = \frac{2}{15} \quad [P(A) = a, \quad P(B) = b]$$
$$\underline{\qquad\qquad} (1)$$

$$P(A \cap B') = \frac{1}{6} \implies P(A) \, P(B') = \frac{1}{6}$$

$$\implies a(1-b) = \frac{1}{6} \qquad —(11)$$

From (1) and (11)
$$b - ab = \frac{2}{15}$$
$$\underline{-\; a - ab = -\frac{1}{6}}$$

$$\implies b - a = \frac{2}{15} - \frac{1}{6} = \frac{4-5}{30} = -\frac{1}{30}$$

$$\implies a = b + \frac{1}{30}$$

$$\left(b + \frac{1}{30}\right)(1-b) = \frac{1}{6} \implies (30b+1)(1-b) = 5$$

$$\implies 30b + 1 - 30b^2 - b = 5 \implies 30b^2 - 29b + 4 = 0$$

$$\implies 30b^2 - 24b - 5b + 4 = 0 \implies 6b(5b-4) - 1(5b-4) = 0$$

$$\implies (5b-4)(6b-1) = 0 \implies b = \frac{4}{5}, \frac{1}{6}$$

$$P(B) = \frac{1}{6}, \frac{4}{5}$$

A student appears for tests I, II, and III. The student is successful if he passes either in tests I and II or test I and test III. The probabilities of the student passing in tests I, II, III are p, q and $\frac{1}{2}$ respectively. If the probability that the student is successful is $\frac{1}{2}$ then

(a) $p = 1, q = 0$

(b) $p = \frac{2}{3}, q = \frac{1}{2}$

(c) $p = \frac{3}{5}, q = \frac{2}{3}$

(d) There are infinitely many values of p and q

P (the student is successful)

$$= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(A \cap B \cap C)$$

$$= P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(\bar{B})\,P(C) + P(A)\,P(B)\,P(C)$$

$$= p\,q\,(1 - 1/2) + p\,(1-q)\,\tfrac{1}{2} + pq \cdot \tfrac{1}{2}$$

$$= \tfrac{1}{2}\left[\,pq + p(1-q) + pq\,\right] = \tfrac{1}{2}\left[\,p + pq\,\right]$$

$$\Rightarrow \tfrac{1}{2} = \tfrac{1}{2}\,(1+q)\,p \Rightarrow p(1+q) = 1$$

There are 10 pairs of shoes in a cupboard, from which 4 shoes are picked at random. The probability that there is at least one pair is ___

Req prob $= 1 - \dfrac{{}^{10}C_4 \times 2^4}{{}^{20}C_4}$