Moments of Binomial distribution

$$\mu_1' = E(X) = \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$${}^nC_x = \frac{\lfloor n}{\lfloor x \lfloor n-x} = \frac{n \lfloor (n-1)}{x \lfloor x-1 \; \lfloor (n-1)-(x-1)}$$

$$= \frac{n}{x} \, {}^{n-1}C_{x-1}$$

$${}^{n-1}C_{x-1} = \frac{\lfloor n-1}{\lfloor x-1 \; \lfloor (n-1)-(x-1)}$$

$$= \frac{(n-1) \lfloor n-2}{(x-1) \lfloor x-2 \; \lfloor (n-2)-(x-2)}$$

$$= \frac{(n-1)}{(x-1)} \, {}^{n-2}C_{x-2}$$

So $${}^nC_x = \frac{n}{x} \, {}^{n-1}C_{x-1} = \frac{n(n-1)}{x(x-1)} \, {}^{n-2}C_{x-2}.$$

$$\mu_1' = \sum_{x=0}^{n} x \, \frac{n}{x} \, {}^{n-1}C_{x-1} \, p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \left({}^{n-1}C_0 \, p^0 q^{n-1} + {}^{n-1}C_1 \, p^1 q^{n-2} + \dots + {}^{n-1}C_{n-1} \, p^{n-1} q^0 \right)$$

$$= np (q+p)^{n-1} = np$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{n} x^n \, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)+x\} \frac{n(n-1)}{x(x-1)} \, {}^{n-2}C_{x-2} \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n(n-1)}{x(x-1)} \, {}^{n-2}C_{x-2} \, p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2} \, p^{x-2} q^{(n-2)-(x-2)}$$

$$+ np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np = n(n-1)p^2 + np$$

$$\mu_3' = E(x^3) = \sum_{x=0}^{n} x^3 p(x)$$

$$= \sum_{x=0}^{n} \{x(x-1)(x-2) + 3x(x-1) + x\} \, {}^nC_x \, p^x q^{n-x}$$

continue similarly

$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\mu_4' = E(x^4)$$
$$= \sum_{x=0}^{n} x^4 p(x)$$

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

§ Central moments of Binomial distribution.

$$\mu_2 = \mu_2' - (\mu_1')^2 = n(n-1)p^2 + np - n^2p^2$$
$$= np - np^2 = np(1-p) = npq$$

$$\cancel{\mu_3'} = \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$= npq(q-p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$
$$= npq\{1 + 3(n-2)pq\}$$

Hence $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{(1-2p)^2}{npq}$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{1-2p}{npq}, \quad \gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$$

Moments ab the Poisson Distribution

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{\lfloor x-1} = \lambda e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{\lfloor 2}+\cdots\right)$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)+x\} \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x(x-1)\lfloor x-2} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{\lfloor x-2} + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{\infty} x^3 p(x)$$

Follow as in Binomial

$$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(x^4) = \sum_{x=0}^{\infty} x^4 p(x)$$

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 3\lambda^2 + \lambda$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{\lambda}, \quad \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda}, \quad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$$

Moment Generating Function of Binomial distribution

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= {}^nC_0 (pe^t)^0 q^n + {}^nC_1 (pe^t)^1 q^{n-1} + \cdots + {}^nC_n (pe^t)^n q^0$$

$$M_X(t) = (q + pe^t)^n$$

$$\mu_r' = \left.\frac{d^r}{dt^r} M_X(t)\right|_{\text{at } t=0}$$

$$\mu_1' = \left.\frac{d}{dt} M_X(t)\right|_{\text{at } t=0}$$

$$\frac{d}{dt}(q+pe^t)^n = n(q+pe^t)^{n-1} pe^t$$

at $t=0$, $\mu_1' = npe^0(q+pe^0)^{n-1}$

$$= np \cdot 1\,(q+p)^{n-1} = np.$$

$$\mu_2' = \frac{d^2}{dt^2} M_X(t) = \frac{d}{dt}\left\{npe^t (q+pe^t)^{n-1}\right\}$$

$$= np\left\{e^t (n-1)(q+pe^t)^{n-2} pe^t + (q+pe^t)^{n-1} e^t\right\}$$

at $t=0$, $\mu_2' = np\{(n-1)p + 1\} = n(n-1)p^2 + np$

Mean $= np$, Var $= npq$

Moment Generating function of the Poisson Distribution

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{\lfloor x}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^2}{\lfloor 2} + \cdots\right)$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

$$\mu_r' = \left\{\frac{d^r}{dt^r} M_X(t)\right\}_{at\ t=0}$$

$$\mu_1' = \left|\frac{d}{dt} M_X(t)\right|_{at\ t=0}$$

$$\frac{d}{dt} e^{\lambda(e^t - 1)} = e^{\lambda(e^t-1)} e^t \cdot \lambda$$

at $t=0$, $\mu_1' = \lambda e^0 e^{\lambda(e^0-1)}$

$$= \lambda \cdot 1\, e^{\lambda(1-1)} = \lambda \cdot e^0 = \lambda$$

$$\frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} \lambda e^t e^{\lambda(e^t-1)}$$

$$= \lambda\left[e^t e^{\lambda(e^t-1)} \cdot \lambda e^t + e^{\lambda(e^t-1)} \cdot e^t\right]$$

at $t=0$, $\mu_2' = \lambda\left[\lambda \cdot e^0 e^0 e^{\lambda(e^0-1)} + e^0 e^{\lambda(e^0-1)}\right]$

$$= \lambda[\lambda + 1) = \lambda^2 + \lambda$$

Mean $= \lambda$, Var $= \lambda^2 + \lambda - \lambda^2 = \lambda$

§ Mode of Binomial distribution

Mode is the value of x for which $p(x)$ is maximum

$$\frac{p(x)}{p(x-1)} = \frac{{}^nC_x\, p^x q^{n-x}}{{}^nC_{x-1}\, p^{x-1} q^{n-x+1}}$$

$$= \frac{\frac{\lfloor n}{\lfloor x \lfloor n-x}\, p^x q^{n-x}}{\frac{\lfloor n}{\lfloor x-1 \; \lfloor n-x+1}\, p^{x-1} q^{n-x+1}} = \frac{(n-x+1)p}{xq}$$

$$= \frac{xq + (n-x+1)p - xq}{xq} = 1 + \frac{(n+1)p - x(p+q)}{xq} \quad \text{—(1)}$$

We discuss the following two cases.

Case I when $(n+1)p$ is an integer.

Let $(n+1)p = (m+f)$, where m is an integer and f is fractional such that $0 < f < 1$.

(1) becomes,
$$\frac{p(x)}{p(x-1)} = 1 + \frac{(m+f) - x}{xq} \quad \text{—(2)}$$

From (2) it is obvious that

$$\frac{p(x)}{p(x-1)} > 1 \quad \text{for } x = 1, 2, \cdots m$$

and $\frac{p(x)}{p(x-1)} < 1$ for $x = m+1, m+2, \ldots, n$.

$\Rightarrow \frac{p(1)}{p(0)} > 1, \frac{p(2)}{p(1)} > 1, \ldots \frac{p(m)}{p(m-1)} > 1$

and $\frac{p(m+1)}{p(m)} < 1, \frac{p(m+2)}{p(m+1)} < 1, \ldots \frac{p(n)}{p(n-1)} < 1$

$\therefore p(0) < p(1) < p(2) \ldots < p(m-1) < \underline{p(m)} > p(m+1)$

$> p(m+2) > \ldots p(n)$

$\Rightarrow p(x)$ is maximum at $x = m$

Thus in this case there exists unique modal value for binomial distribution and it is m, the integral part of $(n+1)p$

<u>Case II</u> When $(n+1)p$ is an integer.

Let $(n+1)p = m$ (an integer)

From (2) $\frac{p(x)}{p(x-1)} = 1 + \frac{m-x}{xq}$ —— (3)

From (3), it is obvious that

$$\frac{p(x)}{p(x-1)} = \begin{cases} > 1 & \text{for } x = 1, 2, \ldots m-1 \\ = 1 & \text{for } x = m \\ < 1 & \text{for } x = m+1, m+2, \ldots n \end{cases}$$

Now Proceeding as in case (I)

$P(0) < P(1) < P(2) - - - < P(m-1) = P(m) > P(m+1)$

$> P(m+2) > - - > P(n)$

Thus in this case the binomial distribution is bimodal and the two modal values are m and $m-1$.

Example: Determine the binomial distribution for which the mean is 4 and variance 3 and find its mode.

Sol: $np = 4$ —(1) $npq = 3$ —(2)

$q = \frac{3}{4}$ (Divide (2) by (1)), $p = 1 - \frac{3}{4} = \frac{1}{4}$

$n = \frac{4}{p} = \frac{4}{\frac{1}{4}} = 16$

$n = 16, \ p = \frac{1}{4}$

$(n+1)p = (16+1)\frac{1}{4} = \frac{17}{4} = 4.25$, which is a fraction.

Hence it is unimodal for $x = 4$

$P(x=4) = {}^{16}C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{12}$

§ Mode of Poisson Distribution

$$\frac{p(x)}{p(x-1)} = \frac{\dfrac{e^{-\lambda}\lambda^{x}}{\lfloor x}}{\dfrac{e^{-\lambda}\lambda^{x-1}}{\lfloor x-1}} = \frac{\lambda}{x} \quad \text{—— (1)}$$

<u>Case I</u> When λ is not an integer. Let us suppose S is an integral part of λ.

$$\lambda = S + f, \quad 0 < f < 1$$

$$\frac{p(x)}{p(x-1)} = \frac{S+f}{x} = \begin{cases} > 1 & \text{if } x = 1, 2, \ldots, S \\ < 1 & \text{if } x = S+1, S+2, \ldots \end{cases}$$

$$\frac{p(1)}{p(0)} > 1, \quad \frac{p(2)}{p(1)} > 1, \quad \ldots \quad \frac{p(S-1)}{p(S-2)} > 1, \quad \frac{p(S)}{p(S-1)} > 1$$

and $\dfrac{p(S+1)}{p(S)} < 1, \quad \dfrac{p(S+2)}{p(S+1)} < 1, \ldots$

Combining these two expression

$$p(0) < p(1) < p(2) < \ldots p(S-2) < p(S-1) < p(S)$$
$$> p(S+1) > p(S+2) > \ldots,$$

which shows that $p(S)$ is maximum value. Hence in this case, the distribution is <u>unimodal</u> and the integral part of λ is the unique modal value.

Case II : when $\lambda = k$ (say) is an integer.

Here as in case I, we have

$$\frac{P(\cancel{x})}{\cancel{P(}} \quad \frac{P(x)}{P(x-1)} = \frac{k}{x}$$

$$\frac{P(1)}{P(0)} > 1, \quad \frac{P(2)}{P(1)} > 1, \; \cdots \; \frac{P(k-1)}{P(k-2)} > 1$$

$$\frac{P(k)}{P(k-1)} = 1, \quad \frac{P(k+1)}{P(k)} < 1, \quad \frac{P(k+2)}{P(k+1)} < 1, \cdots$$

$$P(0) < P(1) < P(2) \cdots P(k-2) < P(k-1) = P(k) > P(k+1) > P(k+2) \cdots$$

In this case we have two maximum values viz, $P(k-1)$ and $P(k)$ and thus the distribution is bimodal and two modes are at $(k-1)$ and k, ie at $(\lambda-1)$ and λ