# Lecture 29

29 October 2021 11:19

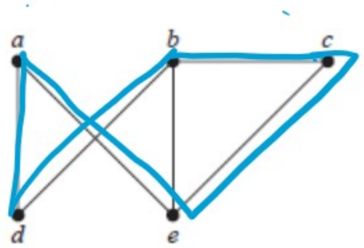
**A path or circuit is simple (Trail)** if it does not contain the same edge more than once.

Q22.

1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

   a) $a, e, b, c, b$ b) $a, e, a, d, b, c, a$

   c) $e, b, a, d, b, e$ d) $c, b, d, a, e, c$



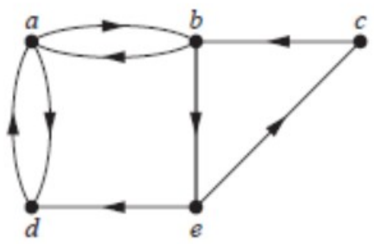

(a) Path, not circuit, length = 4
not simple
(b) Not Path
(c) Not path, b to a
(d) Circuit, Length = 5, Simple.

Q23.

2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

   a) $a, b, e, c, b$ b) $a, d, a, d, a$

   c) $a, d, b, e, a$ d) $a, b, e, c, b, d, a$




(a) Path, Simple, Length = 4
(b) Circuit, length = 4
(c) Not Path, d to b, e to a
(d) Not Path, b to d.

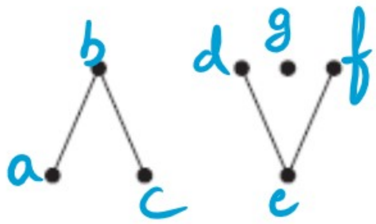
## Connectedness in Undirected graph

An undirected graph is said to be connected it there is a path between every pair of vertices of the graph.

Q24. Determine whether the given graphs are connected?

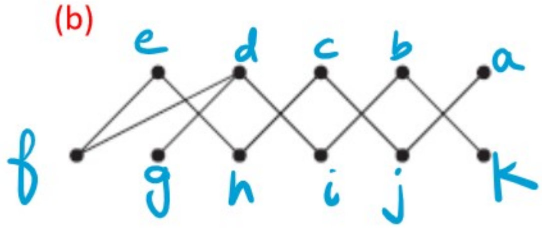
Q24. Determine whether the given graphs are connected?

(a)




Not Connected, b to d

(b)

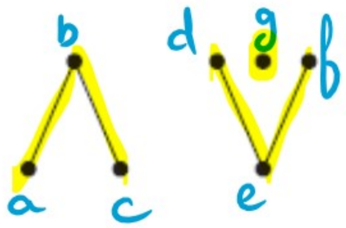



Connected

**Theorem 4: There is a simple path between every pair of vertices of a connected undirected graph.**

**Connected Components:** It is a connected subgraph of $G$ and is not proper subgraph of another connected subgraph of $G$.

A connected component of a graph is maximal connected subgraph of $G$.




3

**Cut Vertices:** The vertex whose removal from connected graph produces a subgraph that is not connected. Removal of cut vertex and all edges incident with it produces a subgraph with more connected components than original graph.

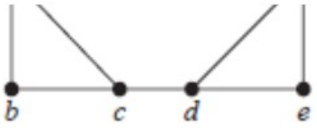
**Cut Edge or Bridge:** The edge whose removal from connected graph produces a subgraph that is not connected.

Q25. Find all the cut vertices and cut edges of the following graphs?

(a)

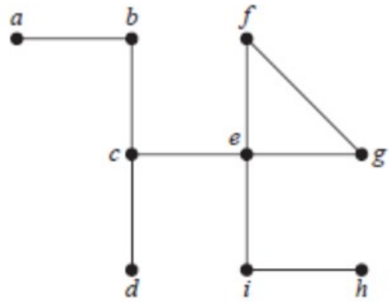

Cut vertices – c, d

Cut edge – {c, d}




Cut edge – {c,d}

(b)




Cut vertices – b, c, e, i

Cut edges – {c,e}, {b,c}, {e,i} {i,h}, {c,d}, {a,b}

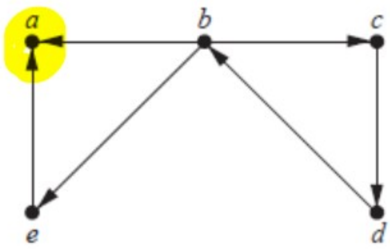
## Connectedness in Directed graph

**Strongly Connected:** A directed graph is strongly connected if there is a path from $a$ to $b$ and from $b$ to $a$.

**Weakly Connected:** A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

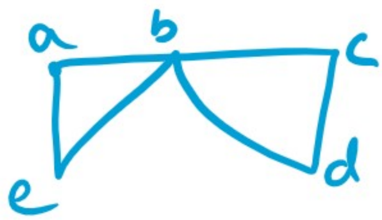
Q26. Determine whether the following graphs are strongly connected or weakly connected?

(i)



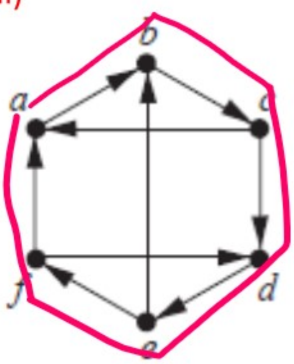

No Strongly connected

No path from a




Weakly Connected.

(ii)



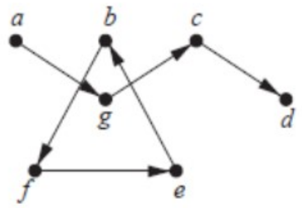

Strongly Connected

(iii)

a b c

Not strongly connected, a to b

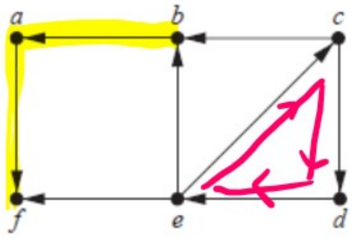
(iii)



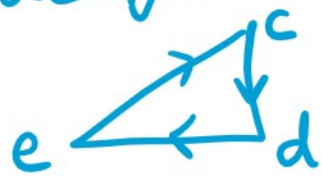

Not strongly connected, a to b
Not weakly connected, a to b

**Strongly Connected Components:** The subgraph of directed graph $G$ that are strongly connected and are not proper subgraph of another strongly connected subgraphs of $G$.

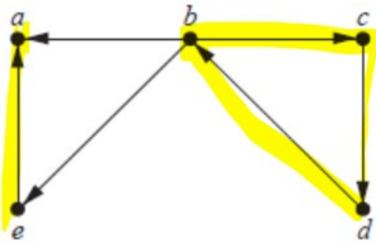
Q27. Find the strongly connected components of the following:




Not strongly connected, no path from f.
weakly connected

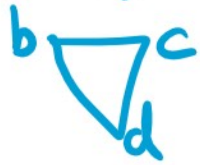


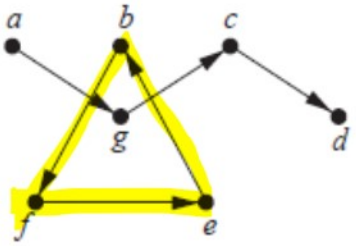

b a f




Not strongly connected, no path from a

3 a e

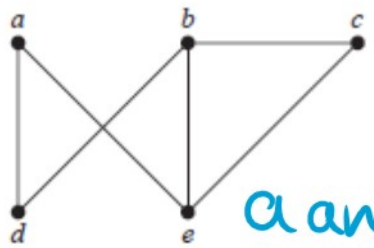



a g c d

5 com

**Counting the paths between vertices**




How many paths b/w a and b of length 2? = adb, aeb

a and b of length 3 = aecb

a and a of length 3 =

a and a of length 3 =

Let $G$ be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length $r$ from $v_i$ to $v_j$, where $r$ is a positive integer, equals the $(i, j)$th entry of $\mathbf{A}^r$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

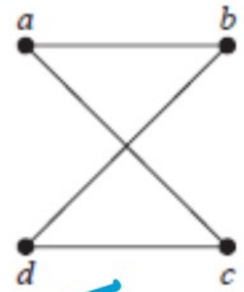
a and d of length 2.

= 0

$$A^2 = \begin{array}{c} \\ a \\ b \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & & & \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 3 \end{bmatrix} \end{array}$$

b and b of length 2 = 3

Q28. Find the no. of paths of length 4 from a to d.




$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

(A) 8 (B) 4 (C) 6 (D) 9

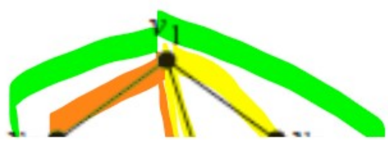
$$A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

**Paths and Isomorphism**

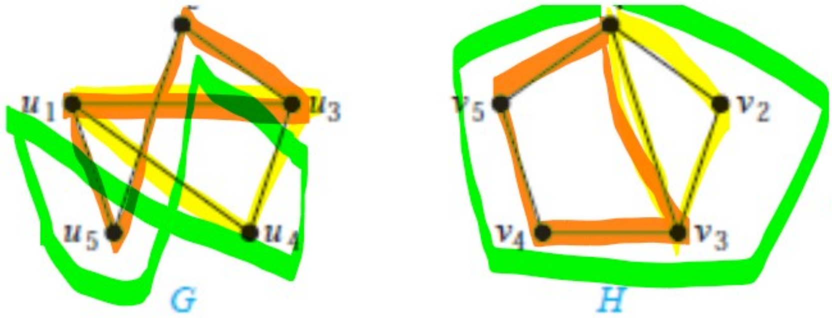
Existence of a simple circuit of particular length is a useful invariant that can be used to show two graphs are isomorphic.

$u_1 u_3 u_4$


$v_1 v_2 v_3$




Simple circuit of length 3 → 1




Simple ci... of length 3 → 1

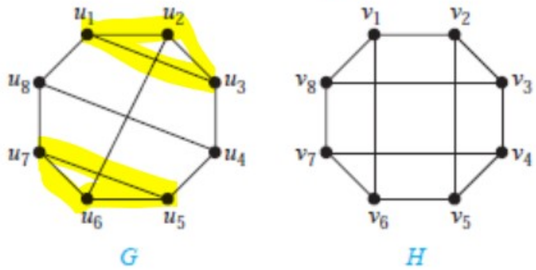
Simple circuit of length 4 → 1

Simple circuit of length 5 → 1

Simple Circuit of length 3 → 1
Simple circuit of length 4 → 1
Simple circuit of length 5 → 1

G and H are isomorphic.

Q29. Check whether the graphs are isomorphic?



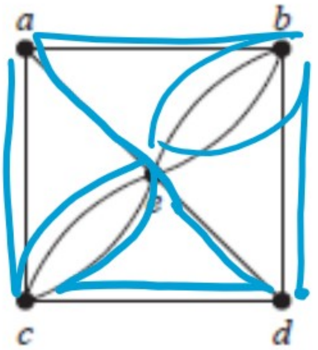

None

not isomorphic

**Euler and Hamilton Paths**

Postman problem: Delivers the post he is able to visit every lawn once.

An *Euler circuit* in a graph $G$ is a simple circuit containing every edge of $G$. An *Euler path* in $G$ is a simple path containing every edge of $G$.




abedcecaebd


**Necessary and Sufficient condition for existence of Euler Circuit and Euler Path**

**Theorem 5:**

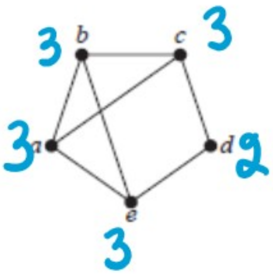
A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

**Theorem 6:**

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

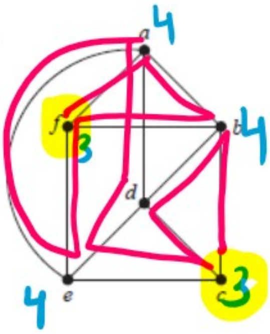
$\downarrow$ end points of path.

Q30. Determine whether the graphs has Euler Circuit, construct such if exists. If no, determine whether graph has Euler path, construct such if exists.



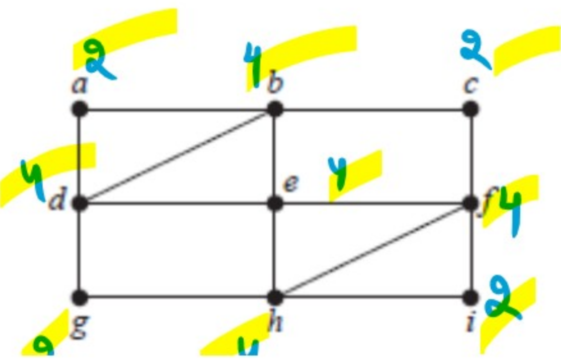

Not all even degrees $\rightarrow$ Euler Circuit No

Four odd degrees – No Euler path.

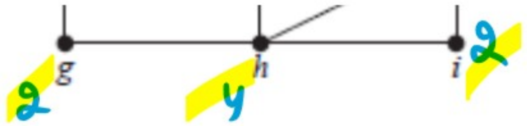


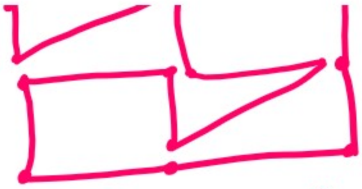

Euler Path

fabfeadecdbc



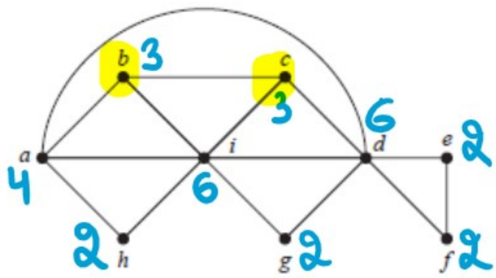

Euler circuit, not Euler Path.



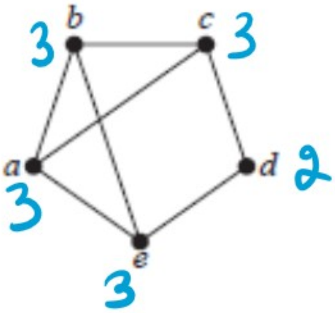



abcfihgdehfebda

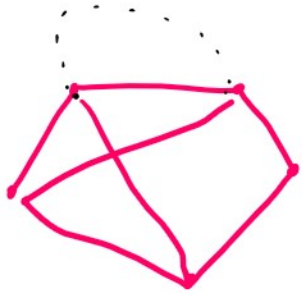
Q31. Find least number of times it is necessary to lift a pencil from the paper when drawing each of the graphs without retracing any part of graph.



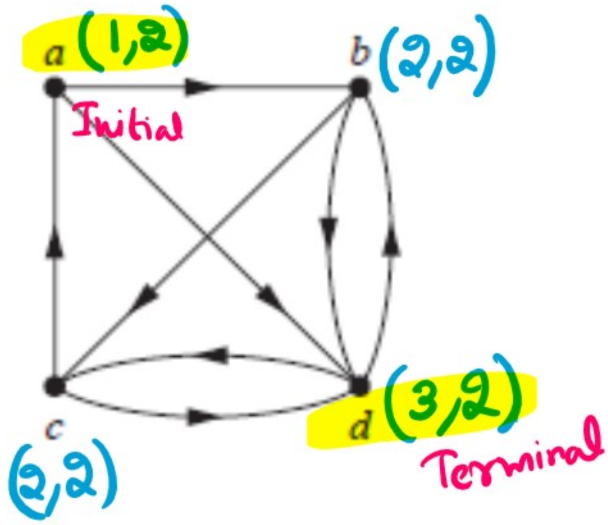

Euler Path
0



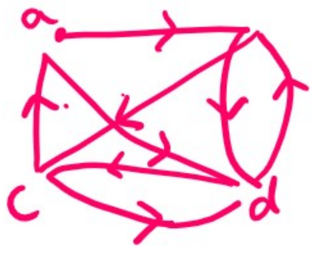

None
1 time

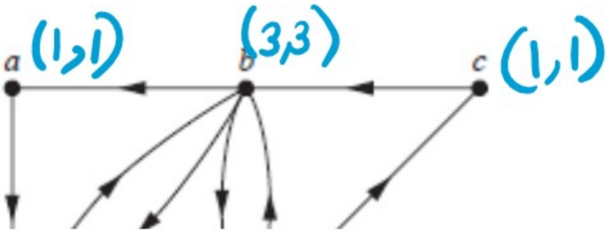


Q32. Determine whether the directed graphs has Euler Circuit, construct such if exists. If no, determine whether graph has Euler path, construct such if exists.

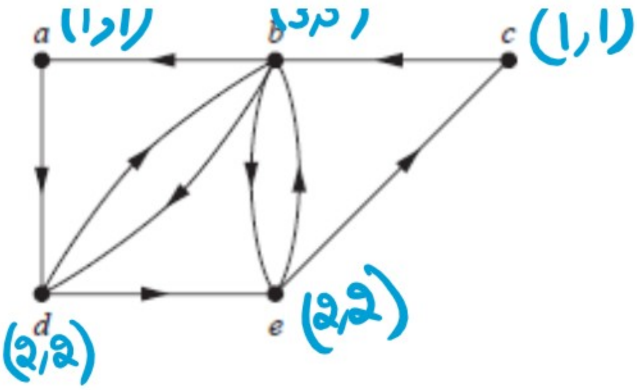



Euler Path
abddcadcd




Euler circuit




Euler circuit

## Euler Path/Circuit in Directed graph

- A directed multigraph having no isolated vertex has Euler circuit iff it is weakly connected and in-degree and out-degree of each vertex are equal.
- A directed multigraph having no isolated vertex has Euler path iff it is weakly connected and in-degree and outdegree of each vertex are equal except for two vertices, where one has in-degree 1 larger than out-degree and other has out-degree 1 larger than in-degree.

Terminal vertex    Initial vertex

## Hamilton Paths and Circuits

A simple path in a graph $G$ that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph $G$ that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

Travelling Salesman

Sufficient Conditions

**DIRAC'S THEOREM** If $G$ is a simple graph with $n$ vertices with $n \geq 3$ such that the degree of every vertex in $G$ is at least $n/2$, then $G$ has a Hamilton circuit.

**ORE'S THEOREM** If $G$ is a simple graph with $n$ vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices $u$ and $v$ in $G$, then $G$ has a Hamilton circuit.

## Important points to remember

**Important points to remember**

- If a graph has pendant vertex, then it doesn't have Hamilton Circuit.
- All edges corresponding to degree 2 vertices will always be considered for constructing Hamilton Circuit.
- No more than two edges will be used corresponding to any vertex in $G$ for constructing Hamilton Circuit.
- Hamilton circuit cannot contain smaller circuit in it.