Lecture 22

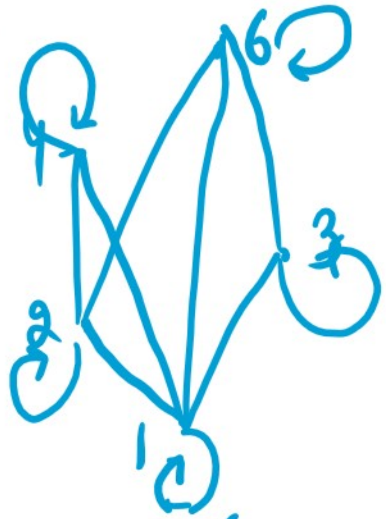
07 October 2021 22:48

Hasse Diagram

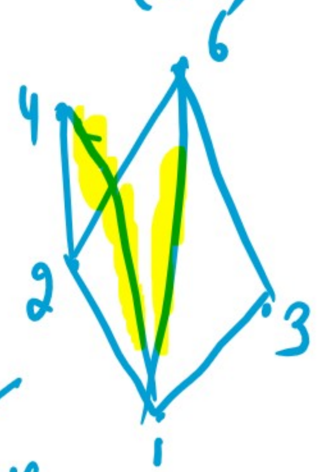
Graphical representation of POSET → Reflexive, Transitive, Anti Symmetric

$(\{1,2,3,4,6\}, |)$

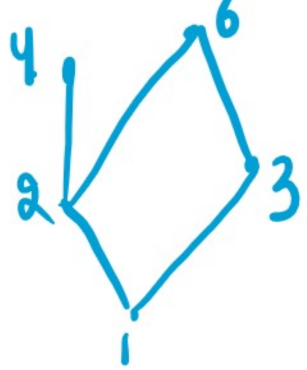
$\{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (6,6), (4,4)\}$



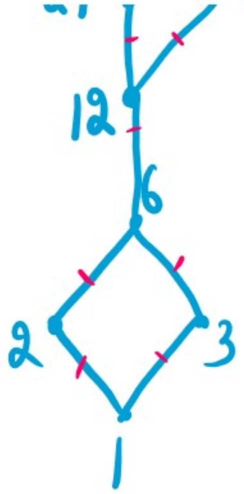
Removed loops →



Remove edges Transitive ↙



Q16. Draw Hasse diagram for divisibility on the set
(a) $\{1, 2, 3, 6, 12, 24, 36, 48\}$



No. of edges = 8

(b) $\{2, 4, 5, 10, 12, 15, 20, 25\}$

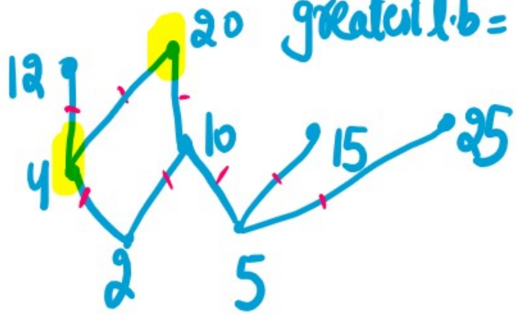
No. of edges in Hasse diagram = 8

Lower bound of $\{4, 20\} = \{2, 4\}$

Upper bound $\{4, 10\} = \{20\}$

Least upper bound $= \{20\}$

greatest l.b $= \{4\}$



Important Definitions

1. **Maximal element:** An element a is called maximal in the poset $(S, \preccurlyeq)$**,** if there is no $b \in S$ such that $a \prec b$.

   No one above it.

2. **Minimal element:** An element a is called minimal in the poset $(S, \preccurlyeq)$**,** if there is no $b \in S$ such that $b \prec a$.

   No one below it.

3. **Greatest element:** a is the greatest element of $(S, \preccurlyeq)$, if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique, if it exists.

   I am above of all.

   More than one maximal elt, No greatest element.

4. **Least element:** a is the least element of $(S, \preccurlyeq)$, if $a \preccurlyeq b$ for all $b \in S$. The least element is unique, if it exists.

   I am below of all

   More than one minimal elt, No least elt.

5. **Upper bound:** If u is an element of S such that $a \preccurlyeq u$ for all the elements $a \in A$, where

5. **Upper bound:** If u is an element of S such that $a \preccurlyeq u$ for all the elements $a \in A$, where A is a subset of S, then u is called upper bound of A.

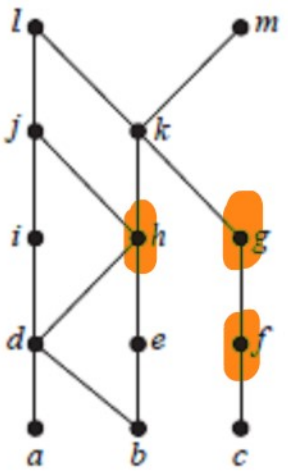
6. **Least upper bound:** The element x, which is upper bound and is less than every other upper bound of A.

7. **Lower bound:** If l is an element of S such that $l \preccurlyeq a$ for all the elements $a \in A$, where A is a subset of S, then l is called lower bound of A.

8. **Greatest lower bound:** The element x, which is lower bound and is greater than every other upper bound of A.

Q17.

Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements. $= \{l, m\}$
b) Find the minimal elements. $= \{a, b, c\}$
c) Is there a greatest element? = No

d) Is there a least element? No
e) Find all upper bounds of $\{a, b, c\}$. $= \{k, l, m\}$
f) Find the least upper bound of $\{a, b, c\}$, if it exists. $= \{k\}$
g) Find all lower bounds of $\{f, g, h\}$. No
h) Find the greatest lower bound of $\{f, g, h\}$, if it exists No

Q18.

Answer these questions for the poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |).

a) Find the maximal elements. {60,48,72,27}

b) Find the minimal elements. {2,9}

c) Is there a greatest element? None

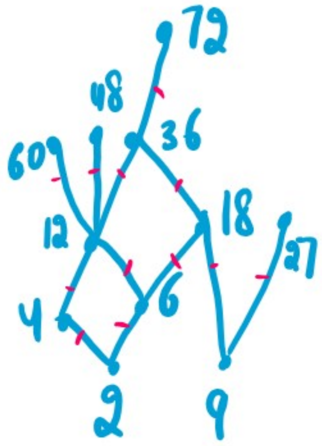
d) Is there a least element? None

e) Find all upper bounds of {2, 9}. {18,36,72}

f) Find the least upper bound of {2, 9}, if it exists. {18}

g) Find all lower bounds of {60, 72}. {2,4,6,12}

h) Find the greatest lower bound of {60, 72}, if it exists. {12}



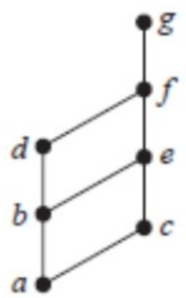
No of edges in Hasse diagram = 12

Lattice

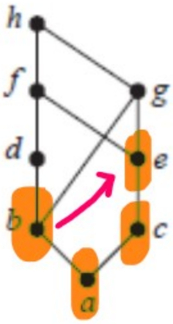
A poset in which every pair of elements has both a least upper bound and a greatest lower bound.

. Determine whether the posets with these Hasse diagrams are lattices.

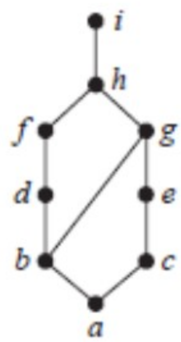
a)



b)



c)



Check it

Lattice ✓

l.u.b {b,g} = h

g.l.b {b,g} = No

Not lattice

Unit III: Counting
Principle of Inclusion-Exclusion
Text Book: Chapter 8

$|A \cup B| = |A| + |B| - |A \cap B|$

Q20.

There are 345 students at a college who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken courses in both calculus and discrete mathematics. How many students have taken a course in either calculus or discrete mathematics?

$|C| = 345, \quad |D| = 212, \quad |C \cap D| = 188$

$|C \cup D| = 345 + 212 - 188 = 369$

Calculus only $= 345 - 188 = 157$

Discrete only $= 212 - 188 = 24$

One subject only $= 157 + 24 = 181 = 369 - 188$

Q21.

Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.

$|S| = 31 \quad , \quad |C| = 10 \quad , \quad |S \cap C| = 3$

$1^3, 2^3, 3^3, \ldots, 10^3$ $\quad$ $1, 64, 729$

$31 + 10 - 3 = 38$

(b) $31+10-3=38$

THE PRINCIPLE OF INCLUSION–EXCLUSION Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

$$|A \cup B \cup C| = |A|+|B|+|C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Q22.

Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if

a) the sets are pairwise disjoint.

b) there are 50 common elements in each pair of sets and no elements in all three sets.

$|A_1|=100$, $|A_2|=100$, $|A_3|=100$

(a) Disjoint $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_1| = |A_1 \cap A_2 \cap A_3| = 0$

$|A_1 \cup A_2 \cup A_3| = 300$

(b) $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_1| = 50$, $|A_1 \cap A_2 \cap A_3| = 0$

$|A_1 \cup A_2 \cup A_3| = 100+100+100-50-50-50+0 = 150$

$|A_1 \cup A_2 \cup A_3| = 100+100+100-50-50-50+0 = 150$

Q23.

In a survey of 270 college students, it is found that 64 like brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussels sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?

$|S| = 64,\ |B| = 94,\ |C| = 58,\ |S \cap B| = 26,\ |S \cap C| = 28$

$|C \cap B| = 22,\ |S \cap B \cap C| = 14$

$|S \cup B \cup C| = 64+94+58-26-28-22+14 = 154$

Don't like any $= \bar{S} \cap \bar{B} \cap \bar{C} = \overline{S \cup B \cup C} = 270-154 = 116$

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Text Book: Chapter 6

Ex. Among any group of 13 people, there must be at least two whose birthday falls in same month.

k = box

$N = 13,\ k = 12 \qquad \left\lceil \frac{13}{12} \right\rceil = \lceil 1. \rceil = 2$

$N=13,\ k=12$

$\left\lceil \frac{13}{12} \right\rceil = \lceil 1. \rceil = 2$

ceiling f^n

$\lceil 3.6 \rceil = 4$

Q24. If there are 71 students in section K20BE, then there are at least ___3___ students whose has their first name starts with same letter.

$k=26$

$\left\lceil \frac{71}{26} \right\rceil = \lceil 2. \rceil = 3$

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Q25. Among a group of 150 integers (not necessarily consecutive), how many minimum integers have same remainder when divided by 4.

$N=150,$

$k=4$

$k=$ remainder when divided by 4

0, 1, 2, 3

$\left\lceil \frac{150}{4} \right\rceil = \lceil 37.5 \rceil = 38$

Q26.

A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color? → Representing box

b) How many balls must she select to be sure of having at least three blue balls?

at least three blue balls?

For part (a) No of boxes = No. of colours = 2.

Output is already given that woman got atleast 3 balls of same colour, means that ceiling f^n answer is given. To find is how many balls she draw i.e. No of objects is to be find.

Imp point: Minimum no. of pigeons required, when it is given that there is at least one box out of k which contains at least m pigeons ………………….

no. of objects $N = \underbrace{(m-1)k}_{\text{assures each box has } (m-1) \text{ pigeons}} + 1$

This says one box contain m pigeons

So

(a) $m = 3$ (atleast 3)

$k = 2$ (2 colours)

$N = (3-1)2+1$

$N = 5$

(b) What is difference b/w (a) and (b)

(b) specifically ask for blue colour only.

only.

Think of worst case. We were getting red balls always until all 10 are drawn, after that next draw will give blue only.

So minimum balls to be drawn to get atleast 3 blue $= 10+3 = 13$

* In part (b), it matters how many balls are given initially while for part (a), it doesn't matter initially how many red and blue are there.

Q27.

What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

k = states = 50

output given $m = 100$

No of students $= (100-1)50+1 = 4951$