Chapter 1

Functions of a Real Variable

1.1 Introduction

In a first course in Mathematics, you have studied limits, continuity, differentiability and integration of functions of one variable $y = f(x)$. We will now discuss the application of derivatives and integration to solve various engineering problems.

1.2 Application of Derivatives

We now discuss some applications of derivatives like finding approximate value of a function, mean value theorems, increasing and decreasing functions, maximum and minimum values of a function.

1.2.1 Differentials and Approximations

Let $y = f(x)$ be a real valued differentiable function and x_0 be a point in its domain. Let $x_0 + \Delta x$ be a point in the neighborhood of x_0. Then, Δx may be considered as an increment in x. The corresponding increment in $f(x)$ is given by

$$\Delta f_0 = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

From the definition of derivative, we have

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f_0}{\Delta x}. \quad (1.1)$$

Since $f'(x_0)$ exists, we can write from Eq. (1.1) that

$$\frac{\Delta f_0}{\Delta x} = f'(x_0) + \alpha \text{ or } \Delta f_0 = f'(x_0)\,\Delta x + \alpha\,\Delta x \quad (1.2)$$

where α is an infinitesimal quantity dependent on Δx and tends to zero as $\Delta x \to 0$. Thus, the increment Δf_0 consists of the following two parts.

(i) Principal part $f'(x_0)\,\Delta x$, which is called the *differential* of f.

(ii) Residual part $\alpha\,\Delta x$ which tends to zero as $\Delta x \to 0$.

In the limit, the differential is also written as

$$df(x_0) = dy_0 = f'(x_0)\, dx. \tag{1.3}$$

Hence, an approximation to $f(x_0 + \Delta x)$ can be written as

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\, dx. \tag{1.4}$$

Differentials have application in calculating errors in functions due to small errors in the independent variable. We define $|dy|$ as the *absolute error*; dy/y as the *relative error* and $(dy/y) \times 100$ as the *percentage error in* computations.

Example 1.1 Find an approximate value of

$$y = 3(4.02)^2 - 2(4.02)^{3/2} + 8/\sqrt{4.02}\,.$$

Solution Let a function be defined as

$$y = f(x) = 3x^2 - 2x^{3/2} + 8/\sqrt{x}\,.$$

Let $x_0 = 4$ and $\Delta x = 0.02$. Then, we need an approximation to $f(x_0 + \Delta x) = f(4.02)$. The approximate value is given by (see Eq. 1,4)).

$$f(4.02) \approx f(4) + (0.02)\, f'(4)$$

We have
$$f(4) = 48 - 2(8) + 8/2 = 36,$$
$$f'(x) = 6x - 3x^{1/2} - 4x^{-3/2} \text{ and } f'(4) = 24 - 6 - 4/8 = 35/2.$$

Therefore, the required approximation is

$$f(4.02) \approx 36 + 0.02\,(35/2) = 36.35.$$

Example 1.2 If there is a possible error of 0.02 cm in the measurement of the diameter of a sphere, then find the possible percentage error in its volume, when the radius is 10 cm.

Solution Let the radius of the sphere be r cm. Volume of the sphere $= V = 4\pi r^3/3$ and $dr = \pm\, 0.01$ when $r = 10$ cm.

Differentiating V, we obtain $dV = 4\pi r^2 dr$.

When $r = 10$, we get from Eq. (1.3), $dV = 4\pi(10)^2\,(\pm\, 0.01) = \pm\, 4\pi$.

Hence, the percentage error in volume is

$$\left(\frac{dV}{V}\right) \times 100 = 100\left[\frac{\pm 12\pi}{4\pi(10)^3}\right] = \pm\, 0.3 \text{ cubic cm.}$$

1.2.2 Mean Value Theorems

We now prove the three basic mean value theorems of the functions of one variable.

Theorem 1.1 (Rolle's theorem) Let a real valued function $f(x)$ be continuous on a closed interval $[a, b]$ and differentiable in the open interval (a, b). If $f(a) = f(b)$, then there exists at least one value c, $a < c < b$ such that $f'(c) = 0$.

Proof Since the function $f(x)$ is continuous on the closed interval $[a, b]$, it is bounded and attains its maximum value M and minimum value m at some points in $[a, b]$. Let $f(x)$ attain respectively its minimum and maximum values at the points c and $d \in [a, b]$, that is

$$f(c) = m \quad \text{and} \quad f(d) = M.$$

If $m = M$, then the function $f(x)$ is constant over $[a, b]$ and therefore, its derivative $f'(x)$ is zero for all x in $[a, b]$.

If $m \neq M$, then both of these cannot be equal to the same quantity $f(a)$ or $f(b)$. We note that $f(a) = f(b)$. Thus, atleast one of these, say m, is different from $f(a)$ and $f(b)$. Hence,

$$f(c) = m \neq f(a), \text{ implies } c \neq a,$$

$$f(c) = m \neq f(b), \text{ implies } c \neq b.$$

Therefore, $c \in (a, b)$. We shall now show that at this point c, $f'(c) = 0$.

If $f'(c) < 0$, then for every x in the interval $(c, c + \varepsilon_1)$, $\varepsilon_1 > 0$,

$$f(x) < f(c) = m$$

which contradicts the assumption that m is the minimum value of $f(x)$.

If $f'(c) > 0$, then for every x in the interval $(c - \varepsilon_2, c)$, $\varepsilon_2 > 0$,

$$f(x) < f(c) = m$$

which is again a contradiction. Hence, $f'(c) = 0$.

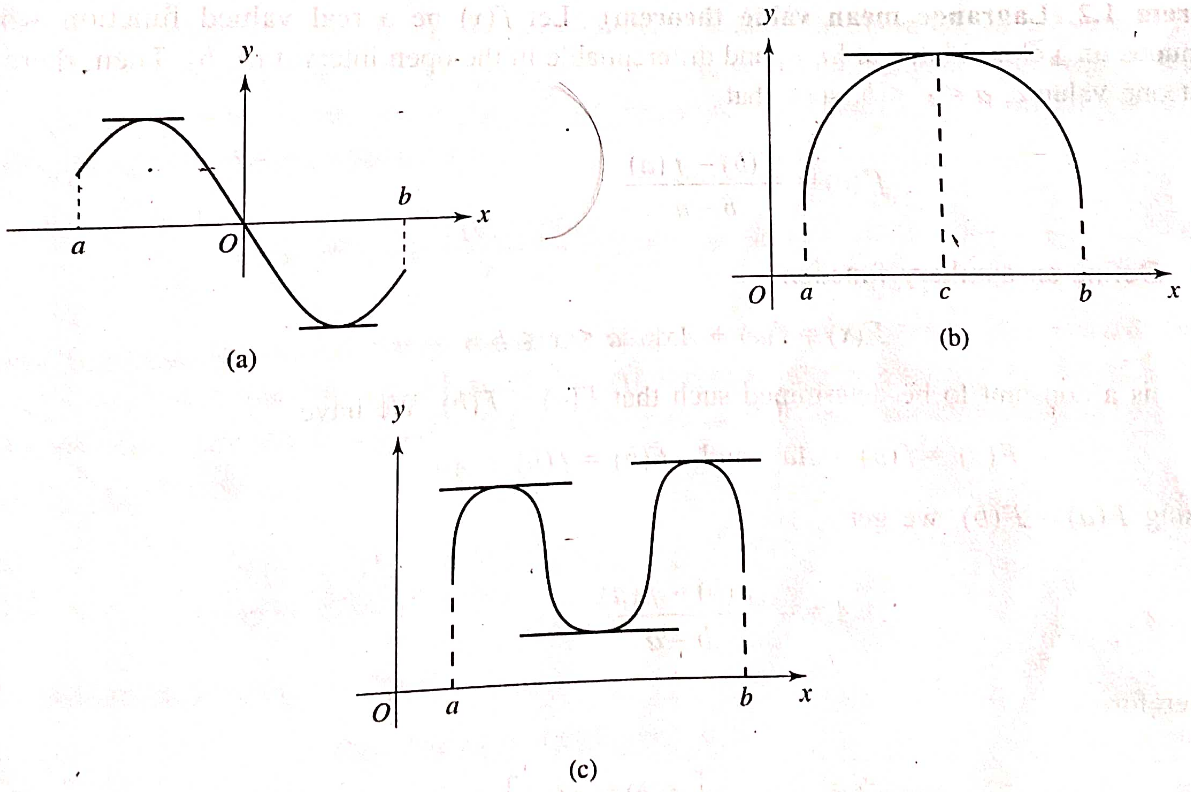

Fig. 1.1. Rolle's theorem.

Remark 1

(a) Differentiability of $f(x)$ in an open interval (a, b) is a necessary condition for the applicability of the Rolle's theorem.

For example, consider the function $f(x) = |x|$, $-1 \le x \le 1$. Now, $f(x)$ is continuous on $[-1, 1]$ and is differentiable at all points in the interval $(-1, 1)$ except at the point $x = 0$. Now,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

does not vanish at any point in the interval $(-1, 1)$. This shows that the Rolle's theorem cannot be applied as the function $f(x)$ is not differentiable in $(-1, 1)$.

(b) Rolle's theorem gives sufficient conditions for the existence of a value c such that $f'(c) = 0$. For example, the function

$$f(x) = \begin{cases} 0, & 1 \le x \le 2 \\ 2, & 2 < x \le 3 \end{cases}$$

is not continuous on $[1, 3]$, but $f'(c) = 0$ for all c in $[1, 3]$.

(c) Geometrically, the theorem states that if a function satisfies the conditions of Rolle's theorem and has the same value at the end points of an interval $[a, b]$, then there exists at least one point $(c, f(c))$, $a < c < b$ where the tangent to the curve $y = f(x)$, $a \le x \le b$ is parallel to the x-axis.

Theorem 1.2 (Lagrange mean value theorem) Let $f(x)$ be a real valued function which is continuous on a closed interval $[a, b]$ and differentiable in the open interval (a, b). Then, there exists atleast one value c, $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1.5}$$

Proof Define an auxiliary function

$$F(x) = f(x) + Ax, \quad a \le x \le b$$

where A is a constant to be determined such that $F(a) = F(b)$. We have

$$F(a) = f(a) + Aa \quad \text{and} \quad F(b) = f(b) + Ab.$$

Using $F(a) = F(b)$, we get

$$A = -\frac{f(b) - f(a)}{b - a}.$$

Therefore,

$$F(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right] x.$$

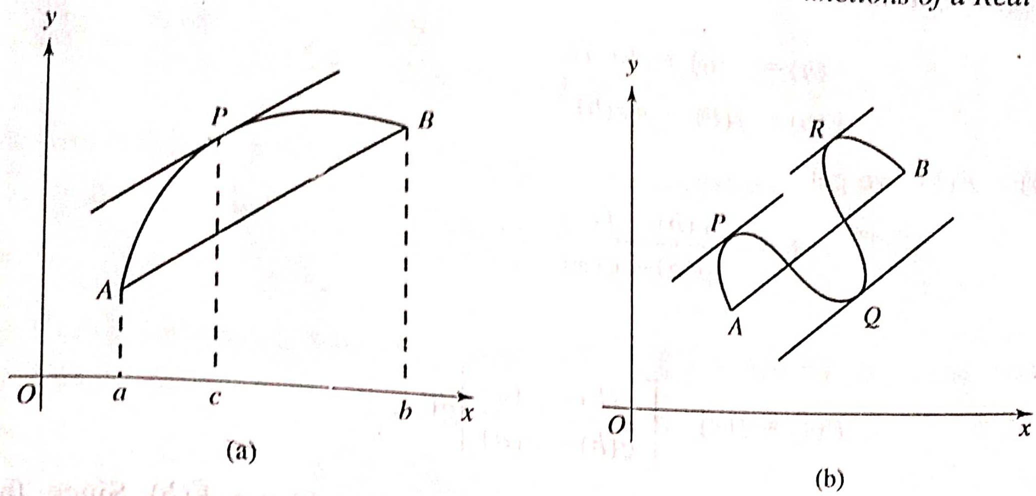

Fig. 1.2 Lagrange mean value theorem.

Now, $F(x)$ is continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b) and $F(a) = F(b)$. Since, the function $F(x)$ satisfies all conditions of the Rolle's theorem, there exists a point $(c, F(c))$, $a < c < b$ such that

$$F'(c) = 0, \quad \text{or} \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark 2

(a) If $f(a) = f(b)$, then Lagrange mean value theorem reduces to the Rolle's theorem.

(b) Geometrically, Lagrange mean value theorem states that there exists a point $(c, f(c))$, $a < c < b$ on the curve C: $y = f(x)$, $a \le x \le b$, such that the tangent to the curve C at this point is parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$ on the curve.

(c) Using Eq. (1.5), we can write

$$\min_{a \le x \le b} f'(x) \le \frac{f(b) - f(a)}{b - a} \le \max_{a \le x \le b} f'(x). \tag{1.6}$$

Theorem 1.3 (Cauchy mean value theorem) Let $f(x)$ and $g(x)$ be two real valued functions defined on a closed interval $[a, b]$ such that (i) they are continuous on $[a, b]$, (ii) they are differentiable in (a, b) and (iii) $g'(x) \neq 0$ for every x in (a, b). Then, there exists at least one value c, $a < c < b$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b. \tag{1.7}$$

Proof Define an auxiliary function

$$F(x) = f(x) + Ag(x), \quad a \le x \le b$$

where A is a constant to be determined such that $F(a) = F(b)$. We have

$$F(a) = f(a) + Ag(a),$$
$$F(b) = f(b) + Ag(b).$$

Using $F(a) = F(b)$, we get

$$A = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Therefore,

$$F(x) = f(x) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(x).$$

Now, $F(x)$ is continuous on $[a, b]$, differentiable in (a, b) and $F(a) = F(b)$. Since, the function $F(x)$ satisfies all conditions of the Rolle's theorem, there exists a point $(c, F(c))$, $a < c < b$, such that

$$F'(c) = 0,$$

or
$$f'(c) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g'(c) = 0,$$

or
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b.$$

Remark 3

(a) For $g(x) = x$, Cauchy mean value theorem reduces to Lagrange mean value theorem.

(b) Let a curve C be represented parametrically as $x = f(t)$, $y = g(t)$, $a \le t \le b$. Then, Cauchy mean value theorem states that there exists a point $(f(c), g(c))$, $c \in (a, b)$ on the curve such that the slope $g'(c)/f'(c)$ of the tangent to the curve at this point is equal to the slope of the chord joining the end points of the curve. Hence, Cauchy mean value theorem has the same geometrical interpretation as the Lagrange mean value theorem.

(c) Cauchy mean value theorem cannot be proved by applying the Lagrange mean value theorem separately to the numerator and denominator on the left side of Eq. (1.7). If we apply the Lagrange mean value theorem to the numerator and the denominator separately, we obtain

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)}, \quad a < c_1 < b, \quad a < c_2 < b, \quad c_1 \neq c_2.$$

Example 1.3 A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is atleast one value ξ, $a < \xi < b$ for which $f''(\xi) < 0$.

Solution Consider the function $f(x)$ defined on $[a, b]$. Since $f''(x)$ exists, both f and f' exist and are continuous on $[a, b]$. Let $a < c < b$. Applying the Lagrange mean value theorem to $f(x)$ on $[a, c]$ and $[c, b]$ separately, we get

$$\frac{f(c)-f(a)}{c-a} = f'(\xi_1), \quad a < \xi_1 < c, \quad \text{and} \quad \frac{f(b)-f(c)}{b-c} = f'(\xi_2), \quad c < \xi_2 < b.$$

Using $f(a) = f(b) = 0$, we obtain from the above equations

$$f'(\xi_1) = \frac{f(c)}{c-a} \quad \text{and} \quad f'(\xi_2) = -\frac{f(c)}{b-c}.$$

Now, $f'(x)$ is continuous and differentiable on $[\xi_1, \xi_2]$. Using the Lagrange mean value theorem again, we obtain

$$\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1} = f''(\xi), \quad \xi_1 < \xi < \xi_2.$$

Substituting the values of $f'(\xi_1)$ and $f'(\xi_2)$, we get

$$f''(\xi) = -\frac{f(c)}{\xi_2-\xi_1}\left[\frac{1}{b-c}+\frac{1}{c-a}\right] = -\frac{(b-a)\,f(c)}{(b-c)\,(c-a)\,(\xi_2-\xi_1)} < 0.$$

Example 1.4 Using the Lagrange mean value theorem, show that

$$|\cos b - \cos a| \le |b - a|.$$

Solution Let $f(x) = \cos x$, $a \le x \le b$. Using the Lagrange mean value theorem to $f(x)$, we obtain

$$\frac{\cos b - \cos a}{b-a} = f'(c) = -\sin c, \quad \text{or} \quad \left|\frac{\cos b - \cos a}{b-a}\right| = |-\sin c| \le 1.$$

Hence, the result.

Example 1.5 Let $f'(x) = 1/(3 - x^2)$ and $f(0) = 1$. Find an interval in which $f(1)$ lies.

Solution Using Eq. (1.6), we obtain for $a = 0$ and $b = 1$

$$\min_{0\le x\le 1} f'(x) \le \frac{f(1)-f(0)}{1-0} \le \max_{0\le x\le 1} f'(x)$$

or

$$\min_{0\le x\le 1}\left[\frac{1}{3-x^2}\right] \le f(1) - 1 \le \max_{0\le x\le 1}\left[\frac{1}{3-x^2}\right]$$

or

$$\frac{1}{3} \le f(1) - 1 \le \frac{1}{2}, \quad \text{or} \quad \frac{4}{3} \le f(1) \le \frac{3}{2}.$$

Example 1.6 Let C be a curve defined parametrically as $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \le \theta \le \pi/2$. Determine a point P on C, where the tangent to C is parallel to the chord joining the points $(a, 0)$ and $(0, a)$.

Solution We have $x = f(\theta) = a\cos^3\theta$ and $y = g(\theta) = a\sin^3\theta$. Using the Cauchy mean value theorem, we have for some θ, $0 \le \theta \le \pi/2$, slope of the tangent to C = slope of the chord joining the points $(a, 0)$ and $(0, a)$

or
$$\frac{g'(\theta)}{f'(\theta)} = \frac{3a\sin^2\theta\cos\theta}{-3a\sin^2\theta\cos\theta} = \frac{g(\pi/2)-g(0)}{f(\pi/2)-g(0)} = \frac{a-0}{0-a}$$

or
$$-\tan\theta = -1, \quad \text{or} \quad \theta = \pi/4.$$

Therefore, the required point is $(a/2\sqrt{2}, a/2\sqrt{2})$.

1.2.3 Indeterminate Forms

Consider the ratio $f(x)/g(x)$ of two functions $f(x)$ and $g(x)$. If at any point $x = a$, $f(a) = g(a) = 0$, then the ratio $f(x)/g(x)$ takes the form 0/0 and it is called an *indeterminate form*. The problem is to determine $\lim_{x\to a}[f(x)/g(x)]$, if it exists. Since $f(a) = g(a) = 0$, we can write

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right hand side exists. This result is known as *L' Hospital's rule.*

L' Hospital's rule Suppose that the real valued functions f and g are differentiable in some open interval containing the point $x = a$ and $f(a) = 0 = g(a)$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}, \quad g'(a) \ne 0. \tag{1.8}$$

Suppose now that $f'(a) = 0 = g'(a)$. Then, we repeat the application of L' Hospital's rule on $f'(x)/g'(x)$ and obtain

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}$$

provided the limits exist. This application of the rule can be continued as long as the indeterminate form is obtained.

When both $f(a) = \pm\infty$ and $g(a) = \pm\infty$, we get another indeterminate form. In this case also, L' Hospital's rule can be applied. We write

$$\lim_{x\to a}\frac{f(a)}{g(a)} = \lim_{x\to a}\frac{[1/g(x)]}{[1/f(x)]}$$

which is of 0/0 form.

Remark 4

(a) L' Hospital rule can be used only when the ratio is of indeterminate form, that is, either it is of form 0/0 or ∞/∞.

(b) The other indeterminate forms are $0 \cdot \infty$, 0^0, ∞^0, 1^∞ and $\infty - \infty$. In each of these cases, we can reduce the ratio function to the form 0/0 or ∞/∞ and use this rule. For the indeterminate forms 0^0, ∞^0 and 1^∞, we take logarithm of the given function and then take the limits.

(c) When the function is of the form 0^∞, $\infty \cdot \infty$, $\infty + \infty$, ∞^∞ or $\infty^{-\infty}$, it is not of indeterminate form and we cannot apply L' Hospital's rule. We note that $0^\infty = 0$, $\infty \cdot \infty = \infty$, $\infty + \infty = \infty$, $\infty^\infty = \infty$ and $\infty^{-\infty} = 0$.

(d) L' Hospital's rule can also be applied to find the limits as $x \to \pm\infty$.

Example 1.7 Evaluate the following limits

(i) $\lim\limits_{x\to 0}\left[\dfrac{\ln(1+x)}{\sin x}\right]$, (ii) $\lim\limits_{x\to 0}[x^n(\ln x)]$, (iii) $\lim\limits_{x\to\infty}\left[\dfrac{e^x}{x}\right]$.

Solution Using L' Hospital's rule, we get

(i) $$\lim_{x\to 0}\left[\frac{\ln(1+x)}{\sin x}\right] = \lim_{x\to 0}\frac{1/(1+x)}{\cos x} = 1.$$

(ii) $$\lim_{x\to 0}[x^n(\ln x)] = \lim_{x\to 0}\frac{[\ln x]}{[1/x^n]} = \lim_{x\to 0}\frac{[1/x]}{[-n/x^{n+1}]} = \lim_{x\to 0}\frac{-x^n}{n} = 0.$$

(iii) $$\lim_{x\to\infty}\left[\frac{e^x}{x}\right] = \lim_{x\to\infty}\left[\frac{e^x}{1}\right] = \infty.$$

Example 1.8 Evaluate $\lim\limits_{x\to 0} x^x$.

Solution The given limit is of the form 0^0 which is an indeterminate form. Let $y = x^x$. Then, $\ln y = x \ln x$. Now,

$$\lim_{x\to 0}[\ln y] = \lim_{x\to 0}[x \ln x] = \lim_{x\to 0}\left[\frac{\ln x}{1/x}\right] = \lim_{x\to 0}\frac{[1/x]}{[-1/x^2]} = -\lim_{x\to 0} x = 0.$$

Therefore, $\lim\limits_{x\to 0} y = e^0 = 1$.

Example 1.9 Evaluate $\lim\limits_{x\to\infty} x\tan(1/x)$.

Solution As $x \to \infty$, the function takes the form $\infty \cdot 0$. We first write it as $\lim\limits_{x\to\infty}\dfrac{x}{\cot(1/x)}$ which is of the form ∞/∞. Applying the L'Hospital's rule, we obtain

$$\lim_{x\to\infty} x \tan(1/x) = \lim_{x\to\infty} \frac{x}{\cot(1/x)} = \lim_{x\to\infty} \frac{1}{(1/x^2)\,\mathrm{cosec}^2(1/x)}$$

$$= \lim_{x\to\infty} \frac{\sin^2(1/x)}{(1/x)^2} = \lim_{y\to 0} \frac{\sin^2 y}{y^2} = \lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2 = 1.$$

1.2.4 Increasing and Decreasing Functions

Let $y = f(x)$ be a function defined on an interval I contained in the domain of the function $f(x)$. Let x_1, x_2 be any two points in I, where x_1, x_2 are not the end points of the interval. On the interval I, the function $f(x)$ is said to be

(i) an *increasing* function, if $f(x_1) \le f(x_2)$ whenever $x_1 \le x_2$.

(ii) a *strictly* increasing function, if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(iii) a *decreasing* function, if $f(x_1) \ge f(x_2)$ whenever $x_1 < x_2$.

(iv) a *strictly* decreasing function, if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

A function which is either increasing or decreasing in the entire interval I is called a *monotonic* function.

Let a real valued function f defined on an interval I, have a derivative at every point x in I. Then, using the Lagrange mean value theorem, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad x_1 < c < x_2.$$

Therefore, we conclude that

(i) f increases in I if $f'(x) > 0$ for all x in I.

(ii) f decreases in I if $f'(x) < 0$ for all x in I.

Thus, a differentiable function increases when its graph has positive slopes and decreases when its graph has negative slopes. Now, if $f'(x)$ is continuous, then $f'(x)$ can go from positive to negative values or from negative to positive values only by going through the value 0. The values of x for which $f'(x) = 0$ are called the *turning points* or the *critical points*. At a turning point, the tangent to the curve is parallel to the x-axis. On the left and right of a turning point, tangents to the curve have different directions.

Example 1.10 Find the intervals in which the function $f(x) = \sin 3x$, $0 \le x \le \pi/2$ is increasing or decreasing.

Solution We have $f'(x) = 3\cos 3x$. Now, $f'(x) = 0$ when $3x = \pi/2, 3\pi/2, \ldots$ for positive x. Hence $x = \pi/6$ is the only turning point in $(0, \pi/2)$. We consider the intervals $(0, \pi/6)$ and $(\pi/6, \pi/2)$. We have in

$0 < x < \pi/6$: $f'(x) = 3 \cos 3x > 0$, $f(x)$ is an increasing function,

$\pi/6 < x < \pi/2$: $f'(x) = 3 \cos 3x < 0$, $f(x)$ is a decreasing function.

Example 1.11 Show that for all $x > 0$

$$1 - x < e^{-x} < 1 - x + \frac{x^2}{2}.$$

Solution Let $f(x) = e^{-x} + x - 1$. Now,

$$f'(x) = 1 - e^{-x} > 0 \text{ for all } x > 0.$$

Hence, $f(x)$ is an increasing function for all $x > 0$. Therefore,

$$f(x) > f(0) = 0, \quad \text{or} \quad e^{-x} + x - 1 > 0 \quad \text{or} \quad e^{-x} > 1 - x.$$

Now, consider $g(x) = e^{-x} - 1 + x - \dfrac{x^2}{2}$.

We have
$$g'(x) = 1 - x - e^{-x} < 0 \text{ for all } x > 0.$$

Hence, $g(x)$ is a decreasing function for all $x > 0$. Therefore,

$$g(x) < g(0) = 0, \quad \text{or} \quad e^{-x} < 1 - x + \frac{x^2}{2}.$$

Combining the above two results, we obtain

$$1 - x < e^{-x} < 1 - x + \frac{x^2}{2}, \quad x > 0.$$

1.2.5 Maximum and Minimum Values of a Function

Let a real valued function $f(x)$ be continuous on a closed interval $[a, b]$. Since a continuous function in a closed interval is bounded and attains these bounds at least once in the interval, we wish to determine the points where $f(x)$ attains these bounds. Let x_0 be a point in (a, b) and $I = (x_0 - h, x_0 + h)$ be an infinitesimal interval around x_0. Then, the function $f(x)$ is said to have a

local maximum (or a *relative maximum*) at the point x_0, if $f(x_0) \geq f(x)$, for all x in I.

local minimum (or a *relative minimum*) at the point x_0, if $f(x_0) \leq f(x)$ for all x in I.

The points of local maximum and local minimum are called the *critical points* or the *stationary points*. The values of the function at these points are called the *extreme values*.

The following theorem gives the necessary condition for the existence of a local maximum or a local minimum.

Theorem 1.4 **(First derivative test)** Let $f(x)$ be differentiable at $x_0 \in (a, b)$. Then, a necessary condition for the function $f(x)$ to have a local maximum or a local minimum at x_0 is that $f'(x_0) = 0$.

At a critical point, $f'(x)$ changes direction. Thus, to find the local maximum/minimum values of the function in an interval I, we find the critical points in I by solving $f'(x) = 0$. By studying the sign of $f'(x)$ as it passes through the critical point, we decide whether it is a point of a local maximum ($f'(x)$ changes sign from positive to negative) or a point of local minimum ($f'(x)$ changes sign from negative to positive).

Example 1.12 Examine the function

(i) $f(x) = x^3 - 3x + 3,\ x \in R$, (ii) $f(x) = \sin^2 x,\ 0 < x < \pi$

for maximum and minimum values.

Solution We have

(i) $f'(x) = 3x^2 - 3$. Now, $f'(x) = 0$ gives $x = 1, -1$.
For $x < 1$, $f'(x) < 0$ and for $x > 1$, $f'(x) > 0$. Since $f'(x)$ changes sign from negative to positive as it passes through the critical point $x = 1$, the function has a local minimum value $f(1) = 1$ at $x = 1$. For $x < -1$, $f'(x) > 0$ and for $x > -1$, $f'(x) < 0$. Since $f'(x)$ changes sign from positive to negative as it passes through the critical point $x = -1$, the function has a local maximum value $f(-1) = 5$ at $x = -1$.

(ii) $f'(x) = 2 \sin x \cos x = \sin 2x = 0$ at $x = \pi/2$.
For $x < \pi/2$, $f'(x) > 0$ and for $x > \pi/2$, $f'(x) > 0$. Since $f'(x)$ changes sign from positive to negative as it passes through the critical point $x = \pi/2$, the function has a local maximum value $f(\pi/2) = 1$ at $x = \pi/2$.

Theorem 1.5 **(Second derivative test)** Let $f(x)$ be differentiable at x_0, $a \le x_0 \le b$ and let $f'(x_0) = 0$. If $f''(x)$ exists and is continuous in a neighborhood of x_0, then

$$f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0)}{h} = \lim_{h\to 0} \frac{f'(x_0-h) - f'(x_0)}{-h},\quad h > 0.$$

Therefore,

(i) $f(x)$ has a local maximum value at $x = x_0$, when $f''(x_0) < 0$,

(ii) $f(x)$ has a local minimum value at $x = x_0$, when $f''(x_0) > 0$.

When $f''(x_0) = 0$, further investigation is needed to decide whether $x = x_0$ is a point of local maximum or local minimum. In this case, we have the following result.

Theorem 1.6 Let $f^{(n)}(x)$ exist for x in (a, b) and be continuous there. Let

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \text{ and } f^{(n)}(x_0) \ne 0.$$

Then,

(i) when n is even, $f(x)$ has a maximum if $f^{(n)}(x_0) < 0$ and a minimum if $f^{(n)}(x_0) < 0$

(ii) when n is odd, $f(x)$ has neither a maximum, nor a minimum.

Absolute maximum/minimum values of a function $f(x)$ in an interval $[a, b]$ are defined as follows:

Absolute maximum value = max $\{f(a), f(b)$, all local maximum values$\}$.

Absolute minimum value = min $\{f(a), f(b)$, all local minimum values$\}$.

Example 1.13 Find the absolute maximum/minimum values of the function

$$f(x) = \sin x(1 + \cos x), \quad 0 \le x \le 2\pi.$$

Solution We have

$$f(x) = \sin x(1 + \cos x) = \sin x + \frac{1}{2}\sin 2x, \quad f'(x) = \cos x + \cos 2x.$$

Setting $f'(x) = 0$, we get

$$\cos x + \cos 2x = 0, \quad \text{or} \quad \cos x + 2\cos^2 x - 1 = 0, \quad \text{or} \quad \cos x = -1, 1/2$$

Therefore, the critical points are $x = \pi/3$, π and $5\pi/3$.

Now,
$$f''(x) = -\sin x - 2\sin 2x.$$

At $x = \pi/3$, $f''(\pi/3) = -3\sqrt{3}/2 < 0$. Hence, $f(x)$ has a local maximum at $x = \pi/3$ and the local maximum value is $f(\pi/3) = 3\sqrt{3}/4$.

At $x = \pi$, $f''(\pi) = 0$. We find that

$$f'''(x) = -\cos x - 4\cos 2x \text{ and } f'''(\pi) = -3 \ne 0.$$

Since, $f^{(n)}(\pi) \ne 0$ and $n = 3$ is odd, the function has neither a maximum nor a minimum at $x = \pi$.

At $x = 5\pi/3$, $f''(5\pi/3) = 3\sqrt{3}/2 > 0$. Hence, $f(x)$ has a local minimum at $x = 5\pi/3$. The local minimum value is $f(5\pi/3) = -3\sqrt{3}/4$.

We also have $f(0) = f(2\pi) = 0$. Therefore,

$$\text{absolute maximum value of } f(x) = \max\{f(0), f(2\pi), \text{local maximum value at } x = \pi/3\}$$
$$= \max\{0, 0, 3\sqrt{3}/4\} = 3\sqrt{3}/4.$$

$$\text{absolute minimum value of } f(x) = \min\{f(0), f(2\pi), \text{local minimum value at } x = 5\pi/3\}$$
$$= \min\{0, 0, -3\sqrt{3}/4\} = -3\sqrt{3}/4.$$

Example 1.14 Find a right angled triangle of maximum area with hypotenuse h.

Solution Let x be the base of the right angled triangle. The area of the right angled triangle is

$$A(x) = \frac{1}{2}x\sqrt{h^2 - x^2}, \quad 0 < x < h.$$

Now,
$$A'(x) = \frac{1}{2}\left[\sqrt{h^2 - x^2} - \frac{x^2}{\sqrt{h^2 - x^2}}\right] = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}.$$

Setting $A'(x) = 0$, we obtain the critical point as $x = h/\sqrt{2}$.

Now, $A'(x) > 0$ for $x < h/\sqrt{2}$ and $A'(x) < 0$ for $x > h/\sqrt{2}$.

Therefore, $A(x)$ is maximum when $x = h/\sqrt{2}$ and the maximum area is $A(h/\sqrt{2}) = h^2/4$.

Leibniz formula Let f and g be two differentiable functions. Then, the nth order derivative of the product fg is given by the Leibniz formula as

$$(f \cdot g)^{(n)} = {}^nC_0 f^{(n)}(x)\, g(x) + {}^nC_1 f^{(n-1)}(x)\, g'(x) + \ldots + {}^nC_r f^{(n-r)}(x)\, g^{(r)}(x) + \ldots + {}^nC_n f(x)\, g^{(n)}(x) \quad (1.9)$$

This formula can be proved by induction.

Example 1.15 Find the fourth order derivative of $e^{ax} \sin bx$ at the point $x = 0$.

Solution Let $f(x) = e^{ax}$, $g(x) = \sin bx$ and $F(x) = f(x)\, g(x)$. Using the Leibniz formula, we obtain

$$F^{(4)}(x) = \frac{d^4}{dx^4}(e^{ax} \sin bx) = {}^4C_0 (e^{ax})^{(4)} \sin bx + {}^4C_1 (e^{ax})^{(3)} (\sin bx)'$$

$$+ {}^4C_2 (e^{ax})'' (\sin bx)'' + {}^4C_3 (e^{ax})' (\sin bx)^{(3)} + {}^4C_4\, e^{ax} (\sin bx)^{(4)}$$

$$= e^{ax}[a^4 \sin bx + 4a^3 b \cos bx - 6a^2 b^2 \sin bx - 4ab^3 \cos bx + b^4 \sin bx]$$

Hence, $\quad F^{(4)}(0) = 4a^3 b - 4ab^3 = 4ab(a^2 - b^2).$

1.2.6 Taylor's Theorem and Taylor's Series

A very useful technique in the analysis of real valued functions is the approximation of continuous functions by polynomials. Taylor's theorem (Taylor's formula) is an important tool which provides such an approximation by polynomials. Taylor's theorem can be regarded as an extension of the mean value theorems to higher order derivatives. Mean value theorems relate the value of the function and its first order derivative, whereas the Taylor's theorem relates the value of the function and its higher order derivatives.

Theorem 1.7 **(Taylor's theorem with remainder)** Let $f(x)$ be defined and have continuous derivatives upto $(n + 1)$th order in some interval I, containing a point a. Then, Taylor's expansion of the function $f(x)$ about the point $x = a$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x) \quad (1.10)$$

where,
$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c), \quad a < c < x \quad (1.11)$$

is the *remainder* or the error term of the expansion.

Proof We first find a polynomial $P_n(x)$, of degree n, which satisfies the conditions

$$P_n(a) = f(a), \quad P_n^{(k)}(a), = f^{(k)}(a), \quad k = 1, 2, \ldots, n.$$

In a certain sense, $P_n(x)$ is a polynomial approximation to $f(x)$. Write the required polynomial as

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n.$$

Using the given conditions, we obtain

$$P_n(a) = f(a) = c_0,\ P_n'(a) = f'(a) = c_1,\ P_n''(a) = f''(a) = 2\,c_2, \ldots,$$
$$P_n^{(n)}(a) = f^{(n)}(a) = (n!)\,c_n.$$

Hence, we have $$c_k = \frac{1}{k!} f^{(k)}(a), \quad k = 0, 1, 2, \ldots, n.$$

Therefore, $$f(x) \approx P_n(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a).$$

The error of approximation is given by $R_n(x) = f(x) - P_n(x)$. Therefore,

$$f(x) = P_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n(x).$$

Now, we derive a form of $R_n(x)$. Write $R_n(x)$ as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} h(x)$$

where $h(x)$ is to be determined.

Consider the auxiliary function

$$F(t) = f(x) - \left[f(t) + (x-t) f'(t) + \ldots + \frac{(x-t)^n}{n!} f^{(n)}(t) + \frac{(x-t)^{n+1}}{(n+1)!} h(x) \right], \quad a < t < x.$$

We have t as a variable and x is fixed. The function $F(t)$ has the following properties:

(i) $F(t)$ is continuous in $a \le t \le x$ and differentiable in $a < t < x$,

(ii) $F(x) = 0$,

(iii) $$F(a) = f(x) - \left[f(a) + (x-a) f'(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} h(x) \right]$$

$$= f(x) - f(x) = 0.$$

Hence, $F(t)$ satisfies the hypothesis of the Rolle's theorem on $[a, x]$. Therefore, there exists a point c, $a < c < x$ such that $F'(c) = 0$. Now,

$$F'(t) = 0 - \left[f'(t) - f'(t) + (x-t) f''(t) - \frac{2(x-t)}{2!} f''(t) + \cdots \right.$$

$$\left. + \frac{(x-t)^n}{n!} f^{(n+1)}(t) - \frac{(n+1)(x-t)^n}{(n+1)!} h(x) \right] = \frac{(x-t)^n}{n!} [h(x) - f^{(n+1)}(t)]$$

and $$F'(c) = 0 = \frac{(x-c)^n}{n!}\,[h(x) - f^{(n+1)}(c)].$$

We obtain $h(x) = f^{(n+1)}(c)$. Therefore,

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}\, f^{(n+1)}(c), \quad a < c < x.$$

The error term can also be written as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}\, f^{(n+1)}\,(a + \theta(x-a)), \quad 0 < \theta < 1 \tag{1.12}$$

which is called the *Lagrange form of the remainder.*

If $a = 0$, we get

$$f(x) = f(0) = \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c), \quad 0 < c < x \tag{1.13}$$

which is called the *Maclaurin's theorem* with remainder.

Writing $x = a + h$ in Eq. (1.10), we obtain

$$f(a+h) \approx f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a). \tag{1.14}$$

The error of approximation simplifies as

$$R_n(x) = \frac{h^{n+1}}{(n+1)!}\, f^{(n+1)}(c), \quad a < c < a + h. \tag{1.15}$$

If we neglect the error term in Eq. (1.10), we obtain

$$f(x) \approx P_n(x) = \sum_{m=0}^{n} \frac{(x-a)^m}{m!}\, f^{(m)}(a) \tag{1.16}$$

which is called the nth degree Taylor's polynomial approximation to $f(x)$.

Since c or θ in the remainder term (see Eqs.(1.11), (1.12)) is not known, we cannot evaluate $R_n(x)$ exactly for a given x in the interval I. However, a bound on the error can be obtained as

$$|R_n(x)| = \left|\frac{(x-a)^{n+1}}{(n+1)!}\, f^{(n+1)}(c)\right| \le \max_{x\in I} \frac{|x-a|^{n+1}}{(n+1)!}\left[\max_{x\in I} |f^{(n+1)}(x)|\right]. \tag{1.17}$$

For a given error bound ε, we can use Eq. (1.17) to determine

(i) n for a given x and a,

(ii) $x = x^*$ for a given n and a such that $|R_n(x^*)| < \epsilon$

Cauchy form of remainder

Consider a function $\phi(x)$ defined on $[a, a + h]$ as

$$\phi(x) = f(x) + (a + h - x)\, f'(x) + \ldots + \frac{(a + h - x)^n}{n!}\, f^{(n)}(x) + A(a + h - x)$$

where A is a constant to be determined such that $\phi(a + h) = \phi(a)$.

The function $\phi(x)$ satisfies all conditions of the Rolle's theorem. Therefore,

$$\phi'(a + \theta h) = 0, \quad 0 < \theta < 1.$$

Now,
$$\phi'(x) = \frac{1}{n!}\,(a + h - x)^n\, f^{(n+1)}(x) - A$$

and
$$\phi'(a + \theta h) = \frac{h^n}{n!}\,(1 - \theta)^n\, f^{(n+1)}(a + \theta h) - A = 0.$$

Hence,
$$A = \frac{h^n}{n!}\,(1 - \theta)^n\, f^{(n+1)}\,(a + \theta h).$$

From $\phi(a + h) = \phi(a)$, we get

$$f(a + h) = f(a) + h\, f'(a) + \ldots + \frac{h^n}{n!}\, f^{(n)}(a) + hA.$$

Therefore, the remainder in the Taylor's theorem is

$$R_n(x) = hA = \frac{h^{n+1}}{n!}\,(1 - \theta)^n\, f^{(n+1)}(a + \theta h), \quad 0 < \theta < 1. \tag{1.18}$$

Integral form of remainder

Consider the result

$$\int_{x_n}^{x_{n+1}} f'(x)\, dx = f(x_{n+1}) - f(x_n),$$

where $x_{n+1} = x_n + h$. Write the transformation $x = x_{n+1} - t$. Then, we have

$$\int_{x_n}^{x_{n+1}} f'(x)\, dx = -\int_h^0 f'(x_{n+1} - t)\, dt = \int_0^h f'(x_{n+1} - t)\, dt.$$

$$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \dots, \quad -1 < x \le 1.$$

Exercise 1.1

Find the approximate values of the following quantities using differentials.

1. $(1005)^{1/3}$.
2. $(999)^{1/3}$
3. $(1.001)^3 + 2(1.001)^{4/3} + 5$.
4. $\sin 60^\circ\ 10'$.
5. $\tan 45^\circ\ 5'\ 30''$.
6. State why Rolle's theorem cannot be applied to the following functions.
   (i) $f(x) = \tan x$ in the interval $[0, \pi]$, (ii) $f(x) = \lfloor x \rfloor$ in the interval $[-1/2, 3/2]$,
   (iii) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2. \end{cases}$
7. It is given that the Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point $x = 4/3$. Find the values of b and c.
8. The functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable in (a, b) such that $f(a) = 4$, $f(b) = 10$, $g(a) = 1$ and $g(b) = 3$. Then, show that $f'(c) = 3g'(c)$, $a < c < b$.
9. Prove that between any two real roots of $e^x \sin x = 1$, there exists atleast one root of $e^x \cos x + 1 = 0$.

10. Let $f'(x)$, $g'(x)$ be continuous and differentiable functions on $[a, b]$. Then, show that for $a < c < b$

$$\frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}, \quad g''(c) \neq 0.$$

11. Let $f(x)$ be continuous on $[a-1, a+1]$ and differentiable in $(a-1, a+1)$. Show that there exists a θ, $0 < \theta < 1$ such that $f(a-1) - 2f(a) + f(a+1) = f'(a+\theta) - f'(a-\theta)$.

12. Using the Lagrange mean value theorem, show that

(i) $1 + x < e^x < 1 + xe^x$; (ii) $\ln(1+x) < x, \quad x > 0$;

(iii) $x < \sin^{-1} x < x/\sqrt{1-x^2}, \quad 0 < x < 1$; (iv) $\dfrac{\pi}{6} + \dfrac{1}{5\sqrt{3}} < \sin^{-1} x < \dfrac{\pi}{6} + \dfrac{1}{8}$.

13. Suppose that $f(x)$ is differentiable for all values of x such that $f(a) = a$, $f(-a) = -a$ and $|f'(x)| \le 1$ for all x. Show that $f(0) = 0$.

14. Let $F(x)$ and $G(x)$ be two functions defined on $[a, b]$ satisfying the hypothesis of the mean value theorem with $G(x) \neq 0$ for any x in $[a, b]$. Show that there exists a point c in (a, b) such that

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)}\left[\frac{G^2(c)}{G(a)G(b)}\right].$$

Evaluate the limits in problems **15** to **28**.

15. $\lim\limits_{x\to 1} \dfrac{x-1}{x^n - 1}$

16. $\lim\limits_{x\to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$

17. $\lim\limits_{x\to \pi/2} \dfrac{\ln(\sin x)}{(\pi - 2x)^2}$

18. $\lim\limits_{x\to 0} \dfrac{\sin^2 x - x^2}{x^2 \sin^2 x}$

19. $\lim\limits_{x\to 1} (1-x)\tan(\pi x/2)$.

20. $\lim\limits_{x\to 2} \left[\dfrac{x-1}{x-2} - \dfrac{1}{\ln(x-1)}\right]$

21. $\lim\limits_{x\to 1} x^{1/(x-1)}$.

22. $\lim\limits_{x\to \pi/2} (\sin x)^{\tan x}$.

23. $\lim\limits_{x\to 0} \dfrac{e^{f(x)} - 1}{f(x)}$, $f(0) = 0$.

24. $\lim\limits_{x\to \infty} \dfrac{x^2}{e^x}$.

25. $\lim\limits_{x\to \infty} [1 + f(x)]^{1/f(x)}$, $\lim\limits_{x\to\infty} f(x) = 0$.

26. $\lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x}\right)^x$.

27. $\lim\limits_{x\to \infty} \sqrt{\dfrac{x + \sin x}{x - \cos^2 x}}$.

28. $\lim\limits_{x\to \infty} \left(\dfrac{x+4}{x+2}\right)^{x+3}$.

In problems **29** to **36**, find the intervals in which $f(x)$ is increasing or decreasing.

29. $\ln(2+x) - 2x/(2+x), \quad x \in \mathbb{R}$

30. $x|x|, \quad x \in \mathbb{R}$.

31. $\tan^{-1} x + x, \quad x \in \mathbb{R}$.

32. $\sin x + |\sin x|, \quad 0 < x \le 2\pi$.

33. $\ln(\sin x), \quad 0 < x < \pi$

34. $(\ln x)/x, \quad x > 0$.

35. $\sin x(1 + \cos x), \quad 0 < x < \pi/2$.

36. $x^x, \quad x > 0$.

37. Let $a > b > 0$ and n be a positive integer satisfying $n \ge 2$. Prove that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$.

In problems **38** to **43**, find the extreme values of the given function $f(x)$.

38. $(x-1)^2(x+1)^3$.

39. $\sin x + \cos x$.

40. $x^{1/x}$.

41. $(\sin x)^{\sin x}$

42. $2\sin x + \cos 2x, \quad 0 \le x \le 2\pi$.

43. $\sin^2 x \sin 2x + \cos^2 x \cos 2x, \quad 0 < x < \pi$.

44. Show that the function $f(x) = (ax+b)/(cx+d)$ has no extreme value regardless of the values of a, b, c, d.

45. Let $f(x) = \begin{cases} -x^3 + [(b^3 - b^2 + b - 1)/(b^2 + 3b + 2)], & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3. \end{cases}$

Find all possible real values of b such that $f(x)$ has minimum value at $x = 1$.

46. If $y = x^3 e^{2x}$, then find $d^n y/dx^n$ at $x = 0$.

47. Find the nth order derivative of $f(x) = \sqrt{ax+b}$.

48. Find the nth order derivative of $f(x) = e^{ax}\sin(bx+c)$.

49. If $y = \cos^{-1} x, \ -(\pi/2) \le x \le (\pi/2)$, then find $d^n y/dx^n$ at $x = 0$.

50. If $y = e^{a\sin^{-1}x}$, then find $d^n y/dx^n$ at $x = 0$.

In problems **51** to **55**, obtain the Taylor's polynomial approximation of degree n to the function $f(x)$ about the point $x = a$. Estimate the error in the given interval.

51. $f(x) = \sqrt{x}$, $n = 3$, $a = 1$, $1 \le x \le 1.5$.

52. $f(x) = e^{-x^2}$, $n = 3$, $a = 0$, $-1 \le x \le 1$.

53. $f(x) = x\sin x$, $n = 4$, $a = 0$, $-1 \le x \le 1$.

54. $f(x) = x^2 e^{-x}$, $n = 4$, $a = 1$, $0.5 \le x \le 1.5$.

55. $f(x) = 1/(1-x)$, $n = 3$, $a = 0$, $0 \le x \le 0.25$.

In problems **56** to **59**, obtain the Taylor's polynomial approximation of degree n to the function $f(x)$ about the point $x = a$. Find the error term and show that it tends to zero as $n \to \infty$. Hence, write its Taylor's series.

56. $f(x) = \sin 3x, \quad a = 0$.

57. $f(x) = \sin^2 x, \quad a = 0$.

58. $f(x) = x^2 \ln x, \quad a = 1$.

59. $f(x) = 2^x, \quad a = 1$.

60. Show that the number θ which occurs in the Taylor's formula with Lagrange form of remainder (given in Eq. 1.12) after n terms approaches the limit $1/(n+1)$ as $h \to 0$ provided $f^{(n+1)}(x)$ is continuous and not zero at $x = a$.

61. Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x = 0$ for the function $\cosh x$ in the interval $[0, 1]$ such that $|\text{Error}| < 0.001$.

62. Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x = 0$ for the function $\sin x \cos x$ in the interval $[0, 1]$ such that $|\text{Error}| < 0.0005$.

63. The function $\ln(1 - x^2)$ is approximated about $x = 0$ by an nth degree Taylor's polynomial. Find n such that $|\text{Error}| < 0.1$ on $0 \le x \le 0.5$.

64. The function $\sin^2 x$ is approximated by the first two non-zero terms in the Taylor's polynomial expansion about the point $x = 0$. Find c such that $|\text{Error}| < 0.005$, when $0 < x < c$.

65. The function $\tan^{-1} x$ is approximated by the first two non-zero terms in the Taylor's polynomial expansion about the point $x = 0$. Find c such that $|\text{Error}| < 0.005$, when $0 < x < c$.

66. Obtain the Maclaurin's series expansion of $y(x) = e^{m\sin^{-1}x}$, where m is a constant.

67. Using Taylor's series, find the approximate value of (i) $\sqrt{15}$, (ii) $\sin 29°$.

Obtain the Taylor's series expansions as given in problems **68** to **75**.

68. $a^x = 1 + x\ln a + \dfrac{(x\ln a)^2}{2!} + \dfrac{(x\ln a)^3}{3!} + \ldots, \quad -\infty < x < \infty$.

69. $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dots, \quad -1 \le x < 1.$

70. $\ln\left(\dfrac{1+x}{1-x}\right) = 2\left[x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right], \quad -1 < x < 1.$

71. $\ln x = 2\left[\left(\dfrac{x-1}{x+1}\right) + \dfrac{1}{3}\left(\dfrac{x-1}{x+1}\right)^3 + \dfrac{1}{5}\left(\dfrac{x-1}{x+1}\right)^5 + \cdots\right], \quad x > 0.$

72. $\ln x = \left(\dfrac{x-1}{x}\right) + \dfrac{1}{2}\left(\dfrac{x-1}{x}\right)^2 + \dfrac{1}{3}\left(\dfrac{x-1}{x}\right)^3 + \dots, \quad x \ge 1/2.$

73. $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = 2\left[x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots\right], \quad -1 < x < 1.$

74. $\tan^{-1}\left(\dfrac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right) = \left[\dfrac{x^2}{2} + \dfrac{x^6}{12} + \cdots\right].$

75. $\sin^{-1}\left(\dfrac{2x^2}{1+x^4}\right) = 2\left[x^2 - \dfrac{x^6}{3} + \dfrac{x^{10}}{5} - \cdots\right].$

1.3 Integration and Its Applications

Let $f(x)$ be defined and be continuous on a closed interval $[a, b]$. Let there exist a function $F(x)$ s that $F'(x) = f(x)$, $a \le x \le b$. Then, the function $F(x)$ is called the *anti-derivative* of $f(x)$. We observe if $F(x)$ is an anti-derivative of $f(x)$, then $F(x) + c$, where c is an arbitrary constant, is also anti-derivative of $f(x)$. We write

$$\int f(x)\, dx = F(x) + c \qquad (1.$$

We note that, not every function is integrable, for example the function $f(x)$ defined on $[0, 1]$ as

$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$$

does not have an antiderivative and hence is not integrable. We note the following:

(a) Every function which is continuous on a closed and bounded interval is integrable.

(b) For integrability, the condition that $f(x)$ is continuous on $[a, b]$ can be relaxed. The func $f(x)$ may only be piecewise continuous on $[a, b]$.

(c) Let m and M be the minimum and maximum values of $f(x)$ on $[a, b]$. Then,