(I) $\frac{d}{dx}[F(x)] = f(x)$

Integral of $f(x)$ or $\int f(x)dx = F(x)$

(II) $\frac{d}{dx}[F(x)+c] = f(x)$

Integral of $f(x)$ or $\int f(x)\,dx = F(x)+c$ → general integral of $f(x)$

Ex $\boxed{y = x^3+3} \Rightarrow \frac{dy}{dx} = 3x^2$

$y = \frac{\cancel{3}x^3}{\cancel{3}} + c$

$y = x^3 + c$

$\int f(x)\,dx = F(x) + c$

$\int$ → sign

$f(x)$ → integrand

dx → Variable x in dx

$F(x)$ → Integral of $f(x)$

c → Constant of integration

$F(x)+c$ → general integral of $f(x)$

Q $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Q $\int K f(x)\,dx = K\int f(x)\,dx$

(I) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, if $x = -1$

(II) $\int \frac{1}{x}\,dx = \log|x| + c$

(III) $\int a^x\,dx = \frac{a^x}{\log a} + c$, where a is +ve & $a \neq 0$

(IV) $\int e^x\,dx = e^x + c$

(V) $\int \sin x\,dx = -\cos x + c$

(VI) $\int \cos x\,dx = \sin x + c$

(VII) $\int \sec^2 x\,dx = \tan x + c$

(VIII) $\int \csc^2 x\,dx = -\cot x + c$

(IX) $\int \sec x \tan x\,dx = \sec x + c$

(IX) $\int \sec x \tan x \, dx = \sec x + C$

(X) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

Q. Write an antiderivative of each of the following functions

Sol (I) $\int \cos 2x \, dx =$

Here $\frac{d}{dx}\left(\frac{1}{2}\sin 2x\right) = \frac{1}{\cancel{2}} \cos 2x \times \cancel{2}$

$= \cos 2x$

∴ anti derivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$

(II) $3x^2 + 4x^3 \longrightarrow$

$\frac{d}{dx}\left[x^3 + x^4\right] = 3x^2 + 4x^3$

∴ anti derivative of $3x^2 + 4x^3$ is $x^3 + x^4$

Q. Find the integrals of following

(I) $\int \frac{x^3 - 1}{x^2} \, dx = \int \left(\frac{x^3}{x^2} - \frac{1}{x^2}\right) dx$

$= \int \left(x - x^{-2}\right) dx$

$= \frac{x^2}{2} - \frac{x^{-1}}{-1} + C$

$= \frac{x^2}{2} + \frac{1}{x} + C$

(II) $\int \left(x^{2/3} + 1\right) dx = \int x^{2/3} dx + \int 1 \, dx$

$= \frac{3}{5} x^{5/3} + x + C$ $\qquad \frac{2}{3} + 1$

(III) $\int \left(x^{3/2} + 2e^x - \frac{1}{x}\right) dx = \int x^{3/2} dx + 2\int e^x \, dx - \int \frac{1}{x} \, dx$

$= \frac{2}{5} x^{5/2} + 2e^x - \log|x| + C$

(IV) $\int (\sin x + \cos x) \, dx = \int \sin x \, dx + \int \cos x \, dx$

$= -\cos x + \sin x + C$

(V) $\int \operatorname{cosec} x \left[\operatorname{cosec} x + \cot x\right] dx = \int \operatorname{cosec}^2 x \, dx + \int \operatorname{cosec} x \cot x \, dx$

$= -\cot x - \operatorname{cosec} x + C$

$$= -\cot x - \operatorname{cosec} x + C$$

(vi) $\displaystyle\int \frac{1-\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos x}\,dx$

$= \int \sec^2 x\,dx \quad \int \tan x \sec x\,dx$

$= \tan x - \sec x + C$

Q. Find the anti derivative ~~of~~ F of f defined by

$f(x) = 4x^3 - 6$, where $\underline{F(0) = 3}$

Sol. $\int f(x)\,dx = F(x) + C$

or $\int (4x^3 - 6)\,dx = F(x) + C$

$x^4 - 6x = F(x) + C$

Put $x = 0$

$0 = F(0) + C$

$0 = 3 + C$ or $\boxed{C = -3}$

$\therefore F(x) = x^4 - 6x - C = \underline{x^4 - 6x + 3}$

Method of Substitution

$I = \int f(g(x))\, \underline{g'(x)\,dx}$

Put $g(x) = u$

$\therefore g'(x)\,dx = du$

$\therefore I = \int f(u)\,du$

Q. $I = \int 2x \sin(x^2+1)\,dx$

Put $x^2 + 1 = u$

$2x\,dx = du$

$\therefore I = \int \sin u\,du$

$= -\cos u + C$

$= -\cos(x^2+1) + C$

Q. $I = \displaystyle\int \frac{\tan^4\sqrt{x}\,\sec^2\sqrt{x}}{\sqrt{x}}\,dx$

Put $\tan(\sqrt{x}) = u$

or $\sec^2\sqrt{x} \times \dfrac{1}{2\sqrt{x}}\,dx = du$

or $\frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx = 2\,du$

$\therefore I = 2\int u^4\,du = 2\frac{u^5}{5} + C = \frac{2}{5}\tan^5\sqrt{x} + C$ Ans

Q. $I = \int \frac{\sin(\tan^{-1}x)}{1+x^2}\,dx$

Put $\tan^{-1}x = u$

$\frac{1}{1+x^2}\,dx = du$

$\therefore I = \int \sin u\,du = -\cos u + C$

$= -\cos(\tan^{-1}x) + C$

① $\int \tan x\,dx = -\log|\cos x| + C$ or $\log|\sec x| + C$

Proof # $I = -\int \frac{-\sin x}{\cos x}\,dx$

$\boxed{\frac{d}{dx}\cos x = -\sin x}$

$= -\log|\cos x| + C$

$= \log 1 - \log|\cos x| + C$

$= \log\left|\frac{1}{\cos x}\right| + C$

$= \log|\sec x| + C$

let $\cos x = t$

$-\sin x\,dx = dt$

or $\sin x\,dx = -dt$

$I = -\int \frac{dt}{t}$

$= -\log|t| + C$

$= -\log|\cos x| + C$

$I = \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$

let $f(x) = t$

$f'(x)\,dx = dt$

$I = \int \frac{dt}{t}$

$= \log|t| + C$

$= \log|f(x)| + C$

② $\int \cot x\,dx = \log|\sin x| + C$

Sol $I = \int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$= \log|\sin x| + C$

③ $\int \sec x\,dx = \log|\sec x + \tan x| + C$

Sol $I = \int \sec x\,dx = \int \frac{\sec x[\sec x + \tan x]}{\sec x + \tan x}\,dx$

$= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$

$= \log|\sec x + \tan x| + C$

$\frac{d}{dx}(\sec x + \tan x) = \sec x\tan x + \sec^2 x$

④ $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

$I = \int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x[\operatorname{cosec} x + \cot x]}{\operatorname{cosec} x + \cot x}\,dx$

$= \int \operatorname{cosec}^2 x + \cot x\operatorname{cosec} x \quad dx$

$$\int \qquad\qquad \int \overline{\operatorname{cosec} x + \cot x}$$

$$= \int \frac{\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x}{\operatorname{cosec} x + \cot x}\, dx$$

Put $\operatorname{cosec} x + \cot x = t$

$(-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x)\, dx = dt$

or $(\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)\, dx = -dt$

$$\therefore I = -\int \frac{dt}{t}$$

$$= -\log|t| + c$$

$$= -\log|\operatorname{cosec} x + \cot x| + c \quad \#$$

$$= \log 1 - \log|\operatorname{cosec} x + \cot x| + c$$

$$= \log\left|\frac{1}{\operatorname{cosec} x + \cot x}\right| + c$$

$$= \log\left|\frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x + \cot x}\right| + c$$

$$= \log\left|\frac{(\operatorname{cosec} x - \cot x)(\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x}\right| + c$$

$$= \log|\operatorname{cosec} x - \cot x| + c \quad \#$$

Q $I = \int \sin^3 x \cos^3 x\, dx$

$$= \int \sin^2 x \cos^3 x \sin x\, dx$$

$$= \int (1 - \cos^2 x) \cos^3 x\, \sin x\, dx$$

Put $\cos x = t$

$\therefore -\sin x\, dx = dt$

$$= -\int (1 - t^2) t^2\, dt = -\int (t^2 - t^4)\, dt = \int (t^4 - t^2)\, dt$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + c$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c \quad \text{An}$$

Q $I = \int \frac{\sin x}{\sin(x+a)}\, dx$

Put $x + a = t$ or $x = t - a$

$dx = dt$

$$\therefore I = \int \frac{\sin(t-a)}{\sin t}\, dt$$

$$\therefore I = \int \frac{\sin(t-a)}{\sin t}\,dt$$

$$= \int \frac{\sin t\cos a - \cos t\sin a}{\sin t}\,dt$$

$$= \int (\cos a - \cot t\sin a)\,dt$$

$$= \cos a\int dt - \sin a\int \cot t\,dt$$

$$= (\cos a)t - \sin a\log \sin t + C$$

$$= (x+a)\cos a - \sin a\log \sin(x+a) + C \quad \underline{\text{Ans}}$$

Q $I = \int \frac{1}{1+\tan x}\,dx = \int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx = \int \frac{\cos x}{\sin x+\cos x}\,dx$

$$= \frac{1}{2}\int \frac{2\cos x}{\sin x+\cos x} = \frac{1}{2}\int \frac{(\sin x+\cos x)+(\cos x-\sin x)}{\sin x+\cos x}\,d$$

$$= \frac{1}{2}\int \left[1 + \frac{\cos x - \sin x}{\sin x+\cos x}\right]dx$$

$$= \frac{1}{2}\left[\int dx + \int \left[\frac{\cos x-\sin x}{\sin x+\cos x}\right]dx\right.$$

$$= \frac{1}{2}\left[x + \log|\sin x+\cos x|\right] + C$$

Integration using Trigonometric Identities

$$I = \frac{1}{2}\int 2\cos^2 x\,dx.$$

$$= \frac{1}{2}\int (1+\cos 2x)\,dx$$

$\cos 2A = 2\cos^2 A - 1$

or $2\cos^2 A = 1+\cos 2A$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$$

$$= \frac{1}{4}\left[2x + \sin 2x\right] + C$$

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

or

$= 2\cos^2 A - 1$ ✓

or

$1 - 2\sin^2 A$

$\sin 3A = 3\sin A - 4\sin^3 A$

① $I = \frac{1}{2}\int 2\sin 2x\cos 3x\,dx$

$$= \frac{1}{2}\int \left[\sin 5x + \sin(-x)\right]dx$$

$\sin(-\theta) = -\sin\theta$

✓ $= \frac{1}{2}\int (\sin 5x - \sin x)\,dx$

$$= \frac{1}{2}\left[-\frac{\cos 5x}{5} + \cos x\right] + C$$

✓ $\sin(A+B) = \sin A\cos B + \cos A\sin B$

✓ $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\sin(A+B) + \sin(A-B)$

$= 2\sin A\cos B$

or $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

or ✓ $I = \frac{1}{2}\int 2\cos 3x\sin 2x\,dx$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2}\int \left[\sin 5x - \sin x\right]dx$$

$I = \frac{1}{2}\int$ ... $- \sin(A-B)$

$= \frac{1}{2}\int [\sin 5x - \sin x]\, dx$

Q $I = \int \sin^3 x\, dx$

$\sin 3A = 3\sin A - 4\sin^3 A$

or $4\sin^3 A = 3\sin A - \sin 3A$

$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

$= \int \left[\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right] dx$

$= \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$

$= \frac{3}{4} \times -\cos x - \frac{1}{4} \times \frac{-\cos 3x}{3} + C$

$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$

Integrals of some particular functions

(1) ✓ $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(2) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(3) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$\boxed{\text{Put } x = a\tan\theta}$

$1+\tan^2\theta = \sec^2\theta$

(4) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$

(5) $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + C$ # Put $x = a\sin\theta$

(6) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$ # $x = a\tan\theta$

Sol (1) $I = \int \frac{dx}{x^2-a^2}$

$\frac{1}{(x-a)(x+a)} = \frac{A}{(x-a)} + \frac{B}{x+a}$

Consider $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)}$

Sub everywhere but there

$= \frac{1}{(x-a)(2a)} + \frac{1}{(2a)(x+a)}$

$x-a=0$

$x=a$ ✓

$x+a=0$

$x=-a$

$= \frac{1}{2a}\left[\frac{1}{(x-a)} - \frac{1}{x+a}\right]$

$\therefore I = \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$

$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right] + C$

$= \frac{1}{2a}\left[\log\left|\frac{x-a}{x+a}\right|\right] + C$

or $\boxed{I = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c}$

Q (iii) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Sol $I = \int \frac{dx}{x^2+a^2}$

$1+\tan^2\theta = \sec^2\theta$

Put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\therefore I = \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$

$= \int \frac{\cancel{a}\,\cancel{\sec^2\theta}}{a^{\cancel{2}}[\cancel{\sec^2\theta}]}\, d\theta$

$= \frac{1}{a}\int d\theta + c = \frac{1}{a}\theta + c$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$x = a\tan\theta$

$\tan\theta = \frac{x}{a}$

$\theta = \tan^{-1}\frac{x}{a}$

(iv) $I = \int \frac{dx}{\sqrt{x^2-a^2}}$

Put $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$\sec^2\theta = 1+\tan^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\, d\theta$

\# $= \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta$

$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + c_1$

$= \log\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c_1$

$= \log|x + \sqrt{x^2-a^2}| - \log a + c_1$

$\boxed{I = \log|x+\sqrt{x^2-a^2}| + c}$ Ans

$x = a\sec\theta$

$\Rightarrow \sec\theta = \frac{x}{a}$

$\tan\theta = \sqrt{\sec^2\theta - 1}$

$= \sqrt{\frac{x^2}{a^2} - 1}$

$= \frac{\sqrt{x^2-a^2}}{a}$

\# $\sin^2\theta + \cos^2\theta = 1$

\# $1 + \tan^2\theta = \sec^2\theta$

\# $1 + \cot^2\theta = \csc^2\theta$

Q $\int \frac{dx}{ax^2+bx+c}$

$ax^2+bx+c \equiv a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$$ax^2+bx+c \equiv a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$

$$\equiv a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]$$

$$\equiv a\left[\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right]$$

$$\equiv a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

$$\equiv a\left[\left(x+\frac{b}{2a}\right)^2-\frac{(b^2-4ac)}{4a^2}\right]$$

$$\equiv a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]$$

$\frac{1}{2}\frac{b}{a}$

$\left(\frac{b}{2a}\right)^2$

$\frac{1}{2}\frac{b}{a}=\frac{b}{2a}$

$\left(\frac{b}{2a}\right)^2$

$(a+b)^2=a^2+2ab+b^2$

$x^2+2x\frac{b}{2a}+\left(\frac{b}{2a}\right)^2$

$=\left(x+\frac{b}{2a}\right)^2$

$$\therefore\ I=\int\frac{1}{a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]}dx$$

$\frac{1}{a}\int\frac{1}{x^2-A^2}$

$=\frac{1}{a}\times\frac{1}{2a}\log\left|\frac{x-A}{x+A}\right|+c$

Q $\displaystyle\int\frac{e^x(1+x)}{\cos^2(e^x x)}dx=$

(a) $-\cot(e x^x)+c$

(b) ✓ $\tan(xe^x)+c$

(c) $\tan(e^x)+c$

(d) $\cot(e^x)+c$

$\frac{d}{dx}e^x x=e^x\cdot 1+xe^x$
$=e^x(1+x)$

$I=\int\frac{dt}{\cos^2 t}$

$=\int\sec^2 t\,dt$

$=\tan t+c$

$=\tan(e^x x)+c$

Q $\displaystyle I=\int\frac{dx}{x^2-16}$

$$=\int\frac{dx}{x^2-4^2}=$$

$$=\frac{1}{2\times 4}\log\left|\frac{x-4}{x+4}\right|+c$$

$\int\frac{1}{x^2-a^2}dx=$

$\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

Q $\displaystyle I=\int\frac{dx}{\sqrt{2x-x^2}}$

$$2x-x^2\equiv -1\left[x^2-2x\right]$$

$$\equiv -\left[x^2-2x+1-1\right]$$

$-\frac{2}{2}=-1$

$(-1)^2=1$

$$2x - x^2 \equiv -1[x^2 - 2x]$$
$$\equiv -[x^2 - 2x + 1 - 1]$$
$$\equiv -[(x-1)^2 - 1^2]$$
$$\equiv [1^2 - (x-1)^2]$$

$-\frac{2}{2} = -1$
$(-1)^2 = 1$

$$\therefore I = \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1^2 - (x-1)^2}}$$
$$= \sin^{-1}\frac{(x-1)}{1} + c$$

Q $\int \frac{px+q}{ax^2+bx+c}\,dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

Consider $px + q = A\frac{d}{dx}(ax^2+bx+c) + B$

Q $I = \int \frac{x+2}{2x^2+6x+5}\,dx$

Here $x + 2 \equiv A\frac{d}{dx}[2x^2+6x+5] + B$

$$x+2 = A(4x+6) + B$$

on solving $A = \frac{1}{4}$ & $B = \frac{1}{2}$

Eq of like
$x)$ $1 = 4A$
$Cons)$ $2 = 6A + B$

$$\therefore x + 2 \equiv \frac{1}{4}(4x+6) + \frac{1}{2}$$

$$\therefore I = \int \frac{\frac{1}{4}(4x+6) + \frac{1}{2}}{2x^2+6x+5}\,dx$$
$$= \frac{1}{4}\int \frac{4x+6}{2x^2+6x+5}\,dx + \frac{1}{2}\int \frac{1}{2x^2+6x+5}\,dx$$
$$= \frac{1}{4}\log|2x^2+6x+5| + \frac{1}{2}I_1 \qquad \text{— (1)}$$

Here $I_1 = \int \frac{1}{2x^2+6x+5}\,dx$
$$= \frac{1}{2}\int \frac{1}{x^2+3x+\frac{5}{2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{x^2+3x+\left(\frac{3}{2}\right)^2 + \frac{5}{2} - \left(\frac{3}{2}\right)^2}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dx$$
$$= \frac{1}{2} \times \frac{1}{1/2}\tan^{-1}\frac{(x+3/2)}{1/2}$$
$$= \cancel{1}\tan^{-1}(2x+3)$$

$$= \cancel{\frac{1}{2}} \tan^{-1}(2x+3)$$

Sub in (1)

$I = \quad ?$

Q $\displaystyle\int \frac{x+3}{\sqrt{5-4x+x^2}}\,dx$

Integration by partial fraction

(1) $\dfrac{px+q}{(x-a)(x-b)}$, $a \neq b$ $\qquad \dfrac{A}{x-a} + \dfrac{B}{x-b}$

(2) $\dfrac{px+q}{(x-a)^2}$ $\qquad \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$

(3) $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ $\qquad \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$

(4) $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$ $\qquad \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b}$

(5) $\dfrac{px^2+qx+r}{(x-a)\,(x^2+bx+c)}$ $\qquad \dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+bx+c}$

Cannot be factorized

Integrate $\displaystyle\int \frac{dx}{(x+1)(x+2)}$

Sol
$$\frac{1}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$$
$$\equiv \frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

or $1 \equiv A(x+2) + B(x+1)$

Equating coeff of like terms

$x)\quad 0 = A + B$

Constant) $\quad 1 = 2A + B$

$\left[\begin{array}{l} ax+by=c \\ a_1x+b_1y=c_1 \end{array}\right]$

on solving $\boxed{A=1 \ \& \ B=-1}$

$$\therefore \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{(-1)}{x+2}$$

Or Perfer

$$\frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)(-1+2)} + \frac{1}{(-2+1)(x+2)}$$

Or Perfer

$$\frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)(-1+2)} + \frac{1}{(-2+1)(x+2)}$$

$x+1=0$
$x=-1$

$x+2=0$
$x=-2$

$$= \frac{1}{x+1} - \frac{1}{x+2} \quad \#$$

$$\therefore I = \int \frac{1}{(x+1)(x+2)}\,dx$$

$$= \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

$$= \int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx$$

$$= \log|x+1| - \log|x+2| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c$$

#

Q $\int \frac{dx}{x^2+2x+2}$ equals $= \int \frac{dx}{(x+1)^2+1}$

$$= \frac{1}{1}\tan^{-1}(x+1) + C$$

(a) $x\tan^{-1}(x+1) + C$

✓ (b) $\tan^{-1}(x+1) + C$

(c) $(x+1)\tan^{-1}x + C$

(d) $\tan^{-1}x + C$

Q $I = \int \frac{x^2+1}{x^2-5x+6}\,dx$

$$x^2-5x+6 \overline{\big)\, x^2+1 \,}\ 1$$
$$\underline{\ -x^2 \ +5x \ -6\ }$$
$$5x - 5$$

1 term

$$\int \frac{x^2+1}{x^2-5x+6}\,dx = \int \left(1 + \frac{5x-5}{x^2-5x+6}\right) dx$$

$$= \int 1\,dx + \int \frac{5(x-1)}{x^2-5x+6}\,dx$$

$$= \int 1\,dx + 5\int \frac{x-1}{x^2-5x+6}\,dx$$

$$= \int 1\,dx + 5\int \frac{(x-1)}{(x-2)(x-3)}\,dx$$

$$= x + 5I_1$$

x^2-5x+6
$\equiv x^2-3x-2x+6$
$= x(x-3)-2(x-3)$
$= (x-2)(x-3)$

$$\therefore I_1 = \int \frac{x-1}{(x-2)(x-3)}\,dx$$

✓ Consider $\frac{(x-1)}{(x-2)(x-3)} \equiv \frac{1}{(x-2)(-1)} + \frac{2}{(1)\ (x-3)}$

Consider $\dfrac{(x-1)}{(x-2)(x-3)} \equiv \dfrac{1}{(x-2)(-1)} + \dfrac{2}{(1)\ (x-3)}$

$$= -\frac{1}{x-2} + \frac{2}{x-3}$$

$$\therefore I_1 = \int\left[-\frac{1}{x-2} + \frac{2}{x-3}\right]dx$$

$$= -\log|x-2| + 2\log|x-3|$$

$$\frac{p(x)}{d(x)} \equiv q(x) + \frac{r(x)}{d(x)}$$

$$\therefore I = x + 5\left[-\log|x-2| + 2\log|x-3|\right] + c \quad \text{Ans}$$

Q $I = \displaystyle\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$

Here $\dfrac{3x-2}{(x+1)^2(x+3)} \equiv \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+3}$

Integration by parts

$$\int \underset{\mathrm{I}}{u}\,\underset{\mathrm{II}}{v}\,dx = u\int v\,dx - \int\left(\frac{d}{dx}u\int v\,dx\right)dx$$

I → Inverse
L → Log
A → Alg
T → Trig
E → Exp

Q $I = \displaystyle\int \underset{\mathrm{I}}{x^2}\,\underset{\mathrm{II}}{e^x}\,dx$

$$= x^2\int e^x dx - \int\left(\frac{d}{dx}x^2\int e^x dx\right)dx$$

$$= x^2e^x - \int 2x\,e^x\,dx$$

$$= x^2e^x - 2\int x\,e^x\,dx$$

$$= x^2e^x - 2\left[x\int e^x - \int\left(\frac{d}{dx}x\int e^x dx\right)dx\right]$$

$$= x^2e^x - 2\left[x\,e^x - \int 1\cdot e^x\,dx\right]$$

$$= x^2e^x - 2xe^x + 2e^x + c \quad \text{Ans} \quad \#$$

Q $\displaystyle\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + c$ Ans

$\displaystyle\int uv\,dx = u(v_1) - u'(v_2) + u''(v_3) - u'''(v_4) + \ldots$

$\int uv\,dx = u(v_1) - u'(v_2) + u''(v_3) - u'''(v_4) + \ldots$

$$\int x^2 e^x\,dx = x^2(e^x) - (2x)(e^x) + (2)(e^x)$$
$$= x^2e^x - 2xe^x + 2e^x + c \quad \#$$

Q $I = \int x\cos x\,dx$
$$= x(\sin x) - (1)(-\cos x) + c$$
$$= x\sin x + \cos x + c$$

Q $I = \int \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Put $\sin^{-1}x = t$ or $x = \sin t$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt$$

$$\therefore \quad I = \int \sin t \; t \; dt$$
$$= \int t\sin t\,dx = t(-\cos t) - (1)(-\sin t) + c$$
$$= -t\cos t + \sin t + c$$
$$= -\sin^{-1}x\cos(\sin^{-1}x) + \sin(\sin^{-1}x) + c$$
$$= -\sin^{-1}x(\cos(\sin^{-1}x)) + x + c$$
#

Q $I = \int e^x[f(x) + f'(x)]\,dx = e^x f(x) + c$

Sol $I = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$
$$= f(x)\int e^x dx - \int \cancel{f'(x)e^x}\,dx + \int \cancel{e^x f'(x)}\,dx + c$$
$$= f(x)e^x + c$$
$$= e^x f(x) + c$$

Q $\int e^x\left(\tan^{-1}x + \dfrac{1}{1+x^2}\right)dx = e^x\tan^{-1}x + c$

Q $I = \int \dfrac{(x^2+1)}{(x+1)^2}e^x\,dx$

$= \int \dfrac{e^x(x^2-1+1+1)}{(x+1)^2}\,dx$

$$= \int e^x \frac{[x^2-1+1+1]}{(x+1)^2}\,dx$$

$$= \int e^x \left[\frac{(x^2-1) + 2}{(x+1)^2}\right] dx$$

$$= \int e^x \left[\frac{(x-1)\cancel{(x+1)}}{(x+1)^{\cancel{2}}} + \frac{2}{(x+1)^2}\right] dx$$

$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

$$= e^x\left[\frac{x-1}{x+1}\right] + c$$

$$\frac{d}{dx}\frac{x-1}{x+1} = \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

① $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$ $\boxed{\text{Put } x = a\sec\theta}$

② $\displaystyle\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$ $\boxed{x = a\tan\theta}$

③ $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$ $\boxed{x = a\sin\theta}$

Sol let $I = \int \sqrt{x^2-a^2}\cdot 1\,dx$

$$= \sqrt{x^2-a^2}\int 1\,dx - \int\left(\frac{d}{dx}\sqrt{x^2-a^2}\int 1\,dx\right)dx$$

$$= \sqrt{x^2-a^2}\;x - \int\left[\frac{1}{2\sqrt{x^2-a^2}}\times 2x \times x\right]dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int\left\{\frac{x^2-a^2}{\sqrt{x^2-a^2}} + \frac{a^2}{\sqrt{x^2-a^2}}\right\}dx$$

$$\frac{(x^2-a^2)}{\sqrt{x^2-a^2}} = (x^2-a^2)^1(x^2-a^2)^{-1/2} = (x^2-a^2)^{1-1/2} = (x^2-a^2)^{1/2}$$

$$I = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$I = x\sqrt{x^2-a^2} - I - a^2\log\left|x+\sqrt{x^2-a^2}\right| + c_1$$

$$2I = x\sqrt{x^2-a^2} - a^2\log\left|x+\sqrt{x^2-a^2}\right| + c_1$$

or $\boxed{I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c}$ $\boxed{c = \frac{c_1}{2}}$

Q $I = \int \sqrt{x^2+2x+5}\,dx$

$$= \int \sqrt{x^2+2x+1+4}\,dx$$

$$= \int \sqrt{x^2+2x+1+4}\, dx$$

$$= \int \sqrt{(x+1)^2+2^2}\, dx \qquad \Bigg|\ \int \sqrt{x^2+a^2}\, dx$$

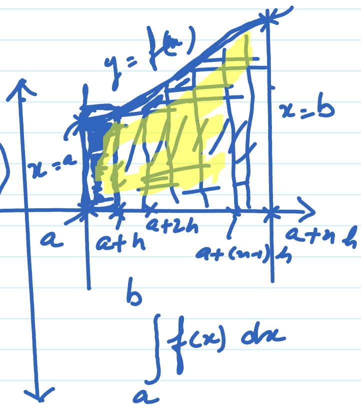
$$= \frac{(x+1)\sqrt{(x+1)^2+2^2}}{2} + \frac{2^2}{2}\log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right| + C$$

$$I = \frac{(x+1)\sqrt{x^2+2x+5}}{2} + 2\log\left|(x+1)+\sqrt{x^2+2x+5}\right| + c$$

Definite Integral

$\boxed{h = \frac{b-a}{n}}$

Sum of area under the curve $\le$ req area $\le$ Sum of areas of rectangles outside the curve



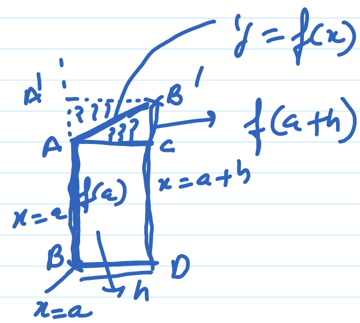
$$\int_a^b f(x)\, dx$$

#

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots \cdots + f(a+(n-1)h)\right]$$

where $\boxed{h = \frac{b-a}{n}}$

area ABCD $\le$ area under AB'BD $\le$ area A'B'DB



$$\int_a^b f(x)\,dx \le \underline{\quad\quad} \le \int_a^b f(x)$$

$$\int_a^b f(x)\,dx = \text{req area}$$

$n \to \infty$

$\boxed{h = \frac{b-a}{n}}$ $n \to \infty$

Q Evaluate $\int_0^2 e^x\, dx$

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + \underline{\quad\quad} + f(a+(n-1)h)\right]$$

where $h = \frac{b-a}{n}$

Here $f(x) = e^x$, $a = 0$, $b = 2$, $h = \frac{2-0}{n} = \frac{2}{n}$

$\therefore\ f(a) = f(0) = e^0 = 1$

$f(a+h) = f(h) = e^h$

$$f(a+\overline{n-1}\,h) = f(\overline{n-1}\,h) = e^{(n-1)h}$$

$$\therefore \int_0^2 e^x\,dx = \lim_{h\to 0} h\left[\underbrace{1 + e^h + e^{2h} + \text{---} + e^{(n-1)h}}\right]$$

$$\boxed{S_n = \frac{a[1-r^n]}{1-r}} \quad e^h$$

$$h = \frac{b-a}{n}$$

$$nh = b-a = 2$$

$$= \lim_{h\to 0} h \,\frac{\cancel{a}\,[1-e^{nh}]}{1-e^h}$$

$$= \lim_{h\to 0} h \times 1\,\frac{[1-e^2]}{1-e^h}$$

$$\boxed{\lim_{h\to 0}\frac{a^h-1}{h} = \log a}$$

$$= -\lim_{h\to 0} \frac{(1-e^2)}{\frac{-1+e^h}{h}}$$

$$\boxed{\lim_{h\to 0}\frac{-1+e^h}{h} = 1}$$

$$= \frac{-(1-e^2)}{1} = e^2 - 1 \ \#$$

Q $\displaystyle\int_0^2 e^x\,dx = \left|e^x\right|_0^2 = e^2 - e^0 = e^2 - 1 \ \#$

2nd Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = \left|F(x)\right|_a^b = F(b) - F(a)$$

$$\# \quad \int_a^b f(x)\,dx = \left|F(x)+c\right|_a^b = F(b) + \cancel{c} - F(a) - \cancel{c}$$
$$= F(b) - F(a)$$

Q $\displaystyle\int_2^3 x^2\,dx = \left|\frac{x^3}{3} + c\right|_2^3$

$$= \left(\frac{27}{3} + \cancel{c}\right) - \left(\frac{8}{3} + \cancel{c}\right) = \frac{19}{3} \text{ Ans}$$

Properties of Definite Integral

P_0 : $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, In particular $\displaystyle\int_a^a f(x)\,dx = 0$

P_2 : $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

✓ P_3 : $\displaystyle\int^b f(x)\,dx = \int^b f(a+b-x)\,dx$

P_3 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P_4 . $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

P_5 $\int_0^{2a} \underline{f(x)\,dx} = \int_0^a \underline{f(x)\,dx} + \int_0^a \underbrace{f(2a-x)\,dx}_{I_1}$

P_6 = $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $\boxed{f(2a-x) = f(x)}$

$= 0$ if $f(2a-x) = -f(x)$

P_7 : $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if f is even fun
ie $f(-x) = f(x)$

$= 0$ if f is odd function
ie $f(-x) = -f(x)$

Q $I = \int_{-\pi/4}^{\pi/4} \sin^2 x\,dx$

Here $f(x) = \sin^2 x$
$f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$
$\Rightarrow f(x)$ is even function

$\therefore I = 2\int_0^{\pi/4} \sin^2 x\,dx$

$\cos 2A = 1 - 2\sin^2 A$
$2\sin^2 A = 1 - \cos 2A$

$= \int_0^{\pi/4} (1 - \cos 2x)\,dx$

$= x - \frac{\sin 2x}{2}\Big|_0^{\pi/4}$

$= \left\{\frac{\pi}{4} - \frac{1}{2}\sin 2\frac{\pi}{4}\right\} - \{0\} =$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}}$

hw Q Evaluate $\int_{-1}^{2} |\underline{x^3 - x}|\,dx$

$|x| = x$ if $x \geq 0$
$= -x$ if $x < 0$

2 0 1 2

$$\int_{-1}^{2}|x^3-x|\,dx = \int_{-1}^{0}|x^3-x|\,dx + \int_{0}^{1}|x^3-x|\,dx + \int_{1}^{2}|x^3-x|\,dx$$

$= -x$ if $x < 0$

$|-1| = -(-1) = +1$

$x^3 - x = 0$

$x(x^2-1) = 0$

$x = 0,\ x = \pm 1$

$x = -1, 0, 1$

Here if $x \in [-1, 0]$, $-1 \le x \le 0$

Now $x^3 - x = x(x^2-1)$

$-ve \quad -ve \quad = +ve$

$-1 \le x \le 0$

S.B.S $1 \geq x^2 \geq 0$

or $0 \le x^2 \le 1$

$\Rightarrow x^2 \le 1$

or $x^2 - 1 \le 0$

$\therefore x^3 - x \geq 0$ in $[-1, 0]$

$\therefore |x^3 - x| = +(x^3 - x)$

if $x \in [0,1]$ $\quad 0 \le x \le 1$

$x^3 - x = x(x^2-1) = -ve$

$+ve \quad -ve$

$|x^3 - x| = -(x^3 - x)$

$0 \le x \le 1$

$0 \le x^2 \le 1$

or $x^2 - 1 \le 0$

if $x \in [1, 2]$, $1 \le x \le 2$

$x^3 - x = x(x^2-1) = +ve$

$= +ve \quad +ve$

$|x^3 - x| = x(x^2-1)$

$1 \le x \le 2$

$\boxed{1 \le x^2} \le 4$

$1 \le x^2$ or $x^2 \geq 1$

or $x^2 - 1 \geq 0$

$$\therefore I = \int_{-1}^{0}(x^3-x)\,dx + \int_{0}^{1}-(x^3-x)\,dx + \int_{1}^{2}(x^3-x)\,dx$$

$$= \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{-1}^{0} - \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{0}^{1} + \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{1}^{2}$$

$$= \frac{11}{4} \quad \text{Ans}$$

Q Evaluate $\displaystyle\int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

Sol $\quad I = \displaystyle\int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

$\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

$$= \int_{0}^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

SL	Af
to	call

$$= \int_{0}^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$$

$$= \int_{0}^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\,dx - \int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$$

$$\text{or } I = \pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - I$$

or $I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - I$

or $2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$0 \le x \le \pi$ $\quad \cos x \to t$

$\cos 0 \le \cos x \le \cos \pi$

$1 \le t \le -1$

Put $\cos x = t$

$-\sin x\, dx = dt$ or $\sin x\, dx = -dt$

when $x = 0$, $t = \cos 0 = 1$

$x = \pi$, $t = \cos \pi = -1$

$\therefore 2I = -\pi \int_1^{-1} \frac{1}{1+t^2} dt$

$= \pi \int_{-1}^{1} \frac{1}{1+t^2} dt$

$\int_a^b = -\int_b^a$

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even

$= \pi \times 2 \int_0^1 \frac{1}{1+t^2} dt$

$f(t) = \frac{1}{1+t^2}$

$f(-t) = \frac{1}{1+(-t)^2} = \frac{1}{1+t^2} = f(t)$

$f(-x) = f(x) \to$ even

$= -f(x) \to$ odd

$= 2\pi \left| \tan^{-1} t \right|_0^1$

or $\cancel{2}I = \cancel{2}\pi \left[\tan^{-1} 1 - \tan^{-1} 0\right]$

$I = \pi \left[\tan^{-1}\tan\frac{\pi}{4} - \tan^{-1}\tan 0\right]$

$\boxed{I = \frac{\pi^2}{4}}$ Ans

Imp Q. Evaluate $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}$

or $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx \quad —(1)$

$\frac{\pi}{3} + \frac{\pi}{6}$

$\frac{2\pi+\pi}{6} = \frac{\pi}{2}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)} + \sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}} dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)} + \sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx$

or $I = \int \frac{\sqrt{\sin x}}{\ldots + \sqrt{\cos x}} dx \quad —(2)$

or
$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad \text{— (2)}$$

on adding (1) & (2) we get
$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$
$$= \int_{\pi/6}^{\pi/3} 1\, dx$$
or $\boxed{I = \pi/12}$ Ans

Q Evaluate $I = \int_{-1}^{1} \sin^5 x \cos^4 x\, dx$
$$= 0$$

$f(x) = \sin^5 x \cos^4 x$
$f(-x) = \sin^5(-x)\cos^4(-x)$
$= -\sin^5 x \cos^4 x$
$= -f(x)$

Q $I = \int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx$ — (1)
$$= \int_0^{\pi/2} \frac{\sin^4(\frac{\pi}{2}-x)}{\sin^4(\frac{\pi}{2}-x) + \cos^4[\frac{\pi}{2}-x]}\, dx$$
$$= \int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\, dx \quad \text{— (2)}$$
on adding $2I = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$

or $\boxed{I = \pi/4}$

Q $I = \int_0^{\pi/2} \log \sin x\, dx$ — (1)
$$= \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$
$$I = \int_0^{\pi/2} \log \cos x\, dx \quad \text{— (2)}$$
on adding $2I = \int_0^{\pi/2} \log \sin x + \log \cos x\,\}\, dx$
$$= \int^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx$$

$$= \int_0 \log\left(\frac{2\sin x\cos x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log \frac{\sin 2x}{2}\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

$$2I = I_1 - \log 2 \int_0^{\pi/2} 1\, dx \quad \text{--- (2)}$$

Here $I_1 = \int_0^{\pi/2} \log \sin 2x\, dx$

put $2x = t$

$2dx = dt$

& when $x = 0$, $t = 0$

$x = \pi/2$, $t = \pi$

$$= \frac{1}{2}\int_0^{\pi} \log \sin t\, dt$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \int_0^{\pi/2} \boxed{\log \sin t\, dt}$$

$\log \sin(\pi - t) = \log \sin t$

from (2)

$$\therefore 2I = \int_0^{\pi/2} \log \sin t\, dt - \log 2 \left|\frac{\pi}{2}\right|$$

$$= \int_0^{\pi/2} \log \sin t\, dt - \frac{\pi}{2} \log 2$$

$$2I = I - \frac{\pi}{2} \log 2$$

or $\boxed{I = -\frac{\pi}{2} \log 2}$ #

Practice

Q $I = \int_4^9 \frac{\sqrt{x}}{(30 - x^{3/2})}\, dx$

put $30 - x^{3/2} = t$

$-\frac{3}{2}\sqrt{x}\, dx = dt$

Q P_0: $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt \rightarrow$

Sol Consider $I = \int_a^b f(x)\, dx$ --- (1)

Sol Consider $I = \int_{a}^{b} f(x)\,dx$ — (1)

Put $x = t$

on diff $dx = dt$

when $x = a$; $t = x = a$

& $x = b$; $t = x = b$

$\therefore$ from (1), $I = \int_{a}^{b} f(t)\,dt$ = RHS

P_1: ✓ $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$

Sol let $\int f(x)\,dx = \phi(x)$

L.H.S $\int_{a}^{b} f(x)\,dx = \left|\phi(x)\right|_{a}^{b} = \phi(b) - \phi(a)$ — (1)

R.H.S $-\int_{b}^{a} f(x)\,dx = -\left|\phi(x)\right|_{b}^{a} = -\left[\phi(a) - \phi(b)\right]$

$= -\phi(a) + \phi(b)$

$= \phi(b) - \phi(a)$ — (2)

$\therefore$ from (1) & (2) $\boxed{\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx}$

Q P_2: $\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$, where c is any point bet a & b

Sol let $\int f(x)\,dx = \phi(x)$

LHS $\int_{a}^{b} f(x)\,dx = \left|\phi(x)\right|_{a}^{b} = \phi(b) - \phi(a)$ — (1)

R.HS $\int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx = \left|\phi(x)\right|_{a}^{c} + \left|\phi(x)\right|_{c}^{b}$

$= \left[\cancel{\phi(c)} - \phi(a)\right] + \left[\phi(b) - \cancel{\phi(c)}\right]$

$= \phi(b) - \phi(a)$ — (2)

from (1) & (2) L.H.S = R.H.S

Q If $f(x) = \int_{0}^{x} t \sin t\,dt$, then $f'(x)$ is

Q If $f(x) = \int_0^x t\sin t\,dt$, then $f(x)$ is

(A) $\cos x + x\sin x$
(B) $x\sin x$
(C) $x\cos x$
(D) $\sin x + x\cos x$
(E) $\sin x - x\cos x$

$$\int_0^x \underset{I}{t}\,\underset{II}{\sin t}\,dt$$

$$\Big|t(-\cos t) - (1)(-\sin t)\Big|_0^x$$

$$= (-x\cos x + \sin x) - (0+0)$$

$$= \sin x - x\cos x$$

Q $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Sol Put $x = a+b-t$

$dx = -dt$

when $x = a$, $\cancel{a} = \cancel{a}+b-t$ or $t = b$

$x = b$, $\cancel{b} = a+\cancel{b}-t$ or $t = a$

$$I = \int_b^a f(a+b-t)\,(-dt) = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-x)\,dx$$

Q $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Sol Put $a=0$ & $b=a$ in P_3, we get

Q $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Sol $$\int_0^{2a} f(x)\,dx = \boxed{\int_0^a f(x)\,dx} + \int_a^{2a} f(x)\,dx$$

$\underbrace{0 \text{ to } 2a}_{a}$

Sol $\int_0^{2a} f(x)\,dx = \boxed{\int_0^{a} f(x)\,dx} + \int_a^{2a} f(x)\,dx$ 0 to 2a, a

let $I_1 = \int_a^{2a} f(x)\,dx$

Put $x = 2a - t$
$dx = -dt$

when $x = a$, $a = 2a - t \Rightarrow t = a$
$x = 2a$, $2a = 2a - t \Rightarrow t = 0$

$$\therefore I_1 = \int_a^0 f(2a-t) \times -dt = -\int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(2a-t)\,dt$$

$$= \int_0^a f(2a-x)\,dx$$

$$\therefore \boxed{I = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx}$$

(P6) ✓ If $\boxed{f(2a-x) = f(x)}$ then from P5

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(ii) If $\boxed{f(2a-x) = -f(x)}$ then from (5)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a -f(x)\,dx = 0$$ ✓ Ans

Q $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even

$= 0$ if $f(x)$ is odd

Sol $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

Consider $I_1 = \int_{-a}^{0} f(x)\,dx$

Put $x = -y$

$dx = -dy$

when $x = -a, \quad y = a$

$x = 0, \quad y = 0$

$\therefore I_1 = \int_{a}^{0} f(-y) \times -dy = -\int_{a}^{0} f(-y)\,dy = \int_{0}^{a} f(-y)\,dy$

$= \int_{0}^{a} f(-x)\,dx$

$\therefore I = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{--- } (*)$

(i) If $f(x)$ is even then $f(-x) = f(x)$

$\therefore$ from $(*)$ $I = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(ii) If $f(x)$ is odd then $f(-x) = -f(x)$

$\therefore$ from $(*)$ $I = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$ #

Q The value of the integral

$I = \int_{-\pi/2}^{\pi/2} (\underbrace{x^3}_{\text{odd}} + \underbrace{x\cos x}_{\text{odd}} + \underbrace{\tan^5 x}_{\text{odd}} + 1)\,dx$

$= 0 + 0 + 0 + \int_{-\pi/2}^{\pi/2} 1\,dx = 2\int_{0}^{\pi/2} dx = 2\left[\frac{\pi}{2} - 0\right]$

$= \pi$ Ans