



# **CSE408**

# **Asymptotic notations**

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## **Lecture #4**

# Asymptotic Notations



- The efficiency analysis framework concentrates on the order of growth of an algorithm's basic operation count as the principal indicator of the algorithm's
- To compare and rank such orders of growth, computer scientists use three notations: *(big oh)*, *(big omega)*, and *(big theta)* efficiency

## *O*-notation

**DEFINITION** A function  $t(n)$  is said to be in  $O(g(n))$ , denoted  $t(n) \in O(g(n))$ , if  $t(n)$  is bounded above by some constant multiple of  $g(n)$  for all large  $n$ , i.e., if there exist some positive constant  $c$  and some nonnegative integer  $n_0$  such that

$$t(n) \leq cg(n) \quad \text{for all } n \geq n_0.$$

## $\Omega$ -notation

**DEFINITION** A function  $t(n)$  is said to be in  $\Omega(g(n))$ , denoted  $t(n) \in \Omega(g(n))$ , if  $t(n)$  is bounded below by some positive constant multiple of  $g(n)$  for all large  $n$ , i.e., if there exist some positive constant  $c$  and some nonnegative integer  $n_0$  such that

$$t(n) \geq cg(n) \quad \text{for all } n \geq n_0.$$

# Example



Here is an example of the formal proof that  $n^3 \in \Omega(n^2)$ :

$$n^3 \geq n^2 \quad \text{for all } n \geq 0,$$

i.e., we can select  $c = 1$  and  $n_0 = 0$ .

## $\Theta$ -notation

**DEFINITION** A function  $t(n)$  is said to be in  $\Theta(g(n))$ , denoted  $t(n) \in \Theta(g(n))$ , if  $t(n)$  is bounded both above and below by some positive constant multiples of  $g(n)$  for all large  $n$ , i.e., if there exist some positive constants  $c_1$  and  $c_2$  and some nonnegative integer  $n_0$  such that

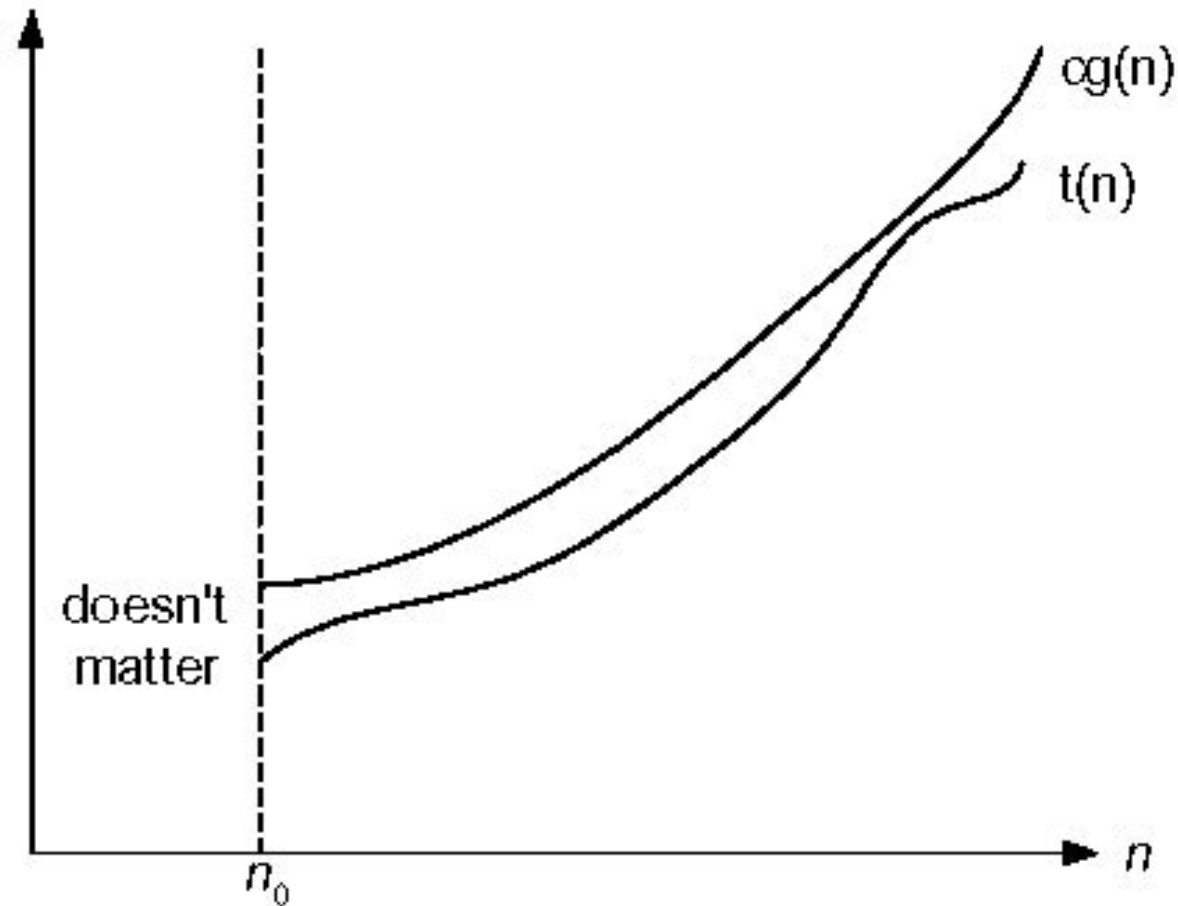
$$c_2g(n) \leq t(n) \leq c_1g(n) \quad \text{for all } n \geq n_0.$$

# Asymptotic order of growth



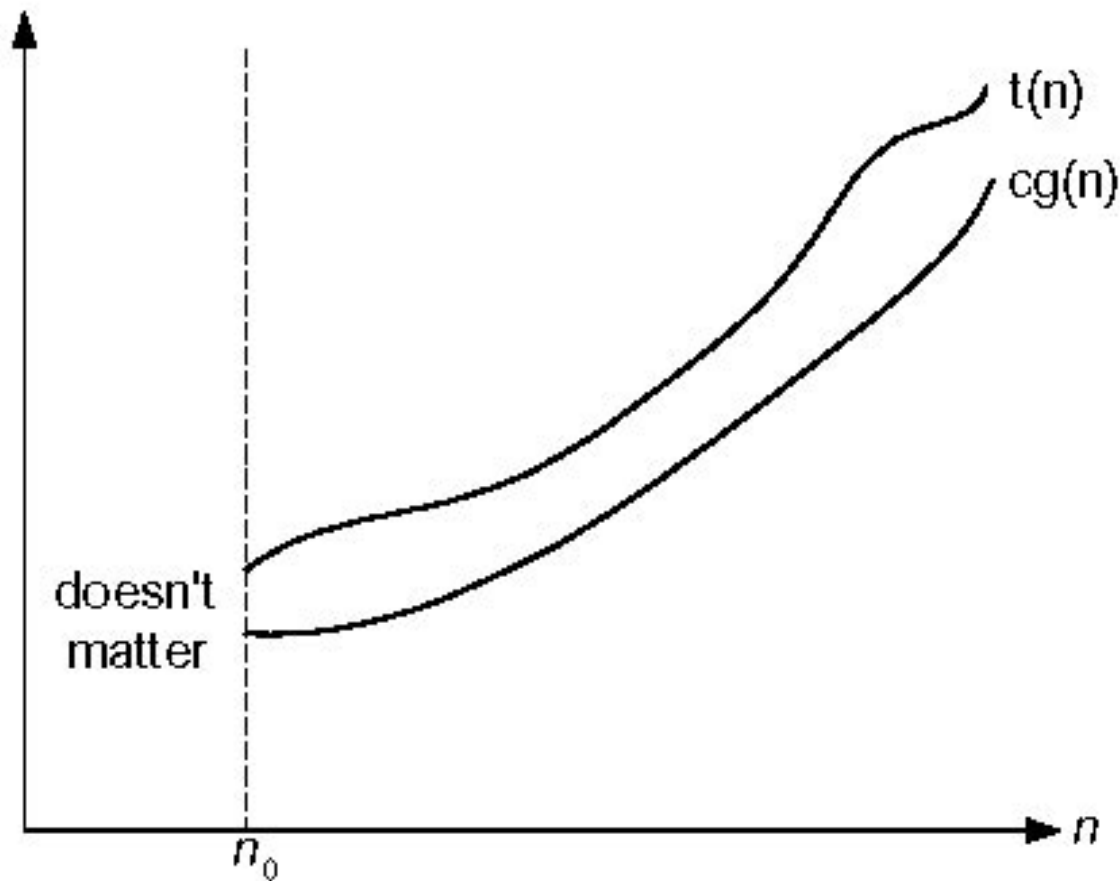
A way of comparing functions that ignores constant factors and small input sizes

- $O(g(n))$ : class of functions  $f(n)$  that grow no faster than  $g(n)$
- $\Theta(g(n))$ : class of functions  $f(n)$  that grow at same rate as  $g(n)$
- $\Omega(g(n))$ : class of functions  $f(n)$  that grow at least as fast as  $g(n)$

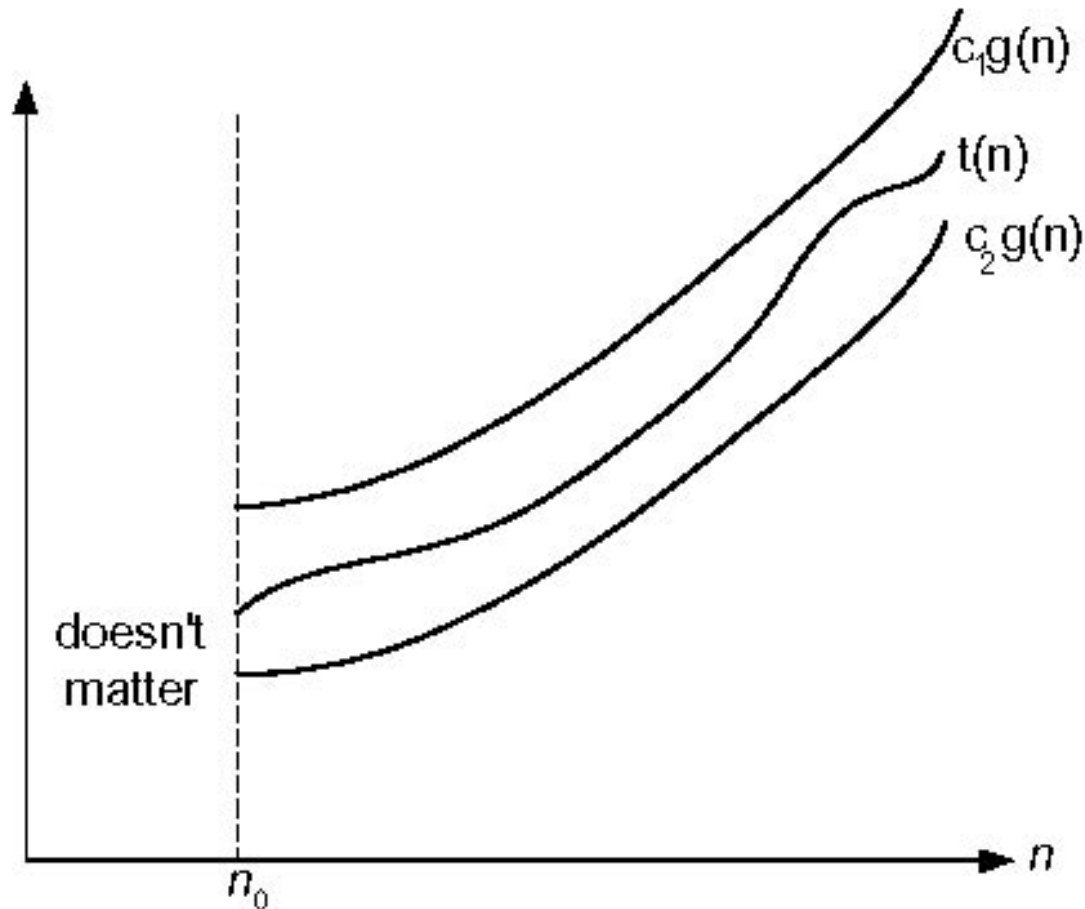


**Figure 2.1** Big-oh notation:  $t(n) \in O(g(n))$





**Fig. 2.2** Big-omega notation:  $t(n) \in \Omega(g(n))$



**Figure 2.3** Big-theta notation:  $t(n) \in \Theta(g(n))$

# Some properties of asymptotic order of growth



- $f(n) \in O(f(n))$
- $f(n) \in O(g(n))$  iff  $g(n) \in \Omega(f(n))$
- If  $f(n) \in O(g(n))$  and  $g(n) \in O(h(n))$ , then  $f(n) \in O(h(n))$

Note similarity with  $a \leq b$

- If  $f_1(n) \in O(g_1(n))$  and  $f_2(n) \in O(g_2(n))$ , then
$$f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$$

# Establishing order of growth using limits



	0	$T(n)$	$g(n)$
$\lim_{n \rightarrow \infty} T(n)/g(n) =$	$c > 0$	$T(n)$	$g(n)$
	$\infty$	$T(n)$	$g(n)$

Examples:

$10n$

vs.

$n^2$

$n(n+1)/2$

vs.

$n^2$

# L'Hôpital's rule and Stirling's formula



L'Hôpital's rule: If  $\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} g(n) = \infty$  and the derivatives  $f'$ ,  $g'$  exist, then

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

Example:  $\log n$  vs.  $n$   
Stirling's formula:  $n! \approx (2\pi n)^{1/2} (n/e)^n$

Example:  $2^n$  vs.  $n!$

**EXAMPLE 1** Compare the orders of growth of  $\frac{1}{2}n(n - 1)$  and  $n^2$ . (This is one of the examples we used at the beginning of this section to illustrate the definitions.)

# Example



$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2}n(n-1)}{n^2} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n^2 - n}{n^2} = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \frac{1}{2}.$$

Since the limit is equal to a positive constant, the functions have the same order of growth or, symbolically,  $\frac{1}{2}n(n-1) \in \Theta(n^2)$ . ■

# Example



**EXAMPLE 2** Compare the orders of growth of  $\log_2 n$  and  $\sqrt{n}$ . (Unlike Example 1, the answer here is not immediately obvious.)



# Example



$$\lim_{n \rightarrow \infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{(\log_2 n)'}{(\sqrt{n})'} = \lim_{n \rightarrow \infty} \frac{(\log_2 e) \frac{1}{n}}{\frac{1}{2\sqrt{n}}} = 2 \log_2 e \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0.$$

Since the limit is equal to zero,  $\log_2 n$  has a smaller order of growth than  $\sqrt{n}$ . (Since  $\lim_{n \rightarrow \infty} \frac{\log_2 n}{\sqrt{n}} = 0$ , we can use the so-called **little-oh notation**:  $\log_2 n \in o(\sqrt{n})$ . Unlike the big-Oh, the little-oh notation is rarely used in analysis of algorithms.)



## *o*-notation

The asymptotic upper bound provided by  $O$ -notation may or may not be asymptotically tight. The bound  $2n^2 = O(n^2)$  is asymptotically tight, but the bound  $2n = O(n^2)$  is not. We use  $o$ -notation to denote an upper bound that is not asymptotically tight. We formally define  $o(g(n))$  (“little-oh of  $g$  of  $n$ ”) as the set

$o(g(n)) = \{f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0\} .$

For example,  $2n = o(n^2)$ , but  $2n^2 \neq o(n^2)$ .

The definitions of  $O$ -notation and  $o$ -notation are similar. The main difference is that in  $f(n) = O(g(n))$ , the bound  $0 \leq f(n) \leq cg(n)$  holds for *some* constant  $c > 0$ , but in  $f(n) = o(g(n))$ , the bound  $0 \leq f(n) < cg(n)$  holds for *all* constants  $c > 0$ . Intuitively, in the  $o$ -notation, the function  $f(n)$  becomes insignificant relative to  $g(n)$  as  $n$  approaches infinity; that is,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 . \tag{3.1}$$

**EXAMPLE 3** Compare the orders of growth of  $n!$  and  $2^n$ .

# Orders of growth of some important functions



- All logarithmic functions  $\log_a n$  belong to the same class  $\Theta(\log n)$  no matter what the logarithm's base  $a > 1$  is
- All polynomials of the same degree  $k$  belong to the same class:  $a_k n^k + a_{k-1} n^{k-1} + \dots + a_0 \in \Theta(n^k)$
- Exponential functions  $a^n$  have different orders of growth for different  $a$ 's
- $\text{order } \log n < \text{order } n^\alpha \ (\alpha > 0) < \text{order } a^n < \text{order } n! < \text{order } n^n$

# Basic asymptotic efficiency classes



$1$	constant
$\log n$	logarithmic
$n$	linear
$n \log n$	$n$ -log- $n$
$n^2$	quadratic
$n^3$	cubic
$2^n$	exponential
$n!$	factorial



**Thank You !!!**