CHAPTER

EIGHT

DISCRETE NUMERIC FUNCTIONS AND GENERATING FUNCTIONS

8.1 (a) The numeric function is given by

$$a_r = 20/2^r,\ r \geq 0$$

i.e. $a_r = 20/2^0, 20/2^1, 20/2^2, \ldots, r \geq 0$

i.e. $a_r = 20, 10, 5, \ldots, r \geq 0.$

(b) $b_r = a_r$

(c) $c_r = 0.3a_r$

8.2 $a_r = \begin{cases} 100 + 2r & 0 \leq r \leq 10 \\ 120 & r \geq 10 \end{cases}$

8.3 (a) $r = 51n, n = 0, 1, 2, \ldots; r = 51n + 7, 51n + 18, 51n + 34, n = 0, 1, 2, \ldots$

(b) $r = 17n$ or $3n$; $r = 51n + 1$ or $51n + 35$, $n = 0, 1, 2, \ldots$

8.4 (a) $S^2\mathbf{a} = \begin{cases} 0 & r = 0, 1 \\ 2 & r = 2, 3, 4, 5 \\ 2^{-(r-2)} + 5 & r \geq 6 \end{cases}$

$$S^{-2}\mathbf{a} = \begin{cases} 2 & r \leq 1 \\ 2^{-(r+2)} + 5 & r \geq 2 \end{cases}$$

(b) $\Delta a_r = \begin{cases} 0 & 0 \leq r \leq 2 \\ 49/16 & r = 3 \\ -2^{-(r+1)} & r \geq 4 \end{cases}$

$$\nabla a_r = \begin{cases} 0 & 0 \leq r \leq 3 \\ 49/16 & r = 4 \\ -2^{-r} & r \geq 5 \end{cases}$$

8.5 (a) $\Delta a_r = 3r^2 - r + 2$

$$\Delta^2 a_r = 6r + 2$$
$$\Delta^3 a_r = 6$$
$$\Delta^4 a_r = 0$$

(b) Show inductively that if a_r is a polynomial in r of degree k then Δa_r is a polynomial of degree $\leq k - 1$. This is easily seen by looking at the high order terms of a_{r+1} and a_r. Since $\Delta a_r = 0$ if a_r is a constant, clearly $\Delta^{k+1} a_r = 0$ for all r.

8.6 (a) $d_r = c_{r+1} - c_r = a_{r+1}\, b_{r+1} - a_r\, b_r$

$$= a_{r+1}[b_{r+1} - b_r] + b_r[a_{r+1} - a_r]$$
$$= a_{r+1}(\Delta b_r) + b_r(\Delta a_r)$$

(b) $\alpha^r[(\alpha - 1)r + 2d - 1]$.

(c) $\Delta d_r = \dfrac{a_{r+1}}{b_{r+1}} - \dfrac{a_r}{b_r} = \dfrac{a_{r+1}\, b_r - a_r\, b_{r+1}}{b_r\, b_{r+1}}$

$$= \frac{(a_{r+1} - a_r)\, b_r - a_r\, (b_r + 1 - b_r)}{b_r\, b_r + 1}$$

(d) $\Delta(a/b)_r = \dfrac{(1-\alpha)^r + 2 - \alpha}{\alpha^{r+1}}$

8.7 No. Note that

$$\Delta^{-1}\mathbf{a} = a_0,\ a_1 + a_1,\ a_0 + a_1 + a_2, \ldots$$
$$\Delta(\Delta^{-1}\mathbf{a}) = a_1,\ a_2,\ a_3, \ldots$$
$$\Delta\mathbf{a} = a_1 - a_0,\ a_2 - a_1,\ a_3 - a_2, \ldots$$
$$\Delta^{-1}(\Delta\mathbf{a}) = a_1 - a_0,\ a_2 - a_3 - a_0, \ldots$$

8.8
$$a_r = \begin{cases} 50 & r = 0, \ldots, 59 \\ 0 & \text{otherwise} \end{cases}$$

$$b_r = (1.005)^r$$

Since
$$\mathbf{c} = \mathbf{a} * \mathbf{b}$$

$$c_r = \begin{cases} \displaystyle\sum_{i=0}^{r} (50)\,(1.005)^i & 0 \leq r \leq 59 \\ \displaystyle\sum_{i=r-59}^{r} 50(1.005)^i & r \geq 60 \end{cases}$$

For $$r = 47,\ c_{47} = \sum_{i=0}^{47} 50(1.005)^i = 2704.89$$

For $$r = 239,\ c_{239} = \sum_{i=180}^{239} 50(1.005)^i = 8561.11$$

8.9 (a) $$(a * b)_r = \begin{cases} 1 & r = 0 \\ 2 & r = 1 \\ 3 & r = 2 \\ 2 & r = 3 \\ 1 & r = 4 \\ 0 & r \geq 5 \end{cases}$$

(b) $$(a * b)_r = \begin{cases} 1 & r = 1, 2 \\ 3 & r = 3, 4 \\ 6 & r = 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

(c) $$(a * b)_r = \begin{cases} (r-2)2^r & r \geq 3 \\ 0 & r \leq 2 \end{cases}$$

8.10 $b_r = (-2)^r$

8.11 $(1 + 3z + 2z^2)\, B(z) = \dfrac{1}{1-5z}$

$$B(z) = \frac{1}{(1+z)\,(1+2z)\,(1-5z)} = \frac{-1/6}{1+z} + \frac{4/7}{1+2z} + \frac{25/42}{1-5z}$$

$$b_r = \frac{1}{6}(-1)^r + \frac{4}{7}(-2)^r + \frac{25}{42}(5)^r$$

8.12 (a) $2^{n+1} - 1$

(b) For $0 \leq n \leq 9,\ b_n = 2^{n+1} - 1.$

$$b_{10} = 2046.$$

For $n \geq 11$, let I_n be the number of newly created particles. At the n^{th} second, one newly created particle is injected and b_{n-1} new particles are created by the b_{n-1} particles inside the reactor. Thus

$$I_n = b_{n-1} + 1 \; n \geq 11.$$
$$b_n = 2b_{n-1} + 1 - I_{n-10}$$
$$b_n = 2b_{n-1} + b_{n-11}$$

Thus, $$\sum_{n=11}^{\infty} b_n z_n = 2\sum_{n=11}^{\infty} b_{n-1} z^n - \sum_{n=11}^{\infty} b_{n-11} z^n$$

$$B(z) - \sum_{n=0}^{10} b_n z^n = 2z\left[B(z) - \sum_{n=0}^{9} b_n z^n\right] - z^{11} B(z)$$

$$B(z) = \frac{(1 + z + \ldots + z^9)}{(1 - 2z - z^{11})}$$

$$= (1 + z + \ldots + z^9) \sum_{i=0}^{\infty} (2z - z^{11})^i$$

$$= (1 + z + \ldots + z^9) \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \binom{i}{k} (-z^{11})^k (2z)^{i-k}$$

$$= (1 + z + \ldots + z^9) \sum_{r=0}^{\infty} \left[\sum_{k=0}^{\infty} \binom{r-10k}{k} (-1)^k \, 2^{r-11k}\right] z^r$$

$$b_n = \sum_{r=n-9}^{n} \left[\sum_{k=0}^{\infty} \binom{r-10k}{k} (-1)^k \, 2^{r-11k}\right]$$

8.13 (a) $$a_r = \begin{cases} 10 & 0 \leq r \leq 4 \\ 2 & 5 \leq r \leq 9 \\ 0 & 10 \leq r \end{cases}$$

$$b_r = \frac{500}{2^r} \quad r \geq 0$$

Total sale, $\mathbf{c} = \mathbf{a} * \mathbf{b}$

Hence $C(z) = A(z)\, B(z)$

$$= \left[10\left(\frac{1-z^5}{1-z}\right) + 2z^5\left(\frac{1-z^5}{1-z}\right)\right] \cdot \left(\frac{500}{1-z/2}\right)$$

$$c_r = \begin{cases} 1000 - 5000/2^3 & 0 \le r \le 4 \\ 2000 + 123000/2^r & 5 \le r \le 9 \\ 1147000/2^r & 10 \le r \end{cases}$$

(b) Let d_r be the thousand dollars invested.

$$d_r = \begin{cases} 2 & 0 \le r \le 9 \\ 0 & 10 \le r \end{cases}$$

$$D(z) = 2\left(\frac{1 - z^{10}}{1 - z}\right)$$

Total sale, $\mathbf{S} = \mathbf{a} * \mathbf{b} * \mathbf{d}$

$$S_r = \begin{cases} (10r + 5/2^r)\,2000 & 0 \le r \le 4 \\ (2r + 40 - 155/2^r)\,2000 & 5 \le r \le 9 \\ (10r + 160 - 6299/2^r)\,2000 & 10 \le r \le 14 \\ (18r + 40 - 124773/2^r)\,2000 & 15 \le r \le 19 \\ (20r + 923803/2^r)\,2000 & 20 \le r \end{cases}$$

8.14 (a) Yes, No, No, No, Yes, Yes
(b) Yes, No
(c) No, Yes, No, Yes
(d) Yes
(e) Yes

8.15 (a) Yes
(b) No
(c) Yes. ($\mathbf{a} * \mathbf{b} = 5^r$)
(d) Yes

8.16 (a) $1^2 + 2^2 + \ldots + r^2 < r \cdot r^2$
Thus **a** is $O(r^3)$.

(b) $1^2 + 2^2 + \ldots + r^2 = \dfrac{r(r+1)(2r+1)}{6}$

$$= \frac{r^3}{3} + \frac{3r^2}{6} + \frac{r}{6}$$

Now, $\dfrac{3r^2}{6} + \dfrac{r}{6} < r^2$

Thus, $\mathbf{a}$ is $\dfrac{r^3}{3} + O(r^2)$.

(c) No. counterexample, $b_r = r^3$ or $b_r = \dfrac{r^3}{6} + r^{2.5}$

(d) Yes

8.17 $\left[\ln r + O\left(\dfrac{1}{n}\right)\right] [n + O(\sqrt{n})]$

$$= n \ln r + O(\sqrt{n}\ \ln r) + O\left(\frac{1}{n}n\right) + O\left(\frac{1}{n}\sqrt{n}\right)$$

$$= n \ln r + O(\sqrt{n}\ \ln r)$$

8.18 Choose $k = 1, m = \varepsilon$.

We show that $r^{\varepsilon} \geq \varepsilon \ln r$ for $r \geq 1$.

Let $f(r) = r^{\varepsilon} - \varepsilon \ln r$

Then, $f(1) = 1 > 0$.

$$f'(r) = \varepsilon r^{\varepsilon-1} - \frac{\varepsilon}{r} = \frac{\varepsilon}{r}(r^{\varepsilon} - 1)$$

$$\geq 0 \quad \text{for} \quad r \geq 1.$$

Thus $f(r) \geq 0$ for $r \geq 1$.

8.19 Suppose $r = 2^i$ ($i = \log r$), then

$$b_r = a_{2^i} + a_{2^i-1} + \ldots + a_2 + a_1$$

$$= \sqrt{2^i} + \sqrt{2^{i-1}} + \ldots + \sqrt{2} + \sqrt{1}$$

$$< \sqrt{2^i} + \sqrt{2^i} + \ldots + \sqrt{2^i} + \sqrt{2^i}$$

$$= (i+1) \cdot \sqrt{2^i}$$

$$= O((i+1)\sqrt{2^i}) = O(i\sqrt{2^i})$$

$$= O(\sqrt{r}\log r)$$

8.20 $a_r = \log_2 r + \log_2\left(\dfrac{2}{3}\right)r + \log_2\left(\dfrac{4}{9}\right)r + \ldots$ ($\lfloor \log_{3/2} r \rfloor$ *terms*)

$< \log_2 r + \log_2 r + \log_2 r + \ldots$ ($\lfloor \log_{3/2} r \rfloor$ *terms*)

$= \log_2 r \cdot \lfloor \log_{3/2} r \rfloor$
$= O(\log_2 r \cdot \log_{3/2} r)$
$= O(\log^2 r)$

8.21 No

8.22 (a) No (b) Yes (c) Yes

8.23 No

8.24 Oh, Omega

8.25 (a) $\dfrac{1}{(1+z)^2}$ (b) $\dfrac{1}{(1-z/3)^2}$

(c) $\dfrac{1+z}{(1-z^2)^2}$ (d) $\dfrac{2z}{(1-z)^3}$

(e) $\dfrac{z}{5(1-z/5)^2}$

8.26 $\dfrac{1}{(1+2z)}$

8.27 (a) $a_r = \begin{cases} 1 & \text{if } r = 3i \\ 0 & \text{otherwise} \end{cases}$

(b) $a_r = \begin{cases} 2\dbinom{n}{r} & r \text{ even} \\ 0 & \text{otherwise} \end{cases}$

(c) $a_r = \begin{cases} 1 & r = 0 \\ 6 & r = 1 \\ \dbinom{r+1}{3} + 2\dbinom{r+2}{3} + \dbinom{r+3}{3} & r \geq 3 \end{cases}$

(d) $a_r = \dfrac{1}{4} - \dfrac{1}{4}\left(\dfrac{1}{5}\right)^r$

(e) $a_r = \begin{cases} 0 & 0 \leq r \leq 4 \\ \dfrac{1}{4} - \dfrac{1}{4} \cdot \left(\dfrac{1}{5}\right)^{r-4} & r \geq 5 \end{cases}$

(f) $a_r = \begin{cases} 0 & 0 \le r \le 1 \\ \dfrac{7}{5}\left[2^{r+1} - (-3)^{r+1}\right] & r \ge 2 \end{cases}$

(g) $a_r = \begin{cases} \dfrac{1}{4} & r = 0 \\ \dfrac{5r-3}{2^{r+2}} & r \ge 1 \end{cases}$

(h) $a_r = \begin{cases} 1 & r = 0 \\ 13/9 & r = 1 \\ \dfrac{r^2}{12} + \dfrac{r}{2} + \dfrac{39}{24} + \dfrac{(-1)^r}{8} & r \ge 2 \text{ ‘and’ } r \ne 0 \; mod \; 3 \\ \dfrac{r^2}{12} + \dfrac{r}{2} + \dfrac{7}{8} + \dfrac{(-1)^r}{8} & r \ge 2 \text{ ‘and’ } r = 0 \; mod \; 3 \end{cases}$

8.28 (a) $A(z) = z^2 + z^3 + 2z^4 + 2z^5 + 3z^6 + 3z^7 + 3z^8 + 2z^9 + 2z^{10} + z^{11} + z^{12}$

(b) $B(z) = (z + z^2 + \ldots + z^6)\, A(z)$

8.29 $a_r = P(r-9, 10)$ for $r \ge 19$.

$$A(z) = \sum_{r=19}^{\infty} (r-9)(r-10)(r-11)\ldots(r-18)z^r$$

$$= z^{19} \sum_{r=0}^{\infty} (r+10)(r+9)\ldots(r+1)z^r$$

$$= \frac{10!z^{19}}{(1-z)^{-11}}$$

8.30 Let $a_r = 2^{r+1}$

$$b_r = \begin{cases} 1 & 0 \le r \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Correspondingly, $A(z) = \dfrac{2}{1-2z}$ $\quad B(z) = \dfrac{1-z^{11}}{1-z}$

Let c_r denote the number of rabbits there are in the rth year.

$$C(z) = A(z)\,B(z) = (1 - z^{11})\left(\frac{-2}{1-z} + \frac{4}{1-2z}\right)$$

Let $$D(z) = \frac{-2}{1-z} + \frac{4}{1-2z}$$

Correspondingly, $d_r = 2^{r+2} - 2$

We have $$c_r = d_r - d_{r-11} = \begin{cases} 2^{r+2} - 2 & 0 \le r < 11 \\ 2^{r+2} - 2^{r-9} & r \ge 11 \end{cases}$$

8.31 $\dfrac{1}{(1-z)^2} \cdot \dfrac{1}{(1-z^2)}$

8.32
$$\begin{aligned} A(z) &= (1 + z + z^2 + \ldots + z^{2r})^3 \\ &= \left(\frac{1-z^{2r+1}}{1-z}\right)^3 \\ &= (1 - z^{2r+1})^3 (1 - z)^{-3} \\ &= (1 - 3z^{2r+1} + 3z^{4r-2} - z^{6r+3}) (1 - z)^{-3} \end{aligned}$$

$$a_{3r} = \binom{3r+2}{2} - 3\binom{r+1}{2}$$

8.33 $(1 + z + z^3 + z^4 + \ldots + z^{100}) (1 + z + z^2 + z^4 + \ldots + z^{50}) (1 + z + z^2 + z^3 + z^5 + \ldots + z^{50})$

$$= \left(\frac{1-z^{101}}{1-z} - z^2\right)\left(\frac{1-z^{51}}{1-z} - z^3\right)\left(\frac{1-z^{51}}{1-z} - z^4\right)$$

8.34 (a) Let

$B(z) = (1 + z^2 + z^4 + \ldots) (z + z^3 + z^5 + \ldots) (1 + z + z^2 + \ldots) (1 + z + z^2 + \ldots)$

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$$= \frac{1}{1-z^2} \cdot \frac{z}{1-z^2} \cdot \frac{1}{1-z} \cdot \frac{1}{1-z}$$

$C(z) = (1 + z + z^2 + \ldots) (1 + z + z^2 + \ldots) (1 + z^2 + z^4 + \ldots) (z + z^3 + z^5 + \ldots)$

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$D(z) = (1 + z^2 + z^4 + \ldots) (z + z^3 + z^5 + \ldots) (1 + z^2 + z^4 + \ldots) (z + z^3 + z^5 + \ldots)$

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$$A(z) = B(z) + C(z) - D(z) = \frac{2z}{(1-z^2)^2 (1-z)^2} - \frac{z^2}{(1-z^2)^4}$$

$$= \frac{2z^3 + 3z^2 + 2z}{(1-z^2)^4}$$

$$= (2z^3 + 3z^2 + 2z)\sum_{r=0}^{\infty}\binom{r+3}{3}z^{2r}$$

Thus, $$a_r = \begin{cases} 3\binom{3+(r-2)/2}{3} & r \text{ even} \\ 2\binom{3+(r-3)/2}{3} + 2\binom{3+(r-1)/2}{3} & r \text{ odd} \end{cases}$$

$$= \begin{cases} 3\binom{(r+4)/2}{3} & r \text{ even} \\ 2\binom{(r+3)/2}{3} + \binom{(r+5)/2}{3} & r \text{ odd} \end{cases}$$

(b) $a_{23} = 1300$

8.35 (a) $A(z) = (z^3 + z^5 + z^7 + \ldots)^4$

$$= \left(\frac{z^3}{1-z^2}\right)^4$$

$$= z^{12} \cdot (1 - z^2)^{-4} = z^{12}\sum_{i=0}^{\infty}\binom{i+3}{i}z^{2i}$$

(b) $$a_r = \begin{cases} \binom{3+\frac{(r-12)}{2}}{\frac{(r-12)}{2}} & \text{if } r \geq 12 \text{ and } r \text{ mod } 2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

8.36 (a) $$\binom{n}{0} + \binom{n}{1}z + \binom{n}{2}z^2 + \ldots + \binom{n}{i}z^i + \ldots + \binom{n}{n}z^n = (1+z)^n$$

Differentiate both sides and set z to 1:

$$\binom{n}{1} + 2\binom{n}{2} + \ldots + i\binom{n}{i} + \ldots + n\binom{n}{n} = n2^{n-1}$$

(b) The total number of occurrences of all letters is

$$\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = n2^{2-1}$$

By symmetry, each letter occurs $\dfrac{n2^1}{n} = 2^{n-1}$ times

8.37 $\sum_{i=0}^{n} 2^i\, C(n,i)\, z^i = (1 + 2z)^n$

Setting $z = 1$, we obtain

$$\sum_{i=0}^{n} 2^i\, C(n,i) = 3^n$$

8.38 $(1 + z)^n\,(1 + z)^m = (1 + z)^{n+m}$. Thus, the sum is $\binom{n+m}{k}$.

8.39 (a) $A(z) = (1 + z)^{2n+1} - z^{n+1}(1 + z)^n$ which easily gives the values for a_r.

(b) $\binom{2n}{n} + \binom{2n-1}{n-1} + \ldots + \binom{n}{0} = \binom{2n}{n} + \binom{2n-1}{n} + \ldots$

$$+ \binom{2n-i}{n} + \ldots + \binom{n}{n}$$

which is $a_n = \binom{2n+1}{n}$.

8.40 (a) $\binom{r}{0}^2 + \binom{r}{1}^2 + \binom{r}{2}^2 + \ldots + \binom{r}{i}^2 + \ldots + \binom{r}{r}^2$ is the constant term of the product

$$\begin{aligned}(1 + z)^r\,(1 + 1/z)^r &= (1 + z)^r\,(z + 1)^r\,z^{-r}\\ &= (1 + z)^{2r}\,z^{-r}\end{aligned}$$

Therefore it is the coefficient of z^r in $(1 + z)^{2r}$ which is $\binom{2r}{r}$.

(b) $(1 - 4z)^{-1/2}$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1/2)\,(-1/2 - 1)\,(-1/2 - 2)\ldots[-1/2 - (r - 1)]}{r!}\,(-4z)^r$$

$$= 1 + \sum_{r-1}^{\infty} \frac{4^r (1/2)\,(3/2)\,(5/2)\ldots[(2r-1)/2]}{r!}\; z^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{2^r [1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2r-1)]}{r!}\; z^r$$

But $\binom{2r}{r} = \dfrac{(2r)!}{r!r!}$

$$= \frac{[(2r)\,(2r-2)\,(2r-4)\ldots(2)]\,[(2r-1)\,(2r-3)\ldots(5)\,(3)\,(1)]}{r!r!}$$

$$= \frac{2^r\,(r!)\,[(2r-1)\,(2r-3)\ldots(5)\,(3)\,(1)]}{r!r!}$$

which is the coefficient of z^r above.