CHAPTER

ELEVEN

BOOLEAN ALGEBRAS

11.1 Suppose $a \wedge b = b$. Then $b \vee a = (a \wedge b) \vee a = a$
Suppose $b \vee a = a$. Then $a \wedge b = (b \vee a) \wedge b = b$

11.2 $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \leq b \wedge (a \vee c)$

11.3 $a \vee (b \wedge c) \leq a \vee b$
$a \vee (b \wedge c) \leq a \vee c$
Thus, $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
The second inequality follows from duality.

11.4 If $b \leq a$, then $a \wedge b = b$ which contradicts the condition $a \wedge b < b$.
If $a \leq b$, then $a \wedge b = a$ which contradicts the condition $a \wedge b < a$.
Thus, if $a \wedge b < a$ and $a \wedge b < b$, then a and b are incomparable.
On the other hand, suppose a and b are incomparable.

Since $a \wedge b \leq a$, if $a \wedge b = a$, then $a \leq b$, which is a contradiction.
Since $a \wedge b \leq b$, if $a \wedge b = b$, then $b \leq a$, which is a contradiction.

11.5 Since $a \vee (a \wedge x) = a$, we have $a \wedge a = a \wedge (a \vee (a \wedge x))$. However, according to the absorption law $a \wedge (a \vee (a \wedge x)) = a$. Thus, $a \wedge a = a$.

11.6 (a) Transitivity: $a \leq b,\ b \leq c \Rightarrow a \wedge b = a,\ b \wedge c = b$
$\Rightarrow a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c)$
$= a \wedge b = a.$

Antisymmerty: $a \leq b,\ b \leq a \Rightarrow a \wedge b = a,\ a \wedge b = b \Rightarrow a = b.$

Reflexivity: Let $a \vee a = b$. Then $a \wedge b = a \wedge (a \vee a) = a$
so $a \vee a = a \vee (a \wedge b) = a$ by absorption.
Thus $a \wedge a = a \wedge (a \vee a) = a$ and hence $a \leq a$.

(b) $a \leq a \vee b$ since $a \wedge (a \vee b) = a$ by absorption. Similarly, $b \leq a \vee b$.
Thus, $a \vee b$ is an upper bound of a and b.

Also, $(a \vee b) \wedge (a \vee b \vee c) = a \vee b$ by absorption.
If $a \leq c$ and $b \leq c$, then $c = c \vee (c \wedge a) = c \vee a$. Similarly, $b \vee c = c$.
Hence, $(a \vee b) \wedge (a \vee b \vee c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$.
Thus, $a \vee b = (a \vee b) \wedge c$ and $a \vee b \leq c$.
Similarly, $a \wedge b \leq a$ and $a \wedge b \leq b$ since $a \wedge (a \wedge b) = a \wedge b$ and $b \wedge (b \wedge a) = b \wedge a$.
If $c \leq a$ and $c \leq b$, then $a \wedge c = c$ and $b \wedge c = c$.
Thus, $c \wedge (a \wedge b) = c \wedge b = c$, so $c \leq a \wedge b$.

11.7 Clearly, $(a \vee b) \wedge c \leq a \vee (b \wedge c) \leq (a \wedge c) \vee (b \wedge c)$. For the reverse inclusion we note that the lattice is modular since if $a \leq c$, then $a \vee (b \wedge c) \geq (a \vee b) \wedge c$ by assumption. While $a \leq a \vee b$, $a \leq c$, and $b \wedge c \leq a \vee b$, $b \wedge c \leq c$, so $a \vee (b \wedge c) \leq (a \vee b) \wedge c$. Thus, since $a \wedge c \leq c$, $(a \wedge c) \vee (b \wedge c) = ((a \wedge c) \vee b) \wedge c \geq ((a \vee b) \wedge c) = (a \vee b) \wedge c$.

11.8 We have $x \vee (a \wedge x) = (x \vee a) \wedge (x \vee x) = x$

However,
$$\begin{aligned} x \vee (a \wedge x) &= x \vee (a \wedge y) \\ &= (x \vee a) \wedge (x \vee y) \\ &= (y \vee a) \wedge (x \vee y) \\ &= (a \wedge x) \vee y \\ &= (a \wedge y) \vee y \\ &= y \end{aligned}$$

11.9 If $(A, \leq)$ is distributive, then

$$\begin{aligned} (a \vee b) \wedge (b \vee c) \wedge (c \vee a) &= [(a \vee b) \wedge (b \vee c) \wedge c] \vee [(a \vee b) \wedge (b \vee c) \wedge a] \\ &= [(a \vee b) \wedge c] \vee [(b \vee c) \wedge a] \\ &= [(a \wedge c) \vee (b \wedge c)] \vee [(b \wedge a) \vee (c \wedge a)] \\ &= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a). \end{aligned}$$

Conversely, if this condition holds, then

$$\begin{aligned} a \wedge (b \vee c) &= a \wedge ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)), \\ &\qquad \text{(since } (a \vee b) \wedge a = a = (a \vee c) \wedge a) \\ &= a \wedge [(b \wedge c) \vee (c \wedge a) \vee (a \wedge b)], \\ &\qquad \text{(by the assumed condition)} \\ &= a \wedge [(b \wedge c) \vee a] \wedge [(b \wedge c) \vee (c \wedge a) \vee (a \wedge b)] \\ &\qquad \text{(since } a \wedge [(b \wedge c) \vee a] = a) \\ &= [a \vee [(c \wedge a) \vee (a \wedge b)]] \vee [(b \wedge c) \wedge a] \vee [(b \wedge c) \wedge [(c \wedge a) \vee (a \wedge b)]] \\ &\qquad \text{(since } a \vee (c \wedge a) \vee (a \wedge b) = a) \\ &= [a \wedge [(c \wedge a) \vee (a \wedge b)]] \vee [(b \wedge c) \wedge a] \vee [(b \wedge c) \wedge [(c \wedge a) \vee (a \wedge b)]] \\ &\qquad \text{(by the assumed condition)} \end{aligned}$$

$= [(c \wedge a) \vee (a \wedge b)] \vee [(b \wedge c) \wedge a] \vee [(b \wedge c) \wedge [(c \wedge a) \vee (a \wedge b)]]$

(since $a \geq (c \wedge a) \vee (a \wedge b)$)

$= [(c \wedge a) \vee (a \wedge b)] \vee [(b \wedge c) \wedge a]$

(since $[(b \wedge c) \wedge [(c \wedge a) \vee (a \wedge b)]] \leq [(c \wedge a) \vee (a \wedge b)]$)

$= (c \wedge a) \vee (a \wedge b)$

(since $(a \wedge b \wedge c) \leq (a \wedge b)$)

11.10 If $(A, \leq)$ is modular, then $a \leq a \vee c$, so $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$.
If the condition holds, let $a \leq c$. Then $c = a \vee c$ and

$$\begin{aligned} a \vee (b \wedge c) &= a \vee [b \wedge (a \vee c)] \\ &= (a \vee b) \wedge (a \vee a \vee c) \\ &= (a \vee b) \wedge (a \vee c) \\ &= (a \vee b) \wedge c. \end{aligned}$$

11.11 Clearly $(c \vee b) \wedge a \geq b \wedge a$ and $(c \vee b) \wedge a \leq a$.
To show $(c \vee b) \wedge a \leq b$ we observe that

$$(c \vee b) \wedge a \leq (c \vee b) \wedge (a \vee b) \leq b \vee [c \wedge (a \vee b)]$$

by the modular law. But by assumption, $c \wedge (a \vee b) = b \wedge c$.

Thus, $(c \vee b) \wedge a \leq b \vee (b \wedge c) = b$. And $(c \vee b) \wedge a \leq a \wedge b$.

11.12 (a) (2) $\Rightarrow$ (2′): If $x \leq a$, then $a \wedge x = x$.

Therefore, $\quad a \wedge x \in I \Rightarrow x \in I$

(2′) $\Rightarrow$ (2): For any $a \in I$ and any $x \in A$, $a \wedge x \leq a$. Thus, $a \wedge x$ is in I.

(b) Let I be an ideal. Clearly, (1″) is satisfied. According to condition (2) if $a \vee b$ is in I, $a = (a \vee b) \wedge a$ is also in I, and $b = (a \vee b) \wedge b$ is also in I. Thus condition (2″) is satisfied.

Now, suppose I is a set satisfying conditions (1″) and (2″). Consider any a in I and $x \leq a$. Note that $a \vee x = a$. According to condition (2″), $a \vee x$ in I implies $a \in I$ and $x \in I$. Thus condition (2′) is satisfied.

11.13 (i) For any c and d in $I(a, b)$

$$\begin{aligned} a \wedge c &= b \wedge c \\ a \wedge d &= b \wedge d \\ a \wedge (c \vee d) &= (a \wedge c) \vee (a \wedge d) \\ &= (b \vee c) \vee (b \wedge d) \\ &= b \wedge (c \vee d) \end{aligned}$$

Thus, $c \vee d$ is in $I(a, b)$.

(ii) For any c in $I(a, b)$ and any y in A

$$\begin{aligned} a \vee (c \wedge y) &= (a \wedge c) \wedge y \\ &= (b \wedge c) \wedge y \\ &= b \wedge (c \wedge y) \end{aligned}$$

Thus, $c \wedge y$ is in $I(a, b)$.

11.14 Let x be a complement of 0. By definition, $0 \vee x = 1$, $0 \wedge x = 0$. Also $0 \le x$, and $0 \vee x = x$. Thus, $x = 1$.

11.15 $a \vee (\bar{a} \wedge b) = (a \vee \bar{a}) \wedge (a \vee b) = a \vee b$

$a \wedge (\bar{a} \vee b) = (a \wedge \bar{a}) \vee (a \wedge b) = a \wedge b$

11.16 $(A, \oplus)$ is certainly closed. Commutativity is clear. Furthermore, 0 is the identity since $a \oplus 0 = a$, and a is its own inverse since $a \oplus a = 0$. As to associativity, we note that

$$\begin{aligned} (a \oplus b) \oplus c &= (((a \wedge \bar{b}) \vee (\bar{a} \wedge b)) \wedge \bar{c}) \vee (\overline{((a \wedge \bar{b}) \vee (\bar{a} \wedge b))} \wedge c) \\ &= (a \wedge \bar{b} \wedge \bar{c}) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee (\overline{(a \wedge \bar{b})} \wedge \overline{(\bar{a} \wedge b)} \wedge c) \\ &= (a \wedge \bar{b} \wedge \bar{c}) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee ((\bar{a} \vee b) \wedge (a \vee \bar{b}) \wedge c) \\ &= (a \wedge \bar{b} \wedge \bar{c}) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee (((a \wedge \mathrm{b}) \vee (\bar{a} \wedge \bar{b})) \wedge c) \\ &= (a \wedge \bar{b} \wedge \bar{c}) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee (a \wedge b \wedge c) \vee (\bar{a} \wedge \bar{b} \wedge c) \end{aligned}$$

A similar expansion for $a \oplus (b \oplus c)$ yields the same expression.

11.17 (a) (i) $a \bigstar a = e$ implies for all a, $a \le a$. (reflexivity)

(ii) $a \le b$ and $b \le a$ mean $a \bigstar b = e$ and $b \bigstar a = e$.
By Problem 12.13 (e), $a = b$. (antisymmetry)

(iii) $a \le b$ and $b \le c$ mean $a \bigstar b = b \bigstar c = e$.

$$\begin{aligned} a \bigstar c &= a \bigstar (e \bigstar c) = a \bigstar [(b \bigstar c) \bigstar c] \\ &= a \bigstar [(c \bigstar b) \bigstar b] = (c \bigstar b) \bigstar (a \bigstar b) \\ &= (c \bigstar b) \bigstar e = e. \text{ Thus, } a \le c \text{ (transitivity)} \end{aligned}$$

Hence, $\le$ is a partial enduring relation.

(b) If $b = x \bigstar a$, then $a \bigstar b = a \bigstar (x \bigstar a) = x \bigstar (a \bigstar a) = x \bigstar e = e$
If $a \bigstar b = e$, then $b = e \bigstar b = (a \bigstar b) \bigstar b = (b \bigstar a) \bigstar a = x \bigstar a$ where $x = b \bigstar a$.

(c) $a \bigstar [(a \bigstar b) \bigstar b] = e \Rightarrow a \le (a \bigstar b) \bigstar b$.
Similarly, $b \le (b \bigstar a) \bigstar a = (a \bigstar b) \bigstar b$.
Thus, $(a \bigstar b) \bigstar b$ is an upper bound for a and b.
Now suppose $a \le c$, $b \le c$. According to (b), $c = x \bigstar a$ for some x.

Also, $\qquad b \bigstar c = e.$

Hence

$$\begin{aligned} [(a \bigstar b) \bigstar b] \bigstar c &= [(a \bigstar b) \bigstar b] \bigstar (x \bigstar a) \\ &= x \bigstar [((a \bigstar b) \bigstar b) \bigstar a] \\ &= x \bigstar (b \bigstar a) \\ &= b \bigstar (x \bigstar a) = b \bigstar c = e. \end{aligned}$$

Hence $(a \bigstar b) \bigstar b \le c$
i.e., $(a \bigstar b) \bigstar b$ is a least upper bound.

11.18 (i) $(a \bigstar b) \bigstar a = \overline{(\bar{a} \vee b)} \vee a = (a \wedge \bar{b}) \vee a = a$

(ii) $(a \bigstar b) \bigstar b = \overline{(\bar{a} \vee b)} \vee b = (a \wedge \bar{b}) \vee b = a \vee b$

$(b \bigstar a) \bigstar a = \overline{(\bar{b} \vee a)} \vee a = (b \wedge \bar{a}) \vee a = a \vee b$

(iii) $a \bigstar (b \bigstar c) = \bar{a} \vee (\bar{b} \vee c)$

$b \bigstar (a \bigstar c) = \bar{b} \vee (\bar{a} \vee c)$

Hence, $(A, \bigstar)$ is an implication algebra.

11.19 $E(x_1, x_2, x_3) = (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3) \vee (\bar{x}_1 \wedge \bar{x}_2 \wedge x_3)$
$\vee (\bar{x}_1 \wedge \bar{x}_2 \wedge x_3)$

$E(x_1, x_2, x_3) = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$
$\wedge (\bar{x}_1 \vee x_2 \vee x_3)$

11.20 $E(x_1, x_2, x_3) = (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \bar{x}_2$
$\wedge \bar{x}_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3)$
$= (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$

11.21 $E(x_1, x_2, x_3, x_4) = (x_1 \wedge x_2 \wedge \bar{x}_3 \wedge x_4) \vee (x_1 \wedge x_2 \wedge \bar{x}_3 \wedge \bar{x}_4) \vee (x_1 \wedge \bar{x}_2 \wedge x_3$
$\wedge x_4) \vee (x_1 \wedge \bar{x}_2 \wedge \bar{x}_3 \wedge x_4) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3 \wedge \bar{x}_4)$
$= (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee x_2 \vee \bar{x}_3$
$\vee x_4) \wedge (x_1 \vee x_2 \vee \bar{x}_3 \vee \bar{x}_4)$
$\wedge (x_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee x_4) \wedge (x_1 \vee \bar{x}_2$
$\vee \bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4)$
$\wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\bar{x}_1 \vee x_2 \vee$
$\bar{x}_3 \vee x_4)$

11.22 (a) $(\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$

(b) $(x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$

11.23 (a) $b \wedge (a \vee c)$

(b) $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$

(c) c

(d) $(a \vee \bar{b}) \wedge c$

11.24 (a) The basis of induction is obvious. Let $E(x_1, x_2,\ldots, x_n)$ be an expression of length $k + 1$. We consider the following cases:

(i) $E(x_1, x_2,\ldots, x_n) = \overline{E_1(x_1, x_2,\ldots, x_n)}$, where $E_1(x_1, x_2,\ldots, x_n)$ is an expression of length k. According to the induction hypothesis:

$$E(x_1, x_2,\ldots, x_n) = \overline{(\bar{x}_i \wedge E_1(x_i = 0)) \vee (x_i \wedge E_1(x_i = 1))}$$
$$= (x_i \vee \overline{E_1(x_i = 0)}) \wedge (\bar{x}_i \vee \overline{E_1(x_i = 1)})$$
$$= (x_i \wedge \overline{E_1(x_i = 1)}) \vee (\bar{x}_i \wedge \overline{E_1(x_i = 0)})$$
$$= (x_i \wedge E(x_i = 1)) \vee (\bar{x}_i \wedge E(x_i = 0))$$

(ii) $E(x_1, x_2,\ldots, x_n) = E_1(x_1, x_2,\ldots, x_n) \wedge E_2(x_1, x_2,\ldots, x_n)$.
According to the induction hypothesis:

$$E(x_1, x_2,\ldots, x_n) = [(\bar{x}_i \wedge E_1(x_i = 0)) \vee (x_i \wedge E_1(x_i = 1))] \wedge$$
$$[(\bar{x}_i \wedge E_{e2}(x_i = 0)) \vee (x_i \wedge E_2(x_i = 1))]$$
$$= [\bar{x}_i \wedge E_1(x_i = 0) \wedge E_2(x_i = 0)] \vee$$
$$[x_i \wedge E_1(x_i = 1) \wedge E_2(x_i = 1)]$$
$$= (\bar{x}_i \wedge E(x_i = 0)) \vee (x_i \wedge E(x_i = 1))$$

(iii) $E(x_1, x_2,\ldots, x_n) = E_1(x_1, x_2,\ldots x_n) \vee E_2(x_1, x_2,\ldots, x_n)$.
(Similar to (ii).)

(b) Repeated applications of the result in (a) yield

$$c_{\delta_1\delta_2\ldots\delta_n} = E(x_1 = \delta_1, x_2 = \delta_2,\ldots, x_n = \delta_n)$$

where $\delta_i = 0$ if $\tilde{x}_i = \bar{x}_i$ and $\delta_i = 1$ if $\tilde{x}_i = x_i$.

(c) $(2 \wedge x_1 \wedge \bar{x}_2) \vee (2 \wedge x_1 \wedge x_2)$.

(d) Determine the disjunctive normal form from the 2^n values of $f(\delta_1, \delta_2,\ldots, \delta_n)$ where $\delta_i = 0$ or 1.

(e) According to $f(0, 0) = 1, f(0, 1) = 0, f(1, 0) = 1, f(1, 1) = 1$ from Figure 11.8, we should have

$$f(x_1, x_2) = (\bar{x}_1 \wedge \bar{x}_2) \vee (x_1 \wedge \bar{x}_2) \vee (x_1 \wedge x_2)$$

However, if that is the case

$$f(0, 2) = (\bar{0} \wedge \bar{2}) \vee (0 \wedge \bar{2}) \vee (0 \wedge 2) = 3$$

which is not consistent with the value of $f(0, 2)$ in Figure 11.8.

(f) Similar to (a).

(g) Similar to (b).

11.25 $((a \vee b) \wedge d) \vee ((a \vee b) \wedge (c \wedge d) \wedge \bar{e}) \vee [((a \vee b) \vee (c \vee d)) \wedge f] \vee g$
$= ((a \vee b) \wedge d) \vee ((a \vee b) \wedge c \wedge \bar{e}) \vee ((a \vee b \vee c \vee d) \wedge f) \vee g$

Condition (2) can be simplified as:

(2) He got a B or better in the mid-term examination and a B in the final examination and did not miss any homework assignment.

11.26 $(a \vee c) \wedge (a \vee b \vee d) \wedge (b \vee c \vee e) \wedge (b \vee e) \wedge d$

$$= (a \vee c) \wedge (b \vee e) \wedge d$$
$$= (a \wedge b \wedge d) \vee (a \wedge e \wedge d) \vee (c \wedge b \wedge d) \vee (c \wedge e \wedge d)$$

11.27 Corresponding to the five conditions, we have

(1) $(a \wedge \overline{b}) \vee (\overline{a} \wedge b)$

(2) $c \vee e$

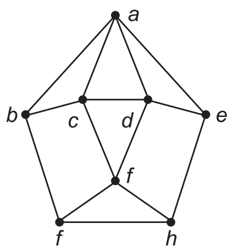
(3) $\overline{d} \vee b$

(4) $(a \wedge c) \vee (\overline{a} \wedge \overline{c})$

(5) $\overline{e} \vee (c \wedge d)$.

Simplifying the conjunction of these five expressions, we obtain $a \wedge \overline{b} \wedge c \wedge \overline{d} \wedge \overline{e}$ as the only way of selection.

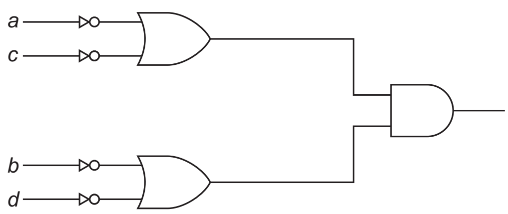
11.28



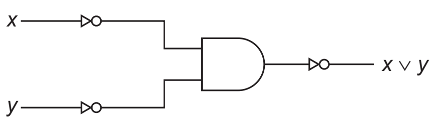
$(a \vee b \vee c \vee d \vee e) \wedge (b \vee a \vee c \vee g) \wedge (c \vee a \vee b \vee d \vee f) \wedge (d \vee a \vee c \vee e \vee f) \wedge (e \vee a \vee d \vee h) \wedge (f \vee c \vee d \vee g \vee h) \wedge (g \vee b \vee f \vee h) \wedge (h \vee e \vee f \vee g)$

$= (a \wedge f) \vee (a \wedge g) \vee (a \wedge h) \vee (b \wedge e \wedge f) \vee \ldots$

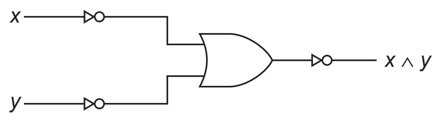
11.29



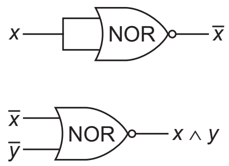
11.30 (a)



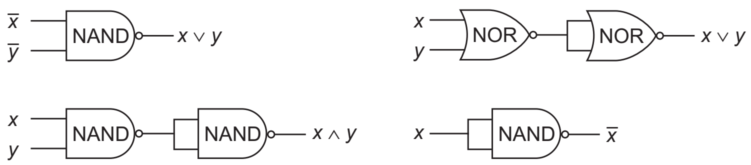
(b)



(c)



(d)



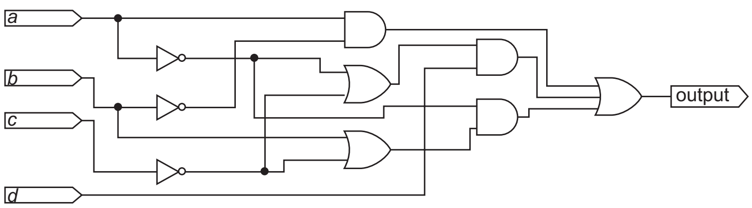
11.31 (a) The three simplified expressions are

(a) $(\overline{a} \wedge (b \vee \overline{c})) \vee (a \wedge \overline{b}) \vee (d \wedge (\overline{a} \vee \overline{c}))$

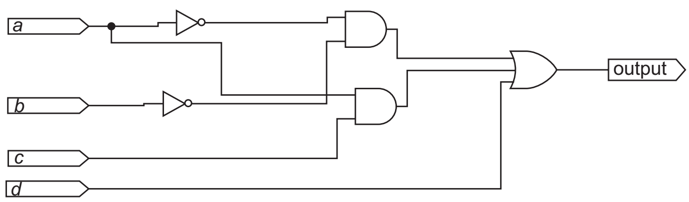
(b) $(\overline{a} \wedge \overline{b}) \vee (a \wedge c) \vee d$

(c) $(\overline{a} \wedge b) \vee (\overline{b} \wedge c)$

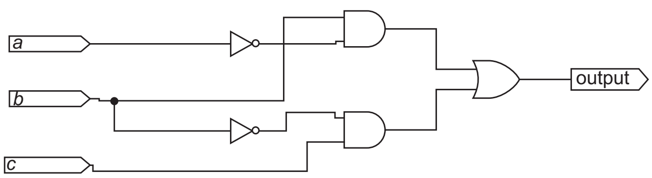
(b)



(a)



(b)



(c)