

Q. Why study Graph Theory?

For graph theory aspect:

1. In Data structure, we are concerned about how to store a graph in an efficient manner.
2. In Algorithms, we are concerned about the computational algorithms in graphs to find the shortest path, minimum spanning tree, etc.
3. In Discrete Mathematics, we are concerned about

(i) The terminologies

- (ii) Special types of graphs - from sets and relations perspective.
- (iii) Properties - we get them with the help of counting / combinatorics.

Some examples of graphs:-

1. Web graph - Web is itself a graph on which page rank algorithm works.
2. Networks (computers) - Computer network as a graph
3. Social graph - Facebook, twitter, etc.
4. Neural network - Graphs are used to mimic human brains.

Note: There are special databases called Graph databases whose primary job is to store large graphs like web graph, social graph, etc., and to operate on them.

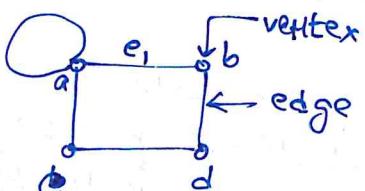
Q. What is a graph?

A graph ( $G$ ) is a 2 tuple, i.e., it consists of ~~two things~~:

- (i) vertices - can be thought of as towns / cities in road networks
- (ii) edges - can be thought of as the roads connecting two towns / cities. Roads can be two way or one way.

$$G_1 = (V, E)$$

$\downarrow$  set of vertices / nodes       $\downarrow$  set of edges



$$e_1 = \{a, b\} \text{ or } \{b, a\}$$

unordered pair

Fig: Undirected graph

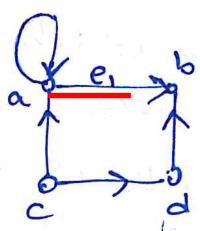


Fig: Directed graph

ordered pair (order is important)  
 $e_1 = (a, b)$   
 $e_1 \neq (b, a)$

The edge  $e_1$  is from a to b.

For directed graphs,  $G_d = (V, E)$ ,

$$E \subseteq V \times V$$

cross product

The set of edges ( $E$ ) is a subset of the cross product of two sets.

From set theory perspective we know that a relation is a subset of the cross product of two sets. Therefore we can think of directed graph edges forming a relation on the vertex set.

In case of undirected graph, we can have  $(a, b)$  or  $(b, a)$ .

Therefore we cannot think of edges forming relations because in relations  $(a, b)$  is different from  $(b, a)$ .

So undirected graphs can be thought of as set of vertices and a set of unordered pairs of vertices (edge set).

And similarly, directed graphs can be thought of as set of vertices and a set of ordered pairs of vertices (edge set).

### Edge-labelled graph:

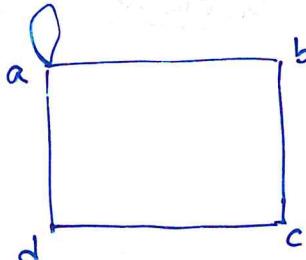


Fig: vertex-labelled graph

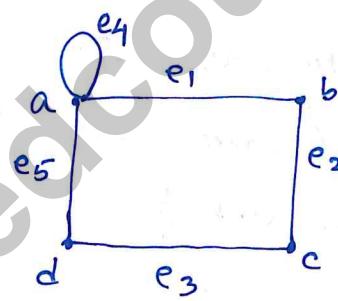
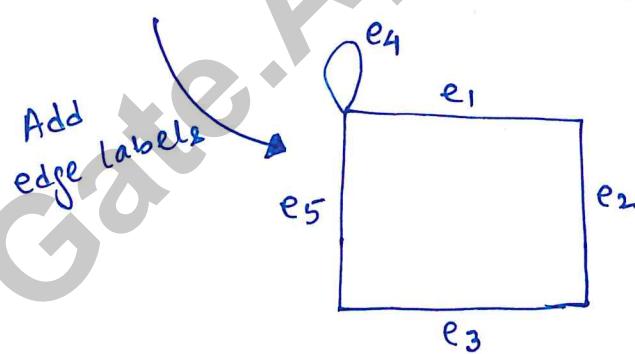


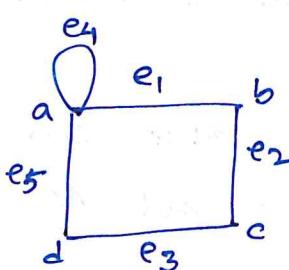
Fig: Labelled graph  
(Both vertex and edge are labelled)



In a vertex labelled graph, if we add labels to the edges then the graph is called as edge-labelled graph.

Adjacent vertices:

Adjacent vertices implies there exists an edge between the vertices.

G<sub>1</sub>:

The vertices a and b are adjacent because there exists an edge  $e_1$  connecting both the vertices.

The vertices a and c are not adjacent because there exists no edge connecting both the vertices.

We can also say, the vertices a and b are adjacent because the edge  $e_1$  is incident on vertices a and b.

Order of a graph:

$$O(G_1) = \# \text{vertices}$$

Means 'Number of'

$$= 4$$

Size of a graph:

$$\text{size } (G_1) = \# \text{edges}$$

= Number of edges

$$= 5$$

Mixed graph: The graph where some edges are directional and some are undirectional. So, the real world road networks are mixed graphs as some roads are one-way while others are bidirectional.

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- ~~(i)~~ No self-loops  
~~(ii)~~ No multiple edges

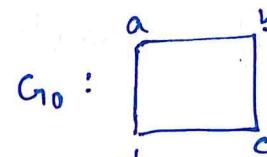
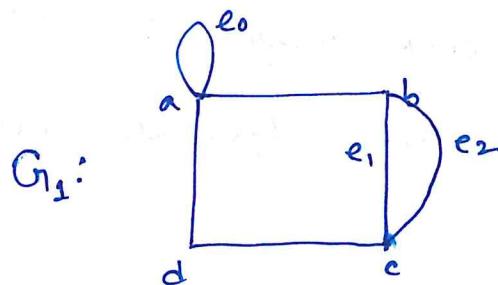


Fig: Simple graph



There exists an edge from vertex a to itself, this edge is called a self loop. Between vertices b and c, there are two edges. These denote the multiple edges.

~~(i)~~ Multi-graph: It is a graph which

- (i) Does <sup>NOT</sup> allow self loops  
(ii) Allows multiple edges.

In the graph  $G_1$ , we can see the multiple edges present between vertices b and c. But  $G_1$  is not a multi-graph since it has a self loop.

~~while~~  $G_2$  is a multi-graph.

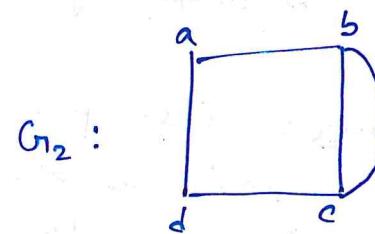


Fig: Multi-graph

The concept of multi-graph comes from multi-set in set theory, where a set containing same element repeating multiple times. Here  $E = \{ (b,c), (b,c), \dots \}$

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The unordered pair  $(b, c)$  is repeating in the edge set  $E$  for graph  $G_2(V, E)$ .

### Pseudo-graph:

A pseudograph is a non-simple graph in which both graph loops and multiple edges are permitted.

### Empty graph: } Null graph: }      Inconsistent terminology                                 Different authors use different definitions.

#### One possible definition:

1. Empty graph: An empty graph on  $n$  ~~no~~ vertices consists of  $n$  isolated vertices with no edges. Such graphs are also called edgeless graphs.
2. An Null graph: The empty graph with  $0$  ~~no~~ vertex.

Note: Null graph is used to refer to:-

- Note: Null graph is used to refer to:-
- (a) empty graph
  - (b) empty graph on  $0$  vertices.

Note: Therefore whenever empty graph / null graph is referred, their definitions should be clearly defined.

The graph  $G_1$  has:

- (i) NO self loops
- (ii) No multiple edges

Therefore graph  $G_1$  is a simple graph.

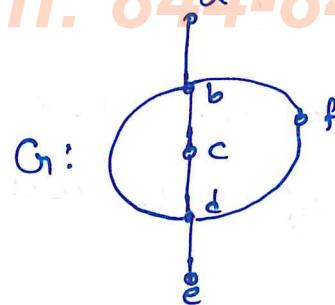


Fig: Simple Graph

### Degree (vertex) :

- Degree of a vertex,  $\deg(v) = \# \text{ edges incident on vertex } v$

Note: 1. Self loops count as 2.  
E.g., vertex b has degree 4.

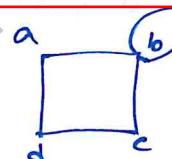


Fig: Undirected graph

2. If vertex has degree = 1, then it is called a pendant vertex.

3. A tree is special type of graph

where the leaf nodes have degree = 1. A leaf node is also called a pendant vertex.

4. A lone vertex (isolated vertex) has degree = 0.

In case of directed graphs, vertex b has indegree = 2 and outdegree = 2.

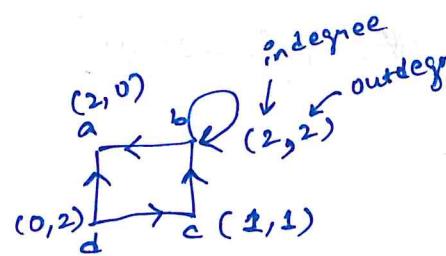


Fig: Directed graph

Note: 1. Sink : only incoming edges. Vertex a is a sink

2. Source : only outgoing edges. Vertex d is a source.

Degree sequence:

Given an undirected graph, we write down the degrees of each vertices in increasing / decreasing order.

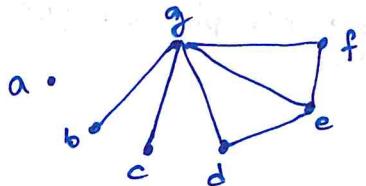


Fig: Undirected graph

$$\begin{aligned}\deg(g) &= 5 \\ \deg(e) &= 3 \\ \deg(f) &= \deg(d) = 2 \\ \deg(c) &= \deg(b) = 1 \\ \deg(a) &= 0\end{aligned}$$

∴ The degree sequence:  $[0, 1, 1, 2, 2, 3, 5]$

(or)

$[5, 3, 2, 2, 1, 1, 0]$

Note: For a given graph  $G_i$ , the degree sequence is unique.

But given a degree sequence, there can be multiple graphs with the same degree sequence.

Min-degree: It is the minimum degree of any graph. It is represented by  $\delta$ .

Max-degree: It is the maximum degree of any graph. It is represented by  $\Delta$ .

Given a simple undirected graph, the  $\delta(G_i) = \delta = \min$ . degree could be 0. And the max.degree  $\Delta(G_i)$  or  $\Delta$  could be  $(n-1)$ .

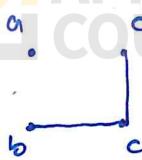


Fig: Graph with  
min-degree = 0  
for vertex a.

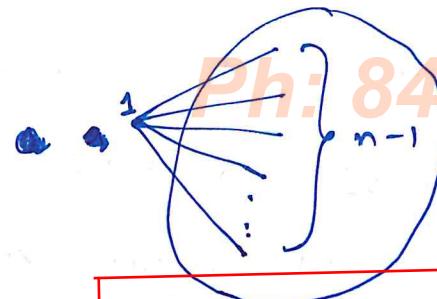
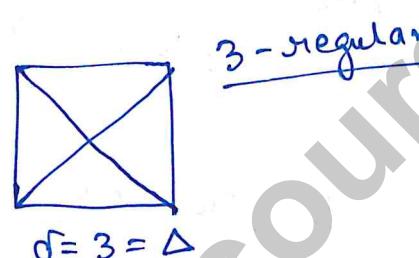
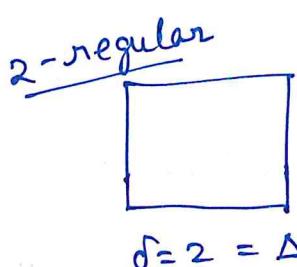


Fig: Graph with  
max-degree = (n-1)  
for vertex 1.

(\*) Regular Graph: A regular graph is a graph where  
min. degree ( $\delta$ ) = max. degree ( $\Delta$ )



(\*) Handshaking Theorem (undirected graph):

The sum of degrees of vertices is equal to the twice the size of the graph. Size of graph is the number of edges.

$$\sum_{v_i \in G} \deg(v_i) = 2 * \text{size}(G) \\ = 2 * |E|$$

This theorem holds for any undirected graph (simple or non simple with self loops or non simple with multiple loops)

Note: Corollaries (results) that follow from the above theorem:-

- (i) Sum of degrees is always even
- (ii) No graph can have odd number of vertices of odd degree. (or) # odd degree vertices is even.

The number of odd degree vertices = 5

By add the 5 odd degree vertices, we don't get even number. Thus the 1st corollary is not satisfied.

### (\*) Maximum number of edges (simple undirected graph):

For combinatorial point of view, we have  $n$  vertices, and we can choose  $n^2$  vertices and create an edge between them, i.e.,  $nC_2$ .

Another way to find max. number of edges:-

$$\begin{aligned} & \text{1. } \quad \begin{array}{c} \cdot 2 \\ \cdot 3 \\ \cdot 4 \\ \vdots \\ \cdot n \end{array} \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} (n-1) \\ & \quad (n-1)+(n-2)+(n-3)+\dots+0 \\ & \quad = \frac{n(n-1)}{2} \\ & \quad = nC_2 \end{aligned}$$

Note: If a simple undirected graph consists of  $nC_2$  edges, i.e., for every pair of vertices there is an edge (or) all the possible edges are present, such a graph is called a complete graph.

$$[(v_i, v_j) \forall v_i, v_j \text{ and } v_i \neq v_j]$$

And a complete graph of  $n$  vertices is called as  $K_n$



For a k-regular graph, every vertex has k-edges.

Therefore, sum of all degrees =  $n * k = 2 * |E|$

$$\therefore |E| = \# \text{ of edges} = \frac{nk}{2}$$

Note: An  $(n-1)$ -regular graph is a complete graph  $K_n$ .

In this case,  $k = (n-1)$

$$\therefore |E| = \frac{nk}{2} = \frac{n(n-1)}{2} = {}^n C_2$$

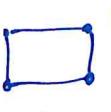
We know that a simple undirected graph with  ${}^n C_2$  edges is a complete graph  $K_n$ .

$$\therefore \boxed{(n-1)-\text{regular graph} = K_n}$$

Note: (1) If a graph is a complete graph  $\Rightarrow$  regular graph

$$K_n \Rightarrow (n-1)-\text{regular graph}$$

(2) A regular graph  $\not\Rightarrow$  complete graph  
 (need not be)

E.g.,  } 2-regular graph  
 But it is not a complete graph.

\* Handshaking Theorem (Directed graph):

{ For a given directed graph, every edge will be counted once as indegree and once as outdegree.

That means the sum of indegrees and sum of outdegrees is same.  $\sum_{v_i \in G} \text{indeg}(v_i) = \sum_{v_i \in G} \text{outdeg}(v_i)$

For any vertex,  $\deg(v) = \text{indegree}(v) + \text{outdegree}(v)$



$$\therefore \sum_{v_i \in G} \text{Indeg}(v_i) = \sum_{v_i \in G} \text{Outdeg}(v_i) = |E|$$

\* Degree sequence of directed graph :-

$(1, 2), (2, 2), (2, y), (x, 2)$   
 ↗ ↗  
 (indegree, outdegree) ... ... (indegree, outdegree)  
 of of  
 1st vertex 4th vertex

To find the relation between  $x$  and  $y$ .

We know that the sum of indegrees = sum of outdegrees

$$\Rightarrow 1 + 2 + 2 + x = 2 + 2 + y + 2$$

$$\Rightarrow x = y + 1$$

\* Hypercube graph ( $Q_n$ ) :-

Order  $O(Q_n) = 2^n = \# \text{ of vertices}$

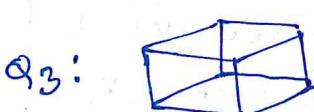
A hypercube graph ( $Q_n$ ) has  $2^n$  vertices and it is  $n$ -regular.

$Q_1:$



$Q_2:$

It is 3-regular  
and it has  $2^3$  vertices



$Q_3:$

It is a structure in 3D Space

Cube is a structure in 4D space which we can't visualize.

$Q_4:$  Hypercube is structure in 4D space which we can't visualize.

$$2 * \# \text{ edges} = \frac{\text{sum of the degrees}}{\# \text{ of vertices}} = \frac{2^n * n}{n}$$

# of vertices      degree of each vertex

$$\Rightarrow \# \text{ edges} = \left( \frac{2^n \cdot n}{2} \right) = (2^{n-1} \cdot n)$$

### (\*) Havell - Hakimi Theorem / Algorithm :

Purpose: It is used to check if a given degree sequence is valid degree sequence of a simple graph or not.

Graphical sequence: It is a term used for degree sequence of a simple graph.

Given a degree sequence in decreasing order, we want to find whether this sequence is a degree sequence of a simple graph.

Eg1.  $(\underline{5}, \underline{3}, \underline{3}, \underline{3}, 2, 2, 1)$

# Odd degree vertices = 5

This is impossible as the number of odd degree vertices should be even, which means this can never be a graphical sequence.

Eg2.  $(\underline{5}, \underline{4}, \underline{3}, \underline{3}, 3)$

# Odd degree vertices = 4

Even though the number of odd degree is even, but this sequence is not a graphical sequence.

Because for any given graph, according to the Handshaking theorem, we will have even # of odd-degree vertices. But if we have even # of odd-degree vertices, this does not imply that it is a valid graph.

Graph  $\Rightarrow$  even # of odd degree vertices  
 $\nLeftarrow$

E.g. S: (5, 3, 3, 3, 2, 2, 1, 1)

The given degree sequence (S) does not violate the handshaking theorem as the number of odd degree vertices is even.

Now let us try to find if the given sequence is a valid degree sequence (graphical sequence) for a simple graph (graphical sequence) or not.

According to Havel-Hakimi theorem:

According to Havel-Hakimi theorem:  
① sort all the vertices in the decreasing order of degrees.

② Pick the largest element. Remove it from the list and construct the list again.

Let the largest element be  $d_1$ .

$\therefore d_1 = 5$   
we remove  $d_1$  from the list and reconstruct the list.

(3, 3, 3, 2, 2, 1, 1)

⑥ Since  $d_1 = 5$ , take the first 5 numbers from the reconstructed list and subtract 1 from 5 numbers.

$$(\overbrace{3, 3, 3, 2, 2}^{\downarrow}, 1, 1)$$

$$S_2: (2, 2, 2, 1, 1, 1, 1)$$

According to Harrel-Hakimi theorem, sequence  $S_1$  is a valid graphical sequence IFF sequence  $S_2$  is a valid graphical sequence.

③ Repeat steps (2a) and (2b) till the time we know whether the subsequent sequences are graphical sequences or not.

$$\therefore S_2: (2, 2, 2, 1, 1, 1, 1)$$

$$\therefore d_1 = 2$$

$$\therefore (2, 2, 1, 1, 1, 1)$$

Now, since  $d_1 = 2$ ,

~~$(\overbrace{2, 2, 1, 1, 1, 1}^{\downarrow})$~~

$\therefore (\overbrace{1, 1, 1, 1, 1, 1}^{\downarrow})$

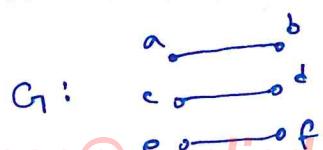
Now, if  $S_3$  is a valid graphical sequence,

then  $S_2$  is a valid graphical sequence.

And if  $S_2$  is a valid graphical sequence,

then  $S_1$  is a valid graphical sequence.

We can draw the graph for the sequence  $S_3$ .



$$\deg(a) = \deg(b) = \deg(c) = \deg(d)$$

$$= \deg(e) = \deg(f) = 1$$

It is a simple graph with no self loops and no multiple paths.

$s_3 : (1, 1, 1, 1, 1, 1)$  is a valid graphical sequence.

$\therefore s_2$  and  $s_2$  are also valid graphical sequences.

Eg. 4.  $s_1 : (1, 0, 0)$

It is not a valid graphical sequence because

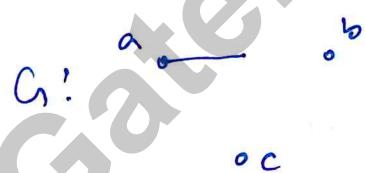
- (i) It violates the handshaking theorem, as the number of odd degree vertices is not even.
- (ii) Using Harwell-Hakimi theorem:-

$$d_1 = 1$$

$$s_2 : (-1, 0)$$

Degree of a vertex can never be less than zero.

$$\deg(v) \geq 0$$



The edge started from vertex 'a' has to end somewhere, but none of the other vertices can have an incident edge since their degree is zero.

$\therefore$  The degree sequence  $s_1$  is invalid.

E.g. 5.  $S_1: (5, 4, 3, 3, 3)$   
 (i)  $S_1$  does not violate Handshaking theorem.  
 (ii) Using Harrell-Hakimi theorem:

$$d_1 = 5$$

$$S_2: (3, 2, 2, 2, \frac{X}{\uparrow})$$

No fifth element

$\therefore S_2$  is not a valid graphical sequence.

$\therefore S_1$  is not a valid graphical sequence.

E.g. 6.  $S_1: (5, 4, 3, 3, 3, 0, 0, 0, 0)$   
 (i) It does not violate Handshaking theorem.

(ii) Using Harrell-Hakimi theorem:

$$d_1 = 5$$

$$S_2: (3, 2, 2, 2, \frac{-1}{X}, 0, 0, 0)$$

We cannot have  $-1$  as a degree of a vertex.  $\therefore S_2$  is not a valid graphical sequence.

$\therefore S_1$  is also not a graphical sequence.

E.g. 7.  $S_1: (5, 5, 4, 4, 3, 2, 2, 1)$

(i) It does not violate Handshaking theorem

(ii) Using Harrell-Hakimi theorem:

$$\therefore d_1 = 5$$

$$S_2: (4, 3, 3, 2, 1, 2, 1)$$

We need to re-arrange  $S_2$  in decreasing order

$$\therefore S'_2: (4, 3, 3, 2, 2, 1, 1)$$

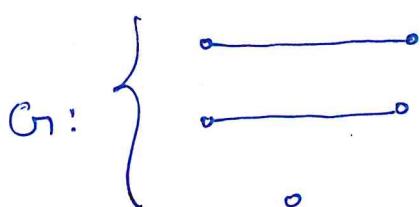
$$\therefore d_1 = 2$$

$$S_3 : (2, 2, 1, 1, 1, 1)$$

we re-arrange  $S_4$  in decreasing order :-

$$S_4' : (1, 1, 1, 1, 0)$$

We can continue ~~too~~ with Harrell-Hakimi (orc)  
we can make an attempt at drawing the graph.



The graph  $G_1$  is a valid simple graph.

$\therefore S_4'$  is a valid graphical sequence.

$\therefore S_3, S_2$  and  $S_1$  are also graphical sequences.

(\*) Harrell-Hakimi Theorem (Definition) :

If  $d_1, d_2, d_3, \dots, d_n$  is a decreasing sequence of non-negative integers, this is a graphical sequence IFF the resultant sequence

$$d_2 - 1, d_3 - 1, d_4 - 1, \dots, d_{\frac{d_1}{2}+1} - 1, d_{\frac{d_1}{2}+2}, d_{\frac{d_1}{2}+3}, \dots, d_n$$

$d_1$  elements

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E.g., For the given ~~graphical~~ sequence

$$S_1: (7, 6, 5, 4, 3, 2, 1, 0)$$

It is not a graphical sequence because by Havel-Hakimi theorem,

$$d_1 = 7$$

$$S_2: (5, 4, 3, 2, 1, 0, \underline{\underline{x}})$$

$\uparrow$  Degree of a vertex

cannot be less than zero.

In sequence  $S_1$ , there is no repetition.

Every graphical sequence must have ~~at least~~ at least one repetition.

But having one repetition does not make the sequence a graphical sequence.

E.g.,  $S_1: (5, 4, 3, 3, 3)$

$S_1$  has repetition but it is not a graphical

sequence by Havel-Hakimi theorem.

As  $d_1 = 5$  and  $S_2: (\underbrace{3, 2, 2, 2}_{\text{only 4 numbers are}}, \underline{\underline{x}})$

present in the sequence

$\text{MIN}$

\* Average Degree Theorem ( $\bar{d}, \Delta$  Theorem) :-

Given a graph  $G_1$ , let  $|V| = n$  and  $|E| = e$ ,

minimum degree  $\leq$  avg. degree  $\leq$  maximum degree

$$\Rightarrow \bar{d} \leq \frac{\text{sum of all the vertices}}{\text{number of vertices}} \leq \Delta$$

$$\Rightarrow \bar{d} \leq \frac{2e}{n} \leq \Delta$$

Given  $O(G) = 10 = n$

$$\delta = 3$$

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what is the minimum number of edges in the graph?

$$\delta \leq \frac{2e}{n}$$

$$\Rightarrow \frac{3 \times 10^5}{2} \leq e$$

$$\Rightarrow e \geq 15$$

$\therefore$  Minimum number of edges is greater than or equal to 15.

E.g. Given  $n=10, e=16$

what is the minimum and maximum degree?

We know,

$$\delta \leq \frac{2e}{n} \leq \Delta$$

$$\Rightarrow \delta \leq \frac{32}{10} \leq \Delta$$

$$\Rightarrow \delta \leq 3.2 \leq \Delta$$

$\therefore$  Maximum possible value of min. degree is 3 and minimum possible value of max. degree is 4

$$\boxed{\delta \leq 3} \text{ and } \boxed{\Delta \geq 4}$$

$$\therefore \delta \leq \left\lfloor \frac{2e}{n} \right\rfloor \text{ and } \Delta \geq \left\lceil \frac{2e}{n} \right\rceil$$

Floor ceil

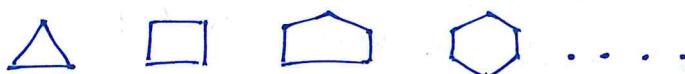
✳ Cycle Graph ( $C_n$ ) :

$C_n$  is a cycle graph with  $n$  vertices and  $n \geq 3$ .

$$n = |V| \text{ and } |E| = n$$

Cycle graphs are 2-regular. Cycle graphs are polygons.

E.g,



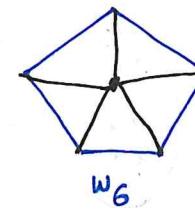
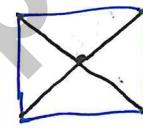
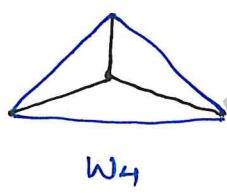
✳ Wheel Graph ( $W_n$ ) :

$W_n$  is a wheel graph with  $n$  vertices and

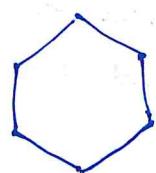
$$n \geq 4 \text{ & } [e = 2(n-1)].$$

Construction of wheel graph :

cycle graph + one more vertex which is adjacent to all other vertices.



To construct  $W_n$ , we use  $C_{n-1}$



$C_i : C_{n-1}$

$C_{n-1}$  has  $(n-1)$  edges. Now we add one more edge which is adjacent to all other vertices.



$G :$

Due to the addition of the new vertex, more

$(n-1)$  edges are added in the graph  $G_i$ .

Thus  $W_n$  has  $2(n-1)$  edges.

#### ④ Bipartite graph:

In a graph  $G_i(V, E)$ , if we can break the vertex set  $V$  in such a way that:

- (i)  $V = V_1 \cup V_2$ , where  $V_1$  &  $V_2$  are sets
- (ii)  $V_1 \cap V_2 = \emptyset$ , where  $V_1$  &  $V_2$  are disjoint sets

and if we have an edge  $(a, b)$  then such that  $a \in V_1$ ,  $b \in V_2$  or vice versa, i.e., vertex  $a$  and  $b$  cannot belong to the same set.

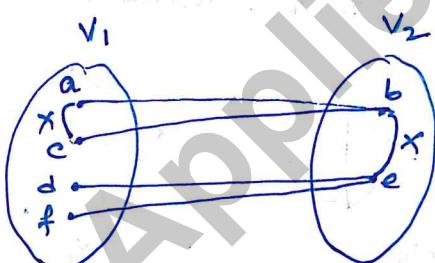


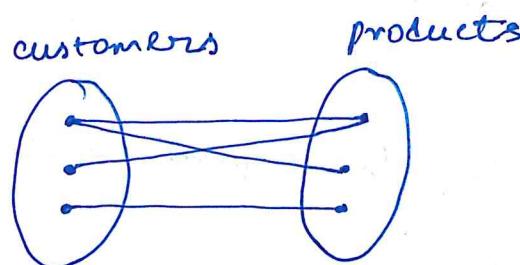
Fig: Bipartite graph  $G_i$ .

In this above figure, there cannot be an edge  $(a, c)$  and  $(b, e)$  since vertex  $a$  and  $c$  belong to set  $V_1$ , and vertex  $b$  and  $e$  belong to set  $V_2$ .

Note: Null Graph ( $\emptyset_n$ ) is a bipartite graph.

Because in Null graph we have  $n$  vertices and no edges. So, we can break/partition the vertices into two sets and no edges violate the edge property of bipartite graph.

- (i) At e-commerce companies lot of interesting data analysis is done by considering all the customers as one set and all the products as the other set.



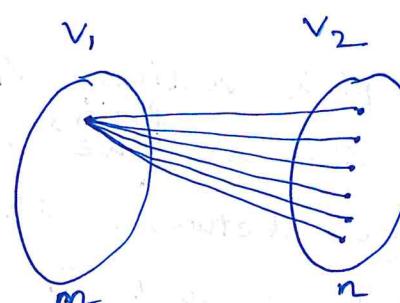
- (ii) We can build a recommendation system using machine learning and graph theory.  
Recommendation system: Given a customer what are the new products that we can suggest. Recommendation systems are very popular and can be seen in Amazon, Facebook websites etc.

### \* Complete Bipartite Graph:

It is a bipartite graph which has total number of vertices of  $(m+n)$ . And total number of edges of  $(m \times n)$ .

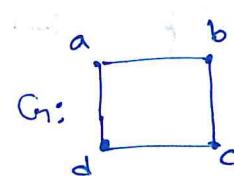
$$\therefore |V| = m+n$$

$$|E| = m \times n$$



### \* Sub Graph:

$$\text{Given } G_1 = (V, E)$$



Say we construct a new graph  $G_1 = (V_1, E_1)$

such that  $V_1 \subseteq V$  and  $E_1 \subseteq E$

We can construct a subgraph  $G_1$  from graph  $G$  in the following ways:-

- (i) we pick edges  $E_1 \subseteq E$  and if the vertex set  $V_1 \subseteq V$  contains the corresponding incident vertices, then the subgraph  $G_1$  is a valid subgraph.

E.g., From graph  $G$ , lets pick edges  $\{e_1, e_4\}$

$$\therefore E_1 = \{e_1, e_4\}$$

Edges  $e_1$  and  $e_4$  are incident on vertices  $(a, b)$  and  $(a, d)$  respectively.

$\therefore$  If  $V_1 = \{a, b, d\}$ , then the subgraph  $G_1$  is a valid subgraph.

E.g.

Note: If  $V_1 = \{a, b\}$ , then the subgraph  $G_1$  is not a valid subgraph.

- (ii) we pick vertices  $V_1 \subseteq V$ .

In this case, for  $V_1$ , we need not have all the edges between the same vertices in graph  $G$  to be present in subgraph  $G_1$  for subgraph  $G_1$  to be a valid subgraph.

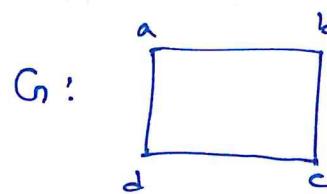
$\therefore$  If  $V_1 = \{a, b, d\}$ , then  $E_1 = \{e_1\}$  or  $\{e_1, e_4\}$  or  $\emptyset$

are all valid subgraphs.

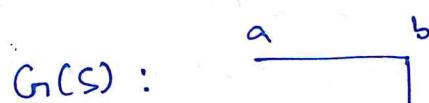
### \* Induced Subgraph:

An induced subgraph is a subgraph where for a subset  $S \subset V$ , all the edges between the vertices of  $S$  should be the same edges between the same vertices of  $V$ .

E.g.



$G_1(S)$  is an induced subgraph

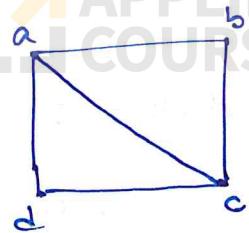
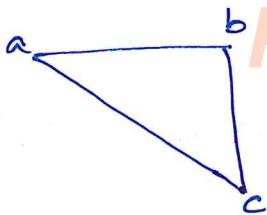


$G_{11}$  is a valid subgraph but not an induced subgraph as graph  $G_1$  has an edge  $(a,d)$  which is missing in the subgraph  $G_{11}$ .

### \* Clique:

A clique is an induced subgraph such that the induced subgraph is complete.

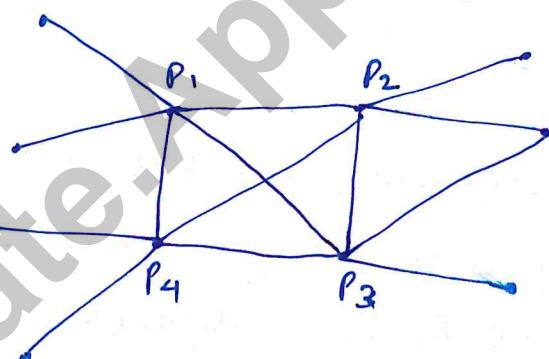
Note: A complete graph is a graph where every pair of vertices has an edge.

Simple graph  $G_2$  $K_3$ 

For the simple graph  $G_2$ ,  $K_3$  is an induced subgraph, and is complete. Therefore  $K_3$  is a clique.

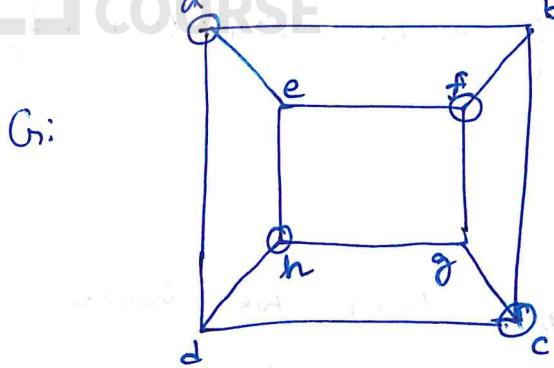
### Application of clique:

In social networks like Facebook, there are billions of people. If we represent each person as a vertex, for form classmates/close friends, we will see a clique being formed. So clique in social networks tell us about the strength of friendship (since every friend in the clique will be connected to others).



### Independent Set:

Independent set can be thought of as anti-clique. It is a subset of vertices such that no two of them are adjacent.

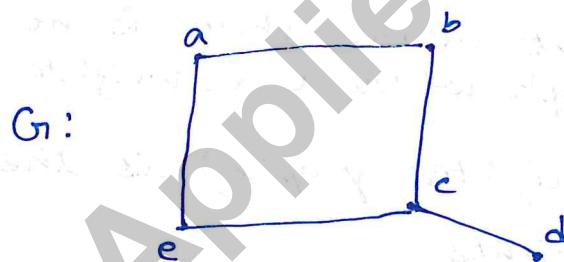


for the graph  $G_1$ , the subset of vertices i.e.,  
 $\{a, f, h, c\}$  form an independent set as no two  
vertices are adjacent to each other.

④ Walk and Paths (Undirected Graph)

A walk is a sequence of vertices such that between every pair of vertices there is an edge.

between every pair of vertices in an undirected graph.



WALK:  edge  
 start a → b → c → e → a → b → c → d end

A path is a walk where vertices do not repeat except when start and end are the same vertex

path : a b c d  
          a b c e a

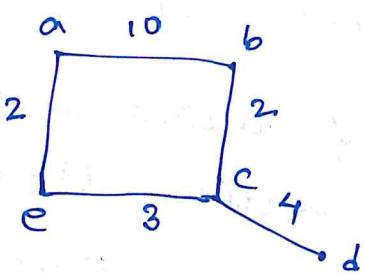
It is also called as a simple path.

path length = # edges

**Note:** For directed graphs, we have directed paths which internally uses directed edges.

### ④ Weighted Graph:

A weighted graph is a graph where we have numeric values given to each edge of the graph. The weights can represent distance between two vertices.



The edge weight can be positive, negative values. If it is zero, that means there is no edge (in the context of a graph where edge weights mean distance).

Path length (in weighted graphs) is the sum of the edge weights

Note: We encounter weighted graphs in Data Structures and Algorithms where we design shortest path Algorithms, minimum spanning tree algorithms etc.

In graph theory mostly we will study unweighted graphs (As far as GATE is concerned).

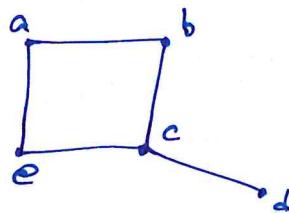
Note: In an unweighted graph, the path length is the number of edges itself.

### \* Shortest Path:

A shortest path between vertices  $u$  and  $v$  is a simple path with minimum # edges.

The length of the shortest path  $u$  to  $v = d(u, v)$

E.g.,



$$\begin{aligned}d(a, d) &= 3 \\d(a, b) &= 1 \\d(a, c) &= 2\end{aligned}$$

### \* Eccentricity:

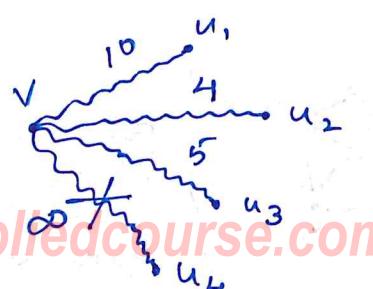
$$e(v) = \max_u d(v, u) \quad \forall u \in G$$

Eccentricity of a vertex  $v$  is defined as the maximum over for all the  $u$ , the shortest distance between  $v$  to  $u$ .

$a \xrightarrow{\text{an edge}} b$

denotes an <sup>edge</sup> between vertices  $a$  &  $b$

$a \xrightarrow{\text{a path}} b$  denotes a path between vertices  $a$  &  $b$



If there is no path between vertices  $v$  and  $u_4$ , the  $e(v) = \infty$  as eccentricity takes the maximum value.

a b denotes a directed path between vertices a and b in a directed graph.

### (\*) Radius ( $G_1$ ):

The radius of a graph is the minimum eccentricity across all the vertices.

$$\text{Radius (Graph } G_1) = \min(\epsilon(v_1), \epsilon(v_2), \epsilon(v_3), \epsilon(v_4), \dots, \epsilon(v_n))$$

$$\boxed{\text{Radius } (G_1) = \min_{v \in V} \epsilon(v)} \quad \text{where } V = \text{set of vertices.}$$

$$\text{Radius } (G_1) = \min_v \left( \max_u d(v, u) \right) \quad \forall u, v \in G_1$$

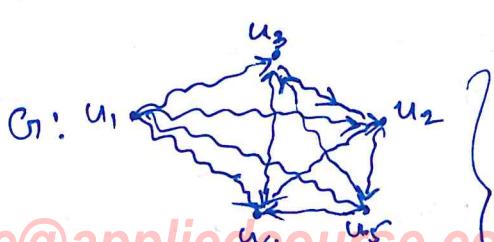
### (\*) Diameter ( $d(G_1)$ ):

The diameter of a graph,  $d(G_1)$ , is the maximum eccentricity across all the vertices.

$$d(G_1) = \max_v \epsilon(v) = \max(\epsilon(v_1), \epsilon(v_2), \dots, \epsilon(v_n))$$

= maximum shortest distance between a pair of vertices  $(u, v) = \max_{u, v} d(u, v)$

where  $u, v \in V$ .



} for all possible pair of vertices  $(u, v)$  find the shortest distances and take the maximum value.

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$d(G)$  = largest of the shortest paths between any pair of vertices.

= shortest path length between two farthest vertices.

vertices which have the largest shortest path

For a null graph with 4 vertices ( $\Phi_4$ ),

$$\text{diameter } (\Phi_4) = \infty$$

Because, shortest path between all vertices to the other vertices is  $\infty$ .  
 $\therefore$  Diameter is  $\infty$ .

$$\therefore \text{Diameter } (\Phi_n) = \infty$$

Likewise, for a complete graph ( $K_n$ ),  
 $\text{diameter } (K_n) = 1$ , because  $d(v_i, v_j) = 1$   
between any two vertices.

In case of cycle graph ( $C_n$ ),

for  $C_3$ : 

$$\text{Diameter } (C_3) = 1 \quad \therefore C_3 = K_3$$

for  $C_4$ : 

$$\text{Diameter } (C_4) = 2$$

for  $C_5$ :



$$\text{Diameter } (C_5) = 2$$

for  $C_6$ :



$$\text{Diameter } (C_6) = 3$$

$$\therefore \text{Diameter } (C_n) = \left\lceil \frac{n}{2} \right\rceil$$

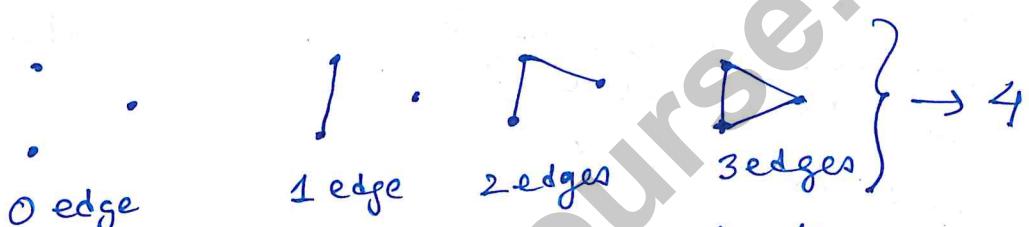
floor



Q. The number of distinct simple graphs with upto three nodes is :

- A. 15
- B. 10
- C. 7
- D. 9

Soln. For 3 vertices :-

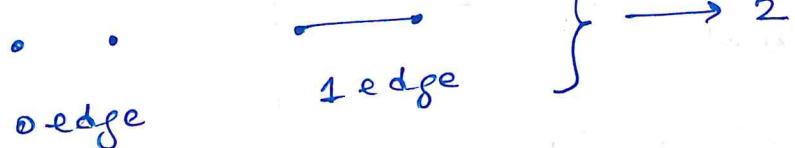


Note: The question does not mention about the vertices being labelled (or labelled graph). In case of a Labelled graph,



we have to count the above cases as different.

For 2 vertices :-



For 1 vertex :-



$$V = \emptyset, E = \emptyset \quad \} \rightarrow 1$$

Q. Total number of distinct simple graphs with upto three nodes (vertices) is 8.

Note: Some authors don't consider 0 vertex to be a graph. Therefore out of the available options, option C, i.e., 7 is the best choice.

∴ Answer is C.

Q. How many undirected graphs (not necessarily connected) can be constructed out of a given set  $V = \{v_1, v_2, v_3, \dots, v_n\}$  of n vertices?

A.  $\frac{n(n-1)}{2}$

B.  $2^n$

C.  $n!$

D.  $2^{\frac{n(n-1)}{2}}$

Soln. Given number of edges =  $n$   
 max # edges =  $\binom{n}{2}$  (as the graph is undirected graph).  
 $= \frac{n(n-1)}{2}$

$\therefore |E|_{\text{max}} = \frac{n(n-1)}{2}$

Now when we are constructing a graph, we can select any subset of edges from  $|E|_{\text{max}}$ . From counting, we know that for a set of size  $K$ , # subsets =  $2^K$ .

$\therefore$  Number of graphs can be constructed

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$$= 2^{\binom{n(n-1)}{2}}$$

Answer is D.

Q. How many graphs on  $n$  labeled vertices exist which have atleast  $\binom{n^2-3n}{2}$  edges?

A.  $\binom{\binom{n^2-n}{2}}{C_{\frac{n^2-3n}{2}}}$

B.  $\sum_{k=0}^{\binom{n^2-3n}{2}} \binom{n^2-n}{2} C_k$

C.  $\binom{\binom{n^2-n}{2}}{C_n}$

D.  $\sum_{k=0}^n \binom{\binom{n^2-n}{2}}{C_k}$

Soln Given  $n$  labeled vertices, i.e., every vertex has a label:

$$v_1, v_2, v_3, \dots, v_n$$

Since it's not mentioned, we are assuming the graph to be a simple and undirected.

For  $n$  vertices, the maximum number of edges =  $\frac{n(n-1)}{2} = \frac{n^2-n}{2} = a$  (let)

Given that the graphs should have atleast  $\binom{n^2-3n}{2}$  edges =  $b$  (let)

# edges =  $e$  (let)

$$\therefore \frac{n^2 - 3n}{2} \leq e \leq \frac{n^2 - n}{2}$$

$$\Rightarrow b \leq e \leq a$$

Note: If we calculate :

$$\begin{aligned} & a - b \\ \Rightarrow & \frac{n^2 - n}{2} - \frac{n^2 - 3n}{2} \\ \Rightarrow & \frac{n^2 - n - n^2 + 3n}{2} \\ \Rightarrow & \frac{2n}{2} \\ \Rightarrow & n \end{aligned}$$

$$\therefore \boxed{a - b = n}$$

Out of  $a$  edges we can select the following edges to construct a graph :

$$\begin{array}{l} b \text{ edges} \rightarrow {}^a C_b = {}^a C_{a-b} \\ (b+1) \text{ edges} \rightarrow {}^a C_{b+1} = {}^a C_{a-b-1} \\ (b+2) \text{ edges} \rightarrow {}^a C_{b+2} \\ \vdots \\ a \text{ edges} \rightarrow {}^a C_a \end{array}$$

$$\begin{aligned} & = {}^a C_n = \frac{n(n-1)}{2} C_n \\ & = {}^a C_{n-1} = \frac{n(n-1)}{2} C_{n-1} \\ & = {}^a C_{n-2} \\ & \vdots \\ & = {}^a C_0 \end{aligned}$$

Note: From combinatorics, we know that,

$$\boxed{{}^n C_k = {}^n C_{n-k}}$$

Now, when we add all the terms :

$${}^a C_n + {}^a C_{n-1} + {}^a C_{n-2} + \dots + {}^a C_0 *$$

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$$\sum_{k=0}^n C_k = \sum_{k=0}^n \frac{n(n-k)}{2} C_k$$

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$$= \boxed{\sum_{k=0}^n \frac{n^2-n}{2} C_k} \leftarrow \underline{\text{Answer}}$$

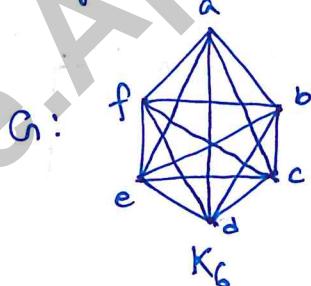
$\therefore$  Option D is the Answer.

Q. Let  $G_2$  be a complete undirected graph on 6 vertices. If vertices of  $G_2$  are labeled, then the number of distinct cycles of length 4 in  $G_2$  is equal to

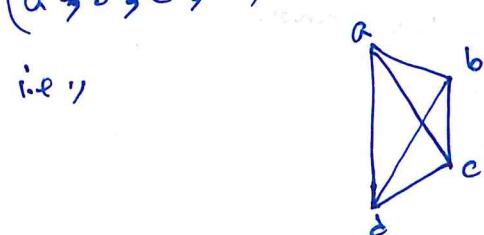
- A. 15
- B. 30
- C. 90
- D. 360

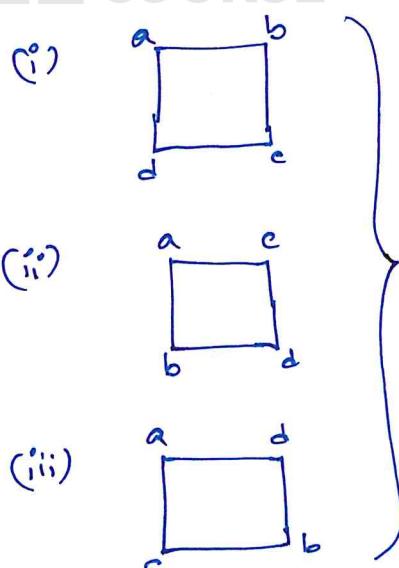
Soln. A cycle is a sequence of edges that start and end at the same vertex.

The given graph is  $K_6$



Let's consider that the cycle is formed using  $(a \rightarrow b \rightarrow c, d)$  vertices (4 vertices).





3 distinct cycles because of the labels.

Now we can choose 4 vertices out of 6 vertices in  $K_6$  in the following ways :-

$${}^6C_4$$

∴ Total number of distinct cycles of length 4 in a complete undirected graph of 6 vertices

$$= {}^6C_4 * 3$$

$$= 15 * 3$$

$$= 45 \leftarrow \text{Answer}$$

Note: None of the options given are correct. And everyone got marks for attempting this question.

Q. A graph  $G = (V, E)$  satisfies  $|E| \leq 3|V| - 6$ . The min-degree of  $G$  is defined as  $\min_{v \in V} \{\text{degree}(v)\}$ . Therefore min-degree of  $G$  cannot be :-

- A. 3
- B. 4
- C. 5
- D. 6

Soln.

$$\delta \leq \text{avg-degree} \leq \Delta$$

$$\therefore \delta \leq \text{avg-degree}$$

$$\delta \leq \frac{2 \cdot e}{n}$$

It is given that  $|E| \leq 3|V| - 6$ ,

$$\therefore \delta \leq \frac{2 \cdot e}{n} \leq \frac{2}{n}(3n - 6)$$

$$\Rightarrow \boxed{\frac{n\delta}{2} \leq 3n - 6}$$

$\therefore$  The graph G has to satisfy this criteria.

Let's try with given options :-

A)  $\delta = 3$

$$\frac{n\delta}{2} \leq 3n - 6$$

$$\Rightarrow \frac{3n}{2} \leq 3n - 6$$

$$\Rightarrow \frac{3n}{2} - 6 \geq 0$$

$$\Rightarrow \frac{3n}{2} \geq 6$$

$$\Rightarrow \boxed{n \geq 4} \leftarrow \text{It is possible}$$

B)  $\delta = 6$

$$\frac{6n}{2} \leq 3n - 6$$

$$\Rightarrow \boxed{3n \leq 3n - 6}$$

$$n=0, 1, 2, 3, \dots$$

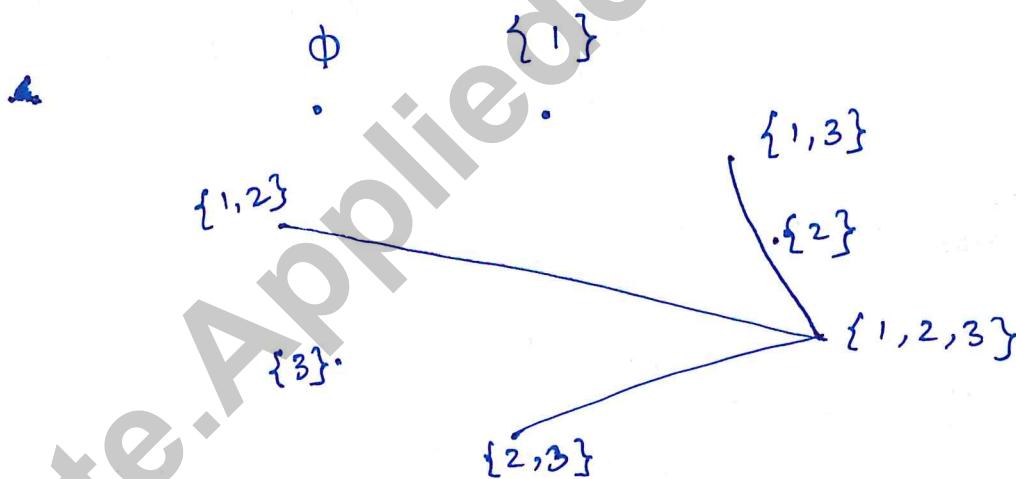
No non-negative integer can satisfy  $\boxed{3n \leq 3n - 6}$

$\therefore$  Option D is the answer.

**Q.** The  $2^n$  vertices of a graph  $G_2$  corresponds to all subsets of a set of size  $n$  for  $n \geq 6$ . Two vertices of  $G_2$  are adjacent if and only if the corresponding sets intersect in exactly two elements. The number of vertices of degree zero in  $G_2$  is :-

- A. 1
- B.  $n$
- C.  $n+1$
- D.  $2^n$

Soln. say  $n = \{1, 2, 3\}$ , i.e., size of  $n = |\{1, 2, 3\}| = 3$ .  
 The subset of a graph  $G_2$  a set of size  $n$  is  $2^n$ .  
 And  $2^n$  vertices corresponds to the  $2^n$  subsets.



We get this from the criteria given that intersection of 2 sets is exactly 2 elements.   
 Note: Two vertices are said to be adjacent iff there exists an edge between them.

We see that vertices  $\emptyset, \{1\}, \{3\}, \{2\}$  have degree zero. Vertices  $\{1, 2\}, \{1, 3\}, \{2, 3\}$  have degree 1 and vertex  $\{1, 2, 3\}$  have degree 3.

Given  $n \geq 6$ ,  
the vertices that will have degree zero are:-

$$\left. \begin{array}{l} \emptyset - \text{null set} \\ \{1\} \\ \{2\} \\ \vdots \\ \{6\} \end{array} \right\} \text{singleton set} \quad \left. \begin{array}{l} \text{degree} = 0 \\ \# \text{vertices} = 1 + 6 = (n+1) \end{array} \right\}$$

$\{1, 2\}$   
 $\vdots$   
 $\{1, 2, 3, 4, 5, 6\}$

All these will be connected to other vertices with edges wherever the edge formation criteria satisfies.  
 i.e.,  $\{1, 2\}$  will be connected to  
 $\{1, 2, 3\}$ ,  $\{1, 2, 3, 4\}$ ,  $\{1, 2, 3, 4, 5\}$   
 and  $\{1, 2, 3, 4, 5, 6\}$  because the intersection of the sets result in exactly 2 elements ( $\{1, 2\}$ ).

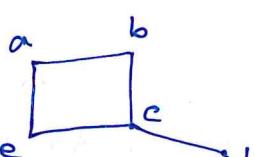
∴ Option c is correct

(17) Bi-pautite graph +

Bipartite graph properties

Path: A path is a walk where vertices don't repeat except when start and end are the same vertex.

$\therefore$  path:  $a \xrightarrow{\uparrow} b \xrightarrow{\uparrow} c \xrightarrow{\uparrow} d \xrightarrow{\uparrow} e \xrightarrow{\uparrow}$   $\frac{a}{\uparrow}$   $\leftarrow$  cycle  
 start end

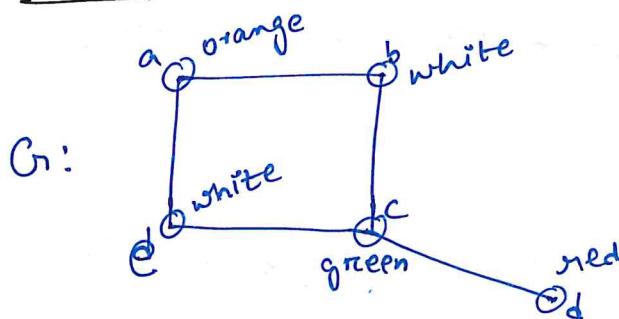


cycle: A path where the start and end vertex is the same, such a path is called as cycle.

**Note:**  
cycle is different from a loop. A loop is from a vertex to itself.

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### ④ Chromatic Number :

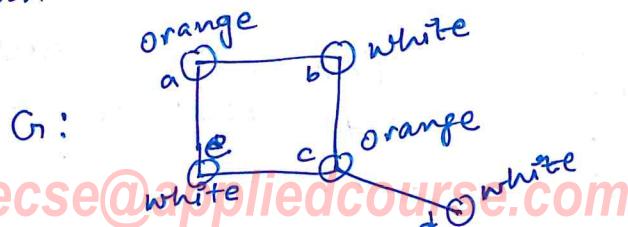


Suppose we want to color the vertices such that no two adjacent vertices are of same color. This is basically called graph coloring.  
The vertex 'a' if colored orange, vertex 'b' and vertex 'd' cannot take the same orange color.  
∴ we can put white color to vertex 'b' and vertex 'd'.  
Vertex 'c' can take orange color as vertex 'c' is not adjacent to vertex 'a'. In fact vertex 'c' can take any color except for white.

Vertex 'd' can even take any color other than green as vertex 'c' has been colored green.

Chromatic Number: What is the minimum # colors using which we can color the graph?

We have colored graph  $G_1$  with 4 colors. So can we color graph  $G_1$  in fewer than 4 colors?



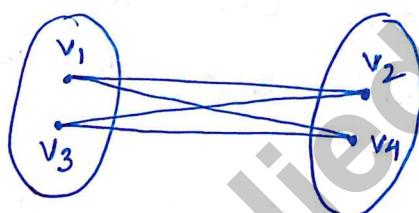
we try to minimize the number of colors required to color the graph by reusing the existing colors wherever possible.

Therefore, we are able to color the graph Or using two colors.  $\therefore$  Chromatic number for the given graph is 2.

### ④ Important properties and theorems of bipartite graph.

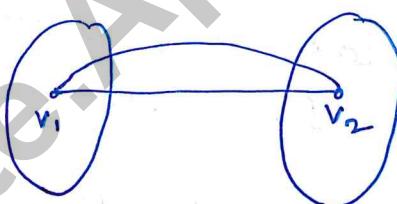
Theorem : A Graph  $G_r$  is bipartite iff  $G_r$  has no odd cycle.

Let's take a bipartite graph and prove  $G_r$  has no odd cycles.



simple graph

There is an even cycle.

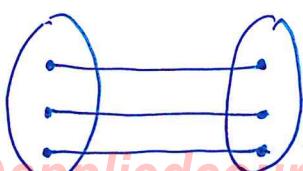


multi-edge graph

There is an even cycle.

Note :

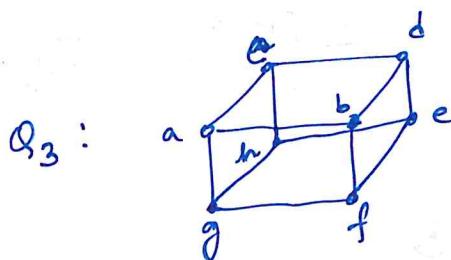
Misconception :  $G_r$  is bipartite iff it has an even cycle.



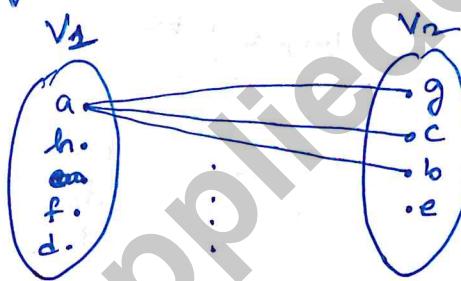
This is a bipartite graph that has no even cycle.

No odd cycle present could also mean no cycles are present. No odd cycle present does not mean that even cycles should be present for a graph to be bipartite graph.

Let's consider the hyper-cuboid graph ( $Q_3$ )



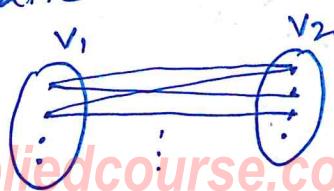
This graph has no odd cycles  $\Rightarrow$  bipartite graph.  
Let's try to draw  $Q_3$  as a bipartite graph.



We are able to divide the vertex set  $V$  into two subsets in such a way that there are no edges between the vertices in the same vertex subset.  
 $\therefore Q_3$  is a bipartite graph.

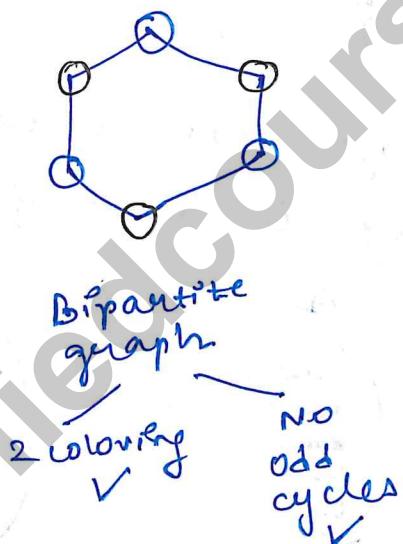
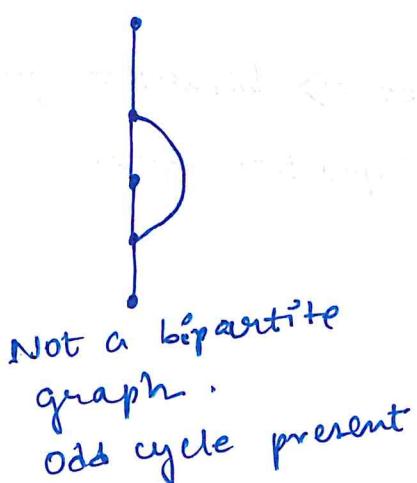
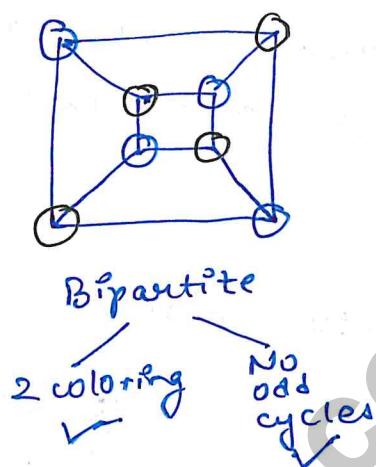
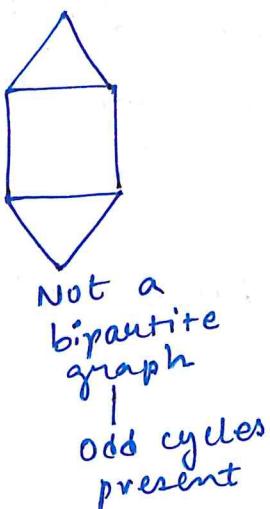
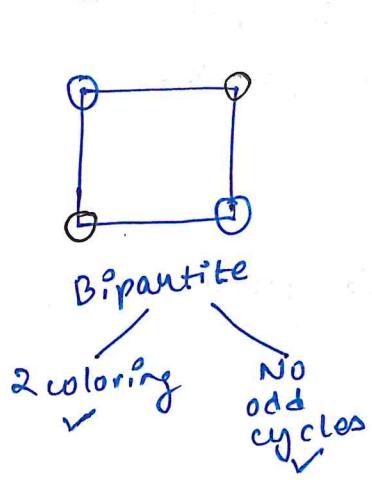
Theorem: A non-null graph is bipartite iff it is bi-chromatic.

Bi-chromatic means the chromatic # = 2.



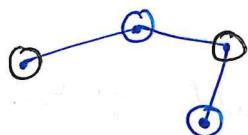
We can color subset  $V_1$  and subset  $V_2$  with different colors and we need only 2 colors.

E.g., Let's find if the following graphs are bipartite or not. Ph. 844-844-0102

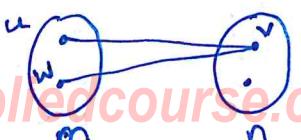


Note! It is easier to check for 2-coloring in comparison to no odd cycles criteria.

\* Observation: An acyclic graph is always bipartite.



\* Observation: The diameter of a complete bipartite graph  $(k_m, n)$  is 2.



diam ( $K_{m,n}$ )

= A diameter is the shortest path between the

Ph: 844-844-0102

farthest point

$$= \max_{u,v} d(u,v)$$

Since we are talking about a complete bipartite graph ( $K_{m,n}$ ), between any two vertices in the same subset, the maximum distance is 2.

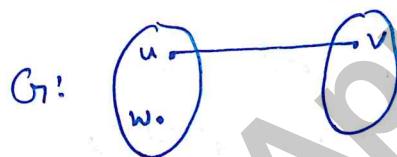
$$\therefore \text{diam } (K_{m,n}) = 2$$

\* Property: Ratio (diameter / chromatic number)

For complete bipartite graph,

$$\text{Ratio } (\text{diam} / \text{chromatic number}) = \frac{2}{2} = 1$$

For any bipartite graph (not complete bipartite graph),



G1: the diameter can be between 2 to  $\infty$ .  
Since u to w, the distance is  $\infty$  as w  
is not reachable from u.

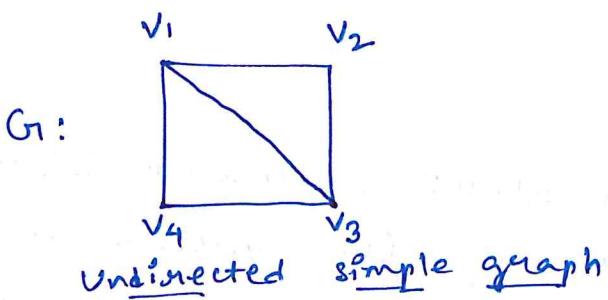
\* Representation of Graphs:

In DS & Algo, we have seen two strategies of representing graphs:

- (i) Adjacency Matrix
- (ii) Adjacency List.
- (iii) Incidence matrix.

① Adjacency Matrix:

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A :

$$A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$n \times n$

for an edge,  
say  $(v_1, v_3)$ ,  
 $v_1$  is the source  
vertex (row) and  
 $v_3$  is the destination  
vertex (column).

The matrix size is  $n \times n$  for  $n$  vertices in graph G<sub>1</sub>. Therefore it is a square matrix.

Since it is an undirected simple graph, the diagonal elements are zero as there are no self loops.

This matrix is symmetric i.e.,  $A_{ij} = A_{ji}$ .

Since it is an undirected simple graph, if there is an edge from  $v_i$  to  $v_j$ , then there is also an edge from  $v_j$  to  $v_i$ .

This matrix is a 0-1 matrix, also called as binary matrix. It is because it is a simple graph with no multi-edges.

Adjacency matrix enable us to take a concept from Graph theory and translate to a concept in Linear Algebra.

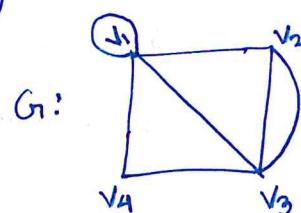
And if we are given a square symmetric matrix, we can imagine it like a graph and map the problems in graph theory.

**APPLIED COURSE** While studying Linear Algebra, we studied about Eigen value and Eigen vector. Using the properties of Eigen value and Eigen vector, we can also understand graph theory. There is an area in graph Theory called spectral graph theory which uses the concepts of eigen value and eigen vector of adjacency matrices to study properties of a graph.

We can think of the whole internet as a graph. One of the first algorithms that the co-founders of google used to rank the search results is called a page rank algorithm. The page rank algorithm actually is based on the concepts of eigen values and eigen vectors of the adjacency matrix of the web graph. So whenever we are using the search we are internally using the inter-linking theory of graph theory and linear algebra. This inter-linking theory is a very powerful tool in mathematics.

Using adjacency matrix we represent :-

(i) loops :

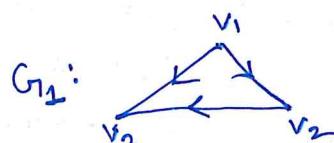


(ii) multi-edges :

	v <sub>1</sub>	v <sub>2</sub>	v <sub>3</sub>	v <sub>4</sub>
v <sub>1</sub>	1	1	1	1
v <sub>2</sub>	1	0	0	0
v <sub>3</sub>	1	0	1	0
v <sub>4</sub>	1	0	1	0

self loop  
multi-edge

(iii) directed edges :



A<sub>1</sub> :

	v <sub>1</sub>	v <sub>2</sub>	v <sub>3</sub>
v <sub>1</sub>	0	0	0
v <sub>2</sub>	0	0	0
v <sub>3</sub>	0	0	0

Directed edges

The adjacency matrix that we get need not be symmetric.

Because in case of directed graph if we have an edge from  $v_1$  to  $v_2$ , there may not be any edge from  $v_2$  to  $v_1$ .

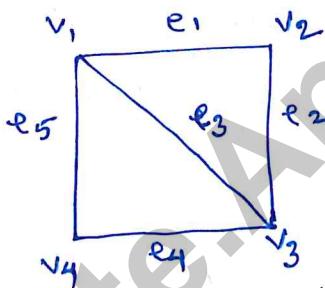
For undirected graph, the rowsum/column sum imply the number of edges that are incident on each vertex, i.e., degree of a vertex.

For directed graph, rowsum imply out-degree of a vertex and column sum imply in-degree of a vertex.

Note: rowsum and column sum are concepts in Linear Algebra and, out-degree and in-degree are concepts in Graph Theory.

## ② Incidence Matrix: (Undirected graph):

For incidence matrix, we need edge labelled graph.



	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$
$v_1$	1	0	1	0	1
$v_2$	1	1	0	0	0
$v_3$	0	1	1	1	0
$v_4$	0	0	0	1	1

We put 1's ~~for~~ and 0's in the incident matrix, depending on whether an edge is incident on a vertex or not.

Here, the row sum = degree of the vertex and the column sum = 2 (because an edge has to be incident on exactly 2 vertices)

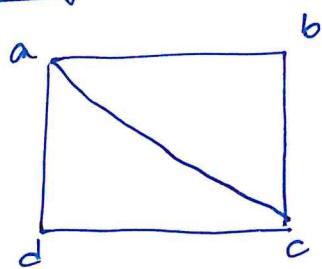
in the incidence matrix.

In case of multi-edges, the incidence matrix will have multiple columns that are identical.

Note:

Incidence matrix for directed graph does not capture the direction information. Therefore for directed graphs we shouldn't use an incidence matrix.

### ③ Adjacency List



For each vertex, we create a list of adjacent vertices

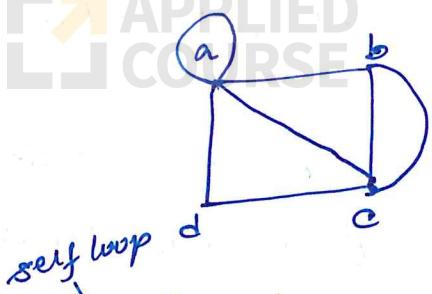
	list of adj. vertices
a	b, c, d
b	a, c
c	a, b, d
d	a, c

we can store this list

using lists and sets (depending on the programming language)

we can represent self-loops using adjacency list.

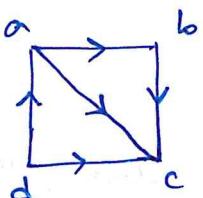
For multi-graphs having multi-edges, we need to use multi-sets because, it allows duplicates while sets does not allow duplicate elements.



	<u>list of adjacent vertices</u>
a	a, b, c, d
b	a, c, c
c	a, b, d, b
d	a, c

multi-set  
(for storing multiple edges)

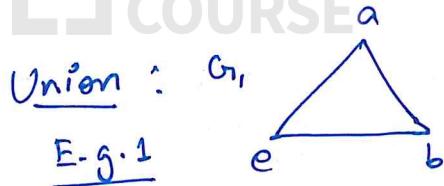
We can also represent directed graphs using adjacency lists.



	<u>list of adjacent vertices</u>
a	b, c
b	c
c	a, c
d	

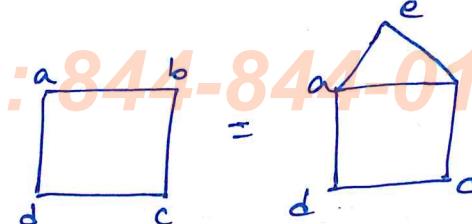
### \* Set-Theoretic operations on Graphs:

In set theory, we have come across operations like Union, Intersection, complement, XOR and subtraction. A graph is also two sets i.e.,  $G = (V, E)$  where  $V$  is the vertex set and  $E$  is the edge set.



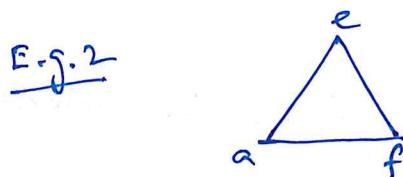
U

$G_2$

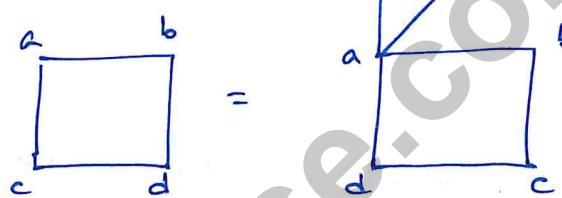


=

$$\left[ \begin{array}{l} V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \\ E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \end{array} \right]$$



U



Note! The graphs given have to be labelled so that union operation can be performed.

The union of two graphs is defined as:

(i) vertex set of the union is the union of the vertex sets i.e.,  $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ .

(ii) Similarly,  $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

The principle of inclusion-exclusion (set theory) can be applied on graphs as graphs are formed using two sets, i.e., vertex set and edge set.

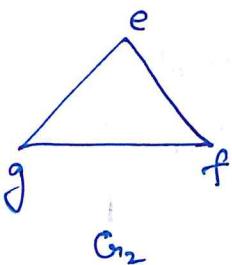
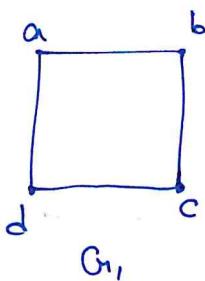
$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$\text{Therefore, } n(G_1 \cup G_2) = n(G_1) + n(G_2) - n(G_1 \cap G_2)$$

$$\text{and, } e(G_1 \cup G_2) = e(G_1) + e(G_2) - e(G_1 \cap G_2)$$

where  $n(G_i) = \# \text{vertices in graph } G_i$   
 $e(G_i) = \# \text{edges in graph } G_i$ .

Properties:  
 1. vertex disjoint: The vertex sets of both the graphs are disjoint.

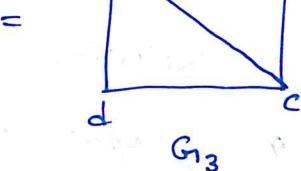
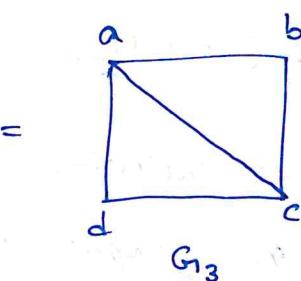
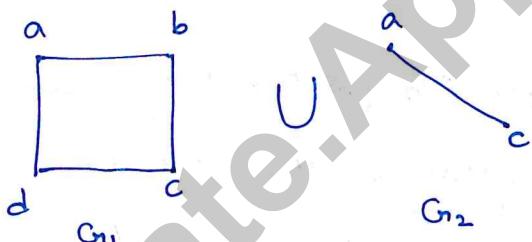


$G_1$  and  $G_2$  are connected but  $G_1 \cup G_2$  is disconnected

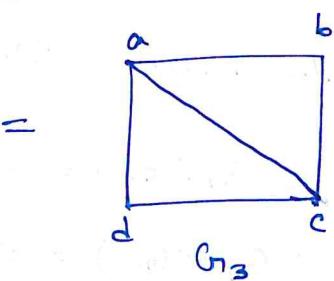
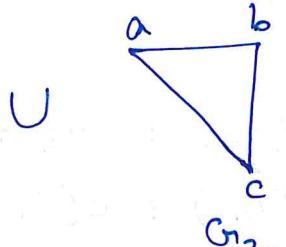
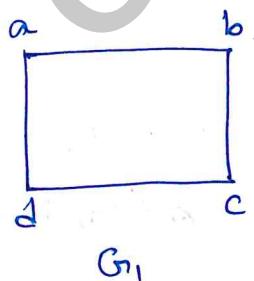
$\exists v_i \sim v_j \forall i, j$   
 path

$\Leftrightarrow V(G_1)$  and  $V(G_2)$  are disjoint sets.  
 (Bidirectional implication)

2. Edge disjoint: The edge sets of both graphs are disjoint.



3. Union does not create multigraphs:

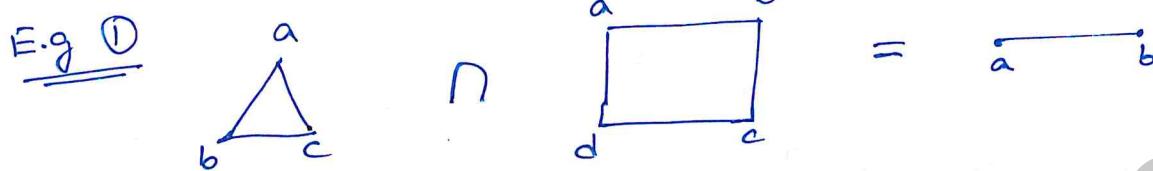


Even though edges  $(a, b)$  and  $(b, c)$  are present in graph  $G_1$ , the union of  $(G_1 \cup G_2)$  does not result into multiple edges between vertices  $a$  and  $b$ , and vertices  $b$  and  $c$ .

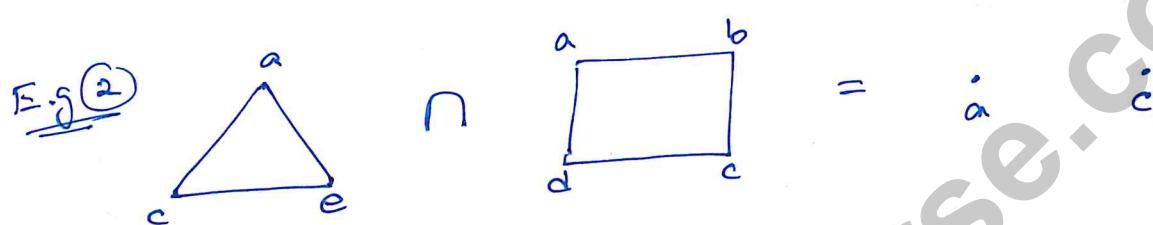
$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$

E.g ①



E.g ②



Properties:

1.  $(V_1 \cap V_2) = \emptyset \iff (G_1 \cap G_2)$  has no vertices  
where  $V(G_1) = V_1$  and  $V(G_2) = V_2$

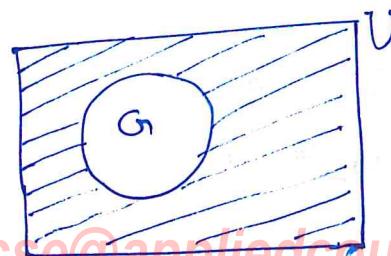
2.  $(G_1 \cap G_2)$  is connected iff

a)  $(V_1 \cap V_2)$  is a singleton set or is a single vertex.

b) If  $|V_1 \cap V_2| > 1$  then

$(E_1 \cap E_2)$  should lead to a connected graph on  $(V_1 \cap V_2)$ .

Complement: Here we look at the edge complements.  
Universe of vertices make no sense.

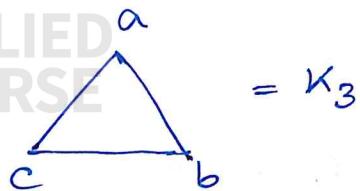


$U$  = Universal set

= all edges in a complete graph.

Eg. ①

Let  $G_1:$



$$= K_3$$

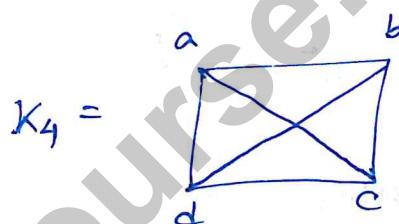
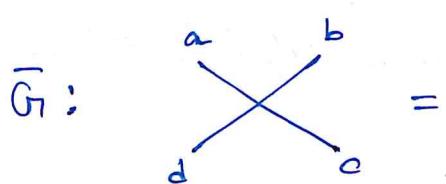
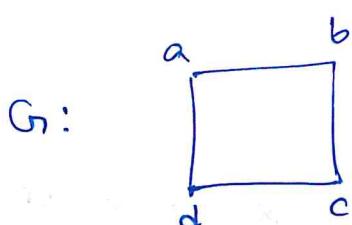
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$$= \phi_3$$

$$\boxed{G_1 = K_n - G_1}$$

E.g. ②



$$= K_4$$

Properties:

①  $e(\bar{G}) = \frac{n(n-1)}{2} - e(G)$

where,  
# edges in  $G$ -complement  $= e(\bar{G})$

②  $G_1, \bar{G}_1:$  Both have same vertex set.  
But,  $E(G_1) \cap E(\bar{G}) = \emptyset$ .

③  $\bar{\Phi}_n = K_n ; \bar{K}_n = \emptyset$

④ If  $G_1 \cup G_2 = K_n$   
 $G_1 \cap G_2 = \emptyset$  } IFF  $\begin{cases} G_1 = \bar{G}_2 \\ G_2 = \bar{G}_1 \end{cases}$

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$$(Q) \quad O(G_1) = 5 = n(G_1)$$

$$e(G_1) = 7$$

$$\therefore O(\bar{G}_1) = O(G_1) = 5$$

And,  $e(\bar{G}_1) = e(K_7) - e(G_1)$

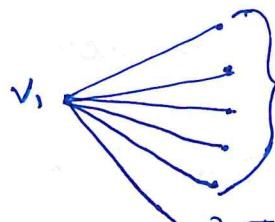
$$= \frac{5 \times 4^2}{2} - 7$$

$$= 3$$

(Q) Degree sequence of  $G_1 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$

$$\therefore n(G_1) = 7.$$

Find the degree sequence of  $\bar{G}_1$ .



These edges are present in  $G_1$ .

} This edge can be present in  $\bar{G}_1$ .

Every vertex  $v \in V$  in  $G_1(V, E)$  can connect to at most  $(n(G_1) - 1)$  vertices in order to form edges.

The  $\bar{G}_1(V, \bar{E})$  will have a degree sequence

of:  $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$

$$\Rightarrow (v_1 + x_1 = n(G_1) - 1, v_2 + x_2 = n(G_1) - 1, v_3 + x_3 = n(G_1) - 1,$$

$$v_4 + x_4 = n(G_1) - 1, v_5 + x_5 = n(G_1) - 1, v_6 + x_6 = n(G_1) - 1,$$

$$v_7 + x_7 = n(G_1) - 1)$$

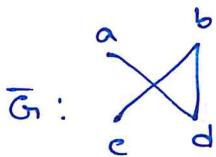
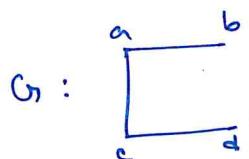
$$\Rightarrow (x_1 = 1, x_2 = 1, x_3 = 2, x_4 = 2, x_5 = 3, x_6 = 3, x_7 = 4)$$

$$\Rightarrow (1, 1, 2, 2, 3, 3, 4)$$

⑤ Complement of a connected graph may/may not

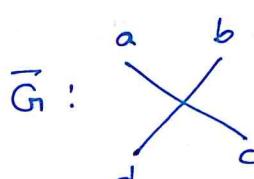
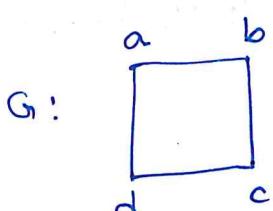
connected.

E.g. ①



Here  $G_1$  is connected and  $\bar{G}_1$  is also connected.

E.g. ②



Here  $G_1$  is connected but  $\bar{G}_1$  is not connected.

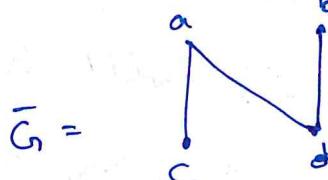
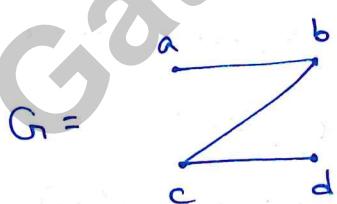
⑥ Complement of a disconnected graph is connected.

If  $G_1$  is disconnected,  $\bar{G}_1$  is connected. This is because

$$G_1 \cup \bar{G}_1 = K_n$$

Self-complementary graphs :

E.g. ①



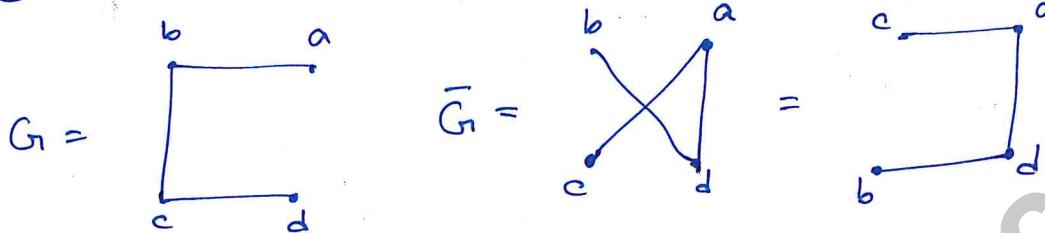
$$G_1 \neq \bar{G}_1$$

If we ignore vertex-labels, the graphs are same, such graphs are called isomorphic

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If a graph and its complementary graph are isomorphic to each other, then the graph is a self-complementary graph.

E.g. ②



If we ignore the vertex labels, then  $G_1 \equiv \bar{G}_1$ .

$G_1 \equiv \bar{G}_1 \Leftrightarrow G_1$  is self-complementary.

⑦ If  $G_1$  is self-complementary, then

$$n(G_1) = n(\bar{G}_1)$$

$$e(G_1) = e(\bar{G}_1)$$

⑧  $\Phi_n$ ,  $K_n$  are not self complementary except when  $n=1$ .

$G_1 = \cdot$        $\bar{G}_1 = \cdot$   
 Null graph &  
 complete graph

⑨  $G_1$  is self complementary  $\Rightarrow n(G_1) = 4x$  or  $n(G_1) = 4x+1$   
 where  $x \in \text{Integers}$

$$\Rightarrow e(G_1) = \frac{n(n-1)}{4} = \text{Integers}$$

$\Rightarrow$  diameter is 2 or 3.

E.g. Given  $n(G) = 6$ , can such graph be even self-complementary? **Ph: 844-844-0102**

$$n(G) = 6$$

$$\begin{aligned}\therefore e(G) \text{ should be} &= \frac{n(n-1)}{4} \text{ edges} \\ &= \frac{6 \times 5}{4} \\ &= \frac{30}{4} \\ &= \frac{15}{2} \neq \text{Integer.}\end{aligned}$$

$\therefore$  we cannot have a self-complementary graph with 6 vertices.

Proof:

We know,

$$\boxed{e(G) = e(\bar{G})}$$

There are no common edges between  $G$  and  $\bar{G}$ .

$$G \cup \bar{G} = K_n$$

$$G \cap \bar{G} = \emptyset$$

$$\therefore \boxed{e(G) + e(\bar{G}) = K_n = \frac{n(n-1)}{2}}$$

$$\text{Thus, } e(G) = e(\bar{G}) = \left( \frac{n(n-1)}{2} \right) / 2 = \frac{n(n-1)}{4} = \text{Integer}$$

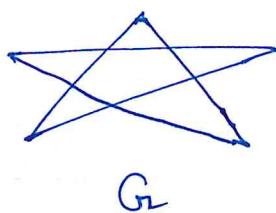
$$\therefore \frac{n(n-1)}{4} = \text{Integer,}$$

therefore, either  $n$  is divisible by 4 or,  
 $(n-1)$  is divisible by 4.

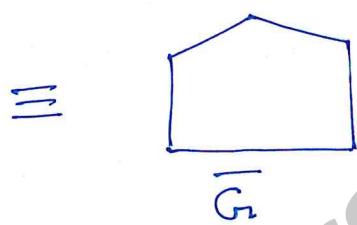
$\therefore n = 4x$   
 $\underbrace{\quad}$   
 n is divisible  
 by 4 with  
 0 remainder

or  $n-1 = 4x$   
 $\Rightarrow n = 4x+1$   
 $\underbrace{\quad}$   
 n is divisible  
 by 4 with  
 1 remainder

E.g. ④



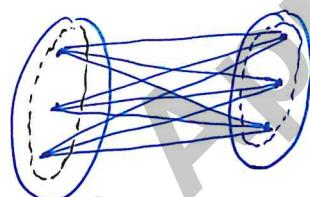
$G_1$



$\overline{G}_1$

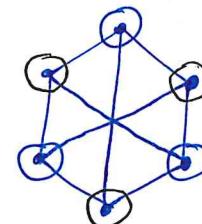
The graph is self-complementary because its complement are isomorphic to each other.

E.g. ⑤



$K_{3,3}$

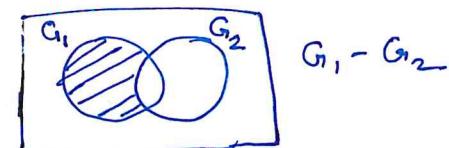
$\equiv$



$K_{3,3}$ .

Subtraction:

$$G_1 - G_2 = G_1 \cap \overline{G}_2$$

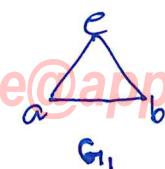


$G_1 - G_2$

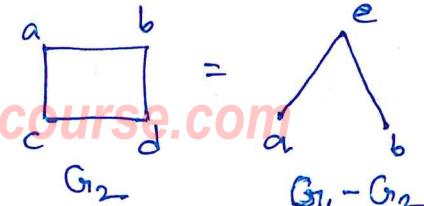
$$V(G_1 - G_2) = V(G)$$

$$E(G_1 - G_2) = E(G_1) - E(G_2)$$

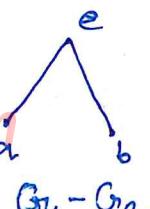
E.g. ①



$G_1$

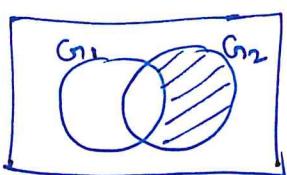
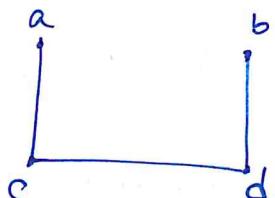


$G_2$



$G_1 - G_2$

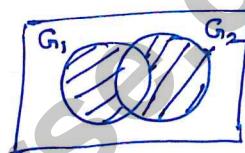
$$G_{r_2} - G_{r_1}$$



$$G_{r_2} - G_{r_1}$$

XOR operation (Ring-sum)  $\oplus$ :

Present in one but not both.

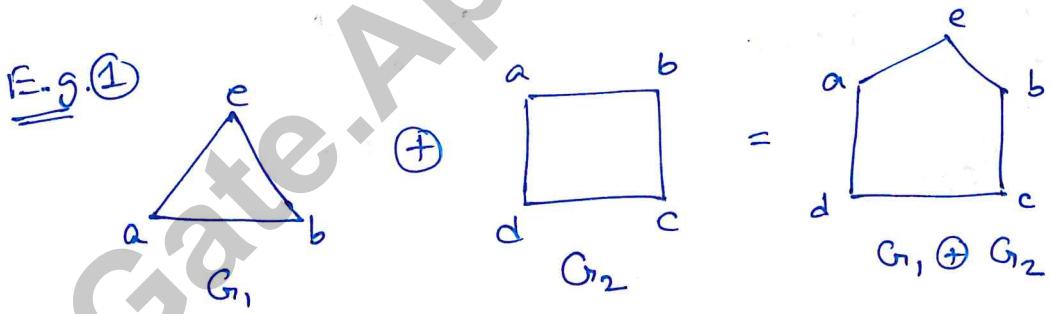


$$\begin{aligned} G_{r_1} \oplus G_{r_2} &= (G_{r_1} - G_{r_2}) \cup (G_{r_2} - G_{r_1}) \\ &= (G_{r_1} \cup G_{r_2}) - (G_{r_1} \cap G_{r_2}) \end{aligned}$$

In terms of vertex sets and edge sets:

$$V(G_{r_1} \oplus G_{r_2}) = V(G_{r_1}) \cup V(G_{r_2})$$

$$E(G_{r_1} \oplus G_{r_2}) = E(G_{r_1}) \oplus E(G_{r_2})$$



The edge (a, b) is present in both  $G_{r_1}$  and  $G_{r_2}$ , therefore edge (a, b) is not included in  $(G_{r_1} \oplus G_{r_2})$ . While other edges are present in only one of the graphs, therefore they are present in  $(G_{r_1} \oplus G_{r_2})$ .

Note: The vertex set of XOR will include all the vertices from both the graphs.

$$Q. \quad n(G_1) = 10$$

$$n(G_2) = 5$$

$$e(G_1 \cap G_2) = 2$$

$$e(G_1) = 8$$

$$e(G_2) = 3$$

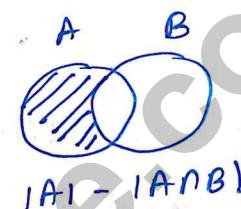
$$n(G_1 \cap G_2) = 3$$

$$\textcircled{1} \quad v(G_1 - G_2) = v(G_1)$$

$$\therefore n(G_1 - G_2) = n(G_1) = 10$$

$$\textcircled{2} \quad e(G_1 - G_2) = ?$$

$$E(G_1 - G_2) = E(G_1) - E(G_2)$$



$$e(G_1 - G_2) = 8 - 2 = 6$$

$$\uparrow = e(G_1) - e(G_1 \cap G_2)$$

$$\textcircled{3} \quad n(G_1 \oplus G_2) = ?$$

$$v(G_1 \oplus G_2) = \frac{v(G_1)}{A} \cup \frac{v(G_2)}{B}$$

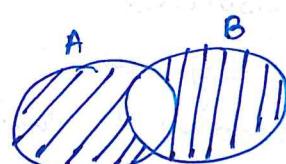
$$|A| = 10 \quad \& \quad |B| = 5 \quad (\text{given})$$

$$|A \cup B| = |A| + |B| - |A \cap B| \\ = 10 + 5 - 3 = 12$$

$$\therefore n(G_1 \oplus G_2) = 12$$

$$\textcircled{4} \quad e(G_1 \oplus G_2) = ?$$

$$\left| \frac{E(G_1)}{A} \oplus \frac{E(G_2)}{B} \right|$$



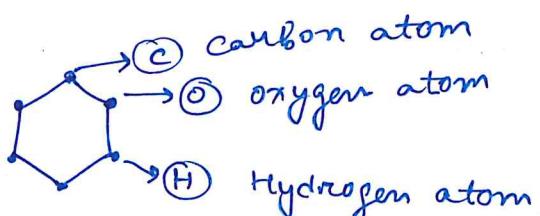
$$e(G_1 \oplus G_2) = e(G_1) + e(G_2) - 2(e(G_1 \cap G_2)) \\ = 8 + 3 - 2(2) \\ = 7$$

(1)

Graph Isomorphisms:

Any chemical compound (organic or inorganic molecule) can be represented as a graph.

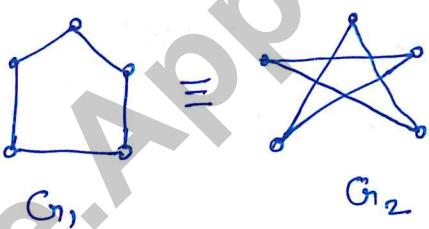
E.g.,



The connections between the atoms are the edges and the atoms are the vertices in the graph.

In organic chemistry, there are very lengthy and complex chemical compounds which can be represented by graph  $G_1$ . The same chemical compound maybe represented as graph  $G_2$  by some one else.

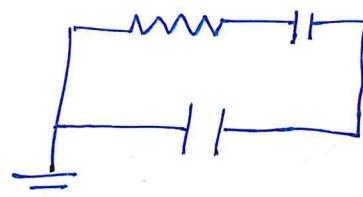
E.g.,



Here  $G_1$  and  $G_2$  are representing the same chemical compound.

In computational chemistry, an area of chemistry, where chemists try to find new molecules or new chemical compounds with some interesting properties and they use computer science to determine the new compound easily. They check if at the graph theoretic level, if the two given compounds are one and the same.

Suppose there is a schematic diagram of an electronic circuit.



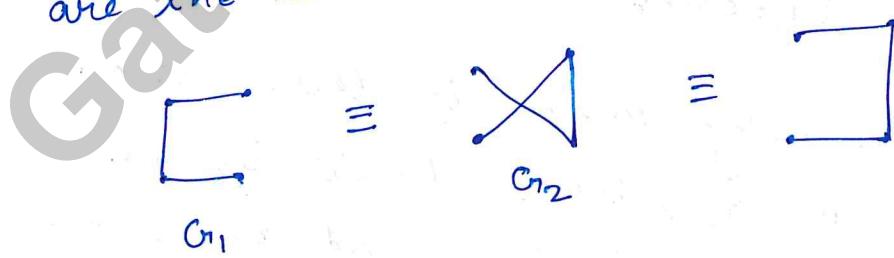
The above schematic diagram can be represented as graph where the resistors, capacitors can be represented as vertices and the connections can be represented as the edges.

In VLSI, circuits can have millions of resistors, capacitors and connections etc. Therefore, given two such circuits we want to determine whether they are one and the same. This is where graph isomorphism comes handy.

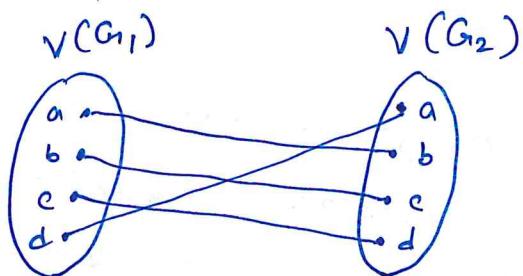
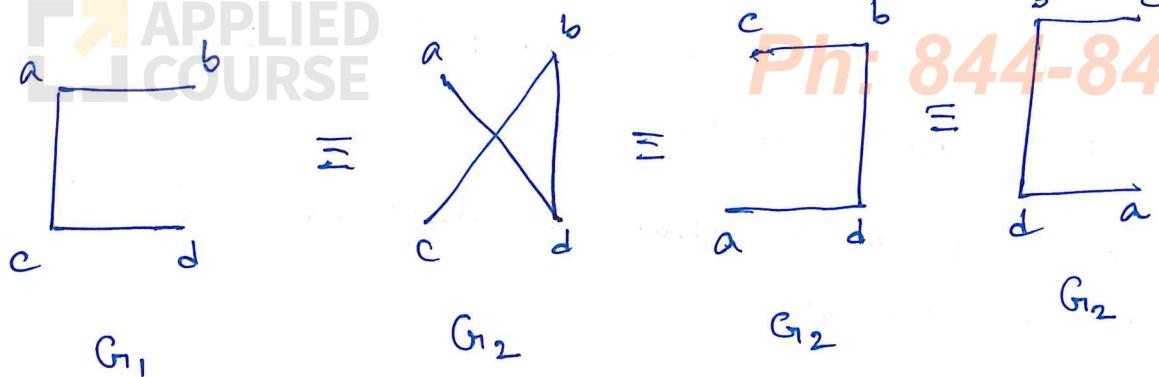
### Isomorphism:

$G_1 \equiv G_2$  when the underlying unlabelled graphs

are the same.



Another way to think is:  
 $G_1 \equiv G_2 \iff \exists$  bijective function  $f: V(G_1) \rightarrow V(G_2)$   
which preserves adjacency.



$$\begin{aligned}f(a) &= b \\f(b) &= c \\f(c) &= d \\f(d) &= a\end{aligned}$$

For every element in  $V(G_2)$ , there is a pre-image in  $V(G_1)$ . Therefore,  $f$  is onto.

Given any element in  $V(G_1)$ , there should be a mapping to exactly one element of  $V(G_2)$ .

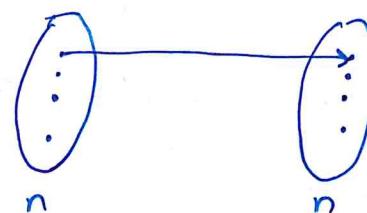
$\therefore f$  is one-one.

Hence  $f$  is bijective.

Adjacency preserving means if there is an edge  $a \rightarrow b$  in  $G_1$ , there should be an edge  $f(a) \rightarrow f(b)$  in  $G_2$  and vice-versa.

In algorithmic perspective, we might have to generate all possible bijections and then test if adjacencies are preserved.

From a set of size  $n$  to another set of size  $n$ , there are  $n!$  possible bijections.



$$\begin{aligned} & n * (n-1) * (n-2) * \dots * 1 \\ & = n! \end{aligned}$$

Therefore to test if two graphs are isomorphic, we have to generate all  $n!$  bijections. And for each bijection we have to check if the adjacencies are preserved. It is a very time consuming method.

Mathematicians came up with some conditions that have to be met if  $G_1$  and  $G_2$  are isomorphic. If any one of those conditions are not met, then we can say that the two graphs are not isomorphic. These conditions are called invariants.

If  $G_1 \equiv G_2$

$$\text{then } ① \quad n(G_1) = n(G_2)$$

Note: This statement is similar to  $A \Rightarrow B$ , then  $\neg B \Rightarrow \neg A$  (in propositional logic)

Using ( $\gamma_B \Rightarrow \gamma_A$ ), we can say that **Pt. 344-844-0102**  
 $n(G_1) \neq n(G_2)$  then  $G_1$  is not isomorphic  
to  $G_2$ .

If  $G_1 \equiv G_2$

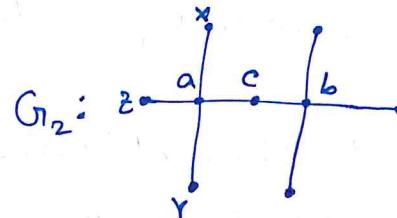
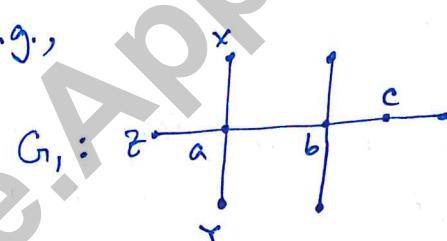
then ②  $e(G_1) = e(G_2)$

③ degree sequence  $G_1 =$  degree sequence  $G_2$

④ #cycles of any length is same  
in  $G_1$  and  $G_2$ .

⑤ Consider vertex  $u \in V(G_1)$  and  
 $v \in V(G_2)$ , all the neighboring  
vertices for  $u$  and  $v$  should  
have same properties.

E.g.,



The vertex a in  $G_1$  has a neighboring vertex b with degree 4.

But in  $G_2$ , none of neighboring vertices of a, i.e., c, x, y, z have degree 4.

Thus neighboring vertices of a in  $G_1$  and  $G_2$  don't have the same properties. Hence  $G_1 \not\equiv G_2$ .

Given a directed graph represented as adjacency matrix. Symmetric matrix represents undirected graph.

$$\begin{array}{c}
 \text{?} \\
 \equiv
 \end{array}
 \begin{array}{l}
 \begin{matrix}
 v_1 & v_2 & v_3 \\
 \begin{bmatrix}
 v_1 & 1 & 1 & 0 \\
 v_2 & 0 & 1 & 1 \\
 v_3 & 0 & 0 & 1
 \end{bmatrix} & \begin{matrix}
 v_1 & v_2 & v_3 \\
 \begin{bmatrix}
 0 & 1 & 0 \\
 0 & 0 & 1 \\
 0 & 1 & 0
 \end{bmatrix} & \begin{matrix}
 v_1 & v_2 & v_3 \\
 \begin{bmatrix}
 1 & 0 & 0 \\
 1 & 0 & 1 \\
 1 & 1 & 0
 \end{bmatrix} & G_2
 \end{matrix}
 \end{matrix}
 \end{array}$$

The above graphs are not symmetric matrices, therefore these are directed graphs.

Vertex  $v_1$  in  $G_1$  has a degree 2 while there are no vertices in  $G_2$  that has degree 2. Therefore, these graphs are not isomorphic. We can also draw the graphs and verify.



Common Isomorphisms :

Eg. ①

$$\begin{array}{c}
 \square \\
 \equiv
 \end{array}
 \begin{array}{c}
 \times \\
 \times
 \end{array}$$

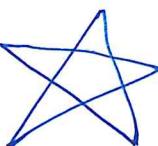
Eg. ②

$$\begin{array}{c}
 \rightarrow \text{cube} \\
 \equiv
 \end{array}
 \begin{array}{c}
 \rightarrow \text{diamond shape}
 \end{array}$$

Eg ③

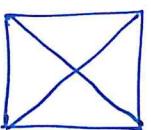


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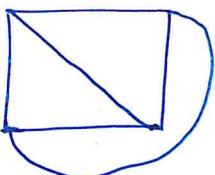


(pentagram)

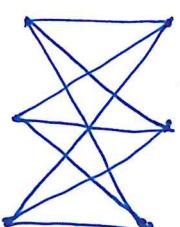
Eg ④



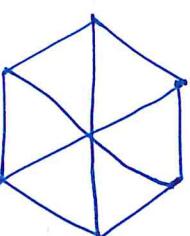
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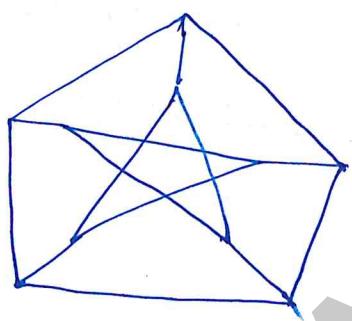
Eg ⑤



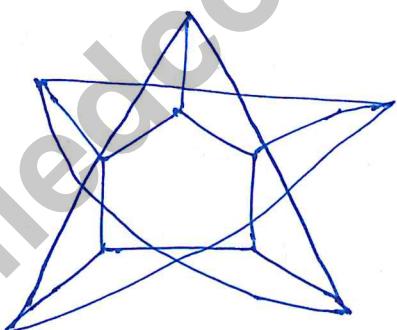
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 $K_{3,3}$ 

Eg ⑥ Peterson graph:



=

✳ Connectivity of Graphs :

Connectivity is a very important topic in the real world. If we think of a road graph, we want connectivity from every town to all the other towns. In computer networks, there should be a path between any computer and a server, using which we can send our data to the server and get it back from the server.



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Connected graph: A graph  $G_1$  is connected iff  $\forall v_i, v_j \in V(G_1)$   $\exists v_i \sim v_j$

That is, a graph  $G_1$  is said to be connected if and only if for all vertices  $v_i, v_j$  that belongs to the vertex set  $V(G_1)$ , there exists a path from  $v_i$  to  $v_j$ .

If the graph is directed then the path will be a directed path. If the graph is undirected then the path will be undirected path.

In the algorithms textbook - 'Introduction to Algorithms by CLR8', the following notations have been used :



smallest connected graph:

If we take a null graph with one vertex ( $\emptyset_1$ ), it is a connected graph.

vertex  $a$  is connected to itself without any loop. Therefore, it is the smallest connected graph.

1. In any graph  $G_1$ , which is connected / disconnected if there exists exactly 2 vertices of odd degree,  $x$  &  $y$ , then there exists a path from  $x$  to  $y$ .

Theorem: In any connected / disconnected  $G_1$ , if  $\exists$  exactly 2 vertices  $x$  &  $y$  of odd degree then  $\exists$   $\text{path}$  from  $x$  to  $y$ .

Proof:

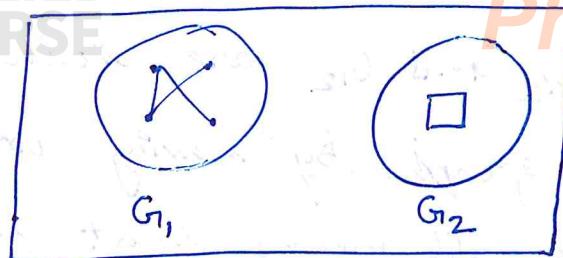
Case I:  $G_1$  is connected.

$G_1$  has even number of odd vertices. Therefore it is a graph.

Given any two vertices  $x$  and  $y$  since  $G_1$  is connected, there is a path from  $x$  to  $y$ .

Case II:  $G_1$  is disconnected.

There is a related concept called connected components. Given any graph, if we can break it into subgraphs, such that, the subgraphs are connected but there exists no edge connecting vertices  $v_1$  and  $v_2$  where  $v_1$  belongs to one subgraph and  $v_2$  belongs to another subgraph. These subgraphs are called connected components.

$G_1:$ Here,  $G_1$  and  $G_2$ 

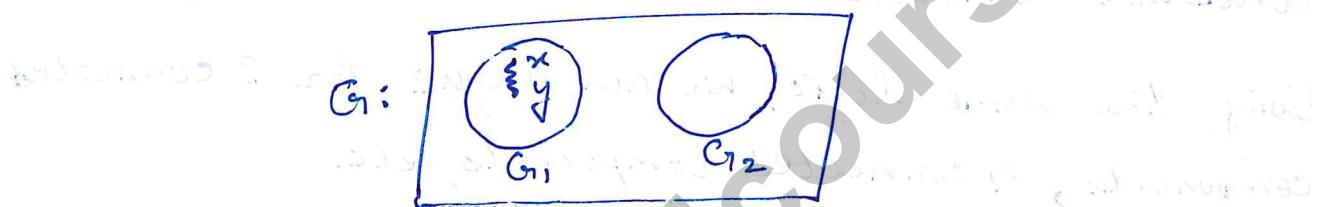
are connected

components. But

 $G_1$  is a disconnected graph.

$\therefore$  If the graph  $G_1$  is disconnected, there are atleast two disconnected components in the graph.

case 2a: vertices  $x$  and  $y$  belong to the same connected component, i.e.,  $x, y \in G_1$ , then



If vertices  $x$  and  $y$  belong to the same connected component then there will be a path between  $x$  and  $y$  as  $G_1$  is itself a connected subgraph.

case 2b: $G_1:$ 

Given that there are exactly two odd degree vertices in graph  $G_1$ . If we assume one such odd degree vertex is in  $G_1$ , and the other is in  $G_2$ . In that case, there cannot be a path from vertex  $x$  and vertex  $y$  since both  $G_1$  and  $G_2$  are connected components. Due to degree 1, these vertices will only

be reachable within their components. Also, the

Ph: 844-844-0102

connected components  $G_{r_1}$  and  $G_{r_2}$  are violating

the property of a graph by having an odd number of vertex with odd degree. It means that this case is never possible.

So, if case 2b is possible, then  $G_{r_1}$  and  $G_{r_2}$  are not graphs. If  $G_{r_1}$  and  $G_{r_2}$  are not graphs, then they can never be connected components.

It means, even if  $G_r$  is disconnected, we cannot construct connected components as  $G_{r_1}$  and  $G_{r_2}$ .

Using the same logic, we can argue for 3 connected components, 4 connected components, etc.

2. Let us assume we have  $e$  edges and  $n$  vertices. Let us assume we have  $k$  components in a graph.

$K$  can be 1.

$$K=1 \Leftrightarrow G_r \text{ is connected}$$

If  $K=1$  then  $G_r$  is connected. If  $G_r$  is connected, there is only one component.

Theorem:  $n-K \leq e \leq \frac{(n-k)(n-k+1)}{2}$

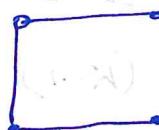
Proof: Let's consider the case when  $K=1$ , i.e.,

it is a connected graph.

$$\therefore n-1 \leq e \leq \frac{(n-1)n}{2}$$

∴ For a connected graph with  $n$  vertices, we need at least  $(n-1)$  edges. If the graph has to be connected.

E.g.,  $n=4$  and we need  $(4-1)$  edges i.e. 3 edges.



Graph  $G_1$  with  $(4-1)$  edges =  $(4-1) = 3$  edges

Graph  $G_1$  is a connected graph.

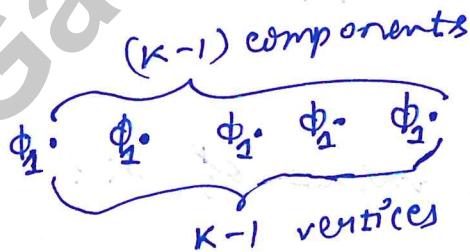
Now,  $\frac{(n-1)n}{2}$  is the number of edges in a complete graph.

Note: We are looking only at the simple graphs, no multi-graphs, no self loops.

∴ Number of edges in a simple graph has to be less than or equal to number of edges in the complete graph  $K_n$ .

Let's try to construct a  $k$ -component graph

with  $n$  vertices



1 connected component  $(K_{n-k+1})$

$(n-k+1)$  vertices

Let's imagine that the single component with  $(n-k+1)$  vertices is completely connected, i.e., it is a complete graph.

then the component has  $\frac{(n-k)(n-k+1)}{2}$  edges

as the other  $(k-1)$  components are each

fore with 1 vertex each and has no

edges. Therefore we have  $(k-1)$  null graphs ( $\emptyset_1$ ).

This is the optimal way to construct a  $k$ -component graph with  $n$  vertices to maximize the number of edges. If we break the graph in any other way, the number of edges will be fewer than the above graph construction.

(Q) Given  $n=10$ ;  $k=3$ , what is the minimum number of edges? And what is the maximum number of edges?

We know,  $e \geq n-k$

$$\therefore \text{minimum # edges} = 10-3 = 7.$$

$$\text{maximum # edges} = \frac{7 \times 8}{2} = 28.$$

(Q) Given  $n=10$  and  $e=6$ , what are the possible values of  $k$ ?

We know,

$$n-k \leq e$$

$$\Rightarrow 10-k \leq 6$$

$$\Rightarrow \boxed{k \geq 4}$$

Again we know,

$$6 \leq \frac{(10-k)(10-k+1)}{2}$$

$$\Rightarrow 2 \times 6 \leq (10-k)(10-k+1)$$

$$\Rightarrow 12 \leq (10-k)(11-k)$$

$$\Rightarrow 12 \leq k^2 - 21k + 110$$

$$\Rightarrow k^2 - 21k + 98 \geq 0$$

Quadratic equation.

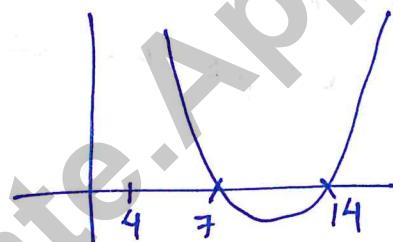
If we solve this for equality, i.e.,

$$k^2 - 21k + 98 = 0,$$

we get,

$$k = 14 \text{ or } 7.$$

The above quadratic equation is actually an equation of parabola.



The equation is  $(\geq 0)$  when  $k$  is less than 7 and greater than 14.

The other equation,  $k \geq 4$   
 $\therefore$  The possible values of  $k \geq 4$ ,  $k \leq 7$  and  $k \geq 14$ .

Now since  $n=10$ , the number of components can never exceed the number of vertices.

(Q) What is the maximum number of edges in a disconnected graph  $G_1$  with  $n$  vertices?

Disconnected graph means  $[k \geq 2]$ .

We know,

$$(n-k) \leq e \leq \frac{(n-k)(n-k+1)}{2}$$

To maximize the number of edges, we need the smallest possible value of  $k$ , which is 2.

$$\therefore e \leq \frac{(n-2)(n-2+1)}{2}$$

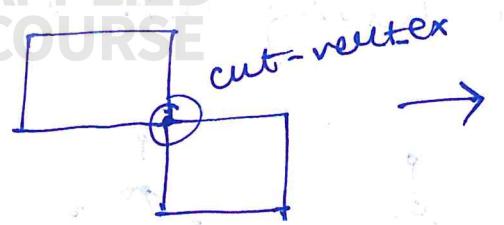
$$\Rightarrow e \leq \frac{(n-2)(n-1)}{2}$$

∴ For a disconnected graph with  $n$  vertices, the maximum possible edges we can have is  $\frac{(n-2)(n-1)}{2}$ .

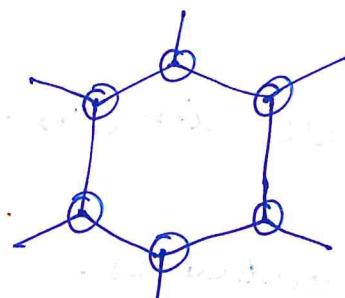
### \* Cut-Vertex:

For  $\forall E$  connected  $G_1$ , such that removal of  $v$  disconnects  $G_1$ , such a vertex is called a cut vertex.

E.g:



E.g:



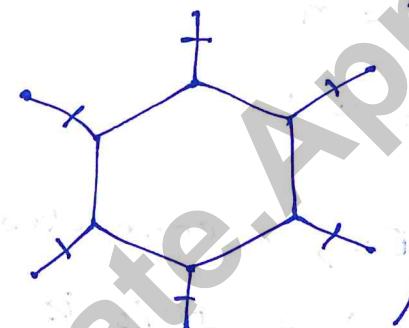
This graph has 6 cut-vertices

④

#### Cut-edge:

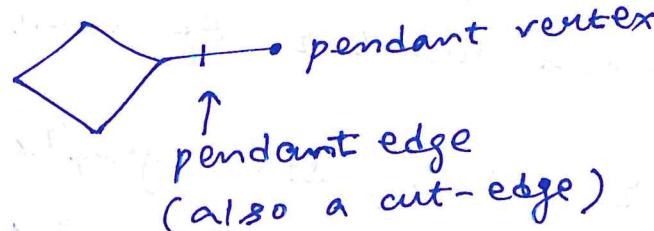
For edge  $e \in$  connected  $G_1$  s.t. the removal of  $e$  disconnects  $G_1$ , such an edge is called a cut-edge.

E.g:

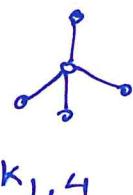
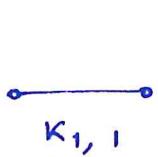


This graph has 6 cut-edges.

E.g.

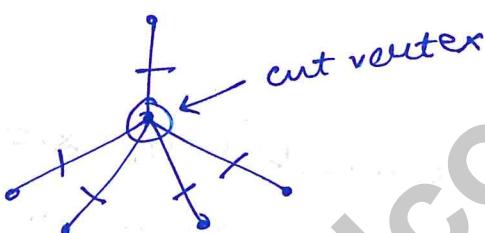


E.g. Star graph:  $(K_{1,n})$  having  $(n+1)$  vertices  
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How many cut-edges are there in  $(K_{1,n})$ ?

For  $K_{1,5}$ , there are  $n$  cut edges.



$K_{1,5}$  has 1 cut vertex.

∴ # cut edges for  $K_{1,n} = n$

# cut vertices for  $K_{1,n} = 1$

On removal of cut vertex,  $n$  connected components are formed. And this is the maximum number of components.

In computer science, star graphs are useful in distributed systems where every message comes to a central server and then it goes to the other places. But if the central server is itself down,

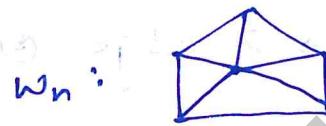
then everything gets disconnected. The star network is based on the concept of star graph.

Also,  $\text{diam}(K_{1,n}) = 2$  if  $n+1 \geq 3$



For  $\text{diam}(K_{1,1}) = 1$

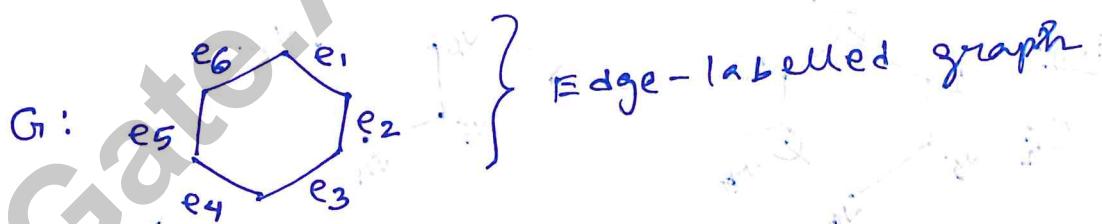
Note: For  $C_n$ ,  $W_n$  and  $K_n$  there are no cut-vertices.



Because if we remove any vertex, the remaining graph is still connected.

④ Cut-set of a connected  $G$ :

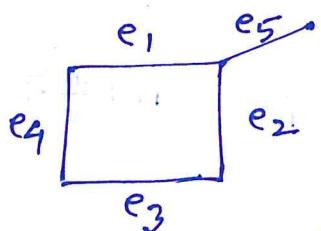
It is a minimal set of edges whose removal disconnects  $G$ .



$\{e_1, e_2\}$ ,  $\{e_2, e_3\}$ ,  $\{e_3, e_4\}$ ,  $\{e_4, e_5\}$ ,  $\{e_5, e_6\}$ ,  $\{e_6, e_1\}$  are the minimal sets whose removal will disconnect the graph  $G$ . Now,  $\{e_1, e_2, e_3\}$  is not a cutset because it is not the minimal set as  $\{e_1, e_2\}$  or  $\{e_2, e_3\}$  exists which are strict or proper subsets of  $\{e_1, e_2, e_3\}$  and  $\{e_1, e_2\} \cap \{e_2, e_3\}$

(Q)

Given



which is not a cutset?

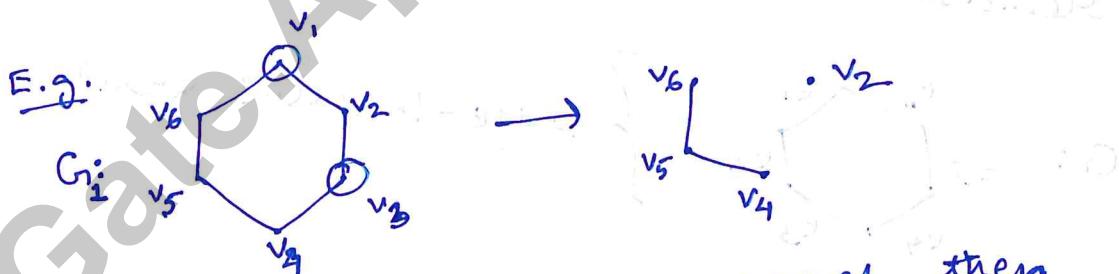
- (a)  $\{e_5\} \rightarrow 1$  edge cutset
- (b)  $\{e_1, e_5\} \rightarrow$  Not a cutset
- (c)  $\{e_1, e_2\} \rightarrow 2$  edge cutset
- (d)  $\{e_3, e_4\} \rightarrow 2$  edge cutset.

### Connectivity - 2

\* vertex connectivity ( $\alpha(G)$  or  $\alpha$ ):

It is the minimum number of vertices to be removed to disconnect a graph  $G$ .

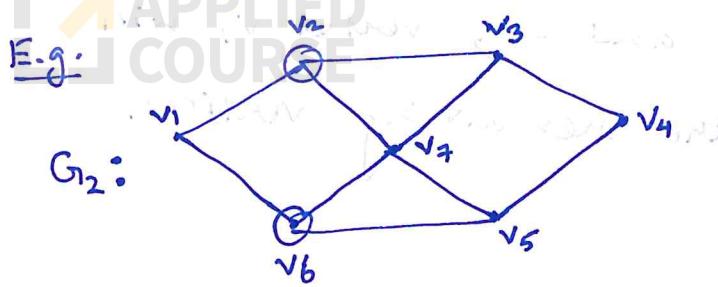
If there exists a cut vertex, then  $\alpha(G) = 1$



If vertices  $v_1$  and  $v_3$  are removed, then

there is no path from  $v_2$  to the remaining vertices.  $\therefore$  For graph  $G_1$ , vertex connectivity or

$$\alpha(G) = 2.$$



In the above graph  $G_2$ ,

$$\alpha(G) \leq \delta(G)$$

where  $\delta(G) = \text{minimum of the vertices degrees}$

If we notice,  $v_1$  has degree 2. And  $\alpha(G) = 2$ .  
 In this graph, as removal of vertices  $v_2$  and  $v_6$  will disconnect  $v_1$  from the remaining vertices.

From average degree theorem :-

$$\alpha \leq \delta \leq \frac{2e}{n} \leq \Delta$$

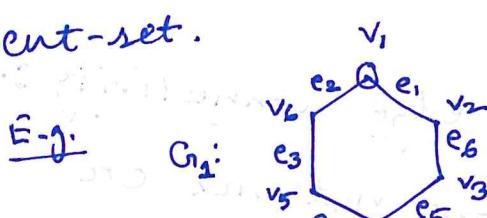
↓ max. degree  
 ↑ min. degree

④ Edge-connectivity ( $\lambda$ ):-

It is the minimum number of edges to be

removed to disconnect  $G_i$ . It is also equal to the size of the smallest

cut-set.



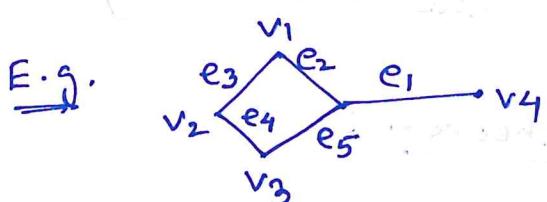
In  $G_3$ , the min. degree ( $\delta$ ) and the max. degree ( $\Delta$ ) are the same.  
 $\delta = \Delta = 2$ .

If we remove edges 1 and 2, vertex  $v_1$  will get disconnected from the remaining vertices.

$$\therefore \lambda(G) \leq \delta$$

since by removing all the edges connected to the vertex with minimum degree, we can disconnect the vertex with minimum degree.

Now, for every vertex that we remove, we also have to remove atleast one edge.



If we remove vertex  $v_4$ , we have to remove edge  $e_1$ .

If we remove vertex  $v_1$ , we have to remove edges  $e_2$  and  $e_3$ .

$$\therefore \lambda \leq \Delta$$

Therefore,

$$\lambda \leq \Delta \leq \delta \leq \frac{2e}{n} \leq \Delta$$

Application of vertex and edge connectivity:  
Telecommunications network or computer network are designed in such a way that if any vertex or edge fails, the network remains connected.

way that the vertex connectivity is as high as possible since vertex connectivity tells us how many cell towers have to go down for the telecommunications network to become a disconnected network. This information is very useful in cases like tsunamis, earthquakes, etc.

And edge connectivity tells us how many of the connections (wired/wireless) can go down the network gets disconnected.

(Q)  $n=10, e=16$   
What are the possible values of  $\alpha$  and  $\lambda$ ?

$$\alpha \leq \lambda \leq \delta \leq \frac{2 \cdot e}{n}$$

$$\frac{2 \cdot e}{n} = \frac{2 \times 16}{10} = 3.2$$

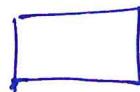
$$\therefore \boxed{\alpha \leq \lambda \leq 3}$$

[since  $\alpha$  and  $\lambda$  has to be integers]

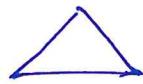
Definition: A graph  $G_i$  is separable iff  $\alpha = 1$   
 $G_i$  is separable iff it is 1-connected  
 or

$G_i$  is separable iff it is a cut-vertex

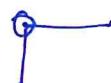
E.g:



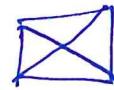
Not separable



Not separable



separable



Not separable

Definition:

A graph  $G_1$  is called  $n$ -connected iff

its vertex connectivity is equal to  $n$ .

$n$ -connected  $G_1 \Leftrightarrow \lambda = n$

If a graph is connected, then it is  $1$ -connected

Similarly, a graph  $G_1$  is called  $k$ -line connected graph iff its edge connectivity is equal to  $k$ .

$k$ -line connected  $G_1 \Leftrightarrow \lambda = k$

For cycle graph  $(C_n)$ ,  $\lambda = 2$  and  $\lambda = 2$ .



For complete graph  $(K_n)$ ,

$\lambda = n - 1$

$\lambda$  is not defined because whatever number of vertices we remove, it will still be connected.



Base

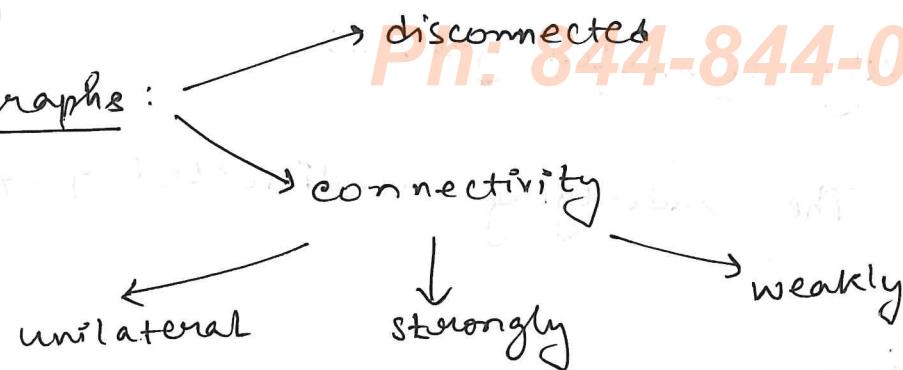


Base



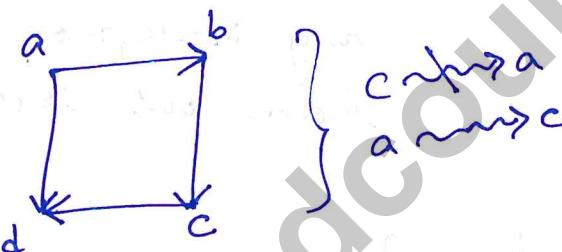
Base  
Median

(\*) Directed graphs:



Unilaterally connected graph: It is a graph where for all vertices  $x$  and  $y$ , there is directed path from  $x \rightarrow y$  or there is a directed path from  $y \rightarrow x$ .

E.g.,

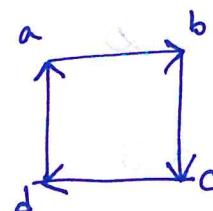


capta  
a  $\rightarrow$  c

Strongly connected graph:

$\forall x, y, x \rightarrow y$  AND  $y \rightarrow x$

E.g.



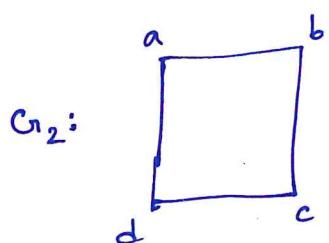
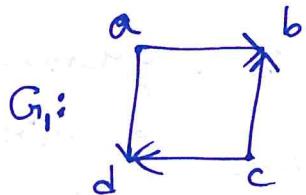
From every vertex to every other vertex, there is a path.

Weakly connected graph:

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The underlying undirected graph is connected.

E.g:



Here vertices b and d are sink i.e., it does not have any outdegree, it only has indegree.  
And vertices a and c are source as they don't have any in-degree, they only have out-degree.

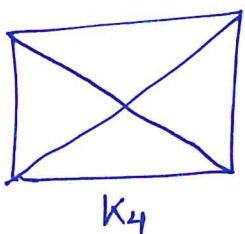
} underlying undirected graph is not unilaterally connected graph as  $a \nrightarrow c$  and  $c \nrightarrow a$ .

Definition

$\text{girth}(G) = \text{length of the shortest cycle in } G.$

E.g: acyclic  $G_1 \rightarrow \text{girth} = \infty$

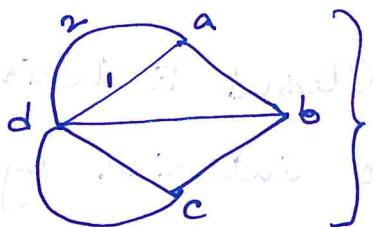
$K_n \rightarrow \text{girth} = 3$



$$\text{girth}(K_4) = 3$$

(\*) Eulerian and Hamiltonian Graphs :

The seven bridges of Königsberg problem:  
Can we devise a walk such that each bridge is crossed exactly once?



The map is drawn as a graph where bridges are edges and land is vertex.

Walk: vertex can be repeated  
edge can be repeated

a b d a b  
1(say using edge 1)

A walk can be open or closed

In closed walk, the starting and the ending vertex is the same. If we don't start and end at the same vertex, then it is called an open walk.

Trail: vertex can repeat  
Edge cannot repeat

open  
closed (circuit)

Path: vertex can't repeat  
Edge can't repeat

open  
closed (cycle)

Eulerian Trail: Trail which visits each edge exactly once. It is also called as Eulerian path.

Eulerian circuit: Closed Eulerian trail.  
It is also called as Eulerian cycle.

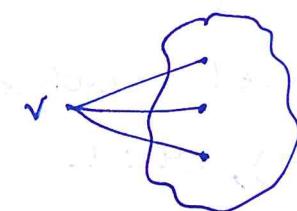
Eulerian Graph:  
Definition:  $G_i$  is Eulerian iff  $\exists$  a Eulerian cycle.

Another definition:  $G_i$  is Eulerian  $\Leftrightarrow$  connected and degree of each vertex is even.

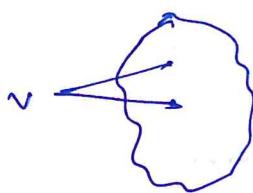


Say we start from vertex  $v$  and reach vertex  $u$ . Suppose there are two paths between vertex  $u$  and  $w$  such that each edge in the component is visited exactly once using the two paths. On return to  $u$  we can't visit vertex  $v$  again since the only edge between  $v$  and  $u$  has already been

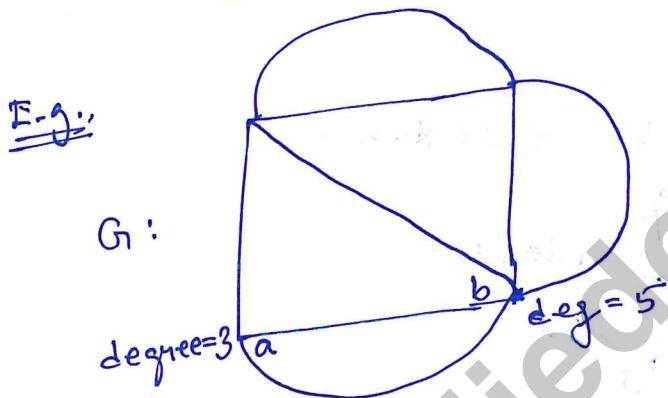
visited. Therefore, if there exists an odd degree vertex, then the graph can't be Eulerian as Eulerian cycle does not exist.



$\deg(v) = 3 \rightarrow$  Not Eulerian graph



$\deg(v) = 2 \rightarrow$  Eulerian graph



Note: In a graph, there may be more than one unique Eulerian cycle.

Note:  $\Phi_n$  : not Eulerian because it is not connected to start with.

$K$ -regular graph: If  $K = \text{even}$ , then the graph is eulerian.

$K_n : n \geq 3$  and  $n$  is odd  $\rightarrow$  Eulerian

Explanation:

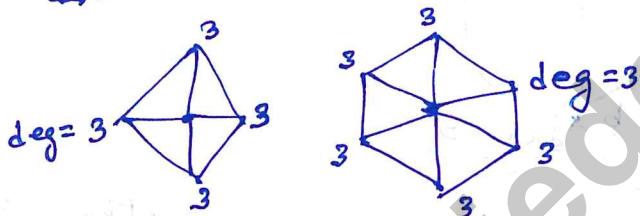
$n \geq 3$  means cycle is present because

$n < 3$  means there can't be a cycle.

If  $n$  is odd, the degree of each vertex in a complete graph is  $(n-1)$ , which means  $(n-1)$  is even.

$C_n : \deg(v) = 2 = \text{even} \rightarrow \text{Eulerian}$

$W_n : \text{It can never be Eulerian.}$



As the degree of some vertices are odd, therefore, it is not Eulerian.

Universal Graph:

A graph is called an universal graph iff it contains Euler path.

OR,

A graph is called an universal graph iff there exists only 2 vertices of odd degree and graph has to be connected.

If a graph is either unicursal or eulerian, then it is said to be traversable or traceable graph.

Given any graph, if the number of odd degree vertices = 0, i.e., all vertices are even degree, then the graph is Eulerian.

If the number of odd degree vertices = 2, then the graph is Unicursal.

Eulerian basically means that a eulerian cycle exists. And unicursal means a eulerian path exists.

### Directed Eulerian Graph:

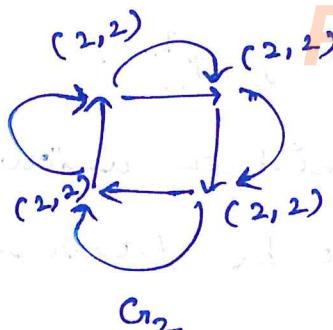
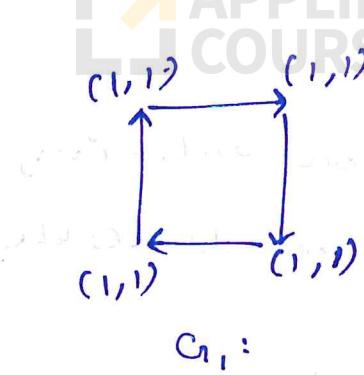
A graph is directed eulerian graph iff

there exists a directed eulerian cycle.

Or, a graph is directed eulerian graph iff

it is unilaterally connected and  $\forall v, \text{indeg}(v) = \text{outdeg}(v)$





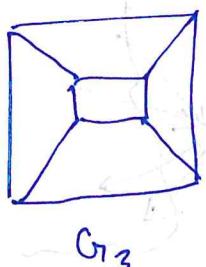
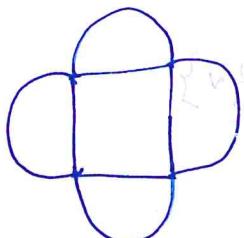
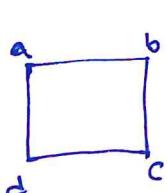
Directed euler graph  $G_1$  and  $G_2$  as they are unilaterally connected and for all vertices the indegree is same as the outdegree.

Note: Eulerian cycle: closed trail, each edge is used exactly once.

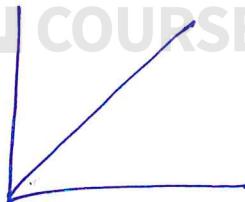
Hamiltonian Path: It is a path with each vertex being visited exactly once.

Hamiltonian cycle: It is a closed Hamiltonian path.

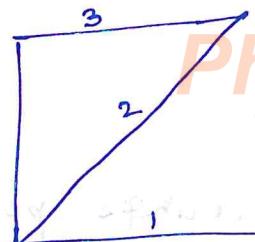
Note: In case of Hamiltonian path and Hamiltonian cycle, all edges may/maynot be traversed but all vertices must be visited.



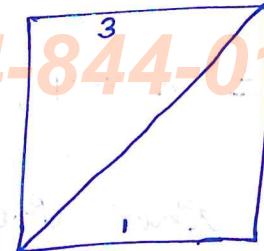
The graphs  $G_1$ ,  $G_2$  and  $G_3$  have Hamiltonian path and Hamiltonian cycle



No Hamiltonian path exists



Hamiltonian path exists



Hamiltonian cycle exists

$\phi_n$ : Not Hamiltonian as it is not connected

$G_0$  The graph  $G_0$  has no edges to traverse between vertices.

K-regular: may or may not be Hamiltonian.

$K_n$ :  $n \geq 3$  is always Hamiltonian

The # Hamiltonian cycles present in a complete graph?

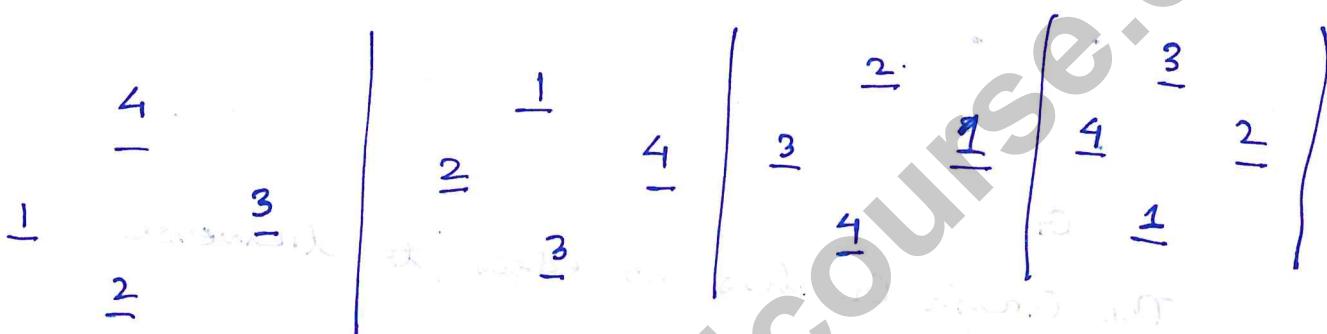
A hamiltonian cycle is basically a circular permutation of the  $n$  vertices, as hamiltonian cycles are built using hamiltonian paths, i.e., we can't repeat a vertex.

the linear orderings possible:

$$\frac{4}{\text{four places}} \quad \frac{3}{\text{}} \quad \frac{2}{\text{}} \quad \frac{1}{\text{}}$$

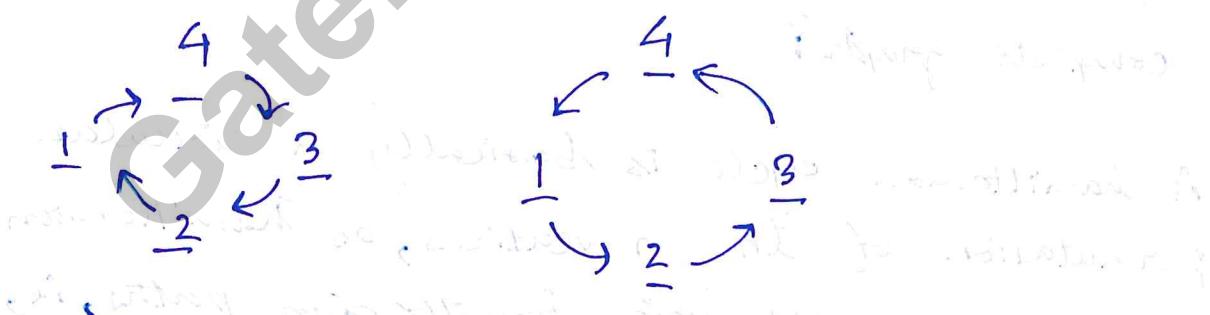
$$\rightarrow 4! = 4 \times 3 \times 2 \times 1$$

the circular orderings possible:



The above orderings are one and the same.

Now each of these orderings can be written clockwise and anticlockwise, which are also one and the same.



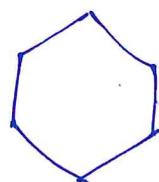
$\therefore$  Total number of Hamiltonian cycles

$$= \left( \frac{4!}{4} \right) / 2$$

# Hamiltonian cycle = circular permutation of  $n$  numbers where clockwise and counter clockwise are the same.

$$= \frac{(n!)}{n} / 2 = \frac{(n-1)!}{2}$$

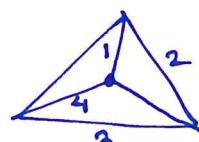
$C_n$ :



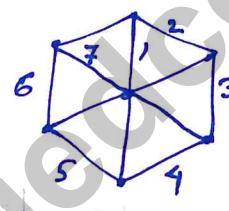
No. of steps:  $\frac{n(n-1)}{2}$

Circular graph is Hamiltonian.

$W_n$ :



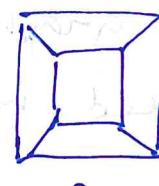
1-2-3-4



1-2-3-4-5-6-7

Wheel graphs are Hamiltonian.

$Q_n$ :



Not all cubes are Hamiltonian.

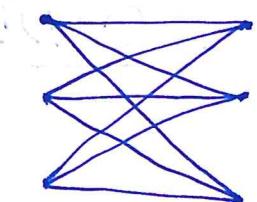
Hypercube graphs  $Q_2$  and  $Q_3$  are Hamiltonian.

Note: Hypercube graphs may or maynot be Hamiltonian.

$K_{m,n}$ : A complete bipartite graph is Hamiltonian.

if  $m = n$  and  $m, n \geq 2$

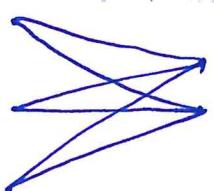
For  $m = n$ ,



$G_{r_1}$

Hamiltonian cycle exists in  $G_{r_1}$ .

For  $m \neq n$ ,



$G_{r_2}$

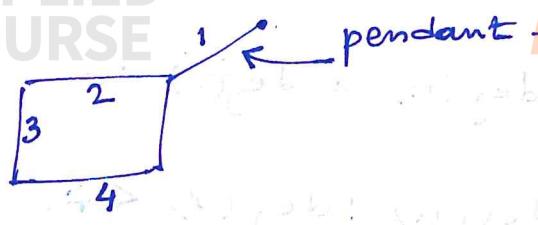
Hamiltonian cycle does not exist in  $G_{r_2}$ .

Theorem: If Graph  $G_r$  has Hamiltonian cycle, it implies  $G_r$  has Hamiltonian path.

But if a graph has Hamiltonian path, it need not have Hamiltonian cycle.

If  $G_r$  has Hamiltonian cycle  $\Rightarrow G_r$  has Hamiltonian path

and converse is not true always so we get:



pendant edge  
Hamiltonian path exists but Hamiltonian cycle not possible.

If graph is Hamiltonian  $\Rightarrow$  No pendant edge



pendant edge  $\Rightarrow$  Not a Hamiltonian graph.

$$\begin{array}{l} \because A \Rightarrow B \\ \therefore \neg B \Rightarrow \neg A \end{array}$$

Theorem:

Dirac Theorem:

If  $G_1$  is connected &  $\forall v$ ,  $\text{degree}(v) \geq n/2$

and  $n \geq 3 \Rightarrow G_1$  is Hamiltonian graph.



Theorem:

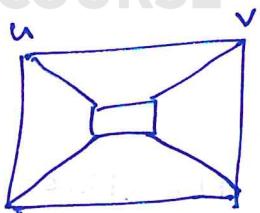
Ore's Theorem:

If  $G_1$  is connected &  $\forall (u, v)$ ,  $\deg(u) + \deg(v) \geq n$

$\Rightarrow G_1$  is Hamiltonian graph.



Note: Every graph that satisfies Dirac Theorem will also satisfy Ore's Theorem. Ore's Theorem is an extension to Dirac Theorem.

Example:

$$\deg(u) = \deg(v) = 3$$

$$\deg(u) + \deg(v) < n$$

 $\text{Q}_3$ 

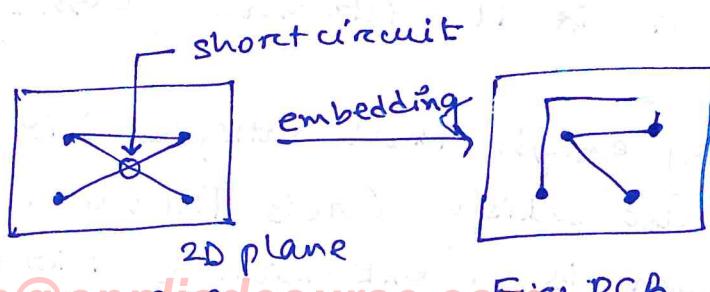
The Hypercube graph ( $\text{Q}_3$ ) is Hamiltonian, has 8 vertices.

$$\therefore \deg(u) + \deg(v) < 8$$

This example shows all Hamiltonian graphs need not satisfy the ( $\Leftarrow$ ) or converse of Dirac or Ore's theorem.

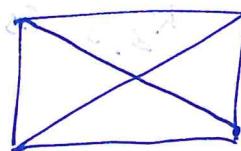
#### ④ Planar Graphs:

Say we are designing a complex electronic circuit. We have to place the electronic circuit on a board called Printed circuit Board (PCB). This board is basically a 2D plane. The open wiring between electronic components should not overlap in order to avoid short circuit.

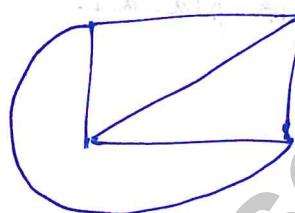


A graph is a planar graph  $\Leftrightarrow$  There exists an embedding in 2D plane such that no edges cross each other.

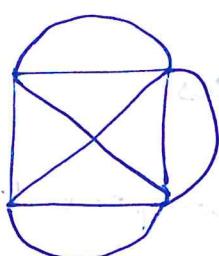
$K_4$ :



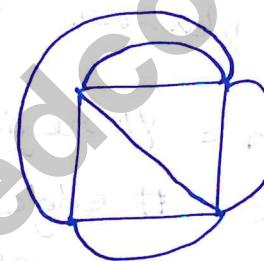
=



$K_4$  is planar



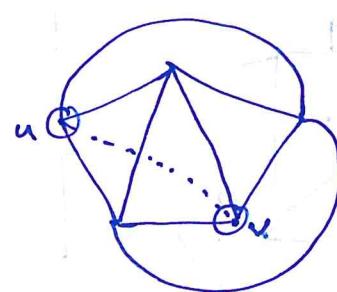
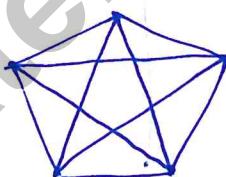
=



$G_1$

Graph  $G_1$  is planar

$K_5$ :



$K_5$  is not planar as we can't connect the edge  $(u, v)$  without overlapping other edges.

Note:  $K_n$  is planar iff  $n \leq 4$ .



$K_{3,2}$  is planar

$K_{3,3}$  is not planar

Euler's formula for planar graph:

If graph is planar  $\Rightarrow n = (e - n + k + 1)$

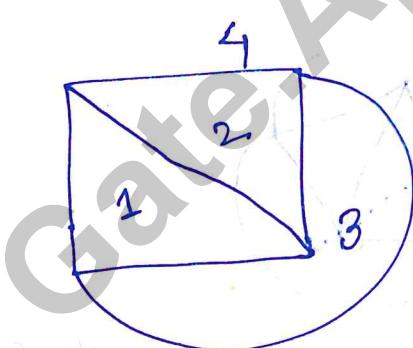
where  $n = \#$  regions/faces

$e = \#$  edges

$k = \#$  connected component

$n = \#$  vertices

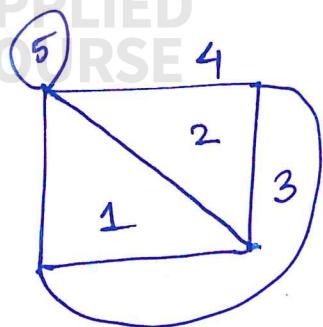
$K_4$ :



$$\begin{array}{|l|l|} \hline n & 4 \\ \hline n & 4 \\ e & 6 \\ k & 1 \\ \hline \end{array}$$

$$\therefore 4 = 6 - 4 + 1 \checkmark$$

1, 2 and 3 are closed regions or faces  
while 4 is an open region/open face.



$$\therefore 5 = 4 + 7 + 1 + 1 \checkmark$$

Note:

If  $G_i$  is connected  $\Leftrightarrow k=1$

If  $G_i$  is connected and planar then,

$$n = e - n + 2$$

E.g. 3-regular graph,  $O(G_i) = n = 10$ , planar

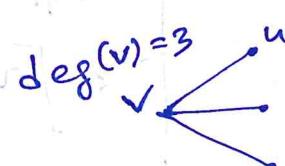
# faces = ?

$$3 = \text{edge} \#$$

$$\therefore n = e - n + 1 + 1$$

$$\Rightarrow n = e - 8$$

For 3-regular graph,



$$e = \frac{(3 \times 10)}{2} = 15$$

because we have counted each edge 2 times.

$$\therefore n = 15 - 8 = 7$$

Eg:

$$n = 10$$

$$\text{size} = e = 15$$

$$K = 3$$

$$n = 15 - 10 + 3 + 1$$

# closed faces = ?

For any graph,

# closed faces = # faces

- # open faces

$$\# \text{ open faces} = 1$$

$$\therefore \# \text{ closed faces} = 9 - 1 = 8$$

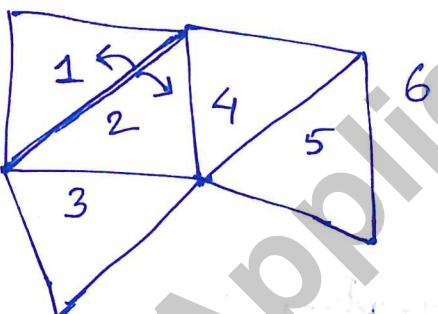
E.g.,

Given a planar graph, every region is bounded by 3 edges.  $O(G) = 10$ ; connected.

$$\# \text{ regions} = ?$$

$$r = e - n + 2$$

$$\Rightarrow r = e - 10 + 2$$



$$\frac{3r}{2} = e$$

$$\Rightarrow 2e = 3r$$

Each region is composed of 3 edges

$$\# \text{ regions} = \frac{\# \text{ edges}}{3}$$

Now, every edge  $e$  is counted 2 times.

$$\therefore \# \text{ edges} = 2e$$

$$\therefore r = \frac{2e}{3}$$

$$\therefore r = e - 10 + 2$$

$$\Rightarrow r = \frac{3r}{2} - 10 + 2$$

$$\Rightarrow r = 16$$

$$\therefore e = 24$$

For any region, we need atleast 3 edges to create the region.

$\therefore 2e \geq 3r$  in a planar graph.

$K_{m,n}$ : For planar, complete, bipartite graph.

$\therefore m \leq 2$  or  $n \leq 2$  to be planar.

$$r = \frac{mn}{mn} - \frac{(m+n)}{\# \text{vertices}} + 2$$

Kuratowski's Theorem: (Test for planarity)

$G_r$  is planar  $\Leftrightarrow G_r$  does not contain a subgraph that is a subdivision of  $K_5$  or  $K_{3,3}$ .

Note! We have seen that  $K_5$  is the smallest complete graph which is non-planar. And  $K_{3,3}$  is smallest complete bipartite graph which is non-planar.

Note! Subdivision / expansion:

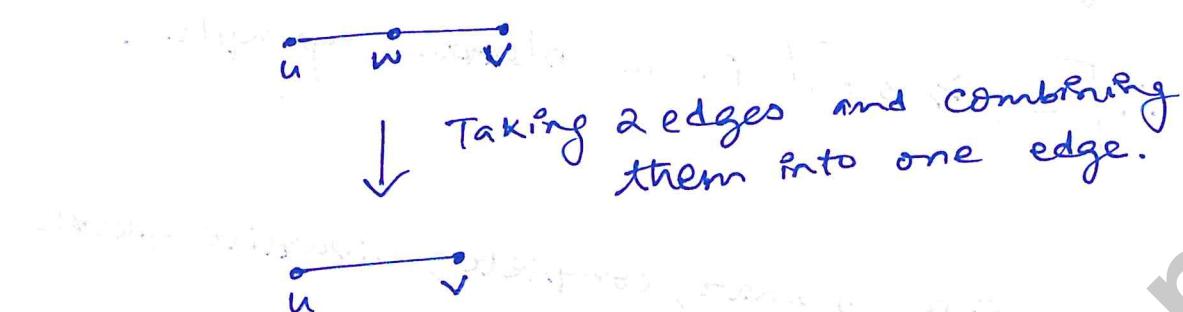
Given an edge  $(u, v)$  in following graph:



Subdivision of an edge is dividing an edge into 2 edges.

Note:

Smoothing:

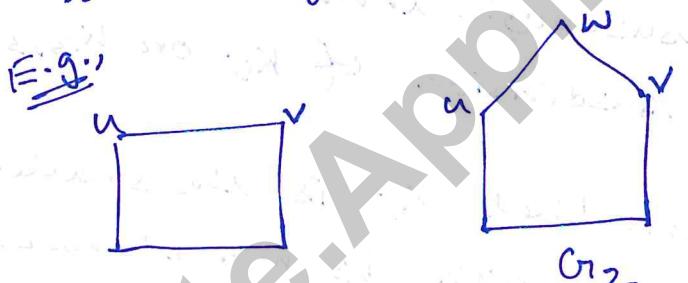


Note:

Homeomorphism:

$$G_1 =_h G_2$$

$G_1$  is homeomorphic to  $G_2$  if we can arrive at  $G_2$  by subdividing some edges in  $G_1$ , then  $G_1 =_h G_2$ .



$$\text{Here, } G_1 =_h G_2$$

Two graphs are homeomorphic if they have the same topological properties.

Two graphs are isomorphic if they have the same topological properties and the same number of edges.

Mail: [gatecse@appliedcourse.com](mailto:gatecse@appliedcourse.com)

**Theorem:** Given connected simple planar graph, **Fig. 844-844-0102**

$$e \leq 3n - 6$$

Proof:  $r = e - n + 2$

for all planar graphs, we know,

$$2e \geq 3r$$

$$\therefore r \leq \frac{2e}{3}$$

$$\Rightarrow e - n + 2 \leq \frac{2}{3}e$$

$$\Rightarrow \frac{1}{3}e \leq n - 2$$

$$\Rightarrow e \leq 3n - 6$$

Theorem:

For connected simple planar graph with no triangles (no 3 edge based regions), then  $e \leq 2n - 4$ .

Proof: Minimum number of edges required to form a region (with no 3 edge regions) is 4.

$$\therefore \# \text{regions} = \frac{\# \text{edges}}{4}$$

Every edge  $e$  is counted 2 times,

$$\therefore \# \text{edges} = 2e$$

$$\therefore \# \text{regions} = \frac{2e}{4} \Rightarrow 2e = 4r$$

For any region, we will need at least 4 edges to create the region.

Ph: 844-844-0102

$$\therefore 2e \geq 4n$$

$$\therefore n = e - n + 2$$

$$\Rightarrow e - n + 2 \leq \frac{2e}{4}$$

$$\Rightarrow \frac{1}{2}e \leq n - 2$$

$$\Rightarrow e \leq 2n - 4$$

Property : If we have a connected simple planar graph, then minimum degree ( $\delta$ ) is less than equal to 5,

$$\text{i.e., } \delta \leq 5$$

Proof:

$$\text{We know } \delta \leq \frac{2e}{n}$$

$$\delta \leq \frac{2(3n - 6)}{n}$$

$$\delta \leq 6 - \frac{6}{n}$$

If  $n$  is finite (considering only finite graphs here),

$(6 - \frac{6}{n})$  will always be less than 6.

$\therefore \delta \leq 5$ , as  $\delta$  is an integer

Property: For complete simple planar graph with no triangles,  $\delta \leq 3$

Proof:  $\delta \leq \frac{2e}{n}$

$$\delta \leq \frac{2(2n-4)}{n}$$

$$\delta \leq 4 - \frac{8}{n}$$

$\therefore \boxed{\delta \leq 3}$  for finite values of  $n$ .

(Q) which cannot be # edges in a complete simple planar graph with  $n = 10$ .

- a) 22
- b) 23
- c) 24
- d) 26

Soln. Triangles can exist, therefore,

$$e \leq 3n - 6$$

$$e \leq (3 \times 10 - 6)$$

$$\boxed{e \leq 24}$$

$\therefore$  Option (d) is not possible.

(\*) Trees:

Tree  $\Leftrightarrow$  a connected acyclic graph

Tree is also called a minimally connected graph, because we can't remove even a single edge without making the graph disconnected.

Tree  $\Leftrightarrow$  connected graph with exactly  $(n-1)$  edges.

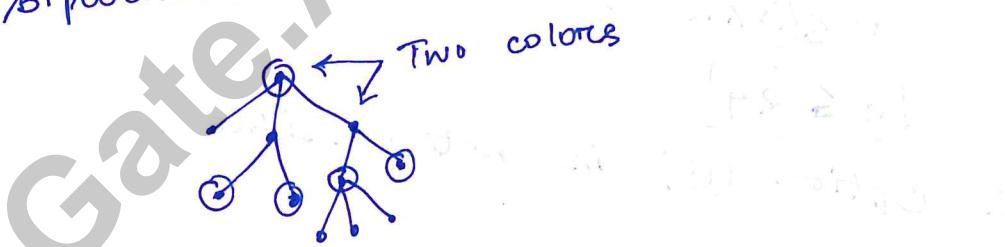
$\Leftrightarrow$  Acyclic graph with  $(n-1)$  edges.

$\Leftrightarrow \forall u, v \exists$  exactly one path  $u \rightsquigarrow v$   
(acyclic) (connected)

$\Leftrightarrow$  one, connected graph.

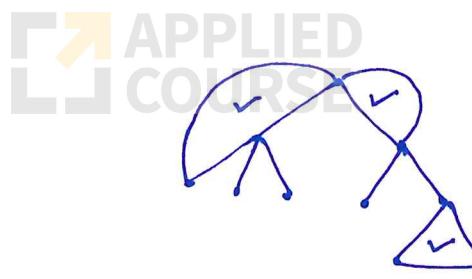
Properties

① Every tree is bi-chromatic and it is bipartite.

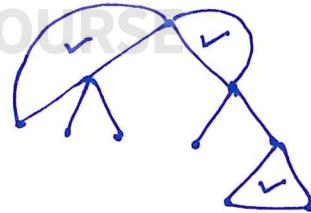


② Fundamental cycle:

Fundamental cycle is the cycle that we obtain by adding exactly one edge



Ph: 844-844-0102



Fundamental cycles  
in the tree.

$$\# \text{ fundamental cycles} = {}^n C_2$$

Rooted Tree: Tree with a specific vertex chosen as a root. In a rooted tree, we have parent-child and siblings relationships; leaf nodes and internal nodes are defined only for rooted trees. Leaf nodes are the nodes that has no child. Anything that is not leaf node is an internal node, including the root node itself.

depth of vertex  $u$  =  $\text{depth}(u) = \text{distance}(u, \text{root})$ . Since there are no cycles, distance is the shortest distance between vertex  $u$  and root.

$$\text{height(Tree)} = \text{height}(T) = \max_u \text{depth}(u)$$

$$\text{level}(u) = 1 + \text{depth}(u)$$

$$\text{level(Tree)} = \text{level}(T) = 1 + \text{height}(T)$$

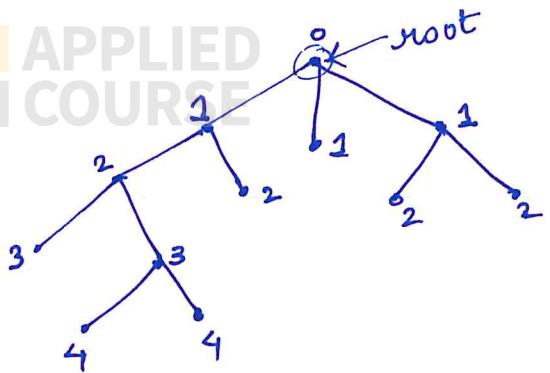


Fig: Tree T.

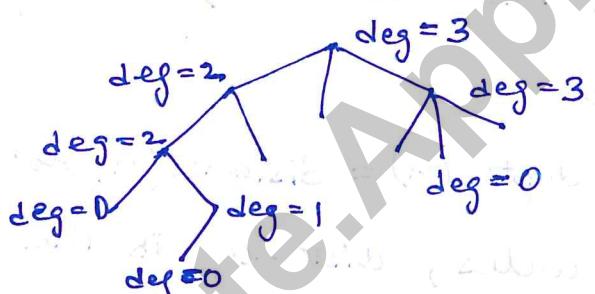
$$\text{height}(T) = 4$$

$$\text{depth}(\text{root}) = 0$$

K-any Tree: A rooted tree where every node has between 0 and  $K$  children.

It is a rooted tree and the #children( $u$ ) is greater than equal to zero and less than equal to  $K$ .

$$0 \leq \# \text{children}(u) \leq K$$



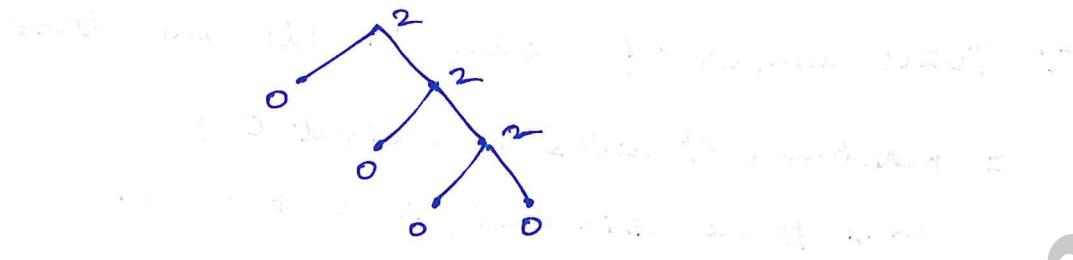
Here  $\text{deg}(u)$  is the number of children.

$$K = \text{max degree} = 3$$

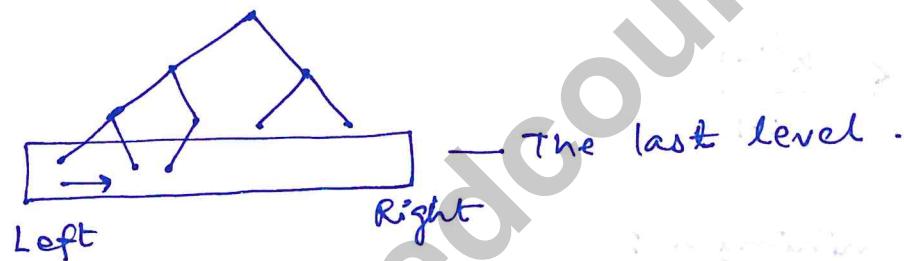
$\therefore$  It is a 3-any tree.

Binary tree = 2-any tree.

# children ( $u$ ) = 0 or  $k$



Complete k-ary Tree : It is a rooted tree and all levels except the last are completely filled. And the last level is left adjusted.



Complete k-ary tree  $\Leftrightarrow$  full k-ary tree

Properties :

K-ary tree

$l = \# \text{ leaves}$

$n = \# \text{ nodes}$

$h = \text{height}$

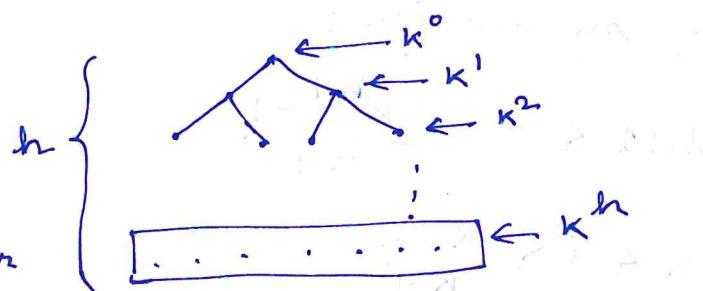
$i = \# \text{ internal nodes}$

To get maximum :-

$$\textcircled{1} \quad l \leq k^h$$

where,

$k^h = \text{maximum number of nodes at a height } h$



$$\textcircled{2} \quad n \leq \left( \frac{k^{h+1} - 1}{k - 1} \right)$$

$\therefore$  Total number of nodes in the tree

= maximum # nodes in height 0 +

max # nodes in height 1 + ...

+ max # nodes in height h

$$= k^0 + k^1 + k^2 + \dots + k^h$$

$$= \frac{k^{h+1} - 1}{k - 1}$$

$$\textcircled{3} \quad i \leq \frac{k^h - 1}{k - 1}$$

To get minimum:

$$\textcircled{1} \quad n \geq h+1$$

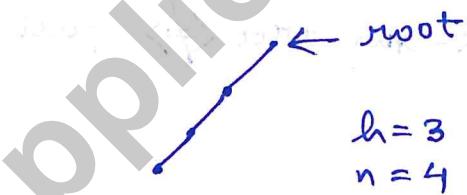


Fig: Chain/line  
(k-ary tree)  
(extremely skewed tree)

$$\textcircled{2} \quad i > h$$

To summarize:

$$h+1 \leq n \leq \frac{k^{h+1} - 1}{k - 1}$$

$$h \leq i \leq \frac{k^h - 1}{k - 1}$$

Note: We know,

$$l \leq k^h$$

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$$\boxed{\log_k l \leq h}$$

$$h = \lceil \log_k l \rceil$$

### Spanning Tree of $G_i$ :

Spanning Tree of a graph is a subgraph

$T(G_i)$  of  $G_i$  —

(i) Tree

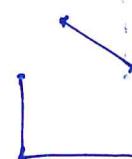
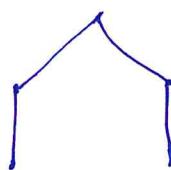
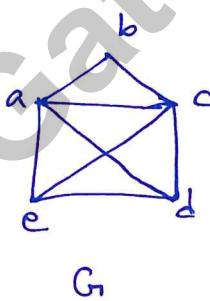
(ii) includes / spans all vertices of  $G_i$

Note: Minimum Spanning Tree is covered in DS/Algo.

### Property:

Every connected graph  $G_i$  has a Spanning Tree (ST)

Note: Spanning Tree need not be a rooted tree.



**APPLIED COURSE** Strategies to count # Spanning Trees: Ph: 844-844-0102

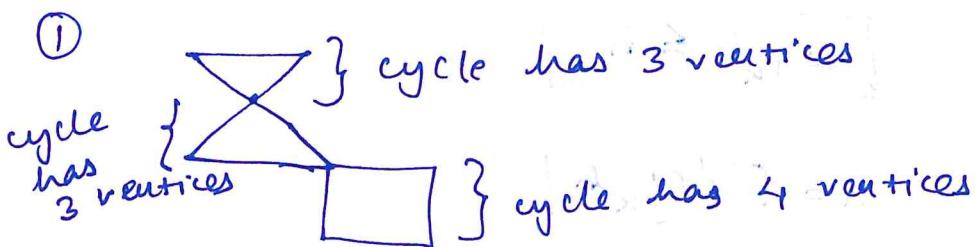


Fig: cycle disjoint graph

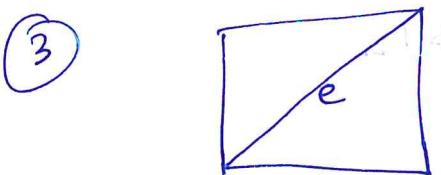
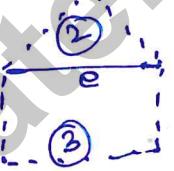
Cycle disjoint graphs have common vertices and no common edges.

$$\therefore \# \text{Spanning Tree} = 3C_2 \times 3C_2 \times 1C_3 = 36$$



We count all ST without  $e \rightarrow 5C_1 = 5$

We count all ST with  $e \rightarrow 3C_1 \times 2C_1 = 6$



Without  $e \rightarrow 4C_1$

$$4C_1$$

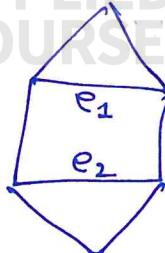
$$2C_1 \quad 2C_1$$

$$\text{With } e \rightarrow 2C_1 \times 2C_1 = 4$$

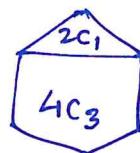
$$\frac{2C_1 \times 2C_1}{8} = 4$$

$$\frac{2C_1 \times 2C_1}{8} = 4$$

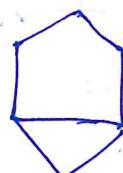
④



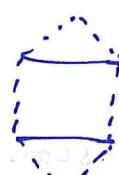
$$\left. \begin{array}{l} e_1 \text{ & } e_2 \text{ both are} \\ \text{not used} \end{array} \right\} \rightarrow 2c_1 = 6$$



$$\left. \begin{array}{l} e_1 \text{ is used but} \\ \text{not } e_2 \end{array} \right\} \rightarrow 2c_1 \times 4c_3 = 8$$



$$\left. \begin{array}{l} e_2 \text{ is used but} \\ e_1 \text{ is not used} \end{array} \right\} \rightarrow 2c_1 \times 4c_3 = 8$$



$$\left. \begin{array}{l} e_1 \text{ and } e_2 \text{ both} \\ \text{are used} \end{array} \right\} \rightarrow 2c_1 \times 2c_1 \times 2c_1 = 8$$

30

Note:

$$\text{Rank}(G) = n - k$$

$$\text{Nullity}(G) = e - \text{rank}(G) = e - n + k$$

$$\text{Rank} + \text{Nullity} = e$$



$$\text{rank} = 4$$

$$\text{nullity} = 2$$

## Case I : Connected graph

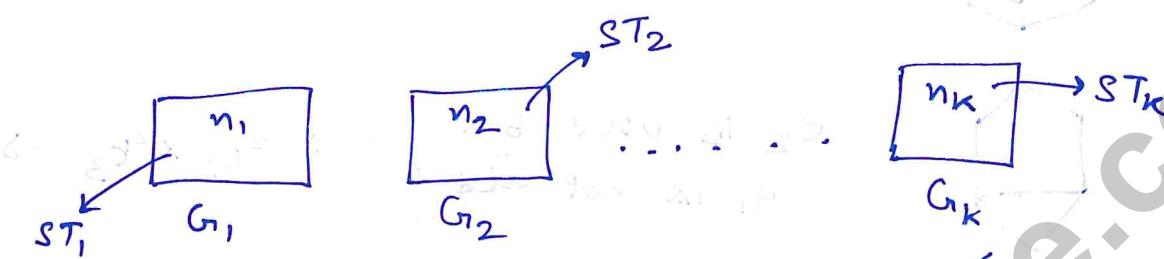
In the context of connected graph :

Ph. 844-844-0102

$$\text{rank}(G) = n - 1$$

= # edges in the S.T of  $G$

## Case II : Disconnected graph $G$



The set of spanning trees in a disconnected graph is called a spanning forest (SF).

The size of  $ST_1 = n_1 - 1$

The size of  $ST_2 = n_2 - 1$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The size of  $ST_k = n_k - 1$

$$(n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1) = \# \text{ edges in the Spanning Forest (SF)}$$

$$\Rightarrow \underbrace{(n_1 + n_2 + \dots + n_k)}_{\text{rank}(G)} - k \cdot 1$$

$$\text{rank}(G) = (n - k) = \# \text{ edges in SF}$$

### Summary:

$\text{rank}(G) = \# \text{ edges in S.T orc (S.F of } G)$   
based on whether the tree is connected or disconnected.

Nullity of graph  $G_1$  = Nullity( $G_1$ ) =  $\#$  edges that need to be removed from the graph to make it a spanning tree or spanning forest.

Nullity( $G_1$ ) is also called cyclomatic complexity which is the number of edges to be removed to break all cycles.

Branch Set is the set of all edges in S.T/S.F

$$|\text{Branch set}| = \text{Rank}(G_1)$$

$$\boxed{|\text{BS}| = \text{Rank}(G_1)}$$

Chord set = set of edges that need to be removed from a graph to make it a spanning tree or spanning forest.

$$|CS| = \text{Nullity}(G_1)$$

#### ④ Graph Counting:

Enumeration of Graphs: Here we count the number of possible graphs (simple and undirected)

$$\textcircled{1} \# \text{labelled graph with } n \text{ vertices} = 2^{nC_2}$$

$$\textcircled{2} \# \text{simple labelled graphs given } n, e = \frac{n(n-1)}{2} C_e$$

- ③ # labelled Trees given  $n = n^{(n-2)}$

Cayley's formula

$$\# \text{ ST in } K_n = n^{(n-2)}$$

- ④ # rooted labelled trees given  $n$  vertices

$$= \frac{n \cdot n^{n-2}}{n} = n^{n-1}$$

n ways  
to select root

- ⑤ # labelled subgraphs of  $K_n$  (labelled)

$$= \sum_{i=1}^n {}^n C_i \underbrace{a^{i(i-1)/2}}_{\substack{i \text{ vertices} \\ \# graphs on \\ i \text{ vertices}}} \quad \begin{array}{l} \text{(i) graphs on } i \text{ vertices} \\ \text{(ii) graphs on } i-1 \text{ vertices} \end{array}$$

Each of such graphs  
are subgraphs of  $K_n$ .

E.g. # graphs with  $n$  vertices and at least

$$\frac{n(n-1)}{4} \text{ edges}$$

$$= {}^n C_2 + \frac{{}^n C_3}{\frac{n(n-1)}{4}} + \dots + \frac{{}^n C_{\frac{n(n-1)}{2}}}{\frac{n(n-1)}{2}}$$

at least

at most  
as number of  
edges with  $n$  vertices

$$= {}^n C_2 = \frac{n(n-1)}{2}$$

# unlabelled graphs with 3 vertices = 4

0 edges

1 edges

2 edges

3 edges



} 4

# unlabelled graphs with n=4

0 edges



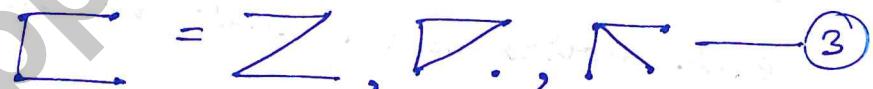
1 edges



2 edges



3 edges



4 edges



5 edges



6 edges



Total distinct possibilities = 11

(Q) # subgraphs with  $e=3$ ,  $n=5$  (labelled) PH. 844-844-0102

$$\begin{aligned}
 3 \text{ vertices & 3 edges} &\rightarrow {}^5C_3 \cdot 1 \quad \triangle \\
 4 \text{ " } \Delta \text{ 3 edges} &\rightarrow {}^5C_4 \cdot {}^{4C_2}_{C_3} \quad : \\
 5 \text{ " } \Delta \text{ 3 edges} &\rightarrow {}^5C_5 \cdot {}^{5C_1}_{C_3} \\
 &\hline
 \end{aligned}$$

Answer :  ${}^5C_3 + {}^5C_4 \cdot {}^{4C_2}_{C_3} + {}^5C_5 \cdot {}^{5C_1}_{C_3}$

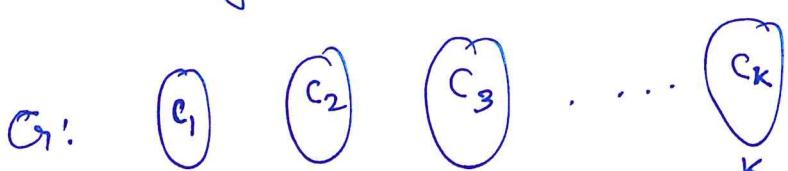
\* Graph coloring, Independence Sets &  
Dominating Sets.

Proper vertex coloring: Assigning a color to each vertex such that no two adjacent vertices have same color.

Min # colors required for vertex-coloring  
= chromatic number  $\chi(G)$

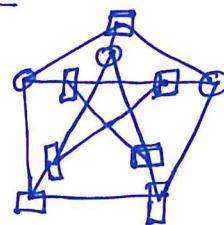
2-coloring  $\Leftrightarrow$  bipartite

$K$ -coloring  $\Leftrightarrow$   $K$ -partite



Here,  
 $C_i$  is the  
 $i$ th color.

Peterson Graph:



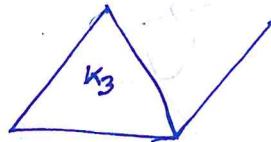
□, ○, ▱ denote different colors.

$$\chi(G) = 3$$

$$\textcircled{1} \quad 1 \leq \chi(G) \leq n$$

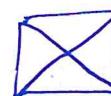
\textcircled{2} Clique: complete subgraph of  $G_1$

$G_1:$



The clique number  $\omega(G_1)$  is the largest size clique in a graph.

$$\chi(K_n) = n$$

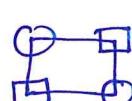


$G_1$  is  $K_4$

$$\boxed{\chi \geq \omega}$$

i.e., we will need atleast  $\omega$  colors to color the complete subgraph of  $G_1$  and remaining vertices may or maynot need new colors.

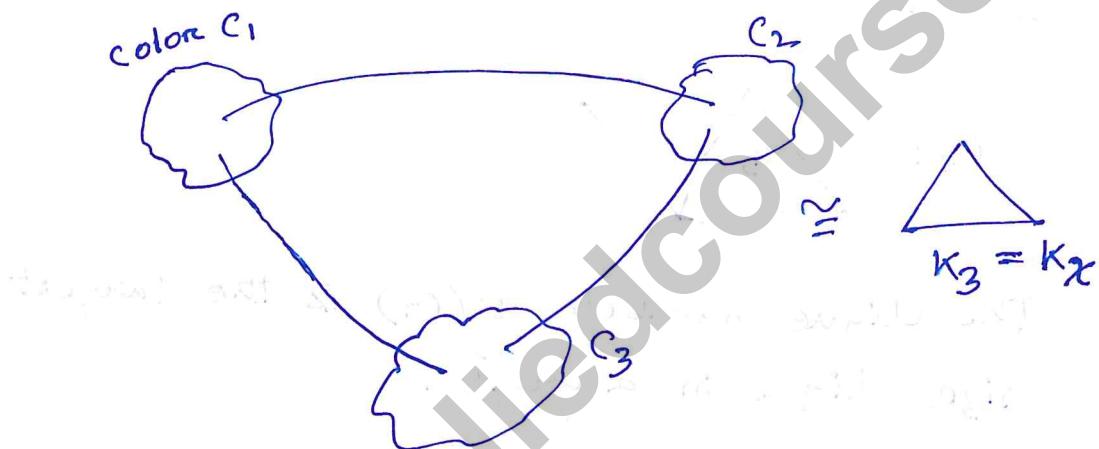
$$\textcircled{3} \quad \chi(C_n) = \begin{cases} 2 & \text{if even cycle} \\ 3 & \text{if odd cycle} \end{cases}$$



$$\textcircled{4} \quad \frac{\chi(\chi-1)}{2} \leq e$$

$\chi_{C_2} \leq e$

Assume a graph is divided in such a way that the same color vertices are drawn together. Say,  $\chi(G_1) = 3$ ,



Now, given a complete graph with chromatic number( $\chi$ ), # edges  $\geq \chi_{C_2}$

$\textcircled{5} \quad \chi \leq \max_{\text{odd } C_n} \text{maximum degree or } \Delta, \text{ except } K_n \text{ and odd } C_n.$

In case of  $K_n$ ,  $\chi(K_n) = n$   
 $\Delta(K_n) = (n-1)$

In case of odd  $C_n$ ,  $\chi(C_n) = 3$

$$\Delta(C_n) = 2$$

Therefore,  $\boxed{\chi \leq (\Delta+1) \leq n}$

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⑥  $\chi(\text{planar graph}) \leq 4$  ~~Ph: 4 color Theorem~~ Ph: 8448440102

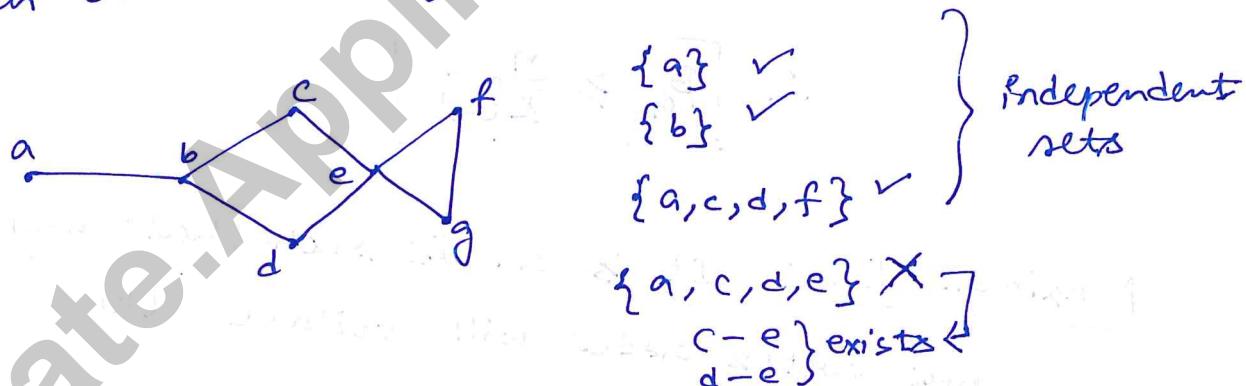
Four color theorem states that the vertices of a planar graph can be colored with at most 4 colors so that no two vertices receive the same color.

⑦  $\chi(G) = 1 \iff \phi_n$

⑧  $\chi(\text{multigraph}) = \chi(\text{simple graph by dropping multiple edges})$

Independence set / Anti-clique (IS):

Set of vertices that are not adjacent to each other in a graph.

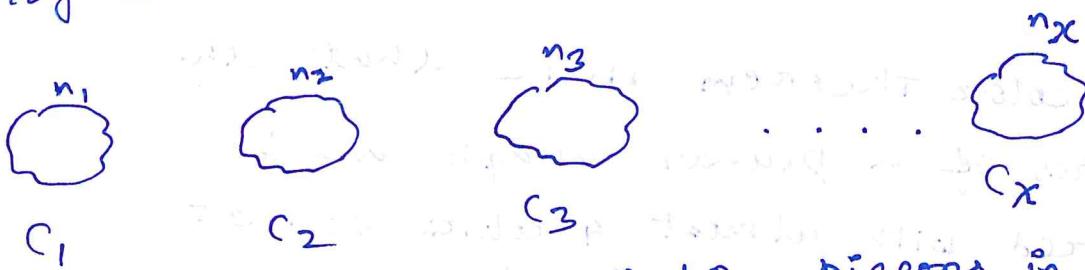


Independence number ( $\beta_G$ ) is the size of the largest sized IS.

$$\beta_G \geq \frac{n}{\chi(G)} \quad [\text{can be proved using Pigeonhole principle}]$$

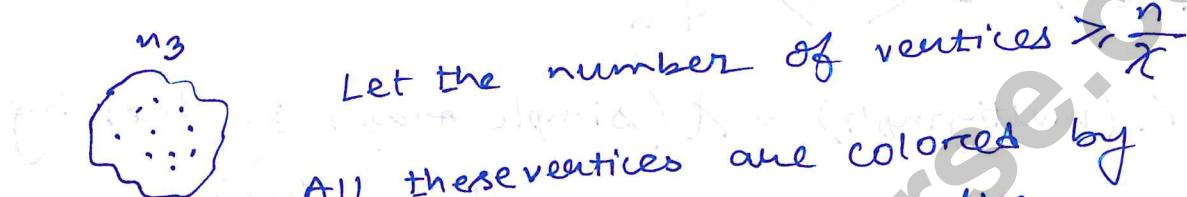
Let vertex represent pigeon and hole represented by color.

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We are trying to distribute  $n$  pigeons in  $x$  holes.

∴ one hole that has  $\geq \frac{n}{x}$  vertices



Let the number of vertices  $\geq \frac{n}{x}$

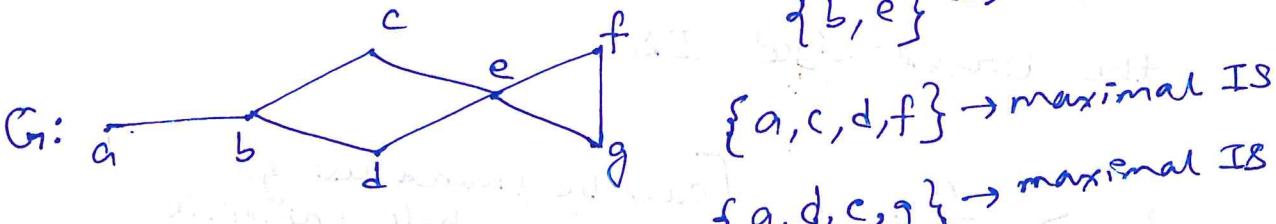
All these vertices are colored by same color. Therefore, they don't have edge between them.

∴ They form independent set.

There exists an independent set which has  $\geq \frac{n}{x}$  elements,

$$\therefore P_G \geq \frac{n}{x(a)}$$

Maximal IS: It is an IS such that any new vertex added will violate the property of IS.



$$P_{G_1} = \text{size of largest maximal IS}$$

① Start with min-degree vertex and ascending order of degree

② Keep adding vertices by checking adjacency and ensuring that it does not violate the property of independent set.

In the graph  $G_1$ ,

$$\deg(a) = 1$$

$$\deg(b) = 2$$

$$\deg(c) = 2$$

$$\deg(d) = 2$$

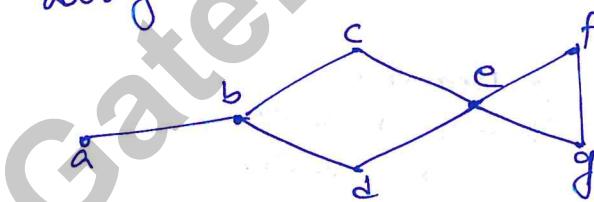
$$\deg(f) = 2$$

$$\deg(g) = 2$$

$$\deg(e) = 4$$

$$\therefore \text{maximal IS} = \{a, c, d, f\}$$

Domination Set (DS): It is a set of vertices from which all vertices are just one step away.



$\{e, b\} \rightarrow$  minimal set

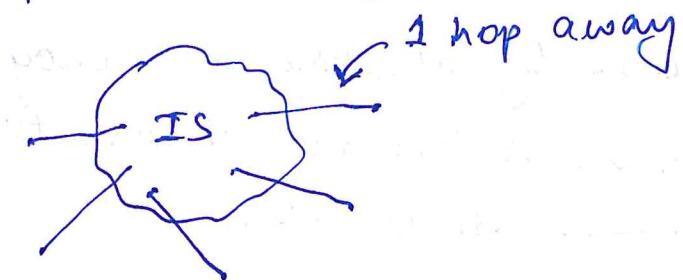
$\{a, c, d, f\} \rightarrow$  minimal set

Domination number ( $\alpha$ ):

= It is the size of the smallest DS.

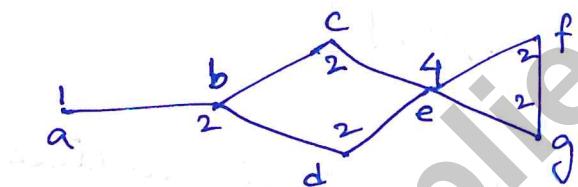
= It is the size of the smallest minimal DS.

$\Rightarrow$  Dominating set



To determine the smallest DS:

- ① start with maximal degree vertex
- ② Keep adding vertices in order as long as the set is a dominating set and you are reaching new vertices which are 1 hop away.



$\{e, b\}$

Theorem:  $\chi_{G_i} \leq \beta_{G_i} \rightarrow$  in Independence number

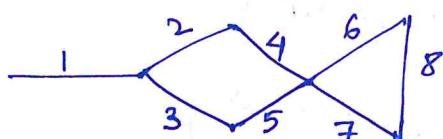
Domination number

$\therefore$  maximal IS  $\Rightarrow$  D.S  
 $\therefore$  Independence #  $\geq$  Domination number

\* Matchings & Coverings:

Note: IS, DS and coloring are related to vertices.

Matchings: set of edges none of which are adjacent

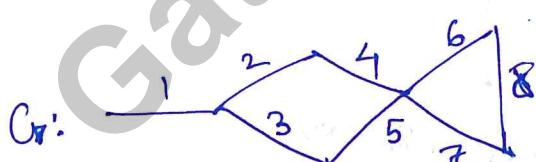


- {1} ✓ — not maximal matching
- {1, 4} ✓ — " "
- {1, 2} ✗ since they have common vertex,
- {1, 4, 8} ✓ — maximal matching

Matching number ( $M_G$ ): size of largest maximal matching.

Q. To obtain a maximal matching:

- ① start with an edge with least # of adjacent edges
- ② keep adding edges without violating the matching property.



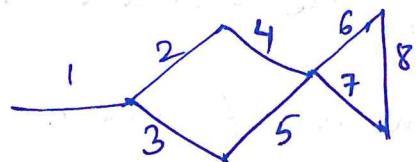
$\{1, 8, 4\}$  → maximal  
 $\{1, 8, 5\}$

$$\therefore M_G = 3$$

Edge 1 is adjacent to 2 vertices

- Edge 2 → 3
- Edge 3 → 3
- Edge 4 → 4
- Edge 5 → 4
- Edge 6 → 4
- Edge 7 → 4
- Edge 8 → 2

Covering: set of edges that covers all vertices



$\{1, 4, 6\}$  — not a covering

$\{1, 2, 3, 4, 5, 6, 7, 8\}$  — covering

$\{1, 4, 8\}$  — not a covering

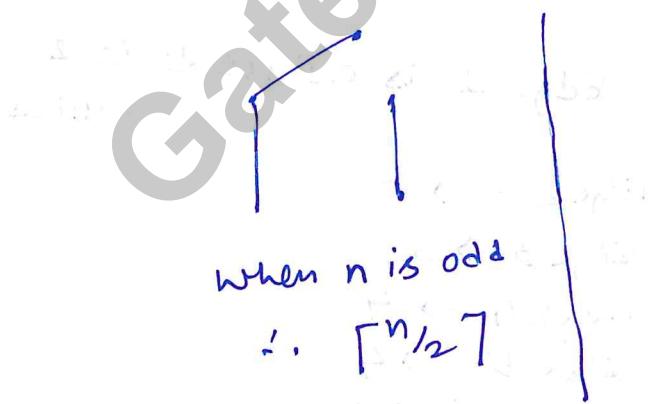
$\{1, 2, 3, 7, 6\}$  — covering (minimal covering)

Covering number ( $C_G$ ): size of smallest minimal covering.

$\{1, 4, 5, 8\}$  — smallest minimal covering

$$\therefore C_G = 4$$

Theorem:  $C_G \geq \lceil \frac{n}{2} \rceil$



when  $n$  is odd

$$\therefore \lceil \frac{n}{2} \rceil$$

when  $n$  is even

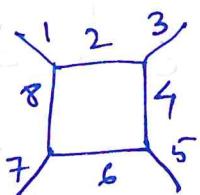
$$\lceil \frac{n}{2} \rceil \text{ holds}$$

Theorem: Pendant edge is always part of the covering.

A Spanning Tree (ST) will always contain the pendant edge, otherwise spanning tree (ST) is not spanning over all the vertices.

Build a covering set!

- ① Include all pendant edges.
- ② Include edges that cover the remaining vertices.



$$\{1, 3, 5, 7\}$$

$$C_G = 4$$

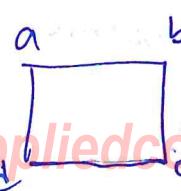
Theorem: (a) Covering need not be a matching

↓  
adjacent edges are allowed.

↓  
adjacent edges are not allowed.

(b) Similarly, a matching need not be a covering. Because a matching need not cover all vertices.

Perfect matching: Set of edges which are both matching and covering.



$\{ab, cd\}$  are not adjacent edges

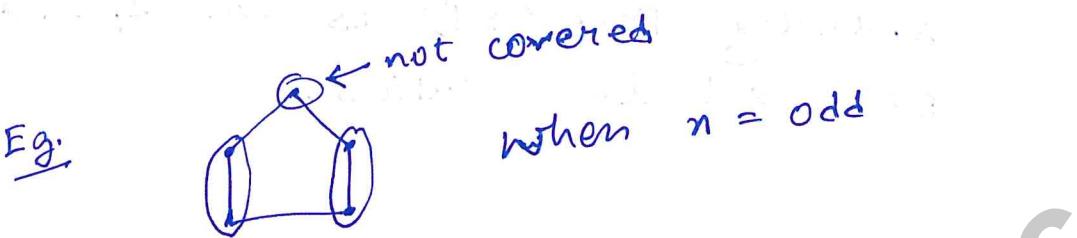
$\{ab, cd\}$  covers all vertices

∴  $\{ab, cd\}$  is perfect matching.

Property: Perfect matching is possible when # vertices is even.

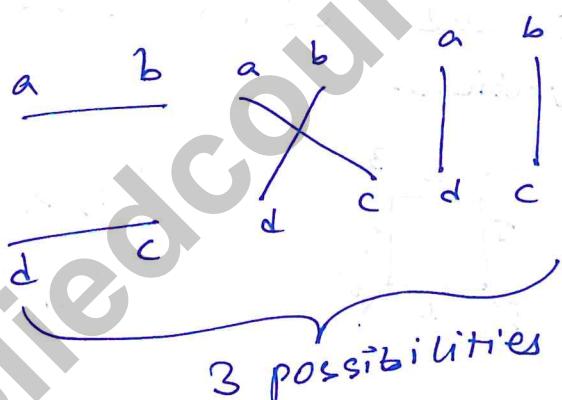
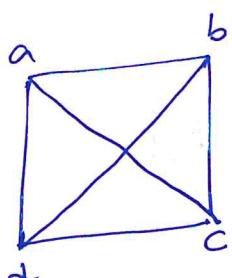
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Note: Every graph with even # vertices need not be perfect matching.



Q. Given  $K_{2n}$ , how many perfect matchings possible?

$K_4$ :



Given  $2n$  vertices,

$\underbrace{1, 2, 3, 4, \dots, n}_{2n}$

- ① order:  $(2n)!$
- ② construct  $n$  pairs.

$n=2$ ,

$\underbrace{1, 2}_{\text{same}},$	$\underbrace{3, 4}_{\text{same}}$
$\underbrace{3, 4}_{\text{same}},$	$\underbrace{1, 2}_{\text{same}}$

$n=4$ ,

$\underbrace{1, 2}_{\text{same}},$	$\underbrace{3, 4}_{\text{same}},$	$\underbrace{5, 6}_{\text{same}},$	$\underbrace{7, 8}_{\text{same}}$
$\underbrace{5, 6}_{\text{same}},$	$\underbrace{3, 4}_{\text{same}},$	$\underbrace{1, 2}_{\text{same}},$	$\underbrace{7, 8}_{\text{same}}$

$n$  pairs can be ordered in  $n!$  ways

$$\therefore \frac{(2n)!}{n!}$$

Note, every pair can be written in 2 ways.

E.g.  $(1, 2)$  is same as  $(2, 1)$ .

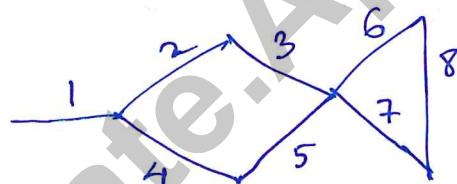
$$\therefore \frac{(2n)!}{n!} \times \left(\frac{1}{2^n}\right)$$

there are  $n$  pairs and each pair can be ordered in 2 ways.

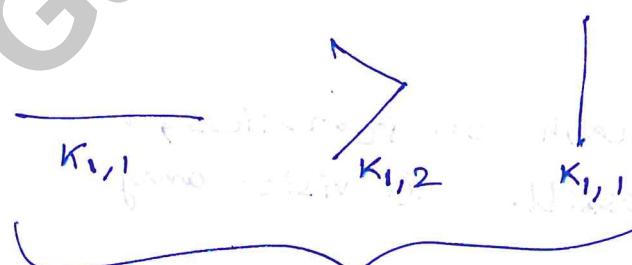
$$= \frac{(2n)!}{n! 2^n}$$

Theorem: A covering component is minimal iff every

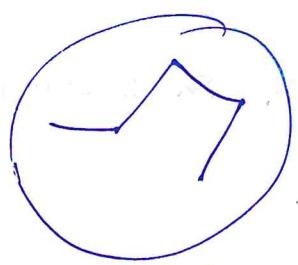
component is a star graph ( $K_{1,n}$ )



$\{1, 3, 5, 8\}$  minimal covering



Each components are star graphs



$K_{1,3}$

This component  
is not a star graph

$\therefore \{1, 2, 3, 8, 5\}$  is not minimal.

Graph Theory      Previous Year      GATE Qn

Q. Let  $G_n$  be an undirected complete graph on  $n$  vertices, where  $n > 2$ . Then, the number of different Hamiltonian cycles in  $G_n$  is equal to : (GATE 2019)

- (A)  $n!$
- (B)  $n-1!$
- (C) 1
- (D)  $(n-1)! / 2$

In an undirected graph on  $n$  vertices,  $n$  permutations are possible to visit every node.

- a)  $n$  different places (nodes) you can start
- b) 2 (clockwise or anti-clockwise) different directions you travel.