CHAPTER

THREE

RELATIONS AND FUNCTIONS

3.1 The set of outfits the man can wear; Coordinated outfits.

3.2 $\{\phi, \{1\}, \{2\}, \{1, 2\}\} \times \{1, 2\} = \{(\phi, 1), (\phi, 2), (\{1\}, 1), (\{1\}, 2), \ldots\}$

3.3 (a) If $(a, b) \in A \times B$, then $a \in A \subseteq C$ and $b \in B \subseteq D$, so $(a, b) \in C \times D$.
(b) If $A = C = \phi$, it is possible that $B \not\subseteq D$.

3.4 (a) Yes, It is ϕ.
(b) Either A or B or both are equal to ϕ.
(c) $\phi \subseteq \phi \times \phi$

3.5 (a) $(a, c) \in (A \cap B) \times (C \cap D)$ implies $a \in A$, $c \in C$. Thus, $(a, c) \in A \times C$.
Also, $a \in B$, $c \in D$ so $(a, c) \in B \times D$.
If $(a, c) \in (A \times C) \cap (B \times D$ then $a \in A \cap B$ and $c \in C \cap D$.
(b) All are false. Any non-trivial sets $A \neq B$ and $C \neq D$ will provide a counter example.

3.6 (a) $(a, c) \in (A \cap B) \times C$ iff $a \in A \cap B$ and $c \in C$, iff $(a, c) \in A \times C$ and $(a, c) \in B \times C$.
(b) They are all true and can be easily shown.

3.7 R_1 and R_2 might be channels that viewers one and two want to watch (or not to watch) at given hours. Then $R_1 \cup R_2$ would be the stations desirable to one or the other. $R_1 \cup R_2$ to both, $R_1 \oplus R_2$ to one or the other but not both, $R_1 - R_2$ to viewer one but not viewer two.

3.8 (a) Yes, as Cartesian coordinates.
(b) The line from (a, b) to (c, d) makes a 45° angle with the coordinate axes.
(c) The points (a, b) and (c, d) are within a distance 10 of each other.

(c, d) is on the line $\{(a + z, b + z) \mid x \in I\}$ or is within distance 10 of (a, b);
(c, d) is on the line segment, $\{(a + x, b + z) \mid |x| \le 7)$;
the rest of the line;
the rest of the line plus the points not on the line within distance 10 of (a, b).

3.9 Let $R_3 = \{(a_1, a_2) \mid$ there exists $b \in B$ such that $(a_1, b), (a_2, b) \in R_1\}$. Must have $R_3 \subseteq R_2$.

3.10 (a) No, No, Yes, Yes.
(b) No, No, No, No.

3.11 (a) No, No, Yes, Yes, No, No.
(b) No, No, No, No, No, No.

3.12 No, No, No, Yes, No, No.

3.13 Yes, Yes, No, Yes, Yes, No.

3.14 Ham sandwich is better than nothing.
Nothing is better than eternal happiness.
By transitivity, ham sandwich is better than eternal happiness.

3.15 (a) 2^{100}
(b) 2^{90}
(c) 2^{55}
(d) 2^{45}
(e) 10!

3.16 If (a, b) in R, by symmetry (b, a) is in R. Then (a, a) is in R by transitivity.

3.17 T is clearly reflexive and symmetric. To show transitivity, let (a, b) and (b, c) be in T. Then (a, b) and (b, a), (b, c), and (c, b) are in R. Since R is transitive, (a, c) and (c, a) are in R so (a, c) is in T.

3.18 Reflexivity: $(a, a) \in R \Rightarrow (a, a) \in S$

Symmetry: $(a, b) \in S \Rightarrow (a, c) \in R$ and $(b, c) \in R$
$\Rightarrow (c, a) \in R$ and $(b, c) \in R$
$\Rightarrow (b, a) \in S$

Transitivity: $(a, q) \in S$ and $(q, b) \in S$
$\Rightarrow (a, n) \in R, (n, q) \in R, (q, x) \in R, (x, b) \in R$
$\Rightarrow (a, q) \in R$ and $(q, b) \in R$
$\Rightarrow (a, b) \in S$

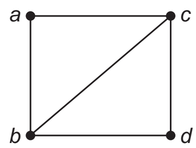
3.19 (If part)

Symmetry: $(a, b) \in R$ and $(a, a) \in R \Rightarrow (b, a) \in R$

Transitivity: $(a, b) \in R$ and $(b, c) \in R \Rightarrow (b, a) \in R$ and $(b, c) \in R$
$\Rightarrow (a, c) \in R$

(Only-if part)
If $(a, b), (a, c) \in R$ then $(b, a) \in R$ and hence $(b, c) \in R$.

3.20 (a), (b), (c) are trivial.
(d) Yes, Yes.
(e) a is related to b if a and b are in the same A_i. A cover is a collection of sets of mutual friends such that each person is in at least one set.
(f) The answer can either be Yes or No. For example, for the compatible relation below, either one of $\{\{a, b, c\}, \{b, c, d\}\}$, and $\{\{a, b\}, \{a, c\}, \{b, c, d\}\}$ is a cover.



3.21 (a) Suppose $(a, b) \in R_1$. Then either $(a, b) \in R$, in which case $(b, a) \in R$, or $(a, c), (c, b) \in R$ for some c. Then (c, a) and $(b, c) \in R$ so $(b, a) \in R_1$ and the transitive extension of a symmetric relation is symmetric. Thus $R_1, R_2, \ldots$ are each symmetric. Suppose $(a, b) \in R^*$ then $(a, b) \in R_i$ for some i. Since R_i is symmetric $(b, a) \in R_i \subseteq R^*$.
(b) No. $\{(0, 1), (1, 2), (2, 0)\}$ is antisymmetric but its transitive closure is not.
(c) From part (a) it is symmetric. $R \subseteq R^*$ so it is reflexive. R^* is transitive since $(a, b), (b, c) \in R^*$ implies $(a, b) \in R_i$, $(b, c) \in R_j$ and (a, b), (b, c) both are in R_k, $k = \max\{i, j\}$. Thus, $(a, c) \in R_{k+1} \subseteq R^*$.

3.22 (a) Yes.
(b) Yes, Yes, Yes.

3.23 Basis: $i = 0$, Trivial.
Induction Step: If $(a, b) \in R_i$, then either $(a, b) \in R_{i-1}$ or $(a, c), (c, b) \in R_{i-1}$ for some c. In the former case, by induction hypothesis, there exists $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, b) \in R$ where $n \le 2^{i-1} - 1$. For the latter case, there exists two chains of pairs $(a, x_1), (x_1, x_2), \ldots, (x_{m-1}, x_n), (x_n, c)$ and $(c, y_1), (y_1, y_2), \ldots, (y_{p-1}, y_p), (y_p, b)$ all in R where $m \le 2^{i-1} - 1$ and $p \le 2^{i-1} - 1$. Thus the combined chain has $m + p + 1 \le 2^i - 1$ elements.

3.24 (a) For $i \ne j$, $A_i \cap A_j = \phi$. Since if $a \in A_i \cap A_j$, $b \in A_i$ and $c \in A_j$ then $(b, a) \in R$ and $(a, c) \in R$. By transitivity, $(b, c) \in R$ and thus b and c

are in the same subset for any $b \in A_i$, $c \in A_j$. Since no inclusion is possible $A_i \cap A_j = \phi$. For any $a \in A$, $(a, a) \in R$ so a is in some subset.

Thus, $A = \bigcup_{i=1}^{k} A_i$.

(b) Since $A = \bigcup_{i=1}^{k} A_i$, then for any $a \in A$, a must be in some A_i. Hence $(a, a) \in R$ and R is reflexive. If $a, b \in A_i$ then both (a, b) and (b, a) are in R so R is symmetric. If $(a, b) \in R$, $(b, c) \in R$ then a and b are in the same A_i and b and c are in the same A_j. Since $A_i \cap A_j = \phi$ for $i \neq j$, a, b, and c are in the same A_i. Thus, $(a, c) \in R$.

3.25 Reflexivity: For $a \in S$, $((f(a), f(a)) \in R_1$, thus $(a, a) \in R_2$.
Symmetry: $(a, b) \in R_2, \Rightarrow ((f(a), f(b)), \in R_1 \Rightarrow (f(b), f(a)) \in R_1 \Rightarrow (b, a) \in R_2$.
Transitivity: $(a, b), (b, c) \in R_1 \Rightarrow (f(a), f(b)), (f(b), f(c)) \in R_1 \Rightarrow ((f(a), f(c)) \in R_1 \Rightarrow (a, c) \in R_2$.

3.26 (a) Yes, Yes, No.
(b) Yes, No.
(c) a, b are in the same block of π_1 and π_2 if and only if they are in the same block of $\pi_1 \cdot \pi_2$. Hence if a, b are in the same block of $\pi_1 \cdot \pi_2$ then $f(a)$ and $f(b)$ are in the same block of π_1 and π_2, hence of $\pi_1 \cdot \pi_2$.
If a and b are in the same block of $\pi_1 + \pi_2$ then there is a chain $a + c_1, c_2, \ldots, c_k = b$ such that for $i = 1, \ldots, k - 1$, c_i and c_{i+1} are in the same block of π_1 or of π_2. Hence $f(c_i)$ and $f(c_{i+1})$ are in the same block of π_1 or of π_2 and hence of $\pi_1 + \pi_2$.

3.27 (a) Transitivity: $a \leq_R b, b \leq_R c \Rightarrow b \leq a, c \leq b \Rightarrow a \leq_R c$
Reflexivity: $a \leq a \Rightarrow a \leq_R a$
Antisymmetry: $a \leq_R a, b \leq_R a \Rightarrow b \leq a, a \leq b, \Rightarrow a = b$
(b) If $glb\ (a, b) = c$ then c is the unique $lub_R\ (a, b)$. If lub $(a, b) = d$ then d is the unique $glb_R(a, b)$.

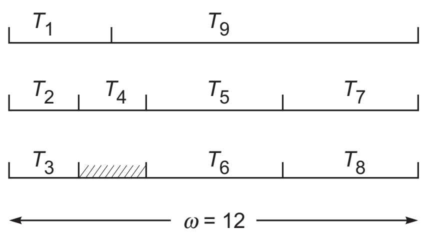
3.28 (a) Reflexive: $a \subseteq a$ for all $a \in P(A) \Rightarrow (a, a) \in R$.
Antisymmetry: (a, b) and $(b, a) \in R \Rightarrow a \subseteq b$ and $b \subseteq a \Rightarrow a = b$.
Transitivity: $(a, b) \wedge (b, c) \in R \Rightarrow a \subseteq b$ and $b \subseteq c \Rightarrow a \Rightarrow c \Rightarrow (a, c) \in R$.
(b) $|A| + 1$.

3.29 Yes.
Proof of claim: Clearly, $(x_1 \vee x_2, y_1 \vee y_2)$ is an upper bound of (x_1, y_1) and (x_2, y_2). Furthermore, it is the least upperbound. Since if (x_3, y_3) is

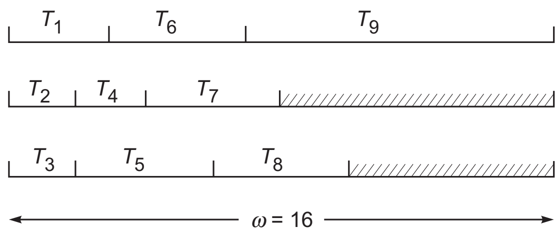
any other upper bound, $x_3 \geq x_1$ and $x_3 \geq x_2 \Rightarrow x_3 \geq x_1 \vee x_2$. Similarly $y_3 \geq y_1 \vee y_2$.

3.30 Let 0 and 1 denote the two elements in L. If $(p, 1) \in A$, then $(p, 0) \notin A$. Replace all $(p, 1) \in A$ by $(p, 0)$, and call the new subset of Q, A'. Clealy, all elements in A' is of the form $(p, 0)$. Moreover, there is no chain of length exceeding 2. It follows that $|B| \geq |A'| = |A|$.

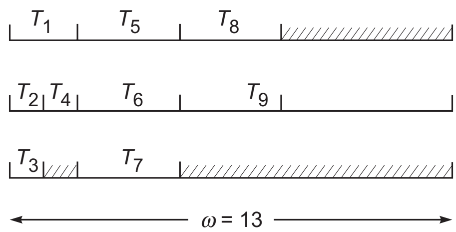
3.31 (a)



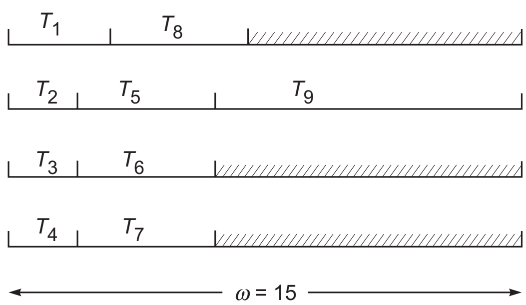
(b)



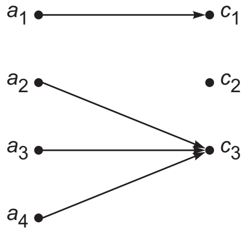
(c)



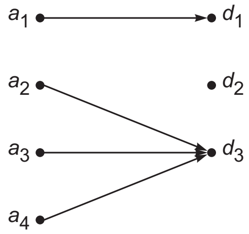
(d)



3.32 (a)



(b) $h \circ (g \circ f) = (h \circ g) \circ f$



(c) For any element $a \in A$,

$$h \circ (g \circ f)\,(a) = h(g \circ f)\,(a) = h(g(f(a))), \text{ and}$$
$$(h \circ g) \circ f(a) = (h \circ g)\,f(a) = h(g(f(a))).$$

(d) (i) g restricted to the range of f must be onto.

(ii) g restricted to the range of f must be one-to-one

(iii) g restricted to the range of f must be one-to-one and onto.

(e) $A = \{a, b\}$, $f(a) = f(b) = a$, $g(a) = g(b) = b$.

3.33 $(f \circ f)\,(n) = n + 2$

$$(f \circ g)\,(n) = 2n + 1$$
$$(g \circ f)\,(n) = 2(n + 1)$$
$$(g \circ h)\,(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$
$$(h \circ g)\,(n) = 0$$
$$(f \circ g) \circ h(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

3.34 (a) $|D| = |R| \cdot i + r \Rightarrow \dfrac{|D|}{|R|} = i + \dfrac{r}{|R|}$

$$\Rightarrow \lfloor |D|/|R| \rfloor = \begin{cases} i+1 & \text{if } r \neq 0 \\ i & \text{if } r = 0 \end{cases}$$

(b) $|D|(\lfloor |D|/|R| \rfloor - 1) = \begin{cases} |D| \cdot i & (<|R|) \quad \text{if } r \neq 0 \\ |D|(i-1) & (<|R|) \quad \text{if } r = 0 \end{cases}$

3.35 There are only 86400 different times in a day and there are 100000 people. By pigeonhole principle, there must be two born at the same time.

3.36 (a) $13 + 13 + 13 + 3 = 42$

(b) $13 + 13 + 13 + 3 = 42$

(c) $13 + 1 + 1 + 1 + 1 = 17$

3.37 (a) $11 + 11 + 11 + 1 = 34$

(b) $10 + 8 + 11 + 11 + 11 + 1 = 52$

(c) $8 + 11 + 11 + 11 + 1 = 42$

3.38 There are 30 numbers each in the class 0, 1, 2 modulo 3. Hence, answer is 61.

$$9 + 1 = 10$$
$$9 + 1 = 10$$

3.39 Total number of courses taken = $4 \times 35000 = 140000$. There are 999 courses. Therefore, there must be a course with at least $\lfloor 140000/999 \rfloor = 141$ students. But the largest classroom can hold only 135 students.

3.40 All 7 tickets should be mutually exclusive.

3.41 (a) Since there is no increasing subsequence of length larger than n, there are at most n distinct labels. If each occurs $\leq n$ times then there could be at most n^2 integers. Hence at least one label occurs $n + 1$ times.

(b) Suppose $x_k = x_l$, for $k < l$. We must have $a_k > a_l$, since $a_k < a_l \Rightarrow x_k \geq x_l + 1$.

3.42 (a) An increasing subsequence. A decreasing subsequence.

(b) There is either a chain of length at least $n + 1$ or an antichain of size at least $n + 1$.

3.43 Consider the remainders of the m integers, $k, k + 1, \ldots, k + m - 1$ when divided by m. If none of the remainders is equal to zero, two of them must be the same. That implies $(k + i) - (k + j)$ is divisible by m, $(i > j)$. However, this is impossible because $i - j < m$.

3.44 Consider the sequence of remainders $r_1, r_2, \ldots$ obtained during the long division, so eventually there must be a repetition which will cause a repetition in the sequence of quotient digits.

3.45 Let $x_1, x_2, \ldots, x_{10}$ denote the numbers of miles he hiked in the ten hours. Assume that

$$x_1 + x_2 < 9$$
$$x_2 + x_3 < 9$$
$$\vdots$$
$$x_9 + x_{10} < 9$$

Then $$x_1 + 2(x_2 + \ldots + x_9) + x_{10} < 81 \text{ and}$$

$$2(x_1 + \ldots + x_{10}) < 90, \text{ since } x_1 = 6, x_{10} = 3.$$

We have $\quad x_1 + \ldots + x_{10} < 45$ which is impossible.

3.46 Let $x_1, x_2, \ldots x_{36}$ denote the numbers assigned to the sectors. Assume that

$$\begin{aligned}
x_1 + x_2 + x_3 &< 55\\
x_2 + x_3 + x_4 &< 55\\
\vdots \quad \vdots \quad \vdots \quad &\vdots\\
x_{34} + x_{35} + x_{36} &< 55\\
x_{35} + x_{36} + x_1 &< 55\\
x_{36} + x_1 + x_2 &< 55
\end{aligned}$$

We have $\quad 3(x_1 + x_2 + \ldots + x_{36}) < 36 \times 55$

or $\quad (x_1 + x_2 + \ldots + x_{36}) < 660$

However, $\quad x_1 + x_2 + \ldots + x_{36} = \dfrac{36 \times 37}{2} = 666.$

3.47 Consider the n sums $x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \ldots + x_n$. Either one of them is divisible by n or two of them have the same remainder when divided by n. In the second case, the difference of the two sums yields the answer.

3.48 Let $a_1, a_2, \ldots, a_{101}$ be the chosen numbers. Write a_i as $b_i \cdot 2^{k_i}$ where b_i is an odd number. By the pigeonhole principle, $b_i = b_j$ for some $i \neq j$. Consequently, we have $\dfrac{a_i}{a_j} = 2^{(k_i - k_j)}$

3.49 If one of the remainders is 0, the result is proved. If not, two of them are the same. That is the two numbers $\underbrace{77\ldots7}_{k\ 7s}$ and $\underbrace{77\ldots7}_{l\ 7s}$ have the same remainder when divided by N. Thus, $\underbrace{77\ldots7}_{l\ 7s} - \underbrace{77\ldots7}_{k\ 7s} = \underbrace{77\ldots7}_{l-k\ 7s}\underbrace{00\ldots0}_{k\ 0s}$ is a multiple of N.

3.50 (a) By the pigeonhole principle, at least two of the numbers have the same remainder when divided by n.

(b) Let r_i denote the remainder of x_i divided by $2n$.

Let $\quad l_i = \begin{cases} r_i & \text{if } r_i \le n \\ 2n - r_i & \text{if } r_i \le n \end{cases}$

Clearly, $0 \le l_i, \le n$. Among the $n + 2$ integers, there are two such that $l_i = l_j$. If $l_i = r_i$ and $l_i, = r_j$, then $2n$ divides $x_i - x_j$. If $l_i = 2n - r_i$ and $l_j = 2n - r_j$ then $2n$ divides $x_i - x_j$. Otherwise, $2n$ divides $x_i + x_j$.

3.51 Let $a_1, a_2, \ldots, a_n$ denote the n distinct integers. Let $b_1, b_2, \ldots b_N$ denote the sequence. We label b_i with an ordered n-tuple $(c_{i1}, c_{i2}, \ldots, c_{in})$ such that $c_{ij} = 1$ if a_j appears an even number of times among $b_1\, b_2 \ldots\, b_i$. Otherwise $c_{ij} = 0$.

Case 1: If for some i b_i is labeled $(0, 0, 0, \ldots, 0)$ then $b_1 \cdot b_2 \ldots\, b_i$ is a perfect square.

Case 2: If $(0, 0, \ldots\, 0)$ never appears, then for $i \neq j$, b_i and b_j have the same label. Thus, $b_{i+1} \cdot b_{i+2} \ldots b_j$ is a perfect square.

3.52 Two of the integers must be consecutive and thus are relatively prime.

3.53 Now $b_i' > b_k'$, for $k = 1, 2, \ldots, i - 1$. Hence, it must be greater than, at least, i of the a_is. Suppose $a_i' > b_i'$, then clearly $a_k' > b_i'$ for all $k = i + 1, i + 2, \ldots, n$. Thus at most $i - 1$ of the a_is can be smaller than b_i. A contradiction.

3.54 There are $i - 1$ numbers less than a_i and $n - i$ numbers bigger than b_i so the larger of the two is, at least, the n^{th} largest while the smaller of the two is at most the $(n - 1)^{\text{th}}$ largest.