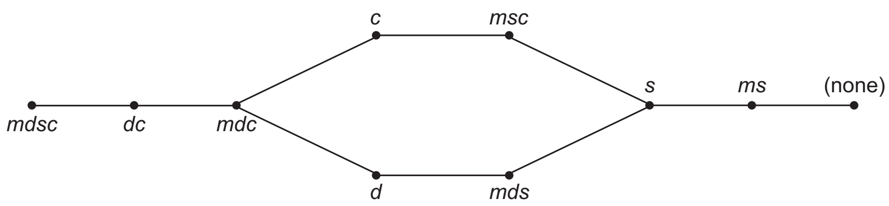
CHAPTER

FOUR

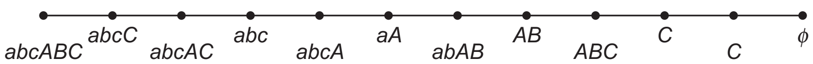
GRAPHS AND PLANAR GRAPHS

4.1



Each of the two paths, from *mdsc* to (none), on this graph represents a solution.

4.2 (a) Let (a, A), (b, B), (c, C) denote the three couples with uppercase letters for the women. Let the labels indicate those left on the west bank (assuming they are going eastward):



11 trips is known to be the minimum number needed.

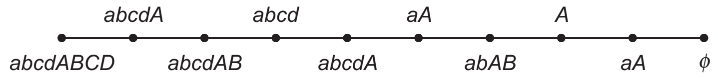
(b) At some point, the number of people on the west bank must decrease from 8 to 3, meaning that there are 3 people on each bank and 2 on the boat.

(i) If either side is all women, then either before the boat leaves or after it arrives, at least one of these women must be in the presence of men without her husband.

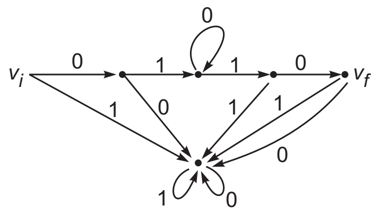
(ii) Neither bank can have one man and two women.

(iii) Either because of (i) or (ii), the possibility of having 3 men on one bank is disallowed.

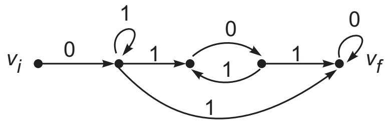
(iv) The only alternative left is to have 2 men and 1 women on both banks. That case, there are two women on the boat. Either before the boat leaves or after it arrives, there will be 2 men and 3 woman.

(c) 

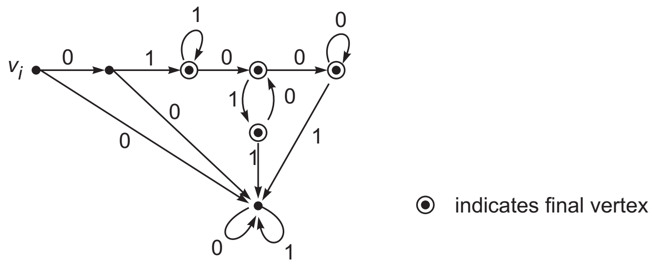
4.3



4.4 (a)

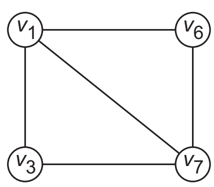


(b)



4.5 (a) An edge between v_i and v_j means all the meetings, i, $i + 1, \ldots, j - 1$ can be scheduled without changing the starting times of v_i and v_j. A complete subgraph containing $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ is a set of meetings, all of which can be scheduled without a change in starting time. A largest complete subgraph signifies the most meetings which can be scheduled without any changes.

(b) Add a node v_7 to the graph and consider it to represent a starting time of 5:00. Construct the resulting graph. Now, we want to find a largest complete subgraph containing both v_1 and v_7. Thus, we delete any nodes from our graph that are not adjacent to either v_1 and v_7 and then find a largest complete subgraph. The resulting graph is



Two largest complete subgraphs are v_1, v_3, v_7 and v_1, v_6, v_7. Thus, a possible scheduling is

Meeting	*Starting*	*Or*
1	8:00	8:00
2	9:30	9:30
3	11:00	10:30
4	1:00	12:30
5	3:00	2:30
6	4:30	4:00

A minimum of 4 changes is necessary.

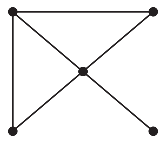
4.6 We show that $e + k \geq n$. When $e = 0$, $k = n$ and so the inequality is trivially satisfied. Each time e increase by 1, the number k of components either remain unchanged, or decrease by 1, thus the above inequality still holds.

4.7 If a maximal connected subgraph has t vertices then the graph has at most

$\binom{t}{2} + \binom{n-t}{2}$ edges.

But $\binom{t}{2} + \binom{n-t}{2} < \binom{n-1}{2} = \dfrac{(n-1)(n-2)}{2}$ for all t, $1 \leq t \leq n - 1$.

4.8 (a)



(b) If there are two vertices of degree 3, there cannot be a vertex or degree 1.

(c) If there is a graph whose degree sequence is ($d_2 - 1$, $d_3 - 1$,…, $d_{d_1} - 1$, $d_{d1+1} - 1$, d_{d1+2},…, d_n), we can add a vertex of degree d_1 to the graph to yield a graph whose degree sequence is (d_1, d_2, d_3,…, d_n).

On the other hand, suppose there is a graph G whose degree sequence is (d_1, d_2, d_3,…, d_n). If there exists a vertex of degree d_1 that is adjacent to vertices of degree d_2,…, d_3, d_{d1+1}, clearly, d_{2-1}, d_{3-1}, …, $d_{d1+1} - 1$, d_{d1+2},…, d_n) is graphical. Suppose there is no such vertex in G. Let v_1 be a vertex of degree d_1 with the sum of the degrees of the vertices it is adjacent to being maximum. Then, there exist vertex v_x of degree d_x and vertex v_y of degree d_y such that $d_x > d_y$, the edge $\{v_1, v_x\}$ is not in G and the edge $\{v_1, v_y\}$ is in G. Since $d_x > d_y$, there exists a vertex v_k such that $\{v_k, v_x\}$ is in G and $\{v_k, v_y\}$ is not in G. Removing $\{v_1, v_y\}$ and $\{v_k, v_x\}$ from and adding $\{v_1, v_x\}$ and $\{v_k, v_y\}$ to G we obtain a graph G_1 whose degree sequence is also $(d_1, d_2,\ldots, d_n)$. Such a step can be repeated until a vertex d_1 that is adjacent to vertices of degree $d_2, d_3,\ldots, d_{d1+1}$ is obtained.

(d) The degree sequences are: (5, 5, 3, 3, 2, 2, 2) (4, 2, 2, 2, 1, 1) (1, 1, 1, 1, 0). Since (1, 1, 1, 1, 0) is graphical, so is (5, 5, 3, 3, 2, 2, 2).

4.9 (a) Every edge contributes 1 indegree and 1 out-degree.

(b) According to (a)

$$\Sigma d^+(v_i) = \Sigma d^-(v_i)$$

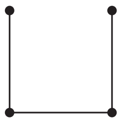
or $$\Sigma[d^+(v_i) - d^-(v_i)] = 0$$

Since $$d^+(v_i) + d^-(v_i) = n - 1$$

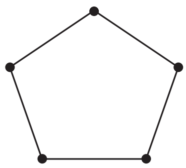
$$\Sigma[d^+(v_i) + d^-(v_i)]\,[d^+(v_i) - d^-(v_i)] = 0$$

$$\Sigma[d^+(v_i)]^2 - \Sigma[d^-(v_i)]^2 = 0$$

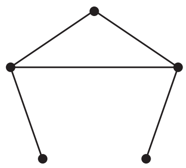
4.10 (a)



(b)



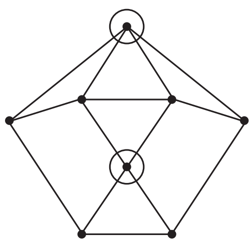
or

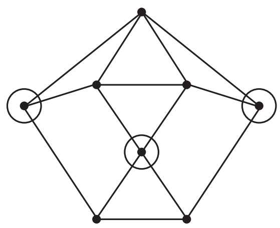


(c) No.

(d) Let G be a self-complementary graph with n vertices and m edges. Then $\dfrac{n\,(n-1)}{2} = 2m$ or $n(n - 1) = 4m$. Thus, either n or $n - 1$ must equal to $4k$ because at most one of them can be even.

4.11 (a)

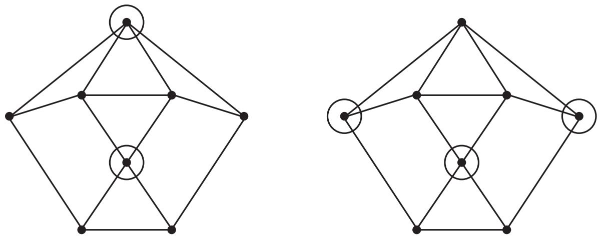




(b) A dominating set is a set of cities at least one of which can communicate with any other city not in the set.

(c) The minimum number of vertices in a dominating set is 5.

4.12 (a)

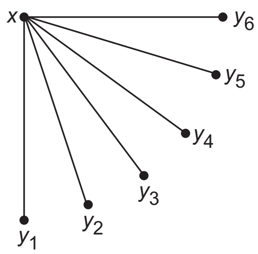


(b) To determine an independent set of size 8 in the graph described in part (c) of Problem 4.11.

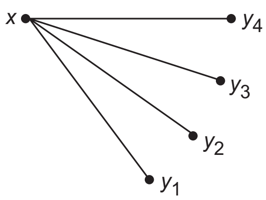
4.13 (a) Let x be one of the vertices. There are at least three red (or blue) edges incident with x. Let us consider only the case that there are at least three red edges, the other case can be taken care of by a symmetrical argument. Let $\{x, y\}$, $\{x, z\}$, $\{x, w\}$ be three red edges incident with x. If any one of the edges $\{y, z\}$, $\{y, w\}$, $\{z, w\}$ is red, we have a red K_3, otherwise, we have a blue K_3.

(b) Let the vertices in K_6 represent six people and red edges represent the friendship relationship and the blue edges represent the "stranger relationship".

(c)



Let $\{x, y_1\}$, $\{x, y_2\}$,…, $\{x, y_6\}$ be six red edges incident with x. If there is a red K_3 among $y_1, y_2,…, y_6$, there is a red K_4. Otherwise, there is a blue K_3 among $y_1, y_2,…, y_6$.



Let $\{x, y_1\}$, $\{x, y_2\}$, $\{x, y_5\}$, $\{x, y_4\}$ be four blue edges incident with x. If there is a blue edge among y_1, y_2, y_3, y_4, there is a blue K_3. Otherwise, there is a red K_4 among y_1, y_2, y_3, y_4.

(d) According to (c), if there are six red edges or four blue edges incident with any vertex in K_9, there is either a red K_4 or a blue K_3. Suppose that there are five red and three blue edges incident with very vertex in K_9. There are $9 \times \frac{5}{2} = 22.5$ red edges in K_9, which is clearly an impossibility.

4.14 (a) 2, 3, 4.

(b) An assignment of time slots to the examinations; the minimum number of examination periods required to schedule all examinations.

4.15 (a) (1) means a child has at most 2 parents.

(2) means a person cannot be his own ancestor.

(3) means we can color mothers and fathers differently, e.g., by coloring all males one color and females another.

(b) The same coloring works for both G and $\hat{G}$, if G is 2-colored, according to (3) then no two adjacent vertices have the same color, and conversely if $\hat{G}$ is properly colored with two colors then no two vertices a and b with (a, c) and (b, c) in E can have the same color.

(c) For a graph that contains no circuits of odd lengths, we pick an arbitrary vertex and paint it with one of the two colors, say red. We then paint the vertices that are adjacent to a red vertex with the other color, say blue, and paint the vertices that are adjacent to a blue vertex with the color red. This procedure is repeated until all the vertices are painted. In this way, no vertex is ever painted with both colors, red and blue, since this will happen only when there are two elementary paths between two vertices, one of the paths having odd length, the other having even length. However, this implies the existence of a circuit of odd length.

On the other hand, if a graph can be properly colored with two colors, we shall find the colors of the vertices alternate when a circuit is traversed. Hence, the length of the circuit must be even.

(d) Use the results in parts (b) and (c).

4.16 (a) 2.

(b) The minimum number of links necessary to communicate between any two computers.

(c) Given any vertex the number of vertices of distance i from it is at most $\delta \cdot (\delta - 1)^{i-1}$ since there are at most δ edges leaving it and $\delta - 1$ other edges from the adjacent vertex. Thus for any $i \geq 1$, the number of vertices of distance $\leq i$ from a given vertex is less than or equal to $1 + \delta + \delta(\delta - 1) + \ldots + \delta(\delta - 1)^{i-1}$. For $i = d$, the result follows.

4.17

Center	Time
b	3:03
c	3:04
d	3:05
e	3:06
f	3:04
g	3:07
h	3:08

4.18 Shortest path has length 14.

4.19 Shortest path has length 6.

4.20 Shortest path has length 23.

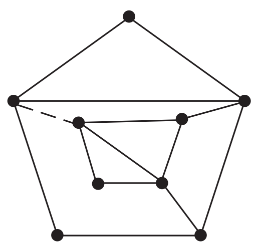
4.21 Shortest path has length 8.

4.22 Shortest path has length 19.

4.23 Consider the directed graph obtained from G by making each edge of G into two edges, one going each way. Then clearly G is connected and each vertex has equal in and out degrees so a eulerian circuit is possible. But that is precisely what the policeman wants.

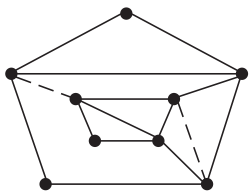
4.24 Consider the mansion as a graph with vertices corresponding to the rooms and edges to the doors. The sum of the degrees of the vertices of the connected subgraph containing the entry must be even, but the outside door adds one. If all rooms had an even number of doors this would be impossible.

4.25 (a)

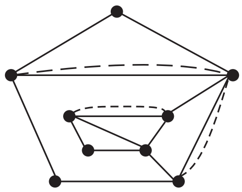


(b) Add $k/2$ edges to the graph, connecting all odd-degree vertices in pairs. Find a eulerian circuit in the resulting graph, and then delete the added edges.

(c) $k/2$. Connect odd degree vertices in pairs so that all vertices are of even degree.



(d)



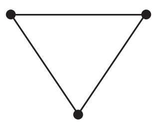
A necessary and sufficient condition is the existence of a subgraph of *G* which contains all the vertices of odd degree. Furthermore, the degrees of these vertices *in the subgraph* are odd while the degree of other vertices in the subgraph are even. (Use the result in part (b).)

4.26 No. Considering the squares as vertices and connecting two vertices by an edge if one can be reached from the other by a knight's move. Since there are eight vertices of degree 3 as shown below, there is no eulerian path in the graph.

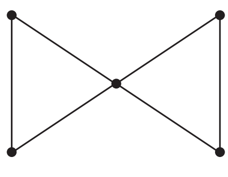
2	3	4	3	2
3	4	6	4	3
4	6	8	6	4
3	4	6	4	3
2	3	4	3	2

4.27 *aaabaacabbbcccacbbaccbabcbc*

4.28 (a)



(b)



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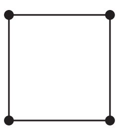
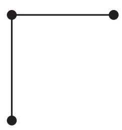


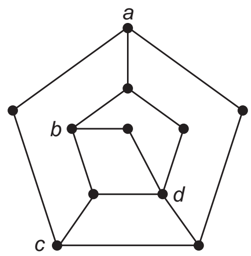
(d)



4.29 (a) Yes, Yes
(b) No, Yes

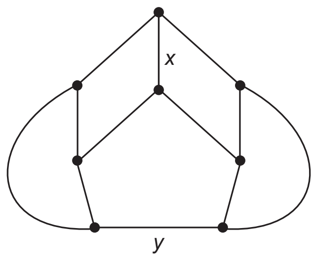
4.30 (a) Yes, No, No
(b) Teaching, Exam, Ruty.
(c) $m = n$
(d) $|m - n| \leq 1$

4.31



The graph has 11 vertices and 15 edges. One of the three edges incident with a, b and c cannot be included in a hamiltonian circuit. Two of the four edges incident with d cannot be included in a hamiltonian circuit. Thus, only 10 edges are left and 11 edges are needed to form a hamiltonian circuit.

4.32



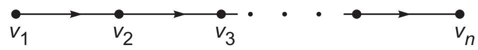
We prove that if we exclude y, then we must exclude x. If we exclude y, then we must include a, b, c and d. We must now exclude e since this will create a 3-cycle. Similarly, we must exclude f. This in turn, means that we must include g, h, i and j. Therefore we cannot include x.

4.33 (a) Number of friends of a person.
(b) Any person can contact another person through a sequence of mutual friends.
(c) m people who are friends of one another.
(d) Pairing off friends.

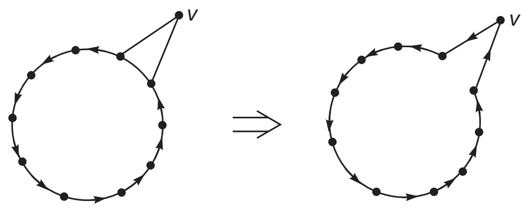
(e) We note that every one actually knows at least $n - 2$ people. (If a person knows only $n - 3$ or fewer people, there are two people who do not know him.) For $n \geq 3$, $2(n - 2) \geq n - 1$. Thus, according to theorem 5.4 there exists a hamiltonian circuit in the graph describing the friendship relation.

(g) Use the same argument in (e) and the result in (f). Note that for $n \geq 4$, $2(n - 2) \geq n$.

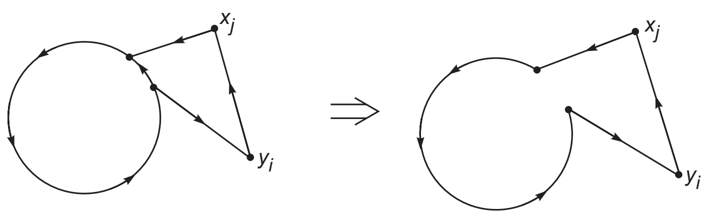
4.34 According to theorem 5.5, there is a directed path



Since v_n is not an outclassed group, there is an edge from v_n to v_i for some i. Thus, we establish the existence of a directed circuit in the graph. This directed circuit can be augmented in a step by step manner as shown below, as long as not all edges are directed from v to the circuit or from the circuit to v:



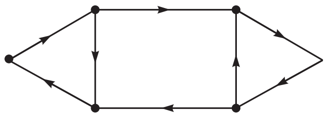
This leaves us with the case that the vertices outside of the directed circuit fall into two types: vertices from which all edges are directed to the circuit and vertices to which all edges are directed from the circuit. Note that there must be vertices of both type because of the nonexistence of outclassed group. Let us denote the first set of vertices $\{x_1, x_2, \ldots\}$ and the second set of vertices $y_1, y_2, \ldots$. We note that there is an edge (y_i, y_j) because otherwise $\{y_1, y_2, \ldots\}$ will be an outclassed group. The directed circuit can then be argumented as shown below:



4.35 Five.

4.36 By induction. Easy for $n = 1, 2$. If G_n is the sequence of vertices of the hamiltonian circuit of an n-cube, then $0G_n$, $1G_n^R$ is a hamiltonian circuit for an $n + 1$ – cube, where $0G_n$ means inserting a 0 as the first component of each vertex of G_n and G_n^R means reversing the sequence G_n.

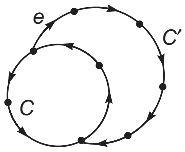
4.37 (a)



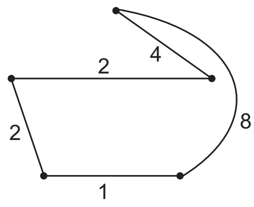
(b) Assign directions for the edges of the eulerian circuit in the order of the edges in the circuit. Since all edges and vertices are reached, the graph is strongly connected.

(c) Assign directions on the hamiltonian circuit in the order of the edges in the circuit and all other edges may be assigned directions arbitrarily. Since one can reach any vertex from any other along the circuit, the resulting graph is strongly connected.

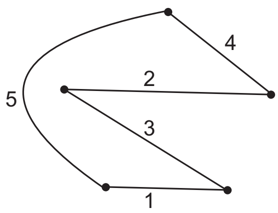
(d) Necessity is clear. To show sufficiency, we choose any circuit, C, and assign directions to the edges in C in the order of the edges in C. If C does not contain all edges, choose an edge e not in C that is incident with a vertex in C. Since e is contained in a circuit, C', let us assign directions to the edges in C' in order of the edges in C' except those edges that are in C, which have already been oriented. Such steps can be repeated to orient all edges.



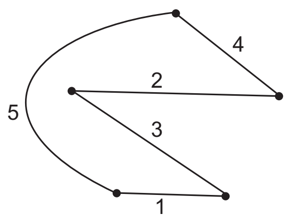
4.38 (a)



(b)



(c)

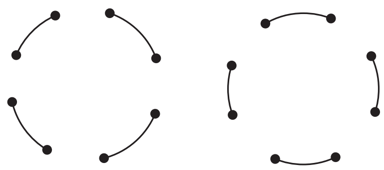


4.39 (a) The number of odd degree vertices in any graph is always even.

(b) All vertices in $T \cap M$ have even degrees.

(c) Deleting any edge from a minimum hamiltonian circuit yields a spanning tree the weight of which is no less than of T.

(d) Suppose the graph has an even number of vertices. The two sets of alternated edges of a minimum hamiltonian circuit are two 1-factors as illustrated below:



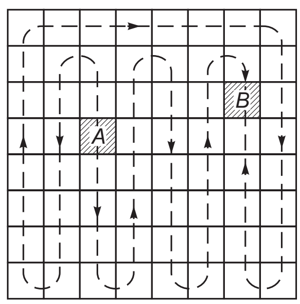
Of these two 1-factors, we choose the one with the smaller weight. The weight of this 1-factor is no less than that of a minimum 1-factor. The case that the graph has an odd number of vertices is similar.

(e) Because of the triangle inequality.

4.40 For $x = -1$, $y = -1$

$$(1 + x)\, f(x, y) + (1 + y)\, g(x, y) = 0$$
$$(1 + x + \ldots + x^7) \cdot (1 + y + \ldots + y^7) - 1 - x^7 y^7 = -2$$

4.41



Let A and B be the removed squares. Clearly the two paths between A and B are always of even length. Consequently, each path can be covered by 2×1 dominos.

4.42 Suppose that every ruling is crossed by at least one domino. For every horizontal ruling, the number of vertical domino crossing it must be even. Thus if it is crossed by a domino, it must be crossed by at least 2 vertical dominos. Hence there must be at least 10 vertical dominos. Similarly, if a vertical ruling is crossed by a domino, then it must be crossed by at least 2 horizontal dominos. Hence, there must be at least 10 horizontal dominos.

Thus the total number of dominos must be at least 20. A contradiction, since there are only 18 dominos (covering 36 squares).

4.43 (a) A 1-factor is an assignment of workmen to jobs such that each workman is qualified for one job and each job has one qualified workman.
(b) A 1-factor is a selection of one senator from each committee such that no senator is selected twice.
(c) A 1-factor is a pairing of boys and girls such that each boy knows the girl he is paired with. A hamiltonian circuit is a chain of boys and girls (alternating) such that each person knows the next which begins and ends with the same person and contains everyone.

4.44 (a) If each vertex has degree at least 6 then there are at least $3v$ edges, but $e \le 3v - 6$.
(b) $\frac{5}{2} \le e \le 3v - 6$

Therefore $\quad v \ge 12 \quad$ and $\quad e \ge \frac{5}{2} \cdot 12 = 30.$

4.45 By Euler's formula, there are 8 regions. Each region must be bounded by at least 3 edges and each edge must be part of the boundary of exactly 2 regions. If there were more than 3 edges bounding any region, more than 12 edges would be required.

4.46 $e + \bar{e} = \binom{v}{2} \le 2(3v - 6)$ if both G and $\overline{G}$ are planar. Thus $v^2 - 13v + 24 \le 0$, but this implies $v \le 10$ since the roots of $v^2 - 13v + 24$ are $\frac{13 \pm \sqrt{73}}{2}$, both less than 11.

4.47 (a) 1
(b) 2, 2
(c) Because each of the planar subgraphs into which G is decomposed can have at most $3v - 6$ edges.
(d) If $\quad p = mq$

$$\left\lceil \frac{p}{q} \right\rceil = m \qquad \left\lfloor \frac{p+q-1}{q} \right\rfloor = \left\lfloor \frac{(m+1)\,q-1}{q} \right\rfloor = m$$

If $\quad p = mq + r, \quad r > 0$

$$\left\lceil \frac{p}{q} \right\rceil = m + 1 \qquad \left\lfloor \frac{p+q-1}{q} \right\rfloor = \left\lfloor \frac{(m+1)\,q+(r-1)}{q} \right\rfloor = m + 1$$

Since $e = \frac{n}{2}(n - 1)$ in K_n, according to (c), we have

$$\theta(K_n) \geq \left\lceil \frac{\frac{n}{2}(n-1)}{3n-6} \right\rceil = \left\lfloor \frac{\frac{n}{2}(n-1)+3n-6-1}{3n-6} \right\rfloor$$

$$= \left\lfloor \frac{n^2+5n-14}{6(n-2)} \right\rfloor = \left\lfloor \frac{n+7}{6} \right\rfloor$$