CHAPTER

TWO

PERMUTATIONS, COMBINATIONS AND DISCRETE PROBABILITY

2.1 (a) $4 \cdot 3 \cdot 6 \cdot 3 = 216$

(b) $5 \cdot 4 \cdot 7 \cdot 4 = 560$.

(c) $4 \cdot 3 \cdot C(6, 2) \cdot 3 = 540$,

$540 + 216 = 756$

2.2 (a) 93

(b) $93 \cdot 7 + 1 + 2 + 3 + 4 + 5 + 6 = 672$

2.3 If the two squares are in the same row then there are 8 different rows and 7 pairs in each row. Thus the answer is $8 \times 7 + 8 \times 7 = 112$.

2.4 $26 + 26 \cdot 10 = 286$

2.5 (a) $5! \cdot 5!$

(b) $5! \cdot 5! \cdot 2$

(c) $9! \cdot 2$

2.6 (a) $10! \cdot P(11, 5)$. There are 10! ways to arrange the boys and $P(11, 5)$ ways to arrange the girls in the spaces on either side of the boys.

(b) $9! \cdot P(10, 5)$.

2.7 There are $P(24, 7)$ ways to choose and arrange the seven letters, 2 ways to place a and b and 18! ways to arrange this string of nine letters and the remainging 17 letters. Total is $P(24, 7) \cdot 18! \cdot 2$.

2.8 (a) 4!, since the four letters b, c, d, e, can be placed in the four spaces separating the a's in any order.

(b) $P(6, 4)$, since the four letters b, c, d, e can be placed in any four of the six blank positions on either side of an a.

2.9 (a) $\dfrac{11!}{4!4!2!}$

(b) $\dfrac{11!}{4!4!2!} - \dfrac{10!}{4!4!}$

2.10 (a) $5!$

(b) By symmetry $\dfrac{1}{2} \times 6!$

2.11 (a) $6 \cdot 5 \cdot 4 \cdot 3 = 360$
(b) $3 \cdot 5 \cdot 4 \cdot 3 = 180$
(c) $2 \cdot 5 \cdot 4 \cdot 3 = 120$
(d) $4 \cdot 5 \cdot 4 \cdot 3 = 240$
(e) $1 \cdot 5 \cdot 4 \cdot 3 = 60$
(f) $4 \cdot 3 \cdot 4 \cdot 3 = 144$

2.12 3^{15}

2.13 26^4

2.14 (a) $26^2 \cdot 10^4$
(b) $26 \cdot 25 \cdot 10^4$

2.15 (a) 2^9
(b) 2^7 of them are symmetrical with respect to the vertical axis. $2^9 - 2^7$ of them are not.

2.16 (a) $9 \times P(52, 9)$
(b) 52×51^9 is the number of sequences the 10th card of which is not a repetition.

$52^{10} - 52 \times 51^9$ is the number of sequences the 10th card of which is a repetition.

2.17 Arrange 5 0's and 3 1's. Where the 1's represent the block. Therefore, the answer is $\dfrac{8!}{5!\,3!} = \dfrac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 8 \cdot 7 = 56$.

2.18 (a) Obvious.
(b) Use a combinatorial argument: In a permutation of $p + 1$ red balls and q white balls, if we throw away the right-most red ball and white balls to the right of it, we obtain a permutation of p red balls and i white balls, $0 \leq i \leq q$.

(c) $$\sum_{i=0}^{p} \frac{(i+q+1)!}{(i+1)!\,q!} = \frac{(q+1)!}{q!\,1!} + \frac{(q+2)!}{9!\,2!} + \ldots + \frac{(q+p)!}{q!\,p!} + \frac{(q+p+1)!}{q!(p+1)!}$$

$$= \frac{(q+p+2)!}{(q+1)!(q+1)!} - 1$$

However, subtract a one corresponding to the case $p = 0$, $q = 0$.

2.19 (a) $C(100, 10)$
(b) $C(40, 5) \cdot C(60, 5)$
(c) $C(40, 6) \cdot C(60, 4) + C(40, 4) \cdot C(60, 6)$

2.20 (a) $C(10, 8)$
(b) $C(7, 5)$
(c) $C(5, 5) \cdot C(5, 3) + C(5, 4) \cdot C(5, 4)$

2.21 (a) $C(12, 4)$
(b) $C(2, 1) \cdot C(10, 3)$
(c) $C(10, 2) + C(10, 4)$
(d) $C(2, 1) \cdot C(8, 3) + C(2, 1) \cdot C(8, 1)$

2.22 (a) $C(200, 30)^2$
(b) $C(200, 5) \cdot C(195, 25) \cdot C(170, 25)$

2.23 $2^{10} - C(10, 1) - C(10, 0) = 1013$.

2.24 (a) $C(15, 5) \cdot C(10, 5)$
(b) $\frac{1}{3!} \cdot C(15, 5) \cdot C(10, 5)$

2.25 $C(15, 5) \cdot 2^{10}$

2.26 $C(7, 3) \cdot 9^4$

2.27 (a) There are $C(50, 2)$ ways to choose both even, $C(50, 2)$ ways to choose both odd, and $C(50, 1) \cdot C(50, 1)$ ways to choose one even and one odd. Thus the answers are $2 \times C(50, 2)$ and $C(50, 1)^2$.
(b) The number of ways to select two integers out of $2n$ integers 1, 2, …, $2n$ is equal to the number of ways to select two even integers, plus the number of ways to select two odd integers, plus the number of ways to select one each.

2.28 Let A: number of male representatives from the junior class
B: number of male representatives from the senior class
C: number of female representatives from the junior class

D: number of female representatives from the senior class

$$\begin{aligned} A + B &= 4 \\ A + C &= 3 \\ A + B + C + D &= 8 \end{aligned}$$

By exhaustion, all the possible solutions to these equations are given in the table below:

A	B	C	D
0	4	3	1
1	3	2	2
2	2	1	3
3	1	0	4

Thus, the total number of ways is:

$$\binom{25}{0}\binom{25}{4}\binom{25}{3}\binom{25}{1} + \binom{25}{1}\binom{25}{3}\binom{25}{2}\binom{25}{2} + \binom{25}{2}\binom{25}{2}\binom{25}{1}\binom{25}{3}$$
$$+ \binom{25}{3}\binom{25}{1}\binom{25}{0}\binom{25}{4}$$

$$= 2 \times 25\left[\binom{25}{4}\binom{25}{3} + \binom{25}{3}\binom{25}{2}\binom{25}{2}\right] = 11,\ 804,\ 750,\ 000$$

2.29 Divide the number into 4 equivalence classes according to the remainder modulo 4. The 3 numbers chosen must have been from classes 0, 0, 0 or 1, 1, 2 or 0, 2, 2 or 0, 1, 3 or 2, 3, 3 with the number of ways for each combination being $C(250, 3)$ or $250 \cdot C(250 \cdot C(250, 2)$ or $250 \cdot C\ (250, 2)$ or 250^3 or $250 \cdot C(250, 2)$. Thus we get $C(250, 3) + 3 \cdot 250 \cdot C(250, 2) + 250^3$.

2.30 Associated with each committee assignment is a collection of positive integers $a_1, a_2, \ldots, a_k$ which are the committee sizes. These positive integers satisfy the two conditions:

$$a_i \geq 2 \qquad i = 1, 2, \ldots, k$$
$$a_i + a_2 + \ldots + a_k = 8$$

Several distinct committee assignments may have the same associated collection of committee sizes. Divide the set of all distinct committee assignments into classes, each class containing all those committee assignments having the same associated collection of committee sizes. The result in Example 2.10 may be applied to compute the number of commitee assignments in each class.

k	$a_1 a_2 \ldots a_k$	Number of Assignments
1	8	$\frac{8!}{8!} = 1$
2	2 6	$\frac{8!}{2!\,6!} = 28$
	3 5	$\frac{8!}{3!\,5!} = 56$
	4 4	$\frac{8!}{(4!)^2\, 2!} = 35$
3	2 2 4	$\frac{8!}{(2!)^2 4!\, 2!} = 210$
	2 3 3	$\frac{8!}{2!(3!)^2 2!} = 280$
4	2 2 2 2	$\frac{8!}{(2!)^4 4!} = 105$
		Total = 715

2.31 $C(100, 20)$, since for every group of 20 people, the two groups are uniquely defined.

2.32 $C(9 + n - 1, n) = C(n + 8, n)$, since for every group of n chosen digits, the sequence are uniquely defined.

2.33 $C(5, 2) \cdot \dfrac{22!}{(5!)^2\,(4!)^2}$ or $\binom{5}{2}\binom{22}{5}\binom{17}{5}\binom{12}{4}\binom{8}{4}$

2.34 The first person can be paired off in $2n - 1$ ways, the next one in $2n - 3$ ways, and so on. Thus, the total number of ways is $(2n - 1)\,(2n - 3) \ldots 5 \cdot 3 \cdot 1$.

2.35 (a) $\dfrac{64 \cdot 49}{2!}$

(b) Since the two squares in a diagonal uniquely define the rectangle, and since there are two diagonals in each rectangle, the answer is $\dfrac{1}{2}\left(\dfrac{64 \cdot 49}{2!}\right)$.

2.36 $\dfrac{(n+k)!}{k! \cdot n!} = \dfrac{(n+k)\,(n+k-1) \ldots (n+1)}{k!}$ which is an integer.

2.37 (a) $C(2n+2, n+1) = C(2n+1, n+1) + C(2n+1, n)$
$= C(2n, n+1) + C(2n, n) + C(2n, n) + C(2n, n-1)$

(b) Suppose among $2n + 2$ objects, there are two special ones. Note that

$C(2n + 2, n + 1)$	number of ways to select $n + 1$ objects from $2n + 2$ objects
$C(2n, n + 1)$	number of ways if none of the special objects is selected.
$2C(n, n)$	number of ways if one of the two special objects is selected.
$C(2n, n - 1)$	number of ways if both special objects are selected.

2.38 (a) $2^n \cdot$ (Select 0, or 1, …, or n of the distinct objects, and make up a total of n objects with the non-distinct ones.)

(b) $\binom{2n+1}{0} + \binom{2n+1}{1} + \ldots + \binom{2n+1}{n} = \frac{1}{2} \cdot 2^{2n+1} = 2^{2n}$

2.39 There are $\lfloor (n-1)/2 \rfloor$ ways to make sum equal n. Since $x + y < n \leftrightarrow (n - x) + (n - y) > n$ there are as many ways to make the sum larger than n as smaller than n. Thus $\frac{1}{2}\left[\binom{n-1}{2} - \lfloor (n-1)/2 \rfloor\right]$ ways to get sum larger than n.

2.40 $C(4 + 5 - 1) = C(8, 5)$ since this is equivalent to placing 5 balls in 4 numbered boxes.

2.41 (a) Place one ball in each box first, and then place the $r - n$ remaining balls. Thus $C((r - n) + n - 1, r - n) = C(r - 1, r - n) = C(r - 1, n - 1)$

(b) $C(14, 4)$

(c) $C(n - ng + r - 1, n - 1)$

(d) $C(5 - 10 + 15 - 1, 4) = C(9, 4)$

2.42 $C(r - 1, n - 1) \cdot C(w - 1, n - 1)$

2.43 $C(2t + 3, 2t) - 2 \cdot C(t + 2, t - 1)$

2.44 (a) $\dfrac{(m+n)!}{m!\,n!}$

(b) Put two 0s into the spaces between the ls. Then there are $m - 2(n - 1)$ 0s left to be distributed into the $n + 1$ space between the ls. Thus, $C(n + 1 + (m - 2(n - 1)) - 1, m - 2(n - 1)) = C(m - n + 2, n)$.

2.45 Exactly one child gets 2 games. There are $C(n, 2)$ ways to choose the 2 games and then $P(n, n-1)$ ways to distribute them to the $n - 1$ children.

Thus, $$C(n, 2)\ P(n, n-1) = \frac{n(n-1)n!}{2}.$$

2.46 (a) $[C(3 + 10 - 1, 10)]^3 = C(12, 2)^3$

(b) $[C(2 + 10 - 1, 10)]^3 = C(11, 1)^3$

(c) $C(12, 2)^3 - 3 \cdot C(11.1)^3 + 3 \cdot C(10, 0)^3$

(d) $C(11, 2)^3 - 3 \cdot C(11, 1) \cdot C(10, 10)^2$.

2.47 (a) We must arrange the r balls and the $n - 1$ box dividers in any permutation with the box dividers in a fixed order $(n + r - 1)!/(n - 1)!$.

(b) As in part (a) with a, b, c as box dividers and placing d, e, f, g, h in the four boxes $(4 + 5 - 1)\ (4 + 5 - 2)\ \ldots\ 5 \cdot 4 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$.

2.48 $7! - 5! - 5! + 3!$

2.49 $10! - 2 \cdot 9! - 2 \cdot 9! + 2.2.8!$

2.50 $26! - 3 \cdot 25! - 4 \cdot 25! + 3 \cdot 4 \cdot 24!$

2.51 $C(26, 12) - C(20, 6) - C(22, 8) - C(23, 9) + C(19, 5) + C(19, 5) + C(18, 4) - C(17, 3)$.

2.52 $10! - \binom{10}{1} 9! + \binom{10}{2} 8! - \ldots + (-1)^{10} \binom{10}{10} 0!$

2.53 There are 6^n numbers containing the digits 2, 3, 4, 5, 6, 7, only. Also there are 5^n of them not containing the digit 2, 5^n of them not containing the digit 7, 4^n of them not containing the digits 2 and 7. Thus by the principle of inclusion and exclusion, the answer is

$$6^n - 5^n - 5^n + 4^n$$

2.54 If we write from 000000 to 999999, we would have written 10^6 digits, out of which 10^5 are 9s. The numbers 0 and 100000 do not contain any 9. Thus the answer is 10^5.

2.55 Intuitively a_p is the smallest element to the right of a_m and (3) ensures that $b_m = a_p$. By (1) the elements $a_{m+1}, \ldots, a_n$ with a_m interchanged with a_p are in order $a_{m+1} < a_{m+2} < \ldots < a_n$. Step (4) reverses the order giving, (iii) of the procedure.

2.56 $\dfrac{7}{7^7} = \dfrac{1}{7^6}$

2.57 $\dfrac{P(20, 10)}{20^{10}}$

2.58 $\dfrac{\sum_{i=1}^{10} C(350, i) \cdot C(i + 50 - i - 1, 50 - i)}{C(350 + 50 - 1, 50)} = \dfrac{\sum_{i=1}^{10} C(350, i) \cdot C(49, i - 1)}{C(399, 50)}$

2.59 Divide the numbers into classes according to their remainder modulo 3. There are 33 in class 0, 34 in class 1 and 33 in class 2. Then required probability is

$$\frac{33 \cdot 32 + 34 \cdot 33 + 33 \cdot 34}{9900} = \frac{1}{3}$$

2.60 (a) 1/10

(b) $\dfrac{9}{10} \cdot \dfrac{1}{9} = \dfrac{1}{10}$

(c) 1/10

2.61 Total number of ways = $C(10, 7)$.
Number of ways in which there are no two adjacent unoccupied spaces = $C(8, 3)$. Thus, probability is $\dfrac{C(10, 7) - C(8, 3)}{C(10, 7)} = \dfrac{8}{15}$

2.62 (a) $\{10, 11, \ldots, 39\}$

(b) $10 \cdot \dfrac{2}{60} = \dfrac{1}{3}$

(c) $\dfrac{(1 + 2 + 3)}{60} = \dfrac{1}{10}$

(d) $10 \cdot \dfrac{2}{60} + \dfrac{1}{60} + \dfrac{3}{60} = \dfrac{2}{5}$

(e) $3 \cdot \dfrac{1}{60} + 3 \cdot \dfrac{2}{60} + 4 \cdot \dfrac{3}{60} = \dfrac{7}{20}$

(f) $2 \cdot \dfrac{1}{60} + 2 \cdot \dfrac{2}{60} + 2 \cdot \dfrac{3}{60} = \dfrac{1}{5}$

(g) $3 \cdot \dfrac{1}{60} + 5 \cdot \dfrac{2}{60} + 4 \cdot \dfrac{3}{60} = \dfrac{5}{12}$

2.63 (a) $25k - 2k\left(\dfrac{1}{22} + \dfrac{1}{23} + \dfrac{2}{23} + \dfrac{1}{24} + \dfrac{2}{24} + \dfrac{3}{24} + \dfrac{1}{25} + \dfrac{2}{25} + \dfrac{3}{25} + \dfrac{4}{25}\right) = 1$

$$k = \frac{2530}{59071} = 0.0428$$

(b) $k \cdot \dfrac{21}{22} = 0.041$

$$5k = 0.214$$

$$\frac{(1-5k)}{2} = 0.393$$

$$\left(\frac{21}{25}+\frac{22}{25}+\frac{23}{25}+\frac{24}{25}+\frac{25}{25}\right)k = \frac{23}{5}k = 0.197$$

(c) Only case where both of them are less than 22 years old is (21, 21). In this case, they are of the same age. Thus, probability = 1.

(d) $P(h = 23) = \left(\dfrac{21}{23}+\dfrac{22}{23}+1+\dfrac{23}{24}+\dfrac{23}{25}\right)k = 0.203$

$$P(h = 23, w = 24) = \frac{23}{24}k = 0.041$$

$$P(h = 23, w > h) = \left(\frac{23}{24}+\frac{23}{25}\right)k = 0.080$$

Thus, $\quad P(w = 24 \mid h = 23) = \dfrac{0.041}{0.203} = 0.203.$

and $\quad P(w > h \mid h = 23) = \dfrac{0.080}{0.203} = 0.394.$

(e) $P(both \geq 22) = 1 - k - 2k\left(\dfrac{21}{22}+\dfrac{21}{23}+\dfrac{21}{24}+\dfrac{25}{25}\right) = 0.650$

$$P(both \geq 24, both \geq 22) = k\left(\frac{24}{24}+\frac{24}{25}+\frac{24}{25}+\frac{25}{25}\right) = 0.168$$

$$P(one\ of\ them \geq 24, both \geq 22) = 2k + 2k\left(\frac{22}{24}+\frac{22}{25}+\frac{23}{24}+\frac{23}{25}+\frac{24}{25}\right)$$

$$= 0.483$$

Thus, $P(both \geq 24 \mid both \geq 22) = \dfrac{0.168}{0.650} = 0.258$

and $P(one\ of\ them \geq 24 \mid both \geq 22) = \dfrac{0.483}{0.650} = 0.743$

(f) $P(one\ of\ them \leq 23) = 1 - 0.168 = 0.832$

$$P(both \leq 22, one\ of\ them \leq 23) = 2k\left(1+\frac{21}{22}\right) = 0.167$$

Thus, $P(both \le 22 \mid one\ of\ them \le 23) = \dfrac{0.167}{0.832} = 0.201.$

2.64 (a) $S = \{(TH)^i TT \mid i \ge 0\} \cup \{(HT)^i HH \mid i \ge 0\} \cup \{H(TH)^i TT \mid i \ge 0\} \cup \{T(HT)^i \mid i \ge 0\}.$

Also, let k be the number of tosses required.

(b) $P(k < 6) = 2 \cdot \left(\dfrac{1}{2^2} + \dfrac{1}{2^3} + \dfrac{1}{2^4} + \dfrac{1}{2^5}\right) = \dfrac{15}{16}$

(c) $P(k\ is\ even) = 2 \cdot \left(\dfrac{1}{4} + \dfrac{1}{4^2} + \dfrac{1}{4^3} + \dots\right) = \dfrac{2}{3}$

(d) Let A be the event the experiment ends with two heads.

$P(A) = 1/2$

$$P(k < 6 \cap A) = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} = 15/32.$$

Thus, $\quad P(k < 6 \mid A) = \dfrac{15/32}{1/2} = 15.16.$

(e) $P(k \ge 3) = 1 - 2(1/4) = 1/2$

$$P(k \ge 7,\ k \ge 3) = 2\left(\frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \dots\right) = 1/32.$$

Thus, $\quad P(k \ge 7 \mid k \ge 3) = \dfrac{1/32}{1/2} = 1/16$

2.65 (i) $\dfrac{\binom{10}{8} \cdot 2^8}{\binom{20}{8}}$ (ii) $\dfrac{\binom{10}{1} \cdot \binom{9}{6} \cdot 2^6}{\binom{20}{8}}$

2.66 (i) $1 - \left(\dfrac{7}{10}\right)^7$

(ii) $\dfrac{1 - \left(\dfrac{7}{10}\right)^7 - 7 \cdot \left(\dfrac{3}{10}\right)\left(\dfrac{7}{10}\right)^6}{1 - \left(\dfrac{7}{10}\right)^7}$

2.67 (a) $\dfrac{50000}{100000}\cdot\dfrac{2}{100}+\dfrac{30000}{100000}\cdot\dfrac{3}{100}+\dfrac{20000}{100000}\cdot\dfrac{5}{100}=\dfrac{1}{100}+\dfrac{9}{1000}+\dfrac{1}{100}$

$= 29/1000.$

(b) $P(\textit{defective and from } A) = \dfrac{1}{100} = 0.01$

$P(\textit{from } A \mid \textit{defective}) = \dfrac{0.01}{0.029} = \dfrac{10}{29}$

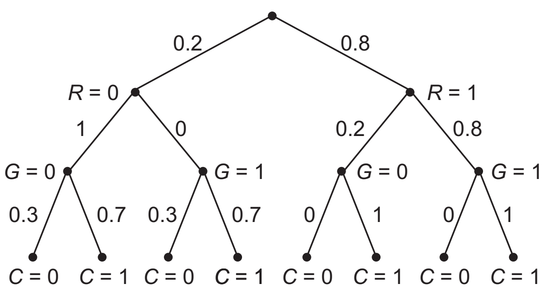
(c) $P(\textit{not from } A) = \dfrac{30000+20000}{100000} = 0.5$

$P(\textit{defective and not from } A) = \dfrac{9}{1000}+\dfrac{1}{100} = 0.019$

Thus, $P(\textit{defective} \mid \textit{not from } A) = \dfrac{0.019}{0.5} = 0.038.$

2.68 $P(\textit{only sub-system number 1 fails} \mid \textit{system fails}) = \dfrac{(0.2)\cdot(0.8)^5}{1-(0.2)^6}.$

2.69 (a) Sample space = $\{(R, G, C) \mid R, G, C \in \{0, 1\}\}$



(b) $C = 0 \Rightarrow R = 0 \Rightarrow G = 0$. Thus, $P(C = 0, G = 0) = 0$.

Thus, $P(C = 0, G = 1) + P(C = 1, G = 0) = P(C = 1, G = 0)$

$= P(R = 0, C = 1, G = 0) + P(R = 1, C = 1, G = 0)$

$= (0.2)\,(0.7) + (0.8)\,(0.2) = 0.3$

(c) $P(R = 1 \mid C = 0, G = 1, G = 0) = \dfrac{P(R=1\cdot C=1, G=0)}{0.3}$

$= \dfrac{(0.2)\,(0.7)}{0.3}$

$= 7/15$

2.70 (a) $-\lg (1/30) = \lg 30$

(b) $-\lg (1/6) = \lg 6$

(c) $-\lg(13/60) = \lg (60/13)$

(d) $-\lg (1/10) = \lg 10$

(e) $P((a) \mid (b)) = 0.$ $\quad P((b) \mid (a)) = 0.$

$P((a) \mid (c)) = \dfrac{1/30}{13/60}.$ $\quad P((c) \mid (a)) = 1.$

$P((a) \mid (d)) = \dfrac{1/30}{1/10}$ $\quad P((d) \mid (a)) = 1.$

$I((a),\ (b)) = \lg 30 + \lg 0 = \infty.$

$= \lg 6 + \lg 0 = \infty.$

$I((a),\ (c)) = \lg 30 + \lg 2/13 = \lg (60/13)$

$= \lg (60/13) + \lg 1 = \lg (60/13)$

$I((a),\ (d)) = \lg 30 + \lg (1/3) = \lg 10$

$= \lg 10 + \lg 1 = \lg 10$

2.71 (a) $P(h > w) = \dfrac{(1-5k)}{2} = 0.393$

$-\lg (0.393)$

(b) $P(|\, h - w \,| \le 2) = 1 - 2k\left(\dfrac{21}{24} + \dfrac{21}{25} + \dfrac{2}{25}\right) = 0.778.$

$-\lg(0.778)$

(c) $P(h = 25 \cup w \ge 22) = 1 - k\left(\dfrac{21}{21} + \dfrac{21}{22} + \dfrac{21}{23} + \dfrac{21}{24}\right) = 0.840$

$-\lg (0.840)$

(d) $P((a) \cup (b)) = k\left(\dfrac{21}{22} + \dfrac{21}{23} + \dfrac{22}{23} + \dfrac{22}{24} + \dfrac{23}{24} + \dfrac{23}{25} + \dfrac{24}{25}\right) = 0.282$

$P((a) \mid (b)) = \dfrac{0.282}{0.778}.$ $\quad P((b) \mid (a)) = \dfrac{0.282}{0.393}$

$P((a) \cap (c)) = k\left(\dfrac{22}{23} + \dfrac{22}{24} + \dfrac{23}{24} + \dfrac{22}{25} + \dfrac{23}{25} + \dfrac{24}{25}\right) = 0.239$

$P((a) \mid (c)) = \dfrac{0.239}{0.840}.$ $\quad P((c) \mid (a)) = \dfrac{0.239}{0.393}$

$$P((b) \cap (c)) = k\left(\frac{21}{22}+\frac{22}{22}+\frac{22}{23}+\frac{22}{24}+\frac{21}{23}+\frac{22}{23}+\frac{23}{23}+\frac{23}{24}+\frac{23}{25}\right)$$

$$+ k\left(\frac{22}{24}+\frac{23}{24}+\frac{24}{24}+\frac{24}{25}+\frac{23}{25}+\frac{24}{25}+\frac{25}{25}\right) = 0.655$$

$$P((b) \mid (c)) = \frac{0.655}{0.840}. \qquad P((c)\ (b)) = \frac{0.655}{0.778}$$

$$I((a),\ (b)) = -\lg(0.393 + \lg\left(\frac{0.282}{0.778}\right) = -\lg(1.084)$$

$$= -\lg(0.778) + \lg\left(\frac{0.282}{0.393}\right) = -\lg(1.084)$$

$$I((a),\ (c)) = -\lg(0.393) + \lg\left(\frac{0.239}{0.840}\right) = -\lg(1.381)$$

$$= -\lg(0.840) + \lg\left(\frac{0.239}{0.393}\right) = -\lg(1.381)$$

$$I((b),\ (c)) = -\lg(0.778) + \lg\left(\frac{0.655}{0.840}\right) = -\lg(0.998)$$

$$= -\lg\ (0.840) + \lg\left(\frac{0.655}{0.778}\right) = -\lg\ (0.998)$$