# CHAPTER NINE

# RECURRENCE RELATIONS AND RECURSIVE ALGORITHMS

9.1 (a) $\alpha_2 - 7\alpha + 10 = 0; \qquad \alpha = 5, 2$

Thus, $a_r = A5^r + B2^r$

$a_0 = A + B = 0; \qquad a_1 = 5A + 2B = 3$

$A = 1, \qquad B = -1$

Hence, $a_r = 5^r - 2^r$

(b) $\alpha_2 - 4\alpha + 4 = 0; \qquad \alpha = 2, 2$

Thus, $a_r = A2^r + Br2^r$

$a_0 = A = 1; \qquad a_1 = 2A + 2B = 6$

$A = 1, \qquad B = 2$

Hence, $a_r = 2^r + 2r2^r = 2^r(1 + 2r)$

9.2 (a) $\alpha^r - 7\alpha^{r-1} + 10\alpha^{r-2} = 0$

$\alpha^2 - 7\alpha + 10 = 0; \qquad \alpha = 5, 2$

Thus, $a_r^{(h)} = A5^r + B2^r$.

We try $a_r^{(p)} = P3^3$.

$P \cdot 3^r - 7P \cdot 3^{r-1} + 10P \cdot 3^{r-2} = 3^r$

$9P - 21P + 10P = 9 \qquad P = -9/2$

$a_r = A \cdot 5^r + B \cdot 2^r - (9/2) \cdot 3^r$

$a_0 = A + B - 9/2 = 0 \qquad \Rightarrow \quad A + B = 9/2$

$a_1 = 5A + 2B - 27/2 = 1 \qquad \Rightarrow \quad 5A + 2B = 29/2$

Thus $\qquad a_r = \dfrac{11}{6}5^r + \dfrac{28}{3}2^r - \dfrac{9}{2}3^r$

(b) $a_r^{(h)} = A(-3)^r + Br(-3)^r$, $a_r^{(p)} = 3/16$

$$a_0 = A + \frac{3}{16} = 0; \qquad a_1 = -3A - 3B + \frac{3}{16} = 1$$

Thus, $\quad a_r = -\frac{3}{16}\,(-3)^r - \frac{1}{12}\,r(-3)^r + \frac{3}{16}$

(c) $\alpha^2 + 1 = 0,\ \alpha = \pm\, i$

$$a_r = A_1(i)^r + A_2(-i)^r = B_1 \cos\frac{r\pi}{2} + B_2 \sin\frac{r\pi}{2}$$

Using boundary conditions gives $B_1 = 0, \qquad B_2 = 2.$

Thus, $\quad a_r = 2 \sin\frac{r\pi}{2}$

9.3 (a) The characteristic equation is $\alpha^2 - \alpha + 1 = 0$

Thus, $\quad \alpha = \frac{1+\sqrt{3i}}{2}$

and $\quad a_r = A\left(\frac{1+\sqrt{3i}}{2}\right)^r + B\left(\frac{1-\sqrt{3i}}{2}\right)^r$

Now, $\quad a_0 = 1 \Rightarrow A + B = 1$

and $\quad a_1 = 1 \Rightarrow \frac{1}{2}(A + B) + \frac{\sqrt{3}}{2}(A - B)i = 1$

Thus, $\quad A = \frac{3+i\sqrt{3}}{6}, \qquad B = \frac{3-i\sqrt{3}}{6}$

and $\quad a_r = \left(\frac{3-i\sqrt{3}}{6}\right)\left(\frac{1+i\sqrt{3}}{2}\right)^r + \left(\frac{3+i\sqrt{3}}{6}\right)\left(\frac{1-i\sqrt{3}}{2}\right)^r$

or $\quad a_r = \cos\frac{r\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{r\pi}{3}$

(b) $\alpha^3 - 2\alpha^2 + 2\alpha - 1 = 0$

$$\alpha = 1,\ \frac{1+i\sqrt{3}}{2},\ \frac{1-i\sqrt{3}}{2}$$

$$a_r = 2 + \frac{i\sqrt{3}}{3}\left(\frac{1+i\sqrt{3}}{2}\right)^r - \left(\frac{i\sqrt{3}}{3}\right)\left(\frac{1-i\sqrt{3}}{2}\right)^r$$

or $\quad a_r = 2 - \dfrac{2}{\sqrt{3}} \sin \dfrac{r\pi}{3}$

9.4 $4 + C_1 = 0 \quad \Rightarrow \quad C_1 = -4$

$12 - 4 \cdot 4 + C_2 = 0 \quad \Rightarrow \quad C_2 = 4$

Thus, $\quad a_r = Ar2^r + B2^r$

$a_0 = 0 \Rightarrow \quad B = 0$

$a_1 = 1 = A_2 + B \quad \Rightarrow A = 1/2$

Therefore, $\quad a_r = r \cdot 2^{r-1}$.

9.5 The characteristic equation is $\alpha^2 - 7\alpha + 12 = 0$ (roots are 3 and 4)

Thus, $\quad K(2 - 14 + 24) = 6 \qquad K = 1/2$

and $\quad C_0 = 1/2, \qquad C_1 = -7/2, \quad C_2 = 6$

9.6 Homogeneous part: $\alpha - A = 0; \qquad \alpha = A = 2.$

$a_0 = C + 3D = 19; \; a_1 = 2C + 9D = 50$

$C = 7 \qquad D = 4$

Particular solution: $4 \cdot 3^{r+1} = 2(4 \cdot 3^r) + B3^r$

$B = 4$

9.7 $a_0 = 4, \quad a_1 = 5.$

Case 1: Characteristic roots are 1, 2. $a_r^{(p)} = -2r$

$4(\alpha - 1)(\alpha - 2) = 0$

$4\alpha^2 - 12\alpha + 8 = 0$

$C_1 = -12, \quad C_2 = 8$

$4(-2r) - 12 \cdot (-2)(r - 1) + 8 \cdot (-2)(r - 2) = f(r)$

$f(r) = 8$

Case 2: Characteristic roots are 1, 1. $a_r^{(p)} = 3 \cdot 2^r$

$4\alpha^2 - 8\alpha + 4 = 0.$

$C_1 = -8, \quad C_2 = 4$

$4(3 \cdot 2^r) - 8(3 \cdot 2^{r-1}) + 4(3 \cdot 2^{r-2}) = f(r)$

$f(r) = 3 \cdot 2^r$

9.8 (a) Same as 9.3 (a).
(b) No. Infinitely many solutions.
(c) No solution.

9.9 (a) $a_r^{(p)} = P\,r\,2^r$

$Pr2^r - 3P(r - 1)2^{r-1} + 2P(r - 2)2^{r-2} = 2^r$

$3P2^{r-1} - 4P2^{r-2} = 2^r \qquad P = 2$

Therefore, $\quad a_r^{(p)} = r \cdot 2^{r+1}$

(b) $a_r^{(p)} = Pr^2\, 2^r$

$Pr^2 2^r - 4P(r-1)^2\, 2^{r-1} + 4P(r-2)^3\, 2^{r-2} = 2^r$

$-4P2^{r-1} + 16P2^{r-2} = 2r \quad P = 1/2$

Therefore, $a_r^{(p)} = r^2 \cdot 2^{r-1}$

9.10 (a) $a_r^{(p)} = Ar + B$

$Ar + B - 2A(r-1) - 2B = 7r$

$A = -7 \qquad B = -14$

(b) $a_r^{(p)} = Ar^2 + Br + C$

$Ar^2 + Br + C - 2A(r^2 - 2r - 1) - 2B(r-1) - 2C = 7r^2$

$A = -7 \qquad B = -28 \qquad C = -42$

(c) $a_r^{(p)} = Ar^2 + Br$

$Ar^2 + B^r - A(r^2 - 2r + 1) - B(r-1) = 7r$

$A = 7/2 \qquad B = 7/2$

(d) $a_r^{(p)} = Ar^3 + Br^2 + Cr$

$Ar^3 + Br^2 + Cr - A(r^3 - 3r^2 + 3r - 1) - B(r^2 - 2r + 1) - C(r-1) = 7r^2$

$A = 7/3 \qquad B = 7/2 \qquad C = 7/6$

(e) $a_r^{(p)} = Ar^{t+1} + Br^t + Cr^{t-1} + \ldots + Yr + Z$

9.11 (a) With boundary conditions $a_0 = a_1 = 0$, we get $a_2 = 1$, $a_3 = -3$, $a_4 = 7$.

Also, $a_r + 3a_{r-1} + 2a_{r-2} = f(r) = 0 \; r \geq 3.$

Solve this homogeneous equation with the calculated $a_3$, $a_4$

$$a_r = (-1)^{r-1} - (-2)^{r-1} \; r \geq 3$$

Thus,
$$a_r = \begin{cases} 1 & r = 2 \\ (-1)^{r-1} - (-2)^{r-1} & r \geq 3 \\ 0 & \text{else} \end{cases}$$

(b) With boundary conditions $a_0 = a_1 = 0$, we get $a_2 = a_3 = a_4 = 0$, $a_5 = 1$, $a_6 = -3$, $a_7 = 7$. Similar to part (a), we obtain

$$a_r = (-1)^{r-4} - (-2)^{r-1} \; r \geq 6$$

Thus,
$$a_r = \begin{cases} 1 & r = 5 \\ (-1)^{r-4} - (-2)^{r-4} & r \geq 6 \\ 0 & \text{else} \end{cases}$$

(c) $\bar{a}_r = \bar{a}_{r-1}$

9.12 (a) $$\sum_{i=0}^{k} C_i\, \bar{a}_{r-i} = \sum_{i=0}^{k} C_i\, \hat{a}_{r-1} + \sum_{i=0}^{k} C_i\, \tilde{a}_{r-1}$$

$$= \hat{f}(r) + \tilde{f}(r) = f(r).$$

(b) Let $\hat{f}(r) = \begin{cases} 0 & r \le 1 \\ 6 & r \ge 2 \end{cases}$

and $\quad \hat{a} + 5\hat{a}_{r-1} + 6\hat{a}_{r-2} = \hat{f}(r)$

and $\quad \hat{a}_0 = \hat{a}_1 = 0$

Hence, $\quad \hat{a}_2 = 6, \ \hat{a}_3 = -24$

$$\hat{a} = -2, -3 \text{ and } \hat{a}_r^{(p)} = 1/2$$

Thus, $\quad \hat{a}_r = \hat{A}(-2)^r + \hat{B}(-3)^r + 1/2 \qquad r \ge 2$

Solve with $\quad \hat{a}_2 = 6$ and $\hat{a}_3 = -24$

$$\hat{A} = -2, \qquad \hat{B} = 3/2$$

Thus, $$\hat{a}_r = \begin{cases} 0 & r \le 1 \\ (-2)^{r+1} - \left(\frac{1}{2}\right)(-3)^{r+1} + \frac{1}{2} & r \ge 2 \end{cases}$$

Now $\quad$ let $\tilde{f}(r) = \begin{cases} 6 & r = 5 \\ 0 & \text{else} \end{cases}$

and $\quad \tilde{a}_r + 5\tilde{a}_{r-1} + 6\tilde{a}_{r-2} = \tilde{f}(r)$

and $\quad \tilde{a}_0 = \tilde{a}_1 = 0$

Hence, $\quad \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0, \quad \tilde{a}_5 = 6, \quad \tilde{a}_6 = -30, \quad \tilde{a}_7 = 114$

Similar to above

$$\tilde{a}_r = \tilde{A}(-2)^r + \tilde{B}(-3)^r \qquad r \ge 6$$

Solve with $\quad \tilde{a}_6 = -30$ and $\tilde{a}_7 = 114$

$$\tilde{A} = 3/8, \qquad \tilde{B} = -2/27$$

$$\tilde{a}_r = \begin{cases} 0 & r \le 4 \\ 6 & r = 5 \\ -3(-2)^{r-3} + 2(-3)^{r-3} & r \ge 6 \end{cases}$$

$$f(r) = \hat{f}(r) - \tilde{f}(r)$$

Thus, $\quad a_r = \hat{a}_r - \tilde{a}_r$

$$a_r = \begin{cases} 0 & r \le 1 \\ \hat{a}_r & 2 \le r \le 4 \\ \hat{a}_r - \tilde{a}_r & 5 \le r \end{cases}$$

$$= \begin{cases} 0 & r \le 1 \\ (-2)^{r+1} - (1/2)\,(-3)^{r+1} + 1/2 & 2 \le r \le 4 \\ 19\,(-2)^{r-3} - (85/2)\,(-3)^{r-3} + 1/2 & 5 \le r \end{cases}$$

9.13 (a) $(A,\ B)$

$(A,\ B)$, $(A,\ C)$, $(A,\ B)$

$(A,\ B)$, $(C,\ D)$, $(A,\ C)$, $(B,\ D)$

(b) Let $A$ be the $r$th person

$(A,\ B)$ (spread gossip among the other $r - 1$ people) $(A,\ B)$

(c) Induction on $r$: $a_r \le 2r - 4$

$$a_{r+1} \le 2r - 4 + 2 = 2(r + 1) - 4$$

9.14 The first element may be in any set of the partition. Then the remainder of the set can be partitioned in $a_i$ ways if the first element is in a set of size $r - i + 1$. That is, if the first element is in a set of size $r - i + 1$, there are $\binom{r}{r-i} a_i = \binom{r}{i} a_i$ ways to choose the partition. Summing over $i$ gives the result.

9.15 (a) $b_r - b_{r-1} = 3\left[1000\left(\dfrac{3}{2}\right)^2 - b_{r-1}\right]$

So $\qquad b_r + 2b_{r-1} = \dfrac{1000 \cdot 27}{4}$

Solving, $\qquad B(z) = \dfrac{1000 \cdot 27}{4} \cdot z\left(\dfrac{1/3}{1-z} + \dfrac{2/3}{1+2z}\right)$

Thus $\qquad b_r = \begin{cases} \dfrac{9000}{4} + \dfrac{9000}{2}(-2)^{r-1} & r \ge 1 \\ 0 & r = 0 \end{cases}$

(b) We first find $b_r$, for $r \le 10$

$$\sum_{r=1}^{\infty} b_r z^r + 2\sum_{r=1}^{\infty} b_{r-1} z^r = 3 \cdot \sum_{r=1}^{\infty} 1000\left(\frac{3}{2}z\right)^r$$

$$(1 + 2z)\, B^{(1)}(z) = 3000\left(\frac{(3/2)\,z}{1 - (3/2)\,z}\right)$$

$$B^{(1)}(z) = \frac{3000 \cdot (3z/2)}{(1-3z/2)\,(1+2z)}$$

$$= \frac{3000 \cdot (3/7)}{1-3z/2} - \frac{3000 \cdot (3/7)}{1+2z}$$

$$b_r = 3000 \cdot \frac{3}{7}\,[(3/2)^r - (-2)^r]$$

Using $b_{10}$ as the starting point

$$\sum_{r=0}^{\infty} (b_{11+r} + 2b_{10+r})z^{11+r} = 3\sum_{r=0}^{\infty} 1000 \cdot (3/2)^{10}\, z^{11+r}$$

$$B^{(2)}(z) = b_{10}\, z^{10} + 2z\, B^{(2)}(z) = 3000 \cdot (3/2)^{10} \frac{z^{11}}{1-z}$$

$$B^{(2)}(z)\,(1 + 2z) = 3000\left[\frac{3}{7}\,((3/2)^{10} - 2^{10}) + (3/2)^{10}\,\frac{z^{11}}{1-z}\right]$$

from which $b_r$ can be determined for $r \geq 10$.

9.16 $a_1 = 1$ and $a_r = 2a_{r-1} + 1$, since the largest ring is transferred once after the other rings have been transferred, and then the other rings are transferred once more. It follows that $a_r = 2^r - 1$.

9.17 Let $g_r$ be the growth rate on the $r$th day.

$$g_r = a_r - 2a_{r-1}$$

$$g_r = 2g_{r-1}$$

Thus,

$$a_r - 2a_{r-1} = 2a_{r-1} - 4a_{r-2}$$

$$a_r - 4a_{r-1} + 4a_{r-2} = 0$$

$$\alpha = 2, 2$$

$$a_r\; A \cdot 2^r + B_r\, 2^r$$

Using boundary conditions (let $a_0 = 1$ to get $a_1 = 4$),

Hence, $$a_r = 2^r + r2^r = (r + 1)2^r$$

9.18 (a) $b_r + b_{r-1} = a_r + a_{r-1} = 3 \cdot 2^{r-1}$

$$B(z) = \frac{3z}{(1-2z)\,(1+z)} = \frac{1}{1-2z} - \frac{1}{1+z}$$

$$b_r = 2^r - (-1)^r$$

(b) $b_r = 2^r - (-1)^r, \quad 0 \leq r \leq 10$

For $\quad r \geq 11, \quad b_r + b_{r-1} = 2^{11}$

So $\quad b_r = 2^{10} - (-1)^r \quad \text{for} \quad r \geq 11$

9.19 $a_r = a_{r-1} + 2(a_{r-1} - a_{r-2}) = 3a_{r-1} - 2a_{r-2}$

$$a_r = B \cdot 2^r + C$$

$$a_0 = B + C = 3; \qquad a_3 = 8B + C = 10$$

So, $B = 1, C = 2$

Thus, $a_r = 2^r + 2$

9.20 $a_r = a_{r-2} + (2r - 3)\ r \geq 2.$

$$a_0 = 0, \qquad a_1 = 0.$$

$$\alpha^2 - 1 = 0; \qquad \alpha = 1, -1$$

$$a_r^{(h)} = A + B(-1)^r$$

Try $a_r^{(p)} = Cr^2 + Dr$

$$Ar^2 + Br - A(r-2)^2 - B(r-2) = 2r - 3$$

$$A = 1/2$$

$$B = -1/2$$

$$a_r = A + B(-1)^r + r^2/2 - r/2$$

$$a_0 = A + B = 0; \qquad a_1 = A - B = 0$$

$$A = B = 0.$$

Thus, $a_r = r^2/2 - r/2 = r(r-1)/2$

9.21 (a) $a_r = a_{r-1} + r - 1 \qquad r \geq 2$

$$a_1 = 0.$$

(b) $\alpha - 1 = 0; \qquad \alpha = 1$

$$a_r^{(h)} = A$$

Try $a_r^{(p)} = Br^2 + Cr$

$$Br^2 + Cr = B(r-1)^2 + C(r-1) + r - 1$$

$$B = 1/2 \qquad C = -1/2$$

$$a_r = A + r^2/2 - r/2$$

$$a_0 = A = 0$$

Thus, $a_r = r^2/2 - r/2 = r(r-1)/2$

9.22 $a_r = a_{r-1} + a_{r-2};\ r \geq 2$

$$a_1 = 2, \qquad a_2 + 3.$$

This is the shifted Fibonacci sequence!

In fact, $a_r = b_{r+1}, \qquad r \geq 1$

where $\{b_r\}$ is the Fibonacci sequence.

Thus, $$a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{r+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{r+2}$$

9.23 $a_n = a_{n-1} + a_{n-2}$

$$a_1 = 1, \qquad a_2 = 2$$

Fibonacci sequence!

Thus, $$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

9.24 $a_r = a_{r-1} + a_{r-2}$

$$a_1 = 2, \qquad a_1 = 3.$$

Fibonacci sequence, as in Problem 9.22.

9.25 $a_r = 100 + 1.1\, a_{r-1}$

$$A(z) = \frac{100z}{(1-z)(1-1.1z)} = \frac{1000}{1-1.1z} - \frac{1000}{1-z}$$

$$a_r = [(1.1)^r - 1]\ 1000 \text{ (in thousands of dollars)}$$

9.26 $a_r - a_{r-1} = 5(a_{r-1} - a_{r-2}) \qquad r \geq 2$

$$a_r - 6a_{r-1} + 5a_{r-2} = 0$$

$$\alpha^2 - 6\alpha + 5 = 0; \qquad \alpha = 1, 5$$

$$a_r = A + B5^5$$

$$a_0 = A + B = 3; \qquad a_1 = A + 5B = 7$$

$$A = 2, \qquad B = 1$$

$$a_r = 2 + 5^r$$

9.27 (a) Each line from the $r^{\text{th}}$ vertex to the remaining nonadjacent vertices creates a new region whenever it intersects a line or reaches the vertex. In addition there are $r - 2$ new regions outside of the convex polygon determined by the first $r - 1$ vertices. The number of lines intersected by a line from vertex $r$ to vertex $i$, $i = 2, \ldots, r - 2$ is $(i - 1)(r - 1 - i)$ if the vertices are numbered in order around the convex polygon.

$$\sum_{i=2}^{r-2} (i-1)(r-1-i) = \frac{(r-1)(r-1)(r-3)}{6}$$

So the number of regions created is $\dfrac{(r-1)(r-2)(r-3)}{6} + r - 2$ and (a) is correct.

(b) $$A(z)(1-z) = \sum_{r=4}^{\infty} \frac{(r-1)(r-2)(r-3)}{6} z^r + \sum_{r=3}^{\infty} (r-2) z^r$$

$$= \frac{z^4}{6} D^{(3)}\left(\frac{1}{1-z}\right) + z^3 D\left(\frac{1}{1-z}\right)$$

$$A(z) = \frac{z^4}{(1-z)^5} + \frac{z^3}{(1-z)^3}$$

(c) $a_r = \dfrac{r(r-1)(r-2)(r-3)}{24} + \dfrac{(r-1)(r-2)}{2}, \quad r \geq 3$

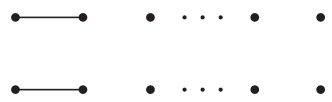
$a_0 = a_1 = a_2 = 0$

9.28 Both $\alpha$ and $\beta$ particles split into three particles each second.

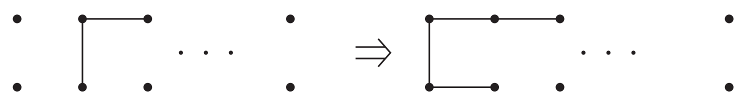
Hence, $p_r = 3p_{r-1}$ where $p_r$ is particles at the $r$th second.
$p_r = 3^r$; and $p_{100} = 3^{100}$

9.29 $a_1 = 1, \quad a_2 = 4.$
Among all spanning trees on $n$ steps, consider those do have the 1st step. Then they must have both the first pair of "side hooks" that is connected to a spanning tree on the remaining $n$ – 1 rungs. Hence,

$$a_n = a_{n-1} + b_n$$



Trees that have the 1st step can be divided into $a_{n-1}$ of those that have the first top only, $a_{n-1}$ of those that have the first bottom hook only, and those that have both hooks. For the last class, consider the $b_{n-1}$ spanning trees on $n$ – 1 steps. If we remove the first step and replace it with a "$C$"-shaped piece as shown, we get a unique spanning tree on $n$ rungs with both hooks.



Thus, $\quad b_n = 2a_{n-1} + b_{n-1}$

$a_n = a_{n-1} + b_n = a_{n-1} + 2a_{n-1} + b_{n-1}$

$a_n = 3a_{n-1} + b_{n-1} = 3a_{n-1} + a_{n-1} - a_{n-2}$

Thus, $\quad a_n - 4a_{n-1} + a_{n-2} = 0$

$\alpha^2 - 4\alpha + 1 = 0; \quad \alpha = 2 \pm \sqrt{3}$

Thus, $\quad a_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$

$a_1 = A(2+\sqrt{3}) + B(2-\sqrt{3}) = 1$

$$a_2 = A(2+\sqrt{3})^2 + B(2-\sqrt{3})^2 = 4$$

$$A = \frac{\sqrt{3}}{6}, \qquad B = -\frac{\sqrt{3}}{6}$$

Thus, $a_n = \dfrac{\sqrt{3}}{6}(2+\sqrt{3})^n - \dfrac{\sqrt{3}}{6}(2-\sqrt{3})^n$.

9.30 (a) Trivially true for $r = 1, 2$. If $k$ is in the first position and 1 is not in the $k$th position, there are $d_{r-1}$, ways to permute the $r - 1$ integers $\{1, 2,\ldots, r\} - \{k\}$. If $k$ is in the first position and 1 is in the $k$th position, there are $d_{r-2}$ ways to permute the $r - 2$ integers $\{1, 2,\ldots, r\} - \{1, k\}$. Since $k$ can be chosen in $r - 1$ ways, the difference equation follows.

(b) $$(r-1)\left[(r-1)!\left(1-\frac{1}{1!}+\frac{1}{2!}-\ldots+\frac{(-1)^{r-1}}{(r-1)!}\right)\right.$$

$$\left.+(r-2)!\left(1-\frac{1}{1!}+\frac{1}{2!}-\ldots+\frac{(-1)^{r-2}}{(r-2)!}\right)\right]$$

$$=(r-1)(r-2)!\left[((r-1)+1)\left(1-\frac{1}{1!}+\ldots+\frac{(-1)^{r-2}}{(r-2)!}\right)\right.$$

$$\left.+\frac{(r-1)(-1)^{r-1}}{(r-1)!}\right]$$

$$=r!\left(1-\frac{1}{1!}+\ldots+\frac{(-1)^{r-2}}{(r-2)!}\right)+(r-1)!(-1)^{r-1}\frac{r(r-1)}{r(r-1)!}$$

$$=r!\left[1-\frac{1}{1!}+\ldots+(-1)^{r-2}\frac{1}{(r-2)!}+(-1)^{r-1}\frac{1}{(r-1)!}+(-1)^r(-1)^r\frac{1}{r!}\right]$$

9.31 $a_r = a_{r-1} + 2^7 a_{r-8}$

$$\sum_{r=8}^{\infty} a_r z^r = \sum_{r=8}^{\infty} a_{r-1} z^r + 2^7 \sum_{r=8}^{\infty} a_{r-8} z^r$$

Also, $$a_0 = a_1 = a_2 = \ldots = a_7 = 1$$

$$A(z) - 1 - z - z^2 - \ldots - z^7 = z[A(z) - 1 - z - z^2\ldots - z^6] + 2^7 z^8 A(z)$$

$$A(z) = \frac{1}{1 - z - 2^7 z^8}$$

9.32 Let $a_r$, $b_r$, $d_r$ be the number of paths of length $r$ ending at vertex $a$, $b$ and $d$ respectively. If the $(r-1)$ long path ends at $a$ or $d$, another '1' will send it to $b$.

Thus, $$b_r = a_{r-1} + d_{r-1}$$

Any sequence ends with '0' will stay in $a$.

$$a_r = \begin{cases} 2^{r-1} & r = 0 \\ 1 & r \geq 1 \end{cases}$$

Any sequence which ends at $b$ will go to $d$ if two more consecutive '1' are added.

$$d_r = \begin{cases} b_{r-2} & r \geq 3 \\ 0 & \text{else} \end{cases}$$

Thus, $$b_r = \begin{cases} 0 & r = 0 \\ a_{r-1} & 1 \leq r \leq 3 \\ a_{r-2} + b_{r-3} & r \geq 4 \end{cases}$$

$$= \frac{4\sqrt{3}}{21} \sin \frac{2r\pi}{3} - \frac{2}{7} \cos \frac{2r\pi}{3} + \frac{1}{7} \cdot 2^{r+1}$$

9.33 (a) Assuming $A(z)$ is well defined (and finite) for $z$ close to 0, then

$$\lim_{z\to 0} A(z) = \lim_{z\to 0} \lim_{r\to\infty} \sum_{i=0}^{r} a_r z^r = \lim_{r\to\infty} \lim_{z\to 0} \sum_{i=0}^{r} a_r z^r = a$$

(b) $A(z) = \dfrac{2}{1-2z}$ $\lim_{z\to 0} A(z) = 2$

9.34 $b_r - 2b_{r-1} = 1$

$$b_r = A2^r + B$$
$$B - 2B = 1$$
$$B = -1$$
$$b_0 = A - 1 = 4$$
$$A = 5$$

Thus, $$b_r = 5 \cdot 2^r - 1$$

and $$a_r = \sqrt{5 \cdot 2^r - 1}$$

9.35 $b_r = b_{r-1} = 2^r$

$$b_r = A(-1)^r + \frac{2}{3} 2^r$$

$$b_0 = -A + 2/3 = 0 \qquad A = 2/3$$

Thus, $$b_r = -\frac{2}{3}\ (-1)^r + \frac{2}{3}2^r$$

and $$a_r = \begin{cases} 273 & r = 0 \\ \dfrac{1}{r}\left(-\dfrac{2}{3}(-1)^r + \dfrac{2}{3}2^r\right) & r \neq 0 \end{cases}$$

9.36 (a) $2 \lg a_r = \lg 2 + \lg a_{r-1}$

$$2b_r - b_{r-1} = 1$$

$$b_r = A(1/2)^r + 1$$

$$b_0 = \lg 4 = 2 \qquad \text{Thus, } A + 1 = 2 \quad A = 1$$

Thus $$a_r = 2^{\left(\frac{1}{2}\right)^r + 1} = 2 \cdot 2^{\left(\frac{1}{2}\right)^r}$$

(b) $a_r^2 = a_{r-1} + \sqrt{a_{r-2} + \sqrt{a_{r-3}}}$

$$= a_{r-1} + a_{r-1}$$

and the solution follows from (a).

9.37 $\dfrac{a_r}{r!} - r\ \dfrac{a_{r-1}}{r!} = \dfrac{r!}{r!}$

$$b_r - b_{r-1} = 1 \qquad \text{for } r \geq 1$$

$$b_r = Ar + B$$

$$Ar - A(r-1) = 1 \qquad A = 1$$

$$b_0 = 2 \qquad \text{Thus, } B = 2$$

$$a_r = r!\ (r+2)\ r \geq 1$$

9.38 By straightforward substitution.