Artificial Intelligence II
Winter term 2020/2021
November 27, 2019

Daniel Wulff
wulff@rob.uni-luebeck.de

Exercise sheet 4

Submission deadline: 10:00 on Friday, December 4, 2020

Task 1: Duality of Linear Programs (20 points)

The lecture discussed how to transform a Linear Program – the *primal problem* or just *primal* – into the dual version of the Linear Program – the *dual problem* or just *dual*. The transformation takes a Linear Program of the form

$$\max(\mathbf{c}^T\mathbf{x}) \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

and gives the dual Linear Program of the form

$$\min(\mathbf{b}^T\mathbf{y}) \text{ subject to } A^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}.$$

Equilibrium Theorem

If the feasible solutions $\mathbf{x}^\star$ and $\mathbf{y}^\star$ are optimal solutions to the primal and dual problem, respectively, then

$$\mathbf{y}_i^\star = 0 \quad \forall i \quad \text{for which} \quad \sum_{j=1}^{n} a_{i,j}x_j^\star < b_i$$

and

$$\mathbf{x}_j^\star = 0 \quad \forall j \quad \text{for which} \quad \sum_{i=1}^{m} a_{j,i}y_i^\star > c_j.$$

Strong Duality Theorem

If $\mathbf{x}^\star$ is an optimal solution to the primal, then there is an optimal solution $\mathbf{y}^\star$ to the dual with

$$\mathbf{c}^T\mathbf{x}^\star = \mathbf{b}^T\mathbf{y}^\star.$$

Weak Duality Theorem

If $\mathbf{x}$ is a feasible solution to the primal and $\mathbf{y}$ is a feasible solution to the dual, then

$$\mathbf{b}^T\mathbf{y} \geq \mathbf{c}^T\mathbf{x}.$$

a) Show that the dual of the dual is the primal again. (6 points)

b) Show that the Weak Duality Theorem follows from the Strong Duality Theorem. (6 points)
(Without proof, you can assume: If there are feasible solutions to the primal and dual, then there is an optimal solution to the primal.)

c) Draw the graphical solution of the Linear Program given below and identify the optimal solution $\mathbf{x}^\star$ as well as the so-called *carrier constraints* (see Equilibrium Theorem). Give the dual representation of the primal problem. With the help of the Equilibrium Theorem and the dual, explicitly determine the optimal solution $\mathbf{y}^\star$ of the dual problem.
(8 points)

maximize $2x_1 + 3x_2$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 6 \\ x_1 &\geq 2 \\ x_1 &\leq 5 \\ x_2 &\leq 4 \end{aligned}$$

where

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Prepare your solution as a .pdf file and submit it in moodle!