as our second postulate the principle of the constancy of the speed of light. It was shown in Section 3.3 that this principle follows if it is assumed that Maxwell's equations obey the principle of relativity. The direct and indirect evidence in favour of the principle of the constancy of the speed of light was reviewed in Section 3.4.

According to the principle of the constancy of the speed of light, the speed of light in empty space has the same numerical value in all inertial reference frames. Thus light should travel in straight lines and have the same speed in all directions of empty space in all inertial reference frames. This principle will be taken as *axiomatic* in this chapter. It will be shown that predictions made on the basis of this postulate are in agreement with the experimental results.

In this chapter radar methods will generally be used to determine the positions and times of events. Some people may prefer to use the principle of the constancy of the limiting speed of particles. Instead of radio signals, they can use fictitious space guns to shoot electrons back and forth with speeds extremely close to c. (It is, however, actually possible to use radio signals in practice.)

4.2 RADAR METHODS

As an example of the use of radar methods, we shall consider the determination of the position and velocity of a spaceship moving directly away from the Earth with uniform velocity, as shown in Fig. 4.1. Let a directional antenna transmit a short radio pulse (or radar signal) at a time t_1, measured by a clock at rest in the radar station. This radar signal travels with speed c in empty space. When it reaches the spaceship, a small fraction of the radio waves are reflected in the backward direction, in the direction of the transmitter. These reflected signals can be received and amplified. Suppose that the reflected signal returns to the Earth at a time t_2, measured by the clock at rest in the radar station. According to the principle of the constancy of the speed of light, the speed of the radar signal should be the same, and equal to c, when the signal is going to the spaceship as when the reflected signal is returning. Since the total time for the radar signal to go to the spaceship and back is $t_2 - t_1$, the time the radar signal

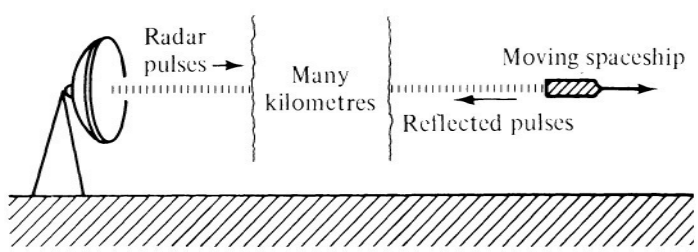

Fig. 4.1 A directional antenna is used to transmit radar pulses, which are reflected by the moving spaceship.

takes to reach the spaceship is $\frac{1}{2}(t_2 - t_1)$ and the time it takes to return is $\frac{1}{2}(t_2 - t_1)$. Since it travels at a speed c, in a time interval of $\frac{1}{2}(t_2 - t_1)$ the radar signal travels a distance $\frac{1}{2}c(t_2 - t_1)$. Since it is transmitted at a time t_1, an observer in the radar station will *calculate* that the radar signal reached the spaceship at a time $t_1 + \frac{1}{2}(t_2 - t_1)$; that is, at a time $\frac{1}{2}(t_2 + t_1)$. Hence, if the radar station is at the origin of a coordinate system, and the spaceship is moving along the x axis, the observer in the radar station will calculate that the radar signal was reflected by the spaceship at a time t at a distance x given by

$$x = \tfrac{1}{2}c(t_2 - t_1), \tag{4.1}$$

$$t = \tfrac{1}{2}(t_2 + t_1), \tag{4.2}$$

By repeating the experiment, the velocity of the spaceship can be determined. For example, let a radar signal be sent from the Earth at 10.00 a.m. Let the reflected signal return to the Earth 2 s later. According to equations (4.1) and (4.2), the coordinates and time of the event of reflection of the signal by the

ignal transmitted at a time 10.20 a.m. returns to the Earth
on by the spaceship then according to equations (4.1) and
as 6.00×10^8 m from the Earth at a time 2 s after 10.20 a.m.
min 1 s (1201 s) the spaceship travels a distance of
^{8} m) or 3.00×10^8 m. Hence, if the spaceship is moving
, its speed is $c/1201$ or about 250 km s^{-1}.

lot the positions of events such as special relativity one is generally
ith speeds comparable to the speed of light. For example, if an
es the origin $x = 0$ at $t = 0$ and travels at a speed $\frac{1}{2}c$ then after a time
s at $x = 1.5$ m. With such high-speed particles, it is not usually
o plot distance in metres against time in seconds. It is a *convention*
ll times by c, the speed of light, before plotting the displacement–
The symbol w will be used for the product ct. Therefore the coor-
e electron, at time $t = 10^{-8}$ s would be ($x = 1.5$ m, $w = ct = 3$ m). The
ct has the dimensions of a length. It is also a convention to use
inate and x as abscissa, as shown in Fig. 4.2(a). Such a displace-
iagram is called a *space–time* diagram. If the time of an event is t it
n as ct on space–time diagrams, but will generally be referred to as
the text.

after 10.00 a.m. If a
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(4.2), the spaceship w
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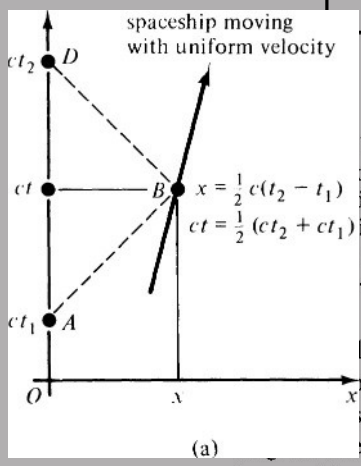
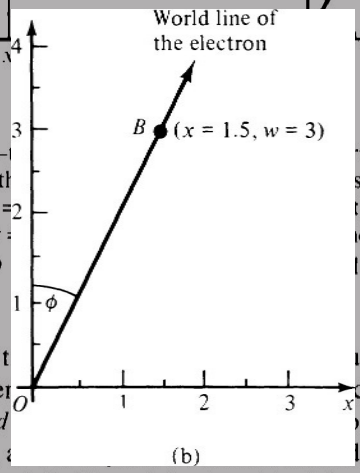
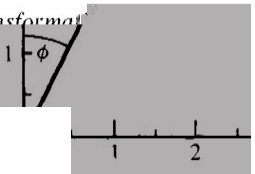


(a)



(b)



(b)

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(4.3)

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thoroughly with the definitions introduced in this section. Since li
signals go at a speed c, they travel a distance $\Delta x = c\Delta t$ in a time
slopes of the world lines of light signals going along the
$c\Delta t/\Delta x = 1$, so that the world lines of light signals going in the pos
tion are at 45° to the x and $w = ct$ axes. Figure 4.2(a) shows the wo
radar signal transmitted from the radar station at the point $x = 0$ of
world point A, which is reflected by the spaceship at world point
to the radar station at $x = 0$ at time t_2 at world point D. The world lin
signal is at 45° to the x axis for the first half of the journey and at
axis for the journey back from the spaceship to the radar station
The x and w coordinates of world point B can be obtained by
parallel to the $w = ct$ and x axes respectively to cut the x axis and t
at the points x and ct respectively, as shown in Fig. 4.2(a). If the
moving with uniform velocity relative to Σ, its world line is s
velocity is v, in a time Δt the spaceship goes a distance $\Delta x = v\Delta$
inclination of its world line to the $w = ct$ axis is equal to ϕ, where

$$\tan\phi = \frac{\Delta x}{c\Delta t} = \frac{v\,\Delta t}{c\Delta t} = \frac{v}{c}\,.$$

Since for all particles of finite mass, $v < c$, the maximum value of $\tan\phi$ is 1, so that the inclinations of the world lines of all moving particles to the $w = ct$ axis must be less than 45°. The world lines of light (radio) signals are at 45° to the $w = ct$ axis.

If the spaceship is accelerating relative to Σ, its world line is curved.

As a numerical example of a space–time diagram, consider again the example of an electron that leaves the origin of Fig. 4.2(b) at world point O at ($x = 0$, $w = 0$) and travels with uniform velocity $v = 0.5\,c$ along the $+x$ axis. The inclination of the world line of the electron to the $w = ct$ axis is given by

$$\phi = \arctan\left(\frac{v}{c}\right) = \arctan 0.5 = 26.57^\circ$$

as shown in Fig. 4.2(b). In a time of 10^{-8} s the electron travels a distance $x = vt = 0.5 \times 3 \times 10^8 \times 10^{-8}$ m = 1.5 m to be at world point B in Fig. 4.2(b), which has coordinates ($x = 1.5$ m, $w = ct = 3$ m).

In this section we have described how one inertial observer with a radar set can determine the positions and times of distant events. We shall now proceed to see how the radar measurements of the coordinates and times of distant events carried out by two inertial observers moving with uniform velocity relative to each other can be related.

4.3 RADAR MEASUREMENTS CARRIED OUT BY TWO ASTRONAUTS IN RELATIVE MOTION

4.3.1 Introduction

Since many people have a mental bias in favour of the laboratory frame, we shall consider two spaceships in outer space moving with different uniform velocities relative to the fixed stars. (If at any stage we want to use the laboratory frame, we can assume that one of the spaceships is at rest on the Earth.) It will be assumed that astronaut John is on one spaceship and astronaut Mary is on the other. Both John and Mary are inertial observers. Let John's spaceship be at rest at the origin of an inertial reference frame Σ, and let Mary's spaceship be at rest at the origin of an inertial reference frame Σ', as shown in Fig. 4.3(b). Suppose that Mary's spaceship moves with uniform velocity v relative to John's spaceship along the x axis of Σ, which is coincident with the x' axis of Σ', as shown in Fig. 4.3(b), so that the inertial reference frame Σ' is moving with uniform velocity v relative to the inertial reference frame Σ.

Suppose that John and Mary construct identical clocks. They must agree to use the same unit of time. It would be sensible for them to use the 1967 definition of the *second* (s), which is still in use, and is

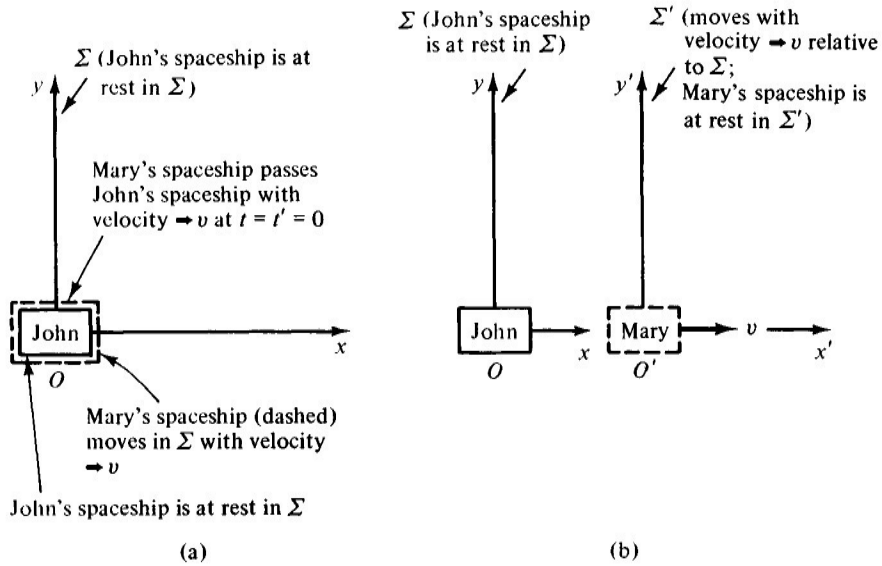

Fig. 4.3 John's spaceship is at rest at the origin of Σ and Mary's spaceship is at rest at the origin of Σ'. Mary's spaceship moves with uniform velocity v relative to John's spaceship along the x axis of Σ, passing John's spaceship at $t = t' = 0$, as shown in (a). The x axis of Σ and the x' axis of Σ' are coincident.

> The duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

John and Mary must also agree to use the same unit of length. In 1983 the 1960 definition of the *metre*, given in Section 1.1, was replaced by a new definition, in which it is assumed that the velocity of light in empty space is given *exactly* by

$$c = 299\,792\,458 \text{ m s}^{-1}.$$

If John and Mary are to carry out their measurements in a way consistent with the principle of the constancy of the speed of light, they must use the same value for c. It would be sensible for them to agree to use the value of 299 792 458 m s^{-1} (in calculations we shall generally use the *approximate* value of $c = 3 \times 10^8$ m s^{-1}). They can then adopt the 1983 definition of the metre to calibrate their metre rules, which is:

> the distance travelled by light in a vacuum in a time of 1/299 792 458 s.

In Chapter 6, when we come back to relativistic mechanics, we shall need a suitable unit of mass. At present, the kilogram is still defined as the mass of a cylinder of platinum–iridium kept at Sèvres near Paris. Clearly it would not be possible for both John and Mary to take the standard kilogram with them. Pending the definition of the kilogram in terms of some atomic quantity, John and

Mary could agree to assign an exact value to the mass of a proton that is consistent within exp origins of Σ and Σ' therefore coincide at $t = t' = 0$.)

f events

at time t_1 on his clock.
the x axis of Σ, and that
a time t_2 on his clock. If
is the same and equal to
rinciple of the constancy
2), he will determine the

(4.4)

(4.5)

ymbols for x, t, t_1 and t_2
vents relative to the iner-
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event on the x' axis of
ip is at rest, and that the
her clock. According to
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ons (4.1) and (4.2), Mary
nt on the x' axis of Σ' to

(4.6)

$'$). (4.7)

determine the positions
ame Σ', in which Mary's

eships

aceship that are reflected
ohn use these signals to

4.3.2 Radar determination of the coordinates o

Let John send out a radar signal from his spaceship
Suppose that this signal is reflected at a distant *event* o
the reflected radar signal returns to John's spaceship a
John assumes that the speed of radar signals (or light)
c in all directions, so as to be in agreement with the p
of the speed of light, then, using equations (4.1) and (4
position x and time t of the event to be

$$x = \tfrac{1}{2}c\,(t_2 - t_1) = \tfrac{1}{2}(ct_2 - ct_1),$$

$$t = \tfrac{1}{2}(t_2 + t_1), \qquad ct = \tfrac{1}{2}(ct_2 + ct_1)$$

Equations (4.4) and (4.5) will be used with different
throughout this chapter to determine the positions of e
tial reference frame Σ in which John's spaceship is at

Similarly, let Mary send out a radar signal from her s
clock. Suppose that this signal is reflected from an
the inertial reference frame Σ', in which Mary's spaces
reflected signal reaches her spaceship at time t_2' on
the principle of the constancy of the speed of light,
axiomatic in this chapter, Mary must use the same valu
of propagation of radar signals in all directions relati
Σ' in which her spaceship is at rest. Hence, using equat
will determine the position x' and the time t' of the ev
be

$$x' = \tfrac{1}{2}c\,(t_2' - t_1') = \tfrac{1}{2}(ct_2' - ct_1')$$

$$t' = \tfrac{1}{2}(t_2' + t_1'), \qquad ct' = \tfrac{1}{2}(ct_2' + c$$

These equations will be used throughout this chapter t
and times of events relative to the inertial reference f
spaceship is at rest at the origin.

4.3.3 Relative speed of John's and Mary's spac

Let John send out a series of radar signals from his sp
back to John's spaceship by Mary's spaceship. Let J

determine a series of values for the position of Mary's spaceship relative to his own spaceship at various times. John can then calculate the speed at which Mary's spaceship is moving away from his spaceship. Similarly, by transmitting radar signals that are reflected by John's spaceship, Mary can determine the speed at which John's spaceship is moving away from her spaceship. John and Mary must agree on their speed of separation. If they did not determine the same numerical value for this speed, it could only be due to motion in one direction of empty space compared with motion in the opposite direction of empty space. This is contrary to our assumption that there are no preferred directions in empty space, i.e. that space is isotropic. Hence John and Mary must agree on their speed of separation, which will be denoted by *v*.

4.3.4 The *K* calculus

A space-time diagram (or displacement-time graph) will be used to represent the positions and times of events relative to the inertial refer

$w = ct$ against x, as shown in Fig. 4.4(a), where x is the position and t is time of the event measured by John using his radar set. The displacement of Mary's spaceship relative to John's spaceship is shown in Fig. 4.4(a). If her spaceship passes his at $t = t' = 0$ then Mary's world line Ow' is a straight line through the origin. Since Mary's spaceship goes a distance $x = vt = (v/c)\, ct = vw/c$ in a time t relative to John, her world line Ow' is at an angle ϕ to the Ow axis, where $\tan\phi = v/c$.

Throughout this chapter until Section 4.11, it will be assumed that all world points and all world lines are plotted relative to the inertial reference frame Σ *in which John's spaceship is at rest, using* Ox *and* Ow *as rectangular axes. We shall, however, take the liberty of labelling the times of events on Mary's world line* Ow' *as the times measured by Mary's clock.*

Let John send out radio signals at times $t = 0$, T_0, $2T_0$, $3T_0$ etc., as measured by his clock. The time interval between the emission of successive signals is T_0 on John's clock. The world lines of these radio signals are at 45° to the x and $w = ct$ axes in Fig. 4.4(a). Mary's spaceship coincides with John's spaceship when he transmits his first signal at $t = 0$, so that she receives his first signal at $t' = 0$ on her clock. Suppose that Mary receives the radio signal transmitted by John at a time $t = T_0$ on his clock at a time T' measured by her clock. Now T' is not equal to T_0, since Mary is moving away from John during the time interval the radio signal takes to travel from his spaceship to hers. Let

measured on her clock. In this case T_0 would be equal to 2 s and T' would be equal to 3 s in equation (4.8), but K would still be equal to $\frac{3}{2}$. This illustrates how, provided that Mary's spaceship continues to move with the same uniform velocity relative to John (Σ), the value of K in equation (4.8) does not depend on the value of the time interval T_0, measured by John in Σ, between the signals transmitted by him.

Let Mary transmit radio signals at times $t' = 0, 1, 2, 3$ s etc. measured by her clock, when she is moving away from John with the same uniform velocity as in the previous numerical example. According to the postulates of special relativity, all directions in space are equivalent and the speed of light is the same in all directions of empty space in all inertial reference frames. There is nothing to differentiate the inertial reference frame Σ' in which Mary is at rest from the inertial reference frame Σ in which John is at rest. John and Mary are just moving apart with uniform velocity relative to each other in outer space. Hence John should receive the radio signals Mary transmits at $t' = 0, 1, 2, 3$ s on her clock at times $t = 0, 1.5, 3, 4.5$ s on his own clock. Thus if Mary measures the interval between John's equally spaced signals to be increased by a factor K when she is going away from him with uniform velocity v John should measure the time interval between Mary's equally spaced signals to be increased by the same factor K, when he is moving away from her with the same uniform velocity v. If Mary transmits radio signals at times $0, T_0', 2T_0', 3T_0'$ etc. measured on her clock, these signals should reach John's spaceship at times $0, T, 2T, 3T$ etc. measured by his clock, where

$$T = KT_0', \tag{4.9}$$

as illustrated in Fig. 4.4(b). Equations (4.8) and (4.9) apply to the time intervals between successive signals. Provided that Mary and John pass each other at $t = t' = 0$, then equations (4.8) and (4.9) can be used to relate the times of single radio signals sent from John to Mary at a time T_0 on his clock, and from Mary to John at a time T_0' on hers.

In summary, provided that Mary's spaceship passes John's spaceship at $t = t' = 0$ and continues to move with uniform velocity v relative to him (Σ), if John transmits a radio signal to Mary at a time T_0 on his clock, and if this signal reaches Mary's spaceship at a time T' on her clock, where

$$T_0 = T'/K. \tag{4.11}$$

If Mary transmits a radio signal to John's spaceship at a time T_0' on her clock, then the signal should reach him at a time T on his clock given by

$$T = KT_0'. \tag{4.12}$$

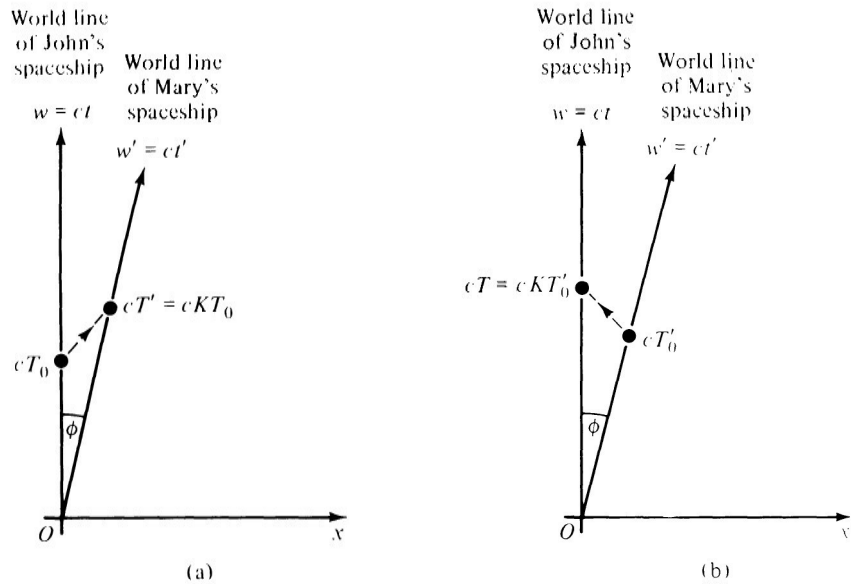


Fig. 4.5 Mary's spaceship passes John's at $t = t' = 0$, so that her world line goes through the origin. (a) If John transmits a radio signals at time T_0 on his clock, it reaches Mary's spaceship at time $T' = KT_0$ on her clock. (b) If Mary transmits a radio signal at a time T_0' on her clock, it reaches John's spaceship at time $T = KT_0'$ on his clock. The space–time diagrams are relative to the inertial reference frame Σ in which John's spaceship is at rest.

This is illustrated in Fig. 4.5(b). Conversely, if we know that Mary's signal reaches John at a time T on his clock then Mary must have transmitted the signal at a time T_0' on her clock, where

$$T_0' = T/K. \tag{4.13}$$

Equations (4.10)–(4.13) are the basic formulae of the K calculus. They hold for all values of T_0 and T_0', provided that Mary passes John at $t = t' = 0$ and they continue to move apart with uniform velocity relative to each other. Readers should familiarize themselves with these equations, and how they can be applied to the conditions illustrated in Figs. 4.5 (a) and (b), *relating the times of transmission and reception of radio or light signals from John to Mary and from Mary to John respectively*. It is assumed that John and Mary coincide at $t = t' = 0$, so that Mary's world line goes through the origin. The value of the constant K will now be determined.

Let John transmit a radar signal at a time T_0 on his clock, as shown in Fig. 4.6, and suppose that this signal reaches Mary at a time T' on her clock where, according to equation (4.10),

$$T' = KT_0. \tag{4.14}$$

$$\frac{v}{c} = \frac{K_2^2 T_0 - T_0}{K_2^2 T_0 + T_0} = \frac{K_2^2 - 1}{K_2^2 + 1},$$

and rearranging,

$$\frac{v}{c} K_2^2 + \frac{v}{c} = K_2^2 - 1,$$

$$K_2^2 \left(1 - \frac{v}{c}\right) = 1 + \frac{v}{c}.$$

So that

$$K = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}. \tag{4.16}$$

Two important expressions involving K can be derived. First, we have

$$K + \frac{1}{K} = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} + \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2}$$

$$= \frac{(1 + v/c) + (1 - v/c)}{[(1 - v/c)(1 + v/c)]^{1/2}}$$

$$= \frac{2}{(1 - v_2^2/c_2^2)^{1/2}} = 2\gamma, \tag{4.17}$$

where

$$\gamma = (1 - v_2^2/c_2^2)^{-1/2}. \tag{4.18}$$

Similarly,

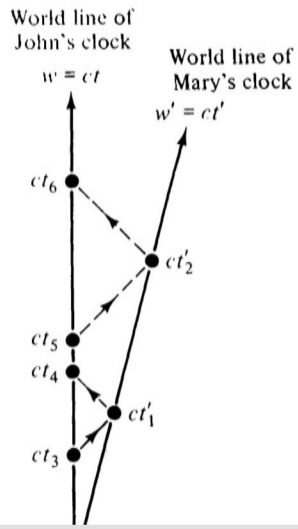
$$K - \frac{1}{K} = \frac{2v/c}{(1 - v_2^2/c_2^2)^{1/2}}$$

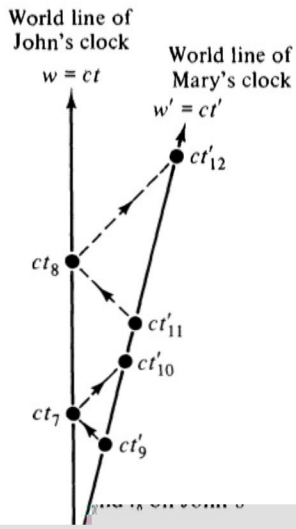
$$= \frac{2\gamma v}{c}. \tag{4.19}$$

4.3.5 Time dilatation

In order to give the reader practice in the use of the K calculus, before deriving the Lorentz transformations, we shall apply the K calculus to derive the formula for time dilatation.

Let two successive ticks on Mary's clock be at times t_1' and t_2' respectively, as measured by her clock on her spaceship, which is moving with uniform velocity v relative to John's spaceship. Let John send a radar signal from his spaceship at





lock.

ohn's spaceship at a time t_4 on his clock, as illustrated in Fig. 4.7(a). Let John ransmit a second radar pulse at a time t_5 on his clock, such that it is reflected rom Mary's spaceship at the time of the second tick on Mary's clock, and uppose that the reflected signal returns to John's spaceship at a time t_6 on his lock (Fig. 4.7(a)). Mary estimates the time interval between the two ticks on her

g radar methods, John estimates the time difference between the two
ry's clock to

$$t_4 = Kt_1', \qquad t_6 = Kt_2'.$$

Substituting in equation (4.20),

$$t_2 - t_1 = \tfrac{1}{2}\left(Kt_2' + \frac{t_2'}{K}\right) - \tfrac{1}{2}\left(Kt_1' + \frac{t_1'}{K}\right)$$
$$= \tfrac{1}{2}\left(K + \frac{1}{K}\right)(t_2' - t_1').$$

But from equation (4.17) $K + 1/K = 2\gamma$. Hence

$$t_2 - t_1 = \gamma(t_2' - t_1') = \frac{t_2' - t_1'}{(1 - v^2/c^2)^{1/2}}. \tag{4.21}$$

Equation (4.21) is the expression for time dilatation. Since $\gamma = (1 - v^2/c^2)^{-1/2}$ is always greater than unity, using radar methods John measures the time interval between the two ticks on Mary's clock to be longer than she does. If $t_2' - t_1'$ is one second on Mary's clock then, by John's reckoning, using radar methods, the time between the two ticks on her clock is γ seconds, whereas the time between two successive ticks on John's own identical clock would be one second. Since γ is always greater than unity, *using radar methods*, John *measures* Mary's clock, which is moving with uniform velocity v relative to him, to go at a slower rate than his own clock. For example, if $v = 0.8\,c$ then $\gamma = \frac{5}{3}$, so that, from equation (4.21), the time interval between the two ticks on Mary's clock, which on her measurements in Σ' is 1 s, is measured to be $\frac{5}{3}$ s by John using radar methods in Σ. Hence John concludes that, relative to Σ, there are 36 ticks on Mary's clock per minute, whereas she concludes that there are 60 ticks per minute on her clock.

The time interval between the two events (the two ticks on Mary's clock), as measured by Mary's clock, *which coincides with both events*, is called the *proper time interval* between the two events. John must use radar methods to measure the time interval between the two events. Even if his clock coincided with the first tick on Mary's clock then, owing to their motion relative to each other, John's clock cannot coincide with the second tick on hers. The measurement of the rate of a moving clock, or the time interval between two events on a moving spaceship, necessitates the use of signals to transmit information to the "stationary" observer.

Let two successive ticks on John's clock be at times t_7 and t_8 measured by his clock, when John's and Mary's spaceships are moving apart with uniform speed v relative to each other. John estimates the time interval between the two events to be $t_8 - t_7$. Let Mary transmit radar signals at times t_9' and t_{11}' on her clock, such that they are reflected from John's spaceship at times t_7 and t_8 respectively on his clock, and suppose that the reflected signals return to Mary's spaceship at times t_{10}' and t_{12}' respectively on her clock, as illustrated in Fig. 4.7(b). Using equation (4.7), Mary estimates the times of the two events to be

$$t_7' = \tfrac{1}{2}(t_{10}' + t_9'), \qquad t_8' = \tfrac{1}{2}(t_{12}' + t_{11}').$$

Hence Mary measures the time intervals between the two ticks on John's clock to be $t_8' - t_7'$, where

$$t_8' - t_7' = \tfrac{1}{2}(t_{12}' + t_{11}') - \tfrac{1}{2}(t_{10}' + t_9').$$

Using equations (4.13) [illegible]

$$(1 - v^2/c^2)^{1/2}. \tag{4.23}$$

$t_8 - t_7$ is the proper time interval between the two events on John's namely the two ticks on his clock. Using radar methods, Mary e time interval between the two ticks on John's clock to be γ times es. Thus, using radar methods, Mary measures John's clock to go at ate than her own clock. Time dilatation is perfectly reciprocal, at Mary and John continue to move apart with uniform velocity ach other. Using radar methods, John measures Mary's clock, which elative to him, to go at a slower rate than his own clock, while, using ds, Mary measures John's clock, which is moving relative to her, to er rate than her own clock. (The case of accelerating clocks will not ed until Chapter 9.) The use of radar signals illustrates how time rises when the information is transmitted by signals travelling at s, and interpreted in a way consistent with the principle of the con- e speed of light. An account of experiments that confirm time dilata- given in Section 4.6.

LORENTZ TRANSFORMATIONS

n be assumed that John's spaceship is at the origin of the inertial ame Σ, that Mary's spaceship is at the origin of the inertial reference nd that Mary's spaceship is moving with uniform velocity v relative paceship, such that Σ' is moving with *uniform* velocity v relative to ir common x axis, as shown previously in Fig. 4.3 (b). Let Mary's

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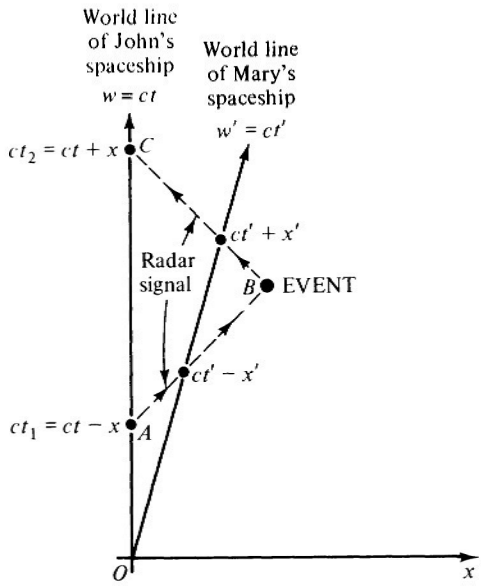

Fig. 4.8 The Lorentz transformations. John transmits a radar signal at a time $t - x/c$ on his clock. The radar signal passes Mary's spaceship at a time $t' - x'/c$ on her clock, and, after reflection at the event, passes her spaceship at a time $t' + x'/c$ on her clock, before reaching John's spaceship at a time $t + x/c$ on his clock.

spaceship pass John's spaceship at $t = t' = 0$, so that the origins of Σ and Σ' coincide at $t = t' = 0$. Suppose that an *event* occurs on the x axis of Σ, at a point farther away from John's spaceship than Mary's spaceship, and that John measures the position and time of this event, using radar methods. Let John transmit the radar signal at a time t_1 on his clock at world point A in Fig. 4.8, and suppose that the signal reflected at the event returns to John's spaceship at a time t_2 measured on his clock at world point C. According to equations (4.4) and (4.5), the x coordinate and time t of the event at world point B in Fig. 4.8, measured by John in Σ using radar methods, are

$$x = \tfrac{1}{2}(ct_2 - ct_1), \tag{4.24}$$

$$ct = \tfrac{1}{2}(ct_2 + ct_1). \tag{4.25}$$

Adding equations (4.24) and (4.25), we find that in Σ

$$ct_2 = ct + x. \tag{4.26}$$

Subtracting equation (4.24) from equation (4.25), we obtain

$$ct_1 = ct - x. \tag{4.27}$$

Hence if John uses x and t to denote the position and time of the event at world point B in Fig. 4.8 then he must have transmitted the radar signal at world point

A at a time $t_1 = t - x/c$ on his clock, and received the signal, reflected at the event at world point B, at a time $t_2 = t + x/c$ on his clock at world point C. As a numerical example, consider an electron moving with velocity $\frac{1}{2}c$ along the x axis of Σ, as shown in the space–time diagram in Fig. 4.2(b). If the position of the electron at world point B having coordinates ($x = 1.5$ m, $w = ct = 3$ m) in Fig. 4.2 (b) is to be measured, using a radar set at rest at the origin, then, according to equation (4.27), the radar signal must be transmitted at a time t_1, given by

$$ct_1 = ct - x = (3 - 1.5)\ \text{m} = 1.5\ \text{m},$$

and, according to equation (4.26), the reflected signal should return to the radar set at a time t_2 given by

$$ct_2 = ct + x = (3 + 1.5)\ \text{m} = 4.5\ \text{m}.$$

Dividing by c gives $t_1 = 0.5 \times 10^{-8}$ s and $t_2 = 1.5 \times 10^{-8}$ s. The reader can check using equations (4.4) and (4.5) that $t_1 = 0.5 \times 10^{-8}$ s and $t_2 = 1.5 \times 10^{-8}$ s do give $x = 1.5$ m and $ct = 3$ m.

Let John's radar signal pass Mary's spaceship at a time $t' - x'/c$ on her clock, and let the signal reflected by the event pass Mary at a time $t' + x'/c$ on her clock, as illustrated in Fig. 4.8, such that using equations (4.6) and (4.7), she estimates the position of the event to be at x' at a time t' relative to Σ'.

Denoting the time from 0 to $t - x/c$ on John's clock by T_0, and the time from 0 to $t' - x'/c$ on Mary's clock by T', since from equation (4.10) $T' = KT_0$, after multiplying both sides by c, we have

$$ct' - x' = K(ct - x) = Kct - Kx. \tag{4.28}$$

Denoting the interval between 0 and $t' + x'/c$ on Mary's clock by T_0' and the interval between 0 and $t + x/c$ on John's clock by T, since from equation (4.13) $T_0' = T/K$, we have

$$ct' + x' = \frac{ct + x}{K} = \frac{ct}{K} + \frac{x}{K}. \tag{4.29}$$

Subtracting equation (4.28) from equation (4.29), we obtain

$$2x' = x\left(\frac{1}{K} + K\right) - ct\left(K - \frac{1}{K}\right).$$

From equation (4.17), $K + 1/K = 2\gamma$, and from equation (4.19), $K - 1/K = 2\gamma v/c$. Hence

$$2x' = 2\gamma x - ct\frac{2\gamma v}{c},$$

Dividing both sides by 2 gives:

$$x' = \gamma(x - vt), \tag{4.30}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2}. \tag{4.31}$$

Adding equations (4.28) and (4.29) gives

$$2ct' = ct\left(\frac{1}{K} + K\right) - x\left(K - \frac{1}{K}\right).$$

Using equations (4.17) and (4.19), we obtain

$$2ct' = ct\,2\gamma - \frac{x\,2\gamma v}{c}.$$

Dividing both sides by $2c$ gives

$$t' = \gamma\,(t - vx/c^2)\,. \tag{4.32}$$

Equations (4.30) and (4.32) are the *Lorentz transformations* relating the positions x and x' and times t and t' of the event, measured by John and Mary respectively using radar methods. The use of radar methods makes it clear that the Lorentz transformations relate the measured positions and times of events. It is left as an algebraic exercise for the reader to rearrange equations (4.30) and (4.32) to show that

$$x = \gamma\,(x' + vt'), \tag{4.33}$$

$$t = \gamma\,(t' + vx'/c^2). \tag{4.34}$$

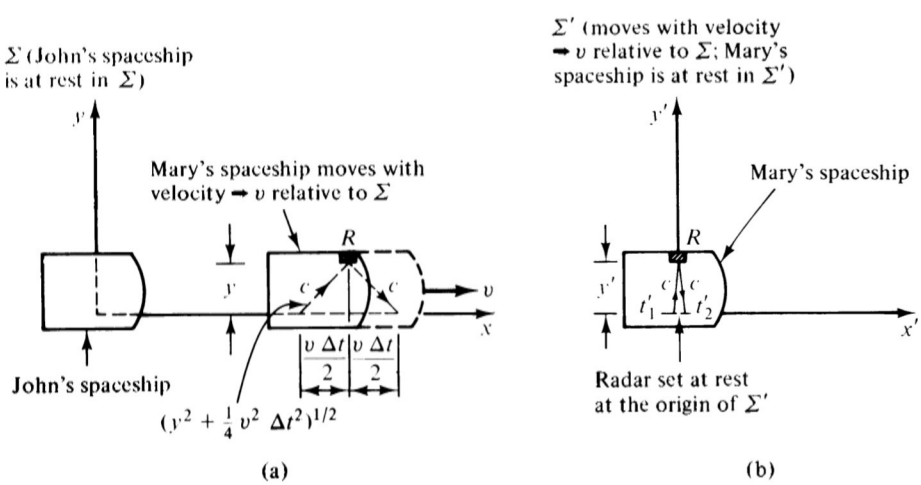

Fig. 4.9 Measurement of a length perpendicular to the direction of relative motion. (a) John's interpretation of Mary's measurements. (b) Mary measures the distance y' between the x' axis and a reflector R on the y' axis of Σ' using radar methods.

These are the *inverse* Lorentz transformations. Note that the inverse transformations can be obtained from equations (4.30) and (4.32) by changing primed into unprimed quantities and unprimed quantities into primed quantities and replacing V by $-V$. This procedure works for all relativistic transformations.

The measurement of lengths in a direction perpendicular to the common v axis, i.e. perpendicular to the direction of motion of Mary's spaceship relative to John's spaceship, will now be discussed. Suppose that Mary places her radar set at a ' = 0 on the a ' axis of the inertial reference frame Z' in which her spaceship is at rest, as shown in Fig. 4.9(b), and uses radar methods to measure the distance y' between her radar set on the a ' axis and a reflector R on her spaceship. Let the line joining Mary's radar set and the reflector R be perpendicular to the direction in which John's spaceship is moving with velocity v along the negative a ' axis of Z'. Let Mary transmit a radar signal at a time tf on her clock, and suppose that this signal is reflected by R and that the reflected signal returns to her radar set at a time on her clock. Since the radar signal travels a total distance 2y' to the reflector R and back in a time interval ^2' - r/ relative to the inertial reference frame Z' in which Mary's spaceship is at rest, she should conclude that in Z'

$$2y' = c\,(tf - tf). \tag{4.35}$$

It follows from equation (4.34) that, since Mary transmits the radar signal from a ' = 0 at a time b' on her clock, John measures this event to be at a time t\ in Z, where

$$T = y(tf - \backslash - ux/r) = ytf$$

Similarly, John measures the event, when the reflected radar signal returns to a ' = 0 at a time tf on Mary's clock, to be at a time ^ in Z, where, according to equation (4.34),

$$i_2 - y\ (I_2 + vxfc\sim) - ytf.$$

Hence, using radar methods, John measures the time interval Ist between the ticks at $t\{$ and tf on Mary's clock, when the radar signal left and returned to the a ' axis respectively, to be $/St = y(tf - tf)$. Using equation (4.35), we have

$$At = 2yyfc. \tag{4.36}$$

Relative to the inertial reference frame Z in which John's spaceship is at rest, Mary's spaceship is moving with uniform velocity v along the -Fa axis, and, relative to Z, the radar signal takes the path shown in Fig. 4.9(a). In the time interval At measured by John, Mary's spaceship goes a distance vAt relative to John (Z). By symmetry, Ma

radar signal relative to Σ is $2\,[y^2 + (\frac{1}{2}\,v\Delta t)^2]^{1/2}$. Since light travels with a speed c during the time Δt it takes to cover this distance, relative to the reference frame Σ shown in Fig. 4.9(a), in which John's spaceship is at rest, we have

$$c\Delta T = 2(y^2 + \tfrac{1}{4}\,v^2\Delta t^2)^{1/2}.$$

Squaring both sides,

$$c^2\Delta t^2 = 4(y^2 + \tfrac{1}{4}\,v^2\,\Delta t^2) = 4y^2 + v^2\Delta t^2.$$

Substituting $2\gamma y'/c$ for Δt from equation (4.36), we have

$$\frac{c^2 4\gamma^2 y'^2}{c^2} = 4y^2 + \frac{v^2 4\gamma^2 y'^2}{c^2}.$$

Rearranging,

$$\gamma^2 y'^2\,(1 - v^2/c^2) = y^2.$$

Since $\gamma^2 = (1 - v^2/c^2)^{-1}$, on taking the square root we have

$$y' = y.$$

Hence John and Mary should agree on the measurements of lengths perpendicular to the x and x' axes, i.e. perpendicular to the directions of their motions relative to each other. Hence, in general, if Σ' moves with uniform velocity v relative to Σ along a common x axis then

$$y' = y. \tag{4.37}$$

Similarly

$$z' = z.$$

Collecting the Lorentz transformations, we have

$$x' = \gamma\,(x - vt), \qquad x = \gamma\,(x' + vt'), \tag{4.38}$$

$$y' = y, \qquad y = y', \tag{4.39}$$

$$z' = z, \qquad z = z', \tag{4.40}$$

$$t' = \gamma\,(t - vx/c^2), \qquad t = \gamma\,(t' + vx'/c^2). \tag{4.41}$$

The Lorentz transformations relate the coordinates and times of *events* in one inertial coordinate system Σ to the coordinates and time of the *same* event measured relative to another inertial coordinate system Σ' that is moving relative to Σ along their common x axis and whose origin coincides with the origin of Σ at $t = t' = 0$. The reader should try to become as familiar as possible with these equations. A few simple problems are given at the end of this chapter.

If we put $v/c = \sin\theta$ then

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}} = \frac{1}{(1 - \sin^2\theta)^{1/2}} = \frac{1}{\cos\theta} = \sec\theta.$$

If $v/c = 0.8000$ then one finds $\theta = 53° 8'$ and $\sec\theta = 1.6667$. By direct calculation, $\gamma = \frac{5}{3}$.

Note that as $c \to \infty$, the Lorentz transformations become

$$x' = x - vt, \qquad y' = y, \qquad z' = z, \qquad t' = t,$$

in agreement with the Galilean transformations of Chapter 1.

So far in this chapter, radar methods have been used when discussing the determination of the coordinates and times of distant events. This has the advantage of making it clear from the outset that one is dealing with the *measured* times of distant events and that one must allow for the propagation times of signals. It is not always convenient to use radar methods. For example, if an astronaut were in outer space in a spaceship *at rest relative to the Earth*, it would be pointless to use radar methods from a base station on the Earth to measure the times of events inside the spaceship. It is far more convenient for the astronaut to use a clock on the spaceship. The question arises as to how the astronaut can synchronize