1 Trees

A graph is called a **tree**, if it is connected and has no cycles. A **star** is a tree with one vertex adjacent to all other vertices.

Theorem 1

For every simple graph G with $n \geq 1$ vertices, the following four properties are equivalent

(A) G is connected and has no cycles;

(B) G is connected and has $n-1$ edges;

(C) G has $n-1$ edges and no cycles;

(D) For every pair u, v of vertices in G, there is exactly one u, v-path.

Proof. A $\Rightarrow$ B $\Rightarrow$ C $\Rightarrow$ D $\Rightarrow$ A

Problem 1 *Let T be a tree and let P and Q be two disjoint paths of the same length in T. Prove that T contains another path longer than P and longer than Q.*

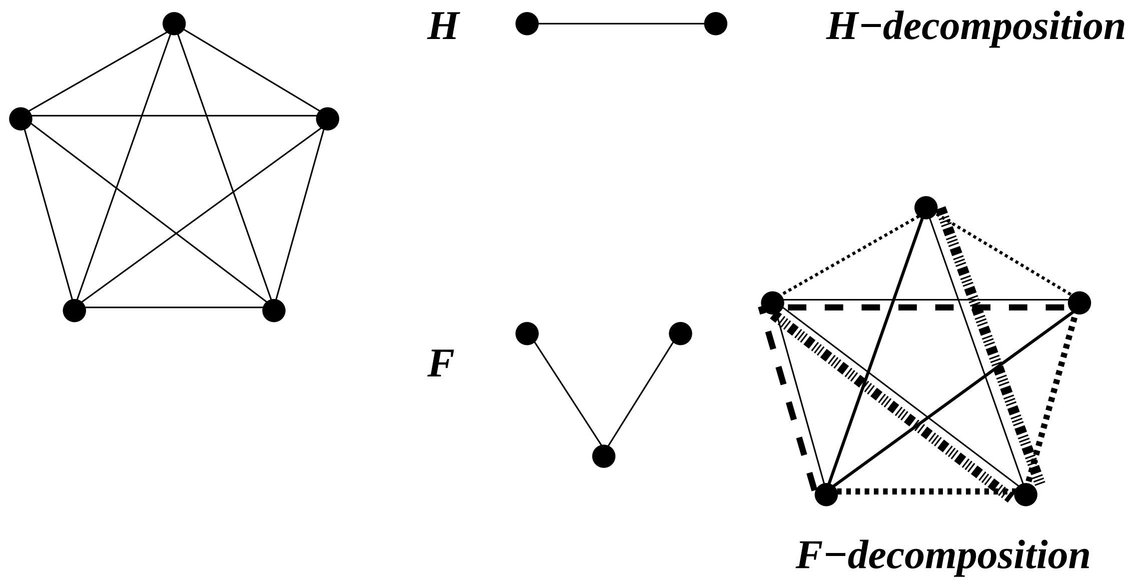
Problem 2 *Let $d_1 \geq d_2 \geq \ldots \geq d_n > 0$ be n integers. Prove that there is a tree T with degrees $d_1, \ldots, d_n$ if and only if*

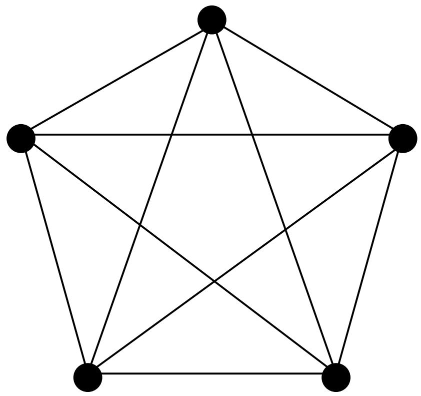
$$d_1 + d_2 + \ldots + d_n = 2n - 2.$$

2 Graceful labeling of trees.

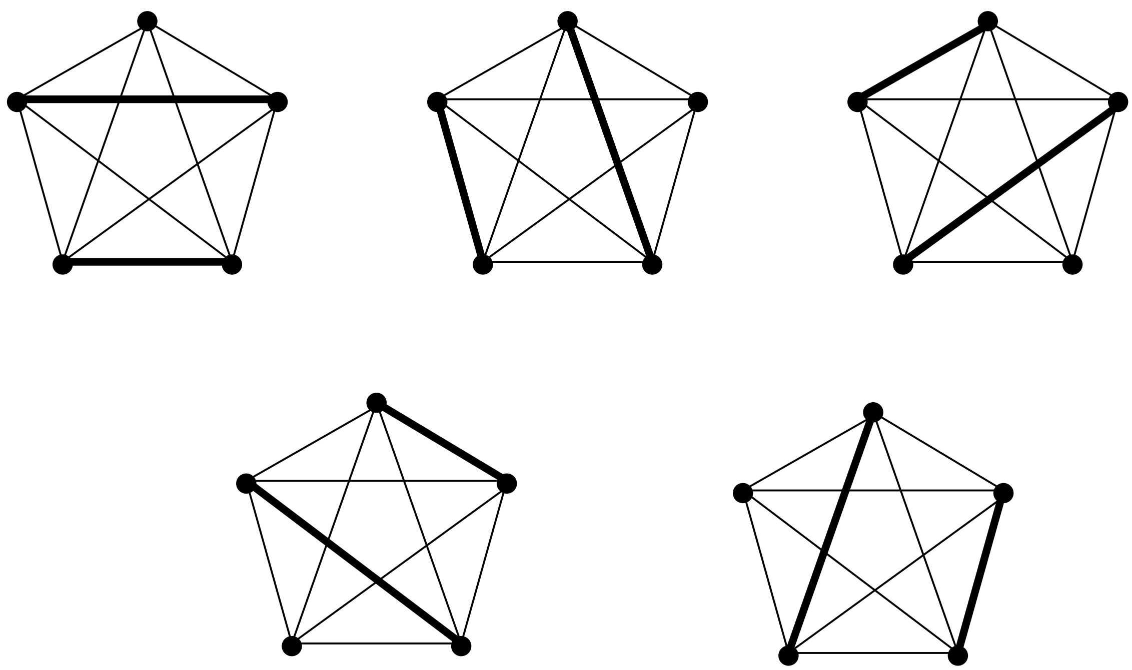
Definition 1

A graph G is said to be **decomposable** into subgraphs $H_1, \ldots, H_m$, if no H_i has an isolated vertex, and $\{E(H_1), E(H_2), \ldots, E(H_m)\}$ is a partitioning of $E(H)$. If all $\{H_i\}$ are isomorphic to a graph H, G is called H-decomposable.



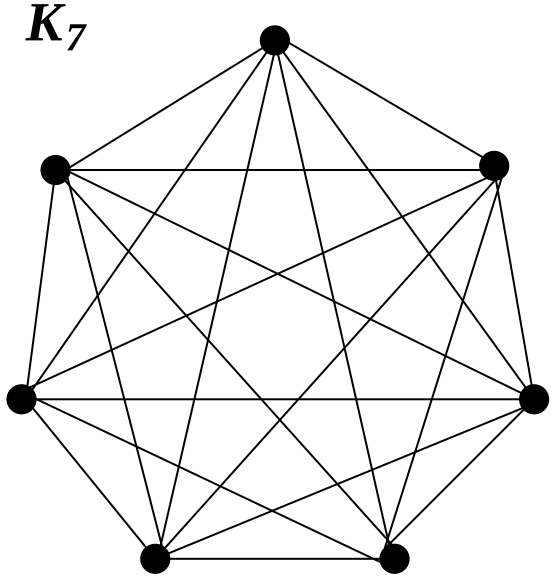


H

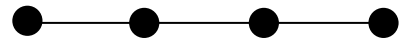


How to construct a T-decomposition of K_7; M-decomposition of K_7?

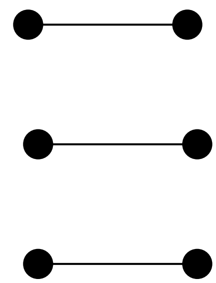
K_7



T



M



Given a labeling ϕ of the vertices of a graph G, for every edge uv, the length of uv is defined as $|\phi(u) - \phi(v)|$.

Definition 2

Given a labeling ϕ of the vertices of a graph G, for every edge uv, the length of uv is defined as $|\phi(u) - \phi(v)|$.

Given a tree $T = (V, E)$ with n vertices, a labeling of its vertices with integers $0, 1, \ldots, n-1$ is called **graceful** if

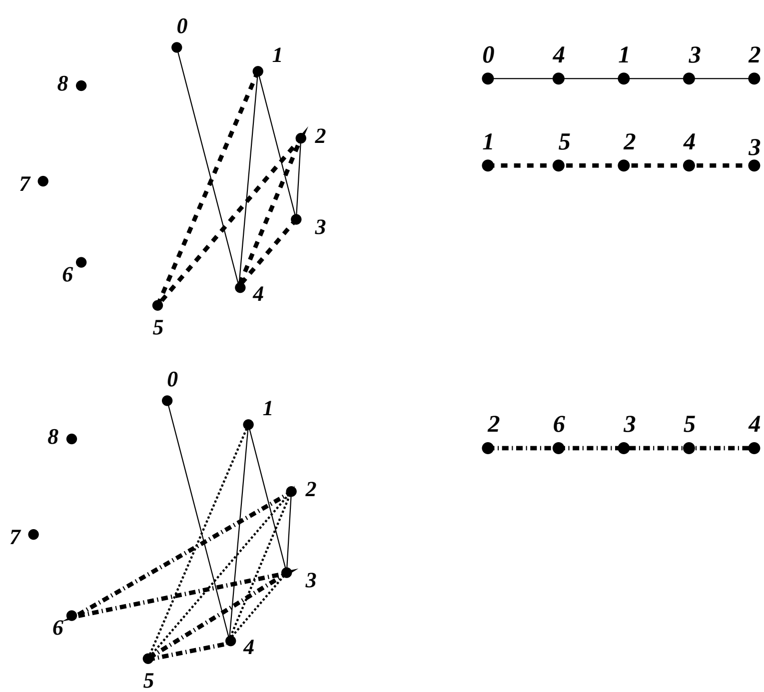
1. different vertices have different labels; and
2. the lengths of the edges are distinct integers $1, 2, \ldots, n-1$.

Conjecture 1 (Ringel, 1964) *For any tree T with m edges ($m \geq 0$), graph K_{2m+1} is T-decomposable.*

Theorem 2 *If T is a tree with m edges which has a graceful labeling, then K_{2m+1} is decomposable into $2m+1$ copies of T.*

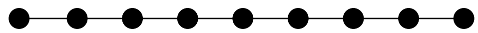
Proof. Label the vertices of K_{2m+1} by $0, 1, 2, \ldots, 2m$; view the vertices as placed around a circle. Let $\phi : V(T) \to \{0, 1, \ldots, m\}$ be a graceful labeling of T. The $2m+1$ copies of T are constructed by the following rule:

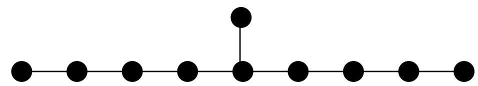
> For $k = 0, \ldots, 2m$, the vertices of k^{th} copy are $k, k+1, \ldots, k+m$, where the addition $k+m$ is understood in the "modular" sense: $2m+1 = 0$. Vertices $k+i$ and $k+j$, in the k^{th} copy of T are adjacent iff i and j are adjacent in the graceful labeling of T.

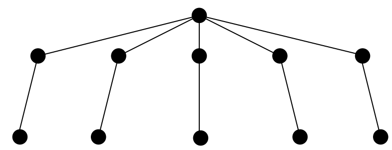


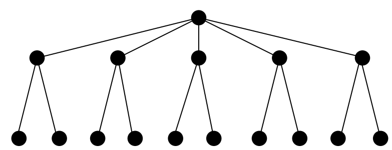
Finish the proof. ∎

Problem 3 *Construct graceful labelings of the trees below.*

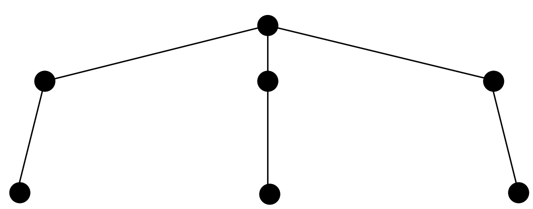




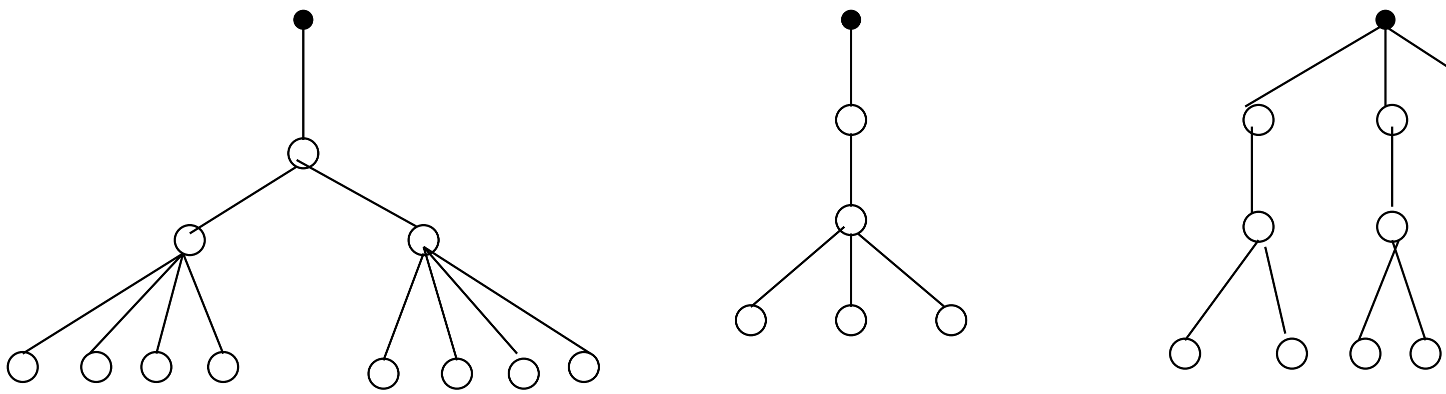




Problem 4 *Construct an S-decomposition of K_{13} for the tree S below:*



Definition 3 *A tree T is called uniform, if $\exists r \in V(T)$ such that all vertices on the same distance from r have the same vertex degrees. Let $(a_0, a_1, \ldots, a_k)$ denote the string(T), k is the largest distance from r to any vertex in T, and a_i denotes the degree of the vertices on the distance i from r*



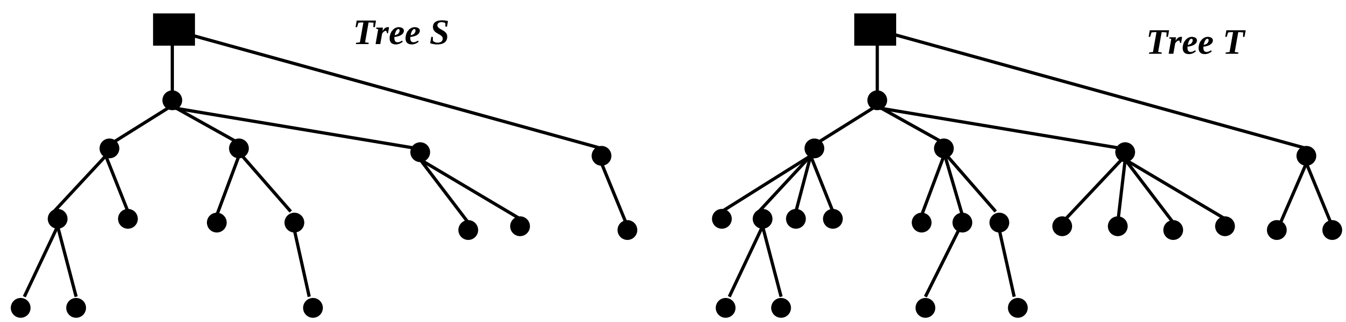
Examples of uniform trees; the shaded vertex is the root.

Problem 5 *Given $(a_0, a_1, \ldots, a_k)$, construct a graceful labeling of the corresponding uniform tree.*

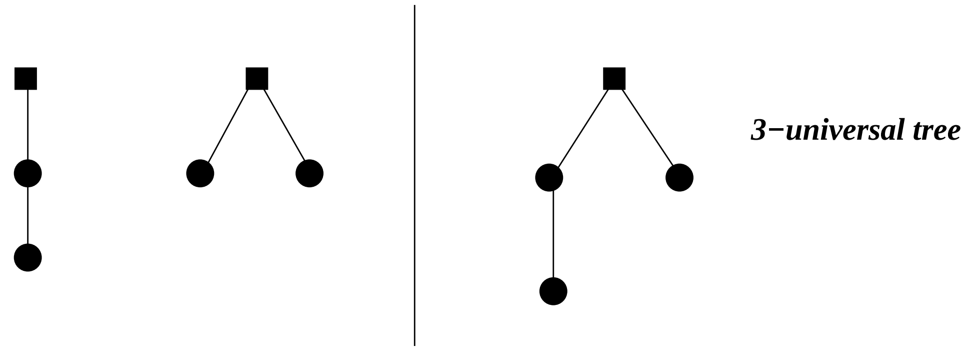
3 Universal rooted trees

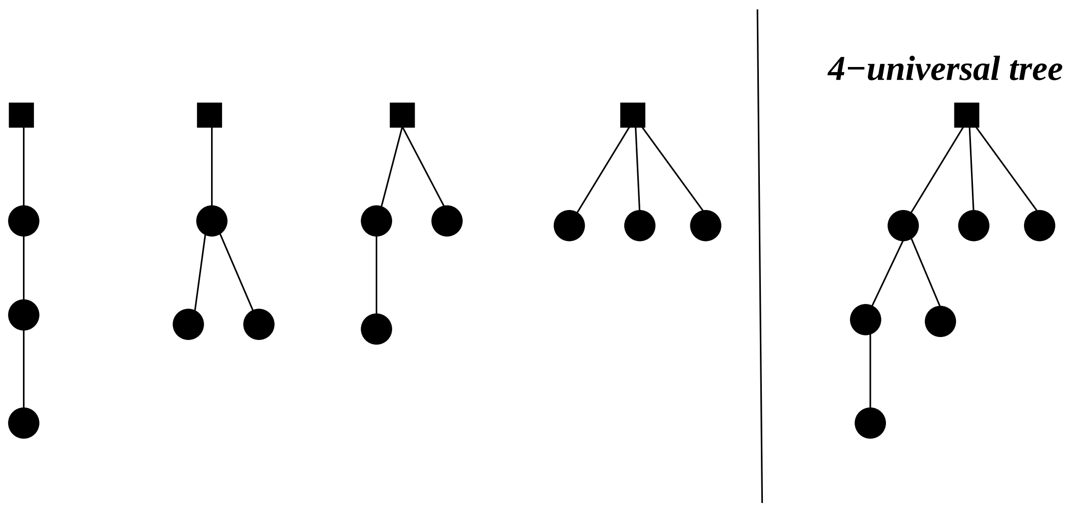
Definition 4 *Let $S(V, E; x_0)$ and $T(U, F; y_0)$ be two directed rooted trees with all edges directed "from" the root. Tree S is said to be embedable into T, if there is a one-to-one mapping $f : V \to U$, from V into U such that*

1. *$f(x_0) = y_0$; and*
2. *for every edge $x'x'' \in E$ of S, the pair $f(x')f(x'')$ is an edge in T.*

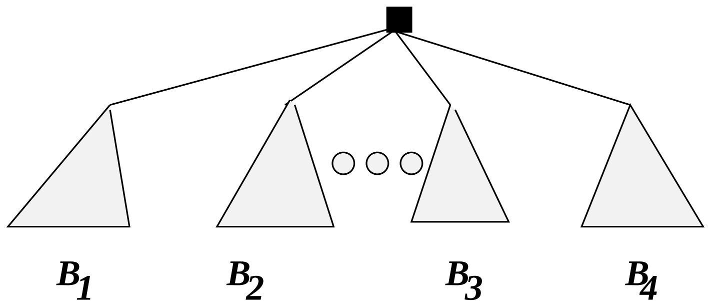


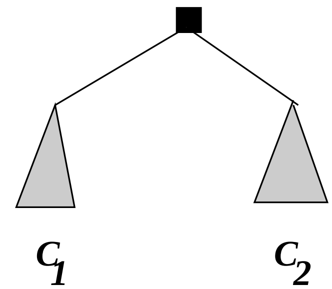
Definition 5 *A rooted tree U is called n-universal if every rooted tree S with at most n vertices can be embedded into U.*

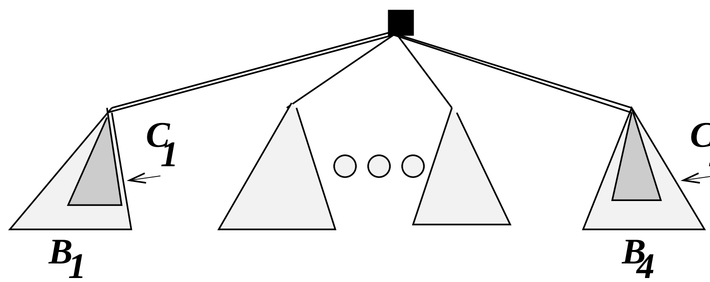




Symbolic representation of rooted trees.

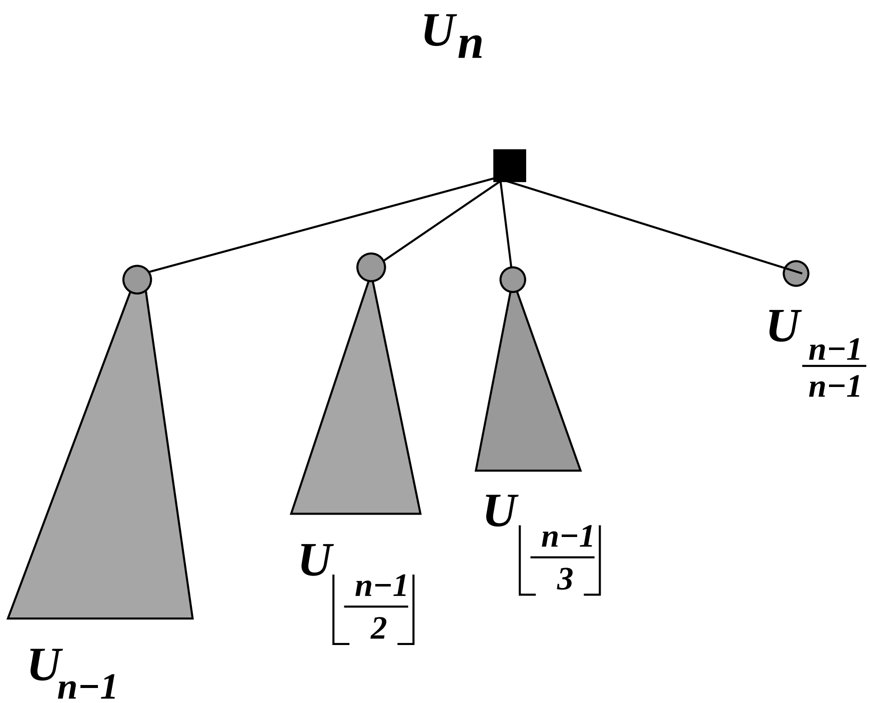






B_1, B_2, B_3, B_4 are the branches of S; all branches are also rooted trees.

How to construct a (smallest) n-universal rooted tree?

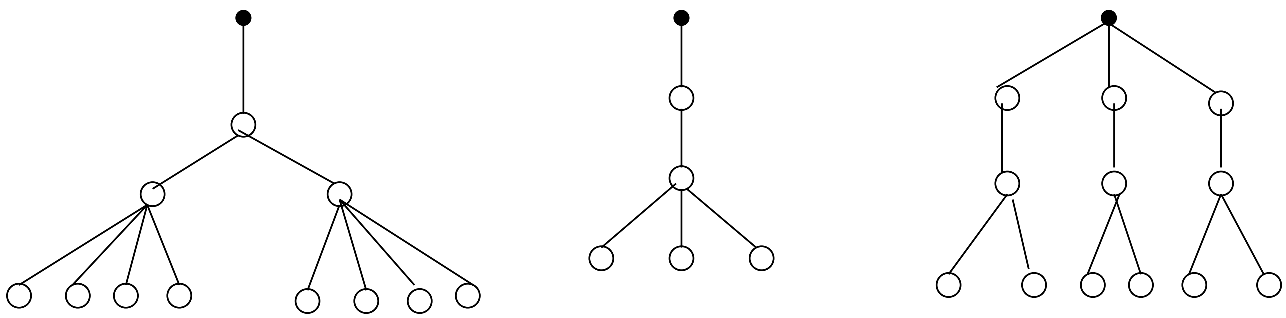


Theorem 3 *Prove that for every $n \geq 1$, U_n is an n-universal rooted tree.*

Let $\alpha(n)$ denote the number of vertices in the tree U_n. Then,

$$\alpha(n) = 1 + \alpha(n-1) + \alpha(\lfloor \frac{n-1}{2} \rfloor) + \alpha(\lfloor \frac{n-1}{3} \rfloor) + \ldots + \alpha(\lfloor \frac{n-1}{n-1} \rfloor).$$

4 Uniform trees



Definition 6 *1. A tree with one vertex is uniform.*

2. If S is a uniform tree and $k \geq 1$ is an integer, then the following tree T is also uniform: the root of T has k branches, each isomorphic to S.

3. The set of uniform trees consists of those trees can be obtained by repeated applications of #2.

Theorem 4 *A tree T is uniform iff the vertex degree of any two vertices on the same distance from the root of T have the same vertex degree.*

Proof. ∎

Theorem 5 *Let $\beta(n)$ denote the number of uniform trees with at most n vertices. Then*

$$\beta(n) = 1 + \beta(n-1) + \beta(\lfloor\frac{n-1}{2}\rfloor) + \beta(\lfloor\frac{n-1}{3}\rfloor) + \ldots + \beta(\lfloor\frac{n-1}{n-1}\rfloor).$$

Proof. ∎

Corollary 1

$$\forall n \geq 1,\ \alpha(n) = \beta(n).$$

Theorem 6 *For an arbitrary tree $T(V,E)$, let $\beta(T)$ denote the number of non-isomorphic uniform trees that can be embedded into T. Then*

$$\beta(T) = |V(T)|.$$

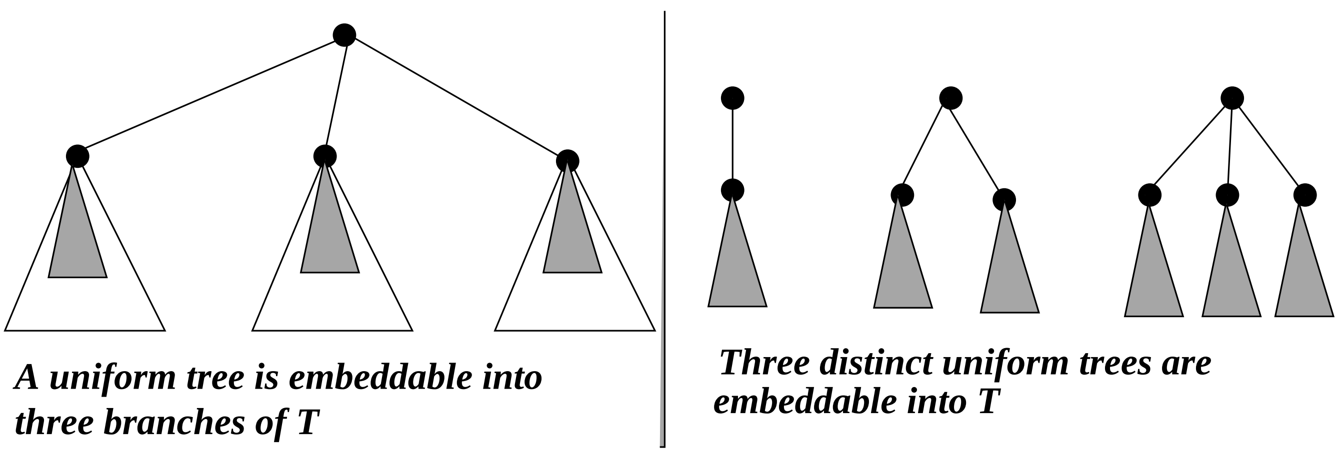
Proof. By induction on the number n of vertices of T.

For $n = 1$, the statement is straightforward.

Let it be correct for all trees with $\leq n-1$ vertices, and let T be a tree with n vertices. Denote $B_1, B_2, \ldots, B_r$ the branches of the root of T. For each of these branches, $|B_i| < n$, therefore

$$\forall i = 1, \ldots, r, \quad \beta(B_i) = |V(B_i)|.$$

For any uniform tree embedable in exactly s branches, we create s non-isomorphic uniform tree embedable into T, as shown on the Figure below.



A uniform tree is embeddable into three branches of T

Three distinct uniform trees are embeddable into T

∎