Knapsack Problem

We are given n objects and a knapsack (or bag). The object i has a weight w_i and the knapsack has a capacity m.

Further, if a fraction x_i, $0 \le x_i \le 1$ of object i is placed into the knapsack, then a profit of $p_i x_i$ is earned.

The objective is to obtain a filling of the knapsack that maximizes the total profit earned.

Formally, maximize $\sum_{1 \le i \le n} p_i x_i$ — (1)

subject to $\sum_{1 \le i \le n} w_i x_i \le m$ — (2)

and $\boxed{0 \le x_i \le 1}$ $1 \le i \le n$ — (3)

Here, a feasible solution is any set $(x_1, x_2 \cdots x_n)$ that satisfies eq^n (2) and (3).

An optimal solution is a feasible solution for which eq^n (1) is maximized.

eg Consider 3 objects with weights $(w_1, w_2, w_3) = (18, 15, 10)$ and profits $(P_1, P_2, P_3) = (25, 24, 15)$.

The capacity of knapsack is 20.

Therefore, $n = 3$

$m = 20$

A few feasible sol^{n}s

	$(x_1\ x_2\ x_3)$	$\sum w_i x_i$	$\sum P_i x_i$
eg	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$	$\Rightarrow \frac{1}{2}(18) + \frac{1}{2}(15) + \frac{1}{4}(10)$ $\Rightarrow 19$	$\Rightarrow \frac{1}{2}(25) + \frac{1}{2}(24) + \frac{1}{4}(15)$ $\Rightarrow 28.25$
eg	$(1, \frac{2}{15}, 0)$	$\Rightarrow 1(18) + \frac{2}{15}(15) + 0$ $\Rightarrow 20$	$\Rightarrow 1(25) + \frac{2}{15}(24)$ $\Rightarrow 28.2$
Infeasible sol^n eg	$(1, 1, 1)$	$\Rightarrow 25 + 24 + 15$ $\Rightarrow 64 \nleq$ knapsack size	

★ A feasible sol^n may or may not be optimal.

sol^n

We can observe following two key points :

(1) if the sum of all weights is $\leq m$, then $x_i = 1$, for $1 \leq i \leq n$. is an optimal sol^n.

(2.) All optimal solutions will fill the knapsack exactly.

(Since we can always increase the contribution of some object i by a fractional amount until the total weight is exactly m)

Greedy strategy 1 (Fill the knapsack by including the object with largest profit, if an object doesn't, then a fraction of it is included to fill the knapsack)

Here,

$(P_1\ P_2\ P_3) = (25, 24, 15)$

$(w_1 w_2\ w_3) = (18, 15, 10)$

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = \frac{2}{15} \\ x_3 = 0 \end{array}\right\} \quad \sum P_i x_i = 1(25) + \frac{2}{15}(24) = 28.2$$

→ This strategy is greedy since at each step (except possibly last) we select an object which increases the objective fn. most.

→ It is not an optimal sol^n.

⇒ Each time an object is included into the knapsack, we obtain largest possible increase in profit value (except possibly when last object is included)

eg $P_i = 4$, $w_i = 4$

$P_j = 3$, $w_i = 2$

so if we want 2 units of space then j is better than i.

Greedy Strategy 2

(Fill the knapsack by including the object with least weight)

$(w_1, w_2, w_3) = (18, 15, 10)$

$(P_1, P_2, P_3) = (25, 24, 15)$

Take
$x_3 = 1$ current weight $= 10$, current profit $= 15$

$x_2 = 10/15 = 2/3$ total weight $= 20$, total profit $= 1(15) + \frac{2}{3}(24)$

$= 15 + 16$

$= 31$

$x_3 = 0$

- it is not an optimal soln

Note

In strategy 1, (non-increasing order of profit) $\boxed{\text{Total Profit}}$

In each step, the objective fn. value takes on large increases, but number of steps is few as knapsack capacity is used up at a rapid rate.

In strategy 2 (non decreasing order of weight) $\boxed{\text{Capacity Used}}$

Here, in each step, even though capacity is used slowly but profits are not coming in rapidly enough.

Greedy Strategy 3

This time, we will try to use an algorithm that strives to achieve a balance between $\begin{cases} \text{the rate at which profit increases} \\ \text{and the rate at which capacity is used} \end{cases}$

Therefore, at each step, we include that object which has maximum profit per unit of capacity used.

⇒ That means objects are considered in order of the ratio p_i/w_i

so $x_1 = 0$ $\quad \sum w_i x_i = 20$

$x_2 = 1$ $\quad \sum p_i x_i = 31.5$ (optimal value)

$x_3 = 1/2$