Discrete Mathematics (MAIR-24)

Unit-1: Discrete Probability

Topic covered-

Finite and Infinite sets

BASIC CONCEPTS

- Infinite set
- Finite set
- Countably Infinite set
- Examples
- Uncountably Infinite set
- Example

Reference- *Elements of Discrete Mathematics* –C. L. Liu.

INFINITE SET

Let us begin be declaring that we have not yet committed ourselves to the precise definitions of finite sets and infinite sets. As the basis of our discussion, we want to construct an example of an infinite set. For a given set A, we define the *successor* of A, denoted A^+, to be the set $A \cup \{A\}$. Note that $\{A\}$ is a set that contains A as the only element. In other words, A^+ is a set that consists of all the elements of A together with an additional element which is the set A. For example, if $A = \{a, b\}$, then $A^+ = \{a, b\} \cup \{\{a, b\}\} = \{a, b, \{a, b\}\}$; and if $A = \{\{a\}, b\}$, then $A^+ = \{\{a\}, b, \{\{a\}, b\}\}$. Let us now construct a sequence of sets starting with the empty set ϕ. The successor of the empty set is $\{\phi\}$, whose successor is $\{\phi, \{\phi\}\}$, and whose successor, in turn, is $\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$. It is clear that we can go on to construct more and more successors. Let us also assign names to these sets. In particular, we use 0, 1, 2, 3, ... as the names of the sets.†
Let

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$$0 = \phi$$

$$1 = \{\phi\}$$

$$2 = \{\phi, \{\phi\}\}$$

$$3 = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

† Using 0, 1, 2, 3, ... as names of sets is just as good as using A, B, C, D, ... As will be seen, it is intentional that we choose 0, 1, 2, 3, ... as names.

We have, clearly, $1 = 0^+$, $2 = 1^+$, $3 = 2^+$, and so on. Let us now define a set N such that

1. N contains the set 0.
2. If the set n is an element in N, so is the set n^+.
3. N contains no other sets.

Since for every set in N its successor is also in N, the reader probably would agree that N is indeed an "infinite set." However, let us proceed in a more precise way.

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Given two sets P and Q, we say that there is a *one-to-one correspondence* between the elements in P and the elements in Q if it is possible to pair off the elements in P and Q such that every element in P is paired off with a distinct element in Q.† Thus, there is a one-to-one correspondence between the elements in the set $\{a, b\}$ and the elements in the set $\{c, d\}$, because we can pair a with c and b with d, or we can pair a with d and b with c. There is also a one-to-one correspondence between the elements in the set $\{a, b, c\}$ and the elements in the set $\{\phi, a, d\}$. On the other hand, there is no one-to-one correspondence between the elements in the sets $\{a, b, c\}$ and $\{a, d\}$.

FINITE SET

We are now ready to introduce some formal definitions. A set is said to be *finite* if there is a one-to-one correspondence between the elements in the set and the elements in some set n, where $n \in N$; n is said to be the *cardinality* of the set. Thus, for example, the cardinalities of the sets $\{a, b, c\}$, $\{a, \phi, d\}$, $\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$ are all equal to 3. Note that it is now precise for us to say that a set is an infinite set if it is not a finite one.

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COUNTABLY INFINITE SET

A set is said to be *countably infinite* (or the cardinality of the set is countably infinite)‡ if there is a one-to-one correspondence between the elements in the set and the elements in N. We observe first of all that the set of all natural numbers $\{0, 1, 2, 3, \ldots\}$§ is a countably infinite set. It follows that the set of all nonnegative even integers $\{0, 2, 4, 6, 8, \ldots\}$ is a countably infinite set because there is an obvious one-to-one correspondence between all nonnegative even integers and all natural numbers, namely, the even integer $2i$ corresponds to the natural number i for $i = 0, 1, 2, \ldots$. Similarly, the set of all nonnegative multiples of 7 $\{0, 7, 14, 21, \ldots\}$ is also a countably infinite set. So is the set of all positive integers $\{1, 2, 3, \ldots\}$. We note that a set is a countably infinite set if starting from a certain element we can sequentially list all the elements in the set

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one after another, because such a listing will yield a one-to-one correspondence between the elements in the set and the natural numbers. For example, the set of all integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is a countably infinite set, since its elements can be listed sequentially as $\{0, 1, -1. 2\ -2, 3, -3, \ldots\}$. This example suggests that the union of two countably infinite sets is also a countably infinite set. It indeed is the case. As a matter of fact, the union of a finite number of countably infinite sets is a countably infinite set and, furthermore, so is the union of a countably infinite number of countably infinite sets

Example Countable and Uncountable sets-

Example- 1. Set of rational numbers. Justify.
Example- 2. Set of positive rational numbers. Justify.
Example- 3. Set of real numbers. Justify.
Example- 4. Set of irrational numbers. Justify.
Example- 5. Set of real numbers in (0,1). Justify.

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DIAGONAL ARGUMENET

Suppose that there are n boys and ice cream of n different flavors. If we were told that there is a boy who disagrees with the first boy on whether the first flavor is delicious, who disagrees with the second boy on whether the second flavor is delicious, and who disagrees with the nth boy on whether the nth flavor is delicious, then we can be certain that this boy is not one of the n boys, because he disagrees with each of them in at least one way. This seemingly frivolous example illustrates a *diagonal argument* in which we assert that a certain object (a new boy) is not one of the given objects (the n boys we know), using the fact that this object is different from each of the given objects in at least one way.

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As an example of infinite sets with cardinalities that are not countably infinite, we now show that the set of real numbers between 0 and 1 is not a countably infinite set. Our proof procedure is to assume that the set is countably infinite and then show the existence of a contradiction. If the cardinality of the set of real numbers between 0 and 1 is countably infinite, there is a one-to-one correspondence between these real numbers and the natural numbers. Consequently, we can exhaustively list them one after another in decimal form as in the following:†

UNCOUNTABLY INFINITE SET

$$0.a_{11}a_{12}a_{13}a_{14}\cdots$$

$$0.a_{21}a_{22}a_{23}a_{24}\cdots$$

$$0.a_{31}a_{32}a_{33}a_{34}\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0.a_{i1}a_{i2}a_{i3}a_{i4}\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

† A number such as 0.34 can be written in two different forms, namely, $0.34000\cdots$ or $0.339999\cdots$. We follow an arbitrarily chosen convention of writing it in the latter form.

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where a_{ij} denotes the jth digit of the ith number in the list. Consider the number

$$0.b_1 b_2 b_3 b_4 \cdots$$

where

$$b_i = \begin{cases} 1 & \text{if} \quad a_{ii} = 9 \\ 9 - a_{ii} & \text{if} \quad a_{ii} = 0, 1, 2, \ldots, 8 \end{cases}$$

for all i. Clearly, the number $0.b_1 b_2 b_3 b_4 \cdots$ is a real number between 0 and 1 that does not have an infinite string of trailing 0s (for example, $0.34000 \cdots$). Moreover, it is different from each of the numbers in the list above because it differs from the first number in the first digit, the second number in the second digit, the ith number in the ith digit, and so on. Consequently, we conclude that the list above is not an exhaustive listing of the set of all real numbers between 0 and 1, contradicting the assumption that this set is countably infinite.

1. A set is countably Infinite if the "exhaustive listing" of the its elements is possible, otherwise uncountably infinite.
2. A set is uncountably Infinite if it contains an uncountably infinite set.
3. A set is finite/countably Infinite if it is contained in a countably infinite set.

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Thank You.