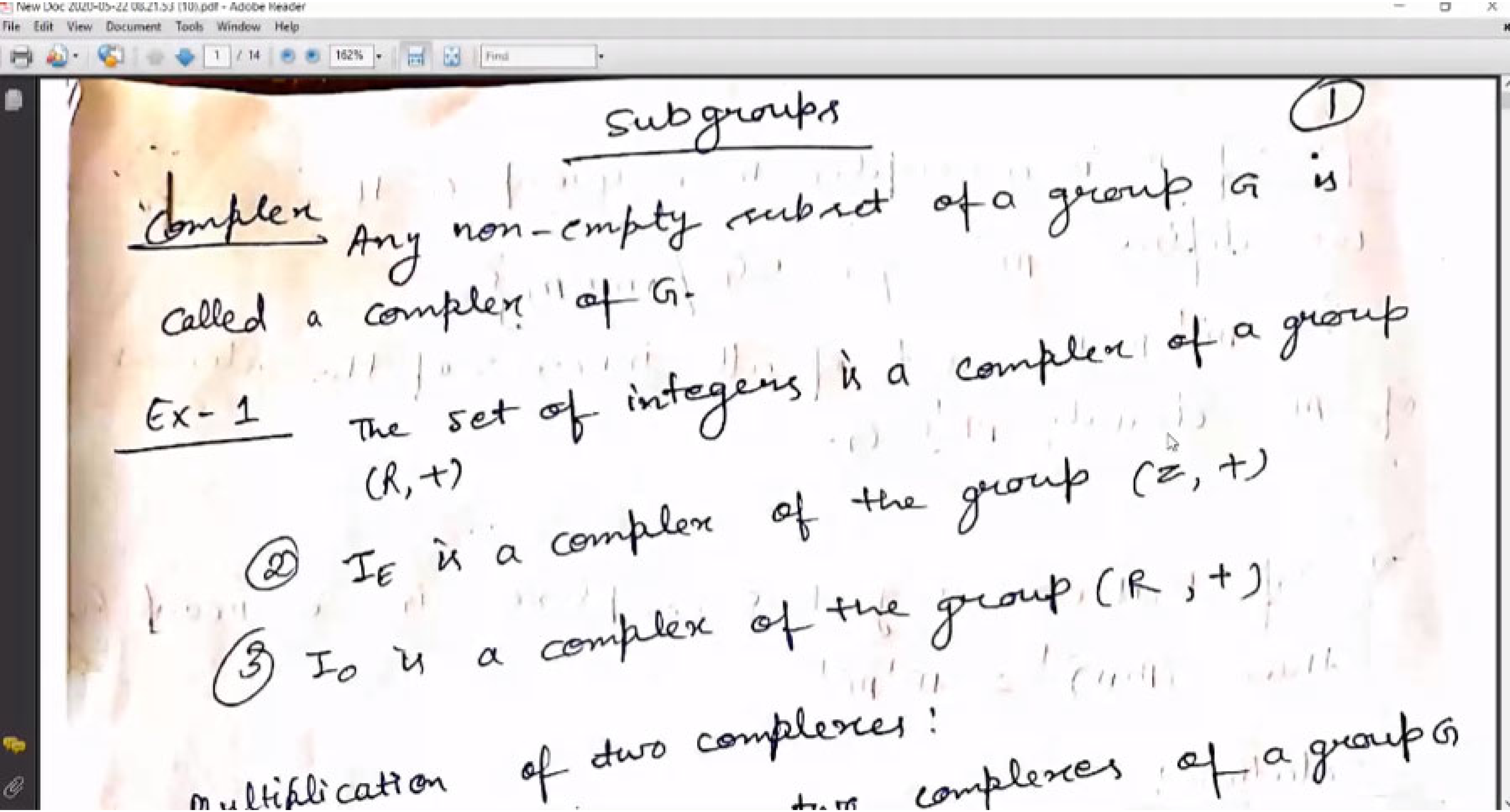


Subgroups

①

Complex Any non-empty subset of a group G is called a complex of G.

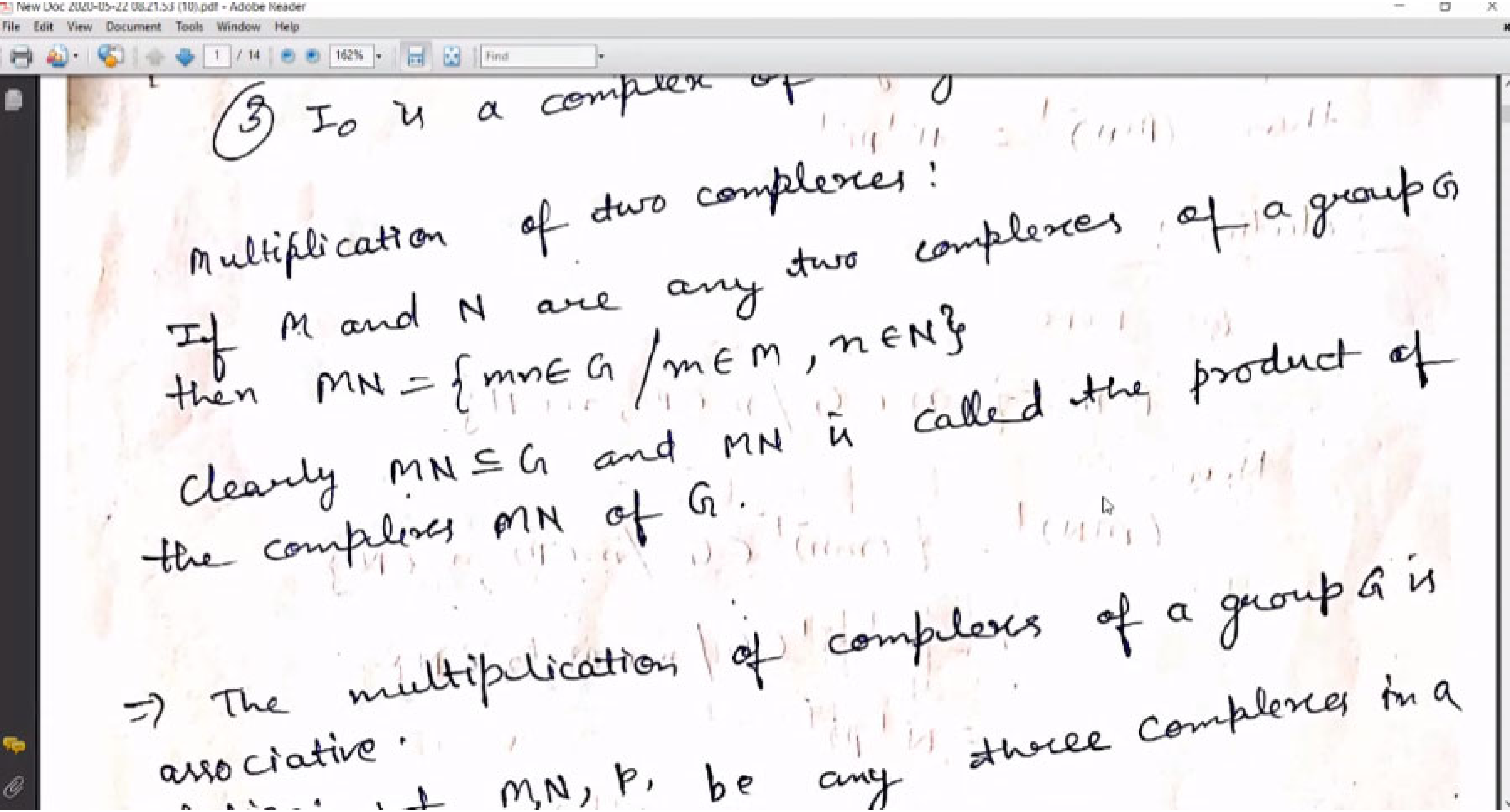
Ex-1 The set of integers is a complex of a group (R,+)

② I_E is a complex of the group (Z,+)

③ I_O is a complex of the group (R,+)

Multiplication of two complexes:

two complexes of a group G



③ Io is a complex of

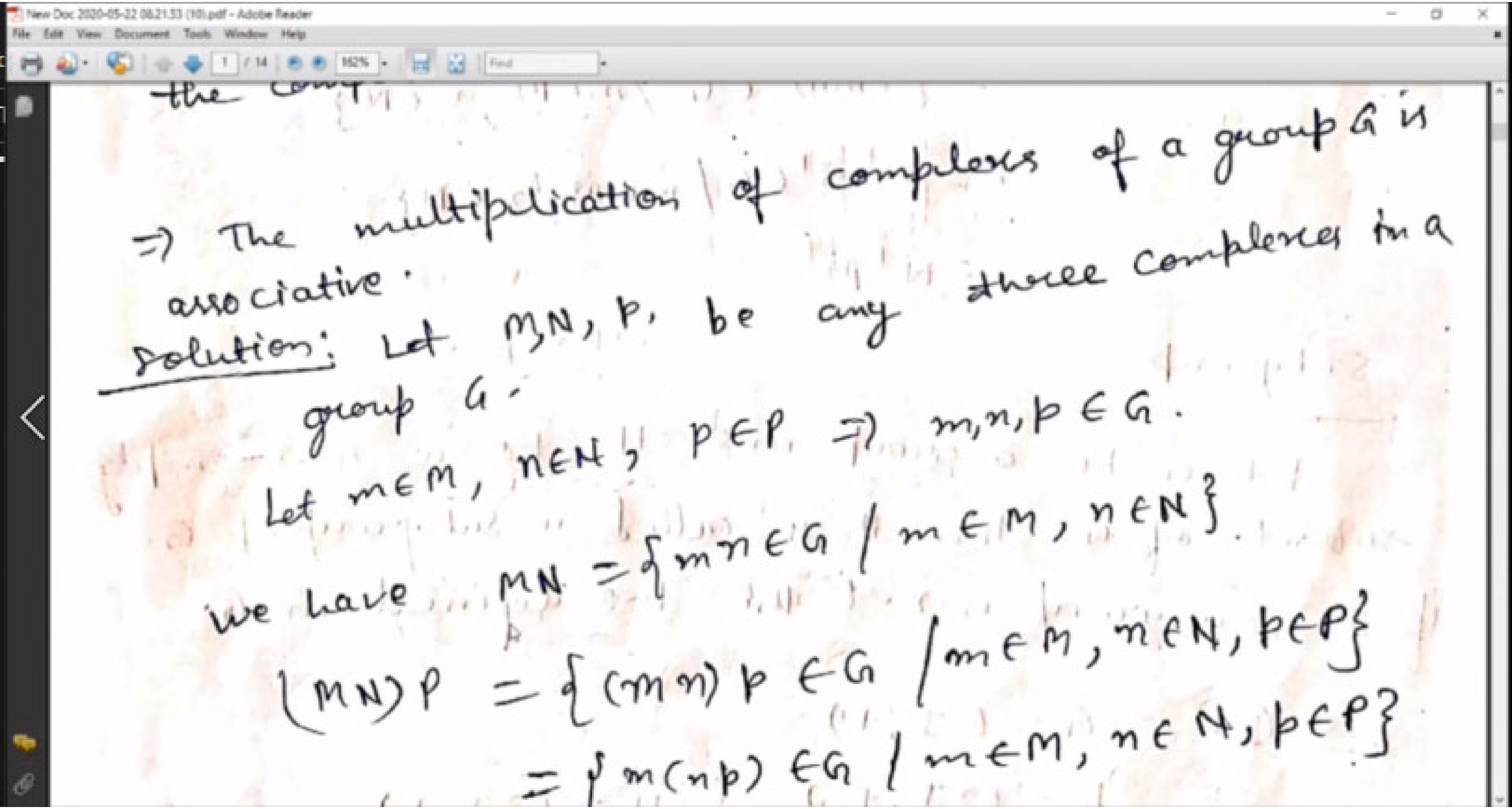
multiplication of two complexes:

If M and N are any two complexes of a group G then $MN = \{mn \in G / m \in M, n \in N\}$

Clearly $MN \subseteq G$ and MN is called the product of the complexes MN of G.

⇒ The multiplication of complexes of a group G is associative.

let M, N, P, be any three complexes in a



the comp

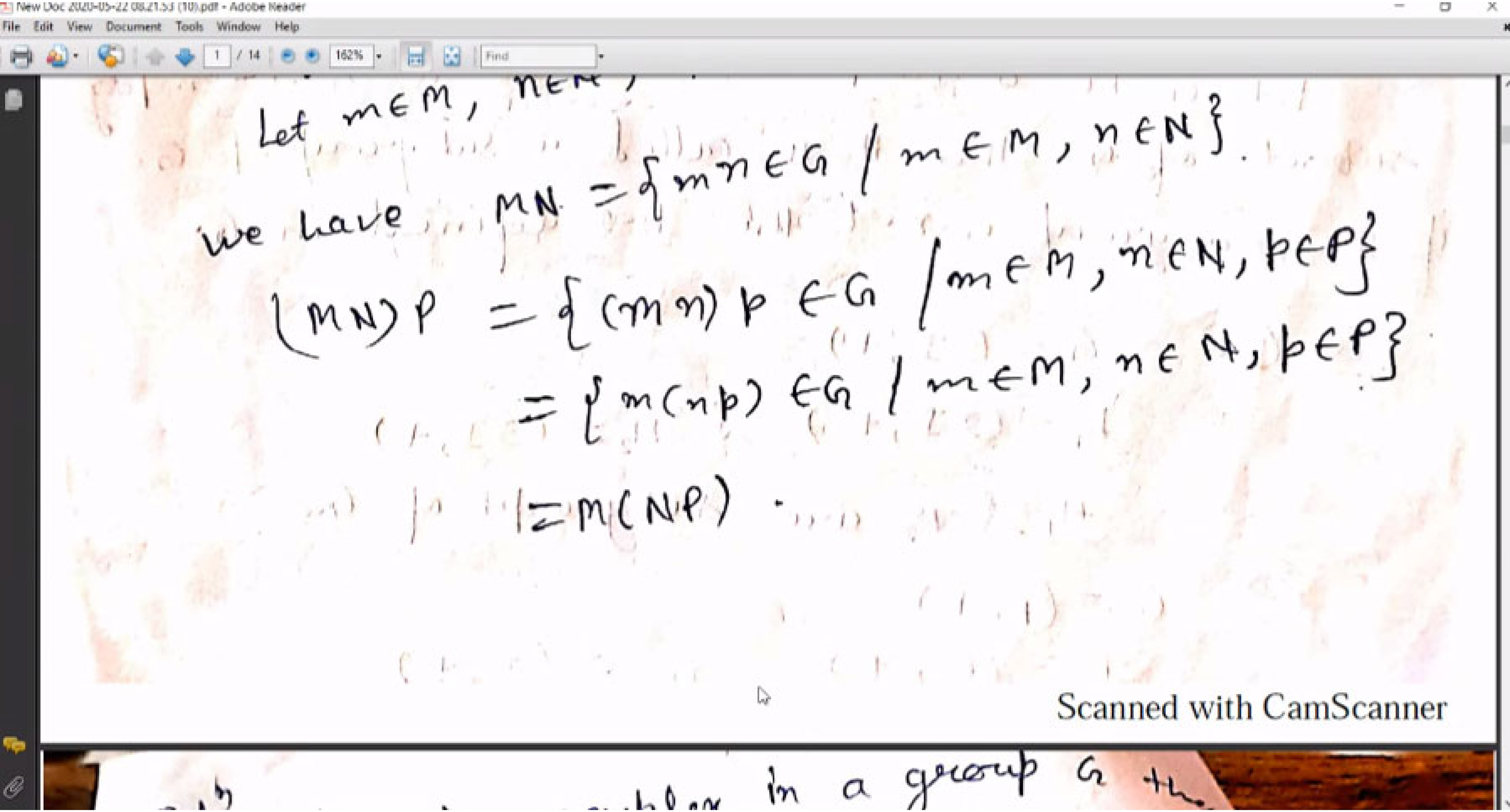
⇒ The multiplication of complexes of a group G is associative.

Solution: Let M, N, P be any three complexes in a group G.

Let $m \in M$, $n \in N$, $p \in P \Rightarrow m, n, p \in G$.

We have $MN = \{mn \in G \mid m \in M, n \in N\}$.

$$(MN)P = \{(mn)p \in G \mid m \in M, n \in N, p \in P\}$$
$$= \{m(np) \in G \mid m \in M, n \in N, p \in P\}$$



Let $m \in M$, $n \in N$

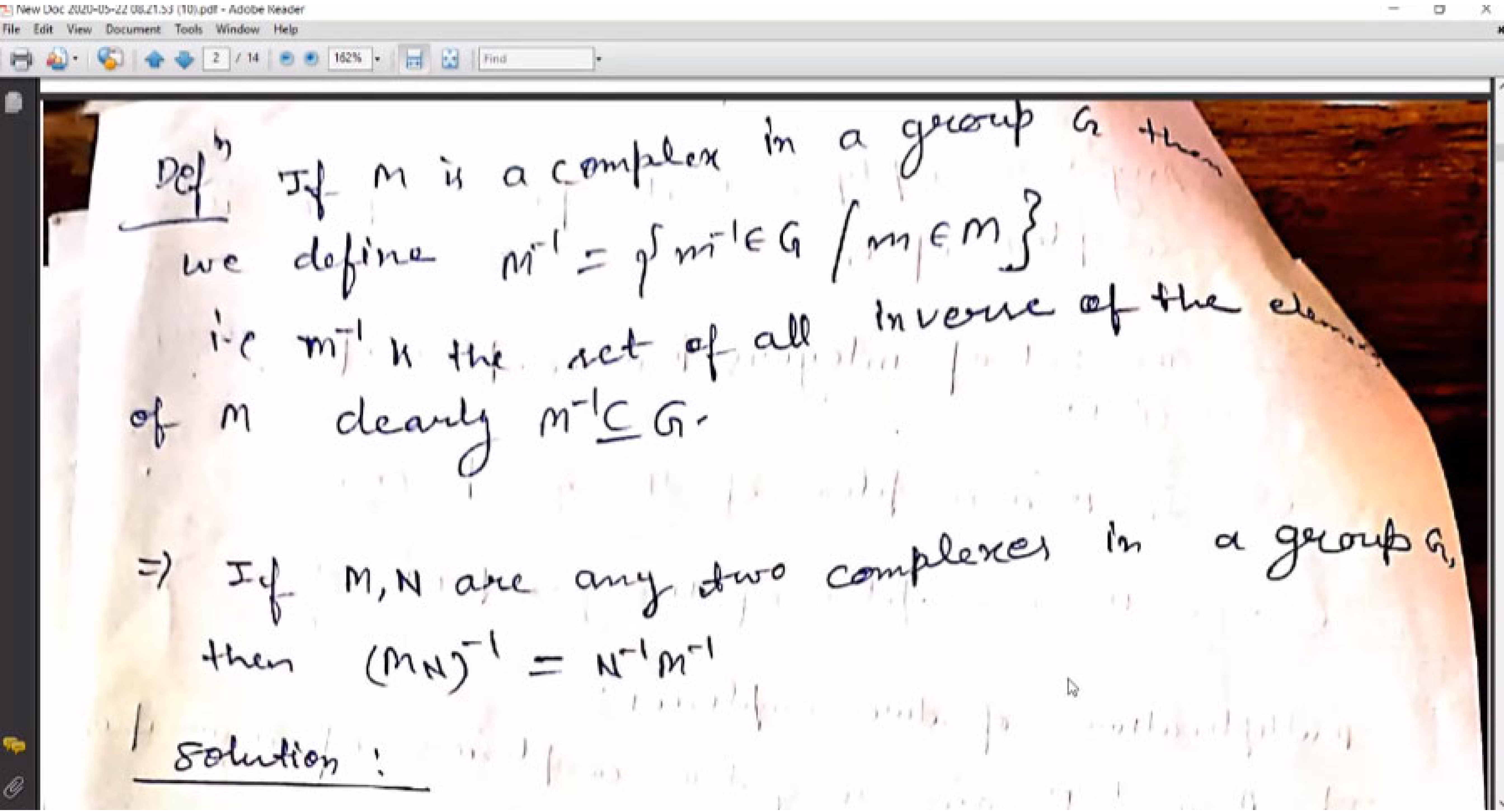
we have $MN = \{mn \in G \,/\, m \in M, n \in N\}$.

$(MN)P = \{(mn)p \in G \,/\, m \in M, n \in N, p \in P\}$

$= \{m(np) \in G \,/\, m \in M, n \in N, p \in P\}$

$= M(NP)$.

in a group G

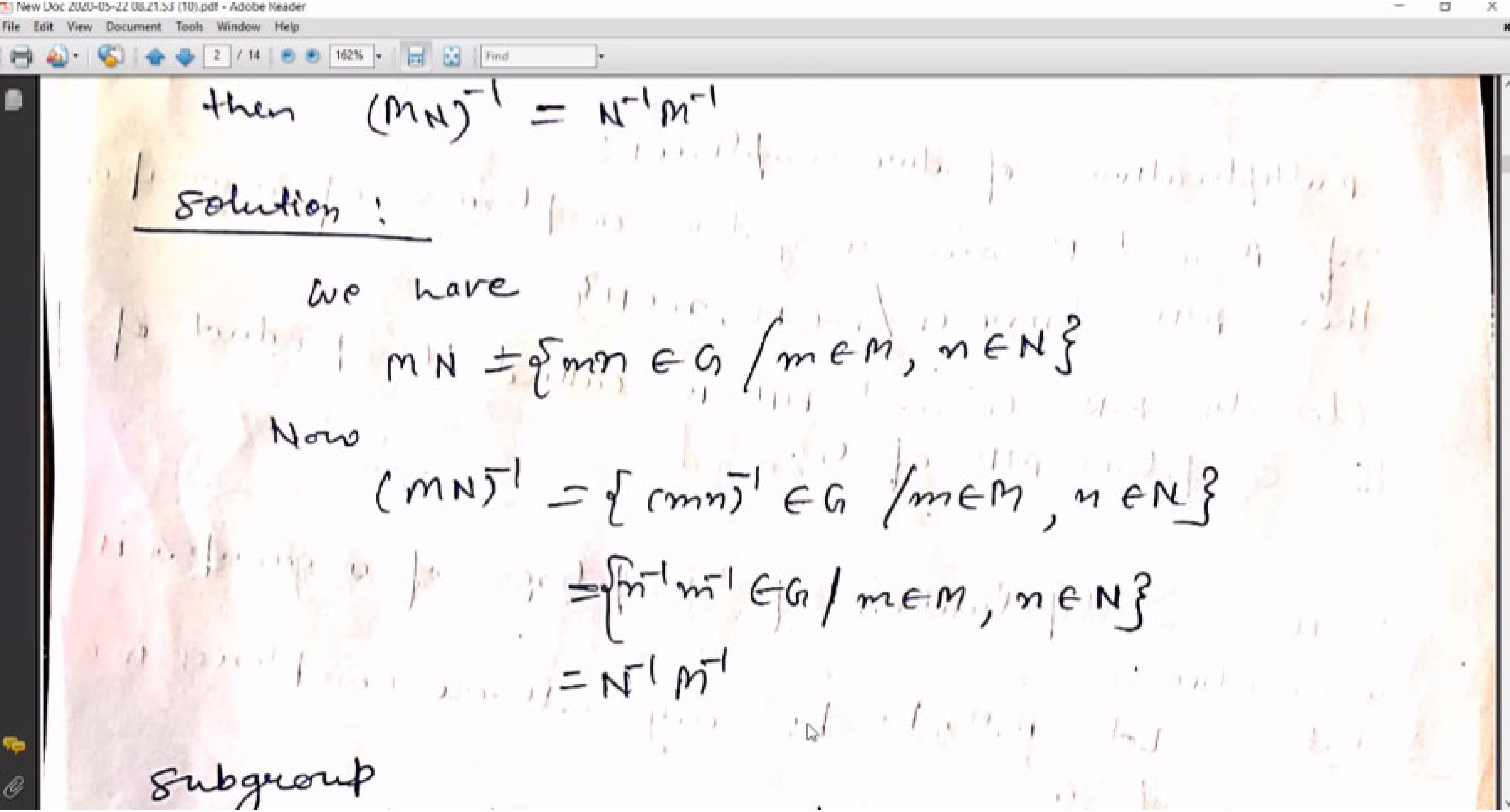


<u>Def^n</u> If M is a complex in a group G then we define $M^{-1} = \{m^{-1} \in G \,/\, m \in M\}$

i.e M^{-1} is the set of all inverse of the ele of M clearly $M^{-1} \subseteq G$.

$\Rightarrow$ If M, N are any two complexes in a group G, then $(MN)^{-1} = N^{-1}M^{-1}$

<u>Solution :</u>



then $(MN)^{-1} = N^{-1}M^{-1}$

Solution :

We have

$$MN = \{mn \in G / m \in M, n \in N\}$$

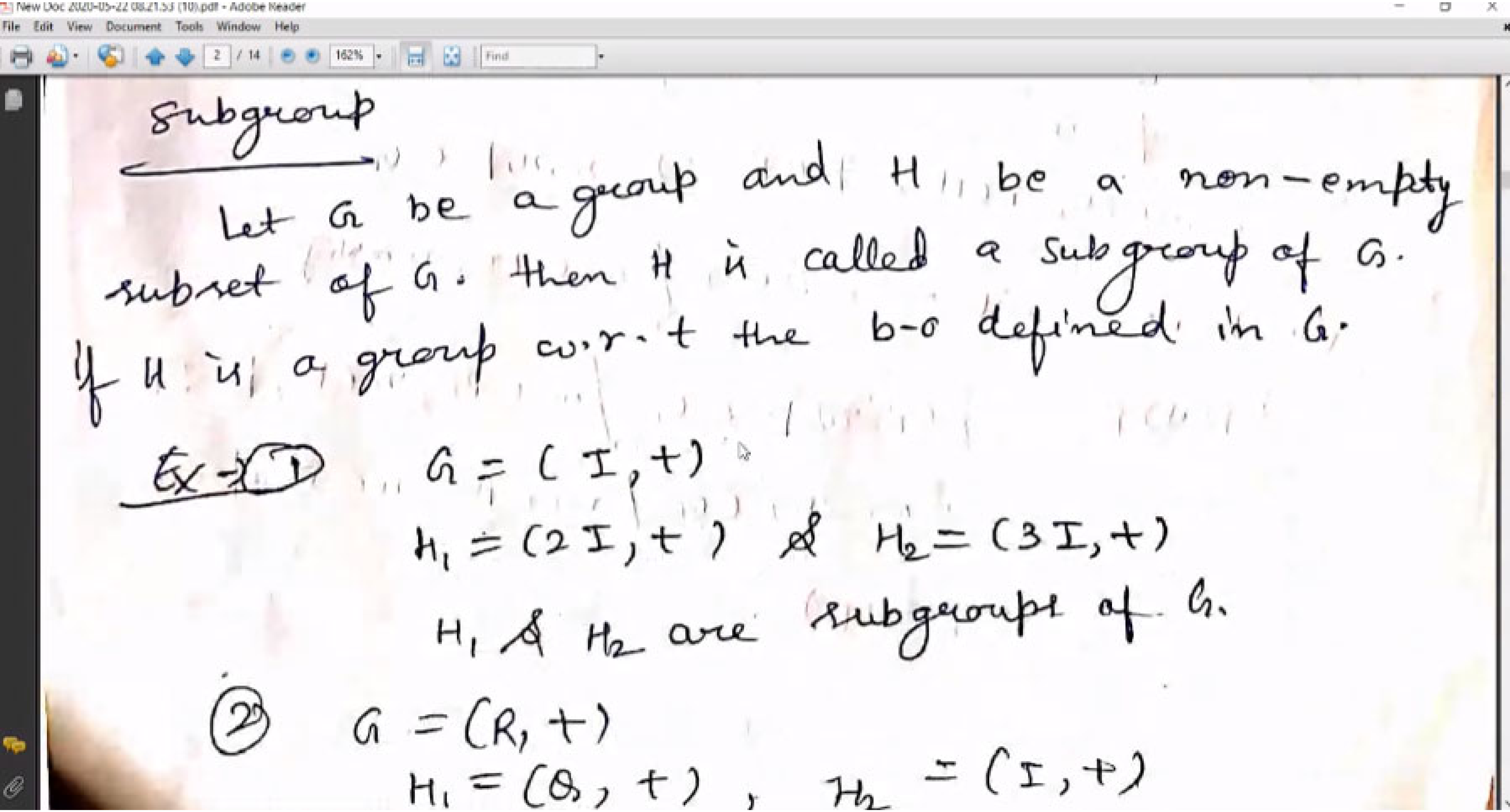
Now

$$(MN)^{-1} = \{(mn)^{-1} \in G / m \in M, n \in N\}$$

$$= \{n^{-1}m^{-1} \in G / m \in M, n \in N\}$$

$$= N^{-1}M^{-1}$$

Subgroup



Subgroup

Let G be a group and H be a non-empty subset of G. then H is called a subgroup of G. If H is a group w.r.t the b-o defined in G.

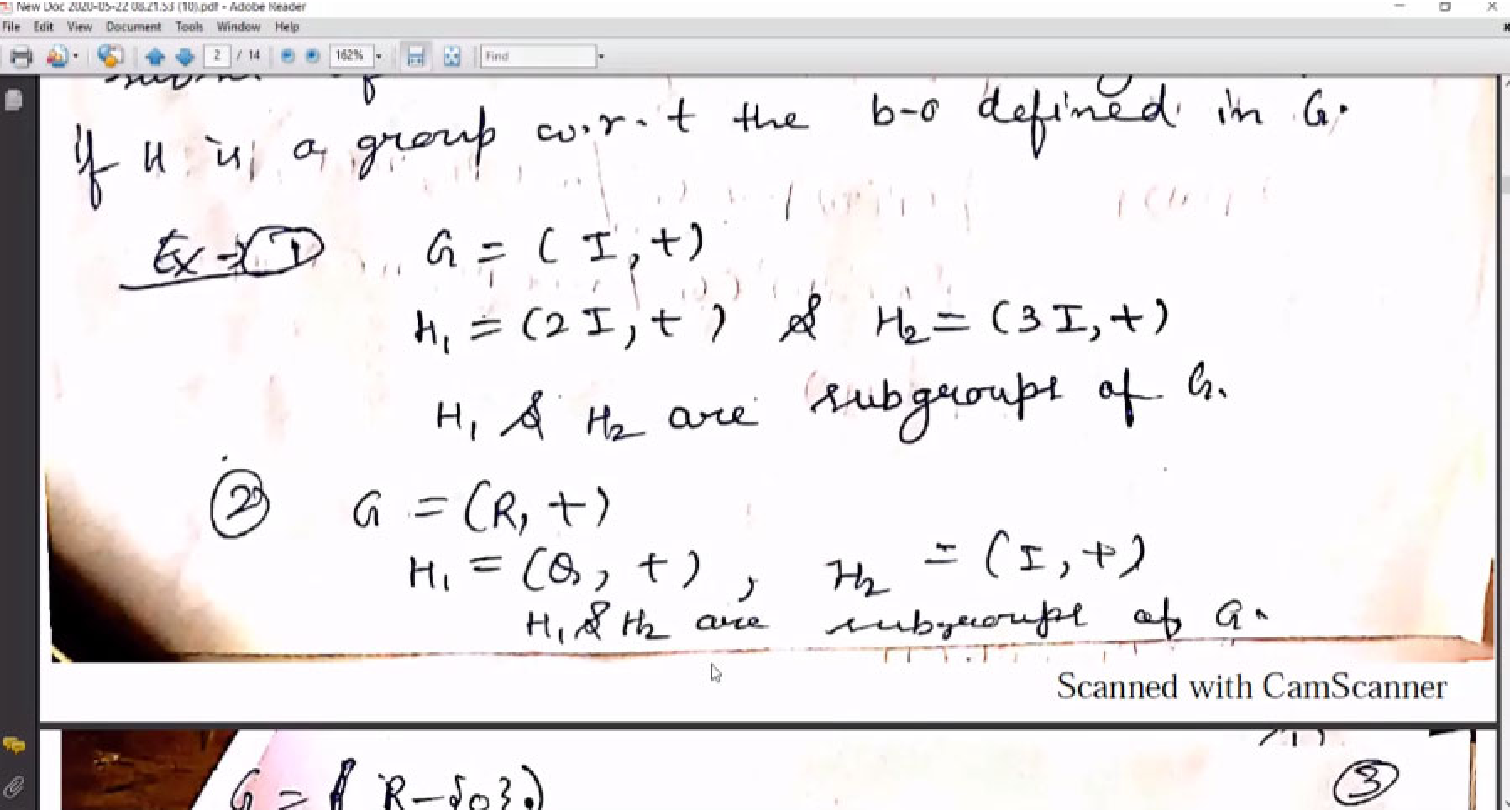
Ex-① $G = (I, +)$

$H_1 = (2I, +)$ & $H_2 = (3I, +)$

H_1 & H_2 are subgroups of G.

② $G = (R, +)$

$H_1 = (Q, +)$, $H_2 = (I, +)$



If H is a group w.r.t the b-o defined in G.

Ex-① $G = (I, +)$

$H_1 = (2I, +)$ & $H_2 = (3I, +)$

H_1 & H_2 are subgroups of G.

② $G = (R, +)$

$H_1 = (Q, +)$, $H_2 = (I, +)$

H_1 & H_2 are subgroups of G.

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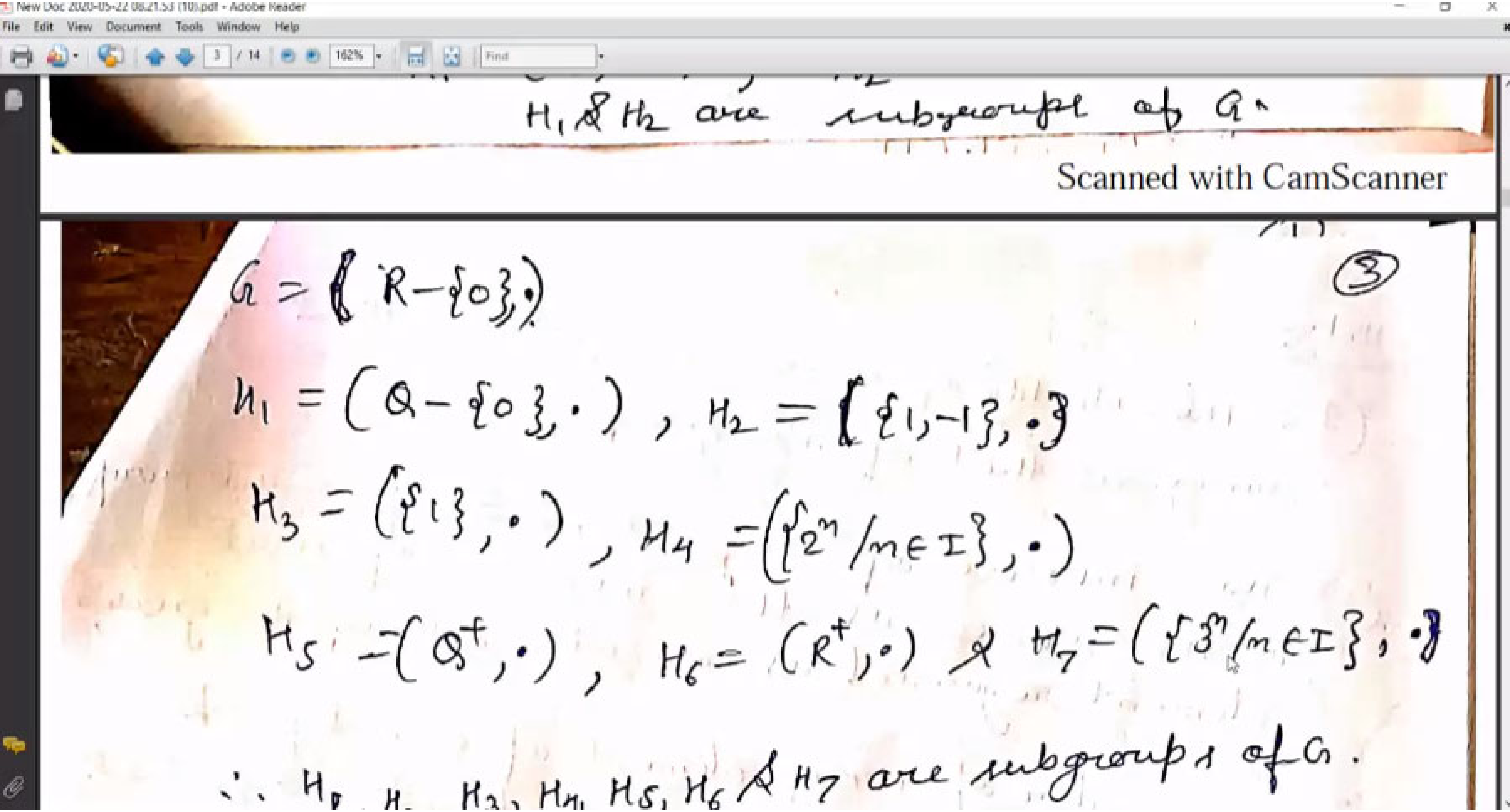
$G = (R - \{0\}, \cdot)$ ③



H_1 & H_2 are subgroups of G.

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③

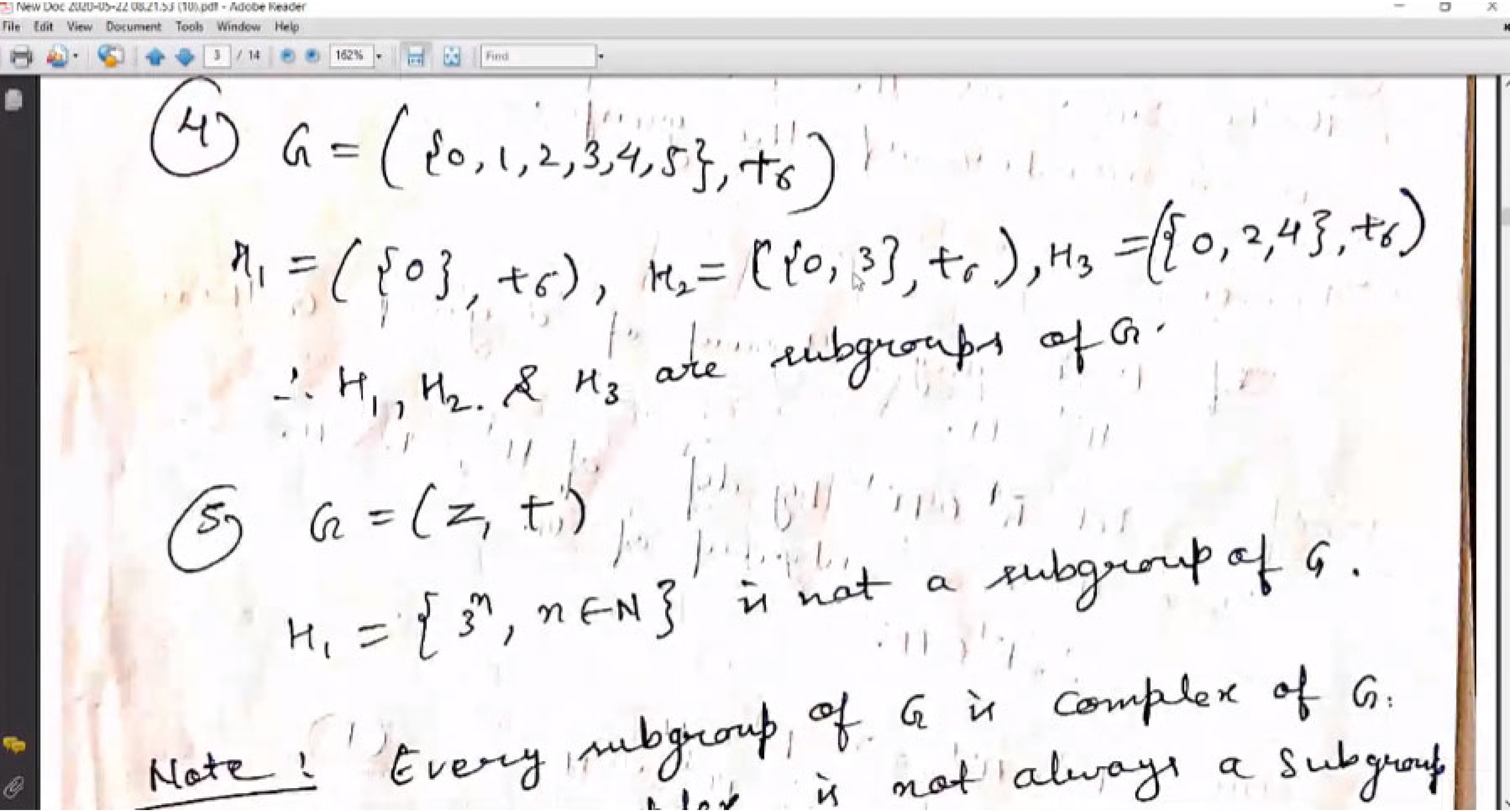
$G = (R-\{0\}, \cdot)$

$H_1 = (Q-\{0\}, \cdot)$, $H_2 = (\{1,-1\}, \cdot)$

$H_3 = (\{1\}, \cdot)$, $H_4 = (\{2^n / n \in I\}, \cdot)$

$H_5 = (Q^+, \cdot)$, $H_6 = (R^+, \cdot)$ & $H_7 = (\{3^n / n \in I\}, \cdot)$

$\therefore H_1, H_2, H_3, H_4, H_5, H_6$ & H_7 are subgroups of G.



④ $G = (\{0,1,2,3,4,5\}, +_6)$

$H_1 = (\{0\}, +_6)$, $H_2 = (\{0,3\}, +_6)$, $H_3 = (\{0,2,4\}, +_6)$

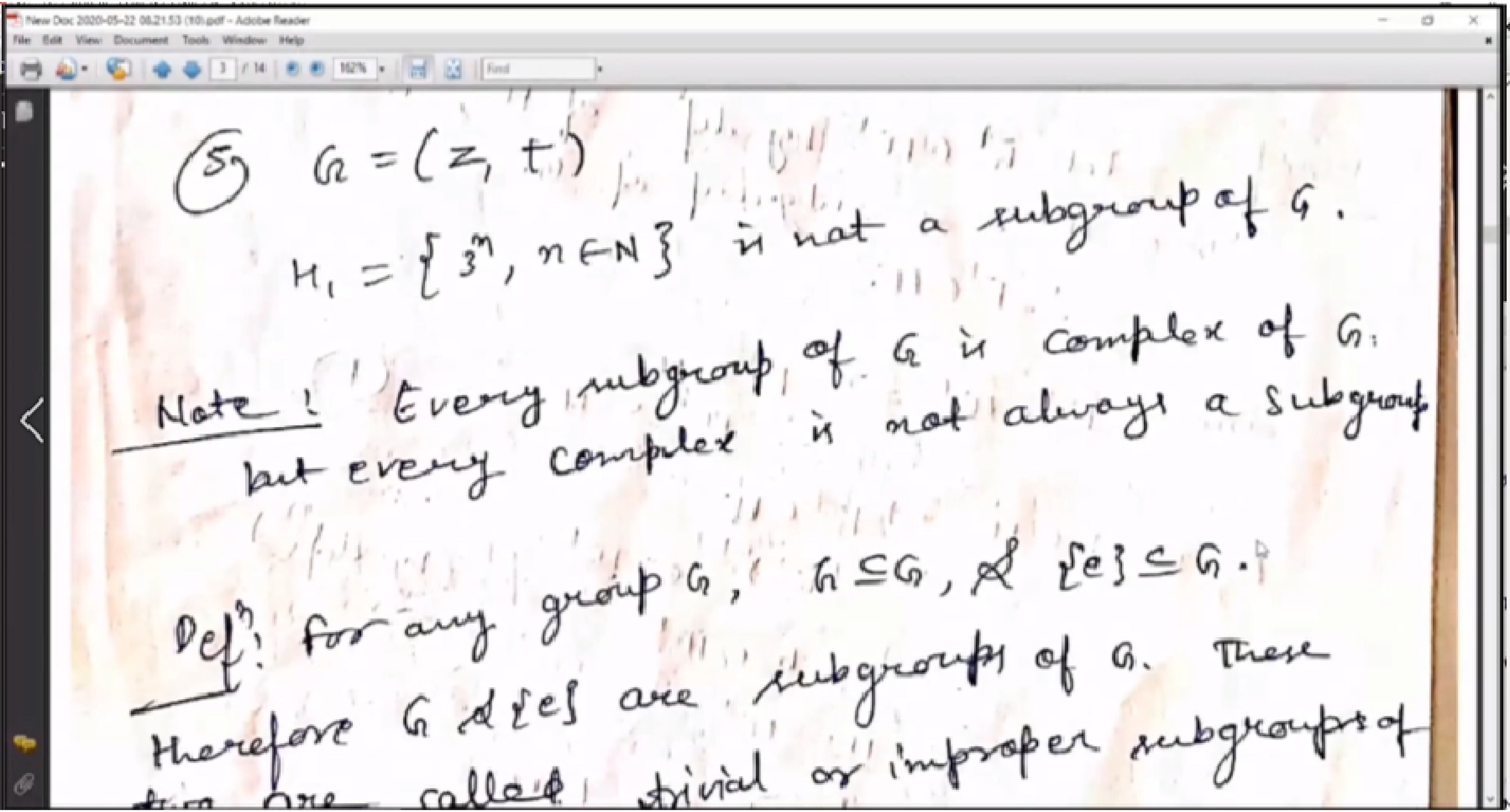
∴ H_1, H_2 & H_3 are subgroups of G.

⑤ $G = (Z, +)$

$H_1 = \{3^n, n \in N\}$ is not a subgroup of G.

Note ! Every subgroup of G is complex of G:

is not always a Subgroup



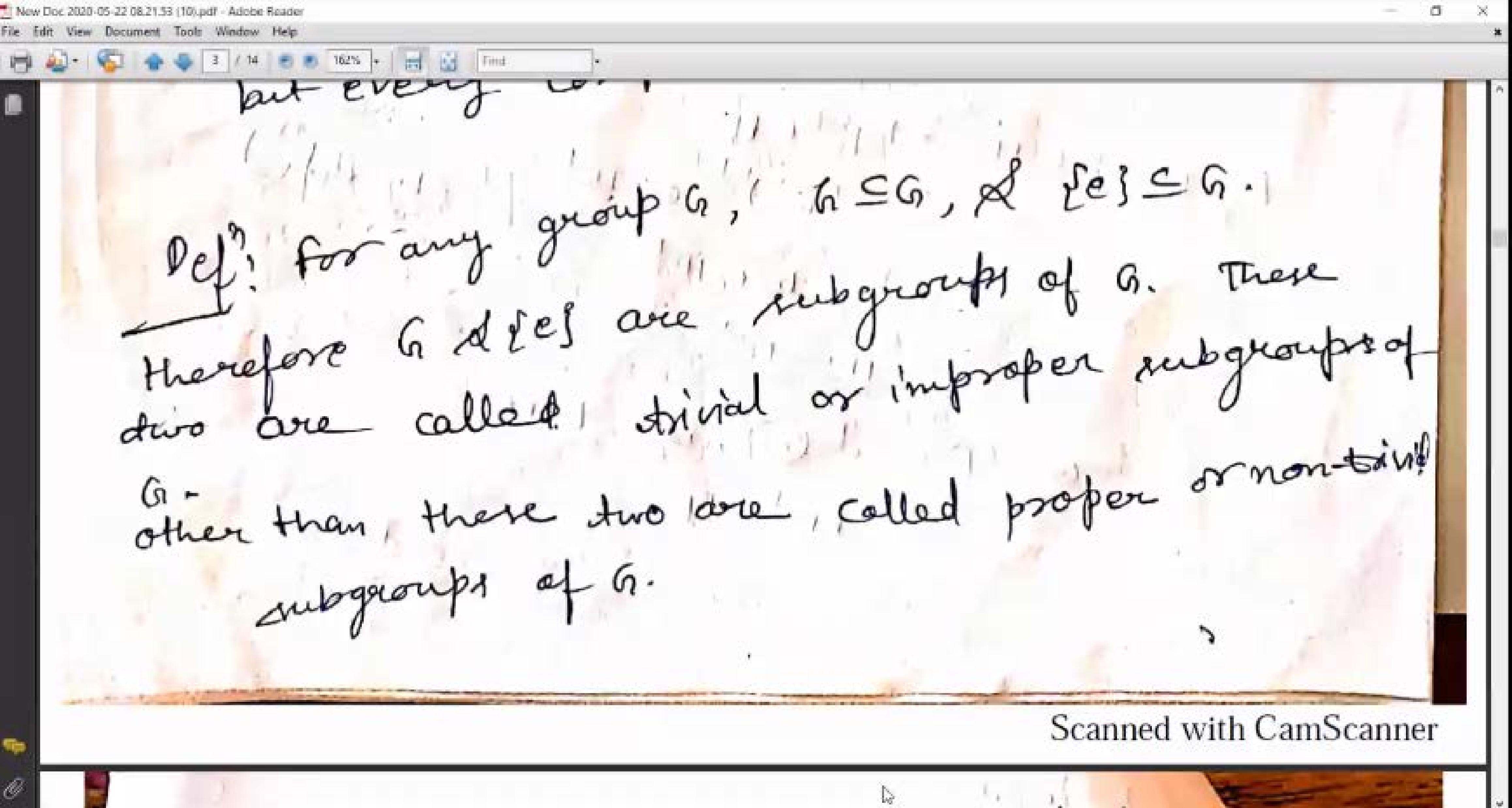
(5) $G = (\mathbb{Z}, +)$

$H_1 = \{3^n, n \in N\}$ is not a subgroup of G.

Note! Every subgroup of G is complex of G. but every complex is not always a subgroup

Def^n! for any group G, $G \leq G$, & $\{e\} \leq G$.

Therefore G & {e} are subgroups of G. These are called trivial or improper subgroups of



but every

Def^n: for any group G, $G \leq G$, & $\{e\} \subseteq G$.

Therefore G & {e} are subgroups of G. These two are called trivial or improper subgroups of G.

other than these two are called proper or non-trivial subgroups of G.

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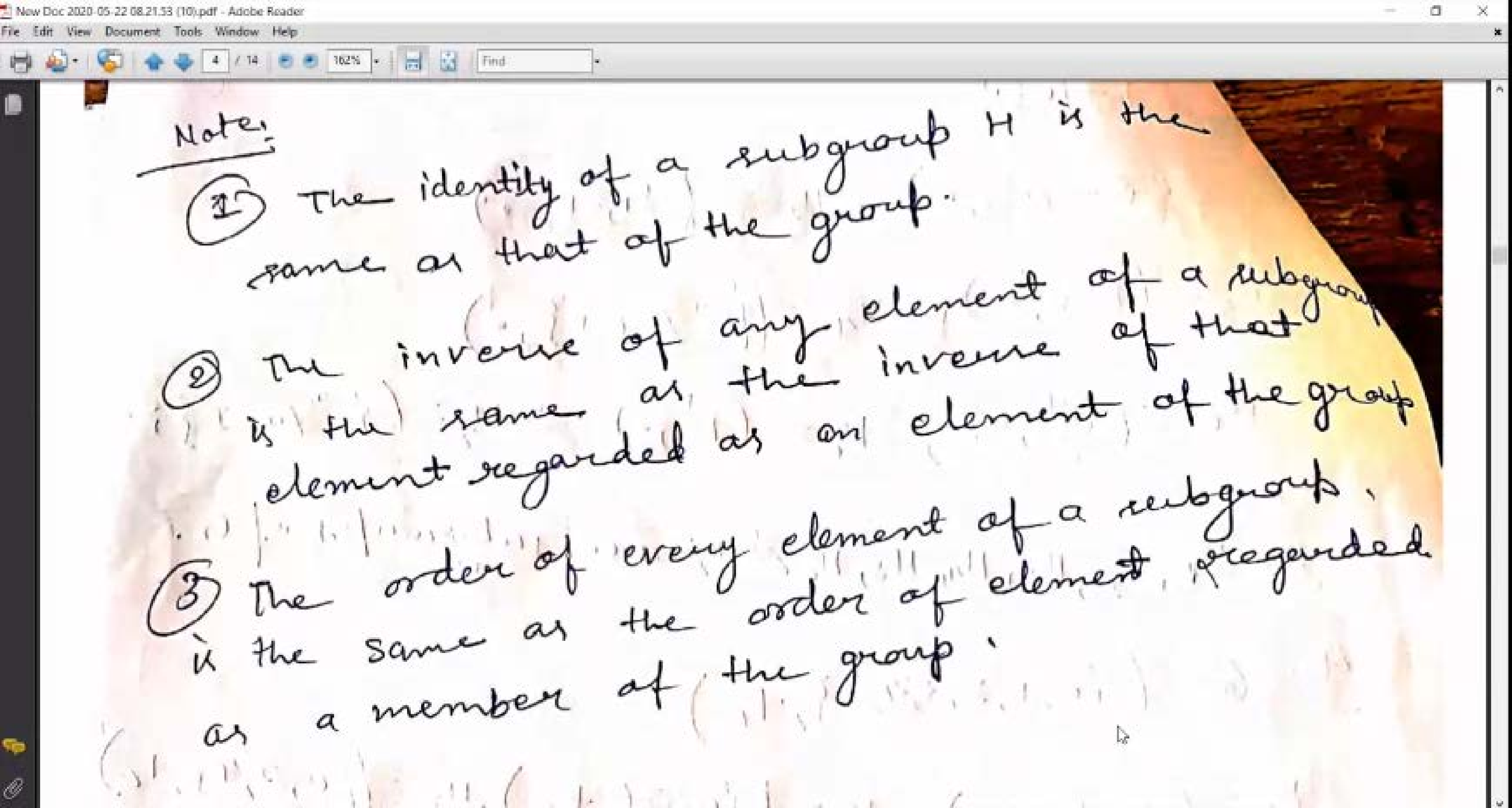
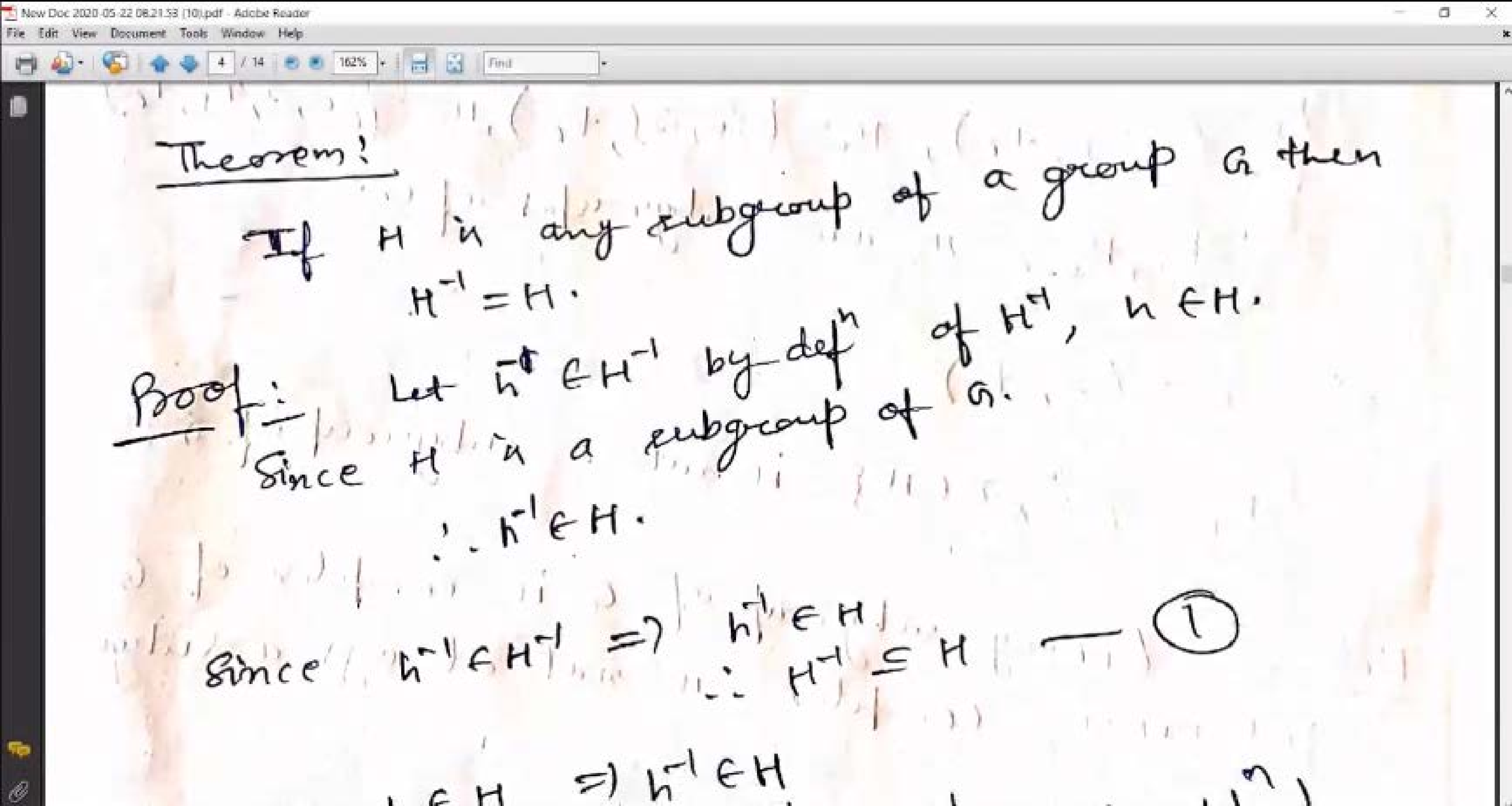


Note:

① The identity of a subgroup H is the same as that of the group.

② The inverse of any element of a subgroup is the same as the inverse of that element regarded as an element of the group.

③ The order of every element of a subgroup is the same as the order of element regarded as a member of the group.



Theorem:

If H is any subgroup of a group G then

$H^{-1} = H$.

Proof: Let $h^{-1} \in H^{-1}$ by def^n of H^{-1}, $h \in H$.

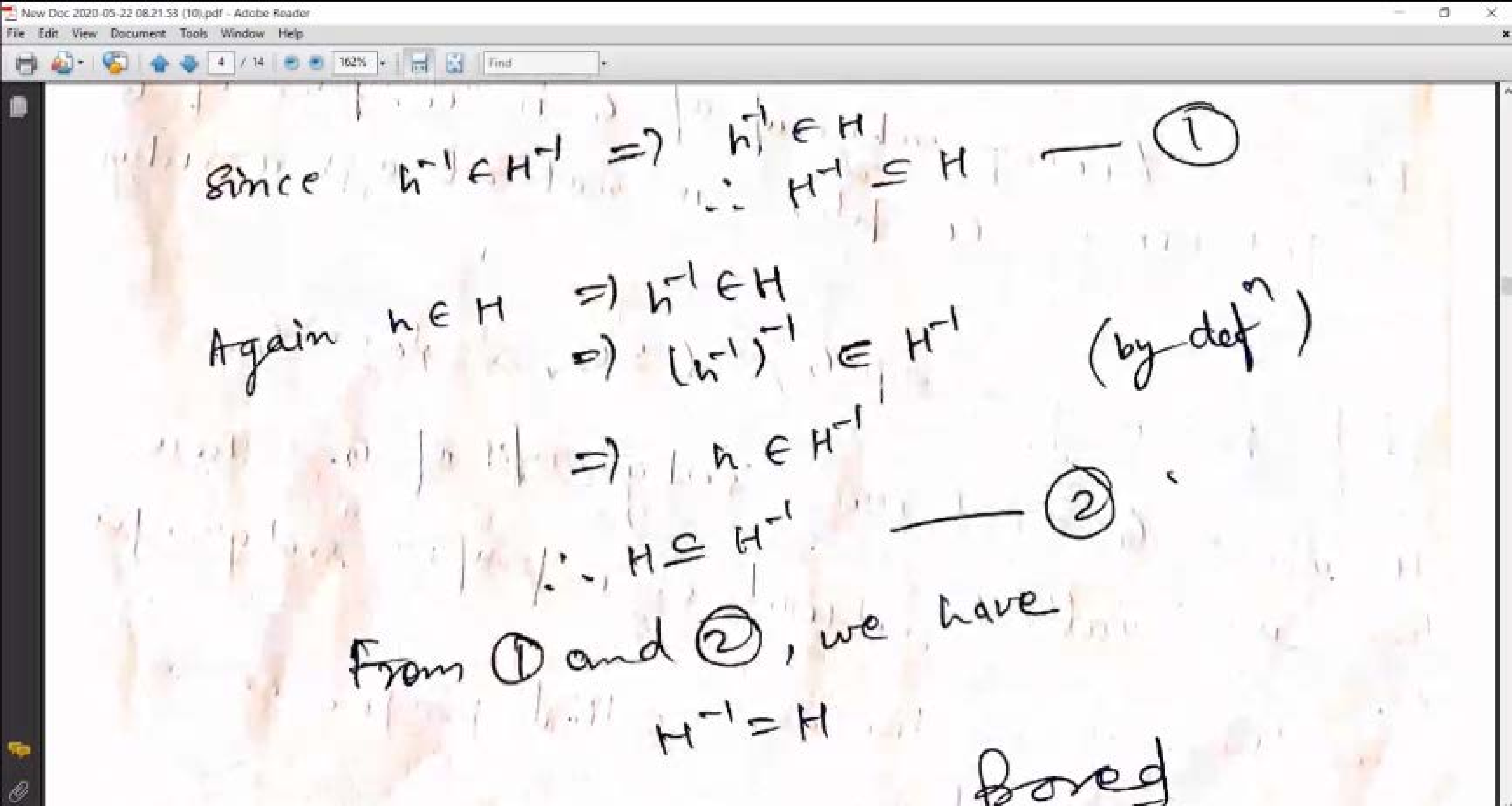
Since H is a subgroup of G.

$\therefore h^{-1} \in H$.

Since $h^{-1} \in H^{-1} \Rightarrow h^{-1} \in H$

$\therefore H^{-1} \subseteq H$ —— ①

$\in H \Rightarrow h^{-1} \in H$



Since $h^{-1} \in H^{-1} \Rightarrow h^{-1} \in H$

$\therefore H^{-1} \subseteq H$ —— ①

Again $h \in H \Rightarrow h^{-1} \in H$

$\Rightarrow (h^{-1})^{-1} \in H^{-1}$ (by def^n)

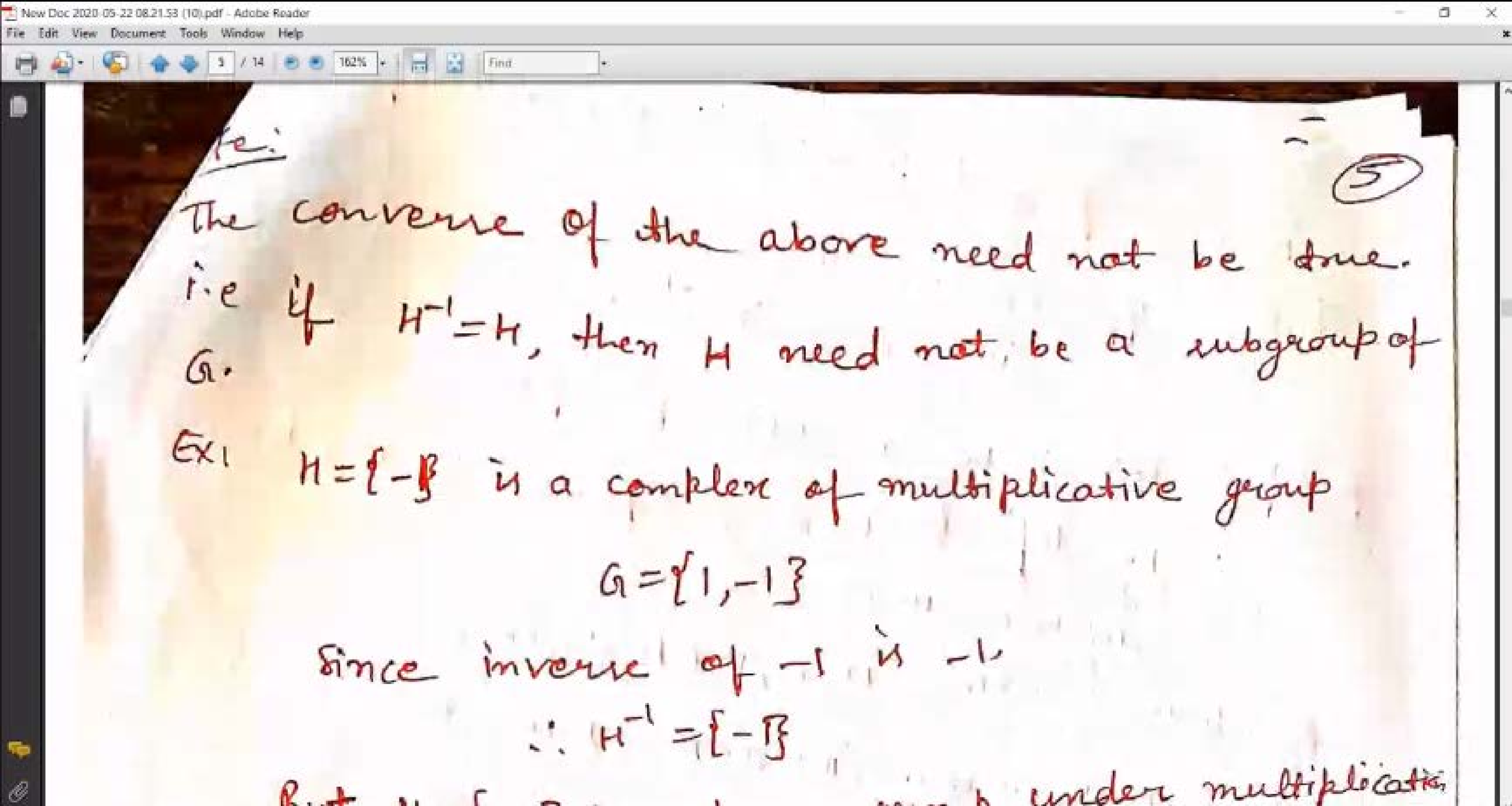
$\Rightarrow h \in H^{-1}$

$\therefore H \subseteq H^{-1}$ —— ②

From ① and ②, we have

$H^{-1} = H$

Proved



te:

The converse of the above need not be true. i.e if $H^{-1}=H$, then H need not be a subgroup of G.

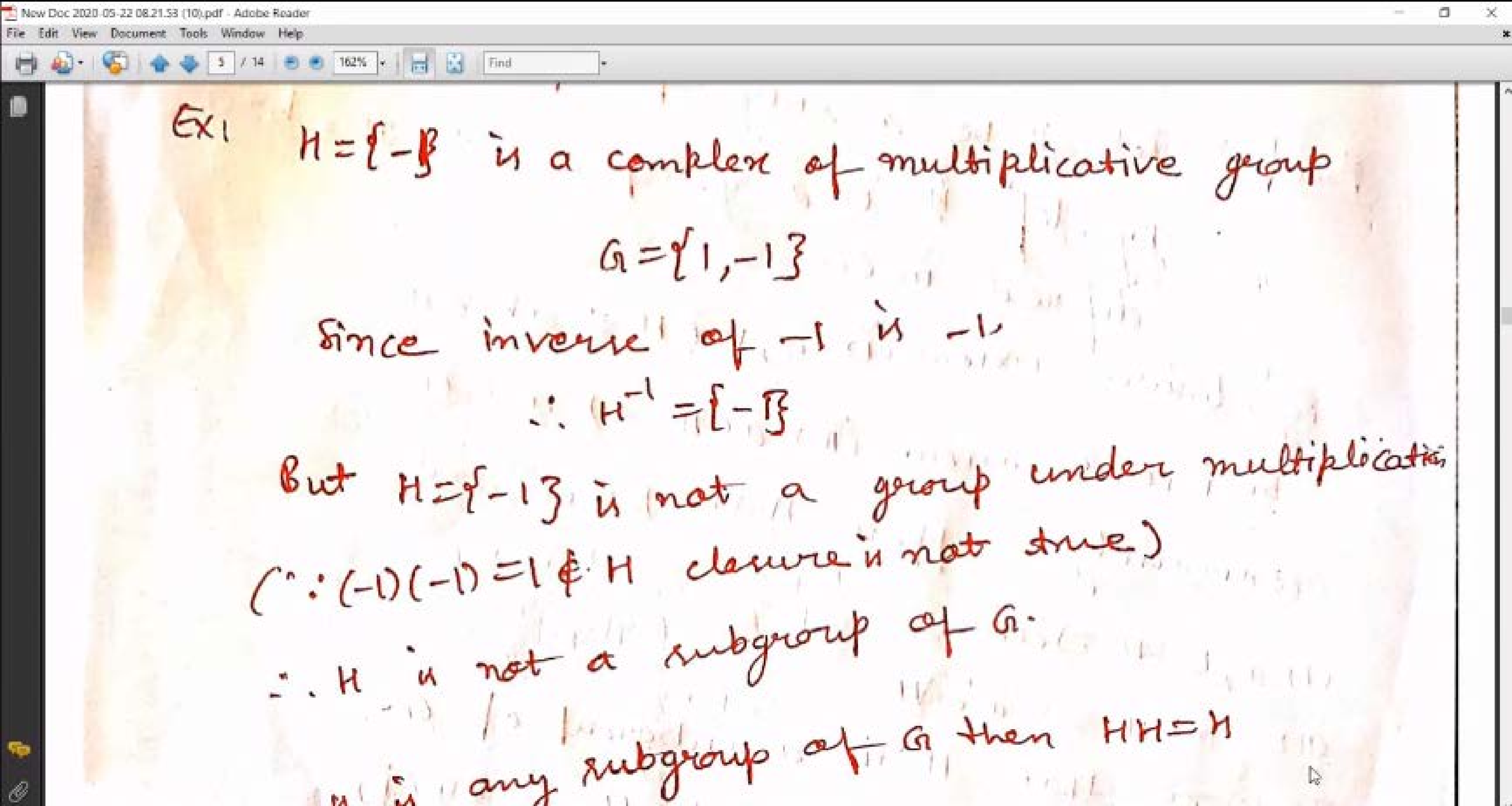
Ex1 $H=\{-1\}$ is a complex of multiplicative group

$$G=\{1,-1\}$$

Since inverse of -1 is -1

$$\therefore H^{-1}=\{-1\}$$

But ... under multiplicati



Ex1 $H = \{-1\}$ is a complex of multiplicative group

$$G = \{1, -1\}$$

Since inverse of -1 is -1

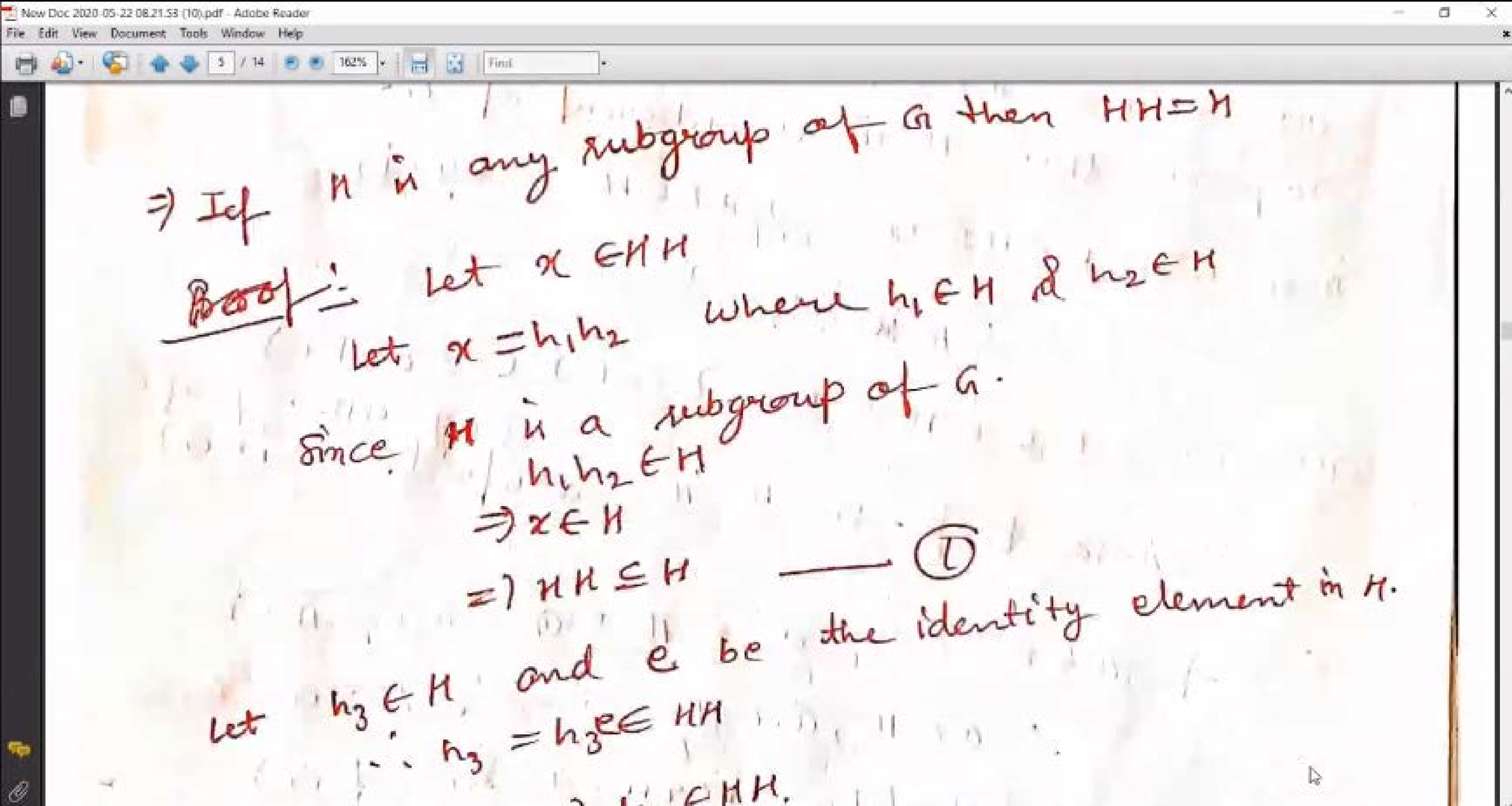
$$\therefore H^{-1} = \{-1\}$$

But $H = \{-1\}$ is not a group under multiplicati

$(\because (-1)(-1) = 1 \notin H$ closure is not true)

$\therefore H$ is not a subgroup of G.

H is any subgroup of G then $HH = H$



⇒ If H is any subgroup of G then $HH = H$

Proof :- Let $x \in HH$

Let $x = h_1 h_2$ where $h_1 \in H$ & $h_2 \in H$

Since H is a subgroup of G.

$h_1 h_2 \in H$

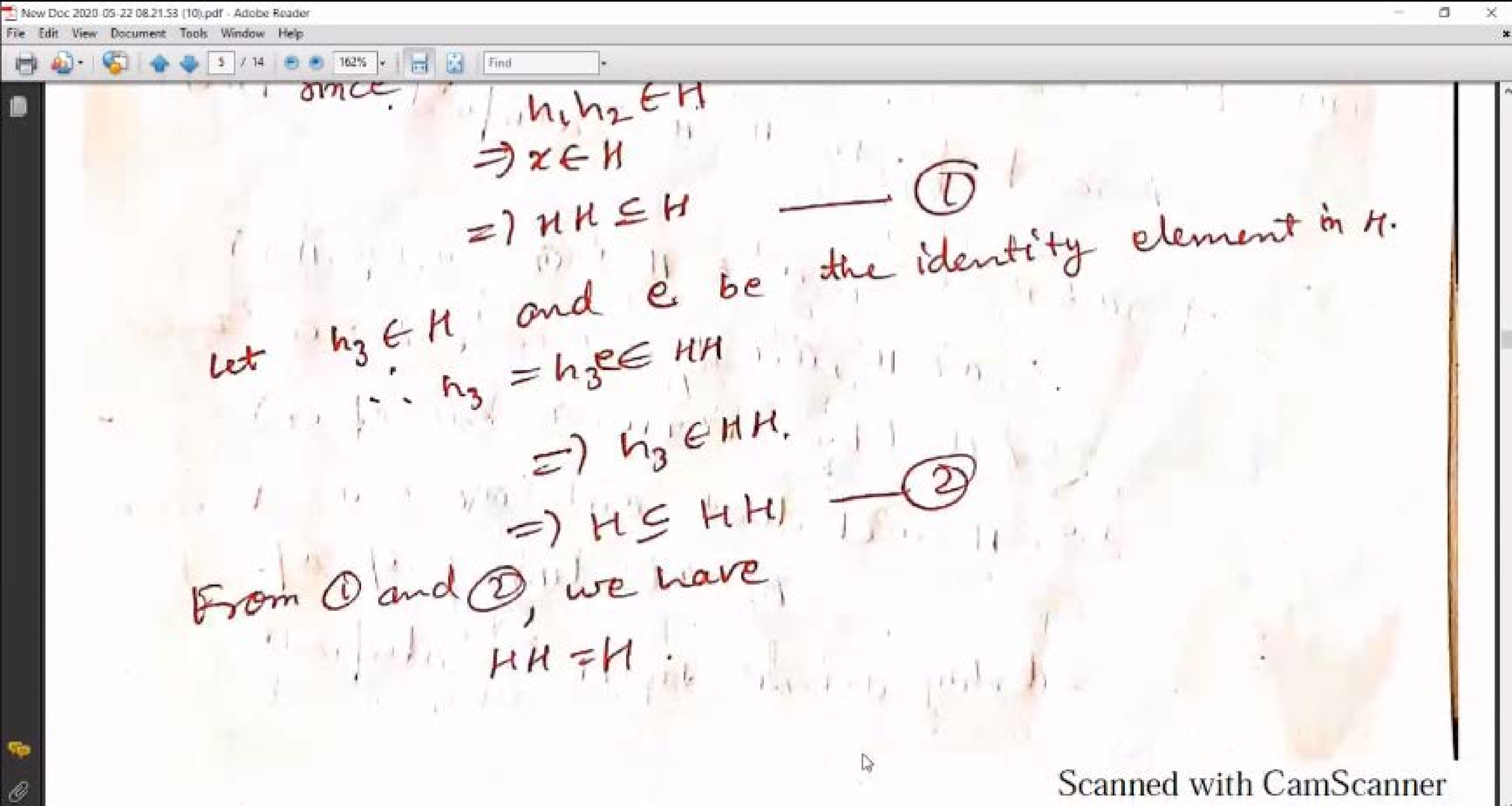
$\Rightarrow x \in H$

$\Rightarrow HH \subseteq H$ ——— (I)

Let $h_3 \in H$ and e be the identity element in H.

$\therefore h_3 = h_3 e \in HH$

$\in HH$.



since $h_1 h_2 \in H$

$\Rightarrow x \in H$

$\Rightarrow HH \subseteq H$ ——— ①

Let $h_3 \in H$, and e be the identity element in H.

$\therefore h_3 = h_3 e \in HH$

$\Rightarrow h_3 \in HH$.

$\Rightarrow H \subseteq HH$ ——— ②

From ① and ②, we have

$HH = H$.

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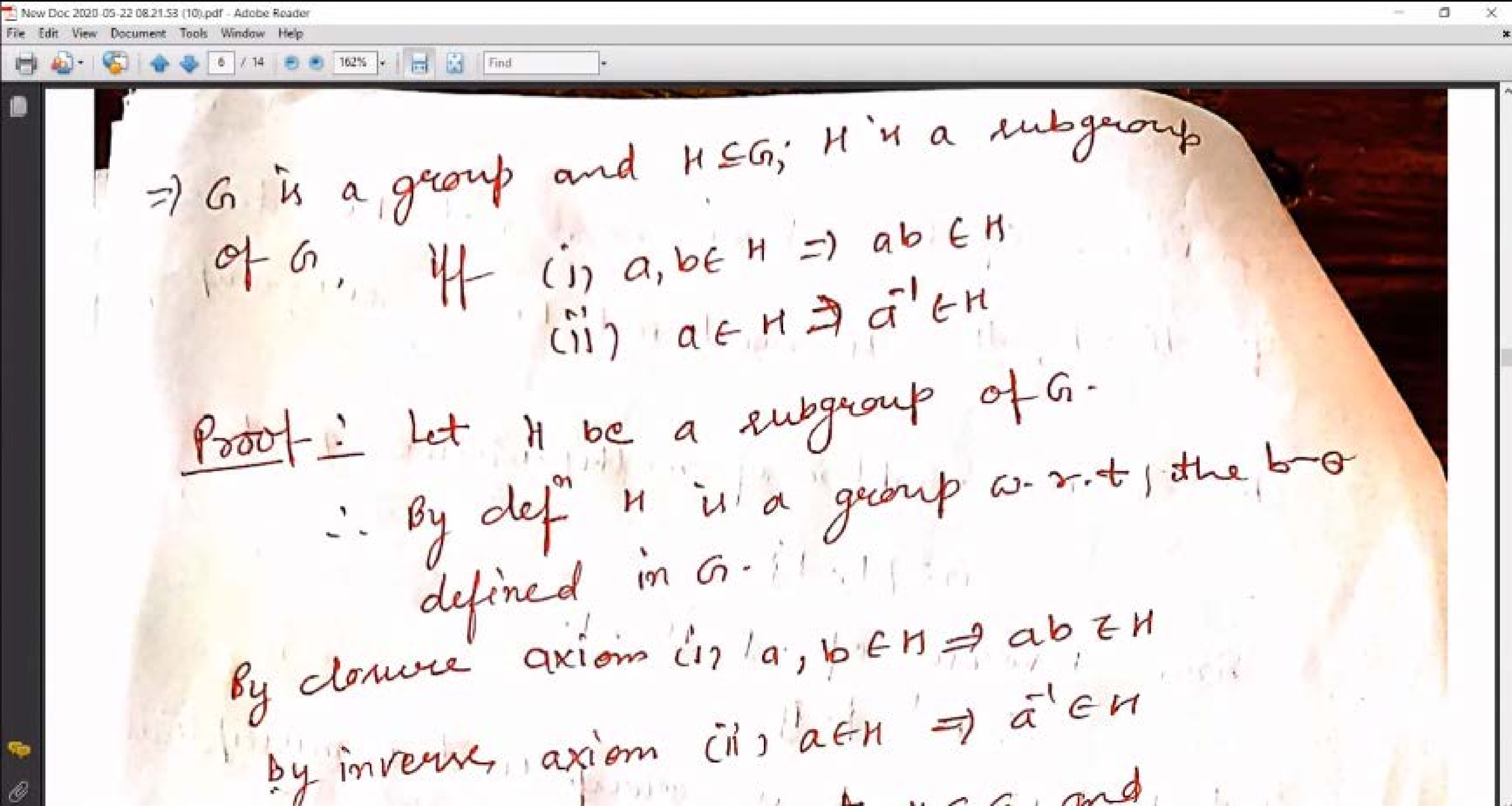


$\Rightarrow$ G is a group and $H \subseteq G$; H is a subgroup of G, iff (i) $a, b \in H \Rightarrow ab \in H$

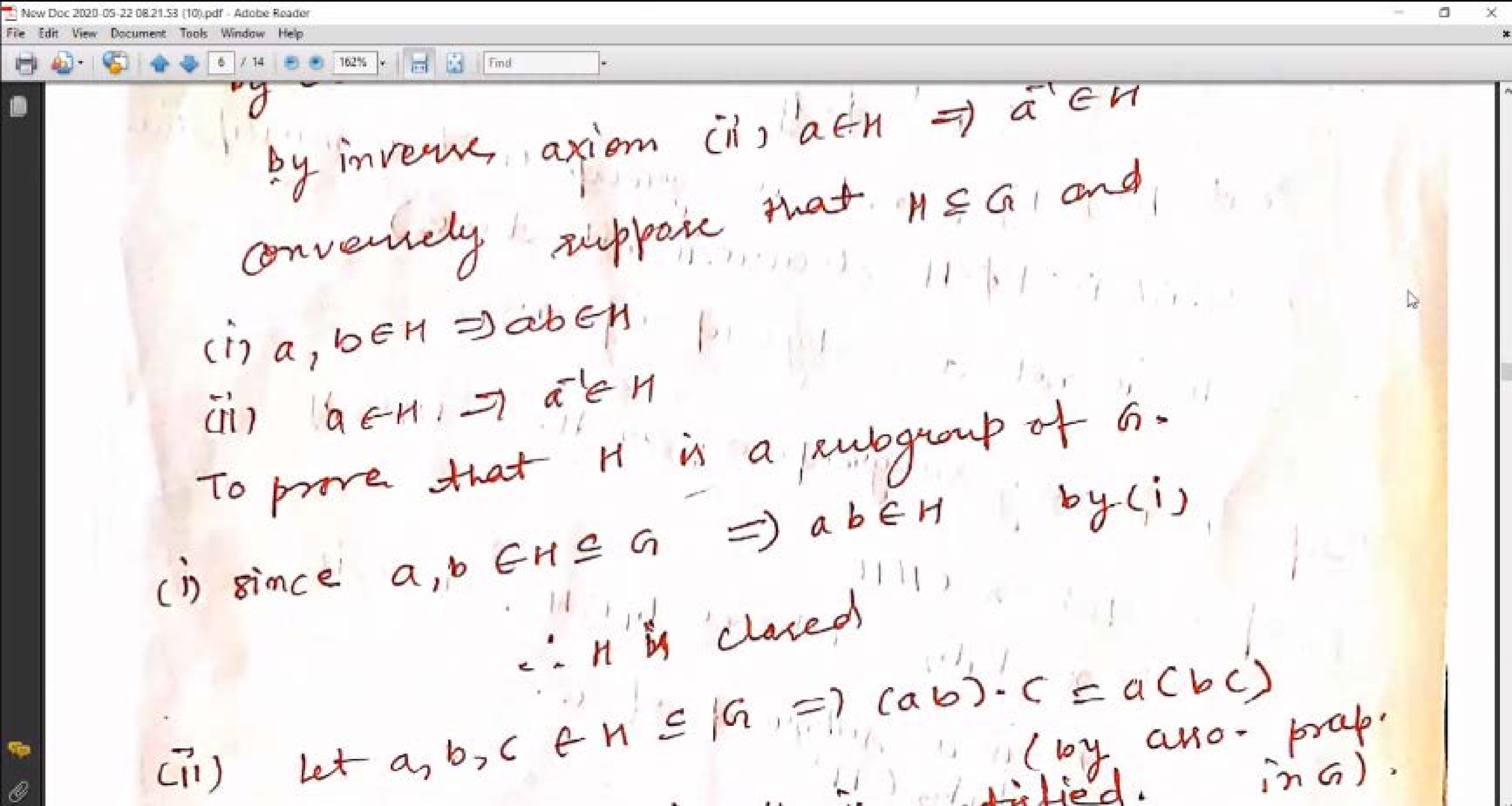
(ii) $a \in H \Rightarrow a^{-1} \in H$

<u>Proof</u>: Let H be a subgroup of G.

$\therefore$ By def^n H is a group w.r.t the b-o defined in G.

By closure axiom (i) $a, b \in H \Rightarrow ab \in H$

By inverse axiom (ii) $a \in H \Rightarrow a^{-1} \in H$



by inverse axiom (ii) $a \in H \Rightarrow a^{-1} \in H$

Conversely suppose that $H \subseteq G$ and

(i) $a, b \in H \Rightarrow ab \in H$

(ii) $a \in H \Rightarrow a^{-1} \in H$

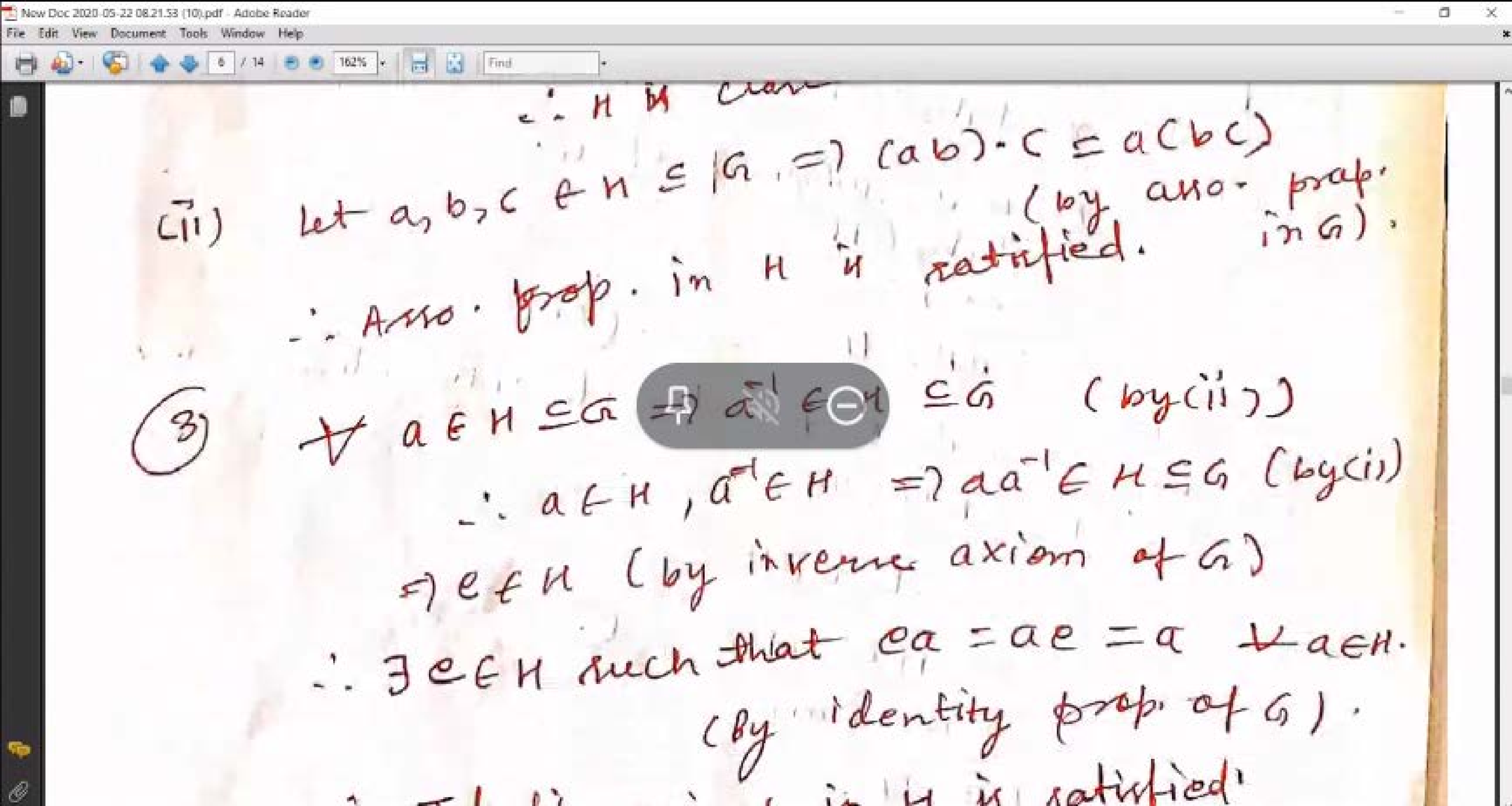
To prove that H is a subgroup of G.

(i) since $a, b \in H \subseteq G \Rightarrow ab \in H$ by (i)

$\therefore$ H is closed

(ii) let $a, b, c \in H \subseteq G \Rightarrow (ab) \cdot c = a(bc)$ (by asso. prop. in G).

tified.



∴ H is cl

(ii) Let $a, b, c \in H \subseteq G \Rightarrow (ab)\cdot c = a(bc)$ (by asso. prop. in G).

∴ Asso. prop. in H is satisfied.

(3) $\forall\ a \in H \subseteq G \Rightarrow a^{-1} \in H \subseteq G$ (by (ii))

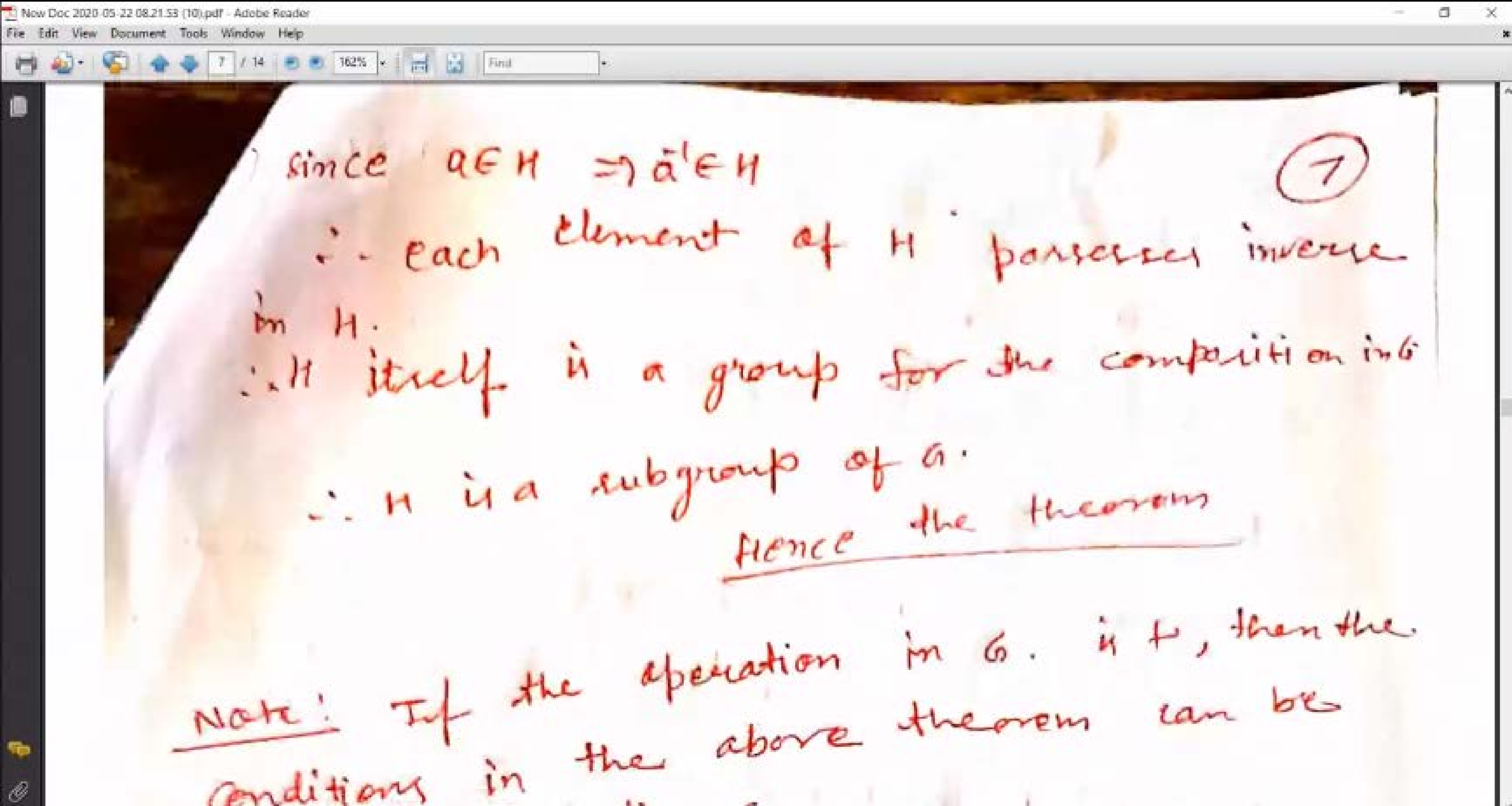
∴ $a \in H,\ a^{-1} \in H \Rightarrow aa^{-1} \in H \subseteq G$ (by (i))

$\Rightarrow e \in H$ (by inverse axiom of G)

∴ $\exists\ e \in H$ such that $ea = ae = a\ \forall\ a \in H$.

(By identity prop. of G).

∴ [illegible] in H is satisfied.



Since $a \in H \Rightarrow a^{-1} \in H$

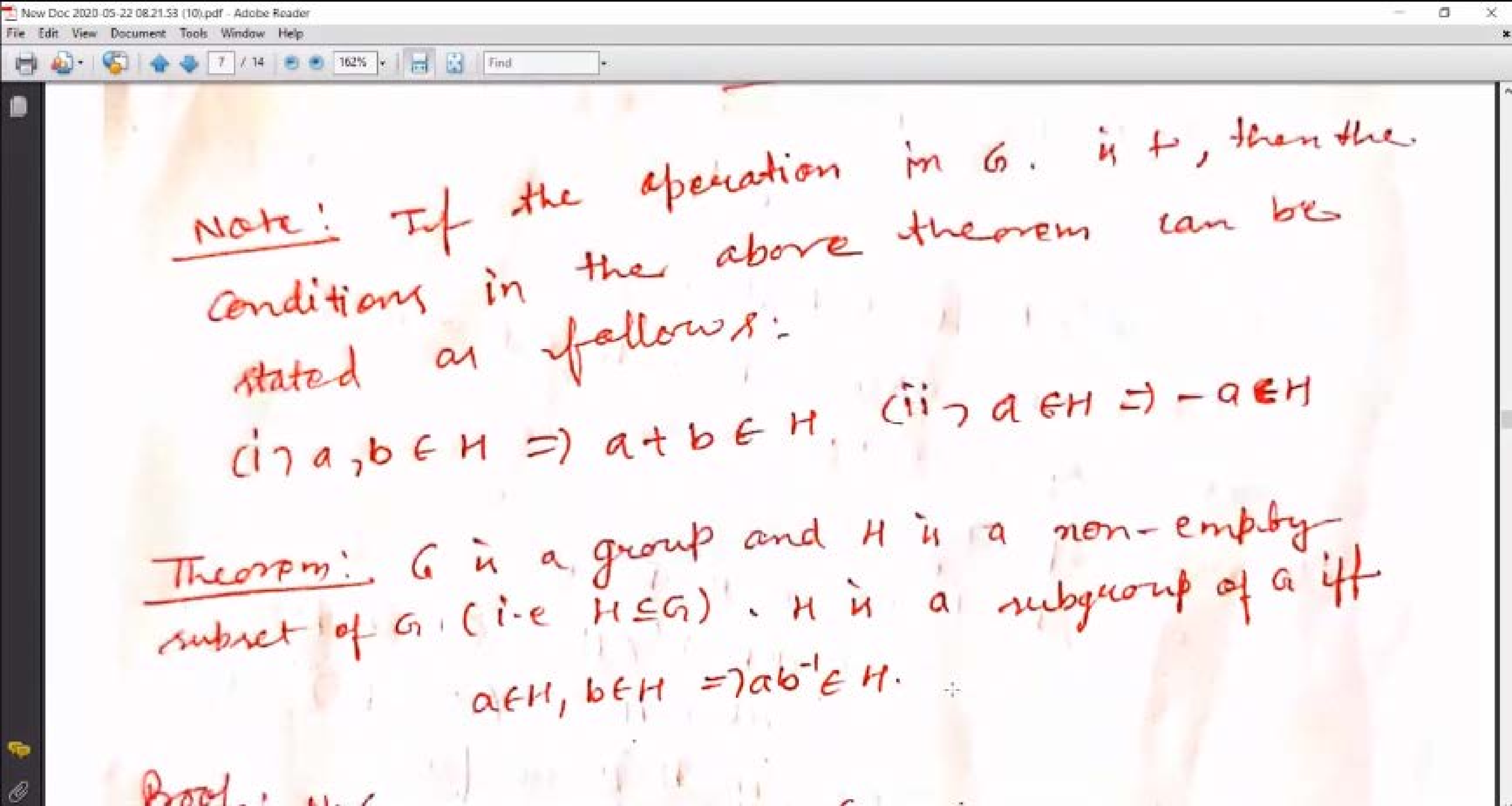
$\therefore$ each element of H possesses inverse in H.

$\therefore$ H itself is a group for the composition in G

$\therefore$ H is a subgroup of G.

Hence the theorem

Note: If the operation in G. is $+$, then the conditions in the above theorem can be



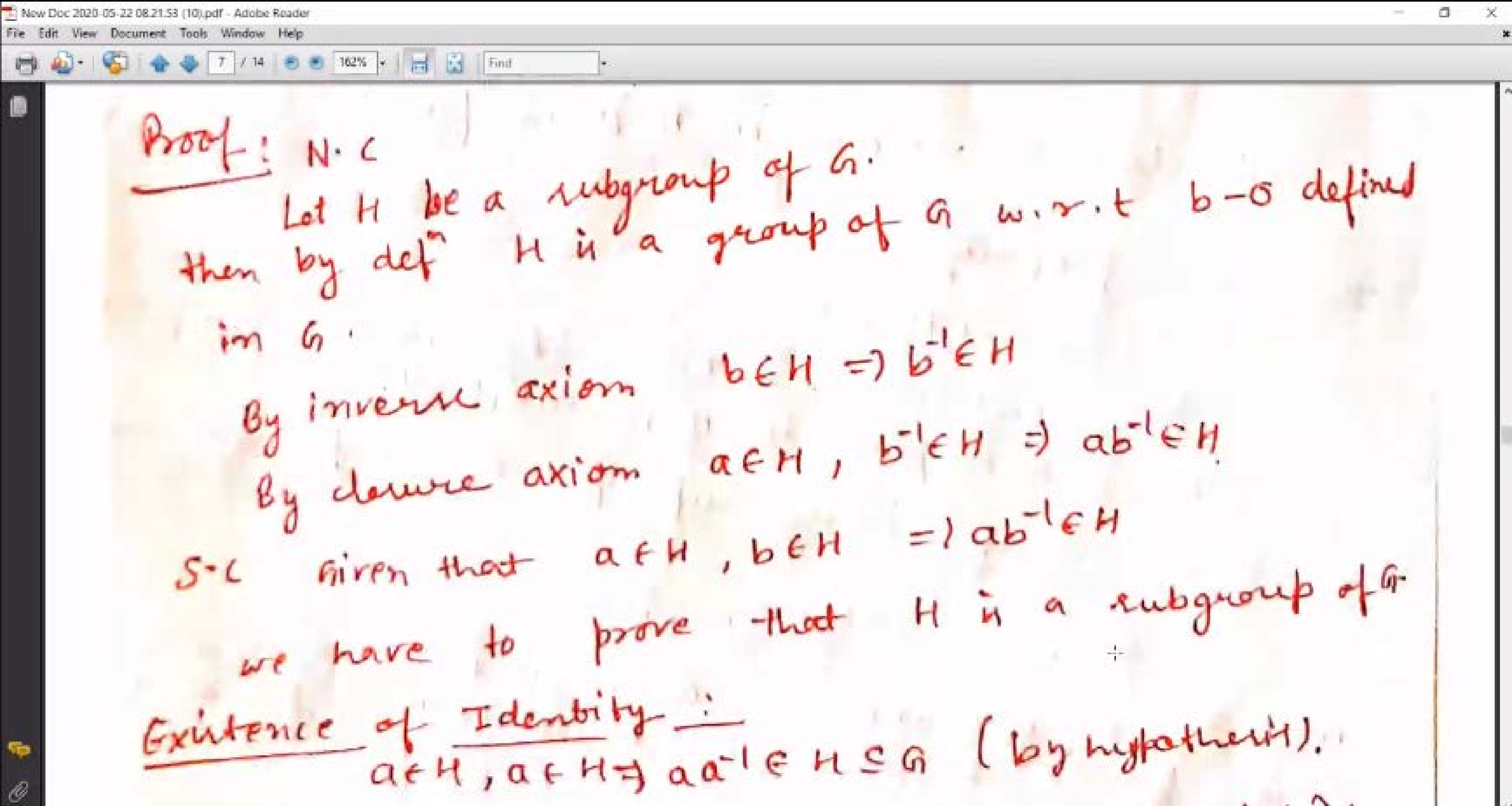
Note: If the operation in G. is +, then the conditions in the above theorem can be stated as follows:-

(i) $a, b \in H \Rightarrow a+b \in H$. (ii) $a \in H \Rightarrow -a \in H$

Theorem: G is a group and H is a non-empty subset of G (i.e $H \subseteq G$). H is a subgroup of G iff

$$a \in H, b \in H \Rightarrow ab^{-1} \in H.$$

Proof:



Proof: N·C

Let H be a subgroup of G.

then by def^n H is a group of G w.r.t b-o defined in G.

By inverse axiom $b \in H \Rightarrow b^{-1} \in H$

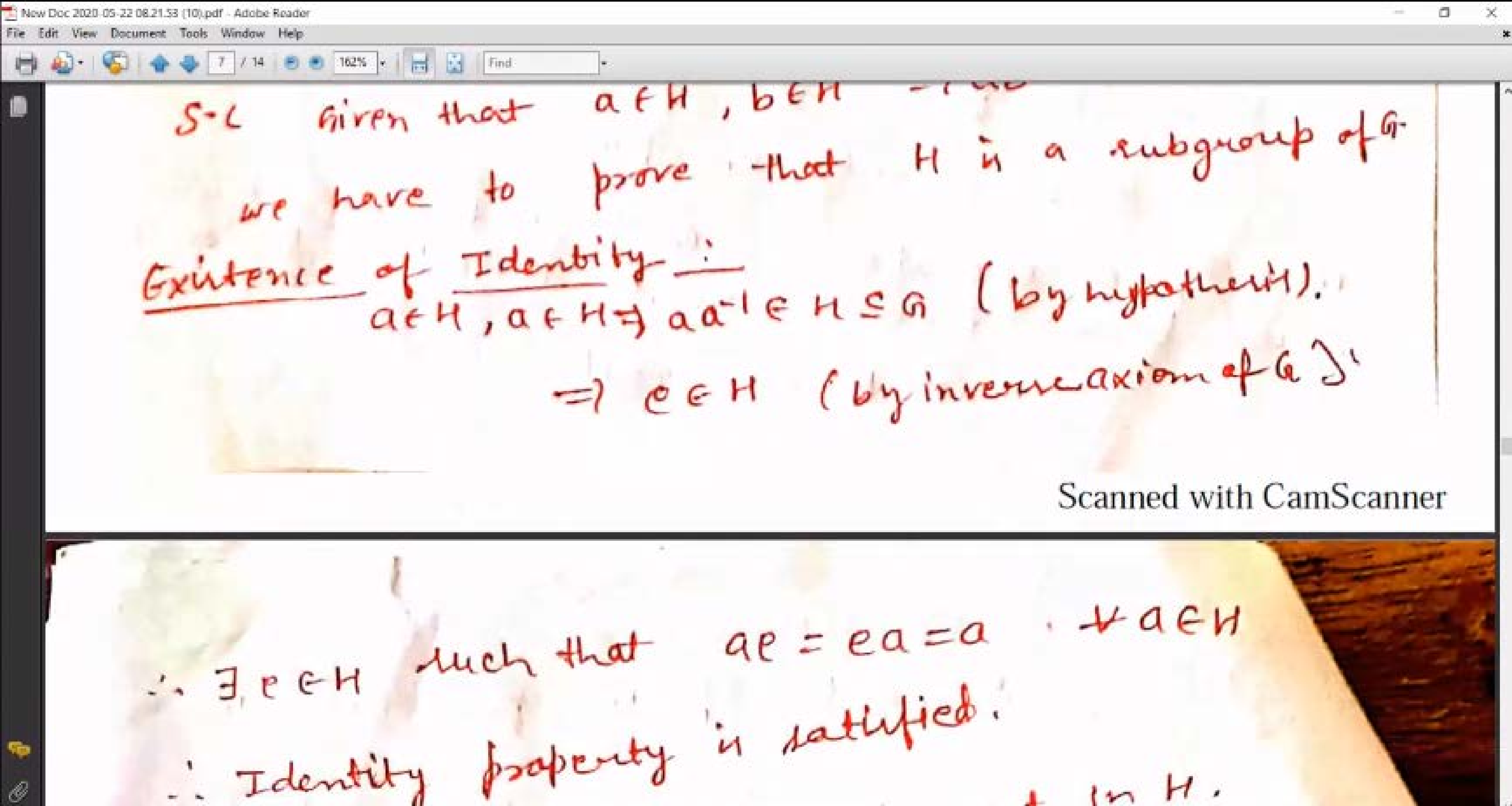
By closure axiom $a \in H,\ b^{-1} \in H \Rightarrow ab^{-1} \in H$

S·C Given that $a \in H,\ b \in H \Rightarrow ab^{-1} \in H$

we have to prove that H is a subgroup of G

Existence of Identity:

$a \in H,\ a \in H \Rightarrow aa^{-1} \in H \subseteq G$ (by hypothesis).



S.L Given that $a \in H, b \in H$

we have to prove that H is a subgroup of G

Existence of Identity :

$a \in H, a \in H \Rightarrow aa^{-1} \in H \leq G$ (by hypothesis).

$\Rightarrow e \in H$ (by inverse axiom of G)

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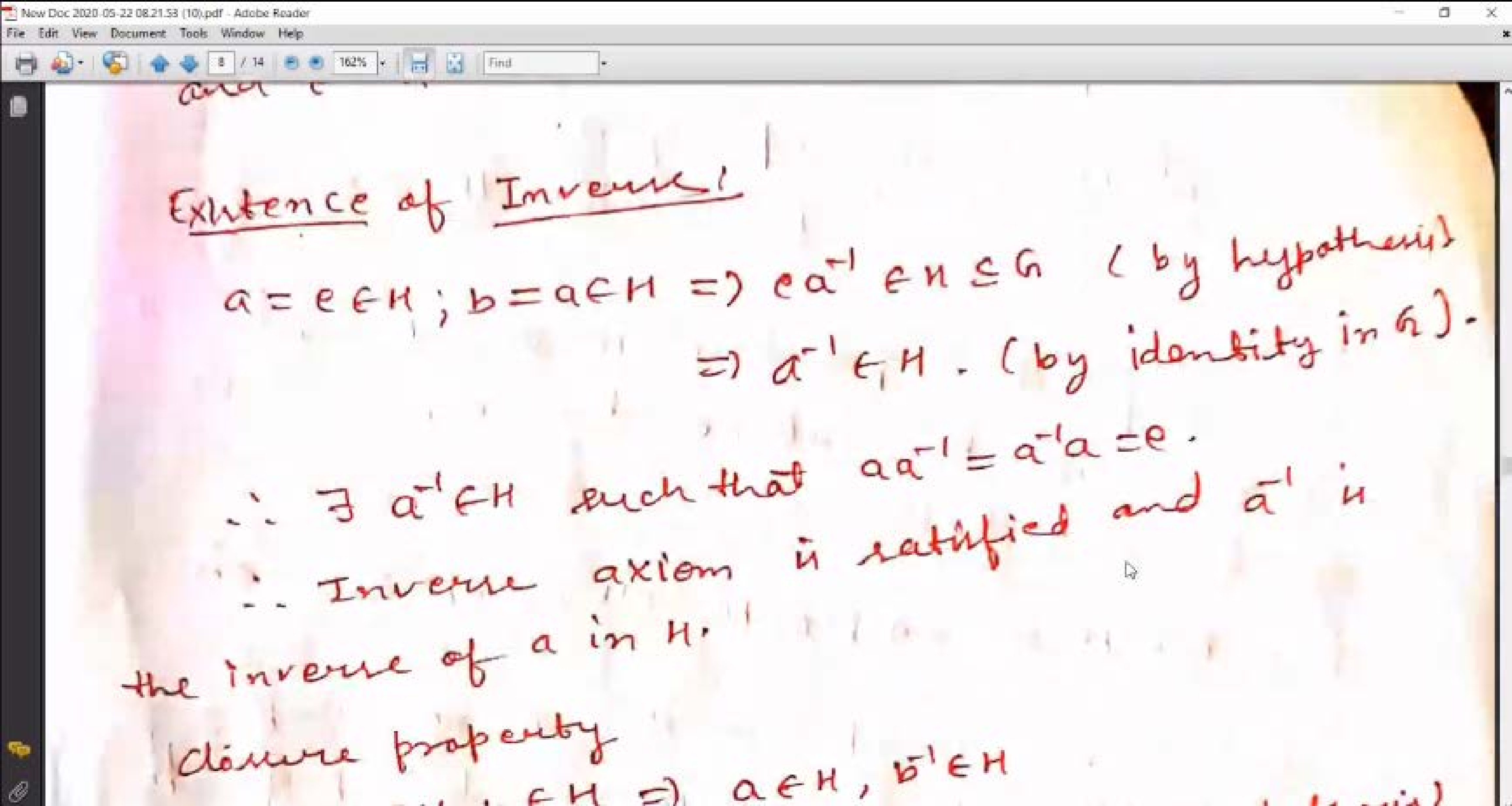
$\therefore \exists\, e \in H$ such that $ae = ea = a \quad \forall a \in H$

$\therefore$ Identity property is satisfied.

in H.



Existence of Inverse:

$a = e \in H ; b = a \in H \Rightarrow ea^{-1} \in H \subseteq G$ (by hypothesis)

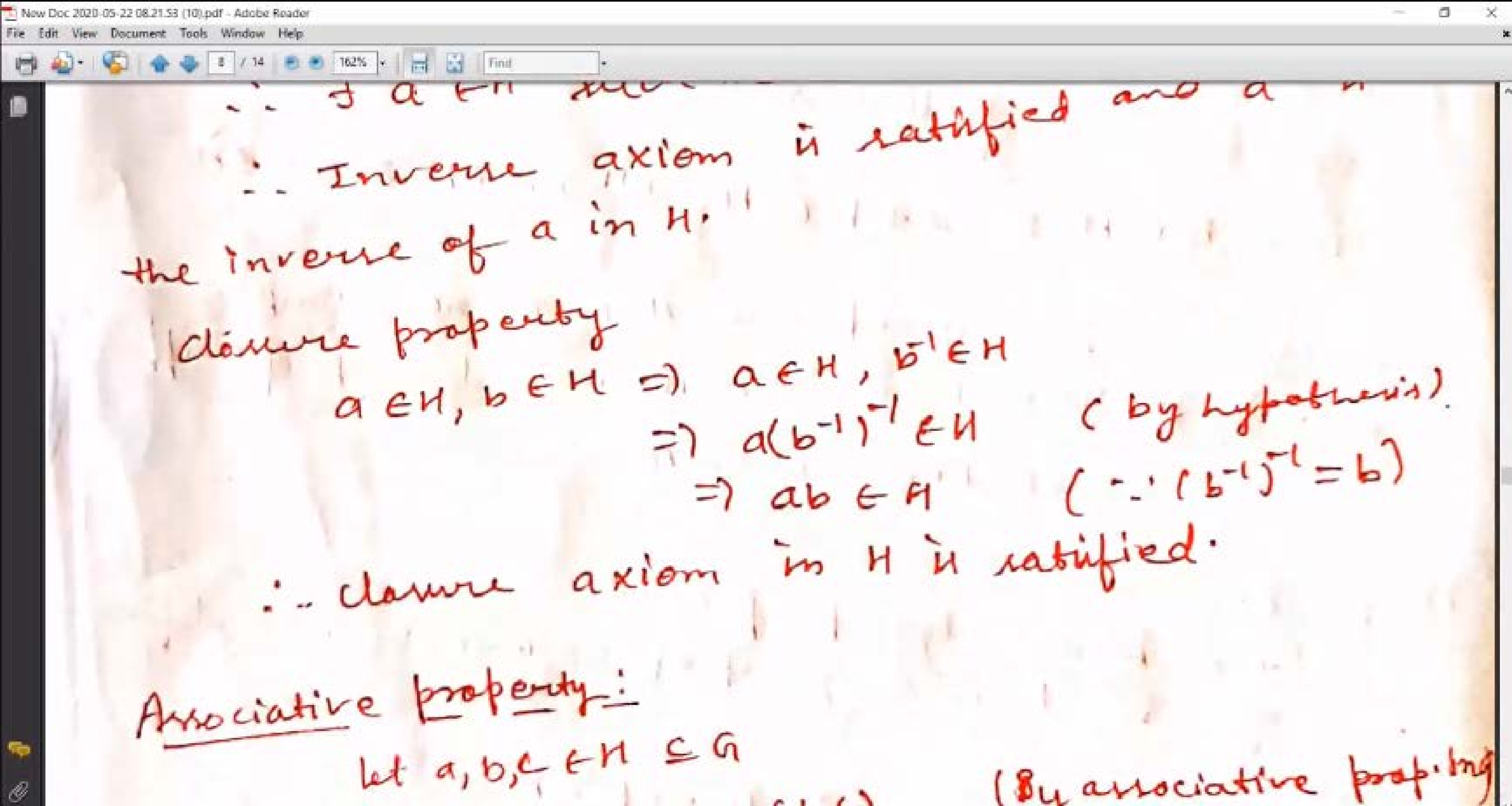
$\Rightarrow a^{-1} \in H$. (by identity in G).

$\therefore \exists\, a^{-1} \in H$ such that $aa^{-1} = a^{-1}a = e$.

$\therefore$ Inverse axiom is satisfied and a^{-1} is the inverse of a in H.

Closure property

$a \in H, b^{-1} \in H$



∴ ∃ $a \in H$ such

∴ Inverse axiom is satisfied and a^{-1} is the inverse of a in H.

Closure property

$a \in H, b \in H \Rightarrow a \in H, b^{-1} \in H$

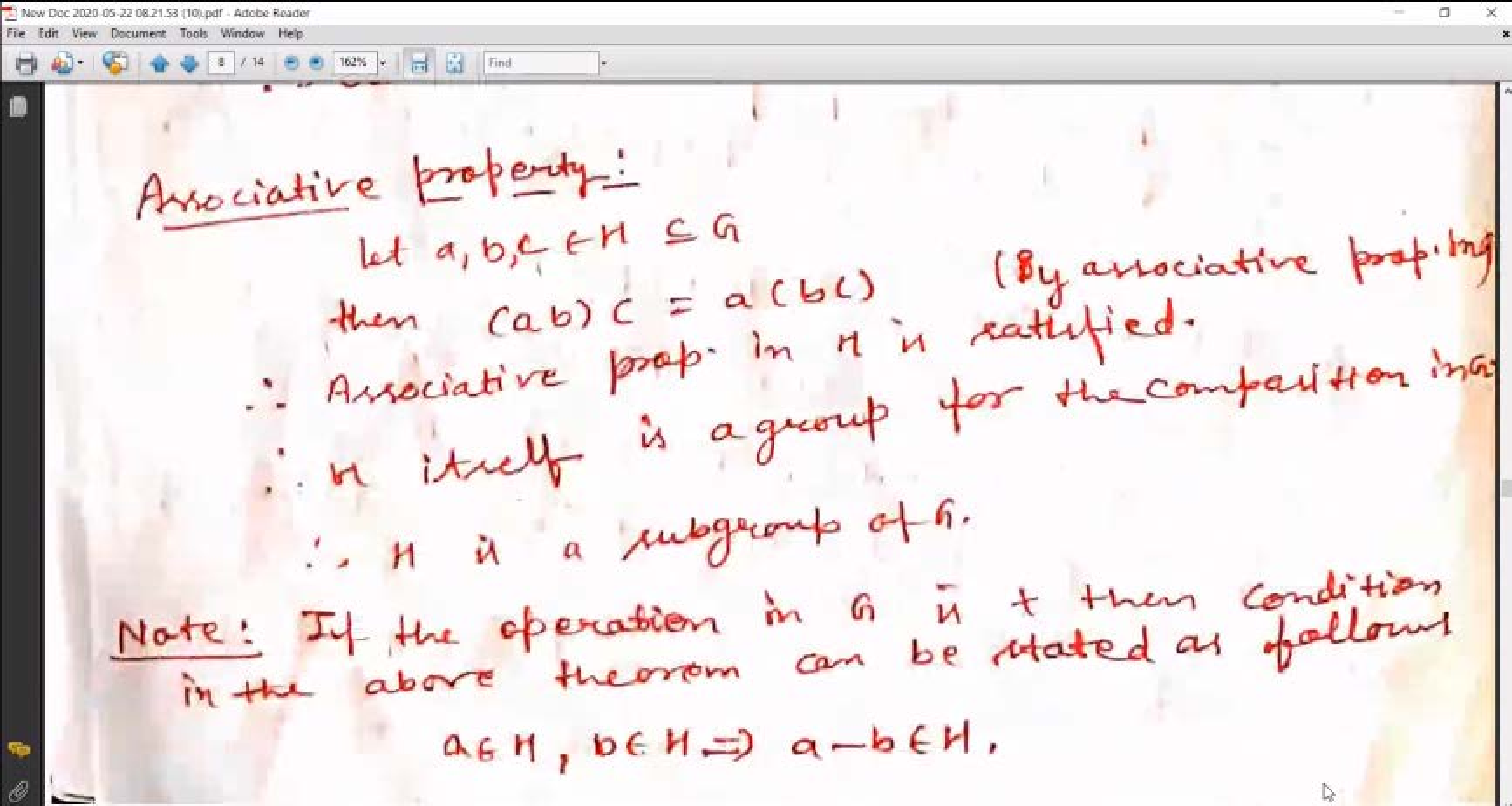
$\Rightarrow a(b^{-1})^{-1} \in H$ (by hypothesis).

$\Rightarrow ab \in H$ ($\because (b^{-1})^{-1} = b$)

∴ closure axiom in H is satisfied.

Associative property:

let $a, b, c \in H \subseteq G$ (By associative prop. in



Associative property:

let $a, b, c \in H \subseteq G$

then $(ab)c = a(bc)$ (By associative prop. in G

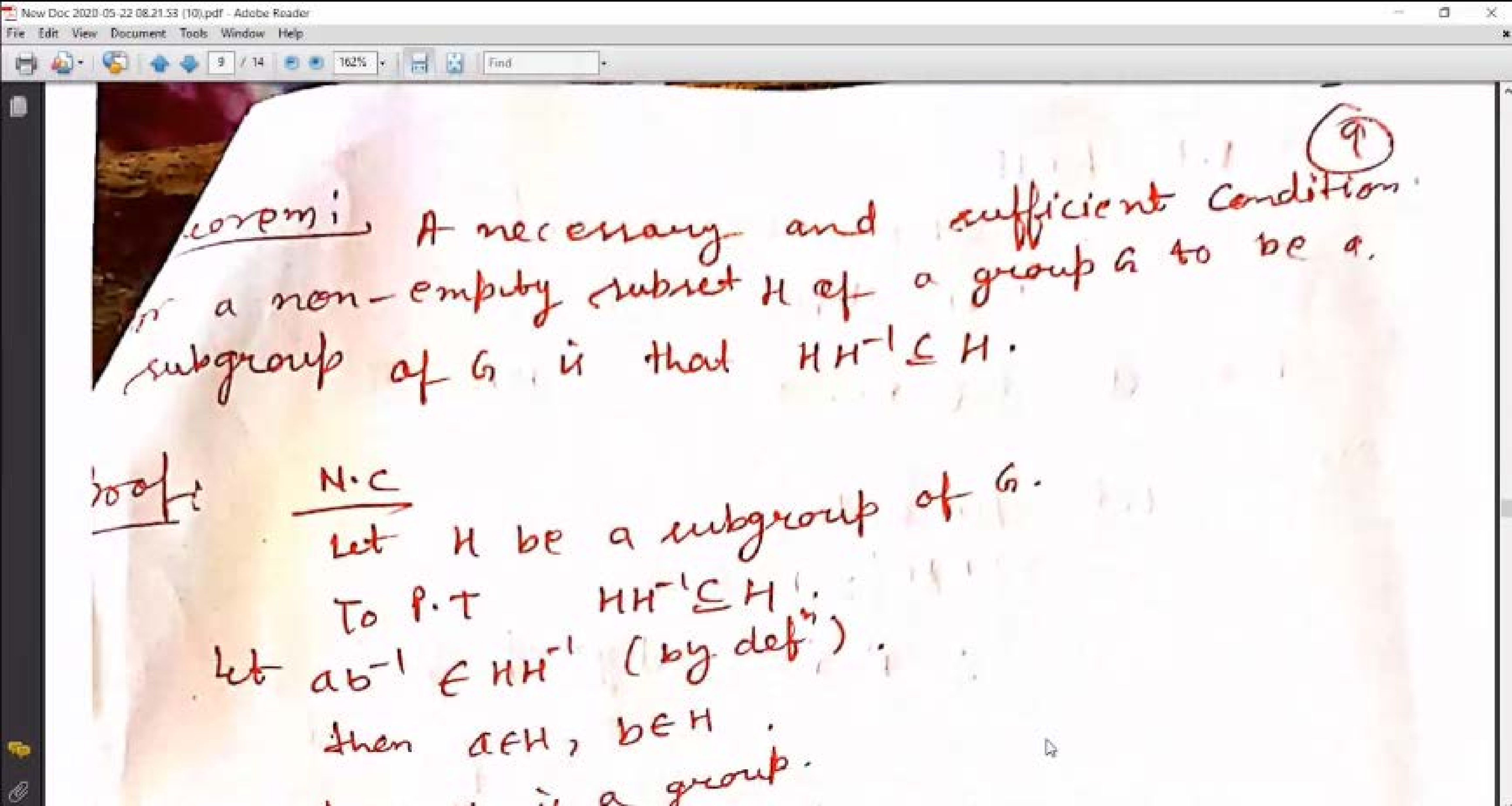
∴ Associative prop. in H is satisfied.

∴ H itself is a group for the composition in G

∴ H is a subgroup of G.

Note: If the operation in G is + then condition in the above theorem can be stated as follows

$a \in H, b \in H \Rightarrow a - b \in H$.



(9)

eorem: A necessary and sufficient condition r a non-empty subset H of a group G to be a subgroup of G is that $HH^{-1} \subseteq H$.

roof: N.C

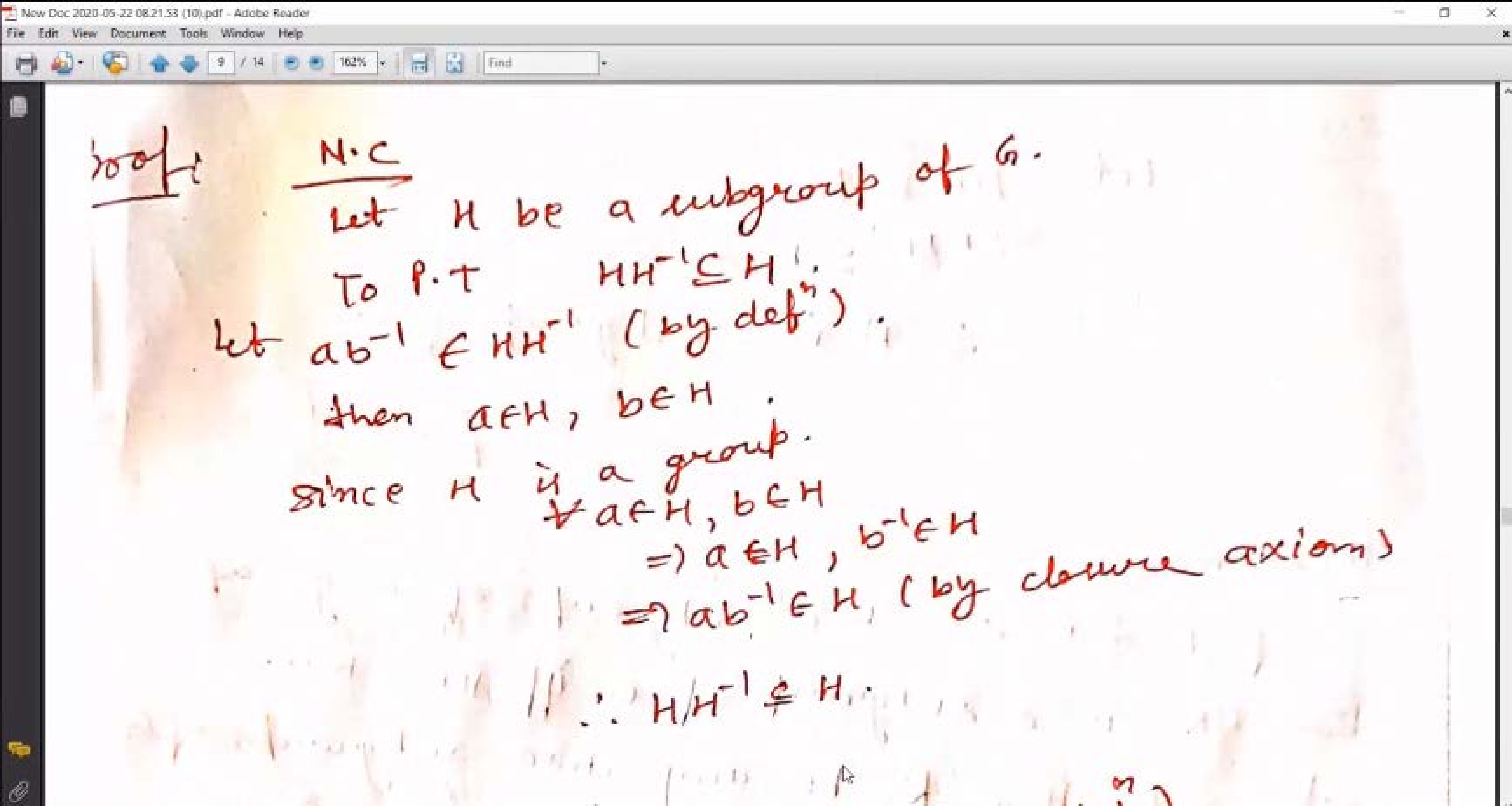
Let H be a subgroup of G.

To P.T $HH^{-1} \subseteq H$.

Let $ab^{-1} \in HH^{-1}$ (by defn).

then $a \in H$, $b \in H$.

is a group.



roof:

N.C

Let H be a subgroup of G.

To P.T $HH^{-1} \subseteq H$.

Let $ab^{-1} \in HH^{-1}$ (by def^n).

then $a \in H$, $b \in H$.

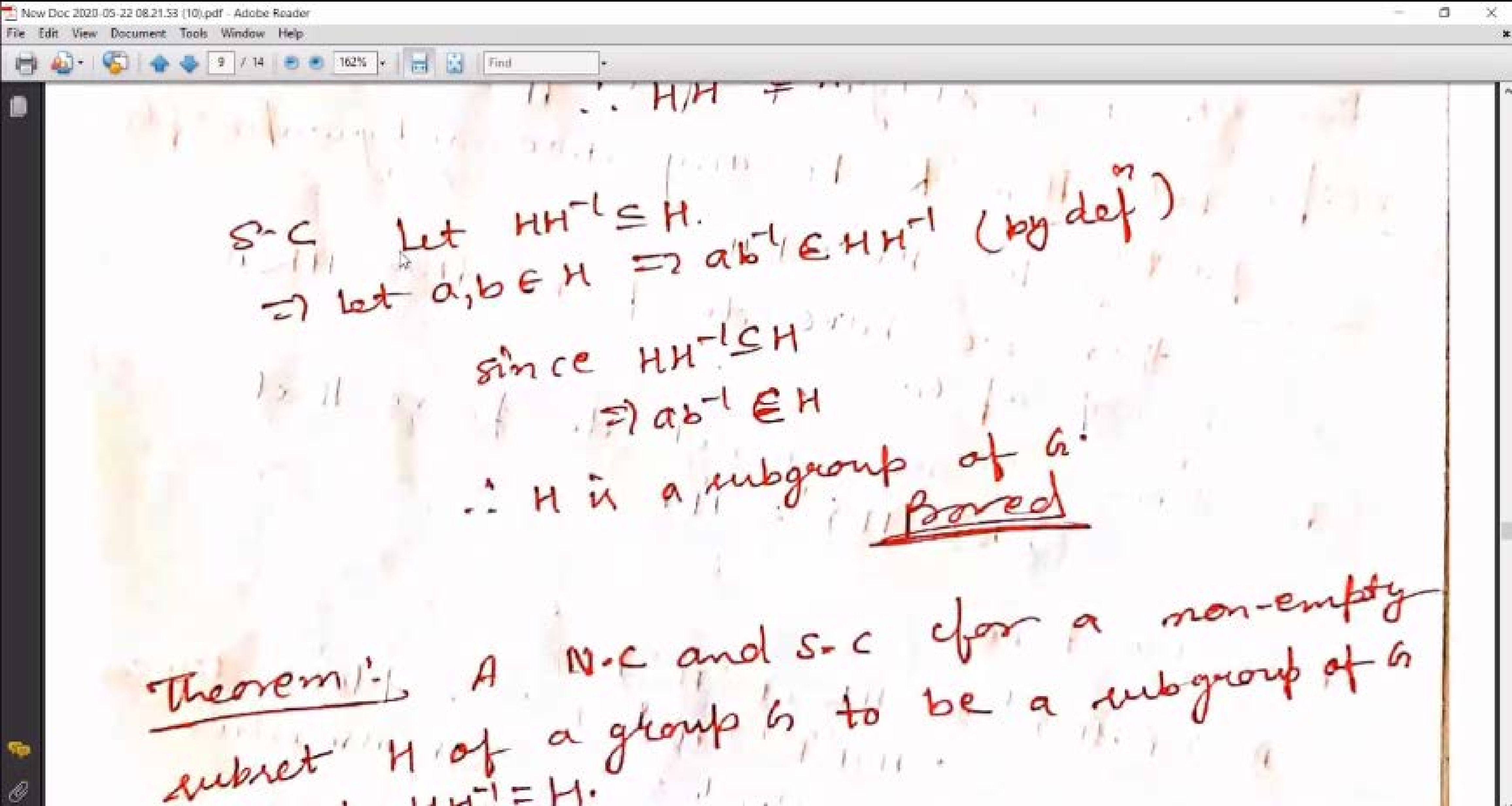
Since H is a group.

$\forall a \in H, b \in H$

$\Rightarrow a \in H, b^{-1} \in H$

$\Rightarrow ab^{-1} \in H$ (by closure axiom)

$\therefore HH^{-1} \subseteq H$.



$\therefore HH$

S-C Let $HH^{-1} \subseteq H$.

$\Rightarrow$ let $a, b \in H \Rightarrow ab^{-1} \in HH^{-1}$ (by defⁿ)

since $HH^{-1} \subseteq H$

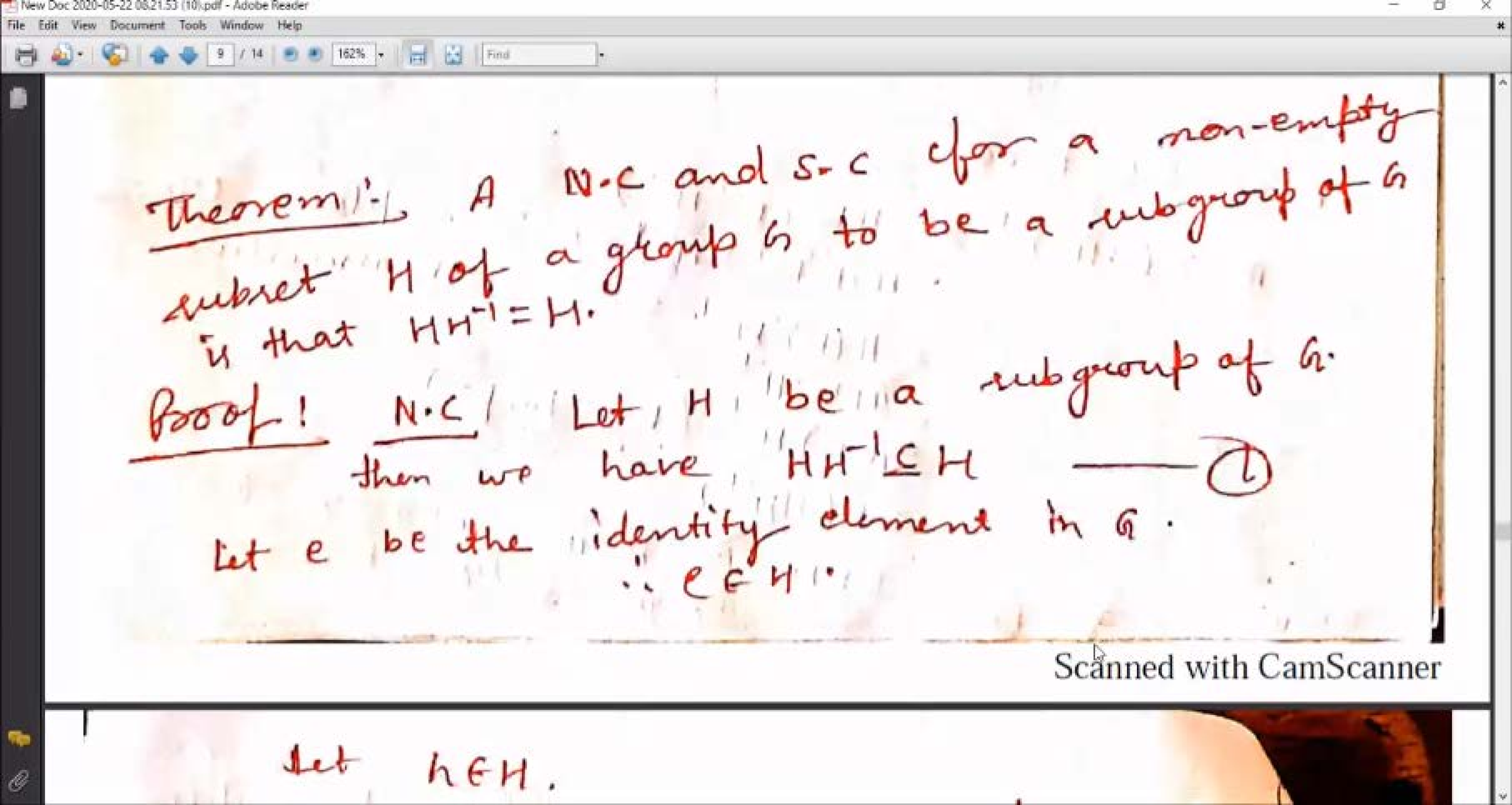
$\Rightarrow ab^{-1} \in H$

$\therefore$ H is a subgroup of G.

Proved

Theorem:- A N.C and S.C for a non-empty subset H of a group G to be a subgroup of G

$HH^{-1} = H$.



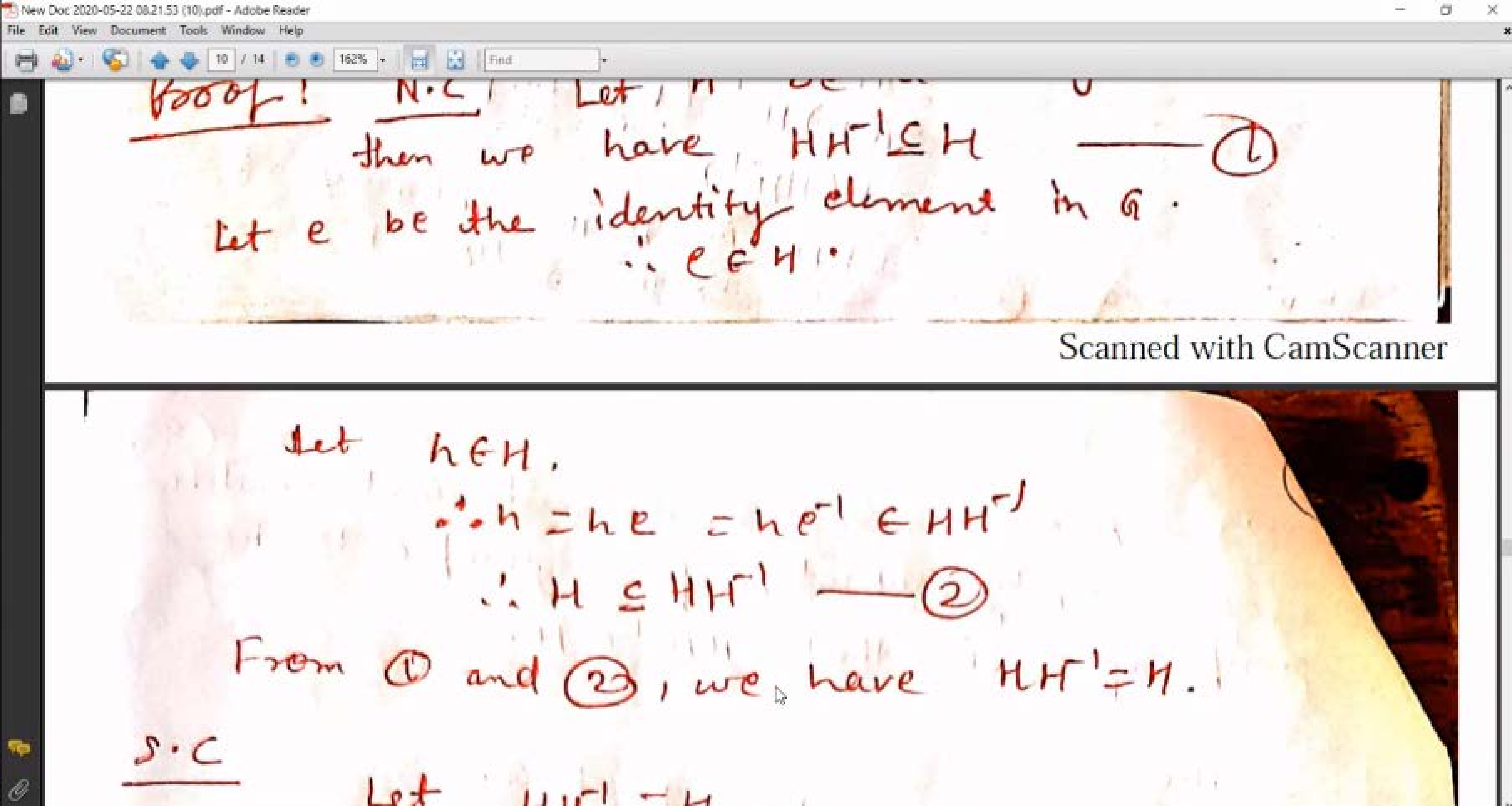
Theorem:- A N.C and S.C for a non-empty subset H of a group G to be a subgroup of G is that $HH^{-1} = H$.

Proof! N.C Let H be a subgroup of G. then we have $HH^{-1} \subseteq H$ —— ①

Let e be the identity element in G.

$\therefore e \in H$.

Let $h \in H$.



Proof! **N.C** Let H

then we have $HH^{-1} \subseteq H$ —— ①

Let e be the identity element in G.

$\therefore e \in H$.

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Let $h \in H$.

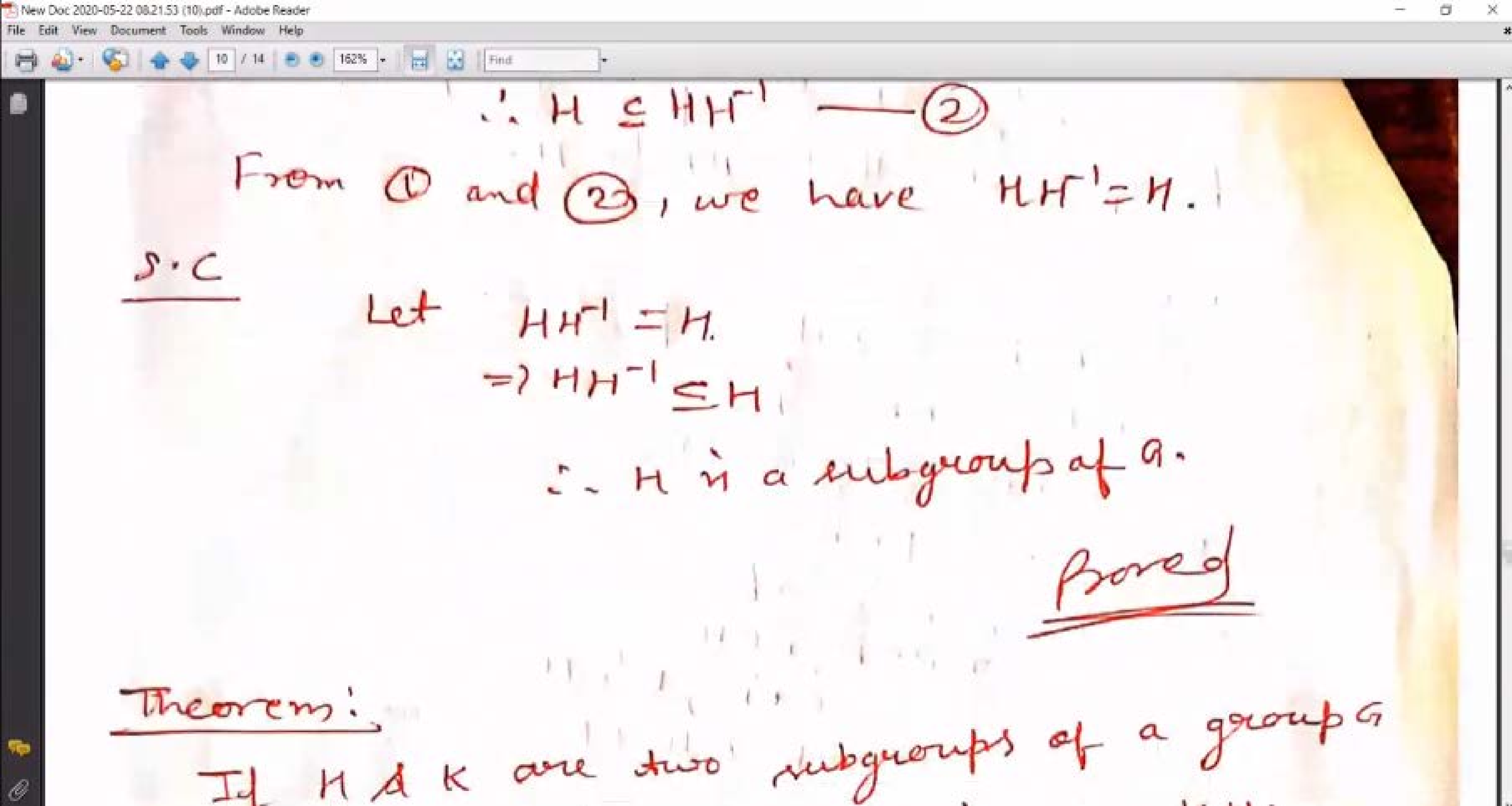
$\therefore h = he = he^{-1} \in HH^{-1}$

$\therefore H \subseteq HH^{-1}$ —— ②

From ① and ②, we have $HH^{-1} = H$.

S.C

Let $HH^{-1} = H$



$\therefore H \subseteq HH^{-1}$ —— ②

From ① and ②, we have $HH^{-1} = H$.

S.C

Let $HH^{-1} = H$.

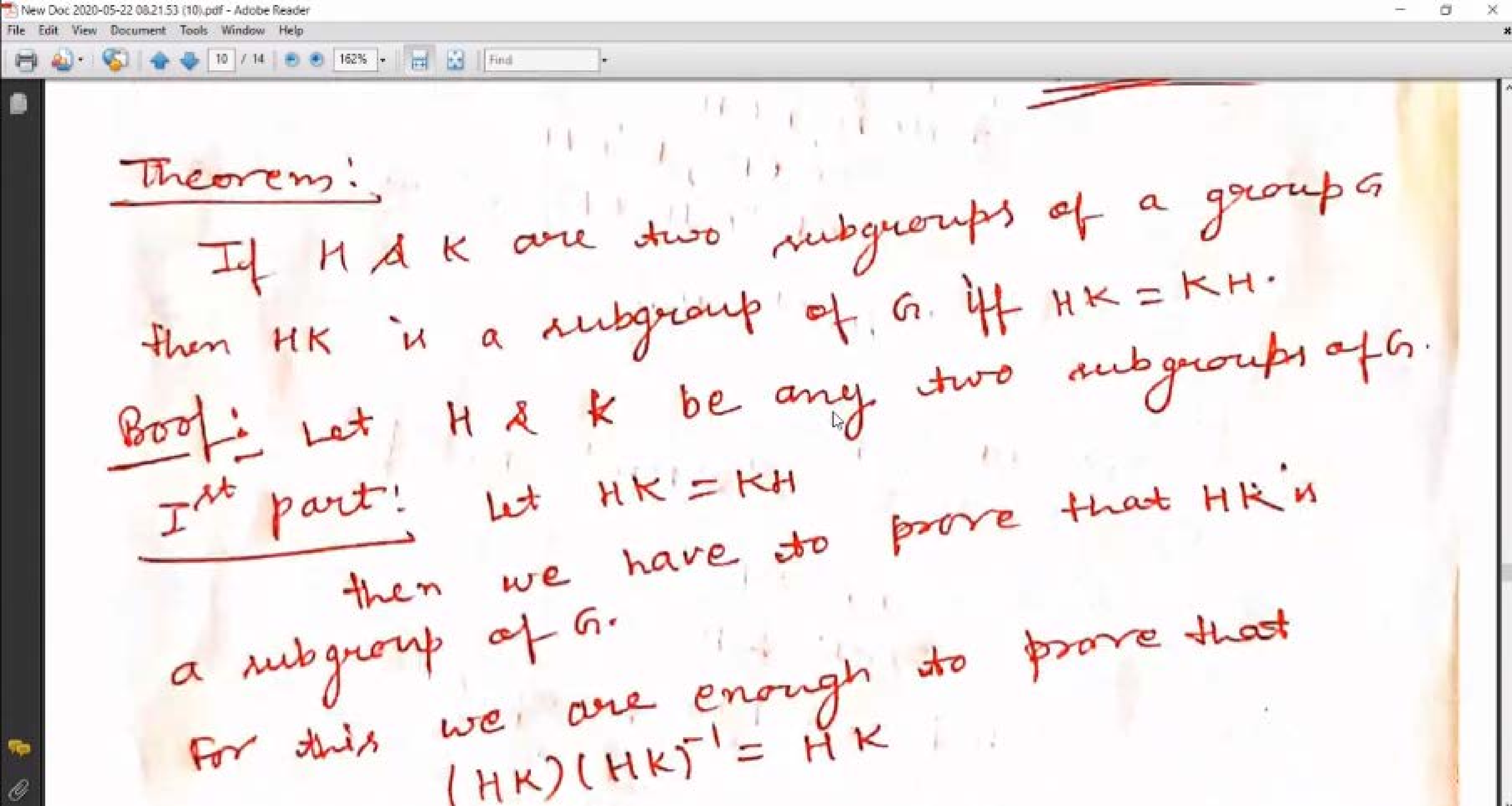
$\Rightarrow HH^{-1} \subseteq H$

$\therefore$ H is a subgroup of G.

Proved

Theorem:

If H & K are two subgroups of a group G



Theorem:

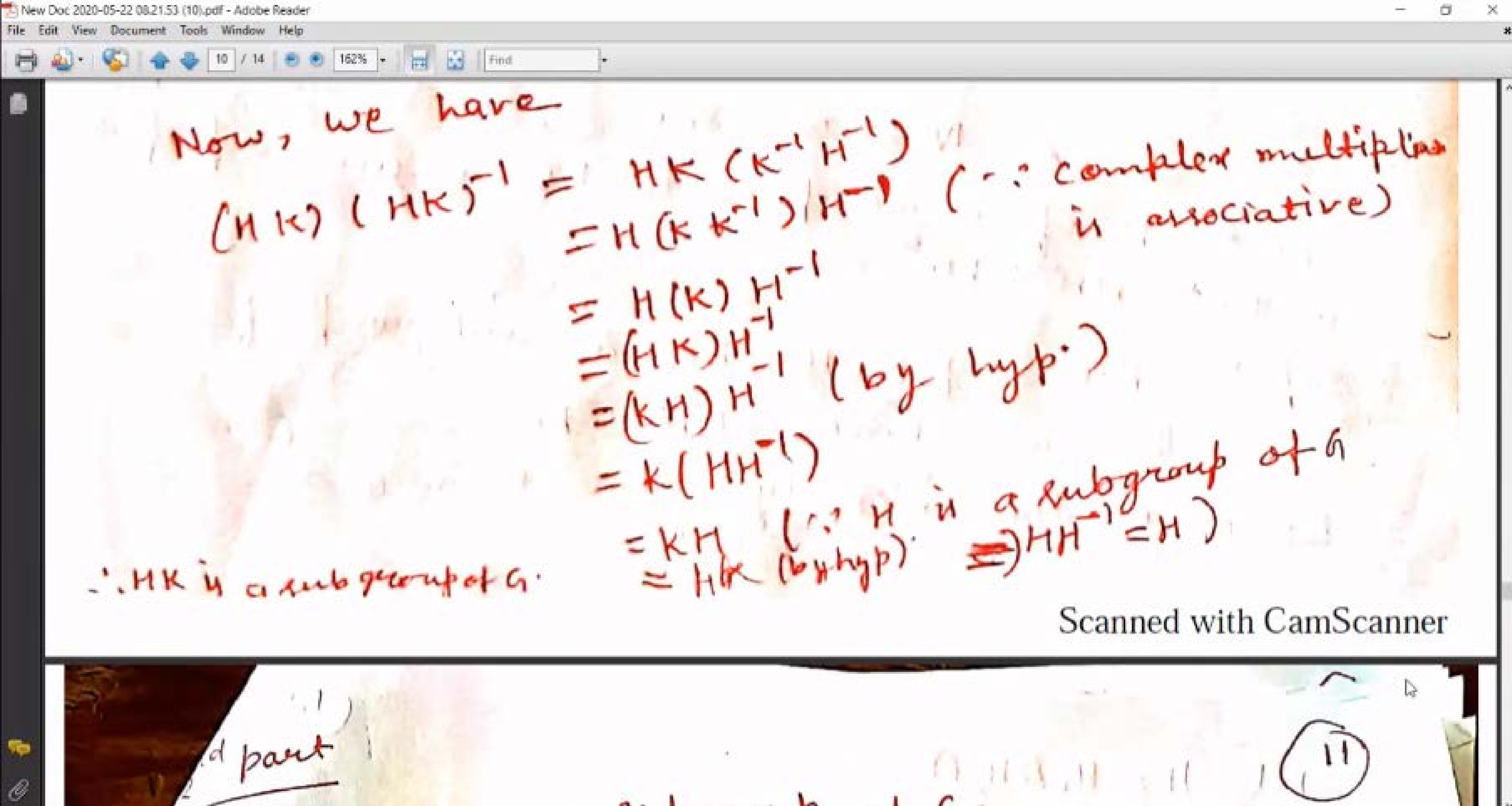
If H & K are two subgroups of a group G then HK is a subgroup of G iff HK = KH.

Proof: Let H & K be any two subgroups of G.

Ist part: Let HK = KH then we have to prove that HK is a subgroup of G.

For this we are enough to prove that

$$(HK)(HK)^{-1} = HK$$



Now, we have

$(HK)(HK)^{-1} = HK(K^{-1}H^{-1})$ (∵ complex multiplication is associative)

$= H(KK^{-1})H^{-1}$

$= H(K)H^{-1}$

$= (HK)H^{-1}$

$= (KH)H^{-1}$ (by hyp.)

$= K(HH^{-1})$

$= KH$ (∵ H is a subgroup of G $\Rightarrow HH^{-1} = H$)

$= HK$ (by hyp)

∴ HK is a subgroup of G.

d part

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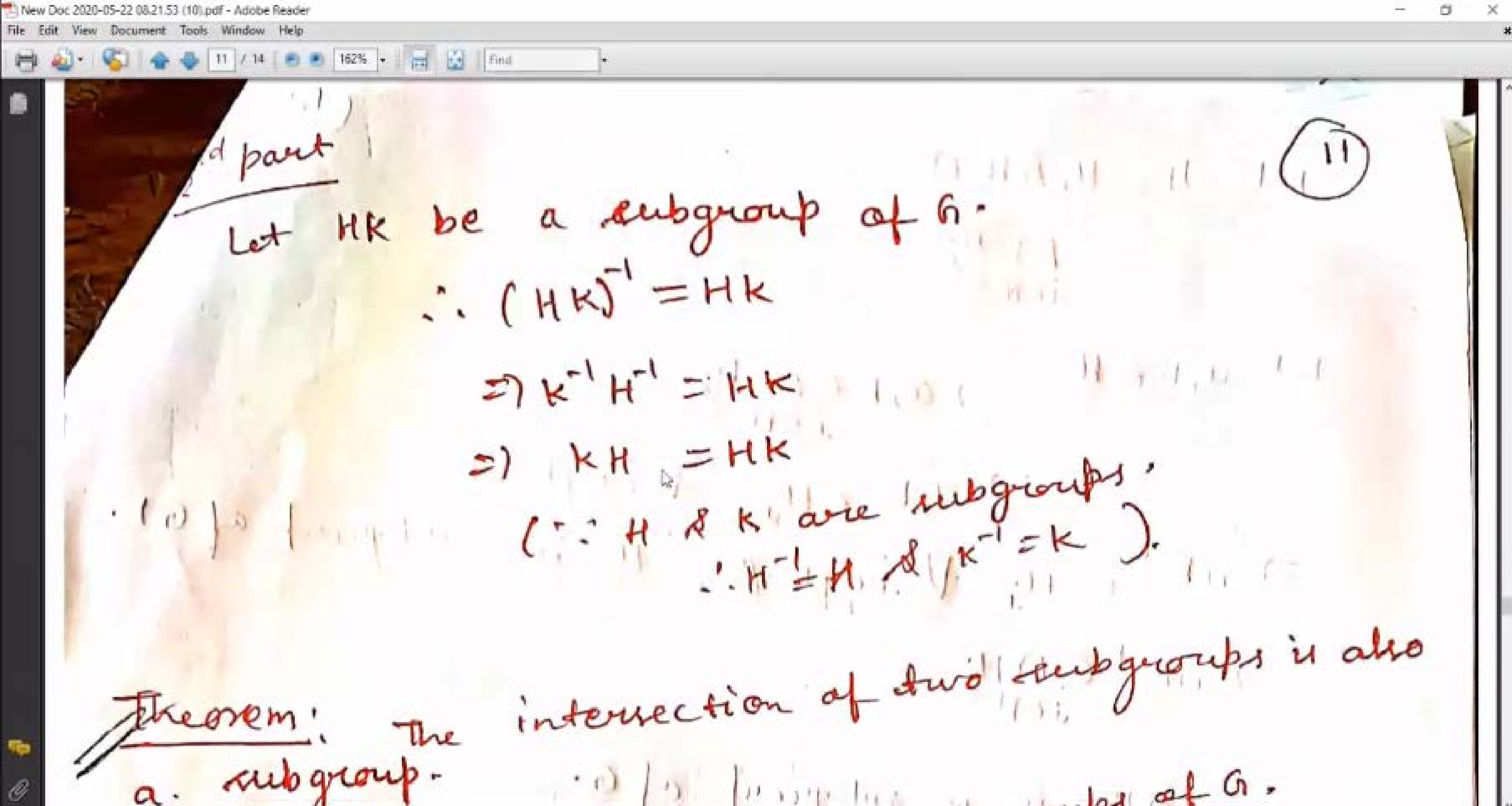


d part

Let HK be a subgroup of G.

$$\therefore (HK)^{-1} = HK$$

$$\Rightarrow K^{-1}H^{-1} = HK$$

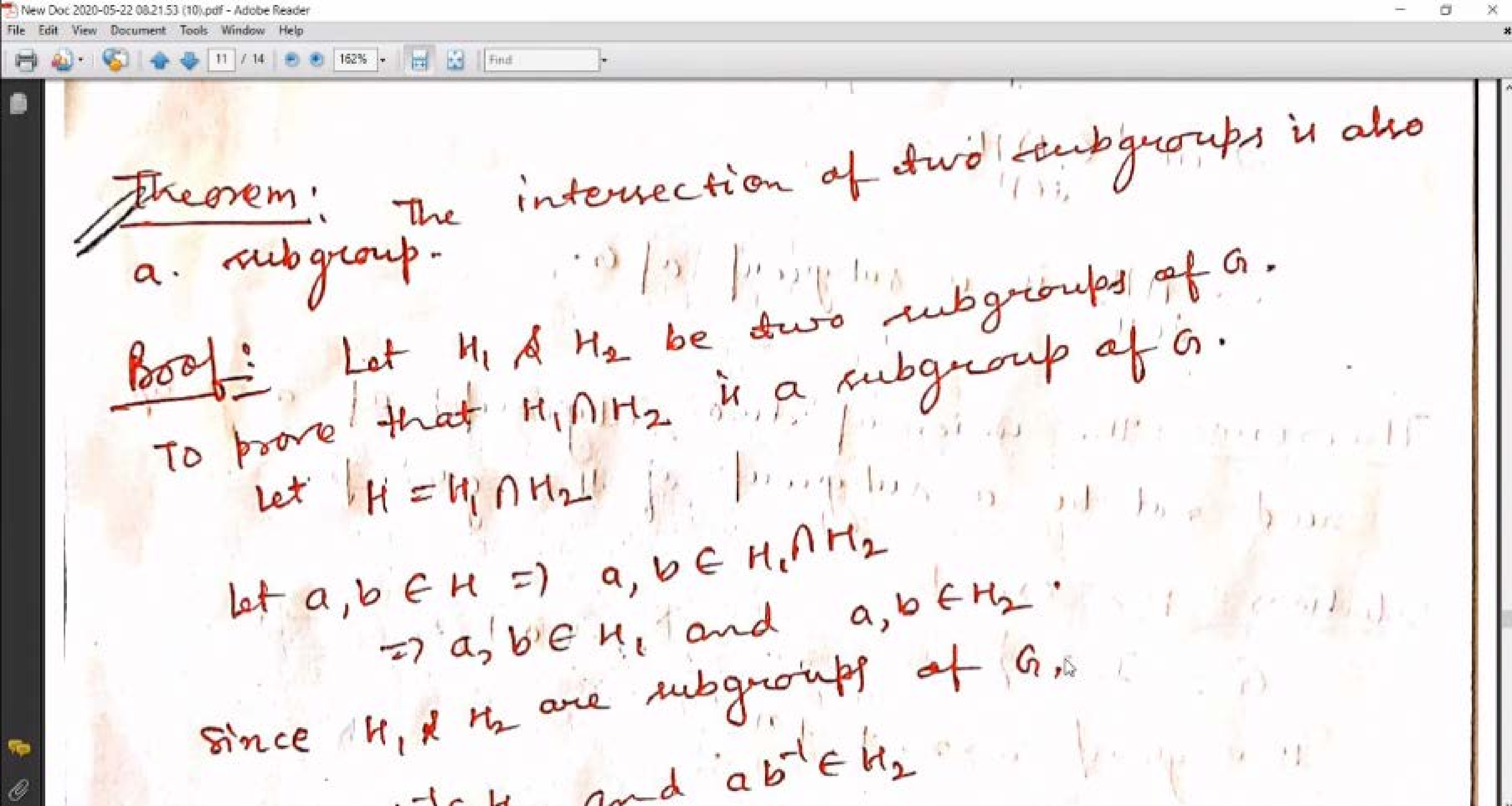
$$\Rightarrow KH = HK$$

($\because$ H & K are subgroups.

$\therefore H^{-1} = H$ & $K^{-1} = K$).

Theorem: The intersection of two subgroups is also a subgroup.

of G.



Theorem: The intersection of two subgroups is also a subgroup.

Proof: Let H_1 & H_2 be two subgroups of G.

To prove that $H_1 \cap H_2$ is a subgroup of G.

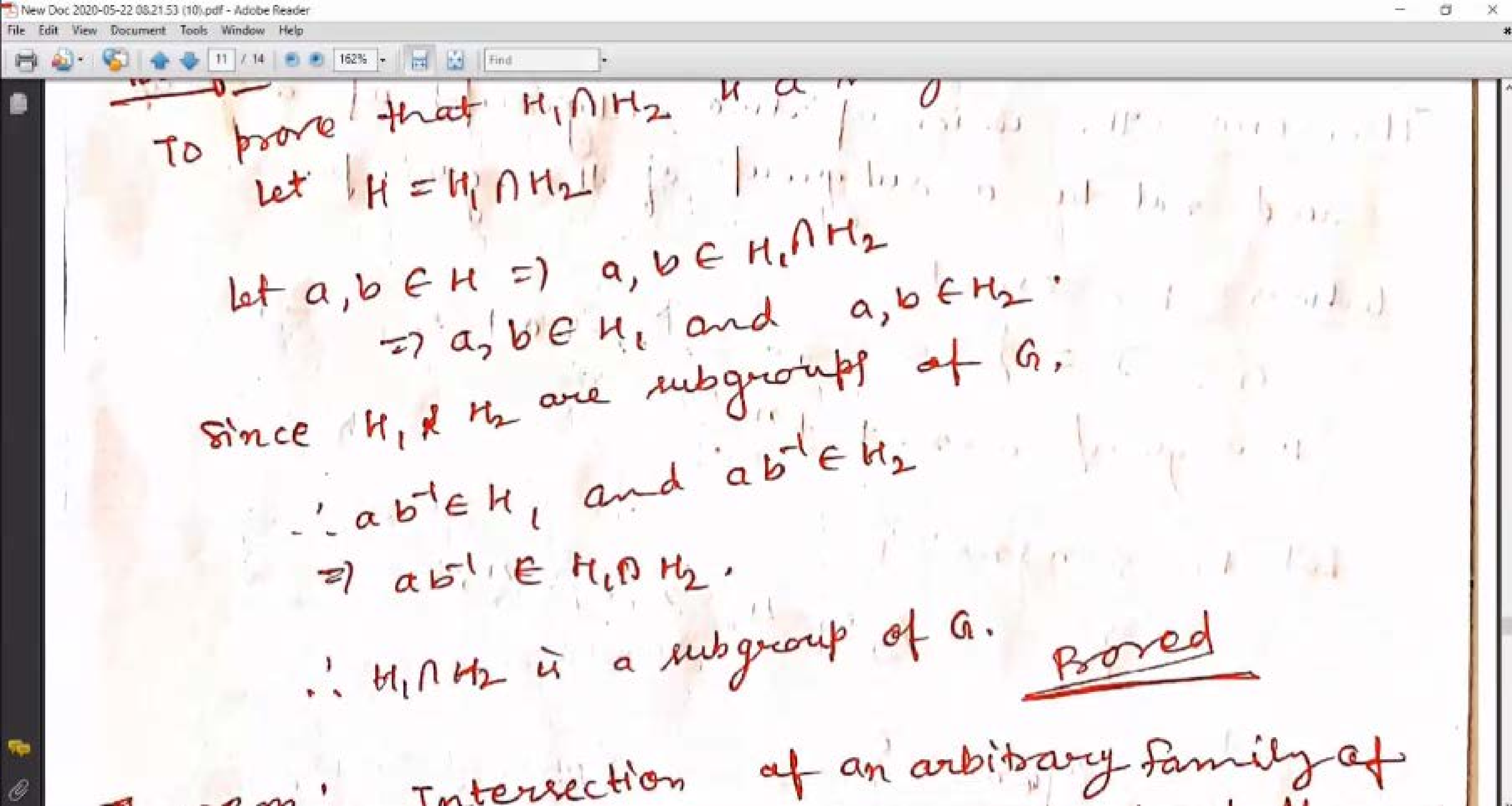
let $H = H_1 \cap H_2$

let $a, b \in H \Rightarrow a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ and $a, b \in H_2$.

Since H_1 & H_2 are subgroups of G,

and $ab^{-1} \in H_2$



To prove that $H_1 \cap H_2$ is a

let $H = H_1 \cap H_2$

let $a, b \in H \Rightarrow a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ and $a, b \in H_2$.

Since H_1 & H_2 are subgroups of G.

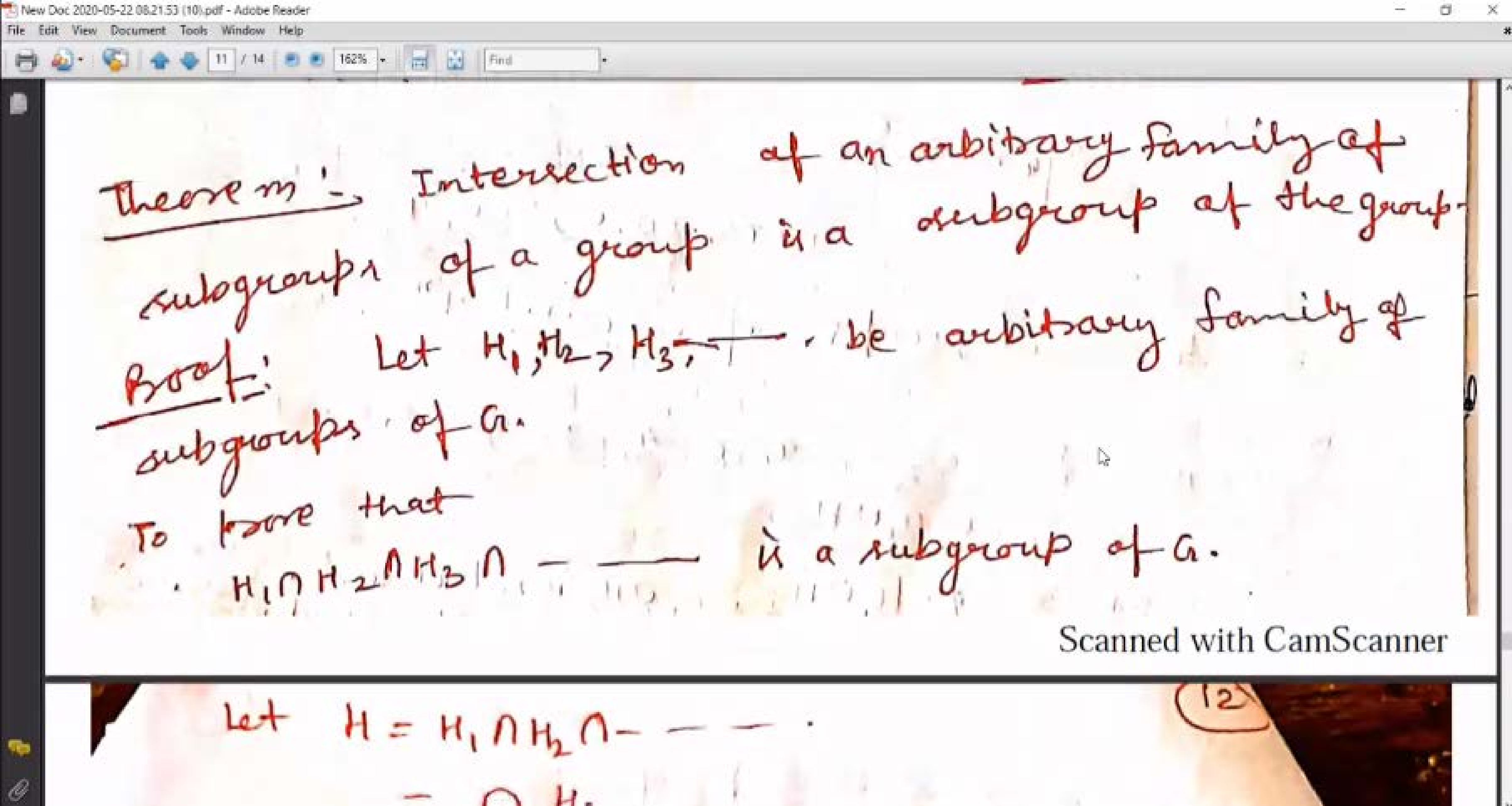
$\therefore ab^{-1} \in H_1$ and $ab^{-1} \in H_2$

$\Rightarrow ab^{-1} \in H_1 \cap H_2$.

$\therefore H_1 \cap H_2$ is a subgroup of G.

Proved

Intersection of an arbitrary family of



Theorem :- Intersection of an arbitary family of subgroups of a group is a subgroup of the group.

Proof :- Let $H_1, H_2, H_3, \text{———}$ be arbitary family of subgroups of G.

To prove that $H_1 \cap H_2 \cap H_3 \cap - \text{———}$ is a subgroup of G.

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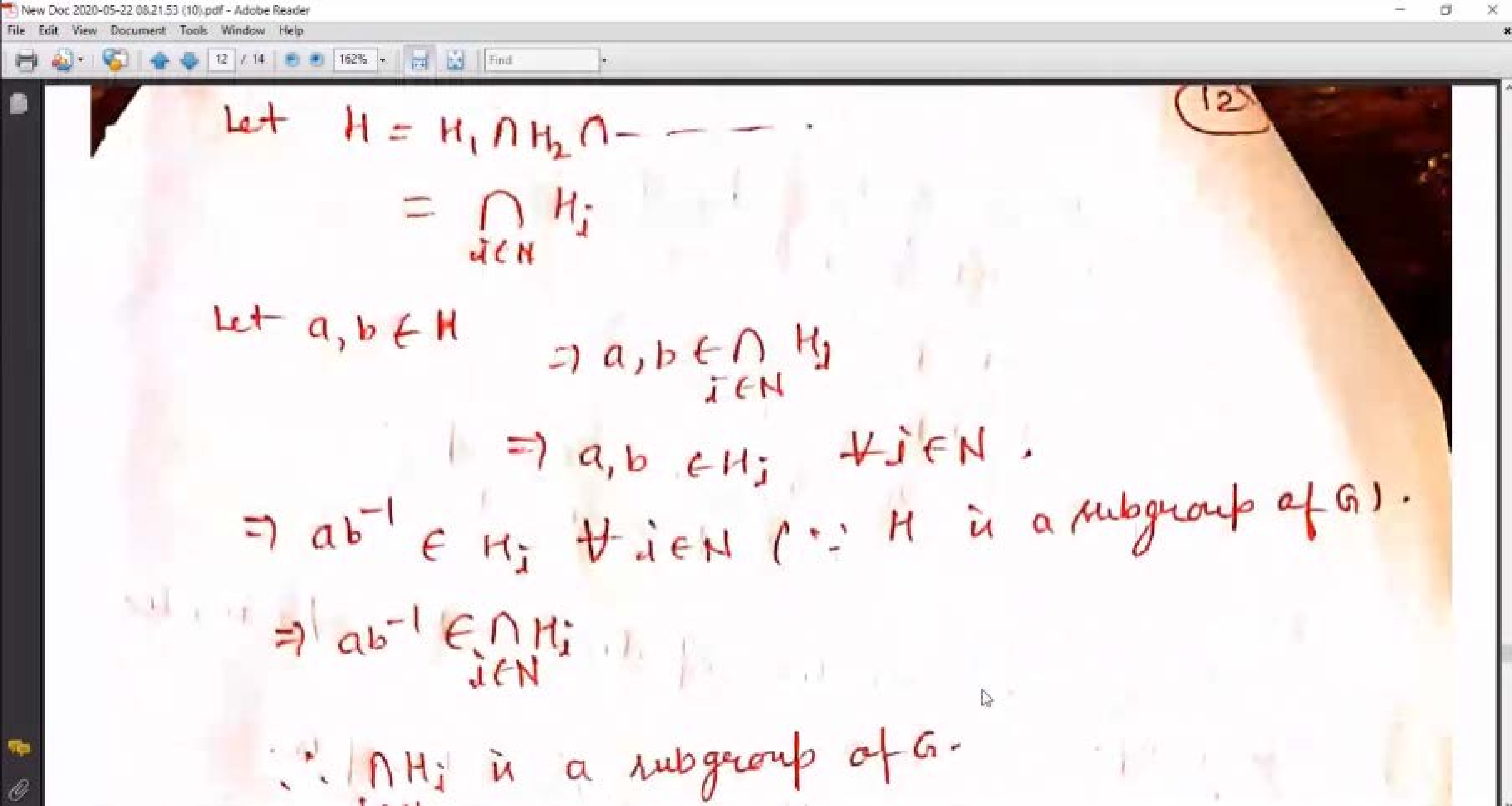
Let $H = H_1 \cap H_2 \cap - - -$.

$= \bigcap H$



Let $H = H_1 \cap H_2 \cap - - - .$

$= \bigcap_{i \in N} H_i$

Let $a, b \in H$

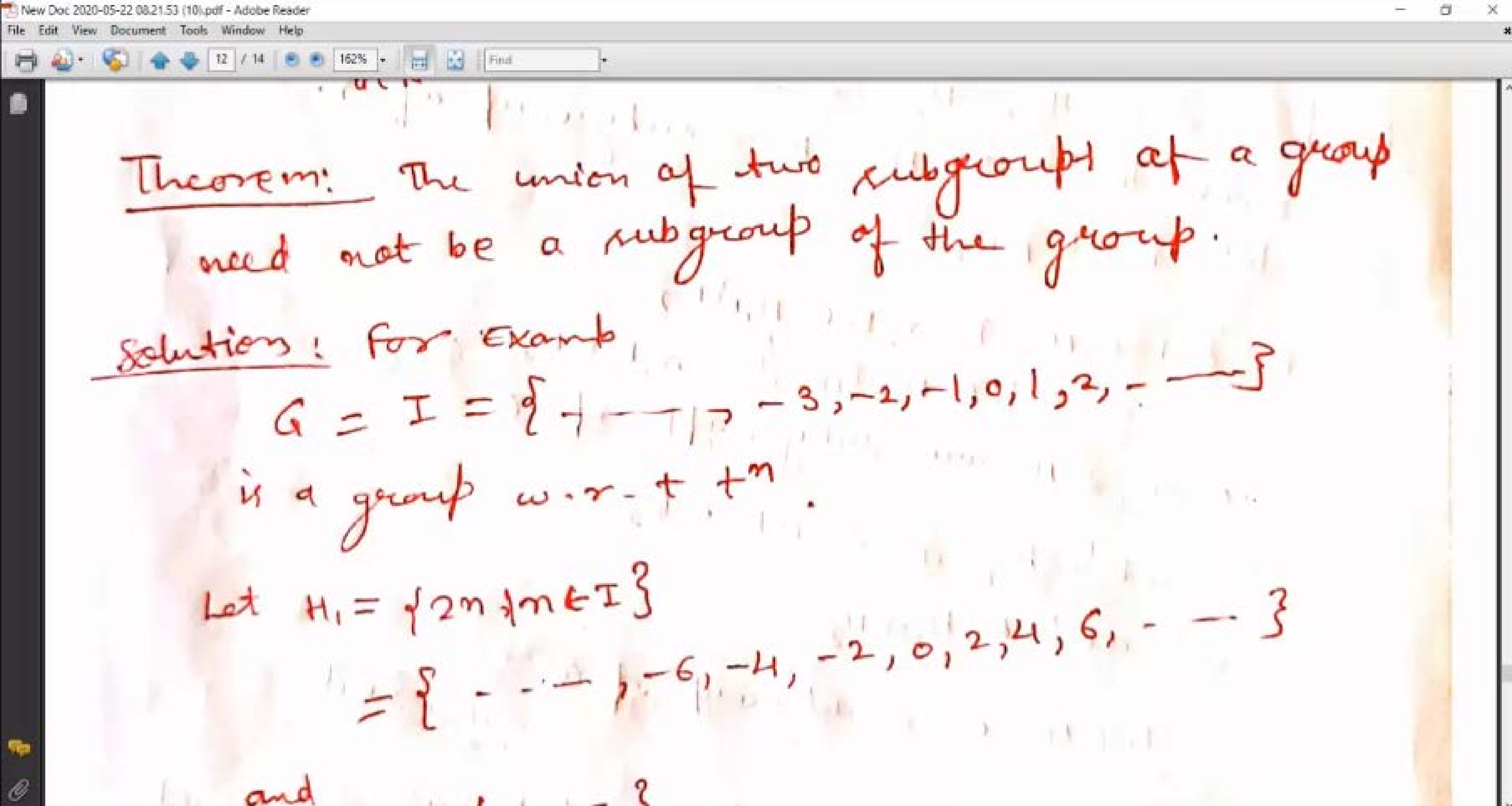
$\Rightarrow a, b \in \bigcap_{i \in N} H_i$

$\Rightarrow a, b \in H_i \quad \forall i \in N.$

$\Rightarrow ab^{-1} \in H_i \quad \forall i \in N$ ($\because$ H is a subgroup of G).

$\Rightarrow ab^{-1} \in \bigcap_{i \in N} H_i$

$\therefore \bigcap H_i$ is a subgroup of G.



Theorem: The union of two subgroups of a group need not be a subgroup of the group.

Solution: For Examb

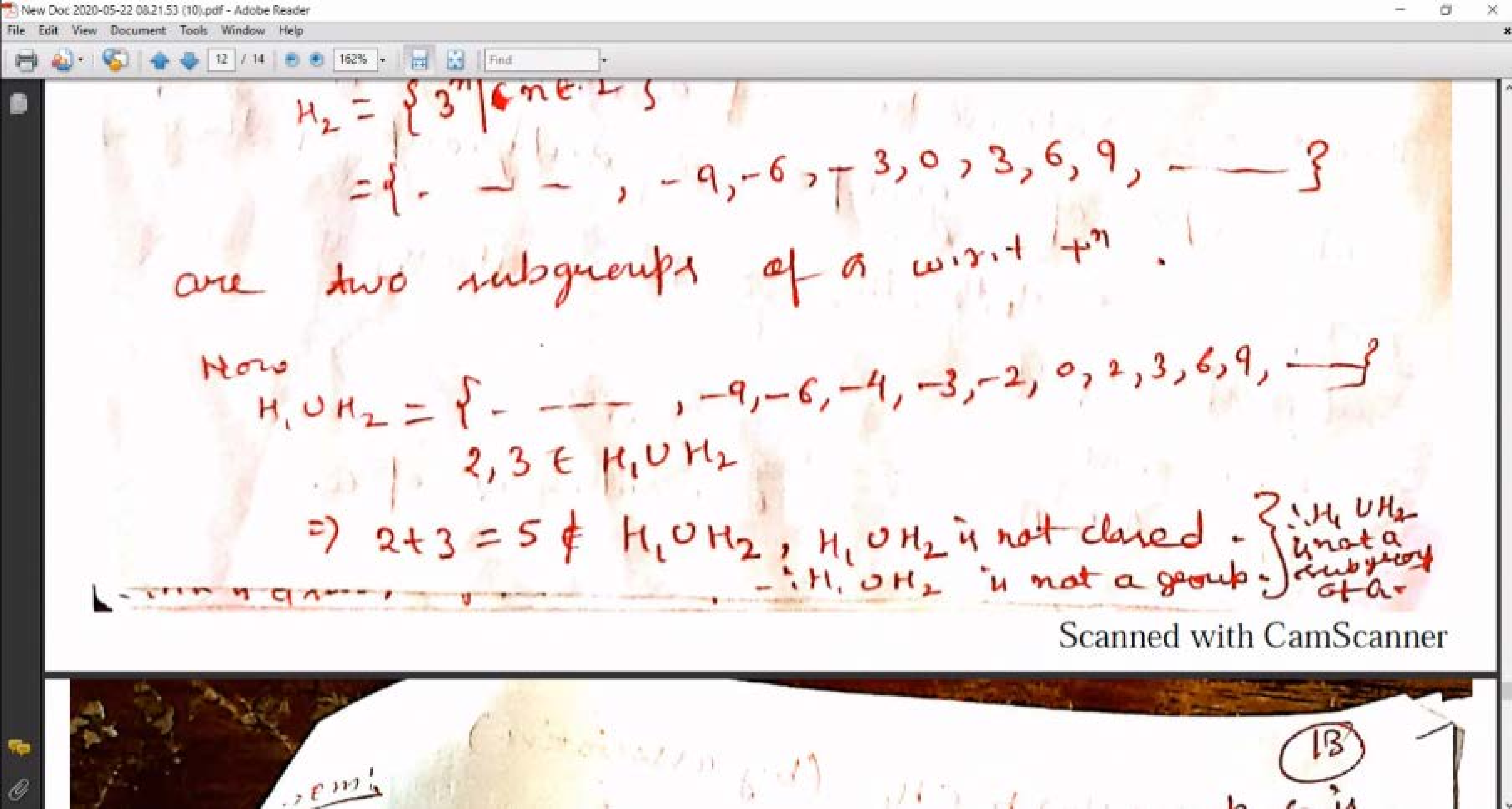
$$G = I = \{ \cdots, -3, -2, -1, 0, 1, 2, \cdots \}$$

is a group w.r.t $+^n$.

Let $H_1 = \{ 2n \mid n \in I \}$

$$= \{ \cdots, -6, -4, -2, 0, 2, 4, 6, \cdots \}$$

and



$H_2 = \{3^n \mid n \in \mathbb{Z}\}$

$= \{. . . , -9, -6, -3, 0, 3, 6, 9, . . . \}$

are two subgroups of a w.r.t $+^n$.

Now

$H_1 \cup H_2 = \{. . . , -9, -6, -4, -3, -2, 0, 2, 3, 6, 9, . . . \}$

$2, 3 \in H_1 \cup H_2$

$\Rightarrow 2+3 = 5 \notin H_1 \cup H_2$, $H_1 \cup H_2$ is not closed.

$\therefore H_1 \cup H_2$ is not a group.

$\left.\right\} \therefore H_1 \cup H_2$ is not a subgroup of a.

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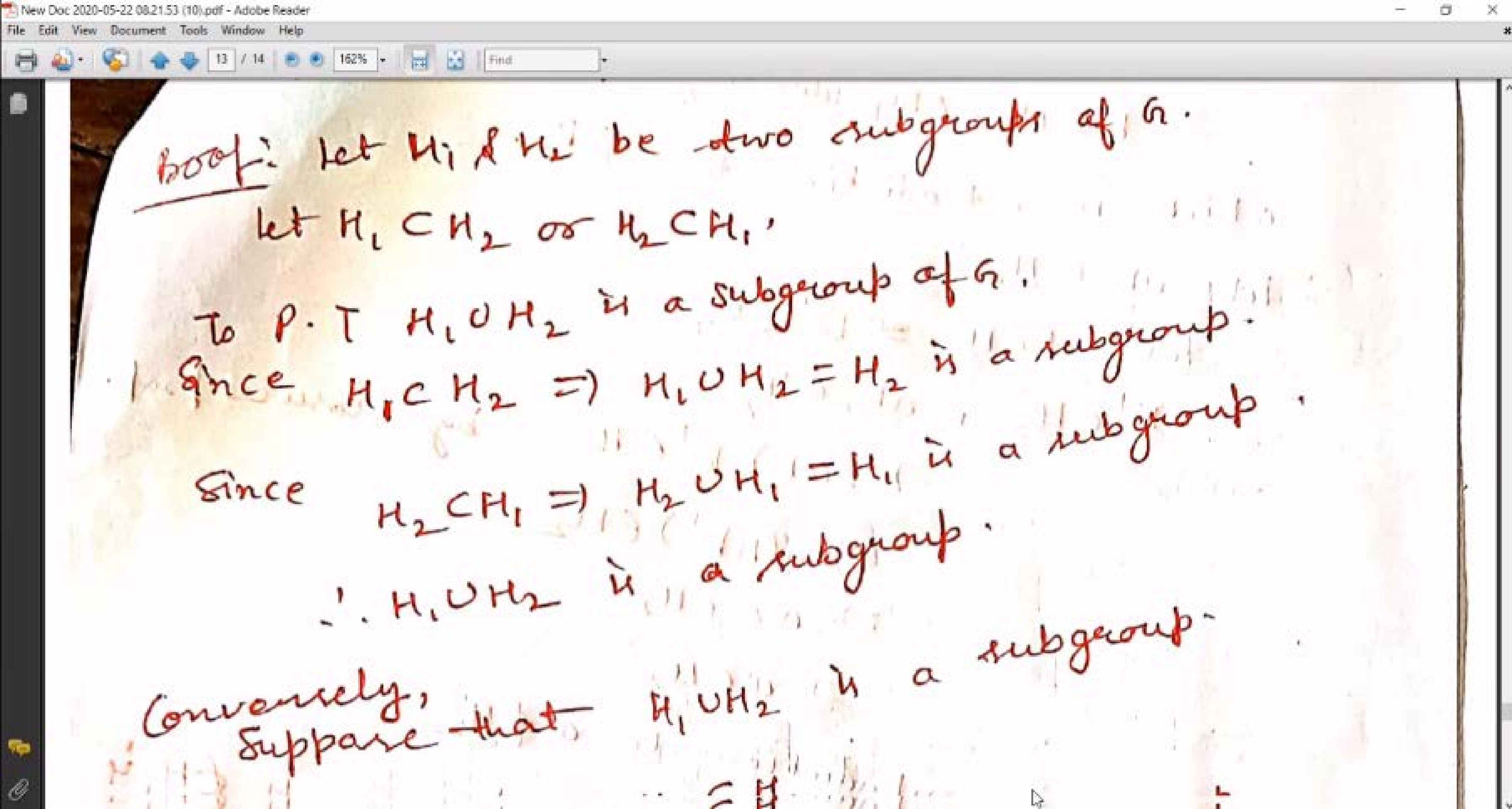


Proof: let H_1 & H_2 be two subgroups of G.

let $H_1 \subset H_2$ or $H_2 \subset H_1$.

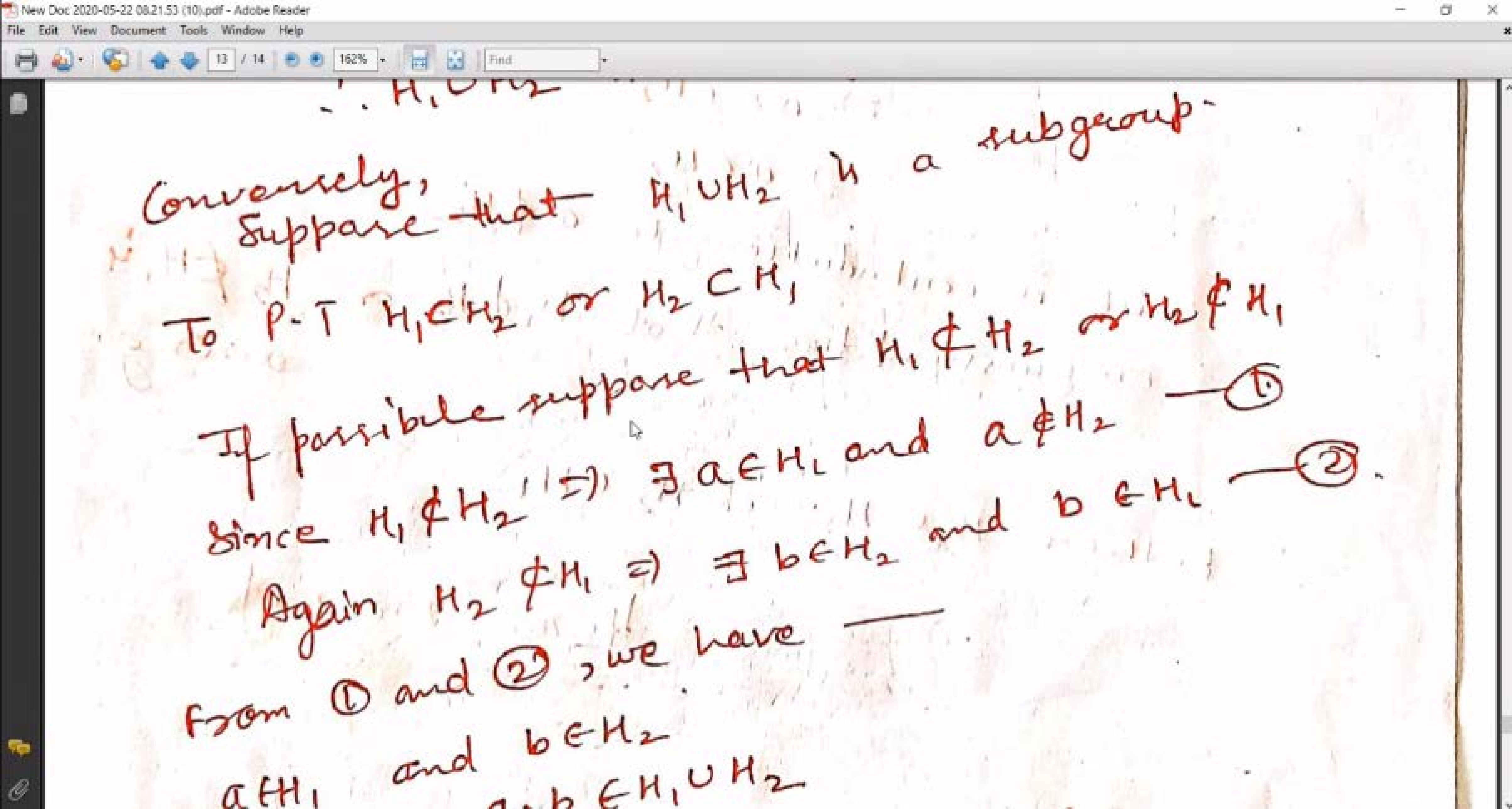
To P.T $H_1 \cup H_2$ is a subgroup of G.

Since $H_1 \subset H_2 \Rightarrow H_1 \cup H_2 = H_2$ is a subgroup.

Since $H_2 \subset H_1 \Rightarrow H_2 \cup H_1 = H_1$ is a subgroup.

$\therefore H_1 \cup H_2$ is a subgroup.

Conversely, suppose ~~that~~ $H_1 \cup H_2$ is a subgroup.



∴ $H_1 \cup H_2$

Conversely, Suppose that $H_1 \cup H_2$ is a subgroup.

To P.T $H_1 \subset H_2$ or $H_2 \subset H_1$

If possible suppose that $H_1 \not\subset H_2$ or $H_2 \not\subset H_1$

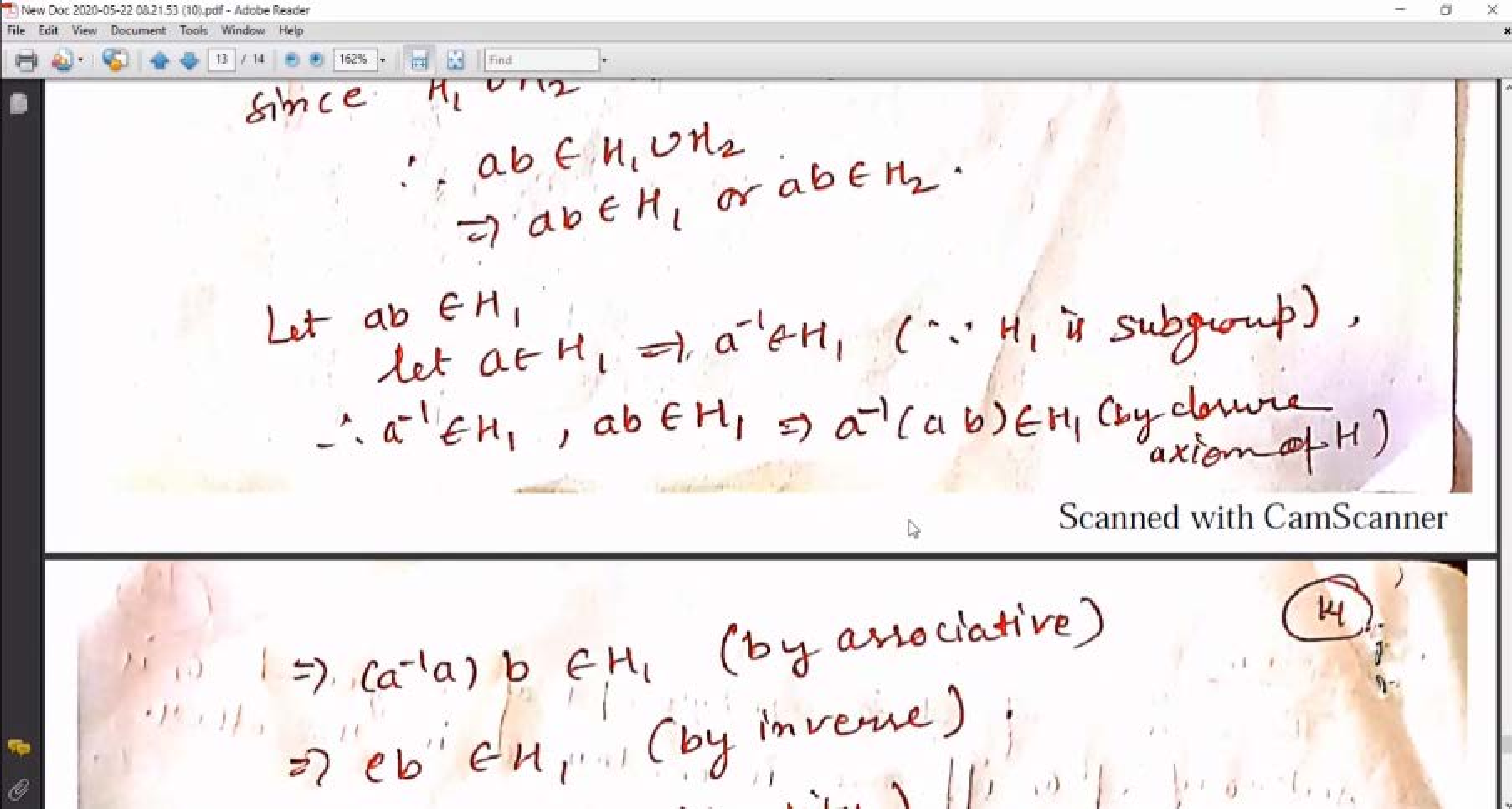
Since $H_1 \not\subset H_2 \Rightarrow \exists\, a \in H_1$ and $a \notin H_2$ — ①

Again $H_2 \not\subset H_1 \Rightarrow \exists\, b \in H_2$ and $b \in H_1$ — ②

From ① and ②, we have —

$a \in H_1$ and $b \in H_2$

$a, b \in H_1 \cup H_2$



Since $H_1 \cup H_2$

$\therefore ab \in H_1 \cup H_2$

$\Rightarrow ab \in H_1$ or $ab \in H_2$.

Let $ab \in H_1$

let $a \in H_1 \Rightarrow a^{-1} \in H_1$ ($\because$ H_1 is subgroup),

$\therefore a^{-1} \in H_1$, $ab \in H_1 \Rightarrow a^{-1}(ab) \in H_1$ (by closure axiom of H)

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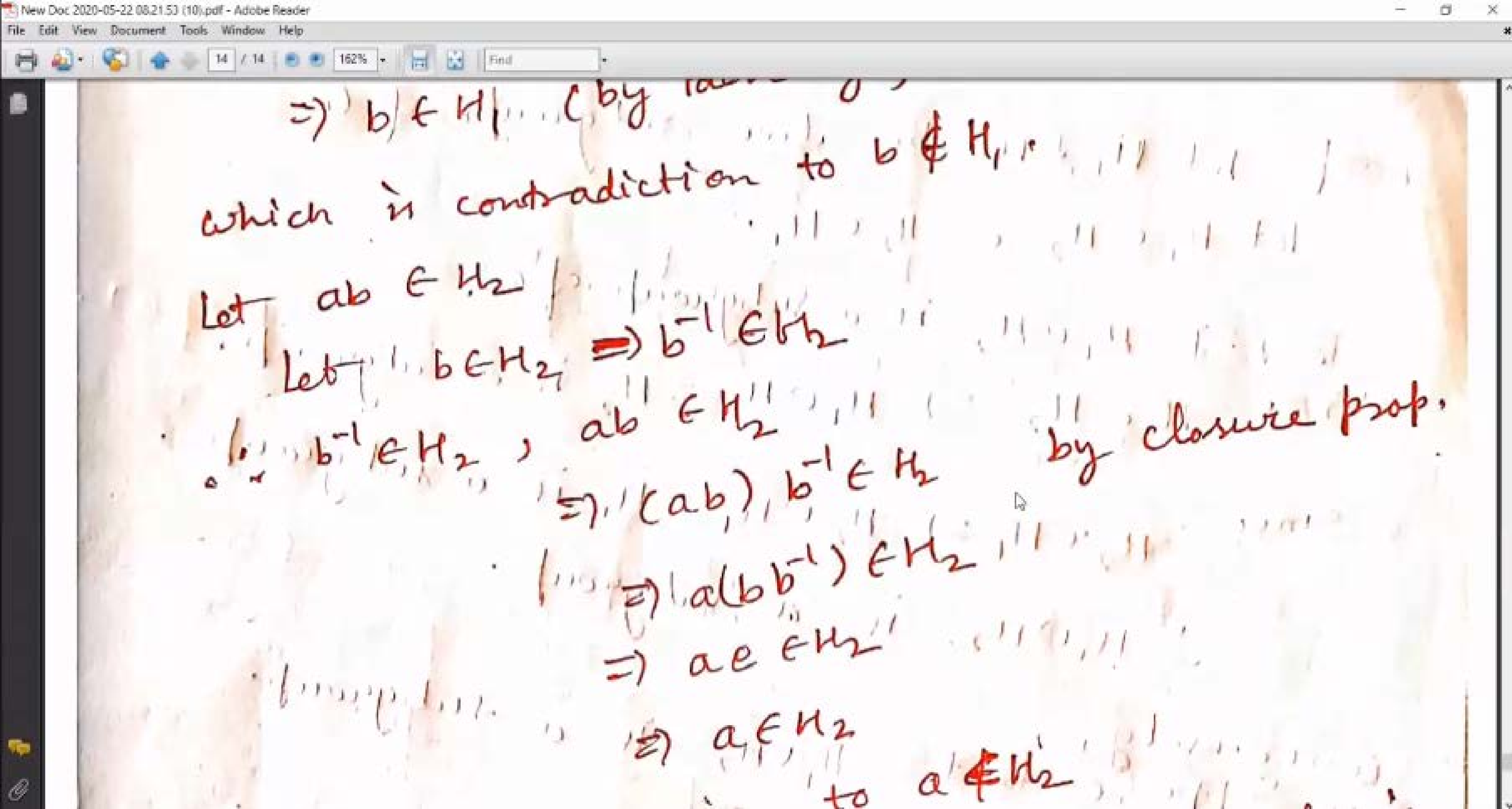
$\Rightarrow (a^{-1}a)\, b \in H_1$ (by associative)

$\Rightarrow eb \in H_1$ (by inverse);



$\Rightarrow b \in H_1$ (by

which is contradiction to $b \notin H_1$.

Let $ab \in H_2$

Let $b \in H_2 \Rightarrow b^{-1} \in H_2$

$\therefore b^{-1} \in H_2$, $ab \in H_2$ by closure prop.

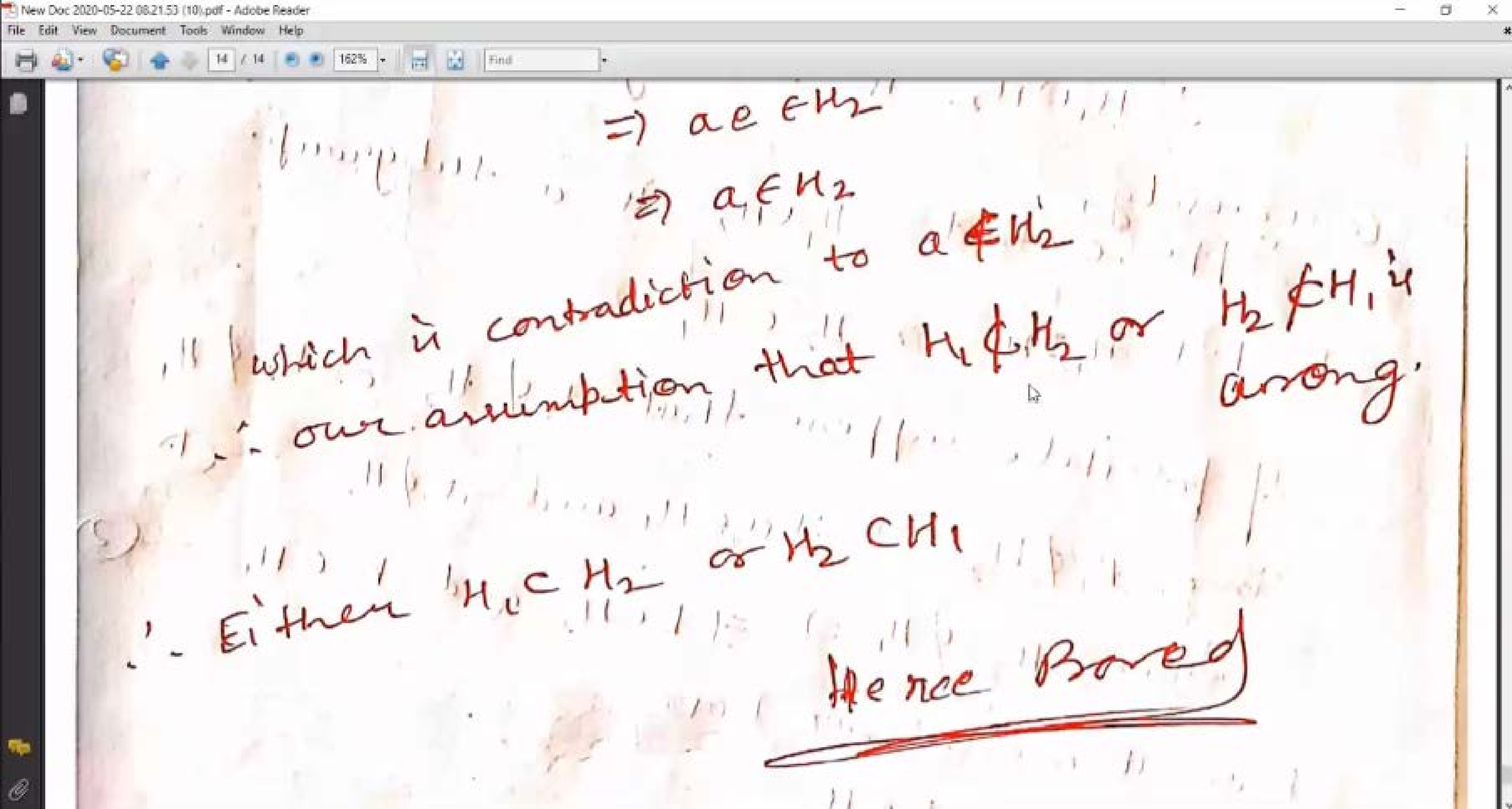
$\Rightarrow (ab)\, b^{-1} \in H_2$

$\Rightarrow a(bb^{-1}) \in H_2$

$\Rightarrow ae \in H_2$

$\Rightarrow a \in H_2$

to $a \notin H_2$



$\Rightarrow a \in H_2$

$\Rightarrow a \in H_2$

which is contradiction to $a \notin H_2$

∴ our assumption that $H_1 \not\subset H_2$ or $H_2 \not\subset H_1$ is wrong.

∴ Either $H_1 \subset H_2$ or $H_2 \subset H_1$

Hence Proved