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Algebraic structure:

G is a non-empty set and $*$ is a b-o on it, G together with the b-o is called an algebraic structure and is denoted by $(G, *)$.

or

A non-empty set equipped with one or more b-o is called an algebraic structure.

Ex: $(N, +)$, $(N, +, \cdot)$, $(I, +, \cdot, --)$ etc.

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is called an algebraic structure.

Ex: $(N, +)$, $(N, +, \cdot)$, $(I, +, \cdot, --)$ etc are algebraic structure.

Groupoid or Quasi group:

An algebraic structure $(G, *)$ is said to be groupoid if it satisfied the closure property.

i.e $\forall\ a, b \in G \Rightarrow a * b \in G$.

Ex: $(N, +)$, $(I, +)$ etc are groupoid.

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Ex: $(N, +)$, ...

Semi Group or Demi group:

If an algebraic structure $(G, *)$ satisfies the closure and associative properties then $(G, *)$ is called semi group.

Ex: $(I, +)$, $(I, \cdot)$ etc are semi-groups, but $(I, -)$ is not a semi group because it is closure but not associative.

$(a * b) * c = a * (b * c)$

1, 2, 3 $\quad (1-2)-3 \neq 1-(2-3)$

Monoid :

A semi-group $(G, *)$ with an identity elem... w.r.t $*$ is known as a monoid.

Ex: A semi group $(I, +)$ is a monoid and the Identity is '0'.

$a * e = a$ $\quad 2 + e = 2$

$2 * e = 2$ $\quad 2 \times 1 = 2$

- A semi group $(I, \cdot)$ is a monoid and the identity element is 1.

- A semi group $(N, +)$ is not monoid ~~and the identity~~ the identity element is 0

Ex: A semi group $(I,+)$ is a mon... (1)

Identity is '0'.

$a * e = a$ $\quad 2 + e = 2$, $e \to 0$

$2 * e = 2$ $\quad 2 \times 1 = 2$

- A semi group $(I,\cdot)$ is a monoid and the identity element is 1. (1)

- A semi group $(N,+)$ is not monoid ~~and the identity is~~ because the identity element is 0 ~~element is 1~~. $0 \notin N$.

$* \rightarrow a * b = \dfrac{a+b-ab}{2ab}$

✓ **Group:** A monoid $(G,*)$ with the inverse element ... group.

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$0 \notin N$.

$* \rightarrow$ 2ab

Group: A monoid $(G, *)$ with the inverse element w.r.t $*$ is known as a group.

or

The algebraic structure $(G, *)$ is said to be a group. if the b-o $*$ on G satisfies the following properties:

1. closure prop: ____ $\forall a, b \in G, \Rightarrow a * b \in G$.

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properties:

(1) closure prop: $\forall a, b \in G, \Rightarrow a * b \in G$.

(2) Associative prop: $\forall a, b, c \in G$

$\Rightarrow (a * b) * c = a * (b * c)$

(3) Existence of Identity:

$\exists e \in G$ s.t $a * e = a = e * a \quad \forall a \in G$.

'e' is called the identity element in G.

$(1-2)-3$ | $1-(2-3)$

$1+1$

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Exintence of Inverse:

for each $a \in G$, $\exists$ $b \in G$, s.t

$$a * b = b * a = e.$$

'b' is inverse of a in G.

$(G, \cdot)$ (3)

$2 \in G$ 1

$2 \cdot \frac{1}{2} = 1$

$3 \cdot \frac{1}{3} = 1$ $\in G$

I;

$2+3 = 3+2$

Abelian or Commutative group: $a*b = b*a$

$2-3 \neq 3-2$

I;

$2+3=3+2$

Abelian or Commutative group:

$a * b = b * a$

$2-3 \neq 3-2$

A group $(G, *)$ which satisfies the commutative property is known as the abelian group.

Otherwise: It is known as non-abelian group.

Finite and Infinite group:

If the number of elements in the group G is finite then the group $(G, *)$ is called a finite

Finite and Infinite group:

If the number of elements in the group G is finite then the group $(G, *)$ is called a finite group, otherwise it is called an infinite group.

Order of a group:

The number of elements in a finite group is called the order of the group. It is denoted by $O(G)$ or

infinite.

the order of the group

$|G|$.

Note 1: The order of infinite group is infinite.

(2) $G = \{e\}$ (i.e the set consisting of identity element e alone). which is a group w.r.t given composition which is known as the smallest group.

8.26 x 11.69 in

Problems: ... where

Problems:

① The algebraic structure $(I, +)$ where $I = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$ is an abelian group.

Note: (i) The set Z^+ under "$+$" is not a group. There is no identity element for "$+$" in Z^+.

(ii) The set of all non-negative integers (including 0) under $+$ is not a group because there is no inverse of $a \in Z$.

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0) under + is not a group

. is no Inverse of $a \in Z$.

(2) The set I_E of all even integers is an abelian group w.r.t $+^n$.

Note! The set I_o of all odd integers is not a group w.r.t $+^n$ because the closure property is not satisfied.

(3) The sets Q, $\mathbb{R}$ and C of all rational, real and complex numbers are abelian groups.

as left the meeting

Note: The ... group w.r.t $+^n$ because the ... is not satisfied.

(3) The sets $\mathbb{Q}$, $\mathbb{R}$ and C of all rational, real and complex numbers are abelian groups under ($+^n$)

(4) G = The set of $m \times n$ matrices is an abelian group w.r.t b-o $+^n$.

... vectors is an abelian group

(3) The sets $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of all ... and complex numbers are abelian groups under '$+$'.

(4) G = The set of $m \times n$ matrices is an abelian group w.r.t b-o '$+$'.

(5) G = The set of vectors is an abelian group w.r.t b-o $+$.

6) The set $G = \{ \ldots, -3m, -2m, -m, 0, m, 2m, 3m, \ldots \}$ of multiple of integers by fixed integer m is an abelian group w.r.t "+".

7) The set N under "$\times$" is not a group because there is no inverse of $a \in N$.

real and

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(7) The set N under $\times^n$ is not a group because there is no inverse of $a \in N$.

(8) The sets Q, R, C of all rational, real and complex numbers are not groups w.r.t $\times^n$. because the inverse of 0 is not defined.

(9) The sets Q^+ and R^+ of all +ive rational and real numbers are abelian groups under $\times^n$.

(10) The sets Q^*, R^* and C^* of all non-zero rational ... abelian groups.

and real numbers ...

(10) The sets Q^*, R^* and C^* of all non-zero rational real and complex numbers are abelian groups w.r.t X^n.

(11) Is the set of all rational number x s.t $0 < x \leq 1$, a group w.r.t X.

Sol^n: Let $G = \{x \mid x$ is a rational number and $0 < x \leq$

then it is not a group under X^n because if $a \in G$ and $0 < a < 1$ then inverse of 'a' is not

(ii) Is the set of all rational numbers x s.t $0 < x \leq 1$, a group w.r.t $\times$.

Sol^n: Let $G = \{x \mid x$ is a rational number and $0 < x \leq 1\}$ then it is not a group under "$\times$" because if $a \in G$ and $0 < a < 1$ then inverse of 'a' is not possible in G.

Ex: Let $a = \frac{1}{5} \in G$ then the inverse of $\frac{1}{5}$ is $5 \notin G$.

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(12) The set of all +ive rational numbers form an abelian group under the composition $*$ defined by $a * b = \frac{ab}{2}$.

Elementary properties of Groups:

If G is a group with $b-0$

then the left and right cancellation laws holding

Elementary properties of Groups:

If G is a group with b-o then the left and right cancellation laws hold in G

i.e $\forall a, b, c \in G$ (i) $ab = ac$.

$\Rightarrow b = c$ (L.C.L)

and (ii) $ba = ca$.

$\Rightarrow b = c$ (R.C.L)

Proof: Given that

w.r.t b-o.

Proof : Given that

G is a group w.r.t b-o.

for each $a \in G$ $\exists$ $a^{-1} \in G$ s.t.

$a^{-1}a = a.a^{-1} = e$ (where 'e' is identity)

Now suppose $a \cdot b = a \cdot c$

Multiplying both sides a^{-1} on left.

$a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$ (Associative Property)

Multiplying both sides a^{-1} on left.

$$a^{-1}(ab) = a^{-1}(ac)$$

$$\Rightarrow (a^{-1}a)b = (a^{-1}a)c \quad \text{(Associative Property)}$$

$$\Rightarrow eb = ec$$

$$\Rightarrow b = c \quad \text{(Identity)}.$$

Similarly $b \cdot a = c \cdot a$.

$$\Rightarrow \boxed{b = c}$$

If G is a group with b-o '+' then the (7)

Note: If G is a group with b-o '+' then the left and right cancellation laws hold in G. (7)

i.e $a+b = a+c \Rightarrow b=c$ (L.C.L)

and $b+a = c+a \Rightarrow b=c$ (R.C.L).

$\forall\ a,b,c \in G$

$\Rightarrow$ If G is a group with b-o and a & b are elements of G, then the linear eqns $ax=b$ and $ya=b$ have unique soln x and y in G.

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i.e $a+b=a+c \Rightarrow b=c$ (L.C.L)

and $b+a=c+a \Rightarrow b=c$ (R.C.L).

$\forall\, a,b,c \in G$'

$\Rightarrow$ If G is a group with b-o and a & b are elements of G, then the linear eq^{n}s $ax=b$ and $ya=b$ have unique sol^n x and y in G.

Proof:- Given that G is a group w.r.t b-o '.'

$\exists\, a^{-1} \in G$ s.t $aa^{-1}=a^{-1}a=e$.

where 'e' is identity,

Proof :- Given that G is a group

for each $a \in G$ $\exists$ $a^{-1} \in G$ s.t $aa^{-1} = a^{-1}a = e$, where 'e' is identity.

Now, we have —

$$ax = b$$

multiplying both sides a^{-1} on left

we get

$$a^{-1}(ax) = a^{-1}b$$

$\Rightarrow (a^{-1}a)x = a^{-1}b$ (by Associative prop)

$\Rightarrow ex = a^{-1}b$ (by inverse)

$\Rightarrow x = a^{-1}b$. (by identity)

... $b \in G$.

we get ... $= a^{-1}b$. (by ...

$$\Rightarrow (a^{-1}a)x = a^{-1}b \quad \text{(by inverse)}$$

$$\Rightarrow ex = a^{-1}b \quad \text{(by identity)}$$

$$\Rightarrow x = a^{-1}b. \quad \text{(by identity)}$$

Now $a \in G$, $b \in G$, $\Rightarrow a^{-1} \in G$, $b \in G$.

$$\Rightarrow a^{-1}b \in G.$$

Now substituting $a^{-1}b$ for x in the left hand side of the eq^n $ax = b$.

we have

$$a(a^{-1}b) = (aa^{-1})b = eb = b$$

$\therefore$ $x = a^{-1}b$ is the sol^n in G of the $ax = b$.

To show that the sol^n is unique.

Now, if possible suppose that

$x = x_1$ and $x = x_2$ are two sol^n of the eq^n

$ax = b$ then $ax_1 = b$, $ax_2 = b$.

$\therefore ax_1 = ax_2 \Rightarrow x_1 = x_2$ (by L.C.L.)

$\therefore$ The sol^n is unique.

Similarly we prove that $ya = b$ has unique sol^n.

and

Note: If G is a group with the b-o $+$ and a & b are two elements of G then the linear equations $a+x=b$ and $y+a=b$ have unique sol^n x and y in G.

Note $\boxed{1}$: cancellation laws hold in a group i.e

$\forall\ a, b, c \in G$.

(i) $ab = ac \Rightarrow b = c$ (LCL)

(ii) $ba = ca \Rightarrow b = c$ (RCL).

(ii) $ba = ca \Rightarrow b = c$ (RCL).

$\boxed{2}$ In a semi group the cancellation laws may or may not hold.

Ex: Let S be the set of all 2×2 matrices with their elements as integers and '$\times$' is b-o on 'S' then 'S' is a semi group but not satisfy the cancellations laws because if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

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their elements, ...

then 'S' is a semi group but not satisfy the cancellations laws because if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

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then $A, B, C \in S$ and $AB = AC$ but $B \neq C$

$\therefore$ left cancellation law is not true in the semi group.

$\boxed{3}$ $(N, +)$ is a semi group.

for $a, b, c \in N$. $a+b = a+c$
and $b+a = c+a$
$\Rightarrow b = c$

But $(N, +)$ is not a group.

$\therefore$ In a semi group even if cancellation laws hold the semi group is not a group.

$\boxed{4}$ A finite semi group $(G, \cdot)$ satisfy the

the semi group is not a group.

$\boxed{4}$ A finite semi group $(G, \cdot)$ satisfy the cancellation laws is a group.

(or)

A finite set G with a binary operation $(\cdot)$ is a group if $(\cdot)$ is associative and cancellation

$\boxed{4}$ A fin...

cancellation laws is a group.

(or)

A finite set G with a binary operation '$\cdot$' is a group if '$\cdot$' is associative and cancellation laws hold in G.

uniqueness of identity:

The identity element in a group is unique (is)

Proof: Let $(G, \cdot)$ be the given group. If possible suppose that e_1 & e_2 are two identity elements in G.

Since e_1 is an identity in G then

$$e_1 e_2 = e_2 = e_2 e_1 \quad \text{--- (1)}$$

Since e_2 is identity in G then $e_1 e_2 = e_1 = e_2 e_1$ --- (2)

From (1) and (2), we have ———

suppose ... in G then

Since e_1 is an identity

$$e_1 e_2 = e_2 = e_2 e_1 \quad \text{--- (1)}$$

Since e_2 is identity in G then $e_1 e_2 = e_1 = e_2 e_1$ --- (2)

From (1) and (2), we have —

$$e_1 = e_1 e_2 = e_2$$

$$\Rightarrow \boxed{e_1 = e_2}$$

Uniqueness of Inverse:

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Uniqueness of Inverse:

Inverse of each element of a group is unique.

Proof: Let $(G, \cdot)$ be the given group.

Now suppose that $a \in G$ has two inverses a' and a''

& a''

of a in G.

Since a' is an inverse of a in G.

$$\therefore aa' = a'a = e \quad \text{--- (1)}$$

Since a'' is an inverse of a in G.

(2)

Now suppose that we ...

Since a' is an inverse of a in G.

$$\therefore aa' = a'a = e \quad \text{---} \; (1)$$

Since a'' is an inverse of a in G.

$$\therefore aa'' = a''a = e \quad \text{---} \; (2)$$

From (1) and (2), we have —

$$aa' = e = aa'' \Rightarrow aa' = aa''$$

$$\Rightarrow a' = a'' \text{ (By LCL)}$$

$\therefore$ ... is unique.

Note : The identity element is its own inverse 11

Since $ee = e \Rightarrow e^{-1} = e$.

$\Rightarrow$ If the inverse of a is a^{-1} then inverse of a^{-1} is a i.e $(a^{-1})^{-1} = a$,

Proof : Let $(G, \cdot)$ be the given group.

$\exists\ a^{-1} \in G$ such that

Note :-

Since $ee = e \Rightarrow e^{-1} = e$.

$\Rightarrow$ If the inverse of a is a^{-1} then inverse of a^{-1} is a i.e $(a^{-1})^{-1} = a$.

Proof :- Let $(G, \cdot)$ be the given group.

for each $a \in G$, $\exists\, a^{-1} \in G$ such that

$aa^{-1} = a^{-1}a = e$.

Now $aa^{-1} = e$

multiplying both sides with $(a^{-1})^{-1}$ on the right —

$$(aa^{-1})\cdot(a^{-1})^{-1} = e(a^{-1})^{-1}$$

$\Rightarrow a(a^{-1}(a^{-1})^{-1}) = (a^{-1})^{-1}$ [by associative and e is identity]

$\Rightarrow a(e) = (a^{-1})^{-1}$ [$\because (a^{-1})^{-1}$ is inverse of a^{-1}]

$\Rightarrow a = (a^{-1})^{-1}$ [$\because$ e is identity]

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$$\Rightarrow a = (a^{-1})^{-1} \qquad [\because e \text{ is identity}]$$

$$\Rightarrow (a^{-1})^{-1} = a.$$

Note: If $(G,+)$ is a group and inverse of a is $-a$ then, inverse of $-a$ is a

i.e $-(-a) = a$.

group:

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Let $(G, \cdot)$ be a group.

P.T $(ab)^{-1} = b^{-1}a^{-1}$ $\forall a, b \in G$.

Proof:- Given that $(G, \cdot)$ is a group.

for each $a \in G$, $\exists a^{-1} \in G$ such that

$aa^{-1} = a^{-1}a = e$ and

for $b \in G$, $\exists b^{-1} \in G \Rightarrow ab \in G$

$a^{-1} \in G$, $b^{-1} \in G \Rightarrow b^{-1}a^{-1} \in G$.

$$aa^{-1} = a^{-1}a = e \quad \text{and}$$

for $b \in G$, $\exists\, b^{-1} \in G \Rightarrow ab \in G$

$$a^{-1} \in G,\ b^{-1} \in G \Rightarrow b^{-1}a^{-1} \in G.$$

Now we have

$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$ (by Associative).

$= aea^{-1}$ by inverse.

$= aa^{-1}$ by identity.

$= e$ by inverse.

$\therefore (ab)(b^{-1}a^{-1})$

Now, we have —

$$(b^{-1}a^{-1})(ab) = e \quad \text{—— (2)}$$

from (1) and (2), we have —

$$(ab)(b^{-1}a^{-1}) = (b^{-1}a^{-1})(ab) = e$$

$\therefore$ The inverse of ab is $b^{-1}a^{-1}$.

i.e. $(ab)^{-1} = b^{-1}a^{-1}$.

Note:

1. Let $(G, +)$ be a group then
$$-(a+b) = (-b) + (-a)$$

2. Generalization ——

$$(a_1, a_2, a_3, \ldots, a_n)^{-1} = a_n^{-1}\, a_{n-1}^{-1} \ldots\, a_2^{-1} a_1^{-1}.$$

Def^n of a group based upon left Axioms (or) Right Axioms:

The algebraic structure $(G, \cdot)$ is said to be group ... following

Def^n of a group based upon Left Axioms (or) Right Axioms:

The algebraic structure $(G, \cdot)$ is said to be group if the binary operation '$\cdot$' satisfies the following properties:

(i) Closure property:

$ab \in G, \ \forall a, b \in G.$

(i) Closure property :

$ab \in G \;,\; \forall a, b \in G.$

(ii) Associative property :

$(ab)c = a(bc)$

$\forall a, b, c \in G.$

(iii) Existence of left identity :

$\exists\, e \in G$ such that $ea = a \;\; \forall a \in G,$

$\therefore$ the element 'e' is called left identity in G.

(iv) Existence of left inverse :

(iii) ...

$\exists\, e \in G$ such that $ea = a \ \forall\, a \in G$,

$\therefore$ the element 'e' is called left identity in G.

(iv) Existence of left inverse :

for each $a \in G$, $\exists\, a^{-1} \in G$ such that

$a^{-1}a$

$$aa^{-1} = e.$$

$\therefore$ The element a^{-1} is called the left inverse of a in G.

Theorem :

... also the right identity ...

Theorem:

The left identity is also the right identity

if 'e' is the left identity then $ae = a \ \forall\, a \in G$

Proof: Let $(G, \cdot)$ be the given group.

and let e be the left identity i.e. $ea = a$

To prove that 'e' is also the right identity.

Let $a \in G$ and e be the left identity then 'a' has the left inverse in G.

if 'e' is the left identity then $ae = a$ ∀ $a \in G$

Proof :- Let $(G, \cdot)$ be the given group.
and let e be the left identity i.e. $ea = a$

To prove that 'e' is also the right identity.

Let $a \in G$ and e be the left identity then
'a' has the left inverse in G.

$\therefore a^{-1}a = e$.

Associative

Let $a \in G$ and e be the

'a' has the left inverse in G.

$\therefore a^{-1}a = e.$

Now we have $a^{-1}(ae) = (a^{-1}a) \cdot e$ (by Associative

$= e \cdot e$ (by inverse)

$= e$ (by identity)

i.e e is left identity

$= a^{-1}a \quad (\because a^{-1}a = e)$

i.e e is left identity

$= a^{-1}a \quad (\because a^{-1}a = e)$

$\therefore a^{-1}(ae) = a^{-1}a$ (by LCL in G).

$\Rightarrow ae = a$

$\therefore$ If e is the left identity then e is also right identity.

Theorem: The left inverse is also right inverse, i.e, if a^{-1} is the left inverse of a then also $aa^{-1} = e$.

Proof: Let $(G, \cdot)$ be the given group.

Let $a \in G$ and e be the left identity in G.

Let a^{-1} be the left inverse of a then $a^{-1}a = e$.

To prove that $aa^{-1} = e$.

Now, we have

Let a be the

To prove that $aa^{-1} = e$.

Now, we have

$a^{-1}(aa^{-1}) = (a^{-1}a)\,a^{-1}$ (by Associative).

$= ea^{-1}$ (by inverse)

$= a^{-1}$ ($\because e$ is the left identity)

$= a^{-1}e$ ($\because e$ is also right identity)

$a^{-1}(aa^{-1}) = a^{-1}e$.

$\Rightarrow aa^{-1} = e$ (by LCL).

$aa^{-1} = e$.

$\Rightarrow aa^{-1} = e$

$\therefore$ If $a^{-1}a = e$ then $aa^{-1} = e$.

<u>Note I,</u> we cannot assume ~~that~~ the existence of left identity and the existence of ~~left~~ right inverse or we cannot assume the existence of right identity and the existence of left inverse.

Problems :

(1) Show that the set $G = \{a + b\sqrt{2} \,/\, a, b \in Q\}$ is a group w.r.t $+^n$.

(2) P.T the set of all $m \times n$ matrices having their elements as integers is an infinite abelian group w.r.t $+^n$ of matrices.

(3) show that the set of all $m \times n$ non-singular matrices having their elements as rational

(3) show that the set of all $m \times n$ non-singular matrices having their elements as rational (real or complex) numbers is an infinite non-abelian group w.r.t matrix multiplication.

Examine if abelian.

Sol^n: Let M be the set of all $m \times n$ non-singular matrices with their elements as rational numbers.

(i) Closure Property:

$|B| \neq 0$ then

singular matrices with their elements numbers.

(i) Closure Property:

Let $A, B \in M$; $|A| \neq 0$, $|B| \neq 0$ then

$AB \in M$ $(\because |AB| = |A||B|)$

Here $|AB| \neq 0$

because $|A| \neq 0, |B| \neq 0)$

(ii) Associative property:

Matrices multiplication is associative.

Here $|AB| \neq 0$
because $|A| \neq 0, |B| \neq 0$)

(ii) Associative property :
matrices multiplication is associative.

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). Existence of left Identity :

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iii). Existence of left Identity:

$\forall A \in M$, $\exists B = I_{n\times n} \in M$, $|I| = 1 \neq 0$

$|A| \neq 0$

Such that $IA = A$

$\therefore B = I_{n\times n}$ is the left identity in M.

(IV) Existence of left Inverse:

$\therefore B = I_{n\times n}$ is the left identity in M.

(iv) Existence of left Inverse:

for each $A_{n\times n} \in M$; $|A| \neq 0$ $\exists\, A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$

$(\because |A| \neq 0)$

such that

$$A^{-1}A = I_{n\times n} \quad \text{(left Identity)}$$

$\therefore A^{-1}$ is the left inverse of A in M with their elements as rational.

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$\therefore A^{-1}$ is the left inverse of A in M with their elements as rational.

(V) Commutative Property:—

$\forall A, B \in M$; $|A| \neq 0$, $|B| \neq 0$ $\Rightarrow AB \neq BA$.

$\therefore (M, \cdot)$ is not an abelian group.

Note:

M is the set of all $n \times n$ non-singular matr with their elements as integers is not a group $\times$" because there is no inverse of all m in the given set.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$; $|A| = -4 \neq 0$

$$\therefore A^{-1} = \frac{\operatorname{adj} A}{|A|} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\operatorname{adj} A}{|A|} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

Now, we have ——

$$A^{-1}A = \begin{bmatrix} -1/2 & \frac{1}{2} \\ 3/4 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

But $A^{-1} \notin M$ because the elements of this matrix are not integers.

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S.T the set of matrices. (19)

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$ where α is a real number forms a group under matrix multiplication.

Sol^n: Let $G = \{A_\alpha \mid \alpha \in \mathbb{R}\}$ and '.' is b-o.

(i) Let $A_\alpha, A_\beta \in G \Rightarrow A_\alpha A_\beta = A_{\alpha+\beta} \in G$.

Closure prop. where $\alpha, \beta \in \mathbb{R}$.

where $\alpha + \beta \in \mathbb{R}$

(1) Let $A_\alpha, A_\beta \in$ ...

Closure prop. where $\alpha, \beta \in \mathbb{R}$.

where $\alpha + \beta \in \mathbb{R}$

Since

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta}$$

prop. is satisfied.

(ii) Associative Prop:-
Matrix multiplication is associative.

(iii) Existence of left Identity.

Since $0 \in \mathbb{R}$

$$\therefore A_0 = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G.$$

(iii) Existence of left Identity.

Since $0 \in \mathbb{R}$

$$\therefore A_0 = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G.$$

Let $A_\alpha \in G$, $\alpha \in \mathbb{R}$ $\exists$ $A_0 \in G$; $0 \in \mathbb{R}$.

such that $A_0 A_\alpha = A_{0+\alpha} = A_\alpha$.

(iv) Existence of left inverse:

Since $\alpha \in \mathbb{R} \Rightarrow -\alpha \in \mathbb{R}$,

$$\therefore A_\alpha \in G \Rightarrow A_{-\alpha} \in G$$

Now

$$A_{(-\alpha)} A_\alpha = A_{-\alpha+\alpha} = A_0 \text{ (left Identity)}.$$

$\therefore A_{(-\alpha)}$ is the left inverse of A_α.

$\therefore$ Each element of G possesses left inverse.

$\therefore G$ is a group under "$\times$".

$\therefore G$ is a group

Note: The sets of all $n \times n$ matrices with the elements as rational, real, complex numbers are not groups. w.r.t matrix multiplication. because the $n \times n$ matrix with entries '0' has no inverse

$\Rightarrow$ S.T $G = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \middle/ a \text{ is any non-zero real number} \right\}$

is a commutative group w.r.t $\times^n$.

are ... $n \times n$ matrix ...
because ...
no inverse.

$\Rightarrow$ S.T $G = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} / a \text{ is any non-zero real number} \right\}$

is a commutative group w.r.t $\times^n$.

$\Rightarrow$ S.T the set $G = \{ x / x = 2^a 3^b \text{ and } a, b \in Z \}$
is a group w.r.t $\times^n$.

...blems

Do the following sets form groups w.r.t the b-o $*$ on them as follows:

(i) The set I of all integers with operation defined by $a * b = a+b+1$.

(ii) The set Q of all rational numbers other than 1 (i.e $Q - \{1\}$) with operation defined by

$a * b = a+b-ab$

... operation

(i) The set I of all integers with

defined by $a * b = a+b+1$.

(ii) The set Q of all rational numbers other than 1

(i.e $Q-\{1\}$) with operation defined by

$$a * b = a+b-ab$$

(iii) The set I of all integers with the operation

defined by $a * b = a+b+2$

$\forall a, b \in Q-\{1\}$.

$a+b-ab$

defined by

$\forall a, b \in Q - \{1\}$.

solution:

(ii) since $a * b = a + b - ab$ ⟵ (A)

(1) closure prop.

Let $a, b \in Q - \{1\}$

$a * b = a + b - ab \in Q - \{1\}$ (by (A))

$\therefore$ $Q - \{1\}$ satisfies closure prop. w.r.t $*$.

(2) Associative property:

$\ldots \in Q - \{1\}$. (by (A))

② Associative property:

$\forall a,b,c \in Q-\{1\}$.

$\Rightarrow (a*b)*c) = (a+b-ab)*c$ (by Ⓐ)

$= a+b-ab+c-(a+b-ab)c$

$= a+b-ab+c-ac-bc+abc$.

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$= a+b+c-(ab+bc+ca)+abc$.

Similarly

$(ab+bc+ca)+abc$.

$$= a+b+c-(ab+bc+ca)+abc$$

Similarly

$$a*(b*c) = a+b+c-(ab+bc+ca)+abc.$$

$$\therefore (a*b)*c = a*(b*c).$$

$\therefore$ $Q-\{1\}$ satisfies Associative prop. w.r.t $*$

③ Existence of left identity prop:

③ Existence of left identity prop:

Let $a \in Q - \{1\}$, $e \in Q - \{1\}$.

then $e * a = a$

Now $e * a = a$

$\Rightarrow e + a - ea = a$

$\Rightarrow e(1-a) = 0$

$\Rightarrow e = 0 \quad (\because a \neq 1)$

$\in Q - \{1\}$

$\therefore e * a = 0 * a$

$\Rightarrow e = 0 \quad (\because a \neq 1)$

$\in \mathbb{Q} - \{1\}$

$\therefore e * a = 0 * a$

$= 0 + a - 0(a)$

$= a.$

$\therefore \forall a \in \mathbb{Q} - \{1\}, \exists\, 0 \in \mathbb{Q} - \{1\}$ such that

$0 * a = a$

$\therefore 0$ is the left identity in $\mathbb{Q} - \{1\}$.

...xistence of Left Inverse:

Let $a \in Q - \{1\}$, $b \in Q - \{1\}$.

then $b * a = e$

Now

$b * a = e$ (by (A))

$\Rightarrow b + a - ba = 0$

$\Rightarrow b(1-a) = -a$

$\Rightarrow b = \frac{-a}{1-a}$ ($\because a \neq 1$)

$= \frac{a}{a-1} \in Q - \{1\}$.

$\Rightarrow b(1-a) = -a$

$\Rightarrow b = \dfrac{-a}{1-a}$ $\quad (\because a \neq 1)$

$= \dfrac{a}{a-1} \in \mathbb{Q} - \{1\}.$

$\therefore b \star a = \dfrac{a}{a-1} \star a$

$= \dfrac{a}{a-1} + a - \dfrac{a}{a-1} a.$

$= \dfrac{a + a(a-1) - a^2}{a-1}. = 0$

For each $a \in \mathbb{Q} - \{1\}$, $\exists\, b = \dfrac{a}{a-1} \in \mathbb{Q} - \{1\}$

for each a ∈

such that $\frac{a}{a-1} * a = 0$

$\therefore b = \frac{a}{a-1}$ is left inverse of a in $\mathbb{Q} - \{1\}$ w.r.t $*$.

$\therefore (\mathbb{Q} - \{1\}, *)$ is a group.