Discrete Mathematics (MAIR-24)

Unit-1: Discrete Probability

Topics covered-

Multiplication Principle, Sum principle, Conditional Probability, Total Probability, Bayes' Rule

BASIC CONCEPTS

- Probability Sum principle
- Probability Multiplication Principle,
- Conditional Probability
- Total Probability
- Bayes’ Rule

Pre requisite-Basic definitions and Examples of discrete probability by Dr. Sarasvati Yadav
https://www.linkedin.com/feed/update/urn:li:activity:6700803049771925504/

Probability Sum principle

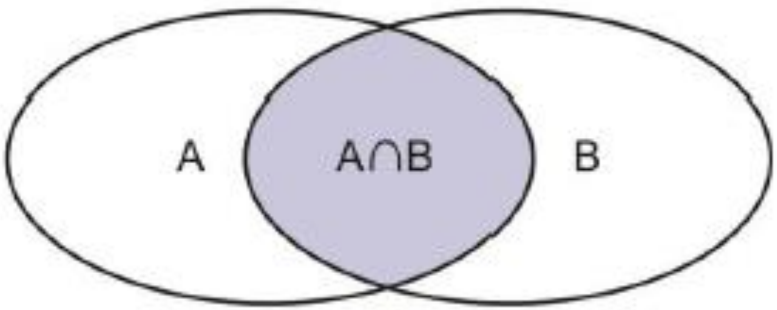
Probability: Addition, Multiplication, and Conditional

Addition Rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The probability of event A or event B can be found by adding the probability of the separate events A and B and subtracting any intersection of the two events.

More intuitively, if we were to take the area of the Venn Diagram below, we would add the areas of figures A and B, and subtract out one overlap's worth of area, which represents the intersection of A and B.



Ref. https://www.middlesex.mass.edu/ace/downloads/tipsheets/prob_adm.pdf

*Example

Let a fair dice be rolled once. The probability of getting an odd face, i.e. {1,3,5} on is $P(A)= 3/6= ½$, and that of getting a number less than 4, i.e. {1,2,3} is $P(B)= 3/6= ½$. *(non mutually exclusive events)*

We have $A \cap B=\{1,3\}$, and $A \cup B= \{1,2,3,5\}$. Then $P(A \cap B)=2/6=1/3$. Probability of getting an odd face on or a number less than 4 is

$P(A \cup B)= P(A)+ P(B) - P(A \cap B)$

$P(A \cup B)= 1/2+ 1/2 -1/3 = 2/3$.

Also, $P(A \cup B)=4/6=2/3$ (verified).

*Example

Let a fair coin be tossed once. The sample space is $S=\{H,T\}$, $|S|=2$. Either Head may turn on or the Tail.

The probability of getting a Tail on the coin is $P(A)= \frac{1}{2}$ and that of getting a Head on the coin is $P(B)= \frac{1}{2}$. (*mutually exclusive events)*

Probability of getting a Tail on the coin or a Head on the coin is

$P(A \cup B)= P(A)+ P(B)$

$P(A\cup B)= \frac{1}{2}+ \frac{1}{2} = 1.$

Probability Multiplication principle

<u>Multiplication Rule:</u>

$$\text{P(A and B)} = P(A) * P(B|A)$$

The probability of events A and B occurring can be found by taking the probability of event A occurring and multiplying it by the probability of event B happening *given* that event A already happened.

If events A and B are independent, simply multiply $P(A)$ by $P(B)$.

Ref. https://www.middlesex.mass.edu/ace/downloads/tipsheets/prob_adm.pdf

*Example

From a box of 50 balls including 10 blue balls, two are selected at random without replacement. What is the probability that the second ball is blue given that the first ball is blue?

P(A)= probability of first ball being blue =10/50=1/5
P(B/A)= probability of second ball being blue, when first ball is blue=9/49. (*conditional probability*). These are *dependent events.*

$P(A \cap B) = P(A).P(B/A)$.
$P(A \cap B)$= probability of first and second ball being blue =(1/5).(9/49)=9/245.

*Example

Let two fair coin be tossed. The sample space is $S=\{H,T\}$, $|S|=2$, for each coin independently.

The probability of getting a Tail on first coin is $P(A)= ½$ and that of getting a Head on second coin is $P(B)= ½$. These are *independent events.*

Probability of getting a Tail on first coin and a Head on second coin is

$P(A \cap B) = P(A).P(B)$.

$P(A \cap B)= ½. ½ = ¼$.

Conditional probability

<u>Conditional Probability:</u>

$$P(B|A) = \frac{P(A \; and \; B)}{P(A)}$$

The probability of event B occurring, given that event A has occurred is equal to the probability of event A and B occurring divided by the probability of event A occurring.

Note that **conditional probability** can be calculated by using **Probability Multiplication principle.**

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Ref. https://www.middlesex.mass.edu/ace/downloads/tipsheets/prob_adm.pdf

Theorem of Total Probability

Law of Total Probability: The "Law of Total Probability" (also known as the "Method of Conditioning") allows one to compute the probability of an event E by conditioning on *cases*, according to a partition of the sample space.

For example, one way to partition S is to break into sets F and F^c, for any event F. This gives us the simplest form of the law of total probability:

$$P(E) = P(E \cap F) + P(E \cap F^c) = P(E|F)P(F) + P(E|F^c)P(F^c).$$

More generally for any partition of S into sets $F_1, \ldots, F_n$,

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i).$$

Example 6 (Parts Inspection) *Consider the parts problem again, but now assume that a one-year warranty is given for the parts that are shipped to customers. Suppose that a good part fails within the first year with probability 0.01, while a slightly defective part fails within the first year with probability 0.10. What is the probability that a customer receives a part that fails within the first year and therefore is entitled to a warranty replacement?*

From before, we know that $P(G) = \frac{90}{92}$ and $P(SD) = \frac{2}{92}$. Let E be the event that a randomly selected customer's part fails in the first year. We are told that $P(E|G) = .01$ and $P(E|SD) = 0.10$. We want to compute

$$P(E) = P(E|G)P(G) + P(E|SD)P(SD) = (.01)\frac{90}{92} + (.10)\frac{2}{92} = \frac{11}{920} = 0.012$$

Ref. www.ams.sunysb.edu/~jsbm/courses/311/conditioning.pdf

Bayes' Rule

Bayes Formula. Often, for a given partition of S into sets $F_1, \ldots, F_n$, we want to know the probability that some particular case, F_j, occurs, given that some event E occurs. We can compute this easily using the definition:

$$P(F_j|E) = \frac{P(F_j \cap E)}{P(E)}.$$

Now, using the Multiplication Rule (i.e., the definition of conditional probability), we can rewrite the numerator: $P(F_j \cap E) = P(E|F_j)P(F_j)$. Using the Law of Total Probability, we can rewrite the denominator: $P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$. Thus, we get

$$P(F_j|E) = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}.$$

This expression is often called *Bayes Formula*, though I almost always just derive it from scratch with each problem we solve. See Proposition 3.1 of Ross.

Example 8 *Urn 1 contains 5 white balls and 7 black balls. Urn 2 contains 3 whites and 12 black. A fair coin is flipped; if it is Heads, a ball is drawn from Urn 1, and if it is Tails, a ball is drawn from Urn 2. Suppose that this experiment is done and you learn that a white ball was selected. What is the probability that this ball was in fact taken from Urn 2? (i.e., that the coin flip was Tails)*

Let T be the event that the coin flip was Tails. Let W be the event that a white ball is selected. From the given data, we know that $P(W|T) = 3/15$ and that $P(W|T^c) = 5/12$. Since the coin is fair, we know that $P(T) = P(T^c) = 1/2$.

We want to compute $P(T|W)$, which we do using the definition (and the same simple manipulation that results in Bayes Formula):

$$P(T|W) = \frac{P(T \cap W)}{P(W)} = \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|T^c)P(T^c)} = \frac{(3/15)(1/2)}{(3/15)(1/2) + (5/12)(1/2)} = \frac{12}{37}$$

Ref. www.ams.sunysb.edu/~jsbm/courses/311/conditioning.pdf

Thank You.