Discrete Mathematics (MAIR-24)

Unit-1: Discrete Probability

Topics covered-

Principle of Inclusion and Exclusion

Outline

- Principle of Inclusion and Exclusion
- Proof
- Example1
- Example2

Reference– **Elements of Discrete Mathematics by C. L. Liu.**

Principle of Inclusion and Exclusion

For two sets, Principle of Inclusion and Exclusion is

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \tag{1.1}$$

For three sets, Principle of Inclusion and Exclusion is

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \tag{1.2}$$

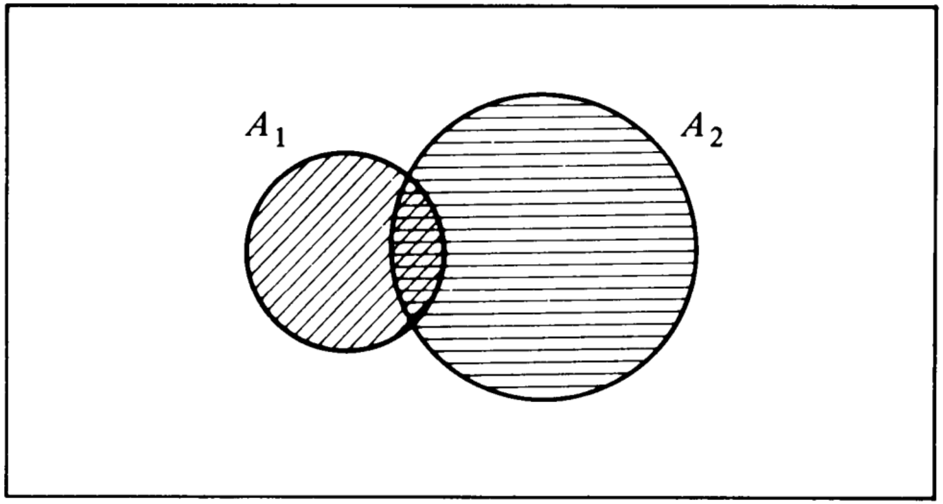
In the general case, for the sets $A_1, A_2, \ldots, A_r$, we have

$$|A_1 \cup A_2 \cup \cdots \cup A_r| = \sum_i |A_i| - \sum_{1 \leqq i < j \leqq r} |A_i \cap A_j| + \sum_{1 \leqq i < j < k \leqq r} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{r-1} |A_1 \cap A_2 \cap \cdots \cap A_r| \tag{1.3}$$

Ref.– **Elements of Discrete Mathematics by C. L. Liu.**

Proof

Note that the sets A_1 and A_2 might have some common elements. To be specific, the number of common elements between A_1 and A_2 is $|A_1 \cap A_2|$. Each of these elements is counted twice in $|A_1| + |A_2|$ (once in $|A_1|$ and once in $|A_2|$), although it should be counted as one element in $|A_1 \cup A_2|$. Therefore, the *double count* of these elements in $|A_1| + |A_2|$ should be adjusted by the subtraction of the term $|A_1 \cap A_2|$ in the right-hand side of (1.1).



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Proof

Although the result in (1.2) is not difficult to visualize, the result in (1.3) is not as obvious. We now prove (1.3) by induction on the number of sets r. Clearly, (1.1) can serve as the basis of induction. As the induction step, we assume that (1.3) is valid for any $r - 1$ sets. We note first that, viewing $(A_1 \cup A_2 \cup \cdots \cup A_{r-1})$ and A_r as two sets, according to (1.1) we have

$$|A_1 \cup A_2 \cup \cdots \cup A_r| = |A_1 \cup A_2 \cup \cdots \cup A_{r-1}| + |A_r| - |A_r \cap (A_1 \cup A_2 \cup \cdots \cup A_{r-1})| \qquad (1.4)$$

Now

$$|A_r \cap (A_1 \cup A_2 \cup \cdots \cup A_{r-1})| = |(A_r \cap A_1) \cup (A_r \cap A_2) \cup \cdots \cup (A_r \cap A_{r-1})|$$

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According to the induction hypothesis, for the $r-1$ sets $A_r \cap A_1$, $A_r \cap A_2, \ldots,$ $A_r \cap A_{r-1}$, we have

$$
\begin{aligned}
&|(A_r \cap A_1) \cup (A_r \cap A_2) \cup \cdots \cup (A_r \cap A_{r-1})| \\
&\qquad = |A_r \cap A_1| + |A_r \cap A_2| + \cdots + |A_r \cap A_{r-1}| \\
&\qquad\quad - |(A_r \cap A_1) \cap (A_r \cap A_2)| - |(A_r \cap A_1) \cap (A_r \cap A_3)| \\
&\qquad\quad - \cdots \\
&\qquad\quad + |(A_r \cap A_1) \cap (A_r \cap A_2) \cap (A_r \cap A_3)| + \cdots \\
&\qquad\quad - \cdots \\
&\qquad\quad + (-1)^{r-2}|(A_r \cap A_1) \cap (A_r \cap A_2) \cap \cdots \cap (A_r \cap A_{r-1})| \\
&\qquad = |A_r \cap A_1| + |A_r \cap A_2| + \cdots + |A_r \cap A_{r-1}| \\
&\qquad\quad - |A_r \cap A_1 \cap A_2| - |A_r \cap A_1 \cap A_3| - \cdots \\
&\qquad\quad + |A_r \cap A_1 \cap A_2 \cap A_3| + \cdots
\end{aligned}
$$

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$$- \cdots$$

$$+ (-1)^{r-2} |A_r \cap A_1 \cap A_2 \cap \cdots \cap A_{r-1}| \tag{1.5}$$

Also, according to the induction hypothesis, for the $r - 1$ sets $A_1, A_2, \ldots, A_{r-1}$, we have

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_{r-1}| = {} & |A_1| + |A_2| + \cdots \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots \\ & + \cdots \\ & + (-1)^{r-2} |A_1 \cap A_2 \cap \cdots \cap A_{r-1}| \end{aligned} \tag{1.6}$$

Substituting (1.5) and (1.6) into (1.4), we obtain (1.3).

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Example 1.13 Thirty cars were assembled in a factory. The options available were a radio, an air conditioner, and white-wall tires. It is known that 15 of the cars have radios, 8 of them have air conditioners, and 6 of them have white-wall tires. Moreover, 3 of them have all three options. We want to know at *least* how many cars do not have any options at all. Let A_1, A_2, and A_3 be the sets of cars with a radio, an air conditioner, and white-wall tires, respectively. Since

$$|A_1| = 15 \qquad |A_2| = 8 \qquad |A_3| = 6$$

and

$$|A_1 \cap A_2 \cap A_3| = 3$$

according to (1.2)

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= 15 + 8 + 6 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + 3 \\ &= 32 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \end{aligned}$$

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Since

$$|A_1 \cap A_2| \geqq |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cap A_3| \geqq |A_1 \cap A_2 \cap A_3|$$

$$|A_2 \cap A_3| \geqq |A_1 \cap A_2 \cap A_3|$$

we have

$$|A_1 \cup A_2 \cup A_3| \leqq 32 - 3 - 3 - 3 = 23$$

That is, there are *at most* 23 cars that have one or more options. Consequently, there are *at least* 7 cars that do not have any options. □

Ref.- Elements of Discrete Mathematics by C. L. Liu.

Example 1.14 Let us determine the number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5, and 7. Let A_1 denote the set of integers between 1 and 250 that are divisible by 2, A_2 denote the set of integers that are divisible by 3, A_3 denote the set of integers that are divisible by 5, and A_4 denote the set of integers that are divisible by 7. Since

$$|A_1| = \left\lfloor \frac{250}{2} \right\rfloor^{\dagger} = 125 \qquad |A_2| = \left\lfloor \frac{250}{3} \right\rfloor = 83$$

$$|A_3| = \left\lfloor \frac{250}{5} \right\rfloor = 50 \qquad |A_4| = \left\lfloor \frac{250}{7} \right\rfloor = 35$$

$$|A_1 \cap A_2| = \left\lfloor \frac{250}{2 \times 3} \right\rfloor = 41 \qquad |A_1 \cap A_3| = \left\lfloor \frac{250}{2 \times 5} \right\rfloor = 25$$

$$|A_1 \cap A_4| = \left\lfloor \frac{250}{2 \times 7} \right\rfloor = 17 \qquad |A_2 \cap A_3| = \left\lfloor \frac{250}{3 \times 5} \right\rfloor = 16$$

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$$|A_2 \cap A_4| = \left\lfloor \frac{250}{3 \times 7} \right\rfloor = 11 \qquad |A_3 \cap A_4| = \left\lfloor \frac{250}{5 \times 7} \right\rfloor = 7$$

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{250}{2 \times 3 \times 5} \right\rfloor = 8 \qquad |A_1 \cap A_2 \cap A_4| = \left\lfloor \frac{250}{2 \times 3 \times 7} \right\rfloor = 5$$

$$|A_1 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2 \times 5 \times 7} \right\rfloor = 3 \qquad |A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{3 \times 5 \times 7} \right\rfloor = 2$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2 \times 3 \times 5 \times 7} \right\rfloor = 1$$

we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 \\ &\quad + 8 + 5 + 3 + 2 - 1 = 193 \end{aligned} \qquad \square$$

Ref.– **Elements of Discrete Mathematics by C. L. Liu.**

Thank You.