Asymptotic notation in equations and inequalities

- When the asymptotic notation stands alone on the right-hand side of an equation, as in $n = O(n^2)$, we have already defined the equal sign to mean set membership: $n \in O(n^2)$.
- In general, when asymptotic notation appears in a formula, we interpret it as standing for some anonymous function (that we do not care to name).
  Eg: For example, $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means $2n^2 + 3n + 1 = 2n^2 + f(n)$ where $f(n)$ is some function in the set $\Theta(n)$. Here, $f(n) = 3n + 1$, which indeed is in $\Theta(n)$.
- Using asymptotic notation in this manner can help eliminate inessential detail and clutter in an equation.
  Eg: For example, the worst-case running time of merge sort as the recurrence
  $T(n) = 2T(n/2) + \Theta(n)$.

- The number of anonymous functions in an expression is understood to be equal to the number of times the asymptotic notation appears.
  Eg: For example, In the expression $\sum_{i=1}^{n} O(i)$ there is only one single anonymous function (a function of i).
  ⇒ This expression is thus *not* the same as $O(1) + O(2) + \cdot \cdot \cdot + O(n)$, which doesn't really have a clean interpretation.
- In some cases, asymptotic notation appears on the left-hand side of an equation:
  Eg: For example, $2n^2 + \Theta(n) = \Theta(n^2)$ such equations are interpreted as follows:
  "*No matter how the anonymous functions are chosen on the left of the equal sign, there is a way to choose the anonymous functions on the right of the equal sign to make the equation valid*

o-notation

- The asymptotic upper bound provided by O-notation may or may not be asymptotically tight. The bound $2n^2 = O(n^2)$ is asymptotically tight, but the bound $2n = O(n^2)$ is not.
- We use o-notation to denote an upper bound that is not asymptotically tight. We formally define o(g(n)) ("little-oh of g of n") as the set:

$$o(g(n)) = \{f(n) : \text{ for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0\} .$$

For example, $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.

- The definitions of O-notation and o-notation are similar. (The main difference is that in $f(n) = O(g(n))$, the bound $0 \le f(n) \le cg(n)$ holds for *some* constant $c > 0$, but in $f(n) = o(g(n))$, the bound $0 \le f(n) < cg(n)$ holds for *all* constants $c > 0$.)

- Intuitively, in the o-notation, the function $f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity; that is:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

ω-notation

- By analogy, ω-notation is to Ω-notation as o-notation is to O-notation.
- We use ω-notation to denote a lower bound that is not asymptotically tight. We formally define ω(g(n)) ("little-omega of g of n") as the set:

$$\omega(g(n)) = \{f(n) : \text{ for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\} .$$

For example, $n^2/2 = \omega(n)$, but $n^2/2 \neq \omega(n^2)$.

- The relation $f(n) = \omega(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$