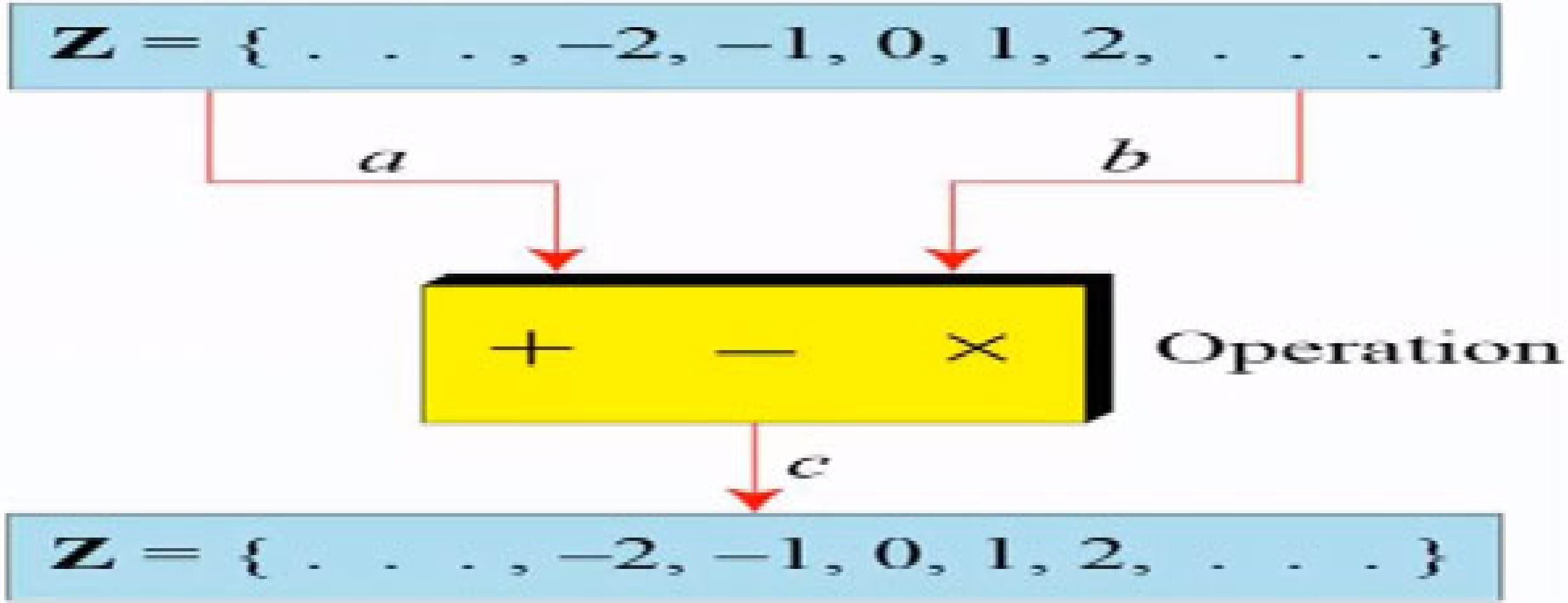
DISCRETE MATHEMATICS
(MAIR-24)
UNIT-III: RELATIONS AND LOGIC

TOPIC COVERED: BINARY RELATIONS & THEIR PROPERTIES
EQUIVALENCE RELATIONS AND PARTITIONS, PARTIAL ORDERING

DR. SMITA SONKER (DISCRETE MATHEMATICS)

BINARY OPERATIONS : Let S be a non void set. A function F from SxS to S is called a binary operation on S. i.e. F: SxS →S is a binary operation on set S. Generally binary operations are represented by the symbols * , instead of letters f, g etc.. Thus a Binary operation * on a set S associates each ordered pair (a, b) of elements of S. (or any two elements of S) to a unique element a*b of S.



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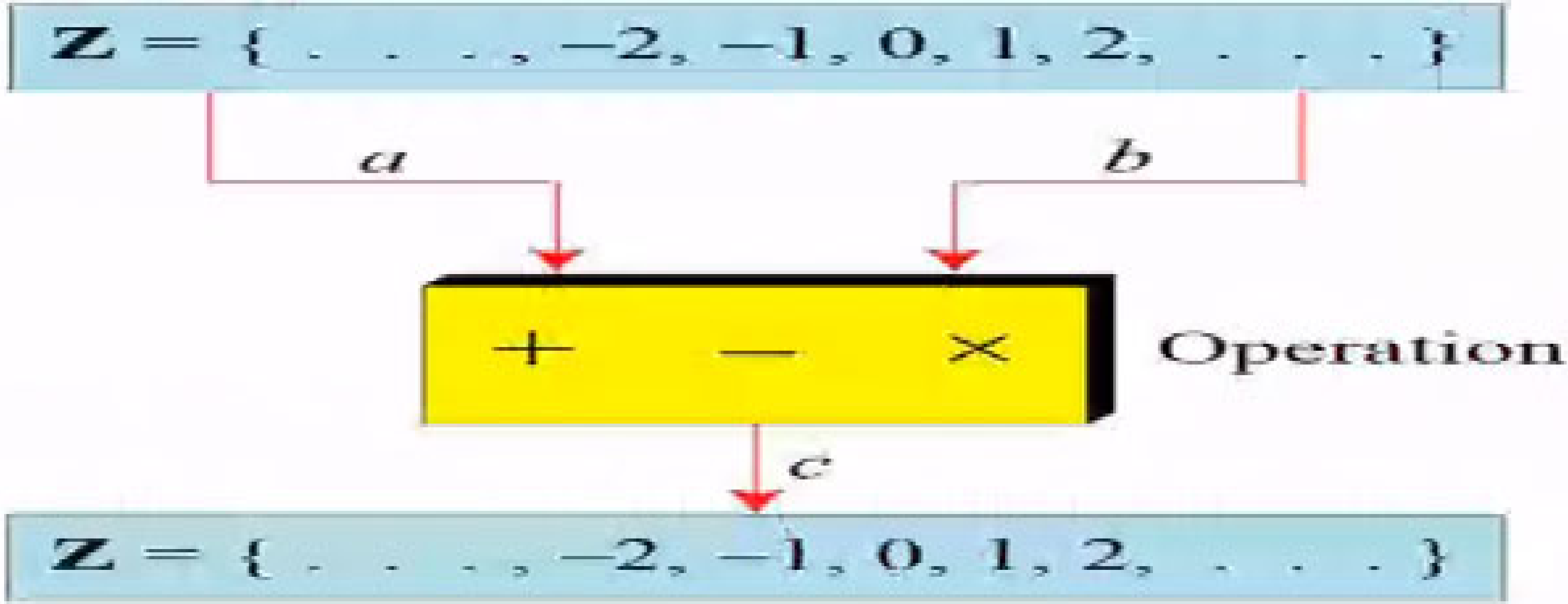

TYPES OF BINARY OPERATIONS:

1. COMMUTATIVE BINARY OPERATION :

A Binary Operation * on a set S is said to be commutative if

$$(a*b) = (b*a) \qquad \forall\ a, b \in S.$$

2. ASSOCIATIVE BINARY OPERATION:

A Binary Operation * on a set S is said to be associative if

$$(a*b)*c = a*(b*c) \qquad \forall\ a, b, c \in S.$$

3. DISTRIBUTIVE BINARY OPERATION:

Let * and 0 be two Binary operations on a set S. Then * is said to be

(i)Left distributive over 0 if $a*(b0c) =(a*b)0(a*c)$ for all a, b, c $\in$ S.

(ii)Right distributive over 0 if $(b0c)*a= (b*a)\ 0(c*a)$ for all a, b, c $\in$ S.

4. IDENTITY BINARY OPERATION:

Let * be a Binary Operation on a set S. An element e ϵ S is said to be an identity element for the binary operation * if a*e=a=e*a, for all a ϵ S.

5. INVERSE BINARY OPERATION :

Let * be a Binary Operation on a set S. An element a ϵ S is said to be

invertible with respect to the operation * , if ∃ an element b in S such that

a*b= =b*a = e and b is called inverse of a.

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NUMBER OF BINARY OPERATIONS:

Let A be a finite set containing m elements then the number of binary operations on A is $m^{m \times m}$

Example:

If A= {a, b} then the no. of binary operations on A = $2^4 = 16$ is equal to number of functions from A × A to A.

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EMPTY RELATION : A relation in a set A is called empty relation , if no element of A is related to any element of A , $R = \phi \subseteq A X A$.

For example: consider a relation R in the set $A = \{1,2,3\}$ given by $R = \{(a, b) : |a - b| = 8\}$.

UNIVERSAL RELATION :

A relation in a set A is called universal relation , if each element of A is related to every element of A , i.e., $R = A X A$.

For example:

consider a relation R in the set $A = \{1,2,3\}$ given by $R=\{(a,b) : |a - b| \geq 0\}$.

NOTE: Both the empty relation and the universal relation are some times called trivial relations.

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EQUIVALENCE RELATION ON A:

Let R be a **relation on a non empty set A, then R is called an equivalence relation iff it is:**

(i) Reflexive ,

(ii) Symmetric &

(iii) Transitive.

Note: If a Relation does not satisfy any of the above properties it is not considered as an equivalence Relation.

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Equivalence Relation

Definition:

A binary relation R on a set A is an **equivalence relation** if and only if

(1) R is reflexive

(2) R is symmetric, and

(3) R is transitive.

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Symmetric Relation

- A relation **R** on a set **A** is symmetric if whenever **a R b** , then **b R a** for all **a,b** ϵ **A**.
- It means if **a** is related to **b** then **b** also related to **a**.

For example:

Let A={1,2,3} and let
R={(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)}

Then R is symmetric relation as (a,b) ϵ R and (b,a) ϵ R.

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Transitive Relation

- We say that a relation **R** on a set **A** is transitive if whenever **a R b** and **b R c**, then **a R c**.
- It means **a** related to **b** and **b** related to **c** then **a** related to **c** for all (a,b,c) ϵ A.

For example:

Let A={1,2,3} and let R={(1,2),(2,3),(1,3)} then R satisfy transitive relation.

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Example of Equivalence relation:

- Let A={1,2,3,4} and let R={(1,1),(1,2),(2,1),(2,2),(2,3),(3,2)(3,4),
  (4,3),(3,3),(4,4),.....}

  from above example we can easily say that R is an Equivalence relation as it is reflexive, symmetric and transitive.

Partitions

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Definition

Let X be a family of sets.

X is **pairwise disjoint** if every two different sets in X are disjoint.

Definition Let X be a set.

A **partition** of X is a family P of sets with the following properties.

Cover: $P = X$,

Disjointness: P is a pairwise disjoint family of sets,

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Example

For $i = 0, 1, 2$, let $X_i = \{n : \exists_{k \in Z} n = 3k + i\}$. The family $\{X_0, X_1, X_2\}$ is a partition of Z.

Exercise

1. List all the partitions of $\{1, 2, 3\}$.
2. List all the partitions of $\{1, 2\}$.
3. List all the partitions of $\{1\}$.
4. List all the partitions of $\emptyset$.

EXAMPLE : *Let $A = \{1,2,3\}$, find the number of relations on A containing (1,2) and (1,3) which are reflexive and symmetric but not transitive.*

SOL. *The smallest relation R_1 on A containing (1, 2) and (1,3) which is reflexive and symmetric but not transitive is $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}$. We are left with two pairs (2,3) and (3,2). Now to get another relation R_2, if we add any pair , say (2,3) to R_1 then we must add the remaining pair (3,2) in order to maintain symmetric of R_2 and then R_2 becomes transitive also. Hence , the number of relation is one.*

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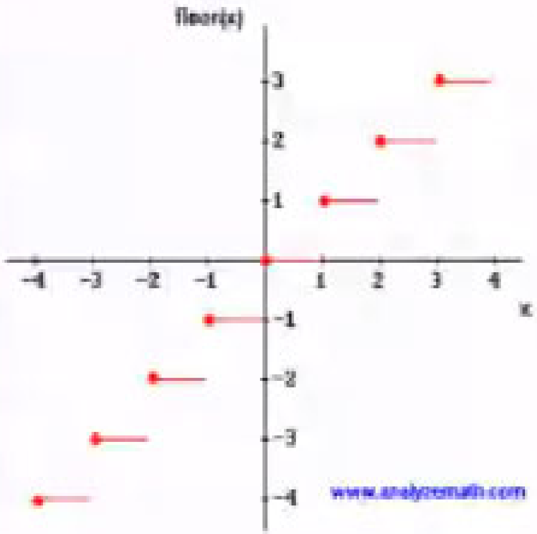
EXAMPLE : Let A ={1,2,3} , find the number of equivalence relations on A containing (1,2) is reflexive and symmetric .

SOL. *The smallest relation R_1 on A containing (1, 2) is {(1,1),(2,2),(3,3),(1,2),(2,1)}. We are left with two pairs (1,3),(3,1),(2,3) and (3,2).*
Now to get another relation R_2 , if we add any pair , say (2,3) to R_1 then we must add the remaining pair (3,2) in order to maintain symmetric of R_2. Also to maintain transitivity we are forced to add (1,3) and (3,1) thus the only bigger that R_1 is the universal relation.
Hence the total number of equivalence relations on A containing (1,2) is two.

EXAMPLE: Let Q be the set all rational numbers and relation on Q defined by R ={(X,Y) : 1 + XY > 0}. Prove that R is reflexive and symmetric but not transitive.

SOL. Consider any x,y ∈ Q , since 1+x.x = 1+x² ≥ 1
(x,x) ∈R ⟹ reflexive
Let (x,y) ∈ R ⟹ 1+xy > 0 ⟹ 1+yx > 0 ⟹ (y,x) ∈R ⟹ symmetric.
But not transitive.
Since (-1,0) and (0,2) ∈ R , because 1 > 0 by putting values. But (-1,2) ∉ R because -1 < 0



Greatest Integer Function (Floor or step Function)
f : R → R defined as f(x) = [x] ∀ x ∈R , {x} = x − [x] where {x} is Fractional part or decimal part of x. For Example, {3.45} = 0.45, {-2.75} = 0.25
From the definition of [x] , greatest integer less than or equal to x we can see that
[x] = -1 for -1≤ x <0
= 0 for 0 ≤ x <1
= 1 for 1 ≤ x <2 and so on.
For smallest Integer Function (ceiling Function) ,we take smallest integer greater than or equal to x. For example, ⌈4.7⌉ =5 , ⌈-7.2⌉ = -7, ⌈0.75⌉ = 1 and so on.
It is neither 1-1 and onto.

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Example : Show that the relation R defined by $(a,b)\ R\ (c,d) \Rightarrow a+d = b+c$ on the set N XN is an equivalence relation.

SOL. Let (a,b) , (c,d) ,$(e,f) \in N\times N$.

(i) reflexive : since $a+b = b+a\ \forall a,b \in N \Rightarrow (a,b)\ R\ (b,a)$

(ii) symmetric : let $(a,b)R(c,d) \Rightarrow a+d = b+c \Rightarrow d+a = c+b$

$\Rightarrow (c,d)\ R\ (a,b)$ [by commutative law in N]

(iii) transitive : let $(a,b)\ R(c,d)$, $(c,d)\ R\ (e,f) \Rightarrow a+d = b+c$ and $c+f = d+e$ by adding we will get

$a+d+c+f = b+c+d+e \Rightarrow (a,b)\ R\ (e,f)$

therefore R is an equivalence relation.

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Example : Show that the relation R defined by $(a,b)\ R\ (c,d) \Rightarrow a+d = b+c$ on the set $N \times N$ is an equivalence relation.

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$a+d+c+f = b+c+d+e \Rightarrow a+f = b+e \Rightarrow a+f = e+b$
$\Rightarrow (a,b)\ R\ (e,f)$

therefore R is an equivalence relation.

EXAMPLE:

Let R be a relation on N × N, defined by (a, b) R (c, d) ⟺ ad=bc ∀ (a, b) ,(c, d) Є N × N. Show that R is an equivalence relation on N × N.

SOL. Let (a, b) be an arbitrary element of N × N. Then (a,b) Є N × N.

⇨ ab = ba (by commutative on N) ⇨ R is reflexive on N × N.

Let (a,b),(c,d) Є N × N such that (a,b) R (c,d) ⇨ ad=bc ⇨ cb=da (by commutative) ⇨(c,d) R (a,b)

Let (a, b),(c, d),(e, f) ЄN × N such that (a, b) R (c, d) and (c, d) R (e, f) then ad=bc and cf=de ⇨(ad)(cf)=(bc)(de) ⇨ af = be ⇨ (a,b) R (e,f)

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Partial Orderings

Definition: A relation R on a set S is called a **partial ordering**, or **partial order**, if it is *reflexive, antisymmetric, and transitive.*

Definition: A set A together with a partial ordering R is called a **partially ordered set** or **poset.**

Example: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution:

Reflexivity: $a \geq a$ for every integer a.

Antisymmetry: If $a \geq b$ and $b \geq a$, then $a = b$.

Transitivity: If $a \geq b$ and $b \geq c$, then $a \geq c$.

Equivalence class

Definition: For an equivalence relation R defined on A and for $a \in A$, the set

$$[a] = \{x \in A \mid (a, x) \in R\}$$

is called the **equivalence class** of a in A.

Definition: Any $b \in [a]$ is called a **representative** of this equivalence class.

Definition: The collection of all equivalence classes of elements of A under an equivalence relation R is called the **quotient set**, denoted by A/R, i.e.

$$A/R = \{[a] \mid a \in A\}.$$

Note: The quotient set A/R is a partition of A.

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