Growth of Functions

Asymptotic notation

1. These notations are used to describe the asymptotic running time of an algorithm.

2. These defined in terms of functions whose domains are the set of natural numbers N = {0, 1, 2, . .}
   It is sometimes convenient, however, to abuse asymptotic notation in a variety of ways. For example, the notation is easily extended to the domain of real numbers or, alternatively, restricted to a subset of the natural numbers. It is important, however, to understand the precise meaning of the notation so that when it is abused, it is not misused.

3. Here, we will define five basic asymptotic notations and will illustrate some common abuses of these notations.

Θ-notation

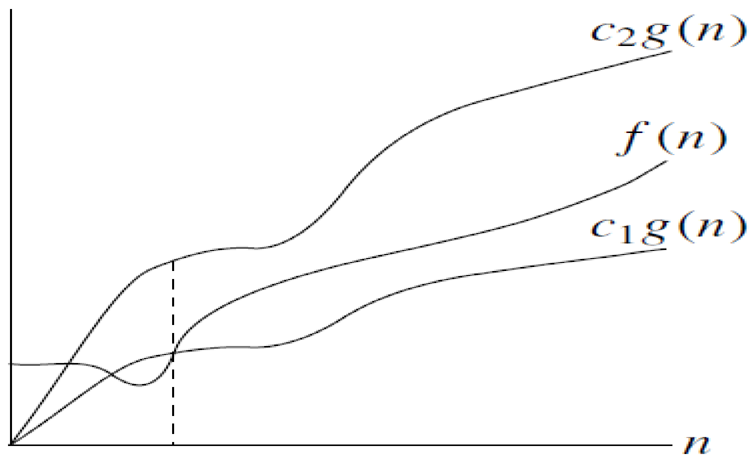
- For a given function g(n), we denote by Θ (g(n)) the set of functions:

$$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\} .^1$$

- A function $f(n)$ belongs to the set **Θ** $(g(n))$ if there exist positive constants c_1 and c_2 such that it can be "sandwiched" between $c1g(n)$ and $c2g(n)$, for sufficiently large n.

- Because **Θ** $(g(n))$ is a set, we could write "$f(n) \in \Theta(g(n))$" to indicate that $f(n)$ is a member of **Θ** $(g(n))$. Instead, we will usually write "$f(n) = \Theta(g(n))$" to express the same notion. This abuse of equality to denote set membership may at first appear confusing, but we shall see later in this section that it has advantages.

- For all values of n to the right of $n0$, the value of $f(n)$ lies at or above $c1g(n)$ and at or below $c2g(n)$. Figure gives an intuitive picture of functions f (n) and g(n), where f (n) = Θ (g(n))



$f(n) = \Theta(g(n))$

- In other words, for all $n \geq n0$, the function $f(n)$ is equal to $g(n)$ to within a constant factor. We say that g(n) is an **asymptotically tight bound** for f (n).

NOTE:

1) The definition of $\Theta(g(n))$ requires that every member $f(n) \in \Theta(g(n))$ be ***asymptotically nonnegative***, that is, that $f(n)$ be nonnegative whenever n is sufficiently large. (An ***asymptotically positive*** function is one that is positive for all sufficiently large n.) Consequently, the function $g(n)$ itself must be asymptotically nonnegative, or else the set $\Theta(g(n))$ is empty.

2) We shall therefore assume that every function used within Θ -notation is asymptotically nonnegative. This assumption holds for the other asymptotic notations defined in this chapter as well.

Example 1: show that $\frac{1}{2}n^2 - 3n = \theta(n^2)$

Example 2: show that $6n^3 \neq \theta(n^2)$

Example 3: show that $6n \neq \theta(n^2)$

Example 4: show that $\frac{1}{2}n^2 - 2n = \theta(n^2)$

Example 5:

As an example, consider any quadratic function $f(n) = an^2 + bn + c$, where a, b, and c are constants and $a > 0$. Throwing away the lower-order terms and ignoring the constant yields $f(n) = \Theta(n^2)$. Formally, to show the same thing, we take the constants $c_1 = a/4$, $c_2 = 7a/4$, and $n_0 = 2 \cdot \max((|b|/a), \sqrt{(|c|/a)})$. The reader may verify that $0 \le c_1 n^2 \le an^2 + bn + c \le c_2 n^2$ for all $n \ge n_0$. In general, for any polynomial $p(n) = \sum_{i=0}^{d} a_i n^i$, where the a_i are constants and $a_d > 0$, we have $p(n) = \Theta(n^d)$ (see Problem 3-1).

Example 6:

Since any constant is a degree-0 polynomial, we can express any constant function as $\Theta(n^0)$, or $\Theta(1)$. This latter notation is a minor abuse, however, because it is not clear what variable is tending to infinity.[2] We shall often use the notation $\Theta(1)$ to mean either a constant or a constant function with respect to some variable.