1.1. Transformation Formulas

Learning objectives:

1. To derive trigonometric product formulas
2. To derive trigonometric sum and difference formulas
   And
3. To practice the related formulas.

Product Formulas

We have already derived the relations

$$sin\,(\alpha+\beta) = sin\alpha\; cos\beta + cos\alpha\; sin\beta$$

$$sin(\alpha-\beta) = sin\alpha\; cos\beta - cos\alpha\; sin\beta$$

Adding and subtracting, we get

$$2\; sin\alpha\; cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta) \quad \ldots (i)$$

$$2\; cos\alpha\; sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta) \quad \ldots (ii)$$

Similarly, from the relations

$$\cos(\alpha+\beta) = cos\alpha\; cos\beta - sin\alpha\; sin\beta \quad \ldots (iii)$$

$$\cos(\alpha-\beta) = cos\alpha\; cos\beta + sin\alpha\; sin\beta \quad \ldots (iv)$$

we get by adding and subtracting

$$2\; cos\alpha\; cos\beta = \cos(\alpha+\beta) + cos\,(\alpha-\beta)$$

$$-2\; sin\alpha\; sin\beta = \cos(\alpha+\beta) - cos\,(\alpha-\beta)$$

These formulas transform products of sines and cosines into sums or differences of sines or cosines.

Sum and Difference Formulas

The sum and difference formulas are obtained from the product formulas by setting

$$\alpha+\beta = A$$

$$\alpha-\beta = B$$

Adding and subtracting, we get

$$\alpha = \frac{A+B}{2}$$

$$\beta = \frac{A-B}{2}$$

Substituting these values into product formulas, we obtain

$$sinA + sinB = 2\; sin\frac{A+B}{2}\; cos\frac{A-B}{2} \quad \ldots (v)$$

$$sinA - sinB = 2\; cos\frac{A+B}{2}\; sin\frac{A-B}{2} \quad \ldots (vi)$$

$$cosA + cosB = 2\,cos\frac{A+B}{2}\,cos\frac{A-B}{2} \qquad \dots(vii)$$

$$cosA - cosB = -2\,sin\frac{A+B}{2}\,sin\frac{A-B}{2} \qquad \dots(viii)$$

These formulas transform sums or differences of sines and cosines into products of sines and cosines.

Example1:

Express each of the following as a sum or difference.

$a.\,2sin40^\circ cos30^\circ$

$b.\,2cos110^\circ sin55^\circ$

Solution:

$$a.\ \ 2sin40^\circ cos30^\circ = sin(40^\circ + 30^\circ) + sin(40^\circ - 30^\circ)$$
$$= sin70^\circ + cos10^\circ$$

$$b.\ \ 2cos110^\circ sin55^\circ = sin(110^\circ + 55^\circ) - sin(110^\circ - 55^\circ)$$
$$= sin165^\circ - sin55^\circ$$

Example 2:

Express each of the following as a product.

$a.\,sin50^\circ + sin40^\circ$

$b.\,sin70^\circ - sin20^\circ$

Solution:

a. $sin50^\circ + sin40^\circ = 2sin\frac{1}{2}(50^\circ + 40^\circ)\,cos\frac{1}{2}(50^\circ - 40^\circ)$

$$= 2sin45^\circ cos5^\circ = \sqrt{2}\,cos5^\circ$$

b. $sin70^\circ - sin20^\circ = 2cos\frac{1}{2}(70^\circ + 20^\circ)\,sin\frac{1}{2}(70^\circ - 20^\circ)$

$$= 2cos45^\circ sin25^\circ = \sqrt{2}\,sin25^\circ$$

Example 3:

Prove $\frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A} = \tan 3A$

Solution:

$$\frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A} = \frac{2\,sin\frac{4A+2A}{2}cos\frac{4A-2A}{2}}{2\,cos\frac{4A+2A}{2}cos\frac{4A-2A}{2}}$$
$$= \frac{2\sin 3A\,cosA}{2\cos 3A\cos A} = tan3A$$

PROBLEM SET

IP1. Prove that $cos^2 76^\circ + cos^2 16^\circ - cos 76^\circ cos 16^\circ = \frac{3}{4}$

Solution:

$cos^2 76^\circ + cos^2 16^\circ - cos 76^\circ cos 16^\circ =$

$= cos^2 76^\circ + 1 - sin^2 16^\circ - \frac{1}{2}(2 cos 76^\circ cos 16^\circ)$

$= 1 + cos^2 76^\circ - sin^2 16^\circ - \frac{1}{2}(\cos(76^\circ + 16^\circ) + \cos(76^\circ - 16^\circ))$

$= 1 + \cos(76^\circ + 16^\circ)\cos(76^\circ - 16^\circ) - \frac{1}{2}(\cos 92^\circ + \cos 60^\circ)$

Since $cos^2 A - sin^2 B = \cos(A + B)\cos(A - B)$

$= 1 + \cos 92^\circ \cos 60^\circ - \frac{1}{2}(\cos 92^\circ + \cos 60^\circ)$

$= 1 + \frac{1}{2}\cos 92^\circ - \frac{1}{2}\cos 92^\circ - \frac{1}{4}$

$= 1 - \frac{1}{4} = \frac{3}{4}$

P1. Prove that $\sin 21^\circ \cos 9^\circ - \cos 84^\circ \cos 6^\circ = \frac{1}{4}$

Solution:

$\sin 21^\circ \cos 9^\circ - \cos 84^\circ \cos 6^\circ =$

$= \frac{1}{2}(2\sin 21^\circ \cos 9^\circ - 2\cos 84^\circ \cos 6^\circ)$

$= \frac{1}{2}(\sin(21^\circ + 9^\circ) + \sin(21^\circ - 9^\circ) - 2\cos(90^\circ - 6^\circ)\cos 6^\circ)$

$= \frac{1}{2}(\sin 30^\circ + \sin 12^\circ - \sin 12^\circ) = \frac{1}{4}$

IP2. Prove that

$$\frac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \cos 2\theta - \sin 2\theta \tan 3\theta$$

Solution:

$$\frac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \frac{2\cos 5\theta + \cos 3\theta + \cos 7\theta}{2\cos 3\theta + \cos\theta + \cos 5\theta}$$

$$= \frac{2\cos 5\theta + 2\cos 5\theta\cos 2\theta}{2\cos 3\theta + 2\cos 3\theta\cos 2\theta}$$

$$= \frac{\cos 5\theta(1 + \cos 2\theta)}{\cos 3\theta(1 + \cos 2\theta)}$$

$$= \frac{\cos 3\theta\cos 2\theta - \sin 3\theta\sin 2\theta}{\cos 3\theta}$$

$$= \cos 2\theta - \tan 3\theta\sin 2\theta$$

P2. Prove that $\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta} = 4\cos 2\theta \cos 4\theta$

Solution:

$$\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = \frac{\sin\ 5\theta \cos 3\theta + \sin 3\theta \cos 5\theta}{\sin\ 5\theta \cos 3\theta - \sin 3\theta \cos 5\theta}$$

$$= \frac{\sin 8\theta}{\sin 2\theta}$$

$$= \frac{2\sin 4\theta \cos 4\theta}{\sin 2\theta}$$

$$= \frac{4\sin 2\theta \cos 2\theta \cos 4\theta}{\sin 2\theta}$$

$$= 4\cos 2\theta \cos 4\theta$$

IP3: Prove that $\frac{sin5x-2sin3x+sinx}{cos5x-cosx} = tanx.$

Solution:

$$\frac{sin5x - 2sin3x + sinx}{cos5x - cosx} = \frac{2\ sin3x\ cos2x - 2sin3x}{-2sin3x\ sin2x}$$

$$= \frac{1 - cos2x}{sin2x}$$

$$= \frac{2\ sin^2 x}{2\ sinx\ cosx} = tanx$$

P3: Prove that $\frac{cos7x+cos5x}{sin7x-sin5x} = cotx.$

Solution:

$$\frac{cos7x + cos5x}{sin7x - sin5x} = \frac{cos7x + cos5x}{sin7x - sin5x}$$

$$= \frac{2cos\frac{7x+5x}{2}\ cos\frac{7x-5x}{2}}{2cos\frac{7x+5x}{2}\ sin\frac{-7x-5x}{2}} = \frac{2cos\ 6x\ cosx}{2cos\ 6x\ sinx}$$

$$= \frac{\cos x}{sinx} = cotx$$

IP4. Show that $\frac{\cos 6\theta+6\cos 4\theta+15\ cos2\theta+10}{\cos 5\theta+5cos3\theta+10\ cos\theta} = 2\ cos\theta$

Solution:

$$\frac{\cos 6\theta + 6\cos 4\theta + 15\ cos2\theta + 10}{\cos 5\theta + 5cos3\theta + 10\ cos\theta}$$

$$= \frac{(\cos 6\theta + \cos 4\theta) + 5cos4\theta + 5\ cos2\theta + 10cos2\theta + 10}{\cos 5\theta + 5cos3\theta + 10\ cos\theta}$$

$$= \frac{2\ \cos5\theta\ cos\theta + 5(2cos3\theta cos\theta) + 10(cos2\theta + 1)}{\cos 5\theta + 5cos3\theta + 10\ cos\theta}$$

$$= \frac{2\ \cos5\theta\ cos\theta + 10cos3\theta cos\theta + 10(2cos^2\theta)}{\cos 5\theta + 5cos3\theta + 10\ cos\theta}$$

$$= \frac{2\ \ cos\theta(\cos5\theta + 5cos3\theta + 10cos\theta)}{\cos 5\theta + 5cos3\theta + 10\ cos\theta} = 2cos\theta$$

P4. Find the value of $\frac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}=\cot 4A$.

Solution:

$$\frac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}$$

$$=\frac{\sin A+\sin 9A-(\sin 5A+\sin 13A)}{\cos A+\cos 13A-(\cos 5A+\cos 9A)}$$

$$=\frac{2\sin 5A\cos 4A-(2\sin 9A\cos 4A)}{2\cos 7A\cos 6A-(2\cos 7A\cos 2A)}$$

$$=\frac{\cos 4A}{\cos 7A}\left(\frac{\sin 5A-\sin 9A}{\cos 6A-\cos 2A}\right)$$

$$=\frac{\cos 4A}{\cos 7A}\left(\frac{2\cos 7A\sin(-2A)}{2\sin 4A\sin(-2A)}\right)$$

$$=\cot 4A$$

Exercises

Prove the following identities.

1) $\frac{sin7\theta-sin5\theta}{cos7\theta+cos5\theta}=tan\theta$

2) $\frac{cos\ 2B+\cos 2A}{cos\ 2B-\cos 2A}=\cot(A+B)\cot(A-B)$

3) $\frac{sin\ A+\sin 2A}{cosA-cos2A}=cot\,\frac{A}{2}$

4) $\cos\,(A+B)+sin(A-B)=2\sin(45^{\circ}+A)\cos(45^{\circ}+B)$

5) $\frac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)+}=\tan\,(A+B)$

6) $\frac{sin\ A+2\sin 3A+sin\ 5A}{\sin 3A+2\ sin\ 5A+\sin 7A}=\frac{\sin 3A}{\sin 5A}$

7) $\frac{sin\ A+\sin B}{sin\ A-\sin B}=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$

8) $\frac{sin\ A+\sin B}{cos\ A-cos\ B}=\tan\frac{A+B}{2}$

9) $\cos 3A+\cos 5A+\cos 7A+\cos 15A$

$$=4\cos 4A\cos 5A\cos 6A$$

II. Express as a sum or difference of the following

1) $2\sin 5\theta \sin 7\theta$
2) $2\cos 7\theta \sin 5\theta$
3) $2\cos 11\theta \cos 3\theta$
4) $2\sin 54^{\circ} \sin 66^{\circ}$

III. Prove the following identities using sum or difference formulas

1) $\sin\frac{\theta}{2}\,\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\,\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$
2) $\sin A \sin(A+2B) - \sin B \sin(B+2A)$
$$= \sin(A-B)\sin(A+B)$$
3) $\frac{\sin A \sin 2A + \sin 3A \sin 6A + \sin 4A \sin 13A}{\sin A \cos 2A + \sin 3A \cos 6A + \sin 4A \cos 13A} = \tan 9A$
4) $\sin(\beta-\gamma)\cos(\alpha-\delta) + \sin(\gamma-\alpha)\cos(\beta-\delta)$
$$+\sin(\alpha-\beta)\cos(\gamma-\delta) = 0$$
5) $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

1.2. Conditional Trigonometric Identities

Learning objectives:

1. To derive conditional trigonometric identities.
   And
2. To practice the related problems.

Conditional Identities

Trigonometric identities are equations involving trigonometric functions of angles, which are satisfied by all values of the angles for which the functions are defined.

If the angles involved in the identities are the three angles A, B and C of a triangle, then they are constrained by the relation $A + B + C = 180^{\circ}$, then these trigonometric identities are known as conditional identities.

Some Identities

i) If $A + B + C = 180^{\circ}$, then prove that

$\sin 2A + \sin 2B + \sin 2C = 4\, sinA\, sinB\, sinC$

Proof: $\sin 2A + \sin 2B + \sin 2C$

$$= 2\sin(A+B) cos\,(A-B) + 2 sinC\, cosC$$

$$= 2\sin(180^{\circ} - C) cos\,(A-B)$$

$$+2 sinC \cos\,(180^{\circ} - (A+B))$$

$$= 2\sin C\, cos\,(A - B) + 2sinC\cos\,(A + B)$$

$$= 2\sin C\,[cos(A - B) - \cos\,(A + B)]$$

$$= 2sinC.2sinA\,\mathrm{sinB}$$

$$= 4sinA\,sinB\,\mathrm{sinC}$$

ii) If $A + B + C = 180^\circ$, then prove that

$$cosA + cosB - cosC = -1 + 4cos\frac{A}{2}\,cos\frac{B}{2}sin\frac{C}{2}\,.$$

Proof:

$cosA + cosB - cosC$

$$= cosA + (cosB - cosC)$$

$$= 2cos^2\frac{A}{2} - 1 - 2sin\frac{B+C}{2}\,sin\frac{B-C}{2}$$

$$= 2cos^2\frac{A}{2} - 1 - 2sin\left(90^\circ - \frac{A}{2}\right)sin\frac{B-C}{2}$$

$$= 2cos\frac{A}{2}\left[cos\frac{A}{2} - sin\frac{B-C}{2}\right] - 1$$

$$= 2cos\frac{A}{2}\left[cos\frac{(180^\circ-(B+C))}{2} - sin\frac{B-C}{2}\right] - 1$$

$$= 2cos\frac{A}{2}\left[sin\frac{(B+C)}{2} - sin\frac{B-C}{2}\right] - 1$$

$$= 2cos\frac{A}{2}\,.2cos\frac{B}{2}sin\frac{C}{2} - 1$$

$$= -1 + 4cos\frac{A}{2}\,cos\frac{B}{2}sin\frac{C}{2}$$

iii) If $\alpha + \beta + \gamma = \pi$, then prove that

$$sin^2\alpha + sin^2\beta - sin^2\gamma = 2\,sin\alpha\,sin\beta\,cos\gamma$$

Proof:

$\alpha + \beta + \gamma = \pi \Longrightarrow \alpha + \beta = \pi - \gamma$

$\Longrightarrow \cos(\alpha + \beta) = \cos(\pi - \gamma) = -\cos\gamma$

$\Longrightarrow \cos\alpha\,cos\beta - sin\alpha\,sin\beta = -\cos\gamma$

$\Longrightarrow (\,sin\alpha\,sin\beta - cos\gamma)^2 = cos^2\alpha\,cos^2\beta$

$\Longrightarrow sin^2\alpha\,sin^2\beta + cos^2\gamma - 2\,sin\alpha\,sin\beta\,cos\gamma$

$$= (1 - sin^2\alpha)(1 - sin^2\beta)$$

$\Longrightarrow sin^2\alpha + sin^2\beta + cos^2\gamma - 1 = 2\,sin\alpha\,sin\beta\,cos\gamma$

$\Longrightarrow sin^2\alpha + sin^2\beta - sin^2\gamma = 2\,sin\alpha\,sin\beta\,cos\gamma$

iv) In a triangle ABC, then prove that

$\tan A + tanB + tanC = tanA\,tanB\,tanC$

Proof:

We have, $A + B + C = \pi \Longrightarrow A + B = \pi - C$

$$\tan(A + B) = \tan(\pi - C) = -\tan C$$

$$\Longrightarrow \tan A + tanB = -(1 - tanA\, tanB)\, tanC$$

$$\Longrightarrow \tan A + tanB + tanC = tanA\, tanB\, tanC$$

v) If $A + B + C = \pi$, then prove that

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$$

Proof:

$$A + B + C = \pi \Longrightarrow A + B = \pi - C$$

$$\Longrightarrow \cot(A + B) = \cot\,(\pi - c)$$

$$\Longrightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C$$

$$\Longrightarrow \cot B \cot C + \cot C \cot A + \cot A \cot B = 1$$

PROBLEM SET

IP1. If A, B, C are angles of a triangle, then prove that

$$\boldsymbol{cos\ 2A + cos\ 2B + cos\ 2C = -4\ cosA\ cosB\ cosC - 1}$$

Solution: We have, $A + B + C = \pi$

$cos\ 2A + cos\ 2B + cos\ 2C$

$$= 2cos(A + B)cos(A - B) + 2\ cos^2C - 1$$

$$= -2cosC\ cos(A - B) + 2\ cos^2C - 1$$

(since $cos(A + B) = \cos(\pi - C) = -cosC$)

$$= -2\cos C\ \{cos(A - B) - cosC\} - 1$$

$$= -2\ cosC\ \{cos(A - B) + \cos\,(A + B)\} - 1$$

$$= -2\ cosC\ (2cosA\ cosB) - 1$$

$$= -4\ cosA\ cosB\ cosC - 1.$$

P1. If A, B, C are angles of a triangle, then prove that

$$\boldsymbol{sin\ 2A + sin\ 2B - sin\ 2C = 4\ cosA\ cosB\ sin\ C}$$

Solution: We have, $A + B + C = \pi$

$sin\ 2A + sin\ 2B - sin\ 2C$

$$= 2sin(A + B)cos(A - B) - 2\ sinC\ cosC$$

$$A + B = \pi - C \Longrightarrow \sin(A + B) = \sin(\pi - C) = sinC$$

$$= 2sinC\ cos(A - B) - 2\ sinC\ cosC$$

$$= 2sinC\ \{cos(A - B) - cosC\}$$

$$A + B = \pi - C \Longrightarrow \cos(A + B) = \cos(\pi - C) = -cosC$$

$$= 2sinC\ \{cos(A - B) + \cos\,(A + B)\}$$

$$= 2\ sinC\ (2\ cosA\ cosB)$$

$$= 4\ cosA\ cosB\ sin\ C$$

IP2. If A, B, C are angles in a triangle, then prove that

$$\boldsymbol{cosA + cosB + cosC = 1 + 4\ sin\frac{A}{2}\ sin\frac{B}{2}\ sin\frac{C}{2}}$$

Solution: We have, $A + B + C = \pi$

$cosA + cosB + cosC$

$$= 2\ cos\left(\frac{A+B}{2}\right)\ cos\left(\frac{A-B}{2}\right) + 1 - \ 2\ sin^2\frac{C}{2}$$

(since $A + B + C = \pi \Longrightarrow \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2} \Longrightarrow cos\frac{A+B}{2} = sin\frac{C}{2}$

$$= 1 + 2\ sin\frac{C}{2}\ cos\left(\frac{A-B}{2}\right) - 2\ sin^2\frac{C}{2}$$

$$= 1 + 2\ sin\frac{C}{2}\left\{cos\left(\frac{A-B}{2}\right) - sin\frac{C}{2}\right\}$$

$$= 1 + 2\ sin\frac{C}{2}\left\{cos\left(\frac{A-B}{2}\right) - cos\left(\frac{A+B}{2}\right)\right\}$$

$$= 1 + 2\ sin\frac{C}{2}\left(2\sin\frac{A}{2}\ \sin\frac{B}{2}\right)$$

$$= 1 + 4\ \sin\frac{A}{2}\ \sin\frac{B}{2}\ sin\frac{C}{2}.$$

P2. If A, B, C are angles in a triangle, then prove that

$$\boldsymbol{sinA + sinB + sinC = 4\ cos\frac{A}{2}cos\frac{B}{2}\ cos\frac{C}{2}}$$

Solution:

We have, $A + B + C = \pi$

$sinA + sinB + sinC$

$$= 2\ sin\left(\frac{A+B}{2}\right)\ cos\left(\frac{A-B}{2}\right) + \ 2\ sin\frac{C}{2}\ cos\frac{C}{2}$$

$$= 2\ cos\frac{C}{2}\ cos\left(\frac{A-B}{2}\right) + \ 2\ sin\frac{C}{2}\ cos\frac{C}{2}$$

(since $sin\frac{A+B}{2} = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = cos\frac{C}{2}$)

$$= 2\ cos\frac{C}{2}\left\{cos\left(\frac{A-B}{2}\right) + sin\frac{C}{2}\right\}$$

$$= 2\ cos\frac{C}{2}\left\{cos\left(\frac{A-B}{2}\right) + cos\left(\frac{A+B}{2}\right)\right\}$$

(since $cos\frac{A+B}{2} = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = sin\frac{C}{2}$)

$$= 2\ cos\frac{C}{2}.\left(2\cos\frac{A}{2}\ \cos\frac{B}{2}\right)$$

$$= 4\cos\frac{A}{2}\ \cos\frac{B}{2}\ cos\frac{C}{2}$$

IP3. If $A+B+C=\pi$, then prove that

$$\sum \cos\frac{A}{2}\cos\frac{B-C}{2} = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Solution:

$$\sum \cos\frac{A}{2}\cos\frac{B-C}{2} = \frac{1}{2}\sum 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} = \frac{1}{2}\sum(sinB + sinC)$$

$$= \frac{1}{2}(sinB + sinC + sinC + sinA + sinA + sinB)$$

$$= sinA + sinB + sinC$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad \text{(As in P2)}$$

P3. In a triangle ABC, then prove that

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$$

Solution: We have, $A+B+C=\pi$

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

$$= \cos\frac{A}{2} + 2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$A+B+C=\pi \Rightarrow \cos\frac{A}{2} = \cos\left(\frac{\pi}{2} - \frac{B+C}{2}\right) = \sin\frac{B+C}{2}$$

$$= \sin\frac{B+C}{2} + 2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$= 2\sin\frac{B+C}{4}\cos\frac{B+C}{4} + 2\cos\frac{B+C}{2}\cos\frac{B-C}{4}$$

$$= 2\cos\frac{B+C}{4}\left(\sin\frac{B+C}{4} + \cos\frac{B-C}{4}\right)$$

$$= 2\cos\frac{B+C}{4}\left[\cos\left(\frac{\pi}{2} - \frac{B+C}{4}\right) + \cos\frac{B-C}{4}\right]$$

$$= 2\cos\frac{B+C}{4}\, 2\cos\frac{\pi-C}{4}\cos\frac{\pi-B}{4}$$

$$= 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$$

IP4. If $A+B+C=\pi$, then prove that

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Solution:

Given, $A+B+C=\pi$

$$A+B = \pi - C \Rightarrow \cos\left(\frac{A}{2} + \frac{B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\frac{C}{2}$$

$$\Rightarrow \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2} = \sin\frac{C}{2}$$

$$\Rightarrow \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{C}{2} = \sin\frac{A}{2}\sin\frac{B}{2}$$

$$\Rightarrow \left(cos\frac{A}{2}cos\frac{B}{2} - sin\frac{C}{2}\right)^2 = sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Rightarrow cos^2\frac{A}{2}cos^2\frac{B}{2} + sin^2\frac{C}{2} - 2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2} = sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Rightarrow \left(1 - sin^2\frac{A}{2}\right)\left(1 - sin^2\frac{B}{2}\right) + sin^2\frac{C}{2} - 2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2} = sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Rightarrow 1 - sin^2\frac{A}{2} - sin^2\frac{B}{2} + sin^2\frac{A}{2}\ sin^2\frac{B}{2} + sin^2\frac{C}{2} - 2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$$
$$= sin^2\frac{A}{2}sin^2\frac{B}{2}$$

$$\Rightarrow sin^2\frac{A}{2} + sin^2\frac{B}{2} - sin^2\frac{C}{2} = 1 - 2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$$

P4. If $A + B + C = \pi$, then prove that

$$\boldsymbol{sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2} = 1 - 2\ \sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}}.$$

Solution:

$$sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2}$$

$$= \frac{1 - cosA}{2} + \frac{1 - cosB}{2} + \frac{1 - cosC}{2}$$

$$= \frac{3}{2} - \frac{1}{2}(\cos A + \cos B + \cos C)$$

$$= \frac{3}{2} - \frac{1}{2}\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) \qquad \text{(As in **IP2**)}$$

$$= \frac{3}{2} - \frac{1}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 - 2\ \sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}$$

Aliter:

We have given, $A + B + C = \pi$

$$\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2} \Rightarrow cos\frac{A+B}{2} = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = sin\frac{C}{2}$$

$$\Rightarrow cos\frac{A}{2}cos\frac{B}{2} - sin\frac{A}{2}sin\frac{B}{2} = sin\frac{C}{2}$$

$$\Rightarrow cos^2\frac{A}{2}cos^2\frac{B}{2} = \left(sin\frac{C}{2} + sin\frac{A}{2}sin\frac{B}{2}\right)^2$$

$$\Rightarrow cos^2\frac{A}{2}cos^2\frac{B}{2} = sin^2\frac{C}{2} + sin^2\frac{A}{2}sin^2\frac{B}{2} + 2sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

$$\Rightarrow \left(1 - sin^2\frac{A}{2}\right)\left(1 - sin^2\frac{B}{2}\right) = sin^2\frac{C}{2} + sin^2\frac{A}{2}sin^2\frac{B}{2} + 2sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

$$\Rightarrow 1 - sin^2\frac{A}{2} - sin^2\frac{B}{2} + sin^2\frac{A}{2}sin^2\frac{B}{2}$$
$$= sin^2\frac{C}{2} + sin^2\frac{A}{2}sin^2\frac{B}{2} + 2sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$
$$\Rightarrow 1 - sin^2\frac{A}{2} - sin^2\frac{B}{2} = sin^2\frac{C}{2} + 2sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$
$$\Rightarrow sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2} = 1 - 2sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

Exercises:

If $A + B + C = 180^{\circ}$, prove that

1. $cos^2A + cos^2B + cos^2C = 1 - 2\cos A\cos B\cos C.$
2. $cos^2A + cos^2B - cos^2C = 1 - 2\sin A\sin B\cos C.$
3. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$
4. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$
5. $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B)$
$= 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\cos\frac{A-B}{2}$
6. $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1 = 4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$
7. $\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2} = 4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}.$
8. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \cos C} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$
9. $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C)$
$= 4\sin A\sin B\sin C.$
10. In a triangle ABC if $\cot A + \cot B + \cot C = \sqrt{3}$, prove that the triangle is equilateral.

1.3Trigonometric Equations

Learning objectives:

1. To find principle solution and general solution of a trigonometric equation.
2. To use different methods to solve trigonometric equations.
   And
3. To practice the related problems.

Solving Trigonometric Equations

Trigonometric equations are equations involving trigonometric functions of unknown angles and they, unlike identities, are satisfied only by particular values of the unknown angles. For example, $sin\, x\, .\, csc\, x = 1$ is an identity, being satisfied by every value of x for which $\sin x$ and $\csc x$ are defined.

$\sin x = 0$; it is not satisfied by $x = \frac{\pi}{4}$ or $\frac{\pi}{2}$. Since it is not satisfied by every value of x for which it is defined, it is not an identity. It is an equation and we will find the particular values of x for which this equation is satisfied.

A ***solution*** of a trigonometric equation is a value of the angle x which satisfies the equation.

If a given equation has one solution, it has in general an unlimited number of solutions due to the periodicity of the trigonometric functions.

Two solutions of $\sin x = 0$ are $x = 0$ and $x = \pi$.

The complete solution of $\sin x = 0$ is given by

$$x = 0 + 2n\pi, \qquad x = \pi + 2n\pi$$

where n is any integer. Both these expressions can be combined into a single expression $x = n\pi$, where n is any integer.

The solution consisting of all possible solutions of a trigonometric equation is called its ***general solution***.

There is no general method for solving trigonometric equations. Several standard procedures are employed in the solution of trigonometric equations.

The numerically least angle of the solution is called the ***principal value or principle solution***.

For example, find the principal value of $\sin x = \frac{1}{2}$.

The numerically least value will be in the first quadrant. Therefore, the principal value is $x = \frac{\pi}{6}$

- Find the principal value of x satisfying $\sin x = -\frac{1}{2}$.

The sine is negative in 3rd or 4th quadrant. Therefore, the principal value is $x = -\frac{\pi}{6}$

- Find the principal value of x satisfying $\tan x = -1$.

Tan is negative in 2^{nd} and 4^{th} quadrants. The principal value is $x = -\frac{\pi}{4}$

- Find the principal value of x satisfying $\cos x = \frac{1}{2}$.

Cosine is positive in first and fourth quadrants. The principal value is $x = \frac{\pi}{3}$

Principal value lies in the first quadrant. It is never numerically greater than π. The clockwise or anticlockwise direction is chosen depending on whether the angle is in 3rd and 4th quadrant or in first and second quadrants.

Factorable Equations

Solve: $sin\, x - 2\, sin\, x\; cos\, x = 0.$

Solution: Factoring,

$$\sin x - 2\sin x \cos x = sinx(1 - 2cosx) = 0$$

Setting each factor equal to zero, we get

$$\sin x = 0 \Longrightarrow x = 0, \pi$$

$$1 - 2\cos x = 0 \Longrightarrow \cos x = \frac{1}{2} \Longrightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Expressible in terms of a Single Function

Solve: $2\, tan^2 x + sec^2 x = 2$

Solution: Given $2\, tan^2 x + sec^2 x = 2$

$$2\, tan^2 x + 1 + tan^2 x = 2 \Longrightarrow\; 3tan^2 x = 1$$

$$\Longrightarrow\; \tan x = \pm\frac{1}{\sqrt{3}}$$

For $\tan x = \frac{1}{\sqrt{3}}, \quad x = \frac{\pi}{6}, \; \frac{7\pi}{6}$

For $\tan x = -\frac{1}{\sqrt{3}}, \quad x = \frac{5\pi}{6}, \; \frac{11\pi}{6}$

Example:

Solve: $\sec x + \tan x = 0$

Solution: Given $\sec x + \tan x = 0$

$$\frac{1}{cos\, x} + \frac{\sin x}{cos\, x} = 0$$

Multiplying by $\cos x$, we have $1 + \sin x = 0,\; \sin x = -1$. Then $x = \frac{3\pi}{2}$

However, neither $\sec x$ nor $\tan x$ is defined when $x = \frac{3\pi}{2}$ and the equation has no solution.

This illustrates that there is a need to check the solution before accepting it as a solution of the equation.

Squaring Both Members of the Equation

Solve: $\sin x + \cos x = 1$

Solution: We write the equation in the form $\sin x = 1 - \cos x$ and square both members. We have

$$sin^2x = 1 - 2\cos x + cos^2x$$

$\Longrightarrow 1 - cos^2x = 1 - 2\cos x + cos^2x$

$\Longrightarrow 2\,cos^2x - 2\cos x = 0 \Longrightarrow 2\cos x(\cos x - 1) = 0$

From $2cos\ x = 0$, $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

From $\cos x = 1$, $\quad x = 0$

Check:

For $x = 0$, $\quad \sin x + cosx = 0 + 1 = 1$

For $x = \frac{\pi}{2}$, $\quad \sin x + cosx = 1 + 0 = 1$

For $x = \frac{3\pi}{2}$, $\quad \sin x + cosx = -1 + 0 \neq 1$

Thus, the solution is $x = 0$ and $\frac{\pi}{2}$.

The value $= \frac{3\pi}{2}$, called an *extraneous solution*, was introduced by squaring the members.

General solution of the equations

$sin\ x = 0, cos\ x = 0$ and $tan\ x = 0$

(i). If $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then $sin\theta = 0$ if and only if $\theta = 0$. Thus the principal solution of $sin\ x = 0$ is 0. Let $\theta \in R$ be any solution of $sin\ x = 0$. Then these exists a $k \in Z$ such that

$$k \leq \frac{\theta}{2\pi} < k + 1$$

$$\Rightarrow 2\pi k \leq \theta < 2\pi k + 2\pi$$

$$i.e., 0 \leq \theta - 2k\pi < 2\pi$$

Since θ and $\theta - 2\pi$ are co terminal angles,

$$\sin\theta = \sin(\theta - 2k\pi) = 0$$

$\Rightarrow \ \theta - 2k\pi = 0$ or $\pi \Rightarrow \theta = 2k\pi$ or $(2k+1)\pi, k \in \boldsymbol{Z}$

i.e., $\theta = n\pi, n \in \boldsymbol{Z}$

Thus, $sin\theta = 0 \iff \theta = n\pi, n \in \boldsymbol{Z}$

Therefore, the general solution of the equation

$$\boldsymbol{sinx = 0\ is\ x = n\pi, n \in Z}$$

(ii). The principal solution of $cosx = 0$ is $x = \frac{\pi}{2}$. Further,

$$cos\ x = 0 \iff \sin\left(x - \frac{\pi}{2}\right) = 0 \iff x - \frac{\pi}{2} = n\pi,\ n \in \boldsymbol{Z}$$

$$\iff x = n\pi + \frac{\pi}{2}$$

$$\iff x = (2n+1)\frac{\pi}{2}, n \in \boldsymbol{Z}$$

Thus, the general solution of $cosx = 0$ is

$$\boldsymbol{x = (2n+1)\frac{\pi}{2}, n \in Z}$$

(iii). The principal value of $tan\ x = 0$ is $x = 0$. Further,

$$tanx = 0 \iff sinx = 0 \iff x = n\pi, n \in \boldsymbol{Z}$$

Thus, the general solution of $tanx = 0$ is

$$\boldsymbol{x = n\pi, \qquad n \in Z}$$

NOTE: The general solution of $cot\ x = 0$ is given by

$$\boldsymbol{x = (2n+1)\frac{\pi}{2}, n \in Z}$$

General solution of $\boldsymbol{sinx = k, |k| \leq 1}$

Since $|k| \leq 1$, there exists a principal solution say α,

i.e., There is a $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $sin\ \alpha = k$

Let θ be any solution of $sin\ x = k$, then

$$sin\theta = sin\alpha \iff sin\theta - sin\alpha = 0$$

$$\iff 2cos\frac{\theta+\alpha}{2}.sin\frac{\theta-\alpha}{2} = 0$$

$$\iff 2cos\frac{\theta+\alpha}{2} = 0 \text{ or } sin\frac{\theta-\alpha}{2} = 0$$

Now, $cos\frac{\theta+\alpha}{2} = 0 \iff \frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}, n \in \boldsymbol{Z}$

$$\iff \theta = (2n+1)\pi - \alpha, n \in \boldsymbol{Z}$$

and, $sin\frac{\theta-\alpha}{2} = 0 \iff \frac{\theta-\alpha}{2} = n\pi, n \in \boldsymbol{Z}$

$$\iff \theta = 2n\pi + \alpha, n \in \boldsymbol{Z}$$

Combining those two, $\theta = n\pi + (-1)^n\alpha, n \in \boldsymbol{Z}$

Thus, the general solution of the equation $sinx = k,\ |k| \leq 1$ is

$$\boldsymbol{x = n\pi + (-1)^n\alpha}$$

Where α is the principal solution (or any solution) of the equation.

By similar argument we prove the following

- The general solution of the equation of

$cosx = k, |k| \leq 1$ is $\boldsymbol{x = 2n\pi \pm \alpha, n \in Z}$

where α is the principal solution (or any solution) of the equation

- The general solution of the equation $tanx = k,\ k \in \boldsymbol{R}$ is $\boldsymbol{x = n\pi + \alpha,\ n \in Z}$**,** where α is the principal solution (or any solution) of the equation.

Summary

The equation $\boldsymbol{f(x) = k}$	Range of $\boldsymbol{k}$	The interval in which the principal solution $\boldsymbol{\alpha}$ lies	General solution
$\sin x = k$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$n\pi + (-1)^n\alpha\ , n \in \boldsymbol{Z}$
$cosx = k$	$[-1,1]$	$[0,\pi]$	$2n\pi \pm \alpha\ , n \in \boldsymbol{Z}$
$tanx = k$	$\boldsymbol{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$n\pi + \alpha\ , n \in \boldsymbol{Z}$
$cscx = k$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	$n\pi + (-1)^n\alpha\ , n \in \boldsymbol{Z}$
$secx = k$	$(-\infty, -1] \cup [1, \infty)$	$[0,\pi] - \left\{\frac{\pi}{2}\right\}$	$2n\pi \pm \alpha\ , n \in \boldsymbol{Z}$
$cotx = k$	$\boldsymbol{R}$	$(0,\pi)$	$n\pi + \alpha\ , n \in \boldsymbol{Z}$

Example:

Solve the equation $\sin x + \sin 5x = \sin 3x$.

$$\Longrightarrow 2\sin 3x \cos 2x = \sin 3x$$

$$\Longrightarrow \sin 3x\,(2\cos 2x - 1) = 0$$

Therefore, $\sin 3x = 0$, or $\cos 2x = \frac{1}{2}$

If $\sin 3x = 0,$ then $3x = n\pi, n \in \boldsymbol{Z}$.

If $\cos 2x = \frac{1}{2},$ then $2x = 2n\pi \pm \frac{\pi}{3}\ , n \in \boldsymbol{Z}$

Hence$x = \frac{n\pi}{3},$ or $n\pi \pm \frac{\pi}{6}\ , n \in \boldsymbol{Z}$.

The general solution of $\boldsymbol{sin^2x = k, 0 \leq k \leq 1}$ is

$\boldsymbol{x = n\pi \pm \alpha, n \in Z}$**, where $\boldsymbol{\alpha}$ is a solution of $\boldsymbol{sin^2x = k}$**

Proof:

The trigonometric equation $sinx = k$ has a solution if and only if $k \in [0,1]$. Thus there exists a solution say $\alpha \in \boldsymbol{R}$ such that $sin^2x = sin^2\alpha$. Now

$sin^2x = sin^2\alpha \Leftrightarrow 1 - 2sin^2x = 1 - 2sin^2\alpha$

$$\Leftrightarrow cos2x = cos2\alpha.$$

$$\Leftrightarrow 2x = 2n\pi \pm 2\alpha,\ n \in \boldsymbol{Z}$$

$$\Leftrightarrow \boldsymbol{x = n\pi \pm \alpha},\ \boldsymbol{n} \in \boldsymbol{Z}$$

where α is a solution of $sin^2x = k$.

By a similar method we prove the following

- The general solution of $cos^2x = k, 0 \le k \le 1$ is

$\boldsymbol{x = n\pi \pm \alpha, n} \in \boldsymbol{Z}$, where α is a solution of $cos^2x = k$

- The general solution of $tan^2x = k, 0 \le k < \infty$ is

$\boldsymbol{x = n\pi \pm \alpha, n} \in \boldsymbol{Z}$, where α is a solution of $tan^2x = k$

Equations of the form $\boldsymbol{a\ cos\ \theta + b\ sin\theta = c}$

We divide both sides of the equation by $\sqrt{a^2+b^2}$, so that it may be written as

$$\frac{a}{\sqrt{a^2+b^2}}\ cos\theta + \frac{b}{\sqrt{a^2+b^2}}\ sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

If we introduce the angle α, so that $\tan\alpha = \frac{b}{a}$

Then $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ $\qquad cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$

Also, we introduce the angle β, so that

$$cos\beta = \frac{c}{\sqrt{a^2+b^2}}$$

The equation can then be written

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \cos\beta$$

The equation is then $\cos(\theta - \alpha) = \cos\beta$.

The solution of this is $\theta - \alpha = 2n\pi \pm \beta$, so that

$$\boldsymbol{\theta = 2n\pi + \alpha \pm \beta}$$

where n is any integer.

Angles, such as α and β, which are introduced to facilitate computation are called **Subsidiary Angles**.

Example: Solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$

We have $\sqrt{a^2+b^2} = \sqrt{1+3} = 2$. We therefore write the equations as

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

Therefore

$$\cos x\ cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4} \Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$$

Taking the positive sign,

$$x = 2n\pi + \frac{\pi}{6} + \frac{\pi}{4} = 2n\pi + \frac{5\pi}{12}$$

Taking the negative sign,

$x = 2n\pi + \frac{\pi}{6} - \frac{\pi}{4} = 2n\pi - \frac{\pi}{12}, n \in \boldsymbol{Z}$

PROBLEM SET

IP1. Find the general solution of $2cos^2u = 1 - cosu$.

Solution:

Given, $2cos^2u = 1 - cosu$

$\Longrightarrow 2cos^2u + cosu - 1 = 0$

$\Longrightarrow (2cosu - 1)(cosu + 1) = 0$

$\Longrightarrow 2cosu - 1 = 0$ or $cosu + 1 = 0$

$\Longrightarrow cosu = \frac{1}{2}$ or $cosu = -1$

$\Longrightarrow u = \frac{\pi}{3}$ or $u = \pi$

General solution is $\left\{2n\pi \pm \frac{\pi}{3} \mid n \in \boldsymbol{Z}\right\} \cup \{2n\pi \pm \pi \mid n \in \boldsymbol{Z}\}$

P1. Find the general solution of $2cos^2x\ tanx = tanx$.

Solution:

Given, $2cos^2x\ tanx = tanx$

$\Longrightarrow tanx(2cos^2x - 1) = 0 \Longrightarrow tanx = 0\ or\ (2cos^2x - 1) = 0$

$\Longrightarrow tanx = 0\ or\ cosx = \frac{1}{\sqrt{2}} \Longrightarrow x = 0, \pi\ or\ x = \frac{\pi}{4}$

$\Longrightarrow x = n\pi\ or\ x = 2n\pi \pm \frac{\pi}{4}, n \in \boldsymbol{Z}$

IP2. Solve: $2cos^2\theta + 11\ sin\theta = 7$

Solution:

$2cos^2\theta + 11\ sin\theta = 7$

$\Longrightarrow 2\ (1 - sin^2\theta) + 11\ sin\theta - 7 = 0$

$\Longrightarrow 2 - 2\ sin^2\theta + 11\ sin\theta - 7 = 0$

$\Longrightarrow 2\ sin^2\theta - 11\ sin\theta + 5 = 0$

$\Longrightarrow (2\sin\theta - 1)(\sin\theta - 5) = 0$

$\Longrightarrow 2\sin\theta - 1 = 0\ or\ sin\theta - 5 = 0$

$\Longrightarrow \sin\theta = \frac{1}{2}$ or $sin\theta = 5 > 1$

$\Longrightarrow \sin\theta = \frac{1}{2} \Longrightarrow \theta = \frac{\pi}{6}$ $(Principle\ solutions)$

The general solution is, $\theta = n\pi + (-1)^n \frac{\pi}{6}$, $n \in \boldsymbol{Z}$

P2. Solve $3\cos 2\theta + 2 = 7\ \sin\theta$

Solution:

$3\cos 2\theta + 2 = 7\ \sin\theta$

$\Longrightarrow 3\,(1 - 2sin^2\theta) + 2 = 7\ sin\theta$

$\Longrightarrow\ 6\ sin^2\theta + 7\ sin\theta - 5 = 0$

$\Longrightarrow (2\sin\theta - 1)(3\sin\theta + 5) = 0$

$\Longrightarrow\ \sin\theta = \frac{1}{2}, -\frac{5}{3}$ and $\sin\theta = -\frac{5}{3} \notin [-1,1]$

$\therefore\ \sin\theta = \frac{1}{2} \Longrightarrow\ \theta = \frac{\pi}{6}$ (Principal solution)

The general solution is $n\pi + (-1)^n \frac{\pi}{6}$ $, n \in \boldsymbol{Z}$

IP3. Solve: $2cosx + 2sinx = \sqrt{6}$

Solution:

Given, $2cosx + 2sinx = \sqrt{6}$

$$\Longrightarrow \frac{cosx}{\sqrt{2}} + \frac{sinx}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Longrightarrow \cos\frac{\pi}{4}cosx + sin\frac{\pi}{4}sinx = cos\frac{\pi}{6}$$

$$\Longrightarrow \cos\left(x - \frac{\pi}{4}\right) = cos\frac{\pi}{6}$$

$\Longrightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{6}$

$\Longrightarrow x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{6}$

$\therefore\ x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{6}, n \in \boldsymbol{Z}$

P3. Solve: $\sqrt{3}\cos x - sinx = 1$

Solution:

Given, $\sqrt{3}\cos x - sinx = 1$

$\Longrightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}sinx = \frac{1}{2}$

$\Longrightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}sinx = \cos\frac{\pi}{3}$

$\Longrightarrow \cos\left(x + \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$

$\Longrightarrow x + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

$$\Rightarrow x = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{3}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{3}, n \in \boldsymbol{Z}$$

IP4. If $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ are the solutions of $\boldsymbol{atan\theta + bsec\theta = c}$, then show that $\tan(\boldsymbol{\alpha + \beta}) = \frac{\boldsymbol{2ac}}{\boldsymbol{a^2 - c^2}}$

Solution:

Given, $a\, tan\theta + b\, sec\theta = c$

$$bsec\theta = c - atan\theta$$

$$\Rightarrow b^2 sec^2\theta = c^2 + a^2 tan^2\theta - 2actan\theta$$

$$\Rightarrow b^2(1 + tan^2\theta) = c^2 + a^2 tan^2\theta - 2ac\, tan\theta$$

$$\Rightarrow (a^2 - b^2)tan^2\theta - 2ac\, tan\theta + (c^2 - b^2) = 0$$

Also given, α and β are the solutions of θ.

$tan\alpha + tan\beta = \frac{2ac}{a^2 - c^2}$ and $tan\alpha \cdot tan\beta = \frac{c^2 - b^2}{a^2 - b^2}$

$$\tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta} = \frac{\left(\frac{2ac}{a^2 - c^2}\right)}{1 - \left(\frac{c^2 - b^2}{a^2 - b^2}\right)} = \frac{2ac}{a^2 - c^2}$$

P4. If $\boldsymbol{\theta_1, \theta_2}$ are solutions of the equation

$\boldsymbol{acos2\theta + bsin2\theta = c, tan\theta_1 \neq tan\theta_2}$ and $\boldsymbol{a + c \neq 0}$, then find the values of

i. $\boldsymbol{tan\theta_1 + tan\theta_2}$
ii. $\boldsymbol{tan\theta_1 \cdot tan\theta_2}$

Solution:

Given, $acos2\theta + bsin2\theta = c\, , a + c \neq 0$

$$\Rightarrow a\left(\frac{1 - tan^2\theta}{1 + tan^2\theta}\right) + b\left(\frac{2tan\theta}{1 + tan^2\theta}\right) = c$$

$$\Rightarrow a - atan^2\theta + 2btan\theta = c + ctan^2\theta$$

$$\Rightarrow (a + c)tan^2\theta - 2btan\theta + c - a = 0$$

$$\Rightarrow (a + c)tan^2\theta - 2btan\theta + c - a = 0$$

This is a quadratic equation in $tan\,\theta$. Since θ_1, θ_2 are roots of the given equation, we get $tan\theta_1$ and $tan\theta_2$ are roots of the equation.

$\therefore$ sum of the roots $tan\theta_1 + tan\theta_2 = \frac{2b}{a+c}$ and

Product of the roots $tan\theta_1 \cdot tan\theta_2 = \frac{c-a}{a+c}$

Exercises

Solve the following equations.

a) $sin\theta + sin7\theta = sin\ 4\theta$
b) $cos\theta + cos7\theta = cos4\theta$
c) $cos\theta + cos3\theta = 2\cos 2\theta$
d) $\sin 4\theta - \sin 2\theta = \cos 3\theta$
e) $cos\theta - sin3\theta = \cos 2\theta$
f) $\sin 7\theta = \sin\theta + \sin 3\theta$
g) $\cos\theta + \cos 3\theta = 0$
h) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$
i) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$
j) $\cos n\theta = \cos(n-2)\theta + \sin\theta$
k) $\sin\frac{n+1}{2}\theta = \sin\frac{n-1}{2}\theta + \sin\theta$
l) $\sin m\theta + \sin n\theta = 0$
m) $\cos m\theta + \cos n\theta = 0$
n) $\sin 3\theta + \cos 2\theta = 0$
o) $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$
p) $\sin\theta + \cos\theta = \sqrt{2}$
q) $sin^2\ n\theta - sin^2\ (n-1)\theta = sin^2\ \theta$

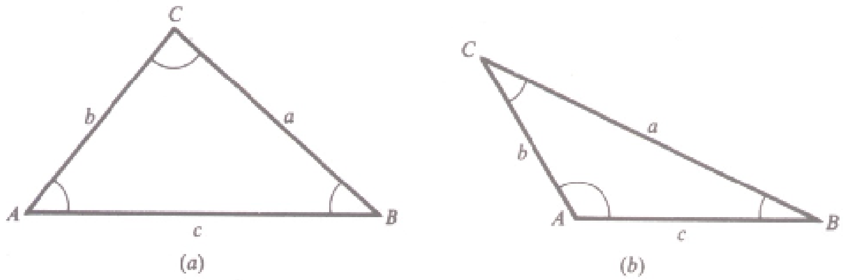
1.4. Relations between the Angles and Sides of a Triangle

Learning Objectives:

1. To derive law of sines, law of cosines and to find sines, cosines, tangents of half angles in terms of sides.
2. To derive tangent rules and projection formulas
   And
3. To practice the related problems.

Notation

A right triangle is the one which has a right angle as one of its angles. On the other hand, a general triangle is one which does not contain a right angle. Such a triangle contains either three acute angles or two acute angles and one obtuse angle.



It is a standard convention to denote the length of sides opposite to angles *A*, *B*, and *C* by *a*, *b*, and *c* respectively.

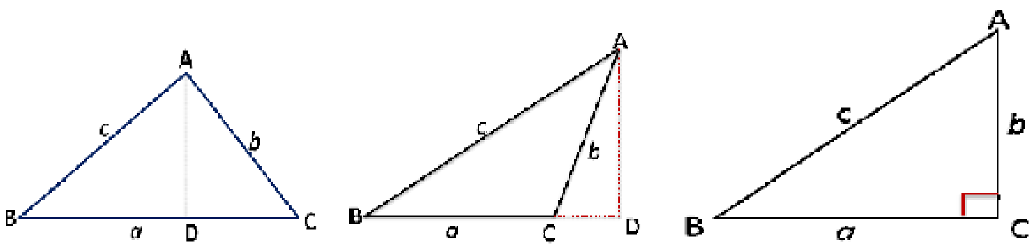
The Law of Sines

Theorem: In any triangle ABC

$$\frac{\boldsymbol{sinA}}{\boldsymbol{a}} = \frac{\boldsymbol{sinB}}{\boldsymbol{b}} = \frac{\boldsymbol{sinC}}{\boldsymbol{c}}$$

i.e., the sines of the angles are proportional to the opposite sides

Proof:



(i) (ii) (iii)

Draw AD perpendicular to the opposite side meeting it, produced if necessary in the point D.

In $\Delta\, ABD$, we have $\frac{AD}{AB} = sinB \Rightarrow AD = c\, sin\, B$

In $\Delta\, ACD$, we have $\frac{AD}{AC} = sinC \Rightarrow AD = b\, sin\, C$

If the angle C is obtuse, then as in the second figure, we have

$$\frac{AD}{AC} = sinACD = \ sin\ (\pi - C) = sinC$$

$$\Longrightarrow AD = b\ sinC$$

Thus $c\ sinB = b\ sinC$, i.e., $\frac{sinB}{b} = \frac{sinC}{c}$

Similarly, by drawing a perpendicular from B onto CA,

we prove that $\frac{sinC}{c} = \frac{sinA}{a}$

If one of the angles, say C is a right angle as in the third figure then $sinC = 1$, $sinA = \frac{a}{c}$, $sinB = \frac{b}{c}$

Therefore, $\frac{sinA}{a} = \frac{sinB}{b} = \frac{1}{c} = \frac{sinC}{c}$

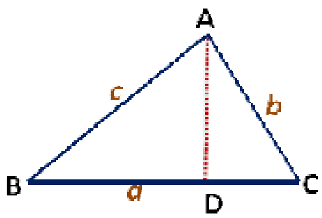
We have, in all cases

$$\boldsymbol{\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}}$$

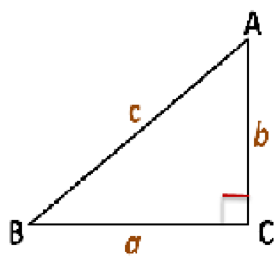
Hence the theorem

The law of Cosines

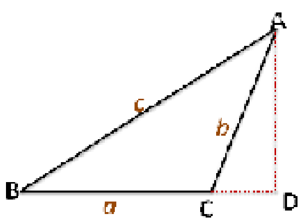
In any triangle ABC, the square of any side is equal to the sum of the squares of the other two sides diminished by twice the product of these sides and the cosine of the included angle.



(i)



(ii)



(iii)

Let ABC be the triangle and let the perpendicular from A on BC meet it, produced if necessary, in the point *D.*

(*i*) Let the angle C be acute, as in fig (i) then

$$BD = BC - DC = a - b\ cosC$$

In the right triangle ABD, we have

$$\begin{aligned} c^2 &= BD^2 + AD^2 \\ &= (a - b\ cosC)^2 + b^2\ sin^2C \\ &= a^2 - 2\ ab\ cosC + b^2\ cos^2C + b^2\ sin^2C \\ &= a^2 + b^2 - 2\ ab\ cosC \end{aligned}$$

(ii) Let the angle C be obtuse as in fig (ii).

Then $\quad BD = BC + CD = a + b\cos(\pi - C)$

$$= a - b\cos C$$

In the right angle ABD,we have

$$c^2 = BD^2 + AD^2$$

$$= (a - b\cos C)^2 + b^2\sin^2 C$$

$= a^2 + b^2 - 2\,ab\cos C$ as in (i) (iii) Let the angle C be a right angle as in fig (iii)

Then $\cos C = 0$ and

$$c^2 = a^2 + b^2 = a^2 + b^2 - 2\,ab\cos C$$

In any case,

$$\boldsymbol{c^2 = a^2 + b^2 - 2\,ab\cos C}$$

Similarly it may be shown that

$$\boldsymbol{a^2 = b^2 + c^2 - 2\,bc\cos A}$$
$$\boldsymbol{b^2 = c^2 + a^2 - 2\,ca\cos B}$$

Example:

The sides of a triangle are 8 cm, 10 cm and 12 cm. Prove that the greatest angle is double the smallest angle.

Solution:

Let $a = 8, b = 10, c = 12$

Here the greatest angle is C and the smallest angle is A and we have to prove C = 2A. Now

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{64+100-144}{2\times8\times10} = \frac{1}{8}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{100+144-64}{2\times10\times12} = \frac{3}{4}$$

and $\quad \cos 2A = 2\cos^2 A - 1 = 2\times\left(\frac{3}{4}\right)^2 - 1 = \cos C$

$$\Longrightarrow 2A = C$$

Sines, Cosines, Tangents of Half Angles

(i). $2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{2bc-b^2-c^2+a^2}{2bc}$

$$= \frac{a^2-(b^2+c^2-2bc)}{2bc} = \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

Let $2s$ stand for $a+b+c$, so that $s=\frac{a+b+c}{2}$, Then $a+b-c=2(s-c)$ and $c+a-b=2(s-b)$

Therefore,

$$2sin^2\frac{A}{2}=\frac{2(s-c)\times 2(s-b)}{2bc} \Longrightarrow \boldsymbol{sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}}$$

Similarly, $\boldsymbol{sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}}$, $\boldsymbol{sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}}$

(ii). $2cos^2\frac{A}{2}=1+cosA=1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}$

$$=\frac{(b+c)^2-a^2}{2bc}=\frac{(a+b+c)(b+c-a)}{2bc}=\frac{2s\times 2(s-a)}{2bc}$$

$$\Longrightarrow \boldsymbol{cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}}$$

Similarly, $\boldsymbol{cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}}$ **and** $\boldsymbol{cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}}$

(iii). $\boldsymbol{tan\frac{A}{2}}=\frac{sin\frac{A}{2}}{cos\frac{A}{2}}=\boldsymbol{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}$

Similarly, $\boldsymbol{tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}}$ and $\boldsymbol{tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$

Since, in a triangle, A is always less than 180°, A/2 is always less than 90°. The sine, cosine, and tangent of A/2 are therefore always positive.

Some Useful Identities

We will express the sine of any angle of a triangle in terms of its sides.

$$sinA=2sin\frac{A}{2}cos\frac{A}{2}=2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \boldsymbol{sinA=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}}$$

The following identity is useful in solving the problems

$$tan\frac{B-C}{2}=\frac{b-c}{b+c}cot\frac{A}{2}$$

We prove this identity as follows.

In any triangle, we have $\frac{b}{c}=\frac{sin\,B}{sinC}$

Therefore,

$$\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}=\frac{2cos\frac{B+C}{2}\,sin\frac{B-C}{2}}{2sin\frac{B+C}{2}\,cos\frac{B-C}{2}}=\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}=\frac{\tan\frac{B-C}{2}}{tan\left(90°-\frac{A}{2}\right)}=\frac{\tan\frac{B-C}{2}}{cot\frac{A}{2}}$$

Hence ,$\boldsymbol{tan\frac{B-C}{2}=\frac{b-c}{b+c}cot\frac{A}{2}}$

This is known as ***Tangent Rule***. Two other formulas can be written in a similar manner.

Projection Formulae

In any triangle ABC, $\boldsymbol{a = bcosC + ccosB}$

$$\boldsymbol{b = ccosA + acosC}$$

$$\boldsymbol{c = acosB + bcosA}$$

Proof: From cosine rules

$cosC = \frac{a^2+b^2-c^2}{2ab}$, $cosB = \frac{a^2+c^2-b^2}{2ac}$

$$\therefore b\ cosC + c\ cosB = \frac{a^2+b^2-c^2}{2a} + \frac{a^2+c^2-b^2}{2a}$$

$$= \frac{2a^2}{2a} = a$$

Similarly, the other laws can be proved.

PROBLEM SET

IP1. If $\boldsymbol{a = 6, b = 5, c = 9}$, then find angle $\boldsymbol{A}$?

Solution:

Given, $a = 6, b = 5, c = 9$

From cosine rule,

$$cosA = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{25+81-36}{2.5.9} = \frac{70}{90} = \frac{7}{9}$$

$$\therefore A = Cos^{-1}\left(\frac{7}{9}\right)$$

P1. Show that $\boldsymbol{(b-c)^2\ cos^2\frac{A}{2} + (b+c)^2\ sin^2\frac{A}{2} = a^2}$

Solution:

$$(b^2 + c^2 - 2bc)cos^2\frac{A}{2} + (b^2 + c^2 + 2bc)sin^2\frac{A}{2}$$

$$= (b^2 + c^2)\left(cos^2\frac{A}{2} + sin^2\frac{A}{2}\right) - 2bc\left(cos^2\frac{A}{2} - sin^2\frac{A}{2}\right)$$

$$= b^2 + c^2 - 2bc\ cosA = a^2$$

IP2. If $\boldsymbol{tan\frac{A}{2} = \frac{5}{6}}$ and $\boldsymbol{tan\frac{C}{2} = \frac{2}{5}}$, determine the relation among $\boldsymbol{a, b, c}$.

Solution:

Given, $tan\frac{A}{2} = \frac{5}{6}$ and $tan\frac{C}{2} = \frac{2}{5}$

$$tan\frac{A}{2}.tan\frac{C}{2} = \frac{5}{6}.\frac{2}{5} = \frac{2}{6}$$

i.e., $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{2}{6}$

$$\Rightarrow \frac{s-b}{s} = \frac{1}{3} \Rightarrow 3s - 3b = s \Rightarrow 2s = 3b$$

$$\Rightarrow a + b + c = 3b \Rightarrow a + c = 2b.$$

Hence a, b, c are in $A.P.$

P2. In $\Delta\ ABC$, find $bcos^2\frac{C}{2} + ccos^2\frac{B}{2}$

Solution:

$$bcos^2\frac{C}{2} + ccos^2\frac{B}{2} = b\left[\frac{s(s-c)}{ab}\right] + c\left[\frac{s(s-b)}{ca}\right]$$

$$= \frac{s(s-c)}{a} + \frac{s(s-b)}{a}$$

$$= \frac{s}{a}[s - c + s - b]$$

$$= \frac{s}{a}[2s - c - b]$$

$$= \frac{s}{a}[a + b + c - c - b]$$

$$= \frac{s}{a}.a = s$$

$$\therefore bcos^2\frac{C}{2} + ccos^2\frac{B}{2} = s$$

IP3. $\frac{a+b}{a-b} = tan\frac{A+B}{2}cot\frac{A-B}{2}$

Solution: Given,

$$\frac{a+b}{a-b} = \frac{sinA + sinB}{sinA - sinB} = \frac{2\ sin\frac{A+B}{2}\ cos\frac{A-B}{2}}{2\ cos\frac{A+B}{2}\ sin\frac{A-B}{2}}$$

$$= \frac{sin\frac{A+B}{2}}{cos\frac{A+B}{2}} \cdot \frac{cos\frac{A-B}{2}}{sin\frac{A-B}{2}}$$

$$= tan\frac{A+B}{2} \cdot cot\frac{A-B}{2}$$

$$\therefore \quad \frac{a+b}{a-b} = tan\frac{A+B}{2}cot\frac{A-B}{2}$$

P3. Prove that $a\ cos\frac{B-C}{2} = (b + c)sin\frac{A}{2}$

Solution: We have,

$$\frac{b+c}{a} = \frac{sinB+sinC}{sinA} = \frac{2\ sin\frac{B+C}{2}\ cos\frac{B-C}{2}}{2\ sin\frac{A}{2}\ cos\frac{A}{2}} = \frac{2\ cos\frac{A}{2}\ cos\frac{B-C}{2}}{2\ sin\frac{A}{2}\ cos\frac{A}{2}}$$

$$= \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$$

$$\therefore a \cos\frac{B-C}{2} = (b+c)\sin\frac{A}{2}$$

IP4. In a Δ ABC, prove that $\sum a^3 \cos(B-C) = 3abc$

Solution: Given,

$$a^3 \cos(B-C) = a^2\, a\cos(B-C) = a^2 k\, sinA \cos(B-C)$$

where, $k = \frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

$$= a^2 k \sin(B+C)\cos(B-C)$$

$(since\ A+B+C=\pi)$

$$= \frac{a^2k}{2}(2sin(B+C)\cos(B-C))$$

$$= \frac{a^2k}{2}(sin2B + sin2C)$$

$$= \frac{a^2k}{2}(2sinB\cos B + 2sinC\, cosC)$$

$$= a^2(b\cos B + c\, cosC)$$

Similarly,

$$b^3\cos(C-A) = b^2(a\cos A + c\, cosC),$$

$$c^3\cos(A-B) = c^2(a\cos A + b\, cosB)$$

$$\sum a^3\cos(B-C) = a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B)$$

$$= a^2(b\cos B + c\, cosC) + b^2(a\cos A + c\, cosC)$$
$$+ c^2(a\cos A + b\, cosB)$$

$$= a^2 b\cos B + a^2 c\, cosC + b^2 a\cos A + b^2 c\, cosC$$
$$+ c^2 a\cos A + c^2 b\, cosB$$

$$= ab(a\, cosB + b\, cosA) + bc(b\, cosC + c\, cosB) + ca(c\, cosA + a\, cosC)$$

$$= abc + bca + cab = 3abc$$

$$\therefore \sum a^3\cos(B-C) = 3abc$$

P4. In Δ ABC, show that $\sum(b+c)cosA = 2s$

Solution: In Δ ABC,

$$\sum(b+c)cosA = (b+c)cosA + (c+a)cosB + (a+b)cosC$$

L.H.S. = $(b+c)cosA + (c+a)cosB + (a+b)cosC$

$$= (bcosA + acosB) + (c\, cosB + bcosC)$$
$$+ (a\, cosC + c\, cosA)$$

$= c + a + b$ (From projection formulas)
$= 2s \qquad \therefore \sum(b+c)cosA = 2s$

Exercises

1. In a triangle ABC, if, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ prove that $\frac{cosA}{7} = \frac{cosB}{19} = \frac{cosC}{25}$
2. In any triangle ABC, if $tan\theta = \frac{2\sqrt{ab}}{a-b} sin\frac{C}{2}$, prove that $c = (a-b)sec\theta$
3. If $a = 13, b = 14, c = 15$ find the trigonometric ratios of the half angles of the triangle.
4. Given $a = \sqrt{3}, b = \sqrt{2}, c = \frac{\sqrt{6}+\sqrt{2}}{2}$, find the angles.
5. In any triangle prove that

$$(b^2 - c^2)cotA + (c^2 - a^2)cotB + (a^2 - b^2)cotC = 0$$

6. In any triangle prove that $(a+b+c)\left(tan\frac{A}{2} + tan\frac{B}{2}\right) = 2c\ cot\frac{C}{2}$
7. In any triangle ABC, prove that

a) $\sin\frac{\mathrm{B-C}}{2} = \frac{\mathrm{b-c}}{\mathrm{a}} \cos\frac{\mathrm{A}}{2}$

b) $b^2 \sin 2C + c^2 \sin 2B = 2\, bc \sin A$

c) $a\,(b \cos C - c \cos B) = b^2 - c^2$

d) $a\,(\cos B + \cos C) = 2\,(b+c)\ sin^2\frac{\mathrm{A}}{2}$

e) $\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$

f) $c^2 = (a-b)^2\ cos^2\frac{C}{2} + (a+b)^2\ sin^2\frac{C}{2}$

g) $\frac{a \sin(B-C)}{b^2 - c^2} + \frac{b \sin(C-A)}{c^2 - a^2} + \frac{c \sin(A-B)}{a^2 - b^2}$

h) $\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0$

i) $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C}$

j) If a, b and c are in H.P., prove that $sin^2\frac{A}{2}, sin^2\frac{B}{2}$ and $sin^2\frac{C}{2}$ are also in H.P.

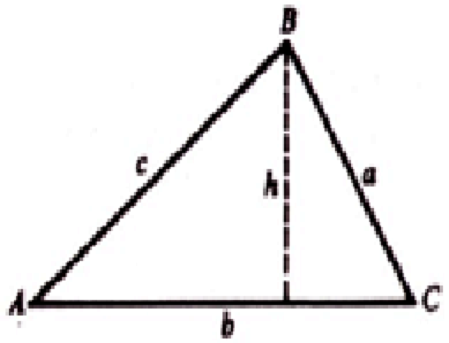
1.5. Properties of Triangles

Learning objectives:

- To derive the formulae for the area of a given triangle.
- To discuss the circles connected with a given triangle.
- To find the radius (R) of the circumcircle of any triangle.
- To find the radius (r) of the incircle of any triangle.
- To find the radii r_1, r_2 and r_3 of the escribed circles opposite the angles A, B and C of a triangle ABC.
  And
- To practice the related problems.

Area of a Triangle:

The area Δ of any triangle equals one-half the product of its base and altitude.



The altitude h is given by $h = c \sin A$

Therefore, the area is given by

$$\Delta = \frac{1}{2} bc \sin A$$

Other expressions similar to this can be derived. Thus,

$$\Delta = \frac{1}{2} \boldsymbol{bc} \sin \boldsymbol{A} = \frac{1}{2} \boldsymbol{ca} \sin \boldsymbol{B} = \frac{1}{2} \boldsymbol{ab} \sin \boldsymbol{C} \quad \text{-------}(1)$$

The area of the triangle is equal to *one-half the product of the two sides times the sine of the included angle.*

Earlier, we derived

$$\sin \boldsymbol{A} = \frac{2}{\boldsymbol{bc}} \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}$$

where s is the semi-perimeter of the triangle.

Substituting this expression, we obtain

$$\Delta = \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})} \quad \text{--------}(2)$$

The formula is known as ***Heron's (or Hero's) formula.***

Given two angles and a side of a triangle ABC, we determine the third angle using the fact that $A + B + C = 180°$. Then, the area of the triangle is calculated using the given side and the two adjacent angles.

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} c \sin A \frac{c \sin B}{\sin C} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

Example:

Find the area of the triangle ABC, given $c = 23\ cm$, $A = 20°$ and $C = 15°$.

Solution:

$B = 180° - (A + C) = 145°$

$$\Delta = \frac{c^2 \sin A \sin B}{2 \sin C} = \frac{23^2 \times \sin 20° \times \sin 145°}{2 \sin 15°} = 200\ cm^2$$

Circles Connected with a Triangle

The circle which passes through the angular points of a triangle ABC is called its ***circumscribing circle*** or ***circum circle***. Its radius is denoted by R.

The circle which can be inscribed within the triangle so as to touch each of the sides is called its ***inscribed circle or incircle***. Its radius is denoted by r.

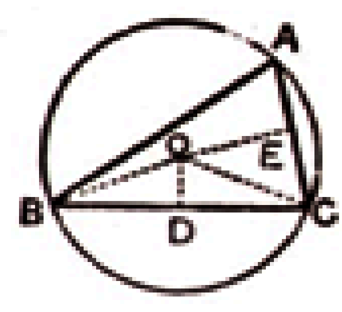
The circle which touches the side BC and the two sides AB and AC produced is called the **escribed circle** opposite to the angle A. Its radius will be denoted by r_1.

Similarly r_2 denotes the radius of the circle which touches the side CA and the two sides BC and BA produced. Also r_3 denotes the radius of the circle touching the side AB and the two sides CA and CB produced.

Circumcircle:

To find R.

In any triangle ABC, bisect the two sides BC and CA in D and E respectively, and draw DO and EO perpendicular to BC and CA.



By geometry, O is the center of the circumcircle. Join OB and OC. The two triangles BOD and COD are equal.

Angle $BOD = ½$ angle $BOC =$ angle $BAC = A$

$$BD = BO \sin BOD$$

$$\frac{a}{2} = R \sin A$$

Thus, we have the relation

$$\boldsymbol{R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}} \qquad \text{-------------}(3)$$

Since

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc} ,$$

we get,

$$\boldsymbol{R = \frac{abc}{4\Delta}} \qquad \text{-----------}(4)$$

This gives the radius of the circumcircle in terms of the sides.

Example:

In a triangle $ABC, a = 2, b = 3, \sin A = \frac{2}{3}$, find angle B.

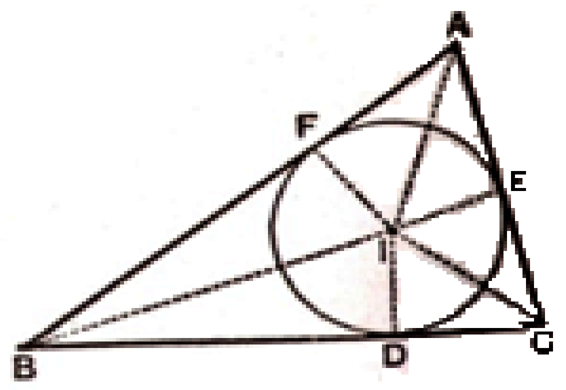
Solution:

$$\sin B = \frac{b \sin A}{a} = \frac{3\times\frac{2}{3}}{2} = 1 \quad \Rightarrow B = 90°$$

Incircle:

To find r.

Bisect the two angles B and C by the two lines BI and CI meeting in I.



By Geometry, I is the center of the **incircle**. Join IA, and draw ID, IE, and IF perpendicular to the three sides.

Then $ID = IE = IF = r$.

We have

Area of triangle $IBC = ½\, ID.BC = \frac{1}{2} r.a$

Area of triangle $ICA = \frac{1}{2} r.b$

Area of triangle $IAB = \frac{1}{2} r.c$

Hence, by addition, we have

$$\frac{1}{2} r.a + \frac{1}{2} r.a + \frac{1}{2} r.a = \Delta \Longrightarrow r\left(\frac{a+b+c}{2}\right) = \Delta \Longrightarrow rs = \Delta$$

$$\Longrightarrow \boldsymbol{r} = \frac{\boldsymbol{\Delta}}{\boldsymbol{s}} \qquad \text{---------}(5)$$

A second value of r:

Notice that the angles IBD and IDB are respectively equal to the angles IBF and IFB. Therefore the two triangles are congruent and $\frac{BD}{BF} = \frac{IB}{IF} = 1 \Longrightarrow BD = BF$ and $2BD = BD + BF$

Similarly

$$2AE = AE + AF$$

$$2CE = CE + CD$$

Hence, by addition, we have

$$2BD + 2AE + 2CE = (BD + DC) + (BF + FA) + (AE + CE)$$

$$2BD + 2AC = BC + CA + AB$$

$$2BD + 2b = a + b + c = 2s \Longrightarrow BD = s - b$$

Similarly

$$CE = s - c\,, AF = s - a$$

In the triangle IBD

$$\frac{ID}{BD} = \tan IBD = \tan\frac{B}{2} \Longrightarrow ID = BD\tan\frac{B}{2}$$

$$\Longrightarrow r = (s - b)\tan\frac{B}{2}$$

and similarly other formulas. We, thus, have

$$\boldsymbol{r} = (\boldsymbol{s} - \boldsymbol{a})\,\mathbf{tan}\,\frac{\boldsymbol{A}}{\mathbf{2}} = (\boldsymbol{s} - \boldsymbol{b})\,\mathbf{tan}\,\frac{\boldsymbol{B}}{\mathbf{2}} = (\boldsymbol{s} - \boldsymbol{c})\,\mathbf{tan}\,\frac{\boldsymbol{C}}{\mathbf{2}} \qquad \text{-------}(6)$$

A third value of r:

A third value for r may be found as follows. We have

$$a = BD + CD = r\cot\frac{B}{2} + r\cot\frac{C}{2} = r\left[\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right]$$

$$\Longrightarrow a\sin\frac{B}{2}\sin\frac{C}{2} = r\left[\sin\frac{C}{2}\cos\frac{B}{2} + \cos\frac{C}{2}\sin\frac{B}{2}\right] = r\sin\left[\frac{B+C}{2}\right]$$

$$= r\sin\left(90° - \frac{A}{2}\right) = r\cos\frac{A}{2}$$

Therefore, $$\boldsymbol{r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}} \qquad \text{----------(7)}$$

Corollary: Since $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$, we have

$$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$\Rightarrow \boldsymbol{r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \qquad \text{------------(8)}$$

Example:

A circle is inscribed in an equilateral triangle of side a. Find the area of any square inscribed in this circle.

Solution: $\Delta = \frac{\sqrt{3}}{4}a^2$ and $s = \frac{3a}{2}$

$r = \frac{\Delta}{s} = \frac{a}{2\sqrt{3}}$

Diagonal of square $= 2r = \frac{a}{\sqrt{3}}$

If x denotes the side of the square, its diagonal $= \sqrt{2}x$

Therefore

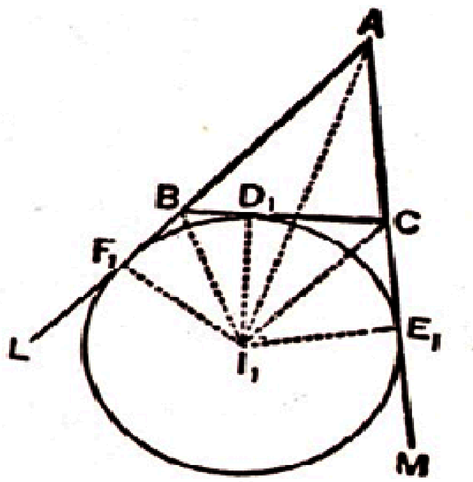
$$\sqrt{2}x = \frac{a}{\sqrt{3}} \Rightarrow x = \frac{a}{\sqrt{6}}$$

The area of square is $x^2 = \frac{a^2}{6}$.

Escribed Circles:

To find r_1, r_2 and r_3:

In any triangle ABC, produce AB and AC to L and M.



Bisect the angles CBL and BCM by the lines BI_1 and CI_1, and let these lines meet in I_1. Draw ID_1, IE_1, IF_1 perpendicular to the three sides respectively.

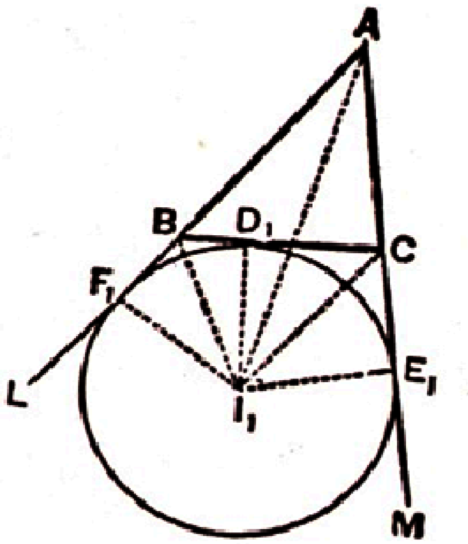
The two triangles I_1D_1B and I_1F_1B are equal, so that

$$I_1F_1 = I_1D_1 \ .$$

Similarly $I_1E_1 = I_1D_1$

Thus, the three perpendiculars I_1D_1, I_1E_1, I_1F_1 are equal. Therefore I_1 is the center of the escribed circle opposite to the angle A of the triangle ABC.

Now the area ABI_1C is equal to the sum of the areas of triangles ABC and I_1BC; it is also equal to the sum of the areas of triangles I_1BA and I_1CA.



Hence

$$\Delta + \frac{1}{2}I_1D_1.BC = \frac{1}{2}I_1E_1.CA + \frac{1}{2}I_1F_1.AB$$

$$\Delta + \frac{1}{2}r_1.a = \frac{1}{2}r_1.b + \frac{1}{2}r_1.c$$

$$\Delta = r_1\left[\frac{b+c-a}{2}\right] = r_1\left[\frac{b+c+a}{2} - a\right] = r_1(s-a)$$

Therefore,

$$\boldsymbol{r_1 = \frac{\Delta}{s-a}} \qquad \text{-------(9)}$$

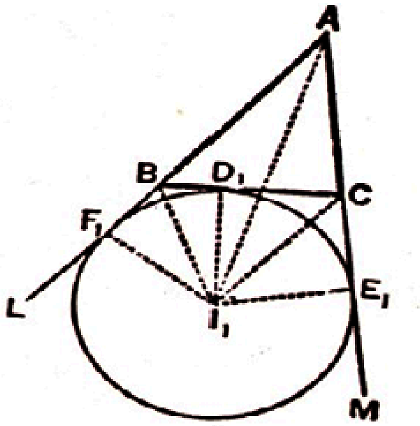
Similarly,

$$\boldsymbol{r_2 = \frac{\Delta}{s-b}} \qquad \text{------(10)}$$

$$\boldsymbol{r_3 = \frac{\Delta}{s-c}} \qquad \text{------(11)}$$

Corollary: $\boldsymbol{rr_1r_2r_3} = \Delta^2$

A second value of r_1, r_2 and r_3.



Since AE_1 and AF_1 are tangents, we have $AE_1 = AF_1$.

Similarly, $BF_1 = BD_1$ and $CE_1 = CD_1$.

Therefore

$$2\,AE_1 = AE_1 + AF_1 = AB + BF_1 + AC + CE_1$$
$$= AB + BD_1 + AC + CD_1 = AB + BC + CA = 2s$$

Therefore, $AE_1 = s = AF_1$.

Also, $BD_1 = BF_1 = AF_1 – AB = s - c$

$CD_1 = CE_1 = AE_1 – AC = s – b$

Therefore, $I_1E_1 = AE_1.\tan I_1AE_1$

Thus, $\boldsymbol{r_1 = s\tan\frac{A}{2}}$ --------(12)

Similarly $\boldsymbol{r_2 = s\tan\frac{B}{2}}$ --------(13)

And $\boldsymbol{r_3 = s\tan\frac{C}{2}}$ --------(14)

A third value of r_1, r_2 *and* r_3:

A third value may be obtained for r_1 in terms of a and the angles B and C.

Since I_1C bisects the angle BCE_1, we have

Angle $I_1CD_1 = ½\,(180° - C) = 90° - C/2$

Similarly, Angle $I_1BD_1 = 90° - B/2$

Therefore

$$a = BC = BD_1 + D_1C = I_1D_1\cot I_1BD_1 + I_1D_1\cot I_1CD_1$$
$$= r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = r_1\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)$$

$$\Longrightarrow\ a\cos\frac{B}{2}\cos\frac{C}{2} = r_1\left[\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right] = r_1\sin\left[\frac{B+C}{2}\right]$$

$$= r_1 \sin\left(90° - \frac{A}{2}\right) = r_1 \cos\frac{A}{2}$$

Therefore, $\boldsymbol{r_1} = \frac{\boldsymbol{a\cos\frac{B}{2}\cos\frac{C}{2}}}{\boldsymbol{\cos\frac{A}{2}}}$ ---------(15)

Corollary:

Since $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$, we have

$$\boldsymbol{r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \quad \text{----------(16)}$$

Similarly we can prove that

$$\boldsymbol{r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}} \quad \text{----------(17)}$$

and $\boldsymbol{r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$ ----------(18)

Example:

The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in H.P. Show that its sides a, b, c are in A.P.

Solution: Given r_1, r_2, r_3 are in HP $\Longrightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in AP.

$\Longrightarrow \frac{1}{r_2} - \frac{1}{r_1} = \frac{1}{r_3} - \frac{1}{r_2}$

$\frac{s-b}{\Delta} - \frac{s-a}{\Delta} = \frac{s-c}{\Delta} - \frac{s-b}{\Delta} \Longrightarrow s - b - s + a = s - c - s + b \; 2b = a + c$

Hence, a, b, c are in A.P.

PROBLEM SET

IP1: If the angles of a triangle are $30°$, $45°$ and included side is $(\sqrt{3}+1)$ units , prove that the area of the triangle is $\frac{\sqrt{3}+1}{2}$ square units.

Solution:

Let $B = 30°$ and $C = 45°$, so that $BC = a = \sqrt{3} + 1$ -------(1)

Then $A = 180° - (30° + 45°) = 105°$

$\Longrightarrow \sin A = \sin 105° = \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{a}{2\sqrt{2}}$ --------(2)

We have, $\frac{a}{\sin A} = \frac{c}{\sin C} \Longrightarrow 2\sqrt{2} = \frac{c}{\sin 45°} \Longrightarrow c = 2$ units.

$\therefore \Delta = \frac{1}{2}ac\sin B = \frac{1}{2}(\sqrt{3}+1)(2)\sin 30° = \frac{\sqrt{3}+1}{2}$ square units.

Aliter:

$$\Delta = \frac{1}{2}.\frac{a^2\sin B\sin C}{\sin A}$$

$= \frac{1}{2} \frac{(\sqrt{3}+1)^2 \sin 30° \sin 45°}{\sin 105°} = \frac{1}{2}(\sqrt{3}+1)^2 \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{2\sqrt{2}}{\sqrt{3}+1}\right)$

$= \frac{\sqrt{3}+1}{2}$ square units.

P1: If the sides of a triangle are $13, 14$ and 15, then find the circum diameter.

Solution:

Let $a = 13$, $b = 14$ and $c = 15$.

Then $2s = a + b + c = 13 + 14 + 15 = 42 \implies s = 21$

Now, $s - a = 8,\ s - b = 7,\ s - c = 6.$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(8)(7)(6)} = 84$$

We have $R = \frac{abc}{4\Delta}$

$$\implies R = \frac{13(14)(15)}{4(84)} = \frac{65}{8}$$

$\therefore$ Circum diameter $(2R) = \frac{65}{4}$.

IP2: In a triangle ABC if $8R^2 = a^2 + b^2 + c^2$, prove that the triangle is right angled.

Solution:

Given $8R^2 = a^2 + b^2 + c^2$

$\implies 8R^2 = 4R^2 \sin^2 A + 4R^2 \sin^2 B + 4R^2 \sin^2 C$

$\implies \sin^2 A + \sin^2 B + \sin^2 C = 2$

$\implies (1 - \cos^2 A) + (1 - \cos^2 B) + \sin^2 C = 2$

$\implies \cos^2 A - \sin^2 C + \cos^2 B = 0$

$\implies \cos(A + C)\cos(A - C) + \cos^2 B = 0$

$(\because \cos(A + C)\cos(A - C) = \cos^2 A - \sin^2 C)$

$\implies -\cos B \cos(A - C) + \cos^2 B = 0$

$(\because \cos(A + C) = \cos(180° - B) = -\cos B)$

$\implies -\cos B\,[\cos(A - C) - \cos B] = 0$

$\implies \cos B\,[\cos(A - C) + \cos(A + C)] = 0$

$\implies 2\cos B \cos A \cos C = 0$

$(\because \cos(A - C) + \cos(A + C) = 2\cos A \cos C)$

atleast one of $\cos A$, $\cos B$, $\cos C$ is zero.

Atleast an angle of triangle ABC is a right angle.

Therefore, the triangle ABC is a right angle triangle.

P2: In a ΔABC, if $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3:5:7$, then find $a:b:c$.

Solution:

We have $\tan\frac{A}{2} = \frac{r}{(s-a)}$, $\tan\frac{B}{2} = \frac{r}{(s-b)}$ and $\tan\frac{C}{2} = \frac{r}{(s-c)}$.

Given $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3:5:7$

$\Longrightarrow \frac{(s-a)}{r} : \frac{(s-b)}{r} : \frac{(s-c)}{r} = 3:5:7$

$\Longrightarrow (s-a):(s-b):(s-c) = 3:5:7$

$\Longrightarrow s-a = 3k,\ s-b = 5k,\ s-c = 7k$

$\Longrightarrow 3s - (a+b+c) = 15k \quad \Longrightarrow \quad s = 15k$

Now, $s - a = 3k \Longrightarrow a = 12k$

Similary $b = 10k,\ c = 8k$.

$\therefore \quad a:b:c = 12k:10k:8k = 6:5:4$.

IP3: Show that $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$.

Solution:

We have $r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$ and $r_3 = \frac{\Delta}{s-c}$.

$\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2}$

$= \frac{s^2}{\Delta^2} + \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2}$

$= \frac{1}{\Delta^2}[s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2]$

$= \frac{1}{\Delta^2}[s^2 + s^2 - 2as + a^2 + s^2 - 2bs + b^2 + s^2 - 2cs + c^2]$

$= \frac{1}{\Delta^2}[4s^2 - 2s(a+b+c) + a^2 + b^2 + c^2]$

$= \frac{1}{\Delta^2}[4s^2 - 2s(2s)] + \frac{a^2+b^2+c^2}{\Delta^2}$

$= \frac{a^2+b^2+c^2}{\Delta^2}$

$\therefore\ \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$

P3: In ΔABC, if $r_1 = 8, r_2 = 12, r_3 = 24$, then find a, b, c.

Solution: We have

$r_1 = 8 \quad \Longrightarrow \frac{\Delta}{s-a} = 8 \quad \Longrightarrow s - a = \frac{\Delta}{8}$ --------(1)

$r_2 = 12 \ \Longrightarrow \frac{\Delta}{s-b} = 12 \ \Longrightarrow s - b = \frac{\Delta}{12}$ --------(2)

$r_3 = 24 \implies \frac{\Delta}{s-c} = 24 \implies s - c = \frac{\Delta}{24}$ --------(3)

$(1) + (2) + (3) \implies$

$$(s-a) + (s-b) + (s-c) = \Delta\left(\frac{1}{8} + \frac{1}{12} + \frac{1}{24}\right)$$

$$\implies 3s - (a+b+c) = \frac{\Delta}{4} \implies 3s - 2s = \frac{\Delta}{4} \implies s = \frac{\Delta}{4}$$

Now, $r = \frac{\Delta}{s} = \frac{\Delta}{\left(\frac{\Delta}{4}\right)} = 4.$

But, $rr_1r_2r_3 = \frac{\Delta}{s}.\frac{\Delta}{s-a}.\frac{\Delta}{s-b}.\frac{\Delta}{s-c} = \frac{\Delta^4}{\Delta^2} = \Delta^2$

$(\because s(s-a)(s-b)(s-c) = \Delta^2)$

$\Delta^2 = rr_1r_2r_3 = 4(8)(12)(24) = 96 \times 96 \implies \Delta = 96$

$s = \frac{\Delta}{r} = \frac{96}{4} = 24$

From (1), $a = s - \frac{\Delta}{8} = 24 - \frac{96}{8} = 12$

From (2), $b = s - \frac{\Delta}{12} = 24 - \frac{96}{12} = 16$

From (3), $c = s - \frac{\Delta}{24} = 24 - \frac{96}{24} = 20$

IP4: Show that $\frac{ab - r_1r_2}{r_3} = \frac{bc - r_2r_3}{r_1} = \frac{ca - r_3r_1}{r_2} = r$.

Solution:

We have $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$,

$r_1 = 4R\, sin\frac{A}{2} cos\frac{B}{2} cos\frac{C}{2}$, $r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ and $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

Now,

$ab - r_1r_2$

$= (2R\sin A)(2R\sin B) - \left(4R\, sin\frac{A}{2} cos\frac{B}{2} cos\frac{C}{2}\right)\left(4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right)$

$= 4R^2\sin A\sin B - 4R^2\left(\cos^2\frac{C}{2}\right)\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)$

$= 4R^2\sin A\sin B - 4R^2\cos^2\frac{C}{2}\sin A\sin B$

$= 4R^2\sin A\sin B\left(1 - \cos^2\frac{C}{2}\right)$

$= 4R^2\sin A\sin B\sin^2\frac{C}{2}$

Now,

$\frac{ab - r_1r_2}{r_3} = \frac{4R^2\sin A\sin B\sin^2\frac{C}{2}}{4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$

$$= \frac{4R^2\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\sin^2\frac{C}{2}}{4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$$

$$= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\frac{ab-r_1r_2}{r_3} = r \qquad \left(\because r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

Similarly we can show that $\frac{bc-r_2r_3}{r_1} = \frac{ca-r_3r_1}{r_2} = r$.

P4: If $r_1 + r_2 = r_3 - r$, then show that $C = 90°$.

Solution:

Given $r_1 + r_2 = r - r_3$

We have,

$r_1 = 4R\,sin\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$, $r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ and $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

Now,

$$4R\,sin\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2} + 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$= 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Longrightarrow \quad 4R\,cos\frac{C}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\right)$$

$$= 4R\sin\frac{C}{2}\left(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$\Longrightarrow \quad 4R\,cos\frac{C}{2}\sin\left(\frac{A+B}{2}\right) = 4R\sin\frac{C}{2}\cos\left(\frac{A+B}{2}\right)$$

$$\Longrightarrow \quad cos\frac{C}{2}\sin\left(90° - \frac{C}{2}\right) = \sin\frac{C}{2}\cos\left(90° - \frac{C}{2}\right)$$

$$\Longrightarrow \quad \cos^2\frac{C}{2} = \sin^2\frac{C}{2} \quad \Longrightarrow \tan^2\frac{C}{2} = 1 \Longrightarrow \frac{C}{2} = 45°$$

$$\Longrightarrow \quad C = 90°$$

Exercises:

1. Find the area of the triangle ABC when

   a. $a = 13,\ b = 14$ and $c = 15$.
   b. $a = 18,\ b = 24$ and $c = 30$.
   c. $a = 25,\ b = 52$ and $c = 63$.
   d. $a = 125,\ b = 123$ and $c = 62$.
   e. $a = 15,\ b = 36$ and $c = 39$.
   f. $a = 287,\ b = 816$ and $c = 885$.
   g. $a = 35,\ b = 84$ and $c = 91$.
   h. $a = \sqrt{3},\ b = \sqrt{2}$ and $c = \frac{\sqrt{6}+\sqrt{2}}{2}$.

i. $a = 5,\ \ b = 7$ and $c = 10.$

2. If $B = 45°, C = 60°$ and $a = 2(\sqrt{3} + 1)\ cm$, prove that the area of the triangle is $6 + 2\sqrt{3}\ sq.cm.$

3. If one angle of a triangle be $60°$, the area $10\sqrt{3}\ sq.cm.$ and the perimeter $20\ cm.$, find the lengths of the sides.

4. Find the area of the triangle ABC, given $c = 23\ cm,\ A = 20°$, and $B = 15°$.

5. Find the area of the triangle ABC, given $a = 112\ m,\ b = 219\ m$ and $A = 20°$.

6. Find the area of the triangle ABC, given $a = 14.27\ cm$, $c = 17.23\ cm$, and $B = 86°14'$.

7. In a triangle ABC, AD is the altitude from A. Given $b > c, C = 23°, AD = \frac{abc}{b^2-c^2}$, find angle B.

8. In a triangle ABC, $a: b: c = 4: 5: 6$. Find the ratio of the circumcircle to that of the incircle.

I.

1. In $\Delta\, ABC$, prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$.
2. Show that $rr_1r_2r_3 = \Delta^2$.
3. If $(r_2 - r_1)(r_3 - r_1) = 2r_2r_3$, then show that $A = 90°$.
4. Prove that $\frac{r_1(r_2+r_3)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}} = a$.
5. Show that $r + r_3 + r_1 - r_2 = 4R\cos B$.
6. Show that $r + r_1 + r_2 - r_3 = 4R\cos C$.
7. Prove that $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$.
8. Prove that $(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) = 4Rs^2$.
9. Prove that $s\left[\frac{r_1-r}{a} + \frac{r_2-r}{b} + \frac{r_3-r}{c}\right] = r_1 + r_2 + r_3$.
10. Prove that $a(rr_1 + r_2r_3) = b(rr_2 + r_3r_1) = c(rr_3 + r_1r_2) = abc$.
11. In $\Delta\, ABC$ if $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, prove that the triangle is a right angled.
12. If $(a - b)(s - c) = (b - c)(s - a)$, prove that $r_1,\ r_2,\ r_2$ are in A.P.
13. If $a = 18, b = 24, c = 30$, show that $r_1 = 12$.
14. If $= 26,\ b = 30, \cos C = \frac{63}{65}$, prove that $R = \frac{65}{4}, r = 3, r_1 = 16, r_2 = 48,$ $r_3 = 4$.
15. If $r_1 = 36, r_2 = 18, r_3 = 12$, show that $a = 30, b = 24, c = 18, R = 15$.
16. If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.
17. The radii r_1, r_2, r_3 of escribed circles of triangle ABC are in harmonic progression. If its area is $24\ sq.cm.$ and its perimeter is $24\ cm.$, find the lengths of its sides.

2.1. Concept of Complex Numbers

Learning Objectives:

- To introduce the concept of a complex numbers through real numbers and the imaginary unit
- To define the conjugate of a complex number

  AND

- To practice the related problems

Imaginary Numbers

The equation, $x^2 + 9 = 0$ ... (1)

has no real number solutions because the square of a real number is always positive. The solution of this equation is given by

$$x = \sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot 3 = 3 \cdot \sqrt{-1}$$

The square root of a negative number, such as $\sqrt{-1}$, $\sqrt{-5}$, is called an *imaginary number*. We may write $\sqrt{-5}$ as $\sqrt{5} \cdot \sqrt{-1}$. It is convenient to introduce the symbol $i = \sqrt{-1}$ and write $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1} = i\sqrt{5}$. Then the solution of the equation (1) is given by

$$x = 3i$$

The symbol i has the property $i^2 = -1$. It is the solution of the equation

$$x^2 + 1 = 0 \quad \ldots \quad (2)$$

For higher integral powers of i we have

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = (-1) \times (-1) = 1$$

$$i^5 = i^4 \cdot i = i$$

We note that i^n has only four possible values: $1, i, -1, -i$. They correspond to values of n which divided by 4 leave the remainders 0,1,2,3.

Complex Numbers

An expression $x + iy$ is called a *complex number* where x and y are real numbers and i is the imaginary unit with the property $i^2 = -1$.

If $z = x + iy$, then the first term x is called the *real part* of the complex number z, denoted by $Re(z)$ and y is called the *imaginary part of* z *denoted by* $Im(z)$.

If $x = 0$, the number is said to be *purely imaginary*; if $y = 0$, it is *real*.

Zero is the only number which is at once real and purely imaginary.

Complex numbers may be thought of as including all real numbers. For example, 8 can be written as $8+0i$. The real numbers occur when $y=0$. When $y\neq 0$, we have complex numbers.

A complex number is denoted by z and the set of all complex numbers is denoted by $\boldsymbol{C}$.

Thus, $\boldsymbol{C}=\{\boldsymbol{z}:\boldsymbol{z}=\boldsymbol{x}+\boldsymbol{iy};\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{R},\boldsymbol{i}=\sqrt{-\boldsymbol{1}}\}$

Two complex numbers are *equal* if and only if they have the same real part and the same imaginary part. Thus, the equality of two complex numbers $a+ib$ and $c+id$ imply $a=c$ and $b=d$.

Example: Find x and y if $3x+4i=12-8yi$.

Solution When two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

$$3x=12 \quad , \quad 4=-8y$$

$$\Rightarrow x=4 \quad , \quad y=-\frac{1}{2}$$

Example: Find x and y if $(4x-3)+7i=5+(2y-1)i$.

Solution Equating the real and imaginary parts,

$$4x-3=5 \ , \ 2y-1=7$$

$$\Rightarrow 4x=8 \quad , \quad 2y=8$$

$$\Rightarrow x=2 \quad , \quad y=4$$

Conjugate of a complex number:

The *conjugate* of a complex number $\alpha+i\beta$ is the complex number $\alpha-i\beta$, obtained by replacing i by $-i$. The process of replacing i by $-i$ is called *complex conjugation*.

For example, $2-3i$ is the complex conjugate of $2+3i$.

The complex conjugate of a complex number z is denoted by $\bar{z}$. If $z=x+iy$,

then $\bar{z}=x-iy$. Notice that $z=x+iy\Rightarrow\bar{z}=x-iy\Rightarrow\bar{\bar{z}}=x+iy=z$. Thus,

$$\bar{\bar{\boldsymbol{z}}}=\boldsymbol{z}\ \textbf{for all}\ \boldsymbol{z}\in\boldsymbol{C}.$$

PROBLEM SET

IP1: Evaluate $i^{49}+i^{68}+i^{89}+i^{110}$.

Solution: We have,

$$i^{49}+i^{68}+i^{89}+i^{110}=i^{4\cdot 12+1}+i^{4\cdot 17+0}+i^{4\cdot 22+1}+i^{4\cdot 27+2}$$

$$=i^{1}+i^{0}+i^{1}+i^{2}$$

$$=i+1+i-1=2i$$

P1: Find the product of $-i, 2i$ and $\left(-\frac{1}{8}i\right)^{3}$.

Solution:

We have, $-i, 2i$ and $\left(-\frac{1}{8}i\right)^3$

$$(-i)(2i)\left(-\frac{1}{8}i\right)^3 = 2 \times \frac{1}{8\times8\times8} \times i^5 = \frac{1}{256} \times (i^2)^2 \times i = \frac{1}{256}i$$

IP2: If $(2x-y)+i = 3-i(2y-x)$, then find the values of x and y.

Solution:

We have $(2x-y)+i = 3-i(2y-x)$

Comparing the real and the imaginary parts on both sides, we get

$$2x - y = 3 \quad \ldots\ldots\ldots \ (1)$$

and $\quad -2y + x = 1 \quad \ldots\ldots\ldots \ (2)$

Solving (1) and (2), we get $x = \frac{5}{3}, y = \frac{1}{3}$.

P2: If $4x + i(3x - y) = 3 - 6i$, where x and y are real numbers, then find the values of x and y.

Solution:

We have, $4x + i(3x-y) = 3 - 6i$

Two complex numbers are equal if their real and imaginary parts are equal.

Thus, $4x = 3$ and $3x - y = -6$

$$\Rightarrow x = \frac{3}{4} \text{ and } 3\left(\frac{3}{4}\right) - y = -6 \Rightarrow y = \frac{9}{4} + 6 \Rightarrow y = \frac{33}{4}$$

$$\therefore x = \frac{3}{4} \text{ and } y = \frac{33}{4}$$

IP3: Evaluate

i) $\left(-\sqrt{-1}\right)^{4n+3}, n \in N$

ii) $i^n + i^{n+1} + i^{n+2} + i^{n+3}, n \in N$

Solution:

i) $\left(-\sqrt{-1}\right)^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3$

$$= \{(-i)^4\}^n(-i)^3 = 1 \times -i^3 = i$$

ii) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1 + i + i^2 + i^3)$

$$= i^n(1 + i - 1 - i) = 0$$

P3: Evaluate $(i^{77} + i^{70} + i^{87} + i^{414})^3$.

Solution:

We have, $(i^{77} + i^{70} + i^{87} + i^{414})^3$

$$= (i^{4\cdot 19+1} + i^{4\cdot 17+2} + i^{4\cdot 21+3} + i^{4\cdot 103+2})^3$$
$$= (i^1 + i^2 + i^3 + i^2)^3 = (i - 1 - i - 1)^3$$
$$= (-2)^3 = -8$$

IP4: Evaluate $1 + i^2 + i^4 + i^6 + \cdots + i^{22}$

Solution:

$$1 + i^2 + i^4 + i^6 + \cdots + i^{22}$$
$$= (1 + i^2) + (i^4 + i^6) + (i^8 + i^{10}) + \cdots + (i^{16} + i^{18}) + (i^{20} + i^{22})$$
$$= (1 + i^2) + i^4(1 + i^2) + i^8(1 + i^2) + \cdots + i^{16}(1 + i^2) + i^{20}(1 + i^2)$$
$$= 0$$

Note: $$1 + i^2 + i^4 + i^6 + \cdots + i^{2n} = \begin{cases} 0\,, & n \text{ is odd} \\ 1\,, & n \text{ is evn} \end{cases}$$

P4: Evaluate $1 + i^2 + i^4 + i^6 + \cdots + i^{20}$

Solution:

$$1 + i^2 + i^4 + i^6 + \cdots + i^{20}$$
$$= (1 + i^2) + (i^4 + i^6) + (i^8 + i^{10}) + \cdots + (i^{16} + i^{18}) + i^{20}$$
$$= (1 + i^2) + i^4(1 + i^2) + i^8(1 + i^2) + \cdots + i^{16}(1 + i^2) + i^{20}$$
$$= i^{20} = 1$$

Exercises:

1. Evaluate the following:
   a. i^{30}
   b. i^{11}
   c. i^{40}
   d. i^{135}
   e. i^{-999}
   f. $i^{37} + \frac{1}{i^{67}}$
   g. $\left(i^{41} + \frac{1}{i^{257}}\right)^9$
   h. $i^{49} + i^{68} + i^{89} + i^{110}$
   i. $i^{30} + i^{80} + i^{120}$
   j. $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$
   k. $\left(i^{18} + \left(\frac{1}{i}\right)^{24}\right)^3$
2. Find x and y such that $2x - iy = 4 + 3i$

3. Find the complex conjugate of $z = a + 2i + 3ib$.

2.2. Algebra of Complex Numbers

Learning Objectives:

- To define addition, subtraction, multiplication and division of complex numbers and to study their properties
  AND
- To practice the related problems

Addition and Subtraction

The addition of two complex numbers z_1 and z_2, in general gives another complex number. The real components and the imaginary components are added separately in a manner similar to the familiar addition of real numbers.

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

For example,

$$(2 + 3i) + (4 - 5i) = (2 + 4) + (3 - 5)i = 6 - 2i,$$

obtained by adding the real parts and the imaginary parts.

By straightforward application of the commutativity and associativity of the real and imaginary parts separately, we can show that the addition of complex numbers is commutative and associative. $z_1 + z_2 = z_2 + z_1$ and $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

Thus it is immaterial in what order complex numbers are added.

Example: Sum the complex numbers $1 + 2i, 3 - 4i, -2 + i$

Solution Summing the real terms we obtain

$$1 + 3 - 2 = 2$$

And summing the imaginary terms we obtain

$$2i - 4i + i = -i$$

Hence , $(1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i$

The subtraction of complex numbers is very similar to their addition. In the subtraction of one complex number from the other, we subtract the real parts and subtract the imaginary parts. For example,

$$(2 + 3i) - (4 - 5i) = (2 - 4) + (3 + 5)i = -2 + 8i$$

As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.

Multiplication

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers z_1 and z_2, is found by multiplying them out in full and using $i^2 = -1$.

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

In the multiplication of two complex numbers, we carry out the multiplication as if the numbers were ordinary binomials and replace i^2 by -1.

$$\begin{aligned} (2+3i)(4-5i) &= 8 + 2i - 15i^2 = 8 + 2i - 15(-1) \\ &= 23 + 2i \end{aligned}$$

The multiplication of complex numbers is both commutative and associative.

$$z_1 z_2 = z_2 z_1$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

Division

The division of two complex numbers z_1 and z_2, may be written as a quotient in the component form

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \quad , z_2 \neq 0$$

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator.

$$\begin{aligned} \frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} &= \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} \\ &= \frac{(x_1x_2+y_1y_2)+i(-x_1y_2+x_2y_1)}{{x_2}^2+{y_2}^2} \\ &= \frac{x_1x_2+y_1y_2}{{x_2}^2+{y_2}^2} + i\,\frac{-x_1y_2+x_2y_1}{{x_2}^2+{y_2}^2} \end{aligned}$$

To divide two complex numbers, multiply both numerator and denominator of the fraction by the conjugate of the denominator. As an example:

$$\frac{2+3i}{4-5i} = \frac{(2+3i)(4+5i)}{(4-5i)(4+5i)} = \frac{8+22i+15i^2}{16+25} = -\frac{7}{41} + \frac{22}{41}i$$

If $\boldsymbol{z \in C}$, then $\boldsymbol{Re(z) = \frac{z+\bar{z}}{2}}$, $\boldsymbol{Im(z) = \frac{z-\bar{z}}{2i}}$

We have, $z = x + iy \Rightarrow \bar{z} = x - iy$

Therefore, $z + \bar{z} = 2x = 2Re(z)$

$$z - \bar{z} = 2iy = 2iIm(z)$$

Thus, $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2i}$

The complex conjugate of the sum (difference) of two complex numbers is equal to the sum (difference) of their complex conjugates. ***i.e.*,**

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$
$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

Similarly, the complex conjugate of the product (quotient) of two complex numbers is equal to the product (quotient) of their complex conjugates. ***i.e.,***

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Example: Show that the complex number $2 + i$ and its conjugate are the roots of the equation $x^2 - 4x + 5 = 0$

Solution For $x = 2 + i$, we have

$$(2+i)^2 - 4(2+i) + 5 = 4 + i^2 + 4i - 8 - 4i + 5$$
$$= 4 - 1 - 8 + 5 = 0$$

Therefore the complex number is a solution of the equation $x^2 - 4x + 5 = 0.$

The conjugate of the complex number $2 + i$ is $2 - i.$

For $x = 2 - i$, we have

$$(2-i)^2 - 4(2-i) + 5 = 4 + i^2 - 4i - 8 + 4i + 5$$
$$= 4 - 1 - 8 + 5 = 0$$

Thus, the complex conjugate $2 - i$ of the complex number $2 + i$ is also a root of the equation $x^2 - 4x + 5 = 0.$

PROBLEM SET

IP1: Find the sum of the complex numbers $-\frac{1}{5} + i$, $1 - \frac{1}{5}i$ and $\frac{1}{7} - \frac{1}{3}i$

Solution: We have, $-\frac{1}{5} + i$, $1 - \frac{1}{5}i$ and $\frac{1}{7} - \frac{1}{3}i$,

$$\left(-\frac{1}{5} + i\right) + \left(1 - \frac{1}{5}i\right) + \left(\frac{1}{7} - \frac{1}{3}i\right)$$
$$= \left(-\frac{1}{5} + 1 + \frac{1}{7}\right) + i\left(1 - \frac{1}{5} - \frac{1}{3}\right)$$
$$= \left(\frac{-7+35+5}{35}\right) + i\left(\frac{15-3-5}{15}\right)$$
$$= \frac{33}{35} + \frac{7}{15}i$$

P1: If sum of the complex numbers $\frac{1}{3} - \frac{1}{2}i$ and $\frac{1}{2} + \frac{1}{3}i$ is $5x + iy$, then find the values of x and y.

Solution: Given, $\left(\frac{1}{3} - \frac{1}{2}i\right) + \left(\frac{1}{2} + \frac{1}{3}i\right) = 5x + iy$

$$\Rightarrow \left(\frac{1}{3}+\frac{1}{2}\right)+i\left(\frac{1}{3}-\frac{1}{2}\right)=5x+iy$$

$$\Rightarrow \qquad \frac{5}{6}+i\left(\frac{-1}{6}\right)=5x+iy$$

Comparing real and the imaginary parts, we get

$$5x=\frac{5}{6} \text{ and } y=\frac{-1}{6}$$

$$\Rightarrow x=\frac{1}{6} \text{ and } y=\frac{-1}{6}$$

IP2: If $z_1=3+5i$, $z_2=\frac{1}{5}-\frac{1}{3}i$, $z_3=\frac{1}{3}-\frac{1}{5}i$ and $z_4=5+3i$, then find $(z_1+z_3)-(z_2+z_4)$.

Solution: We have, $z_1=3+5i$, $z_2=\frac{1}{5}-\frac{1}{3}i$, $z_3=\frac{1}{3}-\frac{1}{5}i$,

and $z_4=5+3i$

$$(z_1+z_3)-(z_2+z_4)$$

$$=\left(3+5i+\frac{1}{3}-\frac{1}{5}i\right)-\left(\frac{1}{5}-\frac{1}{3}i+5+3i\right)$$

$$=\left(\frac{10}{3}+\frac{24}{5}i\right)-\left(\frac{26}{5}+\frac{8}{3}i\right)$$

$$=\left(\frac{10}{3}-\frac{26}{5}\right)+i\left(\frac{24}{5}-\frac{8}{3}\right)$$

$$=-\frac{28}{15}+\frac{32}{15}i$$

P2: If $z_1=7-xi$, $z_2=-3+5i$, $z_3=2y-11i$ and $z_1-z_2=z_3$, then find x and y.

Solution: We have, $z_1=7-xi$, $z_2=-3+5i$, $z_3=2y-11i$,

and $z_1-z_2=z_3$

$$\Longrightarrow (7-xi)-(-3+5i)=2y-11i$$

$$\Longrightarrow \quad (7+3)-i(x+5)=2y-11i$$

$$\Longrightarrow \qquad 10-i(x+5)=2y-11i$$

Comparing real and imaginary parts, we get

$$2y=10 \quad \text{and} \quad -(x+5)=-11$$

$$\Rightarrow y=5 \quad \text{and} \quad x+5=11$$

$$\Rightarrow y=5 \quad \text{and} \quad x=6$$

$$\therefore\ x=6 \text{ and } y=5$$

IP3: Simplify $(1-i)^3(1+i)$.

Solution:

$$(1-i)^3(1+i)=(1-i)^2(1-i)(1+i)$$

$= (1 + i^2 - 2i)(1 - i^2)$

$= (1 - 1 - 2i)(1 + 1) \;\; [\because i^2 = -1]$

$= (-2i)2$

$= -4i$

P3: If the product of $(2b + 3) + i$ and $1 - i(a - 2)$ is $2i$, then find the values of a and b.

Solution:

We have, $[(2b + 3) + i][1 - i(a - 2)] = 2i$

$\Longrightarrow (2b + 3) - i(a - 2)(2b + 3) + i - i^2(a - 2) = 2i$

$\Longrightarrow (2b + 3) - i(a - 2)(2b + 3) + i + (a - 2) = 2i$

$\Longrightarrow (2b + 3 + a - 2) + i\big(1 - (a - 2)(2b + 3)\big) = 2i$

$\Longrightarrow (a + 2b + 1) + i\big(1 - (a - 2)(2b + 3)\big) = 2i$

Comparing real and imaginary parts, we get

$a + 2b + 1 = 0 \;\Rightarrow a = -2b - 1 \;\; ------ (1)$

and $1 - (a - 2)(2b + 3) = 2 \;\; ------ (2)$

From (1) and (2), we get

$1 - (-2b - 1 - 2)(2b + 3) = 2$

$\Longrightarrow b^2 + 3b + 2 = 0$

$\Longrightarrow (b + 1)(b + 2) = 0$

$\Longrightarrow b = -1$ or $b = -2$

If $b = -1, a = -2(-1) - 1 = 1 \Longrightarrow a = 1, b = -1$

If $b = -2, a = -2(-2) - 1 = 3 \Longrightarrow a = 3, b = -2$

IP4: Find the division: $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$

Solution:

$$\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i} = \frac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-\left(\sqrt{2}i\right)^2} = \frac{3+6\sqrt{2}i}{1+2} = 1 + 2\sqrt{2}i$$

P4: Evaluate $\frac{2+5i}{3-2i} + \frac{2-5i}{3+2i}$

Solution:

$$\frac{2+5i}{3-2i} + \frac{2-5i}{3+2i} = \frac{2+5i}{3-2i} \times \frac{3+2i}{3+2i} + \frac{2-5i}{3+2i} \times \frac{3-2i}{3-2i}$$

$$= \frac{6+19i+10i^2}{9-4i^2} + \frac{6-19i+10i^2}{9-4i^2}$$

$$= \frac{6+19i-10}{9+4} + \frac{6-19i-10}{9+4} = \frac{-8}{13}$$

Exercises:

1. Multiply the complex numbers

$$z_1 = 3 + 2i \text{ and } z_2 = -1 - 4i.$$

2. Represent $\frac{1+3i}{2+i}$ as a complex number.
3. Represent $\frac{3-2i}{2-3i}$ as a complex number.
4. Express z in the form $x + iy$, when $z = \frac{3-2i}{-1+4i}$

2.3. Complex Plane

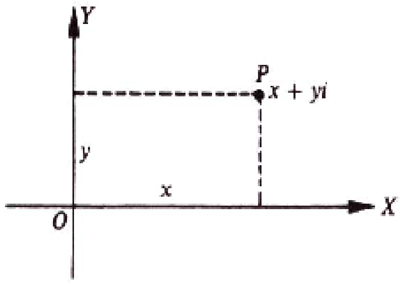
Learning Objectives:

- To study the graphically representation of a complex number in a complex plane.
- To study the graphically representation of the addition and subtraction of two complex numbers using parallelogram law of vectors.
- To define the modulus of a complex number and to study their properties

  AND

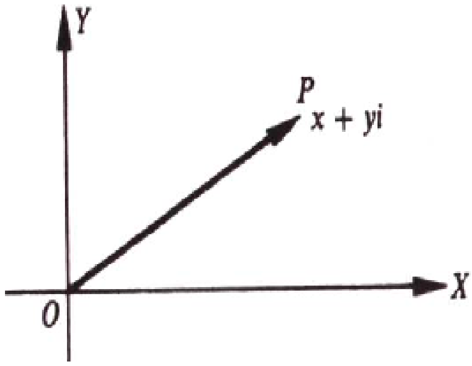
- To practice the related problems

The complex number $x + iy$ may be represented graphically by the point P whose rectangular coordinates are (x, y).

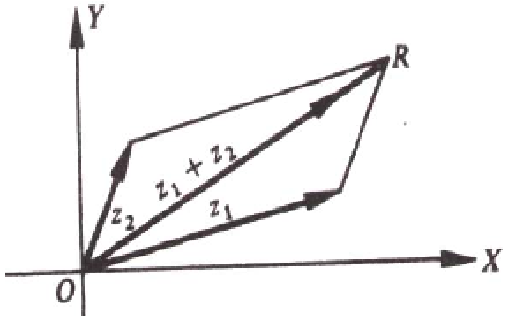


The point O, having the coordinates $(0,0)$, represents the complex number $0 + 0i = 0$. All points on the x-axis have coordinates of the form $(x, 0)$ and correspond to real numbers $x + 0i = x$. For this reason, the x-axis is called the *axis of reals* or *real axis*. All points on the y-axis have coordinates of the form $(0, y)$ and correspond to imaginary numbers $0 + iy = iy$. The y-axis is called the *axis of imaginaries* or *imaginary axis*. The plane on which the complex numbers are assigned to each of its points is called the *complex plane*.

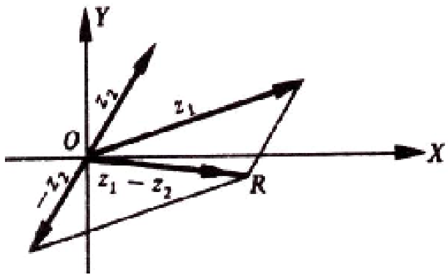
In addition to representing a complex number by a point P on the complex plane, the number may be represented by the directed line segment OP. The directed line segment OP is also called *vector OP.*



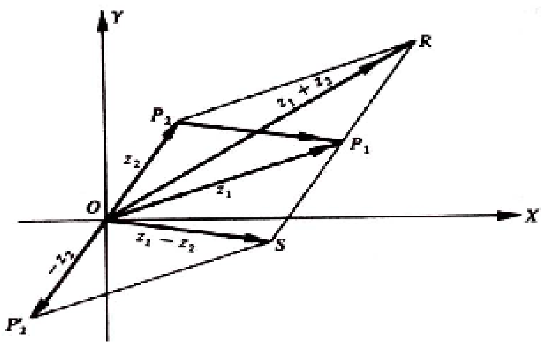
Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. We add these two numbers by using the parallelogram law of vectors.



We can also obtain the difference $z_1 - z_2$ of the two complex numbers by applying the law of parallelogram to z_1 and $-z_2$.



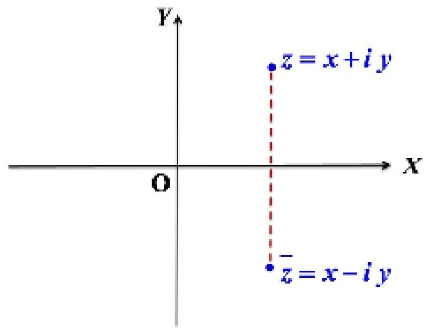
In the figure below, the vector OR gives the sum $z_1 + z_2$ and the vector OS gives the difference $z_1 - z_2$.



It is noted that the segments OS and P_2P_1 are equal. P_2P_1 is the other diagonal of the parallelogram OP_2RP_1.

Complex Conjugate

The complex conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$. Complex conjugate $\bar{z}$ corresponds to the reflection of z in the real axis of the complex plane, as shown below.



Modulus

The modulus of a complex number $z = x + iy$, denoted by $|z|$, is defined to be the nonnegative real number

$$|z| = \sqrt{x^2 + y^2}$$

Example: Find the modulus of the complex number $z = 2 - 3i$

The modulus is given by

$$|z| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$$

The complex conjugate of the complex number $z = x + iy$ is given by $\bar{z} = x - iy$. Its modulus is given by

$$|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$$

Both a complex number and its conjugate has the same modulus.

The product $z\bar{z}$ of a complex number and its conjugate is $|z|^2$.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2$$

The multiplicative inverse of a complex number $z \neq 0$ is denoted by z^{-1},

so that $zz^{-1} = 1$.

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$$

The multiplicative inverse can also be written as

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Note: $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2}$ $(z_2 \neq 0)$ (follows from the above)

The modulus of product of two complex numbers is the product of their moduli.

$$|z_1 z_2| = |z_1||z_2| \qquad \text{...... (1)}$$

This property also applies to division.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{............ (2)}$$

Example: Verify the property $|z_1 z_2| = |z_1||z_2|$ holds for the product of $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.

Solution:

$$|z_1z_2| = |(3+2i)(-1-4i)| = |5-14i|$$

$$= \sqrt{(5)^2+(-14)^2} = \sqrt{221}$$

$$|z_1| = \sqrt{(3)^2+(2)^2} = \sqrt{13}$$

$$|z_2| = \sqrt{(-1)^2+(-4)^2} = \sqrt{17}$$

Hence $\quad |z_1||z_2| = \sqrt{13}\sqrt{17} = \sqrt{221} = |z_1z_2|$

PROBLEM SET

IP1: Express $\frac{2+3i}{(7-i)(4+2i)}$ in the form of $x+iy$.

Solution:

We have, $\frac{2+3i}{(7-i)(4+2i)} = \frac{2+3i}{28+10i-2i^2} = \frac{2+3i}{30+10i}$

$$= \frac{1}{10}\left(\frac{2+3i}{3+i}\right) = \frac{1}{10}\left(\frac{2+3i}{3+i}\times\frac{3-i}{3-i}\right)$$

$$= \frac{1}{10}\left(\frac{6+7i-3i^2}{9-i^2}\right) = \frac{1}{10}\left(\frac{9+7i}{10}\right)$$

$$= \frac{9+7i}{100}$$

P1: Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i are conjugate of each other.

Solution: Since $-3+ix^2y$ is the conjugate of x^2+y+4i,

$$-3+ix^2y = \overline{(x^2+y+4i)}$$

$$\Rightarrow -3+ix^2y = (x^2+y)-4i$$

Equating real and imaginary parts, we get

$$\Rightarrow -3 = x^2+y \text{ and } x^2y = -4$$

$$\Rightarrow -3 = x^2+y \text{ --------------------- (1)}$$

$$\text{and } x^2y = -4 \Rightarrow y = -\frac{4}{x^2} \text{ ----------- (2)}$$

From (1) and (2), we get $-3 = x^2 - \frac{4}{x^2}$

$$\Rightarrow x^4+3x^2-4 = 0 \Rightarrow (x^2+4)(x^2-1) = 0$$

$$\Rightarrow x^2+4 = 0 \text{ or } x^2-1 = 0$$

$$\Rightarrow x = \pm 2i \text{ or } x = \pm 1$$

$$\Rightarrow x = \pm 1 \quad (\text{ since } x \text{ is a real number }) \text{ ------ (3)}$$

From (2) and (3), we get $y = -4$.

Hence, $x=-1, y=-4$ or $x=1, y=-4$.

IP2: If $z_1 = 2-i, z_2 = 1+i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$.

Solution: We have, $z_1 = 2-i, z_2 = 1+i$

$$\therefore \left|\frac{z_1+z_2+1}{z_1-z_2+1}\right| = \left|\frac{2-i+1+i+1}{2-i-1-i+1}\right| = \left|\frac{4}{2-2i}\right| = \left|\frac{2}{1-i}\right| = \frac{|2|}{|1-i|} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

P2: Find $\left|\frac{1}{(2+i)^2} - \frac{1}{(2-i)^2}\right|$.

Solution:

$$\left|\frac{1}{(2+i)^2} - \frac{1}{(2-i)^2}\right| = \left|\frac{(2-i)^2-(2+i)^2}{[(2+i)(2-i)]^2}\right| = \left|\frac{-8i}{(4+1)^2}\right| = \frac{8}{25}$$

IP3: Find the complex conjugate of $\frac{2-i}{(1-2i)^2}$.

Solution:

$$\frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i}$$

$$= \frac{2-i}{1-4-4i} = \frac{2-i}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{-6+11i-4i^2}{9-16i^2}$$

$$= \frac{-6+11i+4}{9+16} = \frac{-2+11i}{25} = -\frac{2}{25} + \frac{11}{25}i$$

Therefore,

the conjugate of $\frac{2-i}{(1-2i)^2}$ = the conjugate of $-\frac{2}{25} + \frac{11}{25}i$

$$= -\frac{2}{25} - \frac{11}{25}i$$

P3: Show that $z_1 = \frac{2+11i}{25}$ and $z_2 = \frac{-2+i}{(1-2i)^2}$ are conjugate to each other.

Solution:

We have, $z_1 = \frac{2+11i}{25}, z_2 = \frac{-2+i}{(1-2i)^2}$

$$z_2 = \frac{-2+i}{(1-2i)^2} = \frac{-2+i}{1+4i^2-4i} = \frac{-2+i}{1-4-4i}$$

$$= \frac{-2+i}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{6-8i-3i+4i^2}{9-16i^2}$$

$$= \frac{6-8i-3i-4}{9+16} = \frac{2-11i}{25} = \overline{z_1}$$

Therefore, z_1 and z_2 are conjugate to each other.

IP4: Find the multiplicative inverse of $\frac{\sqrt{5}+3i}{3+\sqrt{5}i}$.

Solution:

Let $z = \frac{\sqrt{5}+3i}{3+\sqrt{5}i} = \frac{\sqrt{5}+3i}{3+\sqrt{5}i} \times \frac{3-\sqrt{5}i}{3-\sqrt{5}i}$

$$= \frac{3\sqrt{5}-5i+9i-3\sqrt{5}i^2}{9-5i^2} = \frac{3\sqrt{5}+4i+3\sqrt{5}}{9+5}$$

$$= \frac{6\sqrt{5}+4i}{14} = \frac{3\sqrt{5}+2i}{7}$$

$$\therefore z = \frac{3\sqrt{5}+2i}{7}$$

Now, $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\frac{3\sqrt{5}-2i}{7}}{\sqrt{\frac{45}{49}+\frac{4}{49}}} = \frac{\frac{3\sqrt{5}-2i}{7}}{\sqrt{\frac{49}{49}}} = \frac{3\sqrt{5}-2i}{7}$

P4: If $z = \frac{1}{1+\cos\theta+i\sin\theta}$, then find $|z|^2$.

Solution:

We have, $z = \frac{1}{(1+\cos\theta)+i\sin\theta} \times \frac{(1+\cos\theta)-i\sin\theta}{(1+\cos\theta)-i\sin\theta}$

$$\Rightarrow z = \frac{(1+\cos\theta)-i\sin\theta}{(1+\cos\theta)^2-i^2\sin^2\theta} = \frac{(1+\cos\theta)-i\sin\theta}{1+\cos^2\theta+2\cos\theta+\sin^2\theta}$$

$$\Rightarrow z = \frac{(1+\cos\theta)-i\sin\theta}{1+\cos^2\theta+\sin^2\theta+2\cos\theta} = \frac{(1+\cos\theta)-i\sin\theta}{1+1+2\cos\theta}$$

$$\Rightarrow z = \frac{(1+\cos\theta)-i\sin\theta}{2(1+\cos\theta)} = \frac{(1+\cos\theta)}{2(1+\cos\theta)} - \frac{i\sin\theta}{2(1+\cos\theta)}$$

$$\Rightarrow z = \frac{1}{2} - \frac{\sin\theta}{2(1+\cos\theta)}i$$

$$\therefore |z|^2 = \frac{1}{4} + \frac{\sin^2\theta}{4(1+\cos\theta)^2} = \frac{1}{4}\left(\frac{1+\cos^2\theta+2\cos\theta+\sin^2\theta}{(1+\cos\theta)^2}\right)$$

$$= \frac{1}{4}\left(\frac{2(1+\cos\theta)}{(1+\cos\theta)^2}\right) = \frac{1}{2(1+\cos\theta)}$$

Exercise:

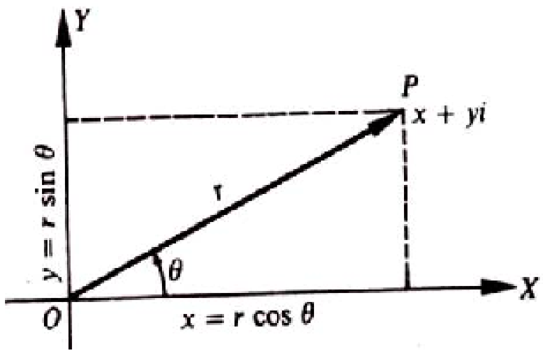
1. If $z = x + iy$, then prove
$$\frac{z}{\bar{z}} = \left(\frac{x^2-y^2}{x^2+y^2}\right) + i\left(\frac{2xy}{x^2+y^2}\right)$$
2. Express $(5-3i)^3$ in the form $x+iy$.
3. Find the multiplicative inverse of $2-3i$.
4. Express $\frac{5+i\sqrt{2}}{1-i\sqrt{2}}$ in the form $x+iy$

2.4. Polar Form

Learning Objectives:

- To represent a complex number in polar form and vice versa
- To study the argument and principle argument of a complex number.
- To study the multiplication and division of two complex numbers in polar form AND
- To practice the related problems

Let the complex number $x + iy$ be represented by the vector OP. This vector may be described in terms of the length r of the vector and any positive angle θ which the vector makes with the positive x-axis. The number $r = \sqrt{x^2 + y^2}$ is called the *modulus* or *absolute value* of the complex number and the angle θ is called the *amplitude* or *argument* of the complex number.



From the figure,

$$x = r\cos\theta \quad , \quad y = r\sin\theta \qquad \dots (1)$$

Then

$$z = x + iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

We call

$$\boldsymbol{z = r(\cos\theta + i\sin\theta)}$$

the ***polar form*** of z and

$$\boldsymbol{z = x + iy}$$

the ***rectangular form*** of the complex number z.

Note: $cos\theta + isin\theta$ is briefly written as $\boldsymbol{cis\theta}$

From (1), we get

$r = \sqrt{x^2 + y^2} = |z|$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$,

where θ is called the $\boldsymbol{argument}$ or the $\boldsymbol{amplitude}$ of z, denoted by $\boldsymbol{arg z}$ or $\boldsymbol{amp z}$

Clearly, θ is defined only when $z \neq 0$. Since θ in (1) can be replaced by the general value $\theta = \theta + 2n\pi, n \in \boldsymbol{Z}.$ We find that $arg z$ *has infinitely many values.*

Principal argument of z

The value of θ which satisfies $-\pi < \theta \leq \pi$ is called the principal value of θ or principal argument of z and is denoted by $Argz$.

Thus, $Argz = \theta, -\pi < \theta \leq \pi$ and

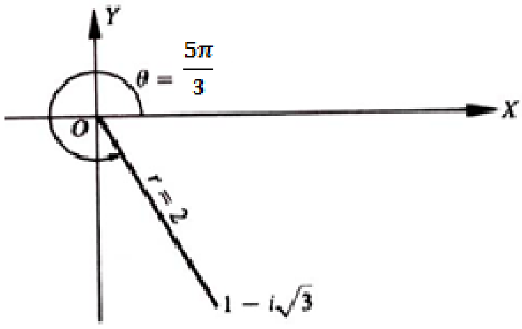
$$argz = Argz + 2n\pi, n \in \boldsymbol{Z}$$

Example:

$$Argi = \frac{\pi}{2}, \quad Arg(1-i) = -\frac{\pi}{4}\ , Arg(-1-i) = -\frac{3\pi}{4}$$

Example: Express $z = 1 - i\sqrt{3}$ in polar form.

Solution:



The modulus $\qquad r = \sqrt{1^2 + (-\sqrt{3})^2} = 2 \qquad \boldsymbol{\frac{5\pi}{3}}$

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

Since $P(-1, \sqrt{3})$ lies in quadrant IV, $\theta \ = \frac{5\pi}{3}$

The principal argument θ of z satisfies $-\pi < \theta \leq \pi, \theta = Argz = -\frac{\pi}{3}$

The required polar form is

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

Example: Express the complex number $\ z = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ in rectangular form.

Solution: Since $\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$, $\sin\frac{7\pi}{6} = -\frac{1}{2}$

We have $\quad z = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 8\left[-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right]$

$$= -4\sqrt{3} - 4i$$

The required rectangular form is $z = -4\sqrt{3} - 4i$

Example: If z is not a negative real number, then $Argz = -Arg\bar{z}$

Solution: Let $z = r(cos\theta + isin\theta)$, where $r = |z|$, $\theta = Argz$, i.e., $-\pi < \theta \leq \pi$. Since z is not a negative real number $\theta \neq \pi$, therefore $-\pi < \theta < \pi$.

Now, $\bar{z} = r(cos\theta - isin\theta) = r(\cos(-\theta) - isin(-\theta))$

Therefore, $arg\bar{z} = -\theta$

Now, $-\pi < \theta < \pi \Rightarrow -\pi < -\theta < \pi$

Thus, $-\theta$ is the principal argument of $\bar{z}$.

$\therefore Arg\bar{z} = -\theta = -Argz$

i.e., $Argz = -Arg\bar{z}$

Multiplication and Division in Polar Form

Multiplication

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $r_1 = |z_1|, r_2 = |z_2|$ and θ_1, θ_2 are the arguments of z_1, z_2 respectively. Then

$$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2\begin{bmatrix}(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\ +i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\end{bmatrix}$$

$$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

Thus, $|z_1z_2| = r_1r_2 = |z_1||z_2|$ and

$arg(z_1z_2) = \theta_1 + \theta_2 + 2n\pi, n \in \boldsymbol{Z}$

i.e., $\boldsymbol{arg(z_1z_2) = arg(z_1) + arg(z_2) + 2n\pi, n \in Z}$

Division

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $r_1 = |z_1|, r_2 = |z_2|$ and θ_1, θ_2 are the arguments of z_1, z_2 respectively. Then

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)} = \frac{r_1(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{r_2(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{\cos^2\theta_2+\sin^2\theta_2}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Thus, $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$ and

$$arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 + 2n\pi, n \in \boldsymbol{Z}$$

$$\boldsymbol{arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) + 2n\pi, n \in Z}$$

Note:

i) $cis\theta_1 . cis\theta_2 = cis(\theta_1 + \theta_2)$

ii) $\frac{cis\theta_1}{cis\theta_2} = cis(\theta_1 - \theta_2)$

iii) In general

$$Arg(z_1 z_2) \neq Arg(z_1) + Arg(z_2)$$

For example, let $z_1 = -1 + i, z_2 = i$, then $z_1 z_2 = -1 - i$. We see that

$$Arg(z_1) = \frac{3\pi}{4}, Arg(z_2) = \frac{\pi}{2}, Arg(z_1 z_2) = -\frac{3\pi}{4}$$

Clearly, $Arg(z_1) + Arg(z_2) = \frac{5\pi}{4} \neq Arg(z_1 z_2)$

Note that $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2) - 2\pi$

Thus, when the principal arguments are added together in a multiplication problem, the resulting argument need not be a principal value.

Remark: $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$ when both z_1 and z_2 lie in the right half plane or on the imaginary axis (not both on the negative imaginary axis).

Example: If $z_1 = i$ and $z_2 = 1 - \sqrt{3}\, i$, then $z_1 z_2 = \sqrt{3} + i$. We find that

$Arg(z_1) = \frac{\pi}{2}$, $Arg(z_2) = -\frac{\pi}{3}$, $Arg(z_1 z_2) = \frac{\pi}{6}$ and

$$Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$$

Example: If $z_1 = -1$ and $z_2 = -\, i$, then $z_1 z_2 = i$. We find that

$Arg(z_1) = \pi$, $Arg(z_2) = -\frac{\pi}{2}$, $Arg(z_1 z_2) = \frac{\pi}{2}$ and

$$Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$$

PROBLEM SET

IP1: Find the polar form of $\sqrt{3} + i$.

Solution: Let $z = \sqrt{3} + i = r(cos\theta + isin\theta)$, where $r = |z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$,

and $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Notice that z lies in quadrant-I. So, $\theta = \frac{\pi}{6}$

$\therefore$ The Polar form of $\sqrt{3} + i$ is $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

P1: Find the polar form of $-1 - i\sqrt{3}$

Solution: Let $z = -1 - i\sqrt{3} = r(cos\theta + isin\theta)$, where

$$r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2\ ,$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \tan^{-1}(\sqrt{3})$.

Notice that z lies in III-quadrant, $\theta = \frac{-2\pi}{3}$

$\therefore$ The Polar form of $-1 - i\sqrt{3}$ is $2(cos(-\frac{2\pi}{3}) + isin(-\frac{2\pi}{3}))$.

IP2: Find the rectangular form of $2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

Solution:

$$2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$
$$= 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \quad (\because\ -\pi < \theta \leq \pi)$$
$$= 2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2} - \sqrt{2}i$$

The required rectangular form is $\sqrt{2} - \sqrt{2}i$.

P2: Find the rectangular form of $2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

Solution:

$$2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$
$$= 2\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) \quad (\because\ -\pi < \theta \leq \pi)$$
$$= 2\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) = 2\left(\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right)$$
$$= 2\left(-sin\frac{\pi}{4} - i\ cos\frac{\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$
$$= -\sqrt{2} - i\sqrt{2}$$

The required rectangular form is $-\sqrt{2} - \sqrt{2}i$.

IP3. Find the principal argument $Arg z$ when $z = \frac{i}{-2-2\,i}$

Solution: Given $z = \frac{i}{-2-2\,i}$

$$arg z = arg\left(\frac{i}{-2-2\,i}\right)$$
$$= arg(i) - arg(-2-2\,i) + 2n\pi, n \in \boldsymbol{Z}$$

We have $rg(i) = \frac{\pi}{2}$,

$$Arg(-2-2\,i) = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$$

$$\therefore\ arg z = \frac{\pi}{2} - \left(-\frac{3\pi}{4}\right) + 2n\pi, n \in \boldsymbol{Z} = \frac{5\pi}{4} + 2n\pi, n \in \boldsymbol{Z}$$

For $n = -1, \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$

Since $-\pi < -\frac{3\pi}{4} < \pi$; $-\frac{3\pi}{4}$ the principal argument of z.

Thus, $Arg z = -\frac{3\pi}{4}$

P3. Find the principal argument $Arg z$ when $z = \frac{-2}{1+\sqrt{3}\,i}$

Solution: Given $z = \frac{-2}{1+\sqrt{3}\,i}$

$$arg z = arg\left(\frac{-2}{1+\sqrt{3}\,i}\right)$$

$$= arg(-2) - arg\left(1+\sqrt{3}\,i\right) + 2n\pi, n \in \boldsymbol{Z}$$

We have $Arg(-2) = \pi, Arg\left(1+\sqrt{3}\,i\right) = \frac{\pi}{3}$

$\therefore argz = \pi - \frac{\pi}{3} + 2n\pi, n \in \boldsymbol{Z} = \frac{2\pi}{3} + 2n\pi, n \in \boldsymbol{Z}$

Since $-\pi < \frac{2\pi}{3} < \pi$; $\frac{2\pi}{3}$ the principal argument of z.

Thus, $Argz = \frac{2\pi}{3}$.

IP4: If $\frac{\pi}{5}$ and $\frac{\pi}{3}$ are the arguments of $\overline{z_1}$ and z_2, then find the value of $Arg(z_1 z_2)$.

Solution: Given, $arg(\bar{z_1}) = \frac{\pi}{5}$ and $arg(z_2) = \frac{\pi}{3}$

Let $z_1 = x_1 + iy_1$ then $\bar{z_1} = x_1 - iy_1$

$arg\bar{z_1} = \tan^{-1}\left(-\frac{y_1}{x_1}\right) = \frac{\pi}{5}$

$\Rightarrow arg\bar{z_1} = -\tan^{-1}\left(\frac{y_1}{x_1}\right) = \frac{\pi}{5} \Rightarrow -arg(z_1) = \frac{\pi}{5}$

$\Rightarrow arg(z_1) = -\frac{\pi}{5}$

$\therefore arg(z_1 z_2) = arg(z_1) + arg(z_2) + 2n\pi, n \in \boldsymbol{Z}$

$$= -\frac{\pi}{5} + \frac{\pi}{3} + 2n\pi, n \in \boldsymbol{Z}$$

$$= \frac{-3\pi + 5\pi}{15} + 2n\pi, n \in \boldsymbol{Z}$$

$$= \frac{2\pi}{15} + 2n\pi, n \in \boldsymbol{Z}$$

Therefore, $Arg(z_1 z_2) = \frac{2\pi}{15}$

P4: If $\frac{\pi}{2}$ and $\frac{\pi}{4}$ are respectively the arguments of z_1 and $\overline{z_2}$,

then find the value of $Arg\left(\frac{z_1}{z_2}\right)$.

Solution: Given, $arg(z_1) = \frac{\pi}{2}$ and $arg(\bar{z_2}) = \frac{\pi}{4}$

$arg\bar{z_2} = \tan^{-1}\left(-\frac{y_2}{x_2}\right) = \frac{\pi}{4}$

$\Rightarrow arg\bar{z_2} = -\tan^{-1}\left(\frac{y_2}{x_2}\right) = \frac{\pi}{4} \Rightarrow -arg(z_2) = \frac{\pi}{4}$

$\Rightarrow arg(z_2) = -\frac{\pi}{4}$

$\therefore arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) + 2n\pi, n \in \boldsymbol{Z}$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{4}\right) + 2n\pi, n \in \boldsymbol{Z}$$

$$= \frac{3\pi}{4} + 2n\pi, n \in \boldsymbol{Z}$$

Therefore, $Arg\left(\frac{z_1}{z_2}\right) = \frac{3\pi}{4}$

Exercises:

1. Express each of the following complex numbers in polar form:
   a. $-1 + i\sqrt{3}$
   b. $6\sqrt{3} + 6i$
   c. $2 - 2i$
   d. $-3 + 0i$
   e. $0 + 4i$
   f. $-3 - 4i$

2. Express each of the following complex numbers in rectangular form:
   a. $4(\cos 240^\circ + i \sin 240^\circ)$
   b. $3(\cos 90^\circ + i \sin 90^\circ)$
   c. $5(\cos 128^\circ + i \sin 128^\circ)$

3. Perform the indicated operations, giving the result in both polar and rectangular form.
   a. $2(\cos 50^\circ + i \sin 50^\circ) \cdot 3(\cos 40^\circ + i \sin 40^\circ)$
   b. $10(\cos 305^\circ + i \sin 305^\circ) \div 2(\cos 65^\circ + i \sin 65^\circ)$
   c. $4(\cos 220^\circ + i \sin 220^\circ) \div 2(\cos 40^\circ + i \sin 40^\circ)$

4. Express each of the numbers in polar form, perform the indicated operation, and give the result in rectangular form.
   a. $(-1 + i\sqrt{3})(\sqrt{3} + i)$
   b. $(1 + i\sqrt{3})(1 + i\sqrt{3})$
   c. $(3 - 3\sqrt{3}i)(-2 - 2\sqrt{3}i)$
   d. $(4 - 4\sqrt{3}i) \div (-2\sqrt{3} + 2i)$
   e. $-2 \div (-\sqrt{3} + i)$
   f. $6i \div (-3 - 3i)$

2.5. De Moivre's Theorem

Learning objectives:

- To state and prove De Moivre's Theorem for rational index.
- To find the n^{th} roots of a given complex number.
- To discuss the n^{th} roots of unity.
  And
- To practice the realted problems.

Theorem: De Moivre's Theorem for integral index

If θ is any real number and n is any integer, then

$$(cos\,\theta + i\,sin\,\theta)^n = cos\,n\theta + i\,sin\,n\theta.$$

Proof:

(i) Let n be a positive integer. For $n = 1$, the theorem is true.

Assume that the theorem is true for $n = m$ where $m \neq 1$ is a positive integer. Now,

$$\begin{aligned}(\cos\theta + i\sin\theta)^{m+1} &= (\cos\theta + i\sin\theta)^m . (\cos\theta + i\sin\theta)\\ &= (\cos m\theta + i\sin m\theta). (\cos\theta + i\sin\theta)\\ &= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + \quad i(\sin m\theta\cos\theta + \\ &\qquad \cos m\theta\sin\theta)\\ &= \cos(m\theta+\theta) + i\sin(m\theta+\theta)\\ &= \cos(m+1)\theta + i\sin(m+1)\theta\end{aligned}$$

Thus the theorem is true for the positive integer $m + 1$ whenever it is true for the positive integer m, and it is also true for $m = 1$. Therefore, by mathematical induction, the theorem is true for any positive integer n.

(ii) Let $n = 0$. Then $(\cos\theta + i\sin\theta)^0 = 1 = \cos 0\theta + i\sin 0\theta$.

Therefore, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for $n = 0$.

(iii) Let n be a negative integer and let $n = -m$, where m is a positive integer.

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m}\\ &= \frac{1}{(\cos\theta+i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}\\ &= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}\\ &= \frac{\cos m\theta - i\sin m\theta}{cos^2 m\theta + sin^2 m\theta} = \cos m\theta - i\sin m\theta\end{aligned}$$

$$= \cos(-m\theta) + i\sin(-m\theta)$$
$$= \cos n\theta + i\sin n\theta$$

Thus, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for all negetive integers n.

From $(i), (ii)$ and (iii),

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for all integers n.

Hence the theorem.

Note:

$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ for all integers n.

Example:

$$\left(\sqrt{3} - i\right)^{10} = [2(\cos 330° + i\sin 330°)]^{10}$$
$$= 2^{10}(\cos 10 \times 330° + i\sin 10 \times 330°)$$
$$= 1024(\cos 60° + i\sin 60°) = 1024\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= 512 + 512i\sqrt{3}$$

Theorem: Demoivre's Theorem for rational index.

If θ is any real number and n is any rational number, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof:

Let n be a rational number and let $n = \frac{p}{q}$, where p and q are integers. Now,

$(\cos n\theta + i\sin n\theta)^q = \cos nq\theta + i\sin nq\theta$

(by the above theorem)

$$= \cos p\theta + i\sin p\theta$$
$$= (\cos\theta + i\sin\theta)^p$$

$\Longrightarrow$ One of the q^{th} roots of $(\cos\theta + i\sin\theta)^p$ is $\cos n\theta + i\sin n\theta$.

i.e., One of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ is $\cos n\theta + i\sin n\theta$.

i.e., One of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$, where n is a rational number.

Hence the theorem.

Roots of Complex Numbers:

Let n be a positive integer and a be a given nonzero complex number. A complex number z is said to be an n^{th} root of a if $z^n = a$ and it is denoted by $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$.

Theorem: Determination of n^{th} roots of a complex number a.

Let z be n^{th} root of a complex number $a \neq 0$. Then $z = a^{\frac{1}{n}}$. To find the n^{th} root of the complex number a, we thus have to solve the equation

$$z^n = a$$

We write $a = r(\cos\varphi + i\sin\varphi)$ and $z = \rho(\cos\theta + i\sin\theta)$

Then $z^n = a \Longrightarrow$

$$\rho^n(\cos n\theta + i\sin n\theta) = r(\cos\varphi + i\sin\varphi) \text{ --------}(1)$$

This equation is fulfilled if $\rho^n = r$ and $n\theta = \varphi$. Hence we obtain the root

$$z = \sqrt[n]{r}\left(\cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}\right)$$

where $\sqrt[n]{r}$ denotes the positive n^{th} root of the positive

number r.

But this is not the only solution. Equation (1) is also fulfilled if $n\theta$ differs from φ by a multiple of the full angle, 2π. Thus, the equation (1) is satisfied if and only if $n\theta = \varphi + k.2\pi$

i.e., $\theta = \frac{\varphi+2k\pi}{n}$

where k is any integer. However, only the values $k = 0,1,\dots,n-1$ give different values of z. Hence the complete solution of the equation (1) is given by

$$\boldsymbol{z = \sqrt[n]{r}\left(\cos\left(\frac{\varphi+2k\pi}{n}\right) + i\sin\left(\frac{\varphi+2k\pi}{n}\right)\right),\ k = 0,1,\dots,n-1}$$

Thus, there are n n^{th} roots of any complex number $a \neq 0$. They have the same modulus $\sqrt[n]{|a|}$, and their arguments are equally spaced.

Geometrically, the n^{th} roots of a complex number $a \neq 0$ are the vertices of a regular polygon with n sides and they lie on the circle with center at the origin and radius $\sqrt[n]{|a|}$.

Special case:

The case $a = 1$ is particularly important. The roots of the equation $z^n = 1$ are called the ***n^{th} roots of unity***.

The n n^{th} roots of unity are given by $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $k = 0,1,2,\dots,n-1$.

If $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$,then all the roots can be expressed by $1, \omega, \omega^2, \dots, \omega^{n-1}$.

If $\sqrt[n]{a}$ denotes any n^{th} root of a, then all the n^{th} roots can be expressed in the form $\omega^k.\sqrt[n]{a}, k = 0,1,\dots,n-1$.

Example: Find all fifth roots of $4 - 4i$.

Solution:

$4 - 4i = 4\sqrt{2}[\cos(315° + k.360°) + i\sin(315° + k.360°)$

A fifth root is given by

$$(4-4i)^{\frac{1}{5}} = \left(4\sqrt{2}\right)^{\frac{1}{5}}\left[\cos\left(\frac{315°+k.360°}{5}\right) + i\sin\left(\frac{315°+k.360°}{5}\right)\right]$$

$$= \sqrt{2}[\cos(63° + k.72°) + i\sin(63° + k.72°)]$$

Assigning in turn the values $k = 0, 1, 2, 3, 4$ we find

$k = 0$: $R_1 = \sqrt{2}[\cos 63° + i\sin 63°]$

$k = 1$: $R_2 = \sqrt{2}[\cos 135° + i\sin 135°]$

$k = 2$: $R_3 = \sqrt{2}[\cos 207° + i\sin 207°]$

$k = 3$: $R_4 = \sqrt{2}[\cos 279° + i\sin 279°]$

$k = 4$: $R_5 = \sqrt{2}[\cos 351° + i\sin 351°]$

The modulus of each of the roots is $\sqrt{2}$; hence these roots lie on a circle of radius $\sqrt{2}$ with center at the origin. The difference in amplitudes of two consecutive roots is 72°; hence the roots are equally spaced on this circle.

PROBLEM SET

IP1: If n is a positive integer, show that

$$\boldsymbol{(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}.}$$

Solution: $1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

Now, $(1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$

$$= (2)^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \qquad \text{---------}(1)$$

$$(1-i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$

$$= (2)^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \qquad \text{---------}(2)$$

$(1) + (2) \implies (1+i)^n + (1-i)^n = (2)^{\frac{n}{2}}\left(2\cos\frac{n\pi}{4}\right) = 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$

P1: Find the value of $(1+i)^{16}$.

Solution: $1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Now, $(1+i)^{16} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{16}$

$$= \left(\sqrt{2}\right)^{16}\left(\cos\frac{16\pi}{4} + i\sin\frac{16\pi}{4}\right)$$

(By De Movire's theorem for integral index)

$$= 2^8(\cos 4\pi + i\sin 4\pi)$$

$$= 256\big(1 + i(0)\big) = 256$$

$\therefore\ (1+i)^{16} = 256$

IP2: Find the sixth roots of $(-i)$.

Solution:

$$-i = 0 + i(-1) = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$= \cos\left(2k\pi - \frac{\pi}{2}\right) + i\sin\left(2k\pi - \frac{\pi}{2}\right) \quad \text{where } k \in Z.$$

A sixth root is given by

$$(-i)^{\frac{1}{6}} = \left[\cos\left(2k\pi - \frac{\pi}{2}\right) + i\sin\left(2k\pi - \frac{\pi}{2}\right)\right]^{\frac{1}{6}}$$

The six sixth roots of $(-i)$ are given by $R_{k+1} = \cos\left(\frac{(4k-1)\pi}{12}\right) + i\sin\left(\frac{(4k-1)\pi}{12}\right)$, $k = 0, 1, 2, 3, 4, 5.$

$k = 0 \Longrightarrow\ R_1 = \cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)$

$k = 1 \Longrightarrow\ R_2 = \cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right)$

$k = 2 \Longrightarrow\ R_3 = \cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)$

$k = 3 \Longrightarrow\ R_4 = \cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)$

$k = 4 \Longrightarrow\ R_5 = \cos\left(\frac{15\pi}{12}\right) + i\sin\left(\frac{15\pi}{12}\right)$

$k = 5 \Longrightarrow\ R_6 = \cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)$

P2: Find all fourth roots of $\sqrt{3} + i$.

Solution:

$$\sqrt{3} + i = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 2\left(\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right) \quad \text{where } k \in Z.$$

A fourth root is given by

$$\left(\sqrt{3} + i\right)^{\frac{1}{4}} = \left[2\left(\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right)\right]^{\frac{1}{4}}$$

$$= (2)^{\frac{1}{4}}\left[\cos\left(\frac{2k\pi + \frac{\pi}{6}}{4}\right) + i\sin\left(\frac{2k\pi + \frac{\pi}{6}}{4}\right)\right]$$

The four fourth roots of $\sqrt{3} + i$ are given by

$$R_{k+1} = (2)^{\frac{1}{4}}\left[\cos\left(\frac{(12k+1)\pi}{24}\right) + i\sin\left(\frac{(12k+1)\pi}{24}\right)\right]$$

$$k = 0, 1, 2, 3.$$

$k = 0 \implies R_1 = (2)^{\frac{1}{4}}\left[\cos\left(\frac{\pi}{24}\right) + i\sin\left(\frac{\pi}{24}\right)\right]$

$k = 1 \implies R_2 = (2)^{\frac{1}{4}}\left[\cos\left(\frac{13\pi}{24}\right) + i\sin\left(\frac{13\pi}{24}\right)\right]$

$k = 2 \implies R_3 = (2)^{\frac{1}{4}}\left[\cos\left(\frac{25\pi}{24}\right) + i\sin\left(\frac{25\pi}{24}\right)\right]$

$k = 3 \implies R_4 = (2)^{\frac{1}{4}}\left[\cos\left(\frac{37\pi}{24}\right) + i\sin\left(\frac{37\pi}{24}\right)\right]$

IP3: If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then prove that

(i) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

Solution:

Let $a = \cos\alpha + i\sin\alpha, = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$.

Then,

$$a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= \left(0 + i(0)\right) = 0$$

$\implies (a + b + c)^2 = 0$

$\implies a^2 + b^2 + c^2 = -2ab - 2bc - 2ca = -2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

Now,

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a^{-1} + b^{-1} + c^{-1}$

$= \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0$

$\therefore\ a^2 + b^2 + c^2 = 2abc(0) = 0$

$\implies (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

$\implies (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta) + (\cos 2\gamma + i\sin 2\gamma) = 0$

$\implies (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$

Equating real parts on both sides, we get:

$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$\implies 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$

$\implies \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

Hence (i) is proved.

Again, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$\Longrightarrow\ 1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$

$\Longrightarrow\ \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

Hence (ii) is proved.

P3: If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then prove that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(iii) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$

Solution:

Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$

Then,

$$a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$= \big(0 + i(0)\big) = 0$$

we have, $a + b + c = 0 \ \Longrightarrow\ a^3 + b^3 + c^3 = 3abc$

$$\Longrightarrow (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

$\Longrightarrow (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$

$\quad = 3[\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

$\Longrightarrow (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$\quad = 3[\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

Equating real and imaginary parts on both sides, we get:

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Now,

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a^{-1} + b^{-1} + c^{-1}$

$$= \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma$$
$$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$= 0$$

$\Longrightarrow\ ab + bc + ca = 0$

$$\Longrightarrow (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) + (\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma) + (\cos\gamma + i\sin\gamma)(\cos\alpha + i\sin\alpha) = 0$$

$$\Rightarrow \cos(\alpha+\beta)+i\sin(\alpha+\beta)+\cos(\beta+\gamma)+i\sin(\beta+\gamma)+\cos(\gamma+\alpha)+i\sin(\gamma+\alpha)=0$$

$$\Rightarrow [\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)]+i[\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)]=0$$

Equating real parts on both sides we get

$(iii)\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)=0$

IP4: Solve $x^{11}-x^7+x^4-1=0$.

Solution: Given $x^{11}-x^7+x^4-1=0$ --------(1)

$$\Rightarrow x^7(x^4-1)+1(x^4-1)=0 \Rightarrow (x^4-1)(x^7+1)=0$$

$$\Rightarrow x^4-1=0 \quad \text{or} \quad x^7+1=0$$

(1) is a polynomial equation of degree 11. By the fundamental theorem of algebra it has 11 roots. They are given by the fourth roots of unity and the seventh roots of -1.

$x^4-1=0 \Rightarrow x^4=1=\cos 0+i\sin 0=\cos 2k\pi+i\sin 2k\pi$

$$\Rightarrow 1^{\frac{1}{4}}=(\cos 2k\pi+i\sin 2k\pi)^{\frac{1}{4}}$$

$$=\cos\left(\frac{2k\pi}{4}\right)+i\sin\left(\frac{2k\pi}{4}\right), k\in \boldsymbol{Z}.$$

The fourth roots of unity are $x_{k+1}=\cos\left(\frac{2k\pi}{4}\right)+i\sin\left(\frac{2k\pi}{4}\right),\ \ k=0,1,2,3.$

$k=0 \Rightarrow x_1=1$; $k=1 \Rightarrow x_2=i$,

$k=2 \Rightarrow x_3=-1$; $k=3 \Rightarrow x_4=-i$.

$x^7+1=0 \Rightarrow x^7=-1=\cos\pi+i\sin\pi$

$$=\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$$

$$=\cos(2k+1)\pi+i\sin(2k+1)\pi\ ,k\in\boldsymbol{Z}.$$

$$\Rightarrow (-1)^{\frac{1}{7}}=(\cos(2k+1)\pi+i\sin(2k+1)\pi)^{\frac{1}{7}}$$

$$=\cos(2k+1)\frac{\pi}{7}+i\sin(2k+1)\frac{\pi}{7}\ ,k\in\boldsymbol{Z}.$$

The seven seventh roots of -1 are

$$y_k=\cos(2k+1)\frac{\pi}{7}+i\sin(2k+1)\frac{\pi}{7}\ ,k=0,1,2,3,4,5,6.$$

$k=0 \Rightarrow y_1=\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}$,

$k=1 \Rightarrow y_2=\cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}$,

$k=2 \Rightarrow y_3=\cos\frac{5\pi}{7}+i\sin\frac{5\pi}{7}$,

$k=3 \Rightarrow y_4=\cos\pi+i\sin\pi$,

$k=4 \Rightarrow y_5=\cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$,

$k=5 \Longrightarrow \quad y_6=\cos\frac{11\pi}{7}+i\sin\frac{11\pi}{7}.$

$k=6 \Longrightarrow \quad y_7=\cos\frac{13\pi}{7}+i\sin\frac{13\pi}{7}.$

The solutions of the given equation are $x_k, 0 \le k \le 3 \quad , \quad y_k, 0 \le k \le 6.$

P4: Solve $x^9-x^5+x^4-1=0.$

Solution: Given $x^9-x^5+x^4-1=0$ --------(1)

$\Longrightarrow x^5(x^4-1)+1(x^4-1)=0 \Longrightarrow (x^4-1)(x^5+1)=0$

$\Longrightarrow x^4-1=0$ or $x^5+1=0$

(1) is a polynomial equation of degree 9. By the fundamental theorem of algebra it has 9 roots. They are given by the fourth roots of unity and the fifth roots of -1.

$x^4-1=0 \Longrightarrow x^4=1=\cos 0+i\sin 0=\cos 2k\pi+i\sin 2k\pi$

$$\Longrightarrow \quad 1^{\frac{1}{4}}=(\cos 2k\pi+i\sin 2k\pi)^{\frac{1}{4}}$$
$$=\cos\left(\frac{2k\pi}{4}\right)+i\sin\left(\frac{2k\pi}{4}\right), k \in \boldsymbol{Z}.$$

The fourth roots of unity are $x_{k+1}=\cos\left(\frac{2k\pi}{4}\right)+i\sin\left(\frac{2k\pi}{4}\right),\ k=0,1,2,3.$

$k=0 \Longrightarrow x_1=1\ ; \quad k=1 \Longrightarrow x_2=i,$

$k=2 \Longrightarrow x_3=-1\ ; \quad k=3 \Longrightarrow \quad x_4=-i\ .$

$$x^5+1=0 \Longrightarrow x^5=-1=\cos\pi+i\sin\pi$$
$$=\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$$
$$=\cos(2k+1)\pi+i\sin(2k+1)\pi\ , k \in \boldsymbol{Z}\ .$$

$$\Longrightarrow (-1)^{\frac{1}{5}}=(\cos(2k+1)\pi+i\sin(2k+1)\pi)^{\frac{1}{5}}$$
$$=\cos(2k+1)\frac{\pi}{5}+i\sin(2k+1)\frac{\pi}{5} \quad , k \in \boldsymbol{Z}.$$

The five fifth roots of -1 are

$$y_k=\cos(2k+1)\frac{\pi}{5}+i\sin(2k+1)\frac{\pi}{5} \quad , k=0,1,2,3,4.$$

$k=0 \Longrightarrow \quad y_1=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}, k=1 \Longrightarrow \quad y_2=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5},$

$k=2 \Longrightarrow \quad y_3=\cos\pi+i\sin\pi\ , k=3 \Longrightarrow \quad y_4=\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5},$

$k=4 \Longrightarrow \quad y_5=\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}.$

The solutions of the given equation are $x_k, 0 \le k \le 3 \quad , \quad y_k, 0 \le k \le 4$

Exercises:

1. Evaluate each of the following using De Moivre's Theorem and express each result in rectangular form:
   a) $(1 + i\sqrt{3})^4$
   b) $\left(\sqrt{3} - i\right)^5$
   c) $(-1 + i)^{10}$
   d) $(1 - i)^8$
2. Show that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = i.$
3. Show that $\left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n = 2^{n+1}.\cos\left(\frac{n\pi}{3}\right).$
4. Show that $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6 + \left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6 = 2.$
5. If n is an positive integer and $z = \cos\theta + i\sin\theta$, show that $\frac{z^{2n}-1}{z^{2n}-1} = i\tan\theta.$
6. Find the indicated roots and express them in rectangular form.
   a) Square roots of $2 - 2i\sqrt{3}$
   b) Fourth roots of $-8 - 8i\sqrt{3}$
   c) Cube roots of $-4\sqrt{2} + 4i\sqrt{2}$
   d) Cube roots of 1
   e) Fourth roots of i
   f) Sixth roots of -1
   g) Fourth roots of $-16i$
7. Solve the following equations
   a) $x^4 - 1 = 0$
   b) $x^4 + 1 = 0$
   c) $x^5 - 1 = 0$

2.6. Quadratic Equations in One Variable

Learning Objectives:

- To find the roots of a quadratic equation
- To find the sum and the product of the roots of the quadratic equation
- To find the nature of the roots of the quadratic equation
  AND

To practice the related problems

Any equation of the form , $ax^2 + bx + c = 0$ … (1)

where a, b and c are real or complex numbers and $a \neq 0$ is called a ***quadratic equation in one variable x***. This form of equation is known as the ***standard form*** of a quadratic equation in one variable. The numbers a, b and c are called the ***coefficients*** of the equation (1), a is called the ***leading coefficient*** of (1) and c is called the ***constant term*** of (1).

A complex number α is said to be a *root* or a *zero* or a *solution* of (1) if $a\alpha^2 + b\alpha + c = 0$.

Theorem 1: The roots of the quadratic equation

The roots of the quadratic equation $ax^2 + bx + c = 0, \quad a \neq 0$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Proof:

α is a root of $ax^2 + bx + c = 0 , a \neq 0$

$$\Leftrightarrow a\alpha^2 + b\alpha + c = 0$$

$$\Leftrightarrow 4a(a\alpha^2 + b\alpha + c) = 0 \ \ (\because a \neq 0)$$

$$\Leftrightarrow (2a\alpha + b)^2 = b^2 - 4ac$$

$$\Leftrightarrow 2a\alpha + b = \pm\sqrt{b^2 - 4ac}$$

$$\Leftrightarrow \alpha = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Theorem: 2

A quadratic equation cannot have more than two roots.

Proof:

Assume that the equation $ax^2 + bx + c = 0, a \neq 0$, have three different roots α, β and γ.

Then $a\alpha^2 + b\alpha + c = 0$ … (i)

$a\beta^2 + b\beta + c = 0$ … (ii)

$a\gamma^2 + b\gamma + c = 0$ … (iii)

Subtract (ii) from (i), we get

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$$

$$\Rightarrow a(\alpha+\beta)+b=0 \text{ ... (iv) (since } \alpha\neq\beta \text{)}$$

Similarly, subtracting (iii) from (ii), we get

$$\Rightarrow a(\beta+\gamma)+b=0 \text{ ... (v)}$$

Subtracting (v) from (iv), we get

$$a(\alpha-\gamma)=0$$

This is not possible since $a\neq 0$ and $\alpha\neq\gamma$.

Therefore, a quadratic equation cannot have more than two roots.

Corollary: A quadratic equation has two roots (not necessarily different).

Theorem: 3

If α and β are the roots of $ax^2+bx+c=0, a\neq 0$, then

$$\alpha+\beta=-\frac{b}{a}\ , \qquad \alpha\beta=\frac{c}{a}$$

Proof: The roots of $ax^2+bx+c=0, a\neq 0$ are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Let $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

Now, $\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}=-\frac{2b}{2a}=-\frac{b}{a}$

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

Hence the result.

Note:

$$\alpha+\beta=-\frac{coefficient\ of\ x}{coefficient\ of\ x^2} \quad ; \quad \alpha\beta=\frac{constant\ term}{coefficient\ of\ x^2}$$

b^2-4ac is called the ***discriminant*** of the quadratic equation $ax^2+bx+c=0, a\neq 0$ and it is denoted by Δ .

Nature of the roots of a quadratic equation

Let a, b and c be real numbers and α, β be the roots of the quadratic equation $ax^2+bx+c=0, a\neq 0$.

(i) If $\Delta=0$, then $\alpha=\beta=-\frac{b}{2a}$ (a repeated root of $ax^2+bx+c=0$)

(ii) If $\Delta>0$, then α and β are real and distinct.

(iii) If $\Delta<0$, then α and β are non-real complex numbers conjugate to each other.

(iv) If Δ is a perfect square, then α and β are rational and unequal.

By applying the above tests the nature of the roots of any quadratic equation can be determined without actually finding the roots.

Example: 1

Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.

Solution:

Here $a = 2, b = -6, c = 7;$

so that $\Delta = b^2 - 4ac = (-6)^2 - 4 \cdot 2 \cdot 7 = -20 \Rightarrow \Delta < 0$

Therefore, the roots are complex numbers. Thus, the given equation cannot be satisfied by any real values of x.

Example: 2

If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.

Solution:

The condition for equal roots is $\Delta = 0$

$$\Rightarrow b^2 - 4ac = 0 \Rightarrow 4(k+2)^2 - 36k = 0$$

$$\Rightarrow (k+2)^2 = 9k \Rightarrow k^2 - 5k + 4 = 0$$

$$\Rightarrow (k-4)(k-1) = 0 \Rightarrow k = 4 \text{ or } 1$$

Example: 3

Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

Solution:

We examine the nature of the discriminant Δ.

Now, $\Delta = (-2p)^2 - 4(p^2 - q^2 + 2qr - r^2)$

$$= 4(q^2 - 2qr + r^2) = 4(q-r)^2$$

Notice that Δ is a perfect square and so the roots are rational.

The following is a necessary and sufficient condition for a given pair of quadratic equations to have a common root.

Theorem: 4

A pair of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root if and only if $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (a_1c_2 - a_2c_1)^2$

Proof: We have a given pair of quadratic equations

$$a_1x^2 + b_1x + c_1 = 0 \text{ --- } (1)$$

and $a_2x^2 + b_2x + c_2 = 0$ --- (2)

Suppose that they have a common root, say α.

Then $a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$.

Now,
$$\begin{array}{cccc} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

By the rule of cross multiplication, we have

$$\frac{\alpha^2}{b_1c_2-b_2c_1} = \frac{\alpha}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1}$$

$$\Rightarrow \alpha^2 = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \quad , \ \alpha = \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$$

$$\Rightarrow \left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)^2 = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$$

$$\Rightarrow (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$$

Conversely, suppose that

$$(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2 \text{ --- (3)}$$

(i) Multiplying (1) by a_2 and (2) by a_1 and subtracting the first from the second, we get

$$(a_1b_2 - a_2b_1)x + (a_1c_2 - a_2c_1) = 0$$

$\Rightarrow x = \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$ is a common root of (1) and (2) if $a_1b_2 - a_2b_1 \neq 0$

(ii) If $a_1b_2 - a_2b_1 = 0$, then from (3), $c_1a_2 - c_2a_1 = 0$.

Thus, $\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{a_1}{a_2} = \frac{c_1}{c_2}$. Therefore, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\Rightarrow$ (1) and (2) have the same roots and so they have a common root.

Hence the theorem

Example: 4

If the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then show that $a^3 + b^3 + c^3 = 3abc$.

Proof: We have, $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

have a common root if and only if

$$(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$$

In this context: $a_1 = a, b_1 = b, c_1 = c$ and

$a_2 = b, b_2 = c, \ c_2 = a$

If the given equations have a common root, then

$$(ac - b^2)(ba - c^2) = (bc - a^2)^2$$

i.e., $a^2bc - ab^3 - ac^3 + b^2c^2 = b^2c^2 - 2a^2bc + a^4$

i.e., $a^4 + ab^3 + ac^3 = 3a^2bc$

i.e., $a^3 + b^3 + c^3 = 3abc \ (\because a \neq 0)$

PROBLEM SET

IP1: Find the nature of the roots of the equation $\frac{x^2}{3} - x = -\frac{1}{2}$ and find the roots.

Solution: We have, $\frac{x^2}{3} - x = -\frac{1}{2} \Rightarrow \frac{x^2-3x}{3} = \frac{-1}{2} \Rightarrow 2x^2 - 6x + 3 = 0,$

Comparing this with the standard form of quadratic equation,

we find $a = 2, b = -6$ and $c = 3$

Discriminant $= \Delta = b^2 - 4ac = (-6)^2 - 4 \cdot 2 \cdot 3$

$$= 36 - 24 = 12 > 0$$

$\therefore$ $\Delta > 0$, the roots are real and distinct.

The roots are $\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-6)\pm\sqrt{12}}{2\cdot 2} = \frac{6\pm2\sqrt{3}}{4} = \frac{3\pm\sqrt{3}}{2}$

P1: Find the nature of the roots of the equation $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$ and find the roots.

Solution: We have, $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0,$

Comparing this with the standard form of quadratic equation,

we find $a = \sqrt{3}, b = -\sqrt{2}$ and $c = 3\sqrt{3}$.

Discriminant $= \Delta = b^2 - 4ac = \left(-\sqrt{2}\right)^2 - 4.\sqrt{3}.3\sqrt{3}$

$$= 2 - 36 = -34 < 0$$

$\therefore$ $\Delta < 0$, the roots are non-real complex numbers conjugate to each other.

The roots are $\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\left(-\sqrt{2}\right)\pm\sqrt{-34}}{2.\sqrt{3}}$

$$= \frac{\sqrt{2}\pm\sqrt{34}i}{2\sqrt{3}} = \frac{1\pm\sqrt{17}i}{\sqrt{6}}$$

IP2: If the equation $x^2 - 15 - m(2x - 8) = 0$ has equal roots, find the values of m.

Solution: We have, $x^2 - 15 - m(2x - 8) = 0$

$x^2 - 2mx + (8m - 15) = 0$ ... (1)

If equation (1) has equal roots, then Discriminant, $\Delta = 0$

$$\Rightarrow (-2m)^2 - 4 \cdot 1 \cdot (8m - 15) = 0$$

$\Rightarrow 4m^2 - 32m + 60 = 0$

$\Rightarrow m^2 - 8m + 15 = 0$

$\Rightarrow (m - 3)(m - 5) = 0$

$\Rightarrow m = 3$ or 5

P2: For what values of m will the equation $\frac{x^2-bx}{ax-c} = \frac{m-1}{m+1}$ have roots equal in magnitude but opposite in sign?

Solution: We have, $\frac{x^2-bx}{ax-c}=\frac{m-1}{m+1}$

$(m+1)x^2-[b(m+1)+a(m-1)]x+c(m-1)=0$ ... (1)

Let α and β be the roots of (1). Then

$$\alpha+\beta=-\frac{coeff.of\ x}{coeff.of\ x^2}=\frac{[b(m+1)+a(m-1)]}{m+1}$$

Since equation (1) has the roots equal in magnitude but opposite in sign, $\alpha+\beta=0$

$$\Rightarrow b(m+1)+a(m-1)=0$$

$$\Rightarrow m(a+b)+b-a=0$$

$$\Rightarrow m=\frac{a-b}{a+b}$$

The value of m for which the equation (1) has roots equal in magnitude but opposite in sign is $m=\frac{a-b}{a+b}$

IP3: Prove that the roots of $x^2-2ax+a^2-b^2-c^2=0$ are real.

Solution:

We have, $x^2-2ax+a^2-b^2-c^2=0$ ... (1)

Discriminant, $\Delta=(-2a)^2-4\cdot 1\cdot(a^2-b^2-c^2)$

$$=4a^2-4a^2+4b^2+4c^2$$

$$=4(b^2+c^2)>0\ (\because\ \forall\, real\ b,c;\ b^2+c^2>0)$$

Therefore, the roots of the given equation are real.

P3: Prove that the roots of $(a-b+c)x^2+4(a-b)x+(a-b-c)=0$ are real.

Solution:

We have, $(a-b+c)x^2+4(a-b)x+(a-b-c)=0$

Discriminant $=\Delta=16(a-b)^2-4\cdot(a-b+c)\cdot(a-b-c)$

$$=16(a-b)^2-4[(a-b)^2-c^2]$$

$$=16(a-b)^2-4(a-b)^2+4c^2$$

$$=12(a-b)^2+4c^2>0$$

$$(\because\ \forall\, real\ a,b,c;\ (a-b)^2>0\ and\ c^2>0)$$

Therefore, the roots of the given equation are real.

IP4: Show that the roots of the equation $x^2-2ax+a^2-b^2+2bc-c^2=0$ are rational.

Solution:

We have, $x^2-2ax+a^2-b^2+2bc-c^2=0$

$$\text{Discriminant} = \Delta = (-2a)^2 - 4 \cdot 1 \cdot (a^2 - b^2 + 2bc - c^2)$$
$$= 4a^2 - 4a^2 + 4b^2 - 8bc + 4c^2$$
$$= 4(b^2 - 2bc + c^2)$$
$$= 4(b-c)^2 = \{2(b-c)\}^2, \text{ a perfect square}$$

Therefore, the roots of the given equation are rational.

P4: Show that the roots of the equation $(a+c-b)x^2 + 2cx + (b+c-a) = 0$ are rational.

Solution:

We have, $(a+c-b)x^2 + 2cx + (b+c-a) = 0$

$$\text{Discriminant}, \Delta = (2c)^2 - 4 \cdot (a+c-b) \cdot (b+c-a)$$
$$= 4c^2 - 4[c + (a-b)][c - (a-b)]$$
$$= 4c^2 - 4[c^2 - (a-b)^2]$$
$$= 4c^2 - 4c^2 + 4(a-b)^2$$
$$= 4(a-b)^2 = \{2(a-b)\}^2, \text{ a perfect square}$$

Therefore, the roots of the given equation are rational.

Exercises:

1. Find the nature of the roots and the roots of the following equations.
   a. $x^2 - 9x + 20 = 0$
   b. $x^2 - 7x + 12 = 0$
   c. $3x^2 + 2x - 5 = 0$
   d. $x^2 - x - 12 = 0$
   e. $x^2 - 32x - 900 = 0$
   f. $3x^2 - 5x - 12 = 0$
   g. $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$
   h. $x^2 + 6x + 34 = 0$
2. If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.
3. If the equation $(2k+1)x^2 + 3(k-1)x + 1 - k = 0$ has equal roots, find k.
4. If the equation $p(q-r)x^2 + q(r-p)x + r(p-q) = 0$ has equal roots, find $\frac{2}{q}$.
5. If the roots of $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal, show that a, b, c are in A.P.
6. If the roots of $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, show that a, b, c are in H.P.
7. If the roots of $(a^2+b^2)x^2 - 2b(a+c)x + (b^2+c^2) = 0$ are equal, show that a, b, c are in G.P.
8. If $x^2 - (5m-2)x + 4m^2 + 10m + 25 = 0$ is a perfect square, then find m.
9. For what values of m will the equation $x^2 - 2x(1+3m) + 7(3+2m) = 0$ have equal roots?

10. Find an appropriate k so that the equation $4x^2 - kx = -9$ has exactly one rational solution.
11. Prove that the roots of $(x-a)(x-b) = h^2$ are always real.
12. If α and β are the roots of $ax^2 + bx + c = 0$, find the values of
    a. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
    b. $\alpha^4\beta^7 + \alpha^7\beta^4$
    c. $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$
    d. $\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2}$
    e. $\frac{\alpha^{-3}+\beta^{-3}}{\alpha^3+\beta^3}$
    f. $\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)^2$
    g. $\alpha^3\beta^3 + \alpha^2\beta^3 + \alpha^3\beta^2$
13. Find the condition that one root of $ax^2 + bx + c = 0$ may be four times the other.
14. Find the condition that one root of $ax^2 + bx + c = 0$ shall be n times the other.
15. Prove the condition that one root of $ax^2 + bx + c = 0$ may be the square of the other is $b^3 + a^2c + ac^2 = 3abc$.
16. Prove the condition that the roots of $ax^2 + bx + c = 0$ will be reciprocal of each other is $a = c$.
17. Find the condition that the roots of $ax^2 + bx + c = 0$ may be
    i. both positive
    ii. opposite in sign, but the greater of them negative.
18. Find the condition that the roots of the equation $ax^2 + bx + c = 0$ should be reciprocals.
19. Find the condition that the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ may have a common root.
20. If $x^2 + 4ax + 3 = 0$ and $2x^2 + 3ax - 9 = 0$ have a common root, then find the values of a and the common roots.
21. If the quadratic equations $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0, (b \neq c)$ have a common root, then show that $a + 4b + 4c = 0$.
22. If $x^2 - 6x + 5 = 0$ and $x^2 - 12x + p = 0$ have a common root, then find p.
23. If $x^2 - 6x + 5 = 0$ and $x^2 - 3ax + 35 = 0$ have a common root, then find a.
24. If the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$, have a common root and the first equation has equal roots, then prove that $2(b + d) = ac$.

2.7. Equations Reducible to Quadratic Equations

Learning Objectives:

- To study the equations reducible to quadratic equations and to solve them by suitable substitutions

  AND

- To practice the related problems

In this module we study some equations which are reducible to quadratic equations by suitable substitutions.

1. Equations of the type $\boldsymbol{ax^{2n} + bx^n + c = 0}$:

If the equation is of the type, $\boldsymbol{ax^{2n} + bx^n + c = 0}$, then we substitute $\boldsymbol{y}$ for $\boldsymbol{x^n}$.

Example: Solve $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 = 0$

Put $y = x^{\frac{1}{3}}$. We get

$$y^2 + y - 2 = 0$$

$$\Rightarrow y = 1 \text{ or } y = -2$$

$$\Rightarrow x^{\frac{1}{3}} = 1, -2 \Rightarrow x = 1, -8$$

2. Equations of type $\boldsymbol{az + \frac{b}{z} = c}$, where $\boldsymbol{a, b}$ and $\boldsymbol{c}$ are constants.

Example: Solve $\sqrt{\frac{2x^2+x+2}{x^2+3x+1}} + 2\sqrt{\frac{x^2+3x+1}{2x^2+x+2}} - 3 = 0$.

Let $y = \sqrt{\frac{2x^2+x+2}{x^2+3x+1}}$. Then

$$y + \frac{2}{y} - 3 = 0 \Rightarrow y^2 - 3y + 2 = 0 \Rightarrow y = 1,2$$

$$\sqrt{\frac{2x^2+x+2}{x^2+3x+1}} = 1 \Rightarrow \frac{2x^2+x+2}{x^2+3x+1} = 1 \Rightarrow x^2 - 2x + 1 = 0$$

$$\Rightarrow x = 1$$

$$\sqrt{\frac{2x^2+x+2}{x^2+3x+1}} = 2 \Rightarrow \frac{2x^2+x+2}{x^2+3x+1} = 4 \Rightarrow 2x^2 + 11x + 2 = 0$$

$$\Rightarrow x = \frac{-11 \pm \sqrt{105}}{4}$$

3. Equations of the type

$$\boldsymbol{(x + a)(x + b)(x + c)(x + d) + k = 0}$$

When sum of two of the quantities a, b, c, d is equal to the sum of the other two, the equation can be solved as shown below.

Example:

Solve $(x+1)(x+2)(x+3)(x+4) = 120.$

Solution:

We have, $(x+1)(x+2)(x+3)(x+4) = 120$

$\Rightarrow [(x+1)(x+4)][(x+2)(x+3)] = 120$

(Note that 1+4 = 2+3)

$\Rightarrow (x^2+5x+4)(x^2+5x+6) = 120$

$\Rightarrow \{(x^2+5x)+4\}\{(x^2+5x)+6\} = 120$

Let $x^2+5x = y$

$\Rightarrow (y+4)(y+6) = 120 \Rightarrow y^2+10y-96 = 0$

$\Rightarrow (y+16)(y-6) = 0$

$\Rightarrow y = -16, 6$

Now, $y = -16 \Rightarrow x^2+5x = -16$

$\Rightarrow x^2+5x+16 = 0$

$\Rightarrow x = \frac{-5\pm\sqrt{25-64}}{2} = \frac{-5\pm i\sqrt{39}}{2}$

and $y = -9 \Rightarrow x^2+5x = 6$

$\Rightarrow x^2+5x-6 = 0$

$\Rightarrow (x+6)(x-1) = 0$

$\Rightarrow x = -6, 1$

Therefore, the solution set for the given equation is $\left\{\frac{-5\pm i\sqrt{39}}{2}, -6, 1\right\}$

4. **The equations of type $\sqrt{\boldsymbol{ax+b}}+\sqrt{\boldsymbol{cx+d}} = \boldsymbol{k}$ or $\sqrt{\boldsymbol{ax+b}}+\sqrt{\boldsymbol{cx+d}} = \sqrt{\boldsymbol{ex+f}}$**

Such equations can be reduced to quadratic equations by getting rid of the square roots by squaring appropriately.

Example:

Solve $\sqrt{x+1}+\sqrt{2x-5} = 3.$

Solution:

We have, $\sqrt{x+1}+\sqrt{2x-5} = 3$

On squaring both sides of the above equation, we get

$(x+1)+(2x-5)+2\sqrt{x+1}\sqrt{2x-5} = 9$

$\Rightarrow 3x-13 = -2\sqrt{x+1}\sqrt{2x-5}$

Again squaring on both sides, we get

$$\Rightarrow (3x-13)^2 = 4(x+1)(2x-5)$$
$$\Rightarrow 9x^2 - 78x + 169 = 4(2x^2 - 3x - 5)$$
$$\Rightarrow x^2 - 66x + 189 = 0$$
$$\Rightarrow (x-63)(x-3) = 0$$
$$\Rightarrow x = 3 \text{ or } 63$$

Since $\sqrt{a} > 0$ for any $a > 0$, 63 does not satisfy the given equation. However 3 satisfies the given equation.

Therefore, the solution set of the given equation is $\{3\}$.

5. Fourth degree equations without odd power terms:

If a fourth degree equation in x does not contain odd power of x terms, then we can solve by the following method:

Example:

Find the solutions of the equation $x^4 - 5x^2 + 6 = 0$.

Solution:

We have, $x^4 - 5x^2 + 6 = 0$.

Put $x^2 = y$, above equation becomes

$$y^2 - 5y + 6 = 0$$
$$\Rightarrow (y-2)(y-3) = 0$$
$$\Rightarrow y = 2 \text{ or } 3$$

Hence $x^4 - 5x^2 + 6 = 0 \Leftrightarrow x^2 = 2$ or $x^2 = 3$

$$\Leftrightarrow x = \pm\sqrt{2} \text{ or } x = \pm\sqrt{3}$$

Therefore, the solution set of the given equation is $\{-\sqrt{2}, \sqrt{2}, -\sqrt{3}, \sqrt{3}\}$.

6. Fourth degree equations with coeff.of x^4 = constant term and coeff.of x^3 =coeff.of x.

If a fourth degree equation in x is such that the coefficient of x^4 is equal to the constant term and the coefficient of x^3 is equal to the coefficient of x, then we solve it by the method illustrated below:

Example:

Solve the equation $x^4 - 2x^3 - x^2 - 2x + 1 = 0$.

Solution:

Since $x = 0$ is not a solution of the given equation, on dividing the given equation by x^2, we get

$$x^2 - 2x - 1 - \frac{2}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow x^2 - 2x - 1 - \frac{2}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) - 1 = 0$$

$$\Rightarrow \left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 2\left(x + \frac{1}{x}\right) - 1 = 0$$

Let $x + \frac{1}{x} = y$

$$\Rightarrow (y^2 - 2) - 2y - 1 = 0$$

$$\Rightarrow y^2 - 2y - 3 = 0$$

$$\Rightarrow y^2 - 2y - 3 = 0$$

$$\Rightarrow (y + 1)(y - 3) = 0$$

$$\Rightarrow y = -1 \text{ or } 3$$

$$x^4 - 2x^3 - x^2 - 2x + 1 = 0 \Leftrightarrow x + \frac{1}{x} = -1 \text{ or } x + \frac{1}{x} = 3$$

$$\Leftrightarrow x^2 + x + 1 = 0 \text{ or } x^2 - 3x + 1 = 0$$

$$\Leftrightarrow x = \frac{-1 \pm i\sqrt{3}}{2} \text{ or } x = \frac{3 \pm \sqrt{5}}{2}$$

Therefore, the solution set of the given equation is $\left\{\frac{-1 \pm i\sqrt{3}}{2}, \frac{3 \pm \sqrt{5}}{2}\right\}$.

7. Miscellaneous:

Example:

Solve $\mathbf{7^{1+x} + 7^{1-x} = 50}$ for real $\boldsymbol{x}$.

Solution:

We have, $\mathbf{7^{1+x} + 7^{1-x} = 50 \Rightarrow 7 \cdot 7^x + \frac{7}{7^x} = 50}$

Let $7^x = y$

$$\therefore 7y + \frac{7}{y} = 50 \Rightarrow 7y^2 - 50y + 7 = 0$$

$$\Rightarrow (7y - 1)(y - 7) = 0 \Rightarrow y = \frac{1}{7}, 7$$

When $y = \frac{1}{7} \Rightarrow 7^x = \frac{1}{7} \Rightarrow 7^x = 7^{-1} \Rightarrow x = -1$

When $y = 7 \Rightarrow 7^x = 7^1 \Rightarrow x = 1$

Therefore, the solution set of the given equation is $\{-1,1\}$.

Example:

Solve $\left(x+\frac{1}{x}\right)^2-\frac{3}{2}\left(x-\frac{1}{x}\right)=4$

Solution:

The given can be written as

$$\left\{\left(x-\frac{1}{x}\right)^2+4\right\}-\frac{3}{2}\left(x-\frac{1}{x}\right)-4=0$$

On taking $x-\frac{1}{x}=y$, the above equation becomes

$$y^2-\frac{3}{2}y=0 \text{ ,i.e., } y\left(y-\frac{3}{2}\right)=0$$

0 and $\frac{3}{2}$ are the roots of the above equation.

Hence

$$\left(x+\frac{1}{x}\right)^2-\frac{3}{2}\left(x-\frac{1}{x}\right)=4 \Leftrightarrow x-\frac{1}{x}=0 \text{ or } x-\frac{1}{x}=\frac{3}{2}$$

$$\Leftrightarrow x^2-1=0 \text{ or } 2x^2-3x-2=0$$

$$\Leftrightarrow (x-1)(x+1)=0 \text{ or } (2x+1)(x-2)=0$$

$$\Leftrightarrow x=1 \text{ or } -1 \text{ or } x=-\frac{1}{2} \text{ or } 2$$

Therefore, the solution set of the given equation is $\left\{-\frac{1}{2},-1,1,2\right\}$.

PROBLEM SET

IP1: Solve: $x^{-2}-2x^{-1}=8$ for real $x\neq 0$.

Solution:

We have, $x^{-2}-2x^{-1}=8 \Rightarrow (x^{-1})^2-2x^{-1}-8=0$

Let $x^{-1}=y$. Then the above equation becomes $y^2-2y-8=0 \Rightarrow (y+2)(y-4)=0 \Rightarrow y=-2$ or 4

Now, $y=-2 \Rightarrow x^{-1}=-2 \Rightarrow x=-\frac{1}{2}$

and $y=4 \Rightarrow x^{-1}=4 \Rightarrow x=\frac{1}{4}$

Therefore, the solution set for the given equation is $\left\{-\frac{1}{2},\frac{1}{4}\right\}$.

P1: Solve: $\sqrt{\frac{x^2+2}{x^2-2}}+6\sqrt{\frac{x^2-2}{x^2+2}}=5.$

Solution:

We have, $\sqrt{\frac{x^2+2}{x^2-2}}+6\sqrt{\frac{x^2-2}{x^2+2}}=5$

Let $\sqrt{\frac{x^2+2}{x^2-2}} = y$. Then the given equation becomes $y + \frac{6}{y} = 5$

$\Rightarrow y^2 - 5y + 6 = 0 \Rightarrow (y-2)(y-3) = 0 \Rightarrow y = 2, 3$

Now, $y = 2 \Rightarrow \sqrt{\frac{x^2+2}{x^2-2}} = 2 \Rightarrow \frac{x^2+2}{x^2-2} = 4 \Rightarrow x^2 + 2 = 4x^2 - 8$

$$\Rightarrow 3x^2 = 10 \Rightarrow x^2 = \frac{10}{3} \Rightarrow x = \pm\sqrt{\frac{10}{3}}$$

and $y = 3 \Rightarrow \sqrt{\frac{x^2+2}{x^2-2}} = 3 \Rightarrow \frac{x^2+2}{x^2-2} = 9 \Rightarrow x^2 + 2 = 9x^2 - 18$

$$\Rightarrow 8x^2 = 20 \Rightarrow x^2 = \frac{5}{2} \Rightarrow x = \pm\sqrt{\frac{5}{2}}$$

Therefore, the solution set for the given equation is $\left\{\pm\sqrt{\frac{10}{3}}, \pm\sqrt{\frac{5}{2}}\right\}$

IP2: Solve the equation $2x^4 + x^3 - 11x^2 + x + 2 = 0$.

Solution:

It is a fourth degree equations with coeff.of x^4 = constant term and coeff.of x^3 =coeff.of x.

Since $x = 0$ is not a solution of the given equation, on dividing the given equation by x^2, we get

$$2x^2 + x - 11 + \frac{1}{x} + \frac{2}{x^2} = 0$$

$$\Rightarrow 2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0$$

$$\Rightarrow 2\left[\left(x + \frac{1}{x}\right)^2 - 2\right] + \left(x + \frac{1}{x}\right) - 11 = 0$$

Let $x + \frac{1}{x} = y$

$$\Rightarrow 2(y^2 - 2) + y - 11 = 0 \Rightarrow 2y^2 + y - 15 = 0$$

$$\Rightarrow (y+3)(2y-5) = 0 \Rightarrow y = -3 \text{ or } \frac{5}{2}$$

$2x^4 + x^3 - 11x^2 + x + 2 = 0 \Leftrightarrow x + \frac{1}{x} = -3$ or $x + \frac{1}{x} = \frac{5}{2}$

$$\Leftrightarrow x^2 + 3x + 1 = 0 \text{ or } 2x^2 - 5x + 2 = 0$$

$$\Leftrightarrow x^2 + 3x + 1 = 0 \text{ or } (2x-1)(x-2) = 0$$

$$\Leftrightarrow x = \frac{-3\pm\sqrt{5}}{2} \text{ or } x = \frac{1}{2} \text{ or } 2$$

Therefore, the solution set of the given equation is $\left\{\frac{-3\pm\sqrt{5}}{2}, \frac{1}{2}, 2\right\}$.

P2: Solve: $2\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+5=0$ when $x \neq 0$.

Solution:

We have, $2\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+5=0$

On taking $x+\frac{1}{x}=y$, the above equation becomes

$$2y^2-7y+5=0 \Rightarrow (2y-5)(y-1)=0 \Rightarrow y=\frac{5}{2} \text{ or } 1$$

Hence

$$2\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+5=0 \Leftrightarrow x+\frac{1}{x}=\frac{5}{2} \text{or } x+\frac{1}{x}=1$$

$$\Leftrightarrow 2x^2-5x+2=0 \text{ or } x^2-x+1=0$$

$$\Leftrightarrow (2x-1)(x-2)=0 \text{ or } x=\frac{1\pm i\sqrt{3}}{2}$$

$$\Leftrightarrow x=\frac{1}{2} \text{ or } 2 \text{ or } x=\frac{1\pm i\sqrt{3}}{2}$$

Therefore, the solution set of the given equation is $\left\{\frac{1\pm i\sqrt{3}}{2},\frac{1}{2},2\right\}$.

IP3: Solve $(2x+1)(2x+3)(2x+5)(2x+7)+12=0$.

Solution:

We have, $(2x+1)(2x+3)(2x+5)(2x+7)+12=0$

$$\Rightarrow [(2x+1)(2x+7)][(2x+3)(2x+5)]+12=0$$

$$\Rightarrow (4x^2+16x+7)(4x^2+16x+15)+12=0$$

Let $4x^2+16x=y$

$$\Rightarrow (y+7)(y+15)+12=0 \Rightarrow y^2+22y+105+12=0$$

$$\Rightarrow y^2+22y+117=0 \Rightarrow (y+9)(y+13)=0$$

$$\Rightarrow y=-13,-9$$

Now, $y=-13 \Rightarrow 4x^2+16x=-13$

$$\Rightarrow 4x^2+16x+13=0$$

$$\Rightarrow x=\frac{-16\pm\sqrt{(16)^2-4.4.13}}{2\cdot 4}=\frac{-16\pm 4\sqrt{16-13}}{8}=\frac{-4\pm\sqrt{3}}{2}$$

and $y=-9 \Rightarrow 4x^2+16x=-9$

$$\Rightarrow 4x^2+16x+9=0$$

$$\Rightarrow x=\frac{-16\pm\sqrt{(16)^2-4.4.9}}{2\cdot 4}=\frac{-16\pm 4\sqrt{16-9}}{8}=\frac{-4\pm\sqrt{7}}{2}$$

Therefore, the solution set for given equation is $\left\{\frac{-4\pm\sqrt{3}}{2},\frac{-4\pm\sqrt{7}}{2}\right\}$

P3: Solve: $(x^2-1)(x+5)(x+3)+16=0.$

Solution:

We have, $(x^2-1)(x+5)(x+3)+16=0$

$\Rightarrow (x-1)(x+1)(x+5)(x+3)+16=0$

$\Rightarrow [(x-1)(x+5)][(x+1)(x+3)]+16=0$

$\Rightarrow (x^2+4x-5)(x^2+4x+3)+16=0$

Let $x^2+4x=y$

$\Rightarrow (y-5)(y+3)+16=0 \Rightarrow y^2-2y-15+16=0$

$\Rightarrow y^2-2y+1=0 \Rightarrow (y-1)^2=0 \Rightarrow y=1$

$\therefore x^2+4x=1 \Rightarrow x^2+4x-1=0$

$\Rightarrow x=\frac{-4\pm\sqrt{16+4}}{2\cdot 1}=\frac{-4\pm 2\sqrt{5}}{2}=-2\pm\sqrt{5}$

Therefore, the solution set for the given equation is $\{-2\pm\sqrt{5}\}$.

IP4: Solve: $4^{1+x}+4^{1-x}=10$ for real x.

Solution:

We have, $4^{1+x}+4^{1-x}=10 \Rightarrow 4.4^x+\frac{4}{4^x}=10$

Let $4^x=y$. Then the given equation becomes $4y+\frac{4}{y}=10$

$\Rightarrow 4y^2-10y+4=0 \Rightarrow (y-2)(4y-2)=0 \Rightarrow y=2,\frac{1}{2}$

Now, $y=2 \Rightarrow 4^x=2 \Rightarrow 2^{2x}=2^1 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$

$y=\frac{1}{2} \Rightarrow 4^x=\frac{1}{2} \Rightarrow 2^{2x}=2^{-1} \Rightarrow 2x=-1 \Rightarrow x=-\frac{1}{2}$

Therefore, the solution set of the given equation is $\left\{-\frac{1}{2},\frac{1}{2}\right\}$

P4: Solve: $\sqrt{2x+1}+\sqrt{3x+2}=\sqrt{5x+3}.$

Solution:

We have, $\sqrt{2x+1}+\sqrt{3x+2}=\sqrt{5x+3}$.

On squaring both sides of the above equation, we get

$(2x+1)+(3x+2)+2\sqrt{2x+1}\sqrt{3x+2}=5x+3$

$\Rightarrow \sqrt{2x+1}\sqrt{3x+2}=0$

Again squaring on both sides, we get

$\Rightarrow (2x+1)(3x+2)=0$

$\Rightarrow x=-\frac{1}{2}$ or $-\frac{2}{3}$

Since $\sqrt{a} > 0$ for any $a > 0$, $-\frac{2}{3}$ does not satisfy the given equation. However $-\frac{1}{2}$ satisfies the given equation.

Therefore, the solution set of the given equation is $-\frac{1}{2}$.

Exercises:

1. Solve the following equations.
   a. $9 + x^{-4} = 10x^{-2}$
   b. $2\sqrt{x} + 2x^{-\frac{1}{2}} = 5$
   c. $x^{\frac{2}{n}} + 6 = 5x^{\frac{1}{n}}$
   d. $3x^{\frac{1}{2n}} - x^{\frac{1}{n}} - 2 = 0$
   e. $1 + 8x^{\frac{6}{5}} + 9\sqrt[5]{x^3} = 0$
   f. $3^x + 3^{-x} - 2 = 0$
   g. $3^{1+x} + 3^{1-x} = 10$
   h. $3^{2x} + 9 = 10 \cdot 3^x$
   i. $2^{2x+8} + 1 = 32 \cdot 2^x$
   j. $2^{2x+3} - 57 = 65(2^x - 1)$
   k. $9^x - 4(3^x) - 45 = 0$
   l. $\sqrt{\frac{x}{x-3}} + \sqrt{\frac{x-3}{x}} = \frac{5}{2}$ when $x \neq 0$ and $x \neq 3$.
   m. $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$ when $x \neq 0$ and $x \neq 1$.
   n. $8\sqrt{\frac{x}{x+3}} - \sqrt{\frac{x+3}{x}} = 2$ when $x \neq 0$ and $x \neq -3$.
   o. $\sqrt{\frac{4x-1}{4x+1}} - \sqrt{\frac{4x+1}{4x-1}} = \frac{8}{3}$ when $x \neq \pm\frac{1}{4}$.
   p. $\sqrt{3x^2 + 1} + \frac{4}{\sqrt{3x^2+1}} = 5$
   q. $\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0$, when $x \neq 0$
   r. $9\left(x^2 + \frac{1}{x^2}\right) - 27\left(x + \frac{1}{x}\right) + 8 = 0$, when $x \neq 0$
   s. $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) = 1$, when $x \neq 0$
   t. $(x + 1)(x + 2)(x + 3)(x + 4) + 1 = 0$
   u. $(x - 1)(x - 3)(x - 5)(x - 7) = 9$
   v. $x(x + 2)(x + 3)(x + 5) = 72$
   w. $x(x - 1)(x + 2)(x - 3) = -8$
   x. $(x - 1)(x + 1)(2x + 3)(2x - 1) = 3$
   y. $(x - 7)(x + 1)(x - 3)(x + 5) = 1680$
   z. $(x + 9)(x - 3)(x - 7)(x + 5) = 385$
   aa. $\sqrt{3x + 1} - \sqrt{x - 1} = 2$
   bb. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}$
   cc. $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$

dd. $6x^4 - 25x^3 + 12x^2 - 25x + 6 = 0$

2.8. Framing Equations with Given Roots

Learning Objectives:

- To find the quadratic equations with the given roots
  AND
- To practice the related problems

In this module we find the quadratic equations with given roots.

Theorem: 1

If α and β are the roots of a quadratic equation, then the equation is

$\boldsymbol{x^2 - (\alpha + \beta)x + \alpha\beta = 0.}$

Proof:

If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0, a \neq 0$, then

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}$$

Now $ax^2 + bx + c = 0 \Rightarrow a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$

$$\Rightarrow a(x^2 - (\alpha + \beta)x + \alpha\beta) = 0$$
$$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0, (\because a \neq 0)$$

- **The quadratic equation for which α and β are the roots is $(x - \alpha)(x - \beta) = 0$**

Example: 1

Find the quadratic equation whose roots are $a + ib$ and $a - ib$.

Solution:

The required quadratic equation is

$$x^2 - [(a + ib) + (a - ib)]x + (a + ib)(a - ib) = 0$$

i.e., $x^2 - 2ax + a^2 + b^2 = 0$

Note:

i) If $a + ib = z_1$, then $a - ib = \overline{z_1}$, then the quadratic equation whose roots are z_1 and $\overline{z_1}$ is

$$x^2 - 2Re(z_1)x + |z_1|^2 = 0$$

Example: 2

Find the quadratic equation whose roots are $7+2\sqrt{5}$ and $7-2\sqrt{5}$.

Solution:

The required quadratic equation is

$$x^2-\left[\left(7+2\sqrt{5}\right)+\left(7-2\sqrt{5}\right)\right]x+\left(7+2\sqrt{5}\right)\left(7-2\sqrt{5}\right)=0$$

i.e., $x^2-14x+29=0$

Example: 3

If α and β are the roots of the equation $lx^2+mx+n=0$ then find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

Solution:

Given α,β are the roots of the equation

$$lx^2+mx+n=0$$

Therefore, $\alpha+\beta=-\frac{m}{l}$, $\alpha\beta=\frac{n}{l}$

The quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is

$$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=0$$

i.e., $x^2-\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x+1=0$

i.e., $\alpha\beta x^2-(\alpha^2+\beta^2)x+\alpha\beta=0$

Here $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{m^2}{l^2}-\frac{2n}{l}=\frac{m^2-2nl}{l^2}$

Thus the required equation is

$$\frac{n}{l}x^2-\frac{m^2-2nl}{l^2}x+\frac{n}{l}=0$$

i.e., $nlx^2-(m^2-2nl)x+nl=0$

Example: 4

If $x=\frac{3+5i}{2}$, then find the value of $2x^3+2x^2-7x+72$. **Solution:**

We first form the quadratic equation whose roots $\frac{3\pm5i}{2}$.

The required equation is

$$x^2-\left(\frac{3+5i}{2}+\frac{3-5i}{2}\right)x+\left(\frac{3+5i}{2}\right)\left(\frac{3-5i}{2}\right)=0$$

i.e., $x^2 - 3x + \frac{17}{2} = 0$

i.e., $2x^2 - 6x + 17 = 0$

Now,

$2x^3 + 2x^2 - 7x + 72$

$= x(2x^2 - 6x + 17) + 8x^2 - 24x + 72$

$= x(2x^2 - 6x + 17) + 4(2x^2 - 6x + 17) + 4$

$= x(0) + 4(0) + 4 = 4$

4 is the value of the given cubic polynomial when $x = \frac{3 \pm 5i}{2}$

Theorem: 2

If α and β are the roots of the equation

$$f(x) = ax^2 + bx + c = 0, a \neq 0 \text{ ---- } (1),$$

then the quadratic equation whose roots are

(i) $k\alpha$ and $k\beta$ (where $k \in R$) is $f\left(\frac{x}{k}\right) = 0$.

(ii) $-\alpha$ and $-\beta$ is $f(-x) = 0$.

(iii) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $f\left(\frac{1}{x}\right) = 0$, when $c \neq 0$ in (1).

(iv) $\alpha + k$ and $\beta + k$ (where $k \in \boldsymbol{R}$) is $f(x - k) = 0$

Proof:

Given, α and β are the roots of the equation

$$f(x) = ax^2 + bx + c = 0, a \neq 0 \ldots (1)$$

Then $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

(i) The quadratic equation whose roots are $k\alpha$ and $k\beta$ where $k \in \boldsymbol{R}$ is

$x^2 - (k\alpha + k\beta)x + (k\alpha)(k\beta) = 0$

i.e., $x^2 - k(\alpha + \beta)x + k^2(\alpha\beta) = 0$

i.e., $x^2 - k\left(-\frac{b}{a}\right)x + k^2\left(\frac{c}{a}\right) = 0$

i.e., $a\left(\frac{x}{k}\right)^2 + b\left(\frac{x}{k}\right) + c = 0$

i.e., $f\left(\frac{x}{k}\right) = 0$

(ii) Taking $k = -1$, we get the result in (ii).

(iii) The quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha} \cdot \frac{1}{\beta} = 0$$

i.e., $x^2 - \left(\frac{\alpha+\beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0$

i.e., $\frac{c}{a}x^2 + \frac{b}{a}x + 1 = 0$

i.e., $cx^2 + bx + a = 0$

i.e., $a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0$

i.e., $f\left(\frac{1}{x}\right) = 0$

The other result follows by a similar argument.

Example: 5

If α, β are the roots of $x^2 + bx + c = 0$ and $\alpha + h$, $\beta + h$ are the roots of $x^2 + qx + r = 0$, then find h.

Solution:

Given, α and β are the roots of $f(x) = x^2 + bx + c = 0$.

The quadratic equation whose roots are $\alpha + h$ and $\beta + h$ is $f(x-h) = 0$; i.e., $(x-h)^2 + b(x-h) + c = 0$

i.e., $x^2 + (b-2h)x + (h^2 - bh + c) = 0$ --- (2)

given that the quadratic equation whose roots are $\alpha + h$ and $\beta + h$ is

$x^2 + qx + r = 0$ --- (3)

Comparing (2) and (3), we get $b - 2h = q$ i.e., $h = \frac{b-q}{2}$

Example: 6

If α and β be the roots of the equation $ax^2 + bx + c = 0$ then find the quadratic equation whose roots are $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$.

Solution:

Given α and β are the roots of the equation

$$f(x) = ax^2 + bx + c = 0$$

The equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $g(x) = 0$ where $g(x) = f\left(\frac{1}{x}\right)$.

The equation whose roots are $\frac{1}{\alpha} - 1$ and $\frac{1}{\beta} - 1$ is $h(x) = 0$ where $h(x) = g(x+1)$.

The quadratic equation whose roots are $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$ is

$h(x) = g(x+1) = f\left(\frac{1}{x+1}\right) = 0$

i.e., $\frac{a}{(x+1)^2} + \frac{b}{x+1} + c = 0$

$\Rightarrow c(x+1)^2 + b(x+1) + a = 0$

$$\Rightarrow cx^2 + (2c + b)x + (a + b + c) = 0$$

PROBLEM SET

IP1: Find the quadratic equation whose roots are $\frac{m}{n}$ and $-\frac{n}{m}$.

Solution:

Let $\alpha = \frac{m}{n}$ and $\beta = -\frac{n}{m}$

The required quadratic equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

i.e., $x^2 - \left(\frac{m}{n} - \frac{n}{m}\right)x + \left(\frac{m}{n}\right)\left(-\frac{n}{m}\right) = 0$

i.e., $x^2 - \left(\frac{m^2-n^2}{mn}\right)x - 1 = 0$

i.e., $mnx^2 - (m^2 - n^2)x - mn = 0$

P1: Find the quadratic equation whose roots are $-\frac{4}{5}$ and $\frac{3}{7}$.

Solution:

Let $\alpha = -\frac{4}{5}$ and $\beta = \frac{3}{7}$

The required quadratic equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

i.e., $x^2 - \left(-\frac{4}{5} + \frac{3}{7}\right)x + \left(-\frac{4}{5}\right)\left(\frac{3}{7}\right) = 0$

i.e., $x^2 + \frac{13}{35}x - \frac{12}{35} = 0$

i.e., $35x^2 + 13x - 12 = 0$

IP2: If α and β are the roots of $x^2 + px + q = 0$, form the equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

Solution: Given α, β are the roots of the equation

$$x^2 + px + q = 0$$

Therefore, $\alpha + \beta = -p$, $\alpha\beta = q$

The quadratic equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$ is

$x^2 - [(\alpha - \beta)^2 + (\alpha + \beta)^2]x + (\alpha - \beta)^2 \cdot (\alpha + \beta)^2 = 0$

$\Rightarrow x^2 - 2(\alpha^2 + \beta^2)x + [(\alpha + \beta)^2 - 4\alpha\beta](\alpha + \beta)^2 = 0$

$\Rightarrow x^2 - 2[(\alpha + \beta)^2 - 2\alpha\beta]x + [(\alpha + \beta)^2 - 4\alpha\beta](\alpha + \beta)^2 = 0$

$\Rightarrow x^2 - 2[(-p)^2 - 2q]x + [(-p)^2 - 4q](-p)^2 = 0$

$\Rightarrow x^2 - 2(p^2 - 2q)x + p^4 - 4p^2q = 0$

P2: If α and β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$.

Solution: Given α, β are the roots of the equation

$$ax^2+bx+c=0$$

Therefore, $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$

The quadratic equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$ is

$$x^2-[(\alpha^2+\beta^2)+(\alpha^{-2}+\beta^{-2})]x+(\alpha^2+\beta^2)(\alpha^{-2}+\beta^{-2})=0$$

$$\Rightarrow x^2-(\alpha^2+\beta^2)\left[\frac{(\alpha\beta)^2+1}{(\alpha\beta)^2}\right]x+\left[\frac{(\alpha^2+\beta^2)^2}{(\alpha\beta)^2}\right]=0$$

$$\Rightarrow x^2-\{(\alpha+\beta)^2-2\alpha\beta\}\left[\frac{(\alpha\beta)^2+1}{(\alpha\beta)^2}\right]x+\left[\frac{\{(\alpha+\beta)^2-2\alpha\beta\}^2}{(\alpha\beta)^2}\right]=0$$

$$\Rightarrow x^2-2\left[\left(-\frac{b}{a}\right)^2-\frac{2c}{a}\right]\left[\frac{\left(\frac{c}{a}\right)^2+1}{\left(\frac{c}{a}\right)^2}\right]x+\left[\frac{\left\{\left(-\frac{b}{a}\right)^2-\frac{2c}{a}\right\}^2}{\left(\frac{c}{a}\right)^2}\right]=0$$

$$\Rightarrow x^2-2\frac{(b^2-2ac)(c^2+a^2)}{a^2c^2}x+\frac{(b^2-2ac)^2}{a^2c^2}=0$$

$$\Rightarrow a^2c^2x^2-2(b^2-2ac)(c^2+a^2)x+(b^2-2ac)^2=0$$

IP3: Find the value of x^3+x^2-x+22 when $x=1+2i$.

Solution:

We first form the quadratic equation whose roots $1\pm 2i$.

The required equation is

$$x^2-[(1+2i)+(1-2i)]x+(1+2i)(1-2i)=0$$

$$\Rightarrow x^2-2x+5=0$$

Now,

$$x^3+x^2-x+22=x(x^2-2x+5)+3x^2-6x+22$$

$$=x(x^2-2x+5)+3(x^2-2x+5)+7$$

$$=x(0)+3(0)+7=7$$

P3: Find the value of $x^3-3x^2-8x+15$ when $x=3+i$.

Solution:

We first form the quadratic equation whose roots $3\pm i$.

The required equation is

$$x^2-[(3+i)+(3-i)]x+(3+i)(3-i)=0$$

$$\Rightarrow x^2-6x+10=0$$

Now,

$$x^3-3x^2-8x+15=x(x^2-6x+10)+3x^2-18x+15$$

$$= x(x^2 - 6x + 10) + 3(x^2 - 6x + 10) - 15$$
$$= x(0) + 3(0) - 15 = -15$$

IP4: If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the roots of $\boldsymbol{9x^2 + 6x + 1 = 0}$, then find the equation with the roots $\boldsymbol{\frac{1}{\alpha}, \frac{1}{\beta}}$.

Solution: Given α and β are the roots of the equation

$$f(x) = 9x^2 + 6x + 1 = 0$$

The equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $g(x) = 0$ where $g(x) = f\left(\frac{1}{x}\right)$.

i.e., $9\left(\frac{1}{x}\right)^2 + 6\left(\frac{1}{x}\right) + 1 = 0$

i.e., $x^2 + 6x + 9 = 0$

P4: If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the roots of $\boldsymbol{ax^2 + bx + c = 0}$, then find the equation with the roots $\boldsymbol{2 + \alpha, 2 + \beta}$.

Solution:

Given α and β are the roots of the equation

$$f(x) = ax^2 + bx + c = 0$$

The equation whose roots are $2 + \alpha$ and $2 + \beta$ is $g(x) = 0$ where $g(x) = f(x - 2)$.

i.e., $a(x - 2)^2 + b(x - 2) + c = 0$

i.e., $a(x^2 - 4x + 4) + bx - 2b + c = 0$

i.e., $ax^2 + (b - 4a)x + 4a - 2b + c = 0$

Exercises:

1. Form the equations whose roots are
   a. $\frac{p-q}{p+q}, -\frac{p+q}{p-q}$
   b. $\pm 2\sqrt{3} - 5$
   c. $-p \pm 2\sqrt{2q}$
   d. $-3 \pm 5i$
   e. $-a \pm ib$
   f. $\pm i(a - b)$
2. If α and β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are $\alpha + \beta$ and $\alpha\beta$.
3. If α and β are the roots of $2x^2 + 3x - 4 = 0$, then find the equation having roots $2\alpha + \frac{3}{\beta}$ and $2\beta + \frac{3}{\alpha}$.
4. If α and β are the roots of $3x^2 + 6x + 2 = 0$, then find the equation having roots $-\frac{\alpha^2}{\beta}$ and $-\frac{\beta^2}{\alpha}$.

5. If α and β are the roots of $x^2 - 2x + 3 = 0$, then find the equation whose roots are $\frac{\alpha-1}{\alpha+1}$ and $\frac{\beta-1}{\beta+1}$.
6. If α and β are the roots of $x^2 - px + q = 0$, then find the equation whose roots are $\alpha\beta + \alpha + \beta$ and $\alpha\beta - \alpha - \beta$.
7. If α and β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $p\alpha$ and $p\beta$ (p is a real number).
8. Form the equation whose roots are the squares of the sum and of the difference of the roots of
$$2x^2 + 2(m+n)x + m^2 + n^2 = 0$$
9. The equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$. Show that $b = -c$.

2.9. Quadratic Expressions

Learning Objectives:

- To discuss the sign of a quadratic expression and to study the change in signs
- To find extreme values of quadratic expressions
  AND
- To practice the related problems

A polynomial of the form $ax^2 + bx + c$, where a, b, c are real or complex numbers and $a \neq 0$, is called a ***quadratic expression in x.***

Throughout this module we consider quadratic expressions with real coefficients. In this module we discuss the sign of a quadratic expression, its change in signs and maximum and minimum values.

Sign of a quadratic expression:

Theorem 1: Let $a, b, c \in \boldsymbol{R}$, $a \neq 0$, then

(i) The roots of $ax^2 + bx + c = 0$ are non-real complex numbers if and only if the quadratic expression $ax^2 + bx + c$ and a have the same sign for all $x \in \boldsymbol{R}$.

(ii) If $ax^2 + bx + c = 0$ has equal roots then the quadratic expression $ax^2 + bx + c$ and a have the same sign for all $x \in \boldsymbol{R}$, $x \neq -\frac{b}{2a}$.

Proof:

We have $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

Thus, $\frac{ax^2+bx+c}{a} = \left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$ --- (1)

(i) If the roots of $ax^2+bx+c=0$ are non-real complex numbers, then $b^2-4ac<0$, i.e., $4ac-b^2>0$ and from (1) $\frac{ax^2+bx+c}{a}>0$ for all $x \epsilon \boldsymbol{R}$
$\Rightarrow ax^2+bx+c$ and a have the same sign for all $x \epsilon \boldsymbol{R}$.
Conversely, suppose that ax^2+bx+c and a have the same sign for all $x \epsilon \boldsymbol{R}$.
$\Rightarrow \frac{ax^2+bx+c}{a}>0 \Rightarrow \left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}>0, \forall x \epsilon \boldsymbol{R}$
Taking $x=-\frac{b}{2a}$, we obtain
$$\frac{4ac-b^2}{4a^2}>0 \Rightarrow 4ac-b^2>0 \Rightarrow b^2-4ac<0$$
Hence the roots of $ax^2+bx+c=0$ are non-real complex numbers.
This proves the first part of the theorem.

(ii) If the equation $ax^2+bx+c=0$ has equal roots, then $b^2-4ac=0$ and from (1)
$$\frac{ax^2+bx+c}{a} = \left(x+\frac{b}{2a}\right)^2 > 0 \text{ for all } x \epsilon \boldsymbol{R}, x \neq -\frac{b}{2a}$$
Thus, $ax^2+bx+c=0$ and a have the same sign for all $x \epsilon \boldsymbol{R}, x \neq -\frac{b}{2a}$.
This proves the second part of the theorem.

Example1: Determine the sign of the quadratic expression x^2-x+2 for $x \in \boldsymbol{R}$.

Solution: The discriminant $=(-1)^2-4\cdot 1\cdot 2=-7<0$.

Therefore, the roots of the quadratic equation $x^2-x+2=0$ are non-real complex numbers.

Therefore, x^2-x+2 and the coefficient of x^2 have the same sign for all $x \in \boldsymbol{R}$. Since the coefficient of x^2 is $1>0$, $x^2-x+2>0$, $\forall x \in \boldsymbol{R}$.

Change in signs of a quadratic expression:

Theorem 2: Let α and β be real roots of $ax^2+bx+c=0$ and $\alpha<\beta$. Then

(i) If $x \epsilon (\alpha,\beta)$, then ax^2+bx+c and a have opposite signs.
(ii) If $x \epsilon (-\infty,\alpha) \cup (\beta,\infty)$, then ax^2+bx+c and a have the same sign.

Proof:

Since α,β are the roots of $ax^2+bx+c=0$;
$$ax^2+bx+c = a(x-\alpha)(x-\beta)$$
Therefore, $\frac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta)$

(i) Suppose $x \epsilon (\alpha,\beta)$. Then $x-\alpha>0, x-\beta<0$ and
$$\frac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta)<0$$
Thus ax^2+bx+c and a have opposite signs.

(ii) Suppose $x \epsilon (-\infty,\alpha)\cup(\beta,\infty)$. Then $x \epsilon (-\infty,\alpha)$ or $x \epsilon (\beta,\infty)$.
a) If $x \epsilon (-\infty,\alpha)$, then $x<\alpha<\beta$ and

$x - \alpha < 0, x - \beta < 0.$

Therefore, $\frac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta) > 0$

Thus $ax^2 + bx + c$ and a have the same sign.

b) If $x \in (\beta, \infty)$, then $\alpha < \beta < x$ and

$x - \alpha > 0, x - \beta > 0.$

Therefore, $\frac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta) > 0$

Thus $ax^2 + bx + c$ and a have the same sign.

Combining the above two cases, the second part of the theorem follows.

Hence the theorem.

Example2: Discuss the sign of the quadratic expression $4x - 5x^2 + 2$ where $x \in \boldsymbol{R}$.

Solution: We have, $-5x^2 + 4x + 2$.

It's discriminant is $b^2 - 4ac = 16 - 4 \cdot (-5) \cdot 2 = 56 > 0.$

The roots are real and they are $\alpha = \frac{2-\sqrt{14}}{5}, \beta = \frac{2+\sqrt{14}}{5}$.

Now $-5x^2 + 4x + 2$ and -5, the coefficient of x^2 have opposite signs if $x \in (\alpha, \beta)$ and have the same sign if $x \in (-\infty, \alpha) \cup (\beta, \infty)$.

Thus $4x - 5x^2 + 2 > 0$ if $x \in (\alpha, \beta)$

and $4x - 5x^2 + 2 < 0$ if $x \in (-\infty, \alpha) \cup (\beta, \infty)$.

Extreme values of a quadratic expression:

We show that the extreme values of a quadratic expression with real coefficients depend on the sign of its leading coefficient.

Theorem 3:

Let $a, b, c \in \boldsymbol{R}, a \neq 0$ and $f(x) = ax^2 + bx + c$.

(i) If $a > 0$, then $f(x)$ has absolute minimum at $x = -\frac{b}{2a}$ and the minimum value is $\frac{4ac-b^2}{4a}$;

(ii) If $a < 0$, then $f(x)$ has absolute maximum at $x = -\frac{b}{2a}$ and the maximum value is $\frac{4ac-b^2}{4a}$.

Proof: We have $f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$ --- (1)

(i) If $a > 0$, then $f(x) \geq \frac{4ac-b^2}{4a}$, $\forall x \in \boldsymbol{R}$ and

$f(x) = \frac{4ac-b^2}{4a}$ when $x = -\frac{b}{2a}$.

This shows that $f(x)$ has absolute minimum at $x = -\frac{b}{2a}$ when $a > 0$ and its minimum value is $\frac{4ac-b^2}{4a}$.

(ii) If $a < 0$, then $f(x) \leq \frac{4ac-b^2}{4a}$, $\forall x \in \boldsymbol{R}$ and

$f(x) = \frac{4ac-b^2}{4a}$ when $x = -\frac{b}{2a}$.

This shows that $f(x)$ has absolute maximum at $x = -\frac{b}{2a}$ when $a < 0$ and its maximum value is $\frac{4ac-b^2}{4a}$.

Hence the theorem.

Example3: Find the maximum and minimum values of the quadratic expressions

(a) $12x - x^2 - 32$
(b) $ax^2 + bx + a\,;\, a, b \in \boldsymbol{R}, a \neq 0.$

Solution:

(a) Let $f(x) = 12x - x^2 - 32$. The coefficient of x^2 negative. Therefore, $f(x)$ has absolute maximum at $x = -\frac{b}{2a} = -\frac{12}{2(-1)} = 6$ and the maximum value is $(6) = \frac{4ac-b^2}{4a} = \frac{4(-1)(-32)-12^2}{4(-1)} = 4$.

(b) Let $f(x) = ax^2 + bx + a\,;\; a, b \in \boldsymbol{R}, a \neq 0.$

If $a > 0$, then $f(x)$ has absolute minimum at $x = -\frac{b}{2a}$ and the minimum value is $\frac{4ac-b^2}{4a} = \frac{4a^2-b^2}{4a}$.

If $a < 0$, then $f(x)$ has absolute maximum at $x = -\frac{b}{2a}$ and the maximum value is $\frac{4ac-b^2}{4a} = \frac{4a^2-b^2}{4a}$.

PROBLEM SET

IP1: Determine the sign of the quadratic expression $\boldsymbol{x^2 + x + 1}$.

Solution: We have, $x^2 + x + 1$.

The discriminant $= (1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0$.

The roots of the quadratic equation $x^2 + x + 1 = 0$ are non-real complex numbers.

Therefore, by the first part of the Theorem 1, $x^2 + x + 1$ and the coeff.of x^2 have the same sign for all $x \in \boldsymbol{R}$. Since the coefficient of x^2 is $1 > 0$, $x^2 + x + 1 > 0$, $\forall\, x \in \boldsymbol{R}$.

P1: Determine the sign of the quadratic expression $\boldsymbol{x^2 - 8x + 16}$.

Solution: We have, $x^2 - 8x + 16$.

The discriminant $= (-8)^2 - 4 \cdot 1 \cdot 16 = 0$.

The roots of the quadratic equation $x^2 - 8x + 16 = 0$ are real and equal.

Therefore, by the second part of the Theorem 1, $x^2 - 8x + 16$ and the coeff.of x^2 have the same sign for all $x \in \boldsymbol{R}, x \neq -\frac{b}{2a} \neq 4$. Since the coefficient of x^2 is $1 > 0$, $x^2 - 8x + 16 > 0$, $\forall\, x \in \boldsymbol{R}, x \neq 4$.

IP2: For what values of $\boldsymbol{x}$, the expression $\boldsymbol{x^2 - 5x + 6}$ is positive.

Solution: We have, $x^2 - 5x + 6$.

It's discriminant is $b^2 - 4ac = 25 - 4 \cdot 1 \cdot 6 = 1 > 0$.

The roots are real and they are $\alpha = 2, \beta = 3$.

Now, by Theorem 2, $x^2 - 5x + 6$ and the coefficient of x^2 have opposite signs if $x \epsilon (\alpha, \beta) = (2,3)$ and have the same sign if $x \epsilon (-\infty, \alpha) \cup (\beta, \infty) = (-\infty, 2) \cup (3, \infty)$.

Since the coefficient of x^2 is $1 > 0$, $x^2 - 5x + 6 < 0$ if $x \epsilon (2,3)$ and $x^2 - 5x + 6 > 0$ if $x \epsilon (-\infty, 2) \cup (3, \infty)$.

Therefore, the expression $x^2 - 5x + 6$ is positive for $x \epsilon (-\infty, 2) \cup (3, \infty)$.

P2: For what values of x, the expression $2x^2 + 5x - 3$ is negative.

Solution: We have, $2x^2 + 5x - 3$.

It's discriminant is $b^2 - 4ac = 25 - 4 \cdot 2 \cdot (-3) = 1 > 0$.

The roots are real and they are $\alpha = -3, \beta = \frac{1}{2}$.

Now, by Theorem 2, $2x^2 + 5x - 3$ and the coefficient of x^2 have opposite signs if $x \epsilon (\alpha, \beta) = \left(-3, \frac{1}{2}\right)$ and have the same sign if $x \epsilon (-\infty, \alpha) \cup (\beta, \infty) = (-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$.

Since the coefficient of x^2 is $2 > 0$, $2x^2 + 5x - 3 < 0$ if $x \epsilon \left(-3, \frac{1}{2}\right)$ and

$2x^2 + 5x - 3 > 0$ if $x \epsilon (-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$.

Therefore, the expression $2x^2 + 5x - 3$ is negative for $x \epsilon \left(-3, \frac{1}{2}\right)$.

IP3: Find the maximum or minimum of the expression $x^2 - 8x + 17$

Solution: We have, $x^2 - 8x + 17$. Comparing this expression with $ax^2 + bx + c$, we have $a = 1, b = -8, c = 17$. Since $a = 1 > 0$, (by the first part of Theorem 3), $x^2 - 8x + 17$ has absolute minimum at

$$x = -\frac{b}{2a} = -\frac{(-8)}{2 \cdot 1} = 4$$

and the minimum value is $\frac{4ac - b^2}{4a} = \frac{4(1)(17) - (-8)^2}{4(1)} = 1$.

Therefore, the given expression has the minimum value 1 at $x = 4$.

P3: Find the maximum or minimum of the expression $x^2 + 5x + 6$.

Solution: We have, $x^2 + 5x + 6$. Comparing this expression with $ax^2 + bx + c$, we have $a = 1, b = 5, c = 6$. Since $a = 1 > 0$, (by the first part of Theorem 3), $x^2 + 5x + 6$ has absolute minimum at

$$x = -\frac{b}{2a} = -\frac{5}{2 \cdot 1} = -\frac{5}{2}$$

and the minimum value is $\frac{4ac - b^2}{4a} = \frac{4(1)(6) - (5)^2}{4(1)} = -1$.

Therefore, the given expression has the minimum value -1 at $x = -\frac{5}{2}$.

IP4: Find the maximum or minimum of the expression $2x - x^2 + 7$.

Solution: We have, $-x^2 + 2x + 7$. Comparing this expression with $ax^2 + bx + c$, we have $a = -1, b = 2, c = 7$. Since $a = -1 < 0$, (by the second part of Theorem 3), $-x^2 + 2x + 7$ has absolute maximum at

$$x = -\frac{b}{2a} = -\frac{2}{2(-1)} = 1$$

and the maximum value is $\frac{4ac-b^2}{4a} = \frac{4(-1)(7)-(2)^2}{4(-1)} = 7$.

Therefore, the given expression has the maximum value 7 at $x = 1$.

P4: Find the maximum or minimum of the expression $2x - 7 - 5x^2$

Solution: We have, $-5x^2 + 2x - 7$. Comparing this expression with $ax^2 + bx + c$, we have $a = -5, b = 2, c = -7$. Since $a = -5 < 0$, (by the second part of Theorem 3), $-5x^2 + 2x - 7$ has absolute maximum at

$$x = -\frac{b}{2a} = -\frac{2}{2(-5)} = \frac{1}{5}$$

and the maximum value is $\frac{4ac-b^2}{4a} = \frac{4(-5)(-7)-(2)^2}{4(-5)} = -\frac{34}{5}$.

Therefore, the given expression has the maximum value $-\frac{34}{5}$ at $x = \frac{1}{5}$.

Exercises:

1. Determine the sign of the following expressions for $x \in \boldsymbol{R}$.
   a. $x^2 - 5x + 6$
   b. $x^2 - 5x + 4$
   c. $x^2 - x + 3$
2. For what values of x, the following expressions are positive?
   a. $3x^2 + 4x + 4$
   b. $4x - 5x^2 + 2$
   c. $4x - 5x^2 + 1$
   d. $x^2 - 5x + 14$
3. For what values of x, the following expressions are negative?
   a. $x^2 - 7x + 10$
   b. $15 + 4x - 3x^2$
   c. $x^2 - 5x - 6$
   d. $2x - 3 - 6x^2$
   e. $-7x^2 + 8x - 9$
4. Find the maximum or minimum of the following expressions as x varies over $\boldsymbol{R}$.
   a. $2x - 7 - 5x^2$
   b. $3x^2 + 2x + 11$
   c. $x^2 - x + 7$
   d. $2x + 5 - 3x^2$

3.1. Rates of Change

Average speed:

First, we consider a familiar concept.

The ***average speed*** of a moving body over a time interval is the distance covered during the time interval divided by the length of the interval.

Example 1:

A rock falls from the top of a 50 m cliff.

Physical experiments show that a solid object dropped from the rest to fall freely near the surface of the earth will fall $y = 5t^2\ m$

during the first $t\ sec.$

i) Find the average speed:

a) During the first 2 sec of fall.

b) During the 1 sec interval between second 1 and second 2.

ii) Find the speed of the rock at $t = 1$ and $t = 2$ sec.

Solution:

The average speed of the rock during a given time interval is the change in distance Δy, divided by the length of the time interval Δt.

i) The average speed

a) For the first 2 sec: $\dfrac{\Delta y}{\Delta t} = \dfrac{5\times 2^2 - 5\times 0^2}{2-0} = 10\,m/sec$

b) From second 1 to second 2:

$$\frac{\Delta y}{\Delta t} = \frac{5\times 2^2 - 5\times 1^2}{2-1} = 15\,m/sec$$

ii) We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$ having length $\Delta t = h$ as

$$\frac{\Delta y}{\Delta t} = \frac{5(t_0+h)^2 - 5t_0^2}{h}$$

We cannot use this formula to calculate the "*instantaneous*" speed at t_0 by substituting $h = 0$, because we cannot divide by 0. But we can use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
0.1	10.5	20.5
0.01	10.05	20.05
0.001	10.005	20.005
0.0001	10.0005	20.0005
.	.	.
.	.	.
.	.	.
0	10	20

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 10 as the length of the interval decreases. This suggests that the rock is falling at a speed of 10 m/sec at $t_0 = 1$ sec. Similarly, the rock's speed at $t_0 = 2$ sec would appear to be $20\ m/sec$.

Average rate of change of a function:

Now, we introduce the concept of the ***average rate of change of a function***. Given a function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in value of y,

$\Delta y = f(x_2) - f(x_1)$, by the length of the interval $\Delta x = x_2 - x_1 = h$ over which the change occurred.

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} \quad , h \neq 0$$

As seen from the figure below, the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$

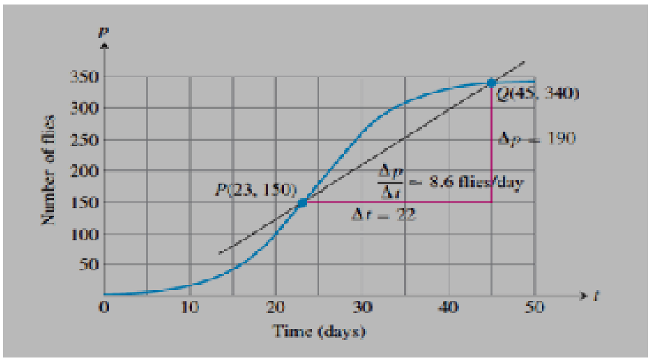
In geometry, *a line joining two points of a curve is called a **secant** to the curve*. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of the secant PQ.

Example 2:

The figure below shows the growth of a population of flies in a 50 –day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve.

(a) Find the average growth rate from day 23 to day 45.

(b) How fast was the number of flies growing on day 23 itself?



Solution

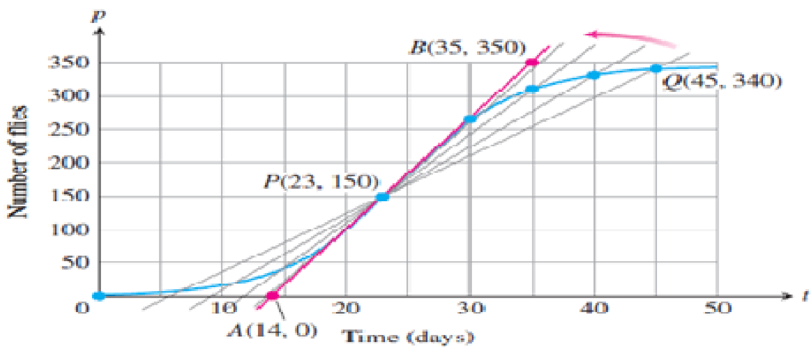
(a) There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by 340 – 150 = 190 in 45 – 23 = 22 days. The average rate of change of the population p from day 23 to day 45 was

Average rate of change: $$\frac{\Delta p}{\Delta t} = \frac{340-150}{45-23} = \frac{190}{22} \approx 8.6 \textbf{ flies / day}$$

This average is the slope of the secant through the points P and Q on the graph.

(b) The average rate of change from day 23 to day 45, $8.6\ flies/day$, does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

We examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by *calculating the slopes of secants from P to Q for a sequence of points Q approaching P along the curve.*



The table below gives the positions of Q and slopes of four secants through the point P on the graph.

Q	Slope of PQ = Δp/Δt (flies/day)
(45,340)	$\frac{340-150}{45-23} \approx 8.6$
(40,330)	$\frac{330-150}{40-23} \approx 10.6$
(35,310)	$\frac{310-150}{35-23} \approx 13.3$
(30,265)	$\frac{265-150}{30-23} \approx 16.4$

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t −coordinate of Q decreases from 45 to 30, and we would expect the slopes to continue rise higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the line PA in the figure, a line that goes through P in the same direction that the curve goes through P. This line is called the ***tangent*** to the curve at P. Since this line passes through the points $(14{,}0)$ and $(35{,}350)$, it has slope

$$\frac{350-0}{35-14} = 16.7 \ flies/day$$

On day 23 the population was increasing at a rate of about $16.7\ flies/day$.

The rate at which the rock in Example 1 was falling at instants $t = 1$ and $t = 2$ and the rate at which the population in Example 2 was changing on day $t = 23$ are called ***instantaneous rates of change***. As the examples suggest, we find instantaneous rates as limiting values of average rates. In Example 2, we also pictured the tangent line to the population curve on day 23 as a limiting position of secant lines.

PROBLEM SET

IP1: Find the average rate of change of the function over the given interval

$$\boldsymbol{R = \sqrt{4\theta + 1};\ [0,\ 2]}$$

Solution:

Step 1: Given, $R(\theta) = \sqrt{4\theta+1}; \theta \in [0,\ 2]$

Step 2: Now $R(2) = 3, R(0) = 1$

Step 3: We have

Average rate of change $= \frac{R(2)-R(0)}{2-0} = \frac{3-1}{2-0} = \frac{2}{2} = 1$

P1: Find the average rate of change of the function over the given interval $f(x) = x^3 + 1; [-1,\ 1]$

Solution: Given, $f(x) = x^3 + 1, x \in [-1,\ 1]$

Now $f(1) = 2, f(-1) = 0$

Average rate of change over $[-1,\ 1]$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(-1)}{1-(-1)} = \frac{2-0}{2} = 1$$

IP2: Find the average rate of change of the function over the given interval

$$f(t) = \tan t\,;\ t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Solution:

Step1: Given, $f(t) = \tan t, t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Step2: $f\left(\frac{\pi}{4}\right) = tan\frac{\pi}{4} = 1,$

$f\left(\frac{3\pi}{4}\right) = tan\frac{3\pi}{4} = -1$

Step3: Average rate of change

$$= \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$$

P2: Find the average rate of change of the function over the given interval

$$f(t) = cot\,t\,;\ t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Solution: Given, $f(t) = \cot t,\ t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$$f\left(\frac{\pi}{4}\right) = cot\frac{\pi}{4} = 1, \quad f\left(\frac{3\pi}{4}\right) = cot\frac{3\pi}{4} = -1$$

Average rate of change

$$= \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$$

IP3: The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in Rs. 1000
1990	6
1991	27
1992	62
1993	111
1994	174

What is the average rate of increase of the profits between 1990 and 1993?

Solution:

Step1: Profit in 1993 $= 111$

Profit in 1990 $= 6$

Step2: Average rate of increase of the profits $=\frac{111-6}{2} = \frac{105}{2} = 55.5$

Step3: The average rate of increase of the profits from 1993 to 1990 is $Rs. 55500$.

P3: The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in Rs. 1000
1990	6
1991	27
1992	62
1993	111
1994	174

What is the average rate of increase of the profit between 1992 and 1994?

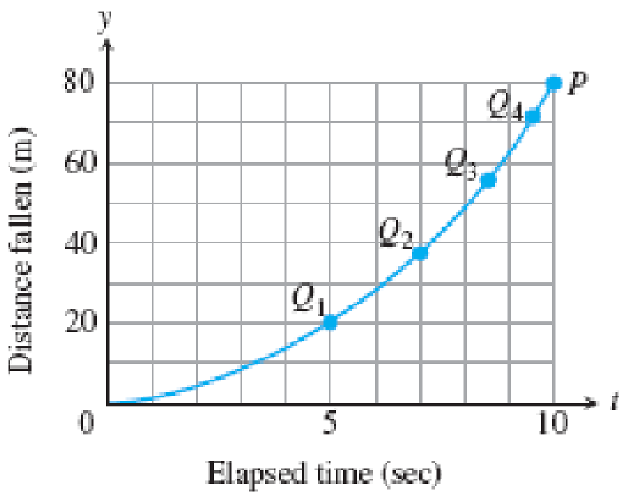
Solution: Profit in 1994 $= 174$

Profit in 1992 $= 62$

Average rate of increase of the profits $= \frac{174-62}{2} = 56$

The average rate of increase of the profits from 1992 to 1994 is $Rs\ 56{,}000$.

IP4: The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80m to the surface of the moon.



Estimate the slope of the secant PQ_2?

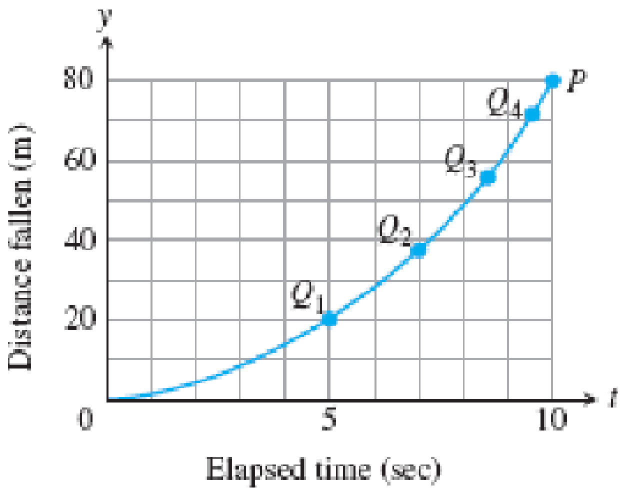
Solution:

Step 1: Given , $P(10{,}80)$, $Q_2(7{,}40)$

Step 2: Slope of the secant $PQ_2 = \frac{y_2-y_1}{x_2-x_1} = \frac{80-40}{10-7} = \frac{40}{3} = 13.33$

P4:

The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80m to the surface of the moon.



Estimate the slope of the secant PQ_1?

Solution:

Given , $P(10,80)$, $Q_1(5,20)$

Slope of the secant $PQ_1 = \frac{y_2-y_1}{x_2-x_1} = \frac{80-20}{10-5} = \frac{60}{5} = 12$

Exercise:

1. Find the average rates of change of the function over the given intervals
   i. $f(x) = x^3 + 1$; $[-1,1], [-2,0]$
   ii. $g(x) = x^2$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   iii. $h(t) = \cot t$; $[-\pi, \pi], [0, \pi]$

2. Find the average rates of change of the function over the given intervals
   i. $f(x) = \sqrt{x}$; $x \geq 0$, $[1,2], [1,1.5], [1,1+h]$
   ii. $f(t) = \frac{1}{t}$; $t \neq 0$, $[2,3], [2,5]$

3.2. Concept of Limit

Learning objectives:

- To understand the concept of the limit of a function through examples and to give an informal definition of the limit of a function

  AND
- To practice related problems.

The concept of limit of a function is one of the fundamental ideas that distinguish calculus from algebra and trigonometry. First, we develop the limit intuitively and then formally.

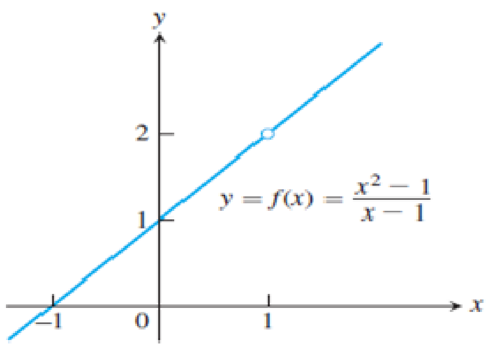
First we look at an example:

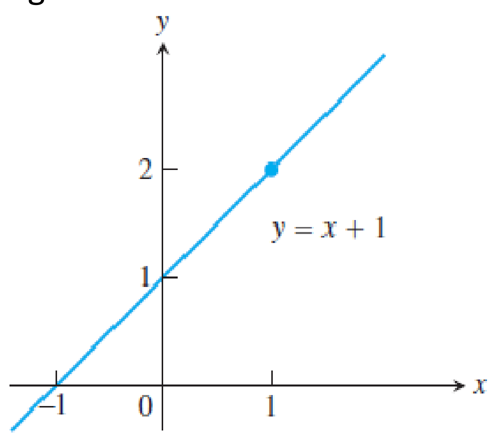
How does the function $f(x)=\dfrac{x^2-1}{x-1}$ behave near $x=1$?

The given function f is defined for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$ we can simplify the function by factoring the numerator and cancelling the common factors:

$$f(x)=\frac{(x-1)(x+1)}{x-1}=x+1 \quad for\ x\neq 1$$

The graph of f is thus the line $y = x + 1$ with one point $(1,2)$ removed. This removed point is shown as a “hole” in the figure.





The graph of f is identical with the line $y = x + 1$ except at $x = 1$ where f is not defined.

Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1.(see the following table)

Values of x Below and above 1	$f(x)=\dfrac{x^2-1}{x-1}=x+1, \quad x\neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We notice that the closer x gets to1, the closer $f(x)$ seems to get 2.

We say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1, or, more simply, ***f(x) approaches the limit* 2 *as x approaches* 1**. We write this as

$$\lim_{x\to 1} f(x)=2 \quad or \quad \lim_{x\to 1}\frac{x^2-1}{x-1}=2$$

Definition

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself*. If there exists a real number L and $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, then we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x\to x_0} f(x)=L$$

The definition says that the value of $f(x)$ are close to L

whenever x close to x_0 on either side of x_0.(either from right or from left of x_0)

This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise. A formal definition will be given later.

The existence of a limit as $x \longrightarrow x_0$ does not depend on how the function may be defined at x_0.

A) The function f in the figure above has limit 2 as $x \longrightarrow 1$ even though f is not defined at $x = 1$.

B) The function g has limit 2 as $x \longrightarrow 1$ even though $2 \neq g(1)$.

C) The function h is the only one whose limit as $x \longrightarrow 1$ equals its value at $x = 1$. For h we have $\lim\limits_{x \to 1} h(x) = h(1)$.

Sometimes, $\lim\limits_{x \to x_0} f(x)$ can be evaluated by calculating $f(x_0)$. This holds, for example, whenever $f(x)$ is an algebraic combination of polynomials and trigonometric functions for which $f(x_0)$ is defined.

Example 1:

a) $\lim\limits_{x \to 2} (4) = 4$

b) $\lim\limits_{x \to -13} (4) = 4$

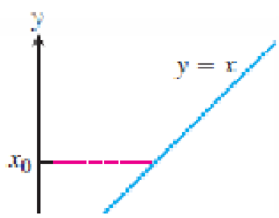
c) $\lim\limits_{x \to 3} (x) = 3$

d) $\lim\limits_{x \to 2} (5x - 3) = 10 - 3 = 7$

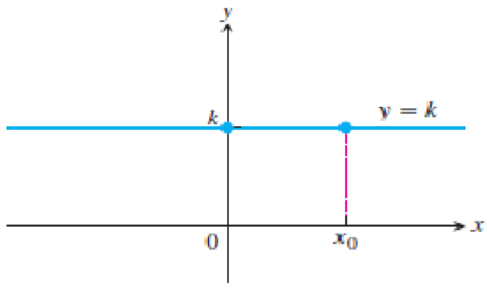
e) $\lim\limits_{x \to -2} \dfrac{3x+4}{x+5} = \dfrac{-6+4}{-2+5} = -\dfrac{2}{3}$

The Identity and Constant Functions Have Limits at Every Point

1. If f is the **identity function** $f(x) = x$, then for any value of x_0,
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

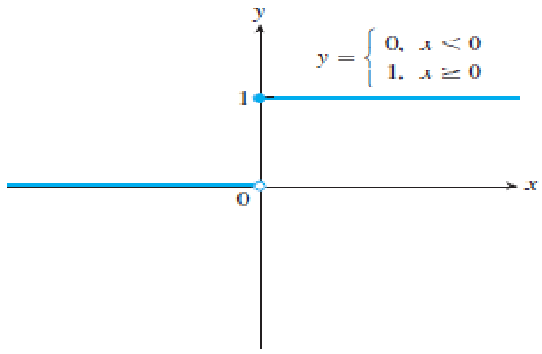


2. If f is the **constant function** $f(x) = k$, then for any value of x_0,
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$$



Constant function

3. The function $u(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$ has the following graph



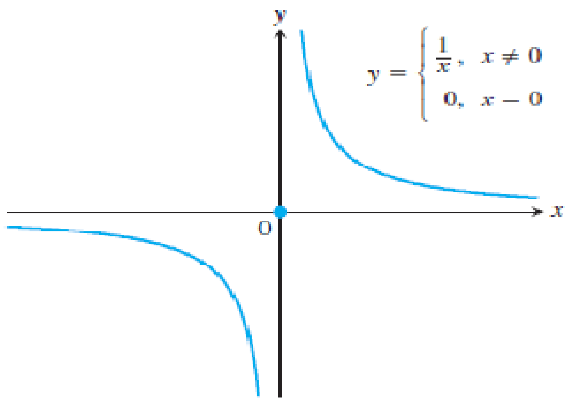
Unit step function $U(x)$

The **unit step function** *u*(*x*) has no limit as $x \to 0$

because its values jump at $x = 0$.

for negative values of *x* arbitrarily close to zero, where $u(x)=0$. For positive values of *x* arbitrarily close to zero, we have $u(x)=1$. There is no *single* value *L* approached by $u(x)$ *as* $x \to 0$.

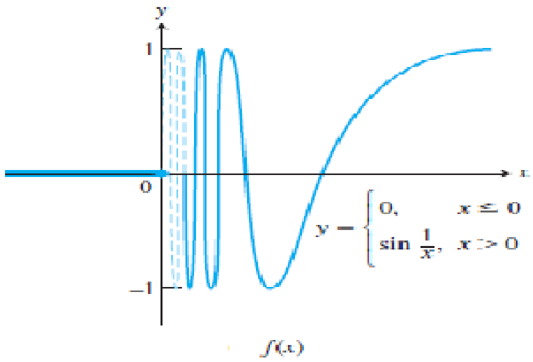
4. The function $g(x)=\begin{cases}\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ has the following graph



g(x)

$g(x)$ has no limit as $x \to 0$ because the values of g grow ***arbitrarily*** large in absolute value as $x \to 0$ and do not stay close to *any* real number.

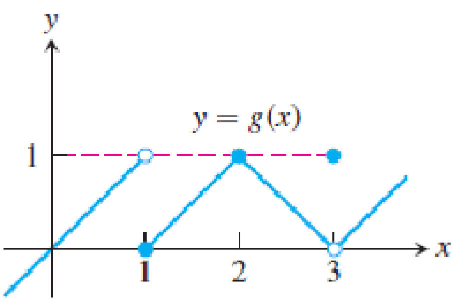
5. The function $f(x)=\begin{cases} 0 & , \quad x \le 0 \\ \sin\dfrac{1}{x} & , \quad x > 0 \end{cases}$ has the following graph



$f(x)$ has no limit as $x \to 0$ because the function's values oscillate too much between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.

PROBLEM SET

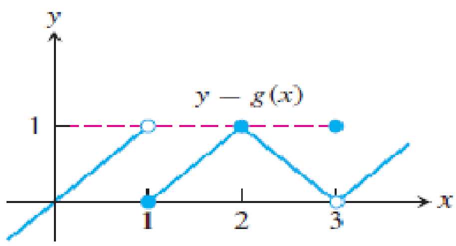
IP1: The function $g(x)$ graphed here



Find limit or explain why limit doesn't exist at $\lim\limits_{x \to 1} g(x)$

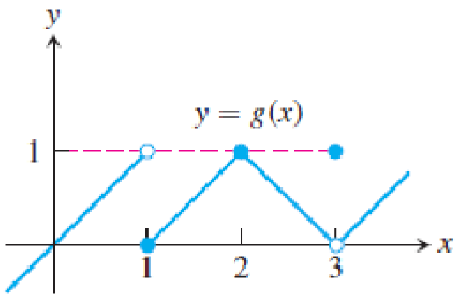
Solution:

$$\lim_{x \to 1} g(x)$$



From the graph it is clear that $g(x)$ approaches 0 when x approaches 1 from the right. Also $g(x)$ approaches 1 when x approaches 1 from the left. Thus there is no single number L such that $g(x)$ get arbitrarily close to L when x is sufficiently close to 1. There fore, $\lim_{x\to 1} g(x)$ doesn't exist.

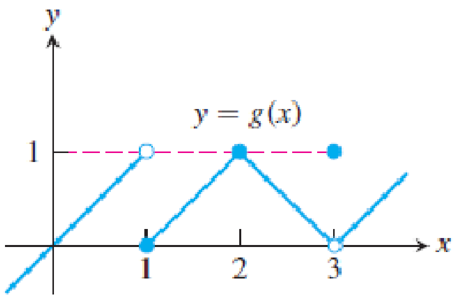
P1: The function $g(x)$ graphed here



Find limit or explain why limit doesn't exist at $\lim_{x\to 1} g(x)$

Solution:

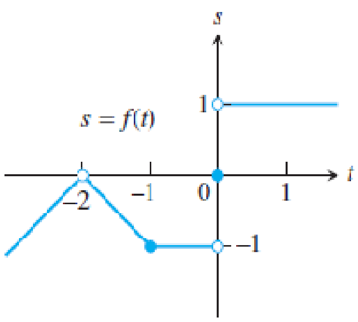
$\lim_{x\to 1} g(x)$



From the graph it is clear that $g(x)$ approaches 1 when x approaches 2 from the right. Also $g(x)$ approaches 1 when x approaches 2 from the left. Thus $g(x)$ approaches 1 when x approaches 2 from either side. Therefore, $\lim_{x\to 2} g(x)$ exists and $\lim_{x\to 2} g(x) = 1$.

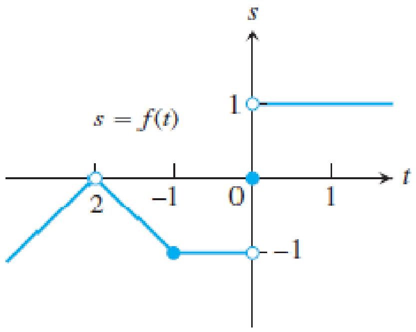
IP2:

For the function $f(t)$ graphed here.



Find limit or explain why limit doesn't exist for the function $\lim_{t\to -2} f(t)$.

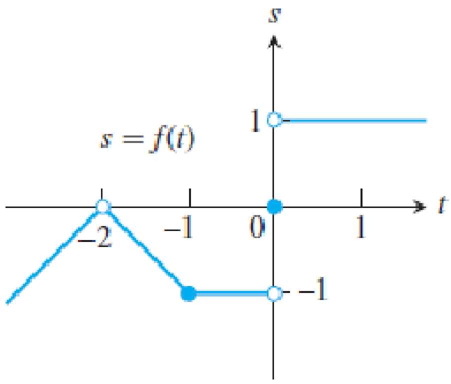
Solution: $\lim_{t\to -2} f(t)$



From the graph it is clear that $f(t)$ approaches 0 when t approaches -2 from the right. Also $f(t)$ approaches 0 when t approaches -2 from the left. Thus $f(t)$ approaches 0 when t approaches -2 from either side. Therefore, $\lim_{t\to -2} f(t)$ exists and $\lim_{t\to -2} f(t) = 0$.

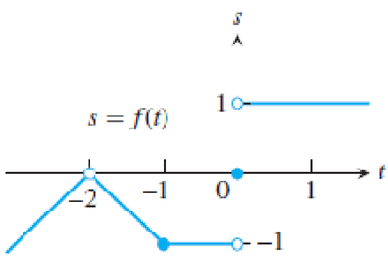
Notice that $f(-2)$ is not define.

P2: For the function $f(t)$ graphed here.



Find limit or explain why limit doesn't exist for the function $\lim_{t\to 0} f(t) =$

Solution: $\lim_{t\to 0} f(t)$



From the graph it is clear that $f(t)$ approaches 1 when t approaches 0 from the right. Also $f(t)$ approaches -1 when t approaches 0 from the left. Thus there is no single number L such that $f(t)$ get arbitrarily close to L when t is sufficiently close to 0. There fore , $\lim_{t\to 0} f(t)$ doesn't exist.

IP3: $\boldsymbol{lim_{x\to -1} \frac{3x^2}{2x-1} =}$

Solution:

Let $f(x) = \frac{3x^2}{2x-1}$. $f(x)$ is an algebraic quotient function and $f(-1)$ is defined. And

$$\lim_{x\to -1}\left(\frac{3x^2}{2x-1}\right) = f(-1) \;= \frac{3(-1)^2}{2(-1)-1} = \frac{3}{2(-1)-1} = \frac{3}{-3} = -1$$

P3: $\boldsymbol{lim_{x\to\ -1} 3x(2x-1) =}$

Solution:

Let $f(x) = 3x(2x-1)$. $f(x)$ is an algebraic polynomial and $f(-1)$ is defined. And

$\lim_{x\to -1} 3x(2x-1) = f(-1) = 3(-1)\{2(-1)-1\} = -3(-3) = 9$

IP4: $\boldsymbol{lim_{x\longrightarrow\pi} \frac{cosx}{1-\pi} =}$

Solution:

Let $f(x) = \frac{cosx}{1-\pi}$. $f(x)$ is a trigonometric function and $f(\pi)$ is defined. And

$$\lim_{x\longrightarrow\pi}\frac{cosx}{1-\pi} = f(\pi) = \frac{cos\pi}{1-\pi} = \frac{-1}{1-\pi} = \frac{1}{\pi-1}$$

P4: $\boldsymbol{\lim_{x\longrightarrow\frac{\pi}{2}} x\; sinx \;=}$

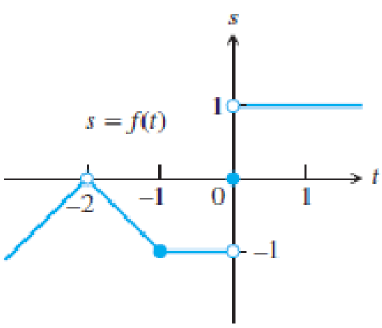
Solution:

Let $f(x) = x\; sinx$. $f(x)$ is a trigonometric function and $f\left(\frac{\pi}{2}\right)$ is defined. And

$$\lim_{x\to\frac{\pi}{2}} x\; sinx = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Exercise:

1) For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

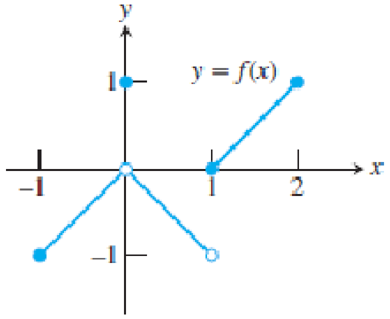


(a) $\lim_{t\to -2} f(t)$

(b) $\lim_{t\to -1} f(t)$

(c) $\lim_{t \to 0} f(t)$

2) Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



(a) $\lim_{x \to 0} f(x)$ exists

(b) $\lim_{x \to 0} f(x) = 0$

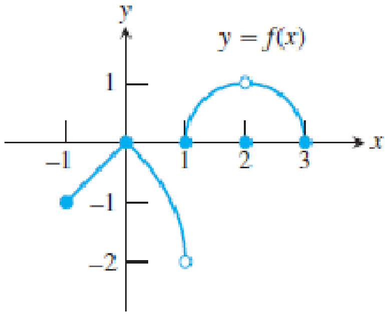
(c) $\lim_{x \to 0} f(x) = 1$

(d) $\lim_{x \to 1} f(x) = 1$

(e) $\lim_{x \to 1} f(x) = 0$

(f) $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(-1,1)$

3) Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



(a) $\lim_{x \to 2} f(x)$ does not exist

(b) $\lim_{x \to 2} f(x) = 2$

(c) $\lim_{x \to 1} f(x)$ does not exist

(d) $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(-1,1)$

(e) $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(1,3)$

4) Find the limits of following functions.

(a) $\lim_{x \to 2} 2x$

(b) $\lim_{x \to 0} 2x$

(c) $\lim_{x \to 1/3} (3x-1)$

(d) $\lim_{x \to 1} \dfrac{-1}{3x-1}$

3.3. Rules for Finding Limits

Learning objectives:

- To state the properties of limits and to apply them to polynomial and rational functions
- To state the Sandwich Theorem

And

- To find the limits of functions by different techniques

Properties of Limits

Here we state, rules to calculate the limits of functions that are the arithmetic combination of functions whose limits are known.

If L, M, *c* and *k* are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then,

1. **Sum Rule:** $\lim_{x \to c} [f(x) + g(x)] = L + M$

i.e., The limit of the sum of two functions is the sum of their limits

2. **Difference Rule:** $\lim_{x \to c} [f(x) - g(x)] = L - M$

i.e., The limit of the difference of two functions is the difference of their limits.

3. **Product Rule:** $\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M$

i.e., The limit of the product of two functions is the product of their limits.

4. **Constant Multiple Rule:** $\lim_{x \to c} kf(x) = kL \qquad (\text{any number } k)$

i.e., The limit of a constant times a function is that constant times the limit of the function.

5. Quotient Rule: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \qquad M \neq 0$$

i.e., The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. Power Rule: If *m* and *n* are integers, then

$$\lim_{x \to c} [f(x)]^{\frac{m}{n}} = L^{\frac{m}{n}}$$, provided $L^{\frac{m}{n}}$ **is a real number.**

i.e., The limit of any rational power of a function is that power of the limit of the function

Example 1: Find $\lim_{x \to c} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$

Solution:

$$\lim_{x \to c} x^3 + 4x^2 - 3 = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$

(Sum and difference rule)

$$= \lim_{x \to c} x^3 + 4\lim_{x \to c} x^2 - 3$$

(Constant multiple rule)

$$= \left(\lim_{x \to c} x\right)^3 + 4\left(\lim_{x \to c} x\right)^2 - 3$$

(Power rule or product rule)

$$= c^3 + 4c^2 - 3 \quad \left(\because \lim_{x \to c} x = c, \lim_{x \to c} k = k\right)$$

$$\lim_{x \to c} x^2 + 5 = \lim_{x \to c} x^2 + \lim_{x \to c} 5$$ (Sum rule)

$$= \left(\lim_{x \to c} x\right)^2 + \lim_{x \to c} 5$$ (Power rule)

$$= c^2 + 5 \quad \left(\because \lim_{x \to c} x = c, \lim_{x \to c} k = k\right)$$

Now $\lim_{x\to c}\dfrac{x^3+4x^2-3}{x^2+5}=\dfrac{\lim_{x\to c} x^3+4x^2-3}{\lim_{x\to c} x^2+5}$ (quotient rule)

$$=\frac{c^3+4c^2-3}{c^2+5}\quad \left(\because c^2+5\neq 0\right)$$

Example 2: Find $\lim_{x\to -2}\sqrt{4x^2-3}$

Solution:

$\lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{\lim_{x\to -2}\left(4x^2-3\right)}$ (Power rule)

$=\sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3}$ (Difference rule)

$=\sqrt{4\lim_{x\to -2}x^2-3}$ (Constant multiple rule)

$=\sqrt{4\left(\lim_{x\to -2}x\right)^2-3}$ (Power rule)

$$=\sqrt{4(-2)^2-3}=\sqrt{13}$$

Limits of Polynomials and Rational Functions

The properties of limits simplify the computation of limits of polynomials and rational functions.

Limits of Polynomials

The limit of a polynomial can be found by substitution. If

$p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$, then

$$\lim_{x\to c}p(x)=a_nc^n+a_{n-1}c^{n-1}+\cdots+a_0=p(c)$$

Limits of Rational Functions

The limit of a rational function can be found by substitution if the limit of the denominator is not zero. If $p(x)$ and $q(x)$ are polynomials and $q(c)\neq 0$, then

$$\lim_{x\to c}\frac{p(x)}{q(x)}=\frac{p(c)}{q(c)}$$

Example 3:

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}=\frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}=\frac{0}{6}=0$$

The above rule applies only when the denominator of the rational function is not zero at *c*. If the numerator and denominator of a rational function of *x* are both zero at *x* = *c*, then (*x* − *c*) is a common factor. Canceling the common factors in the numerator and denominator will reduce the fraction to one whose denominator is no longer zero at *c*. When this happens, we can find the limit by substitution in the simplified fraction.

Example 4: Cancelling a common factor

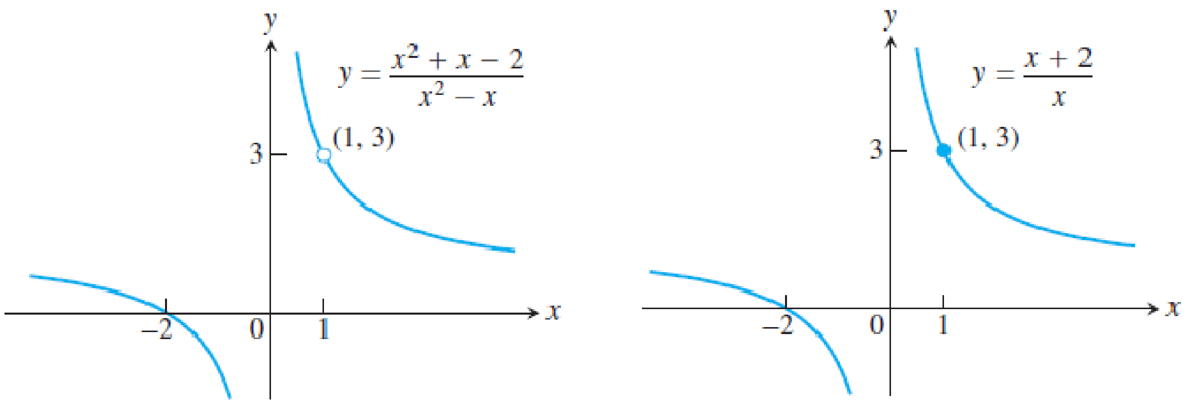
Evaluate $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$

Solution: We cannot just substitute *x* = 1, because it makes the denominator zero. However, we can factor the numerator and denominator and cancel the common factor to obtain

$$\frac{x^2+x-2}{x^2-x}=\frac{(x-1)(x+2)}{x(x-1)}=\frac{x+2}{x},\quad if\ x\neq 1$$

Thus

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3$$



Example 5: **Creating and cancelling a common factor**

Find $\lim\limits_{h\to 0}\dfrac{\sqrt{2+h}-\sqrt{2}}{h}$

Solution: We cannot find the limit by substituting *h* = 0, and the numerator and denominator do not have obvious factors. However, we can create a common factor as shown below.

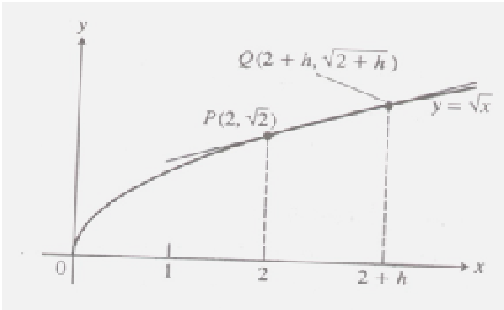
$$\frac{\sqrt{2+h}-\sqrt{2}}{h}=\frac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}=\frac{2+h-2}{h\left(\sqrt{2+h}+\sqrt{2}\right)}$$

$$=\frac{h}{h\left(\sqrt{2+h}+\sqrt{2}\right)} \quad \text{we have created a common factor of } h$$

$$=\frac{1}{\sqrt{2+h}+\sqrt{2}}$$

Therefore,

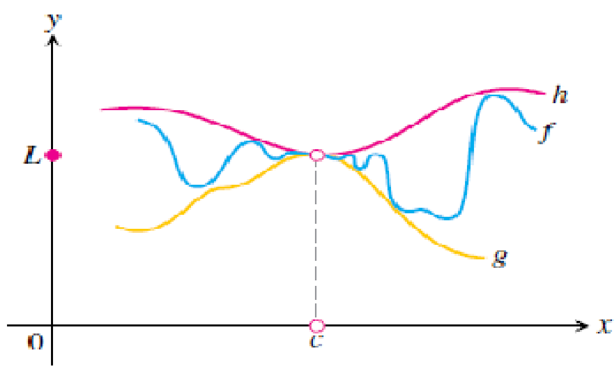
$$\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}=\lim_{h\to 0}\frac{1}{\sqrt{2+h}+\sqrt{2}}=\frac{1}{\sqrt{2+0}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$



The fraction $\frac{\sqrt{2+h}-\sqrt{2}}{h}$ is the slope of the secant through the point $P\left(2,\sqrt{2}\right)$ and the point $Q\left(2+h,\sqrt{2+h}\right)$ nearby on the curve $y=\sqrt{x}$. Our calculation shows that the limiting value of this slope as $Q\to P$ along the curve is $\frac{1}{2\sqrt{2}}$.

Sandwich Theorem

The Sandwich Theorem refers to a function *f* whose values are sandwiched between the values of two other functions *g* and *h* that have the same limit *L* at a point *c*. Being trapped between the values of two functions that approach L, the value of *f* must also approach L.



Theorem:

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

Example 6: Given that Find $\lim_{x\to 0} f(x)$.

Solution: we have,

$$\lim_{x\to 0}\left(1-\frac{x^2}{4}\right)=1 \quad and \quad \lim_{x\to 0}\left(1+\frac{x^2}{2}\right)=1$$

The Sandwich Theorem implies that $\lim_{x\to 0} f(x) = 1$

Example 7: Show that if $\lim_{x\to c}|f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$ **Solution**:

Since $-|f(x)| \le f(x) \le |f(x)|$, and

$-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches c,

there fore, $\lim_{x\to c} f(x) = 0$ by the Sandwich Theorem.

PROBLEM SET

IP1: Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = 5$. Then $\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{\frac{2}{3}}} =$

Solution:

$$\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{\frac{2}{3}}} = \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\lim_{x\to 0} (f(x)+7)^{\frac{2}{3}}} \text{ (Quotient rule)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}(f(x)+7)\right)^{\frac{2}{3}}} \text{ (Power and product rule)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{\frac{2}{3}}}$$

(difference and constant multiples rule)

$$= \frac{2(1)-5}{(1+7)^{\frac{2}{3}}} = \frac{-3}{(1+7)^{\frac{2}{3}}} = -\frac{3}{4}$$

P1: $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$ *and* $\lim_{x\to 1} r(x) = 2$ *then*

$$\lim_{x\to 1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))} =$$

Solution:

$$\lim_{x\to 1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1} p(x)(4-r(x))} \text{(Quotient rule)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1}(4-r(x))\right)} (Difference\ and\ product\ rule)$$

$$= \frac{\sqrt{5\lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4-\lim_{x\to 1} r(x)\right)} \text{(Constant multiple rules)}$$

$$= \frac{\sqrt{5.5}}{(1)(4-2)} = \frac{5}{2}$$

IP2: $\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2} =$

Solution:

If we put x=0 in the expression $\frac{0}{\sqrt{0+4}-2} = \frac{0}{\sqrt{4}-2} = \frac{0}{2-2} = \frac{0}{0}$

So it is indeterminate form

Rationalize the denominator, we will get

$$\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2} = \lim_{x\to 0}\frac{x}{\left(\sqrt{x+4}-2\right)} \times \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{\left(\sqrt{x+4}\right)^2-2^2}$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{x+4-4}$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{x}$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{x}$$

$$= \lim_{x\to 0}\left(\sqrt{x+4}+2\right)$$

$$= \sqrt{0+4}+2 = \sqrt{4}+2 = 2+2 = 4$$

P2: Evaluate $\boldsymbol{lim_{x\to\sqrt{2}} \frac{x^3-2\sqrt{2}}{x-\sqrt{2}} =}$

Solution:

If we substitute $x = \sqrt{2}$ in the expression $\frac{x^3-2\sqrt{2}}{x-\sqrt{2}}$ it becomes

$$\frac{x^3-2\sqrt{2}}{x-\sqrt{2}} = \frac{(\sqrt{2})^3-2\sqrt{2}}{\sqrt{2}-\sqrt{2}}$$

$$= \frac{(\sqrt{2})^2\sqrt{2}-2\sqrt{2}}{0} = \frac{2\sqrt{2}-2\sqrt{2}}{0} = \frac{0}{0}$$

It is indeterminate form.

$$\lim_{x\to\sqrt{2}} \frac{x^3 - 2\sqrt{2}}{x - \sqrt{2}} = \lim_{x\to\sqrt{2}} \frac{x^3 - (\sqrt{2})^3}{x - \sqrt{2}}$$

$$\left(2\sqrt{2} = (\sqrt{2})^2\sqrt{2} = (\sqrt{2})^3\right)$$

$$= \lim_{x\to\sqrt{2}} \frac{(x - \sqrt{2})(x^2+\sqrt{2}x+2)}{x - \sqrt{2}}$$

Since $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ Here $a = x$ and $b = \sqrt{2}$

$$= \lim_{x\to\sqrt{2}} \frac{(x - \sqrt{2})(x^2+\sqrt{2}x+2)}{x - \sqrt{2}}$$

$$= \lim_{x\to\sqrt{2}} (x^2 + \sqrt{2}x + 2)$$

$$\therefore \lim_{x\to\sqrt{2}} \frac{x^3-2\sqrt{2}}{x-\sqrt{2}} = 6$$

IP3: If $\boldsymbol{\lim_{x\to 2} \frac{f(x)-3}{x-2} = 3}$**, find** $\boldsymbol{\lim_{x\to 2} f(x)}$

Solution: Given that, $\lim_{x\to 2} \frac{f(x)-3}{x-2} = 3$

$$\Longrightarrow \lim_{x\to 2} (f(x) - 3) = \lim_{x\to 2} \left((f(x) - 3) \times \frac{x-2}{x-2}\right)$$

$$= \lim_{x\to 2} \left(\frac{f(x)-3}{x - 2} \times x - 2\right)$$

$$= \lim_{x\to 2} \frac{f(x)-3}{(x-2)} \times \lim_{x\to 2} (x - 2)$$

(by using the product rule on limits)

$$= 3 \times (2 - 2) = 3 \times 0 = 0$$

$$\Longrightarrow \lim_{x\to 2} (f(x) - 3) = 0$$

$$\Longrightarrow \lim_{x\to 2} f(x) - \lim_{x\to 2} 3 = 0$$

(by using the difference rule on limits)

$$\Longrightarrow \lim_{x\to 2} f(x) - 3 = 0 \Longrightarrow \boldsymbol{\lim_{x\to 2} f(x) = 3}$$

P3: If $\lim\limits_{x \to 4} \frac{f(x)-5}{x-4} = 1$, find $\lim\limits_{x \to 4} f(x) =$

Solution: Given that $\lim\limits_{x \to 4} \frac{f(x)-5}{x-4} = 1$

$$\Rightarrow \lim_{x \to 4}(f(x)-5) = \lim_{x \to 4} \frac{f(x)-5}{(x-4)}(x-4)$$

$$= \lim_{x \to 4} \frac{f(x)-5}{(x-4)} \times \lim_{x \to 4}(x-4)$$

$$= 1 \times 0 = 0$$

$$\Rightarrow \lim_{x \to 4} f(x) - \lim_{x \to 4} 5 = 0$$

$$\Rightarrow \lim_{x \to 4} f(x) = 5$$

IP4: If $\sqrt{9 - 4x^2} \leq g(x) \leq \sqrt{9 - 3x^2}$ for all $-1 \leq x \leq 1$, find $\lim\limits_{x \to 0} g(x) =$

Solution:

We have $\sqrt{9 - 4x^2} \leq g(x) \leq \sqrt{9 - 3x^2}$ for all $-1 \leq x \leq 1$ and,

$\lim\limits_{x \to 0} \sqrt{9 - 4x^2} = \sqrt{9} = 3$, $\lim\limits_{x \to 0} \sqrt{9 - 3x^2} = \sqrt{9} = 3$

By Sandwich theorem $\lim\limits_{x \to 0} g(x) = 3$

P4: If $4 - x^2 \leq g(x) \leq 4\cos x$ for all x, find $\lim\limits_{x \to 0} g(x)$

Solution:

We have $4 - x^2 \leq g(x) \leq 4\cos x \ \forall \ x \in \boldsymbol{R}$, and

$\lim\limits_{x \to 0} 4 - x^2 = 4 - 0 = 4$, $\lim\limits_{x \to 0} 4\cos x = 4.1 = 4$

By Sandwich theorem $\lim\limits_{x \to 0} g(x) = 4$

Exercise:

1. Evaluate

$$a) \quad \lim_{x \to -2}\left(x^3 - 2x^2 + 4x + 8\right)$$

$$b) \quad \lim_{h \to 0} \frac{3}{\sqrt{3h+1}+1}$$

$$c) \quad \lim_{t \to 2} \frac{t+3}{t+6}$$

$$d) \quad \lim_{t \to c} f(t) = 5, \quad \lim_{t \to c} g(t) = -2 \ \ find \ \lim_{t \to c} \frac{f(t)}{f(t)-g(t)} =$$

2. Evaluate

a) $\lim\limits_{x\to 5} \frac{x-5}{x^2-25}$

b) $\lim\limits_{x\to -5} \frac{x^2+3x-10}{x+5}$

c) $\lim\limits_{x\to 2} \frac{x^2-7x+10}{x-2}$

d) $\lim\limits_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

3. Evaluate

a) $\lim\limits_{h\to 0} \frac{\sqrt{3h+1}-1}{h}$

b) $\lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

c) $\lim\limits_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3}$

d) $\lim\limits_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

4. Evaluate $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ for the following cases

a. $f(x)=x^2 \qquad x=1$

b. $f(x)=\dfrac{1}{x} \qquad x=-2$

c. $f(x)=\sqrt{x} \qquad x=7$

d. $f(x)=\sqrt{3x+1} \qquad x=0$

5.

a) If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ **for** $-1\le x\le 1$, find $\lim\limits_{x\to 0} f(x)$

b) Find $\lim\limits_{x\to 0} \dfrac{x\sin x}{2-\cos 2x}$ given that the inequalities

$1-\dfrac{x^2}{6} \le \dfrac{x\sin 2x}{2-2\cos 2x} \le 1$ hold for all values of x close to zero.

c) Find $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$ given that the inequalities

$\dfrac{1}{2}-\dfrac{x^2}{24} \le \dfrac{1-\cos x}{x^2} \le \dfrac{1}{2}$ hold for all values of x close to zero.

6. Evaluate

a) If $\lim_{x\to 4}\frac{f(x)-5}{x-2}=1$, find $\lim_{x\to 4} f(x)$

b) If $\lim_{x\to 2}\frac{f(x)-5}{x-2}=3$, find $\lim_{x\to 2} f(x)$

c) If $\lim_{x\to 2}\frac{f(x)-5}{x-2}=4$, find $\lim_{x\to 2} f(x)$

d) If $\lim_{x\to 1}\frac{f(x)-8}{x-1}=10$, find $\lim_{x\to 1} f(x)$

3.4. Formal Definition of Limit

Learning objectives

- To give a formal definition of the limit of a function and to prove some properties of limits.
  And
- To solve related problems.

Formal Definition of Limit

We look at an example of determining the input values x that ensure the output y near a ***target* value**.

Example 1:

How close to $x_0 = 4$ must we hold the input x to be sure that the output $y = 2x - 1$ lies within 2 units of $y_0 = 7$?

Solution: The problem simply means:

For what values of x is $|y - 7| < 2$.

We have $|y-7|=|2x-1-7|=|2x-8|$.The question becomes:

What values of x satisfy the inequality $|2x - 8| < 2$?

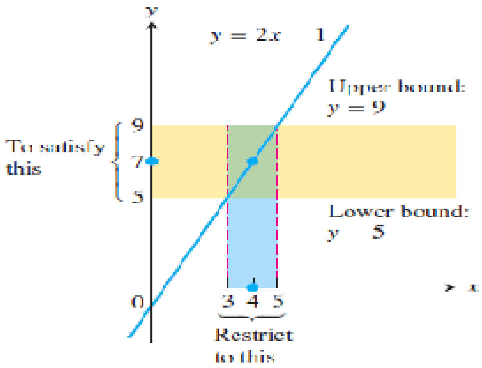
We solve the inequality:

$$|2x-8|<2$$

$$\Rightarrow\ -2<2x-8<2\ \Rightarrow\ 6<2x<10$$

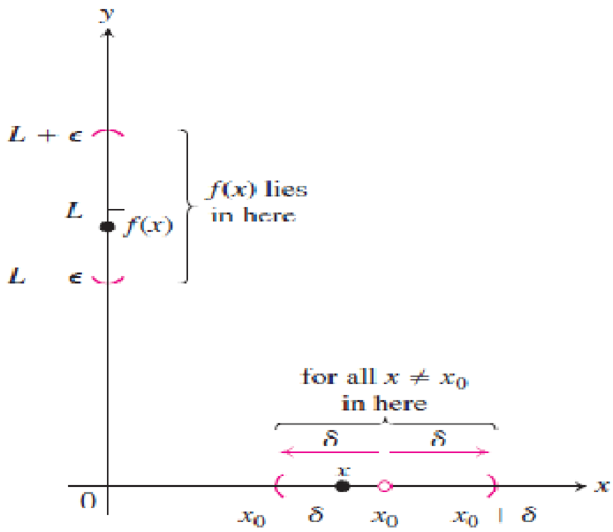
$$\Rightarrow\quad 3<x<5\qquad \Rightarrow -1<x-4<1$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$.



In a target-value problem, we determine how close to hold a variable x to a particular value x_0 to ensure that the outputs $f(x)$ lie within a prescribed interval about a target value L.

To show that the limit of $f(x)$ as $x \to x_0$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than any prescribed error, no matter how small, by holding x close enough to x_0.



For every ε, if we can find a δ, then we say $f(x)$ has limit L as x tends to x_0 .

Definition

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write $\lim_{x \to x_0} f(x) = L$, if,

for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Note: If $\lim_{x \to x_0} f(x) = L$ exists then it is unique.

The now accepted ε, δ definition of limit was formulated by German mathematician Weierstrass in the middle of the nineteenth century.

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified.

Example 2: Show that $\lim\limits_{x \to 1} (5x-3) = 2$.

Solution:

For a given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if

$$0 < |x-1| < \delta \quad \text{then} \quad |f(x)-2| < \varepsilon .$$

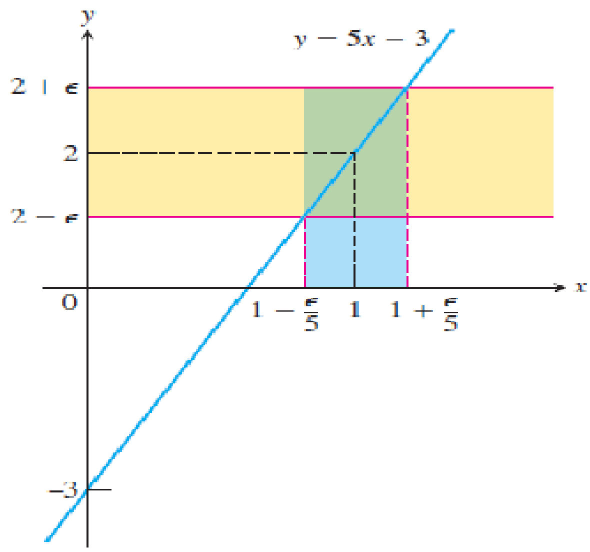
We find δ by working backwards from the ε inequality:

$$|f(x)-2| = |(5x-3)-2| = |5x-5| < \varepsilon$$

$$i.e., \ 5|x-1| < \varepsilon$$

$$i.e., \quad |x-1| < \varepsilon / 5$$

Thus, we can take $\delta = \frac{\varepsilon}{5}$ or any smaller positive value. This proves that $\lim\limits_{x \to 1} (5x-3) = 2$.



Note:

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \varepsilon$$

can be accomplished in two steps.

1) We solve the inequality $|f(x)-L|<\varepsilon$ to find an open interval (a,b) about x_0 on which the inequality holds for all $x \neq x_0$.

2) We find a value of $\delta>0$ that places the open interval $(x_0-\delta, x_0+\delta)$ centered at x_0 inside the interval (a,b). The inequality $|f(x)-L|<\varepsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Proving the rule for the limit of a sum:

The formal definition of limit is not usually employed to determine specific limits. Rather general theorems on limits are used to determine the limits of functions. The theorems are proved using the formal definition of the limit.

Theorem:

The limit of the sum of two functions is equal to the sum of their limits.

Given that $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$, to prove that $\lim_{x\to c}\left(f(x)+g(x)\right)=L+M$

***Proof*:** Let $\varepsilon>0$ be given. We want to find a positive number δ such that for all x

$$0<|x-c|<\delta \Rightarrow |f(x)+g(x)-(L+M)|<\varepsilon$$

Regrouping terms, we get

$$|f(x)+g(x)-(L+M)|=|(f(x)-L)+(g(x)-M)|$$
$$\leq |(f(x)-L)|+|(g(x)-M)|,$$
$$(\textbf{By triangle inequality } |a+b|\leq|a|+|b|)$$

Since $\lim_{x\to c} f(x)=L$, for the above $\varepsilon>0$ there exists a number $\delta_1 > 0$ such that for all x

$$0<|x-c|<\delta_1 \quad \Rightarrow \quad |f(x)-L|<\varepsilon/2$$

Similarly, since $\lim_{x\to c} g(x)=M$, for the above $\varepsilon>0$ there exists a number $\delta_2>0$ such that for all x

$$0<|x-c|<\delta_2 \quad \Rightarrow \quad |g(x)-M|<\varepsilon/2$$

Let $\delta=\min(\delta_1,\delta_2)$. If $0<|x-c|<\delta$ **then**

$$0<|x-c|<\delta_1, \text{ so } |f(x)-L|<\varepsilon/2$$
$$\text{and } 0<|x-c|<\delta_2, \text{ so } |g(x)-M|<\varepsilon/2$$

Therefore,

$$\left|f(x)+g(x)-(L+M)\right|<\varepsilon/2+\varepsilon/2=\varepsilon,\textbf{when ever } 0<\left|x-c\right|<\delta$$

This shows that

$$\lim_{x\to c}\left(f(x)+g(x)\right)=L+M$$

PROBLEM SET

IP1: Given, $f(x)=\sqrt{19-x}$, $L=3$, $x_0=10, \epsilon=1$. Find an open interval about x_0 on which the inequality $|f(x)-L|<\epsilon$

holds and give a value of $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$, the inequality $|f(x)-L|<\epsilon$ holds .

Solution: Given, $f(x)=\sqrt{19-x}\,, L=3\,, x_0=10, \boldsymbol{\epsilon}=1$

$$i.e., \text{given } \lim_{x\to 10}\sqrt{19-x}=3$$

Now $|f(x)-L|<\in \implies \left|\sqrt{19-x}-3\right|<1$

$$3-1<\sqrt{19-x}<3+1 \implies 2<\sqrt{19-x}<4$$

$\implies 4<19-x<16 \implies -15<-x<-3 \implies 15>x>3$

$\implies 3<x<15 \implies x\in(3,15)$

The open interval about $x_0=10$ on which the inequality $|f(x)-L|<$ holds on (3,15).

Now $3<x<15 \Leftrightarrow 3-10<x-10<15-10$
$\Leftrightarrow -7<x-10<5$

Notice that, $-7<-5<x-10<5$. Take $-5<x-10<5$, i.e., $|x-10|<5$

$\therefore\ \delta=5$ such that for all x satisfying $0<|x-10|<5$, the inequality $|f(x)-L|<\boldsymbol{\epsilon}$ holds.

P1: Show that $\displaystyle\lim_{x\to 1}\left(\frac{3}{2}x-1\right)=\frac{1}{2}$

Solution: Let $f(x)=\frac{3}{2}x-1$.For any given $\epsilon>0$,

we have to find a suitable $\delta>0$ such that

$$0<|x-1|<\delta \Rightarrow \left|f(x)-\frac{1}{2}\right|<\epsilon$$

To find δ, we work backwards from ϵ -inequality.

Now, $\left|\left(\frac{3}{2}x-1\right)-\frac{1}{2}\right|=\left|\frac{3}{2}x-\frac{3}{2}\right|<\epsilon$

$$\Rightarrow \frac{3}{2}|x-1|<\epsilon \implies |x-1|<\frac{2}{3}\epsilon$$

Take $\delta = \frac{2}{3}\epsilon$. If $0 < |x-1| < \delta = \frac{2}{3}\epsilon$ then $\left|f(x) - \frac{1}{2}\right| < \epsilon$

Thus $\lim_{x\to 1}\left(\frac{3}{2}x - 1\right) = \frac{1}{2}$

IP2: Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ and that $f(x) \le g(x)$ for all x in an open interval containing c (except possibly c itself), prove that $L \le M$.

Solution: We use the method of proof by contradiction. Suppose the contrary, that is, $L > M$. Then by the limit of difference property, $\lim_{x\to c}\big(g(x) - f(x)\big) = M - L$.

Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$|(g(x) - f(x)) - (M - L)| < \epsilon \ whenever\ 0 < |x - c| < \delta.$

Since $L - M > 0$ by hypothesis, we take $\epsilon = L - M$ in particular and we have a number $\delta > 0$ such that

$|(g(x) - f(x) - (M - L)| < L - M \ whenever\ 0 < |x - c| < \delta.$

Since $a \le |a|$ for any number a, we have

$|(g(x) - f(x) - (M - L)| < L - M whenever\ 0 < |x - c| < \delta.$

This simplifies to

$$g(x) < f(x) \quad whenever\ 0 < |x - c| < \delta$$

But this contradicts $f(x) \le g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \le M$.

P2: Limits of the Identity and Constant Functions

Prove: $(a) \lim_{x\to x_0} x = x_0 \qquad\qquad (b) \lim_{x\to x_0} k = k\,(k \text{ constant}).$

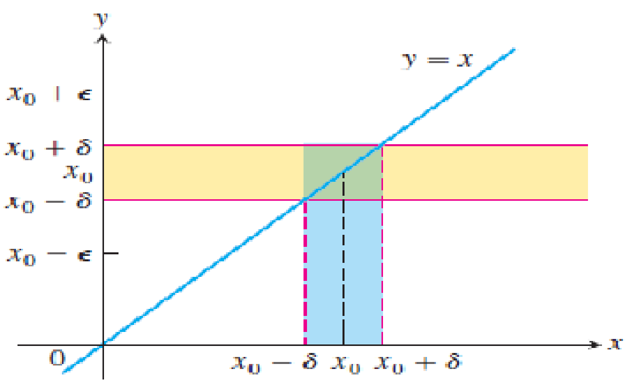
Solution:

A). Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that for all x.

$$0 < |x - x_0| < \delta \quad implies \quad |f(x) - L| < \epsilon$$

$$i.e., \quad 0 < |x - x_0| < \delta \quad implies \quad |x - x_0| < \epsilon$$

The implication will hold if δ equals ϵ or any smaller positive number. It proves that $\lim_{x\to x_0} x = x_0$

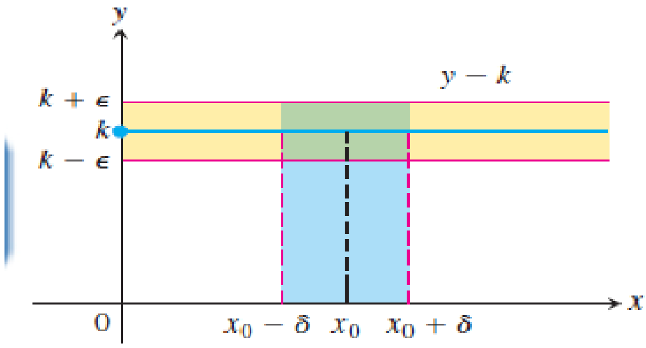


B). Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that for all x.

$$0 < |x - x_0| < \delta \quad implies \quad |f(x) - L| < \epsilon$$

$$0 < |x - x_0| < \delta \quad implies \quad |k - k| < \epsilon$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold . This proves that $\lim_{x \to x_0} k = k$



IP3: $f(x) = -3x - 2$, $x_0 = -1$, $\epsilon = 0.03$. Find $\lim_{x \to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$, the inequality $|f(x) - L| < \epsilon$ holds.

Solution: $\lim_{x \to -1} (-3x - 2) = (-3)(-1) - 2 = 1$

$$|(-3x - 2) - 1| < 0.03 \Rightarrow -0.03 < -3x - 3 < 0.03$$
$$\Rightarrow 0.01 > x + 1 > -0.01$$
$$\Rightarrow -0.01 < x + 1 < 0.01$$
$$\Rightarrow |x + 1| < 0.01$$
$$\Rightarrow \delta = 0.01$$

P3: Given $f(x) = 3 - 2x$, $x_0 = 3$, $\epsilon = 0.02$. Find $\lim_{x \to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$, the inequality $|f(x) - L| < \epsilon$ holds.

Solution:

Given, $\lim_{x \to 3} (3 - 2x) = 3 - 2(3) = -3$

$$|(3 - 2x) — 3) < 0.02$$
$$\Rightarrow -0.02 < 6 - 2x < 0.02$$
$$\Rightarrow -6.02 < -2x < -5.98$$
$$\Rightarrow 3.01 > x > 2.99 \quad \text{or} \quad 2.99 < x < 3.01.$$
$$\Rightarrow 2.99 - 3 < x - 3 < 3.01 - 3.$$
$$\Rightarrow -0.01 < x - 3 < 0.01 \Rightarrow |x - 3| < 0.01$$

$$\therefore \delta = 0.01$$

IP4: Prove the limit statement

$\lim_{x\to1} f(x) = 2$ if $f(x) = \begin{cases} 4-2x, & x<1 \\ 6x-4, & x\geq 1\end{cases}$

Solution: Let $x < 1$, then

$|(4-2x)-2| < \epsilon \implies 0 < 2-2x < \epsilon$ (since $x < 1$)

Thus, $1-\frac{\epsilon}{2} < x < 0$.

Let $x \geq 1$, then

$|(6x-4)-2| < \epsilon \implies 0 \leq 6x-6 < \epsilon$ (since $x \geq 1$)

Thus, $1 \leq x < 1+\frac{\epsilon}{6}$.

Now, $1-\frac{\epsilon}{2} < x < 0 < 1 \leq x < 1+\frac{\epsilon}{6}$

$\implies 1-\frac{\epsilon}{2} < x < 1+\frac{\epsilon}{6} \implies x \in \left(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{6}\right)$

Notice that $\left(1-\frac{\epsilon}{6}, 1+\frac{\epsilon}{6}\right) \subset \left(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{6}\right)$

$\therefore$ $|f(x)-2| < \epsilon$ will hold for all $x \neq 1$when $|x-1| < \delta = \frac{\epsilon}{6}$

Therefore, $\lim_{x\to1} f(x) = 2$

P4: Prove the limit statement: $lim_{x\to1}\left(\frac{x^2-1}{x-1}\right) = 2$

Solution: Given , $\epsilon > 0$ we have to find a $\delta > 0$ such that

$\left|\left(\frac{x^2-1}{x-1}\right)-2\right| < \epsilon$, Whenever $|x-1| < \delta$

Now, $\left|\left(\frac{x^2-1}{x-1}\right)-2\right| < \epsilon \implies |(x+1)-2| < \epsilon, x \neq 1$

$\implies |x-1| < \epsilon, x \neq 1$

$\therefore$ $\delta = \epsilon$ and $|x-1| < \delta \implies \left|\left(\frac{x^2-1}{x-1}\right)-2\right| < \epsilon$

Exercise:

1. Define the limit of a function.
2. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist then prove that

   $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
3. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ and $f(x) \leq g(x)$ for all x in an open interval containing c.

   (Except possibly c itself), Prove that $L \leq M$.
4. Verify

   a) $\lim_{x\to x_0} x = x_0$

b) $\lim\limits_{x \to x_0} k = k$, $\quad k$ is a constant

5. Prove that $\lim\limits_{x \to 5} \sqrt{x-1} = 2$.

6. Prove that $\lim\limits_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$

3.5. Extension of the Limit concept

Learning objectives:

- To define right and left hand limits
- To define the limit in terms of one sided limits
  And
- To practice related problems.

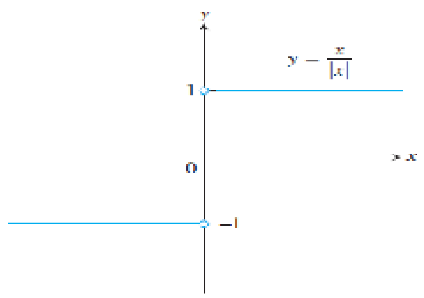
Extension of the Limit concept

Now we extend the concept of limit to ***one-sided limits***, which are limits as x approaches a from the left-hand side(where $x < a$) or the right-hand side (where $x > a$) only.

One Sided Limits

To have a limit L as x approaches a, a function f must be defined on both sides of a, and its value $f(x)$ must approach L as x approaches a from either side. Because of this, ordinary limits are sometimes called **two-sided** limits.

It is possible for a function to approach a limiting value as x approaches a from only one side, either from the right or from the left. In this case we say that f has a **one-sided** limit at a. The function $f(x) = \dfrac{x}{|x|}$ graphed below has limit 1 as x approaches zero from the right, and limit -1 as x approaches zero from the left.



Definition

Let $f(x)$ defined on an interval (a,b) where $a < b$. If $f(x)$ approaches arbitrarily close to L as $\boldsymbol{x}$ approaches a from within that interval, then we say that f has **right-hand limit** L at a, and we write

$$\lim_{x \to a^+} f(x) = L$$

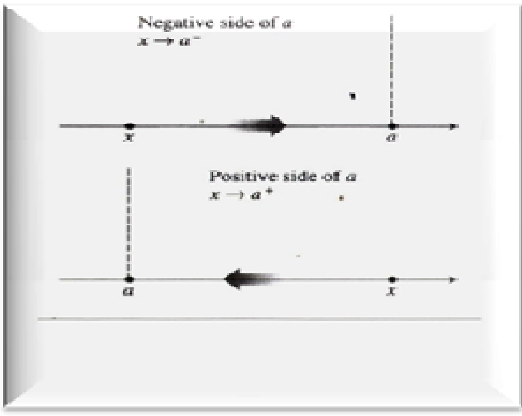
Let $f(x)$ be defined on an interval (c,a) where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within that interval, then we say that f has **left-hand limit** M at a, and we write

$$\lim_{x \to a^-} f(x) = M$$

For the function $f(x) = \dfrac{x}{|x|}$ in the figure above, we have

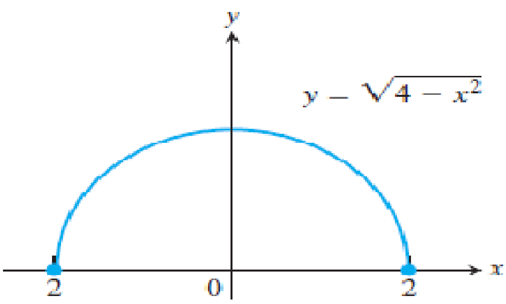
$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$

A function cannot have an ordinary limit at an endpoint of its domain, but it can have a one-sided limit.



The symbol $x \to a^-$ means *x* approaches *a* from the negative side of *a*, through values less than *a*. The symbol $x \to a^+$ means *x* approaches *a* from the positive side of *a*, through values greater than *a*.

Example 1: The domain of $f(x) = \sqrt{4 - x^2}$ *is* $[-2, 2]$; its graph is semi-circle shown below.



$$\lim_{x\to -2^+} f(x)=0 \ , \lim_{x\to 2^-} f(x)=0$$

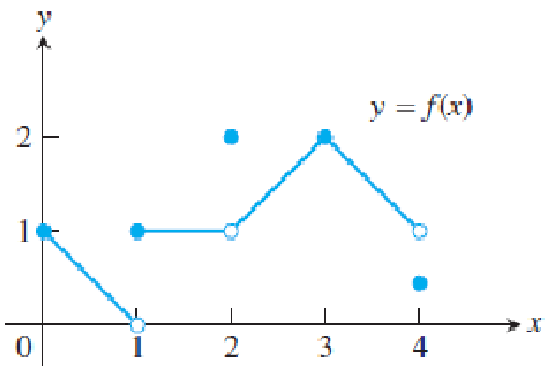
The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2.

One-sided versus two-sided Limits

A function $f(x)$ has a limit as *x* approaches *c* if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal:

$$\lim_{x\to c} f(x)=L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x)=L \ \ and \ \ \lim_{x\to c^+} f(x)=L$$

Example 2: All of the following statements about the function graphed in the figure below are true.



At $x = 0$

$$\lim_{x\to 0^+} f(x)=1$$

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$

$\lim_{x\to 1^+} f(x)=0$ even though $f(1) = 1$

$$\lim_{x\to 1^+} f(x)=1$$

$\lim_{x\to 1} f(x)$ does not exist. The right-hand and left-hand limits are not equal.

At $x = 2$

$$\lim_{x\to 2^-} f(x)=1$$

$$\lim_{x\to 2^+} f(x)=1$$

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$$

At $x = 4$

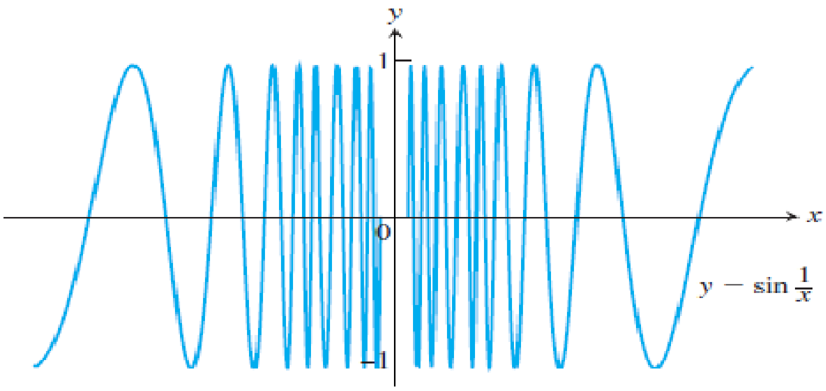
$\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$.

$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point *a* in $[0,4]$, *f*(*x*) has limit $f(a)$.

The function $y = \sin\left(\frac{1}{x}\right)$ has neither a right-hand nor a left-hand limit as x approaches zero. This can be seen from the following observations.

As x approaches zero, its reciprocal $\frac{1}{x}$ grows without bound and the value of sin(1/x) cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive or negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.



The formal definition of two-sided limits can be easily modified for one-sided limits.

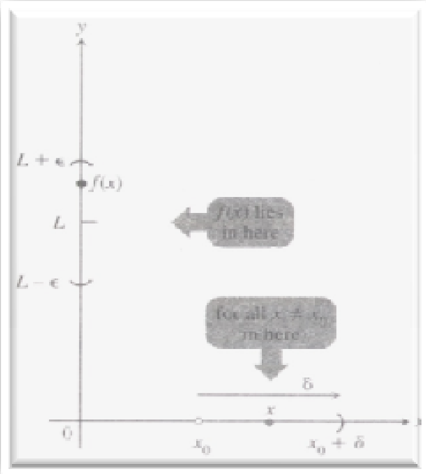
Right-hand Limit

We say that $f(x)$ has right-hand limit L at x_0, and write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \quad |f(x) - L| < \varepsilon$$



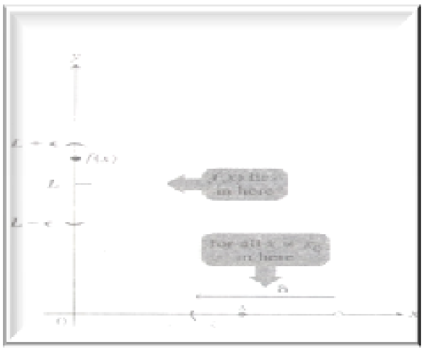
Left-hand Limit

We say that $f(x)$ has left-hand limit L at x_0, and write

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \quad |f(x) - L| < \varepsilon$$



PROBLEM SET:

IP1: Evaluate the right hand limit of the function

$$f(x) = \begin{cases} \frac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad at \;\; x = 4.$$

Solution:

Given, $f(x) = \begin{cases} \frac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad at \;\; x = 4$

If $x > 4$ then $x - 4 > 0$ and $|x - 4| = (x - 4)$

$$\lim_{x\to4^+} f(x) = \lim_{x\to4^+} \frac{|x-4|}{x-4} = \lim_{x\to4^+} \frac{(x-4)}{x-4} = 1$$

P1: Evaluate the left hand limit of the function

$$f(x) = \begin{cases} \frac{|x-4|}{x-4} & ,x \neq 4 \\ 0 & ,x = 4 \end{cases} \quad at\ \ x = 4.$$

Solution:

Given, $f(x) = \begin{cases} \frac{|x-4|}{x-4} & ,x \neq 4 \\ 0 & ,x = 4 \end{cases} \quad at\ \ x = 4$

If $x < 4$ then $x - 4 < 0$ and $|x-4| = -(x-4)$

$$\lim_{x\to4^-} f(x) = \lim_{x\to4^-} \frac{|x-4|}{x-4} = \lim_{x\to4^-} \frac{-(x-4)}{x-4} = -1$$

IP2: $\lim_{x\to1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$

Solution:

$$\lim_{x\to1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \lim_{x\to1^+} \sqrt{2x}\ .\ \lim_{x\to1^+} \frac{(x-1)}{|x-1|} \text{ (Product rule)}$$

$$= \lim_{x\to1^+} \sqrt{2x}\ .1 \quad \left(\text{Since } \lim_{x\to1^+} \frac{(x-1)}{|x-1|} = 1\right)$$

$$= \lim_{h\to0} \sqrt{2(1+\mathrm{h})}$$

$$= \sqrt{\lim_{h\to0} 2(1+\mathrm{h})} \qquad \text{(Power rule)}$$

$$= \sqrt{2}$$

P2: $f(x) = \begin{cases} 1 + x^2, & if \quad 0 \leq x \leq 1 \\ 2 - x\ , & if \quad x > 1 \end{cases} \quad at\ \ x = 1.$

show that $\lim_{x\to1} f(x)$ does not exist.

Solution: We have,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (1 + x^2)$$

$$= \lim_{h\to0} (1 + (1-h)^2) = 1 + 1 = 2$$

and,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2 - x)$$

$$= \lim_{h\to0} (2 - (1+h)) = 1$$

Thus $\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$

Therefore, $\lim_{x \to 1} f(x)$ does not exist.

IP3: $f(x) = \begin{cases} mx^2 + n\,, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m\,, & x > 1 \end{cases}$ **. For what values of integers m, n does the limits $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exist.**

Solution:

$\lim_{x \to 0} f(x)$ exist

$\Longrightarrow \quad \lim_{x \to 0^-} f(x)\,, \lim_{x \to 0^+} f(x)$ both exist. And they are equal.

Now , $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

$\Longrightarrow \lim_{x \to 0^+}(nx + m) = \lim_{x \to 0^-}(mx^2 + n)$

$\Longrightarrow \lim_{h \to 0}(n(0 - h) + m) = \lim_{h \to 0}(m(0 + h)^2 + n)$

$\Longrightarrow \lim_{h \to 0}(-nh + m) = \lim_{h \to 0}(mh^2 + n) \Longrightarrow m = n$

Therefore, $\lim_{x\to 0} f(x)$ exists for all values of m, n $such\ that\ m = n$.

$\lim_{x \to 1} f(x)$ exist

$\Longrightarrow \lim_{x \to 1^-} f(x),\ \lim_{x \to 1^+} f(x)$ both exist, and they are equal.

Now , $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Longrightarrow \lim_{x\to 1^-}(nx + m) = \lim_{x\to 1^+}(nx^3 + m)$

$\Longrightarrow \lim_{h\to 0}(n(1 - h) + m) = \lim_{h\to 0}(n(1 + h)^3 + m) \Longrightarrow n + m = n + m$

Therefore, $\lim_{x \to 1} f(x)$ exists for all values of m, n .

Now the values of m, n for which the limits $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exists is $m = n$

P3: $f(x) = \begin{cases} \cos x\,, & if \quad x \ge 0 \\ x + k, & if \quad x < 0 \end{cases}$ **. Find the value of constant k, given that $\lim_{x\to 0} f(x)$ exists.**

Solution: We have,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-}(x + k) = \lim_{x\to 0^+} \cos x$$

$\Longrightarrow \lim_{h\to 0}(0-h+k) = \lim_{h\to 0}\cos(0+h)$

$\Longrightarrow k = \lim_{h\to 0}\cos(h) \Longrightarrow k = 1$

IP4: If $f: R \to R$ is an even function, then prove that $\lim_{x\to 0} f(x)$ exists.

Solution:

We have, $\boldsymbol{f(-x) = f(x)}$, $\forall\, \boldsymbol{x}$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} f(h) \qquad [\because f(x) \text{is even }]$$

$$= \lim_{h\to 0} f(0+h) = \lim_{x\to 0^+} f(x).$$

Therefore $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

P4: If $f: R \to R$ is an odd function and $\lim_{x\to 0} f(x)$ exists, then prove that $\lim_{x\to 0} f(x) = 0$.

Solution: It is given that, $f(-x) = -f(x), \forall\ x$

$\lim_{x\to 0} f(x)$ exists

$\Longrightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Longrightarrow \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h)$

$\Longrightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h)$

$\lim_{h\to 0} (-f(h)) = \lim_{h\to 0} f(h)$

$[\because f(x) \text{is odd }]$

$\Longrightarrow -\lim_{h\to 0} f(h) = \lim_{h\to 0} f(h)$

$\Longrightarrow 2\lim_{h\to 0} f(h) = 0 \Longrightarrow \lim_{h\to 0} f(h) = 0 \Longrightarrow \lim_{x\to 0} f(x) = 0$

Exercise:

1. Which of the following statements about the function $\boldsymbol{y = f(x)}$ graphed here are true, and which are false?

a. $\lim_{x\to1^+} f(x)=1$

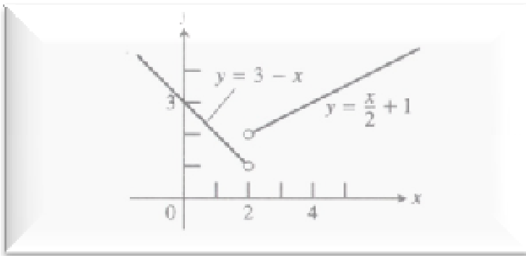
b. $\lim_{x\to0^-} f(x)=0$ $\lim_{x\to0^-} f(x)=1$ $\lim_{x\to0^-} f(x)=\lim_{x\to0^+} f(x)$

c. $\lim_{x\to0} f(x)$ exists $\lim_{x\to0} f(x)=0$ $\lim_{x\to0} f(x)=1$

d. $\lim_{x\to1} f(x)=1$ $\lim_{x\to1} f(x)=0$

e. $\lim_{x\to2^-} f(x)=2$ $\lim_{x\to-1^-} f(x)$ does not exist $\lim_{x\to2^+} f(x)=0$

2. Let $f(x)=\begin{cases} 3-x & x<2 \\ \dfrac{x}{2}+1 & x>2 \end{cases}$



a. Find $\lim_{x\to2^+} f(x)$ and $\lim_{x\to2^-} f(x)$

b. Does $\lim_{x\to2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to4^-} f(x)$ and $\lim_{x\to4^+} f(x)$

d. Does $\lim_{x\to4} f(x)$ exist? If so, what is it? If not, why not?

3. Let $f(x)=\begin{cases} 0 & x\le 0 \\ \sin\dfrac{1}{x} & x>0 \end{cases}$

a. Does $\lim_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

4.

a. Graph $f(x) = \begin{cases} x^3 & x \neq 1 \\ 0 & x = 1 \end{cases}$

b. Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

5. Graph the function $f(x) = \begin{cases} \sqrt{1 - x^2} & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \\ 2 & x = 2 \end{cases}$. Answer the following questions.

a. What are the domain and range of f ?

b. At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

c. At what points does only the left-hand limit exist?

d. At what points does only the right-hand limit exist?

6. Find the limits

a. $\lim_{x \to 0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

b. $\lim_{x \to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

c. $\lim_{h \to 0^+} \dfrac{\sqrt{h^2+4h+5} - \sqrt{5}}{h}$

d. $\lim_{x \to -2^+} (x+3)\dfrac{|x+2|}{x+2}$ $\qquad$ $\lim_{x \to -2^-} (x+3)\dfrac{|x+2|}{x+2}$

e. $\lim\limits_{\theta\to 3^+} \frac{\lfloor\theta\rfloor}{\theta}$ $\qquad$ $\lim\limits_{\theta\to 3^-} \frac{\lfloor\theta\rfloor}{\theta}$

3.6. Infinite Limits

Learning objectives:

- To study $\lim\limits_{\theta \to 0} \frac{sin\theta}{\theta}$.
- To define limits of a function at $\pm\infty$
- To define infinite limits of functions

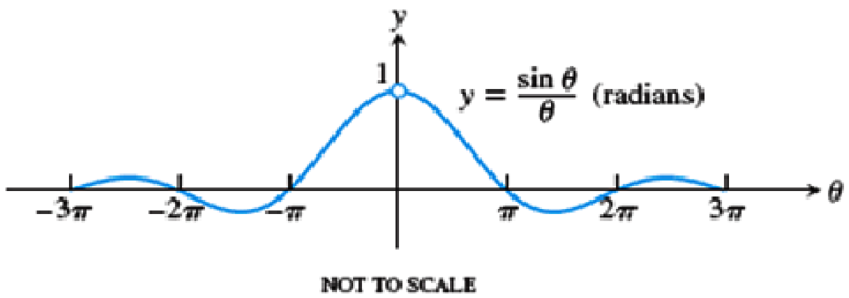
 AND

- To practice related problems.

Limits involving $\frac{\sin\theta}{\theta}$

We have already noted that $\lim\limits_{\theta\to 0} sin\ \theta = 0$ and $\lim\limits_{\theta\to 0} cos\ \theta = 1$.

We now, take up $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}$, where θ is measured in Radian measure. It may be seen $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ from the following figure



The graph of $f(\theta) = \frac{(\sin\theta)}{\theta}$

Notice that $\sin\theta$ and θ are odd functions. Therefore

$f(\theta) = \frac{\sin\theta}{\theta}$ is an even function with a graph symmetry about the y $-$axis (see the above fig). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$$

Therefore, $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

We prove the above result algebraically in a subsequent module.

Example 1:

Find (i) $\lim\limits_{x\to 0}\frac{\cos x-1}{x}$ **(ii)** $\lim\limits_{x\to 0}\frac{\sin 3x}{2x}$

Solution:

$$(i)\ \lim_{x\to 0}\frac{cosx-1}{x}=\lim_{x\to 0}\frac{-2\,sin^2\frac{x}{2}}{x}=-\lim_{x\to 0}\frac{sin\frac{x}{2}}{\left(\frac{x}{2}\right)}.\lim_{x\to 0}sin\frac{x}{2}$$

$$=-1.0=0$$

$$(ii)\ \lim_{x\to 0}\frac{\sin 3x}{2x}=\lim_{x\to 0}\left(\frac{3}{2}\right)\frac{\sin 3x}{3x}$$

$$=\frac{3}{2}.\lim_{x\to 0}\frac{sin\,3x}{3x}$$

Put $\theta=3x$. Now, $\theta\to 0$ as $x\to 0$.

$$=\left(\frac{3}{2}\right).\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{3}{2}.1=\frac{3}{2}$$

Finite limits as $x\longrightarrow\pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range out grow all finite bounds.

For example the function $f(x)=\frac{1}{x}$ is defined for all $x\in\boldsymbol{R}$,

$x\neq 0$. If x is positive and becomes increasingly large then $\frac{1}{x}$ becomes increasingly small. Further, If x is negative and its magnitude becomes increasingly large then $\frac{1}{x}$ again becomes small. Summarizing these observations, we say that $f(x)=\frac{1}{x}$ has limit 0 as $x\longrightarrow\pm\infty$(or that 0 is the limit of $f(x)=\frac{1}{x}$ at infinity and negative infinity.)

Definition: Limits as $x\longrightarrow\infty$ $or-\infty$

1. We say that $f(x)$ **has the limit L** as x **approaches infinity** and write
$$\lim_{x\to\infty}f(x)=L$$
if for every number $\epsilon>0$, there exists a corresponding number M such that for all x
$$x>M\Rightarrow|f(x)-L|<\epsilon.$$

2. We say that $f(x)$ **has the limit L** as x **approaches minus infinity** and write
$$\lim_{x\to-\infty}f(x)=L$$
if for every number $\epsilon>0$, there exists a corresponding number N such that,
$$x<N\Rightarrow|f(x)-L|<\epsilon.$$

That is,

(1) $\lim_{x\to\infty} f(x) = L$ if $f(x)$ gets arbitrarily close to L whenever x moves increasingly far from the origin in the positive direction.

(2) $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ gets arbitrarily close to L whenever x moves increasingly far from the origin in the negative direction.

The calculation of limits of functions as $x \to \pm\infty$ is similar to the one for finite limits, as discussed in the earlier modules. Further the limits at infinity have properties similar to those of finite limits. We have,

$$\lim_{x\to\pm\infty} k = k \text{ and } \lim_{x\to\pm\infty} \frac{1}{x} = 0$$

Example 2: Find $\lim_{x\to\infty}\left(5+\frac{1}{x}\right)$

Solution: $\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = 5+0 = 5 \quad (sum\ \ rule)$

Example 3: Find $\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2}$

Solution:

$$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \pi\sqrt{3}.\lim_{x\to-\infty}\frac{1}{x^2}$$

$$= \pi\sqrt{3}.\left(\lim_{x\to-\infty}\frac{1}{x}\right)\left(\lim_{x\to-\infty}\frac{1}{x}\right) \quad \text{(Product rule)}$$

$$= 0$$

Limits at infinity of rational function

To determine the limit of a rational function as $x \to \pm\infty$, we divide the numerator and denominator by the highest power of x in the denominator and apply the limits.

Numerator and denominator of same degree

Example 4: Find $\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}$

Solution: $\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}}$

(by dividing numerator and denominator by the highest power of x in the denominator ,i.e., x^2).

$$= \frac{\lim_{x\to\infty}\left(5+\frac{8}{x}-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(3+\frac{2}{x^2}\right)} = \frac{5}{3}$$

Degree of Numerator less than degree of denominator

Example 5: Find $\lim_{x\to-\infty} \frac{11x+2}{2x^3-1}$

Solution:

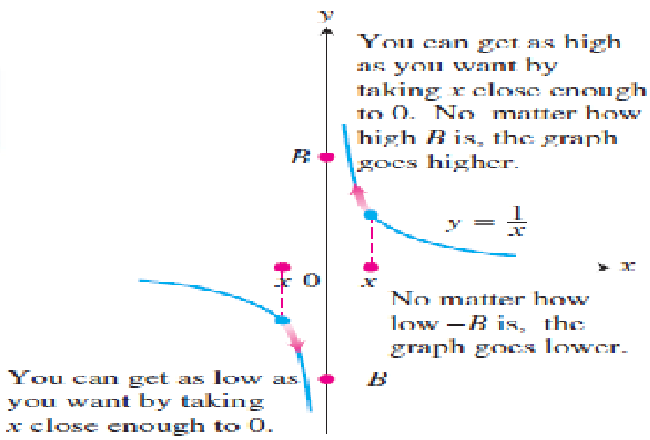
$$\lim_{x\to-\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to-\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}$$

(by dividing numerator and denominator by the highest power of x in the denominator i.e., x^3).

$$= \frac{\lim_{x\to-\infty}\left(\frac{11}{x^2}+\frac{2}{x^3}\right)}{\lim_{x\to-\infty}\left(2-\frac{1}{x^3}\right)} = \frac{0+0}{2-0} = 0.$$

Infinite limits:

Consider the function $f(x) = \frac{1}{x}$. Its graph is shown below.



As $x \to 0^+$, the values of f grow without bound. That is, given any positive real number B, however large, the values of f become larger still. Thus f *has no limit as* $x \to 0^+$. However, *it is conventional to say that* $f(x)$ *approaches* ∞ *(although there is no such number* ∞*) as* $x \to 0^+$. *We write*

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{x} = \infty$$

As $x \to 0^-$, the values of $f(x) = \frac{1}{x}$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. Thus f *has no limit as* $x \to 0^-$. We write

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

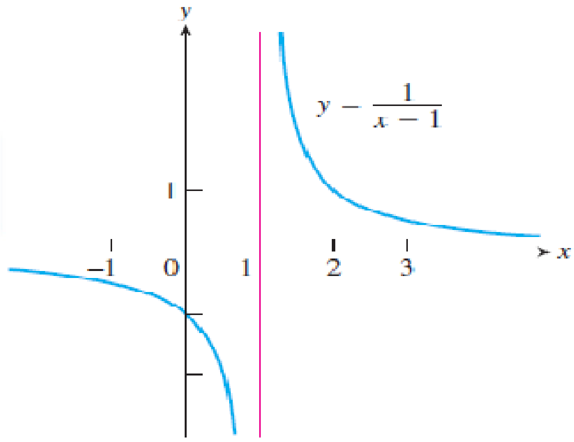
The notion of infinite limit and the symbol ∞ facilitate the description of the behaviour of functions whose values become arbitrarily large.

One - sided infinite limits

Example 6: Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$

Solution: The graph of $y = \frac{1}{x-1}$ is the graph of of $y = \frac{1}{x}$ shifted 1 unit to the right.

Therefore, the behaviour of $\frac{1}{x-1}$ near $x = 1$ is exactly same as the behaviour of $\frac{1}{x}$ at $x = 0$.



Therefore, $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$ and $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

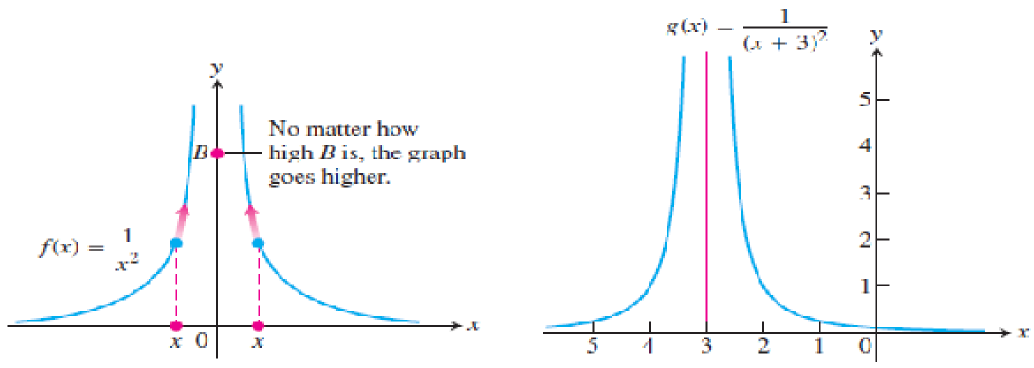
We can also think: As $x \to 1^+$, we have $(x-1) \to 0^+$ and $\frac{1}{x-1} \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $\frac{1}{x-1} \to -\infty$.

Two sided infinite limits

Example 7: Discuss the behaviour of

a) $f(x) = \frac{1}{x^2}$, near $x = 0$,

b) $g(x) = \frac{1}{(x+3)^2}$, near $x = -3$



(a) (b)

Solution:

a) As $x \longrightarrow 0$ *from either side*, the values of $\frac{1}{x^2}$ are positive and becomes arbitrarily large. Therefore,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{x^2} = \infty$$

b) The graph of $g(x) = \dfrac{1}{(x+3)^2}$ is the graph of $f(x) = \frac{1}{x^2}$ shifting 3 units to the left. Therefore the behaviour of g near $x = -3$ is exactly same as $f(x) = \frac{1}{x^2}$ near x=0.

Therefore, $\displaystyle \lim_{x\to -3} g(x) = \lim_{x\to -3}\frac{1}{(x+3)^2} = \infty$

Note:

1) The function $y = \frac{1}{x}$ shows no consistent behaviour as $x \longrightarrow 0$. We have seen that $\frac{1}{x} \longrightarrow \infty$ as $x \longrightarrow 0^+$ and $\frac{1}{x} \longrightarrow -\infty$ as $x \longrightarrow 0^-$. In this sense we say that $\lim_{x\to 0} f(x)$ doesn't exist.
2) The values of the function $y = \frac{1}{x^2}$ approaches ∞ as $x \longrightarrow 0$ from either side. In this sense we say that $\lim_{x\to 0}\frac{1}{x^2} = \infty$.

Rational functions can behave in various ways near zeros of their denominators.

Example 8: We consider the limits of certain rational functions near zeros of their denominators.

a) $$\lim_{x\to 2}\frac{(x-2)^2}{x^2-4} = \lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{x-2}{x+2} = 0$$

b) $$\lim_{x\to 2}\frac{(x-2)}{x^2-4} = \lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{4}$$

c) $\lim\limits_{x\to 2^+} \dfrac{(x-3)}{x^2-4} = \lim\limits_{x\to 2^+} \dfrac{(x-3)}{(x-2)(x+2)} = -\infty$

(The values are –ve for x>2, x close to 2)

d) $\lim\limits_{x\to 2^-} \dfrac{(x-3)}{x^2-4} = \lim\limits_{x\to 2^-} \dfrac{(x-3)}{(x-2)(x+2)} = \infty$

(The values are +ve for x<2, x close to 2)

e) $\lim\limits_{x\to 2} \dfrac{(x-3)}{x^2-4} = \lim\limits_{x\to 2} \dfrac{(x-3)}{(x-2)(x+2)}$ **does not exist** $\big($**by c and d**$\big)$

f) $\lim\limits_{x\to 2} \dfrac{2-x}{(x-2)^3} = \lim\limits_{x\to 2} \dfrac{-(x-2)}{(x-2)^3} = \lim\limits_{x\to 2} \dfrac{-1}{(x-2)^2} = -\infty$

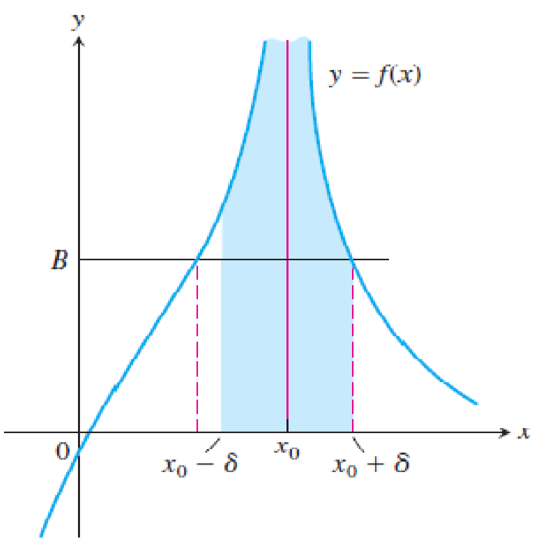
Precise definitions of infinite limits can also be formulated in the same way, as we did in the previous module. *Instead of requiring* $\mathbf{f(x)}$ *to lie arbitrarily close to a finite number* $\mathbf{L}$ *for all* $\mathbf{x}$ *sufficiently close to* $\mathbf{x}_0$*, the definitions of infinite limits require* $\mathbf{f(x)}$ *to lie arbitrarily far from the origin.*

Infinite Limits

We say that $f(x)$ ***approaches infinity*** as $\mathbf{x}$ ***approaches*** $\mathbf{x}_0$, and write

$\lim\limits_{x\to x_0} f(x) = \infty$, if for every positive real number *B* there exists a corresponding number $\delta > 0$ such that for all $\mathbf{x}$

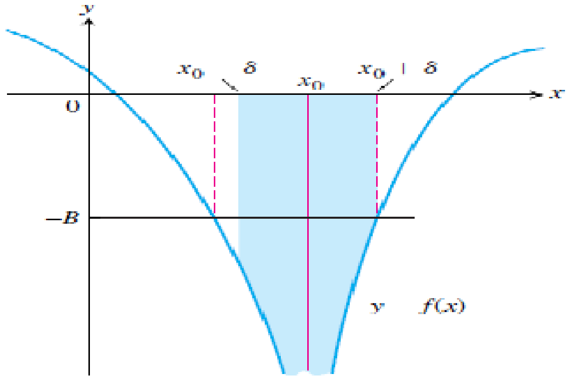
$$0 < |x - x_0| < \delta \;\Rightarrow\; \mathbf{f(x) > B}$$



We say that $\mathbf{f(x)}$ ***approaches minus infinity*** as $\mathbf{x}$ ***approaches*** $\mathbf{x}_0$, and write

$\lim\limits_{x \to x_0} f(x) = -\infty$, if for every negative real number $-B$ there exists a corresponding number $\delta > 0$ such that for all $\mathbf{x}$

$$0 < |x - x_0| < \delta \Rightarrow \quad f(x) < -B$$



PROBLEM SET

IP1: Evaluate: $\lim\limits_{x \to 0} \frac{\tan x}{x}$

Solution:

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1.1 = 1$$

P1: Evaluate: $\lim_{x \to 0} \frac{\sin 4x}{\sin 2x}$

Solution:

$$\lim_{x \to 0} \frac{\sin 4x}{\sin 2x} = \lim_{x \to 0} \left[2.\frac{\sin 4x}{4x} \cdot \frac{2x}{\sin 2x}\right]$$

$$= 2 \cdot \lim_{x \to 0} \left(\left[\frac{\sin 4x}{4x}\right] \div \left[\frac{\sin 2x}{2x}\right] \right)$$

Let $u = 4x$ *and* $v = 2x$. Now, $u \longrightarrow 0, v \longrightarrow 0$ as $x \longrightarrow 0$

$$= 2 \cdot \lim_{u \to 0} \left[\frac{\sin u}{u}\right] \div \lim_{v \to 0} \left[\frac{\sin v}{v}\right]$$

$$= 2.1.1 = 2$$

IP2: Evaluate: $\lim\limits_{x \to 0} \frac{\sin x^0}{x} =$

Solution: $\lim\limits_{x \to 0} \frac{\sin x^0}{x} =$

We have to convert degrees into radians and

$$1^o = \frac{\pi}{180} \text{ therefore, } x^o = \frac{\pi x}{180}$$

Now, $\lim\limits_{x\to 0}\frac{\sin x^0}{x} = \lim\limits_{\left(\frac{\pi x}{180}\right)\to 0}\frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}} \times \frac{\pi}{180} = 1 \times \frac{\pi}{180} = \frac{\pi}{180}$

P2: Evaluate: $\lim\limits_{x\to 0}\frac{\sin x^2}{|x|} =$

Solution: $\lim\limits_{x\to 0}\frac{\sin x^2}{|x|} =$

As $x \to 0, |x| \to 0^+ and\ |x|^2 \to 0^+$ $\lim\limits_{x\to 0}\frac{\sin x^2}{|x|} = \lim\limits_{x\to 0}\frac{\sin |x|^2}{|x|} = \lim\limits_{x\to 0}\frac{\sin |x|^2}{|x|^2} \times |x|$

put $u = |x|^2$. Now, $u \longrightarrow 0^+$ as $x \longrightarrow 0$

$$= \left[\lim_{u\to 0^+}\frac{\sin u}{u}\right] \times \left[\lim_{x\to 0}|x|\right] = 1 \times 0 = 0 \ \left(\text{since } \lim_{u\to 0}\frac{\sin u}{u} = 1 \ and \ \lim_{x\to 0}|x| = 0\right.$$

IP3: If $f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$ with $a_n > 0, b_m > 0$ then show that
$\lim\limits_{x\to\infty} f(x) = \infty$ **if $n > m$.**

Solution:

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$$

$$= \frac{x^n\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)}{x^m\left(b_m + \frac{b_{m-1}}{x} + \cdots + \frac{m_1}{x^{m-1}} + \frac{b_0}{x^m}\right)} = x^{n-m}\frac{\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)}{\left(b_m + \frac{b_{m-1}}{x} + \cdots + \frac{m_1}{x^{m-1}} + \frac{b_0}{x^m}\right)}$$

As $x \to \infty$, all the quotients $\frac{a_{n-j}}{x^j}, \frac{b_{m-i}}{x^i}$ approach to zero and $\lim\limits_{x\to\infty} x^{n-m} = \infty$

$$\therefore \lim_{x\to\infty} f(x) = \infty$$

P3: $\lim\limits_{n\to\infty}\frac{1^2+2^2+3^2+4^2+\cdots+n^2}{n^3} =$

Solution:

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+4^2+\cdots+n^2}{n^3} \Rightarrow \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} \Rightarrow \lim_{n\to\infty}\frac{\left(\frac{1}{n}+1\right)(2+\frac{1}{n})}{6}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\left(\frac{1}{n}+1\right)(2+\frac{1}{n})}{6} \Rightarrow \lim_{n\to\infty}\frac{1\times 2}{6} = \frac{1}{3}$$

IP4: Evaluate: $\lim\limits_{x\to 1^+} \dfrac{x}{x^2-1} =$

Solution: $\lim\limits_{x\to 1^+} \dfrac{x}{x^2-1}$

$$= \lim_{x\to 1^+} \frac{x}{(x-1)(x+1)}$$

$$= \lim_{x\to 1^+} x . \lim_{x\to 1^+} \frac{1}{(x+1)} . \lim_{x\to 1^+} \frac{1}{(x-1)} = \infty$$

$$\left(\text{since } \lim_{x\to 1^+} \frac{1}{(x-1)} = \infty \right)$$

P4: Evaluate: $\lim\limits_{x\to 2^+} \dfrac{1}{x^2-4} =$

Solution:

$$\lim_{x\to 2^+} \frac{1}{x^2-4}$$

$$= \lim_{x\to 2^+} \frac{1}{(x-2)(x+2)}$$

$$= \lim_{x\to 2^+} \frac{1}{(x+2)} . \lim_{x\to 2^+} \frac{1}{(x-2)}$$

Let $x-2=u$ then $u \to 0^+ \, as \; x = 2^+$

$$= \lim_{h\to 0} \frac{1}{(h+2+2)} . \lim_{h\to 0} \frac{1}{(h+2-2)} = \frac{1}{4} . \lim_{h\to 0} \frac{1}{h} = \infty$$

Exercise:

1. Find the limits of

   a) $\lim\limits_{x \to 0} \dfrac{sinx}{xcos2x}$

   b) $\lim\limits_{x \to a} \dfrac{\sin(x-a)}{(x^2-a^2)}$

   c) $\lim\limits_{x \to 0} \dfrac{\tan ax}{\sin bx}$

d) $\lim\limits_{x \to 0} \frac{x+x\cos x}{\sin x.\cos x}$

e) $\lim\limits_{x \to 1} \frac{\sin(x-1)}{(x^2-1)}$

2. Find the limits

a) $\lim\limits_{x \to \infty} \frac{11x^3-3x^2+4}{13x^3-5x-7}$

b) $\lim\limits_{x \to \infty} \frac{3x^3+4x+5}{2x^2+3x-7}$

c) $\lim\limits_{x \to -\infty} \frac{2x^2-x+3}{x^2-2x+5}$

d) $\lim\limits_{x \to -\infty} \frac{2x+3}{\sqrt{x^2-1}}$

e) $\lim\limits_{x \to \infty} \{\sqrt{x^2+ax+b}-x\}$

f) $\lim\limits_{x \to \infty} \frac{(3x-1)(2x+5)}{(x-3)(3x-7)}$

3. Find the limits

a. $\lim\limits_{x \to 0^+} \frac{1}{3x}$

b. $\lim\limits_{x \to 2^-} \frac{3}{x-2}$

c. $\lim\limits_{x \to -8^+} \frac{2x}{x+8}$

d. $\lim\limits_{x \to 7} \frac{4}{(x-7)^2}$

e. $\lim\limits_{x \to 0^+} \frac{2}{3x^{\frac{1}{3}}}$, $\lim\limits_{x \to 0^-} \frac{2}{3x^{\frac{1}{3}}}$

f. $\lim\limits_{x \to 0} \frac{4}{x^{\frac{2}{5}}}$

g. $\lim\limits_{x \to (\pi/2)^-} \tan x$

h. $\lim\limits_{\theta \to 0^-} (1+\csc\theta)$

3.7. Some Special Limits

Learning objectives:

- To prove $-|\theta| < \sin\theta < |\theta|$, $-|\theta| < 1 - \cos\theta < |\theta|$ and $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, when θ is measured in radians.
  And
- To practice the related problems.

The inequalities and limits given below are useful in establishing the derivatives of trigonometric functions.

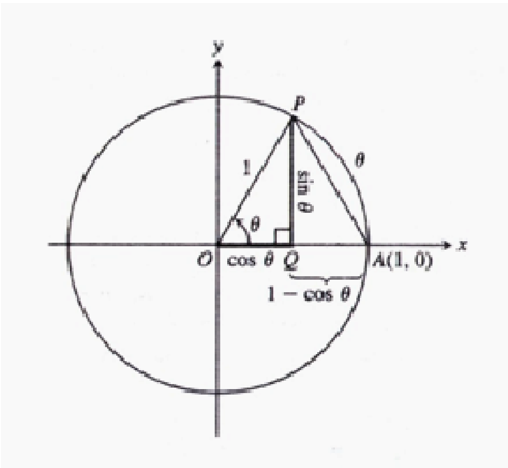
Theorem 1:

If θ is measured in radians, then

$-|\theta| < \sin\theta < |\theta|$ and $-|\theta| < 1 - \cos\theta < |\theta|$

Proof

Consider θ as an angle in standard position



The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right angled triangle with sides of length

$$QP = |\sin\theta| \quad , \quad AQ = |1 - \cos\theta|$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get,

$$\sin^2\theta + (1 - \cos\theta)^2 = AP^2 < \theta^2$$

The terms on the left side are both positive, so each is smaller than their sum and hence is less than θ^2:

$$\sin^2\theta < \theta^2 \quad , \quad (1 - \cos\theta)^2 < \theta^2$$

We take square roots.

$$|\sin\theta| < |\theta| \quad , \quad |1-\cos\theta| < |\theta|$$

This is equivalent to saying

$-|\theta| < \sin\theta < |\theta|$ and $-|\theta| < 1-\cos\theta < |\theta|$

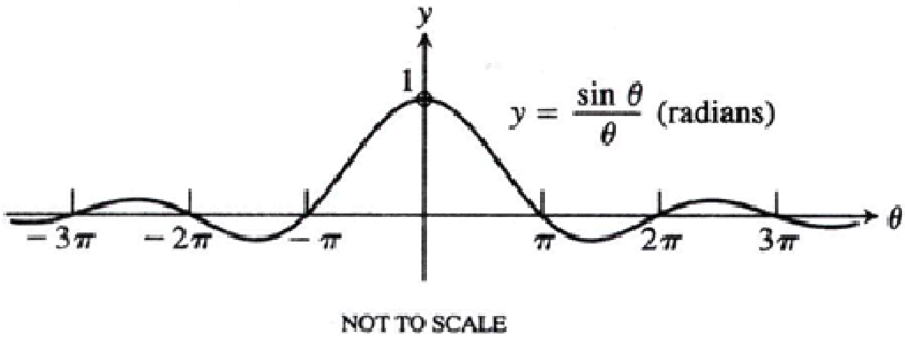
Thus the result is proved.

Example 1: Show that $\sin\theta$ and $\cos\theta$ are continuous at $\theta = 0$.

Solution

We need to show that $\lim_{\theta\to 0}\sin\theta = 0$ and $\lim_{\theta\to 0}\cos\theta = 1$. As $\theta \to 0$, both $|\theta|$ and $-|\theta|$ approach zero. The result follows immediately from the theorem 1 and the Sandwich Theorem.

The function $f(\theta) = (\sin\theta)/\theta$ graphed below appears to have a removable discontinuity at $\theta = 0$. As the figure suggests, $\lim_{\theta\to 0} f(\theta) = 1$.

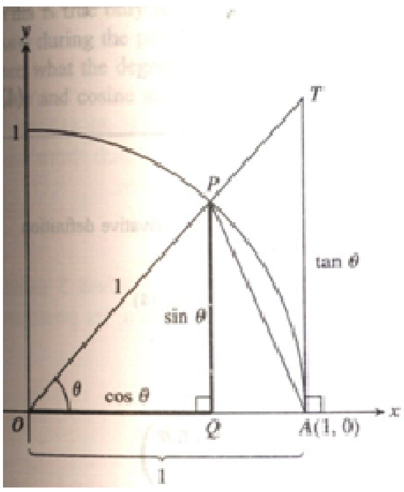


Theorem 2:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 (\theta \text{ in radians })$$

Proof:

We prove this by showing that the right-hand and left-hand limits are both 1. To show that the right-hand limit is 1, we begin with values of θ that are positive and less than $\pi/2$.



We notice that

Area of triangle OAP < area of sector OAP < area of triangle OAT

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

We divide by the positive number $(1/2)\sin\theta$.

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

We take reciprocals, which reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

We observe that $\sin\theta$ and θ are both odd functions. Therefore, $f(\theta) = (\sin\theta)/\theta$ is an even function, with a graph symmetric about the y $-$axis. This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$$

Therefore, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Thus the result is proved.

Theorem 2 can be combined with limit rules and known trigonometric identities to yield other trigonometric limits.

Example 2: Show that $\boldsymbol{\lim_{h\to 0}\frac{\cos h-1}{h} = 0}$

Solution:

Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\lim_{h\to 0}\frac{\cos h-1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta,$$

where $\theta = h/2$

$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}.\lim_{\theta\to 0}\sin\theta = -(1)(0) = 0$$

Example 3: Show that $\lim_{x\to 0}\frac{\sin 3x}{2x} = \frac{3}{2}$

Solution:

$$\lim_{x\to 0}\frac{\sin 3x}{2x} = \lim_{x\to 0}\left(\frac{3}{2}\right)\frac{\sin 3x}{3x}$$

$$= \frac{3}{2}.\lim_{\theta\to 0}\frac{\sin\theta}{\theta}, \quad \text{where } \theta = 3x$$

$$= \frac{3}{2}$$

PROBLEM SET

IP1: Evaluate $\lim_{x\to 0} \sec\left(\cos x + \pi \tan\left(\frac{\pi}{4\sec x}\right) - 1\right)$.

Solution: We have,

$$\lim_{x\to 0} \sec\left(\cos x + \pi \tan\left(\frac{\pi}{4\sec x}\right) - 1\right)$$

$$= \sec\left(\lim_{x\to 0} \cos x + \pi \lim_{x\to 0} \tan\left(\frac{\pi}{4\sec x}\right) - 1\right)$$

$$= \sec\left(1 + \pi \tan\left(\frac{\pi}{4\lim_{x\to 0} \sec x}\right) - 1\right) \qquad (\because \lim_{x\to 0} \cos x = 1)$$

$$= \sec\left(\pi \tan\left(\frac{\pi}{4}\right)\right) \qquad (\because \lim_{x\to 0} \sec x = 1)$$

$$= \sec \pi = -1$$

$$\therefore \lim_{x\to 0} \sec\left(\cos x + \pi \tan\left(\frac{\pi}{4\sec x}\right) - 1\right) = -1$$

P1: $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) =$

Solution: $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) = \cos\left(\lim_{\theta\to 0}\left(\frac{\pi\theta}{\sin\theta}\right)\right)$

$$= \cos\left(\pi \lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right)\right)$$

$$= \cos\left(\pi . \frac{1}{\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)}\right)$$

$$= \cos \pi = -1 \qquad \left(\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right)$$

$$\therefore \lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) = -1$$

IP2: Evaluate $\lim_{x\to 0} \frac{5x\cos x + 3\sin x}{3x^2 + \sin x}$.

Solution: We have,

$$\lim_{x\to 0} \frac{5x\cos x + 3\sin x}{3x^2 + \sin x}$$

$$= \lim_{x\to 0} \frac{x\left(5\cos x + 3\frac{\sin x}{x}\right)}{x\left(3x + \frac{\sin x}{x}\right)}$$

$$= \lim_{x\to 0} \frac{5\cos x + 3\frac{\sin x}{x}}{3x + \frac{\sin x}{x}} = \frac{5.\lim_{x\to 0} \cos x + 3.\lim_{x\to 0} \frac{\sin x}{x}}{3.\lim_{x\to 0} x + \lim_{x\to 0} \frac{\sin x}{x}}$$

$$= \frac{5(1) + 3(1)}{3(0) + 1} = 8 \qquad \left(\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right)$$

$$\therefore \lim_{x\to 0} \frac{5x\cos x + 3\sin x}{3x^2 + \sin x} = 8$$

P2: $\lim_{x\to 0}\frac{x\cos x+2\sin x}{x^2+\sin x} =$

Solution: We have,

$$\lim_{x\to 0}\frac{x\cos x+2\sin x}{x^2+\sin x}$$

$$= \lim_{x\to 0}\frac{x\left(\cos x+2\frac{\sin x}{x}\right)}{x\left(x+\frac{\sin x}{x}\right)}$$

$$= \lim_{x\to 0}\frac{\cos x+2\frac{\sin x}{x}}{x+\frac{\sin x}{x}} = \frac{\lim_{x\to 0}\cos x+2.\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}x+\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{1+2(1)}{0+1} = 3 \qquad \left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right)$$

$$\therefore \lim_{x\to 0}\frac{x\cos x+2\sin x}{x^2+\sin x} = 3$$

IP3: Evaluate $\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$.

Solution: We have,

$$\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$$

$$= \lim_{x\to 0}\left[\frac{1-\cos x\sqrt{\cos 2x}}{x^2}\times\frac{1+\cos x\sqrt{\cos 2x}}{1+\cos x\sqrt{\cos 2x}}\right]$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x(\cos 2x)}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x(2\cos^2 x-1)}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$$

$$= \lim_{x\to 0}\frac{1-2\cos^4 x+\cos^2 x}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$$

$$= \lim_{x\to 0}\frac{(1-\cos^2 x)(1+2\cos^2 x)}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$$

$$= \lim_{x\to 0}\frac{\sin^2 x\,(1+2\cos^2 x)}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$$

$$= \lim_{x\to 0}\frac{\sin^2 x\,(1+2\cos^2 x)}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times\lim_{x\to 0}\frac{(1+2\cos^2 x)}{\left(1+\cos x\sqrt{\cos 2x}\right)}$$

$$= \frac{1+2}{1+1} = \frac{3}{2}$$

$$\therefore \lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2} = \frac{3}{2}$$

P3: $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x} =$

A. $\mathbf{0}$
B. $\mathbf{1}$

C. **2**

D. **3**

Answer: B

Solution:

We have,

$$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$$

$$=\lim_{x\to 0}\left[\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right]$$

$$=\lim_{x\to 0}\frac{(1+\sin x)-(1-\sin x)}{x\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\lim_{x\to 0}\frac{2\sin x}{x\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=2\lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\frac{1}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=2\left(\frac{1}{1+1}\right)=1$$

$$\therefore \lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}=1$$

IP4:

Evaluate $\boldsymbol{\lim_{x\to 0}\frac{\cot 2x-\csc 2x}{x}}$**.**

Solution:

We have,

$$\lim_{x\to 0}\frac{\cot 2x-\csc 2x}{x}$$

$$=\lim_{x\to 0}\frac{\frac{\cos 2x}{\sin 2x}-\frac{1}{\sin 2x}}{x}$$

$$=\lim_{x\to 0}\frac{\cos 2x-1}{x\sin 2x}$$

$$=\lim_{x\to 0}\frac{-2\sin^2 x}{x\,(2\sin x\cos x)}$$

$$=-\lim_{x\to 0}\frac{\sin x}{x\cos x}$$

$$=-\lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{1}{\cos x}=-1$$

$$\therefore \lim_{x\to 0}\frac{\cot 2x-\csc 2x}{x}=-1$$

P4:

$$\boldsymbol{\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=}$$

Solution: We have,

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left(\frac{\sin x}{\cos x}-\sin x\right)$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left(\frac{\sin x-\sin x\cos x}{\cos x}\right)$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}$$

$$=\lim_{x\to 0}\left[\frac{\sin x(1-\cos x)}{x^3\cos x}\times\frac{1+\cos x}{1+\cos x}\right]$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos^2 x)}{x^3\cos x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin x\sin^2 x}{x^3\cos x(1+\cos x)}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3\times\lim_{x\to 0}\frac{1}{\cos x(1+\cos x)}=\frac{1}{2}$$

$$\therefore \lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\frac{1}{2}$$

Exercise:

1. Find the limits
   a. $\lim_{x\to 2}\sin\left(\frac{1}{x}-\frac{1}{2}\right)$
   b. $\lim_{x\to 0}\sec\left[\cos x+\pi\tan\left(\frac{\pi}{4\sec x}\right)-1\right]$
   c. $\lim_{t\to 0}\tan\left(1-\frac{\sin t}{t}\right)$
   d. $\lim_{x\to 0}\frac{1+\sin x}{1+\cos x}$
2. Find the limits
   a. $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$
   b. $\lim_{x\to 0}\frac{1-\cos 2x}{x}$
   c. $\lim_{x\to 0}\frac{\sin x\cos x}{3x}$
   d. $\lim_{x\to 0}\frac{3\sin x-4\sin^3 x}{x}$
   e. $\lim_{x\to 0}\frac{2x-\sin x}{\tan x+x}$
   f. $\lim_{x\to 0}\frac{x^2+1-\cos x}{x\sin x}$
   g. $\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{x^2}$
   h. $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}$
   i. $\lim_{x\to 0}\frac{x\tan x}{1-\cos 2x}$
   j. $\lim_{x\to 0}\frac{\csc x-\cot x}{x}$

k. $\lim\limits_{x\to 0}\frac{x\cos x+\sin x}{x^2+\tan x}$

l. $\lim\limits_{x\to 0}\frac{ax+x\cos x}{\sin x}$

m. $\lim\limits_{x\to 0}\frac{\sec x-1}{x^2}$

n. $\lim\limits_{x\to 0}\frac{\sec^2 x-1}{\cot x-1}$

o. $\lim\limits_{x\to 0}\frac{1-\cos^3 x}{\sin^2 x}$

3.8. Continuity at a Point

Learning objectives:

- To study the concept of continuity of a function at a point and to present continuity test
- To study the types of discontinuities through examples

  And

- To practice related problems

A continuous function is a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Several physical processes proceed continuously, and they are represented by functions of a real variable and have domains that are intervals or unions of separate intervals.

We study the continuity of a function at a point. There are three kinds of points to consider: ***interior points, left endpoint(s),*** and ***right endpoint(s)***.

Definition: continuity at a point

A function f is ***continuous at an interior point*** $x=c$ of its domain if

$$\lim_{x\to c} f(x)=f(c)$$

Continuity at end points is defined by taking one-sided limits.

A function f is **continuous at a left endpoint** $x=a$ of its domain if

$$\lim_{x\to a^+} f(x)=f(a)$$

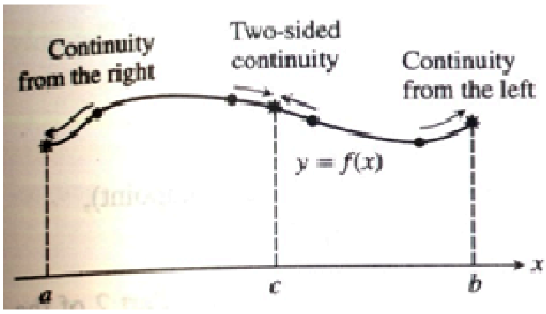
and **continuous at a right endpoint** $x=b$ of its domain if

$$\lim_{x\to b^-} f(x)=f(b)$$

In general, a function f is **right-continuous** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous** at c if $\lim_{x \to c^-} f(x) = f(c)$.

Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b.

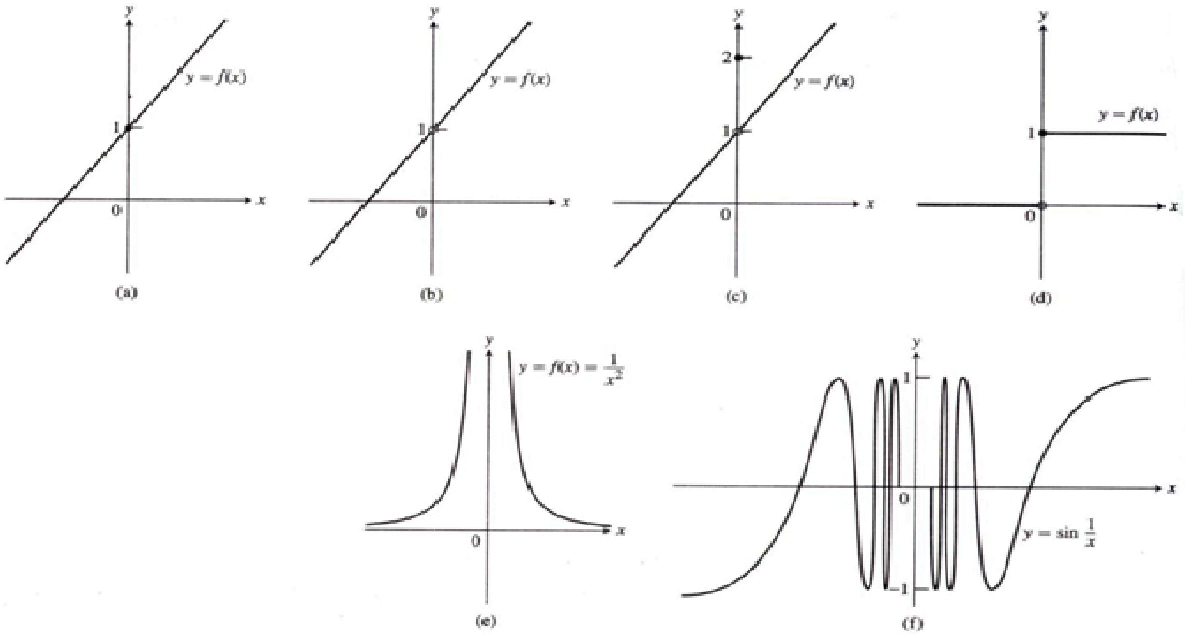
A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c.



If a function f is not continuous at a point c, then we say that f is ***discontinuous*** at c and c is called a ***point of discontinuity***

of f.

Types of discontinuities:



The function in (a) is continuous at $x = 0$.

The function in (b) would be continuous if it had $f(0) = 1$.

The function in (c) would be continuous if $f(0)$ were 1 instead of 2.

The discontinuities in (b) and (c) are ***removable.*** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

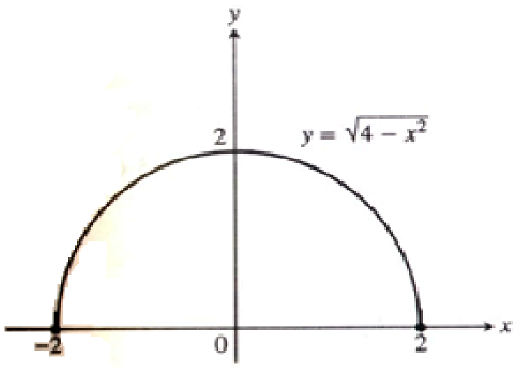
The discontinuities in parts (d) to (f) are of different nature: $\lim_{x \to 0} f(x)$ does not exist.

The step function in (d) has a **jump discontinuity**: the one-sided limits exist but have different values.

The function $f(x) = \frac{1}{x^2}$ in (e) has an **infinite discontinuity**. These discontinuities are the ones most frequently encountered in applications. The function in (f) has an **oscillating discontinuity** at the origin because it oscillates too much to have limit as $x \to 0$.

Example 1: *A function continuous through out its domain.*

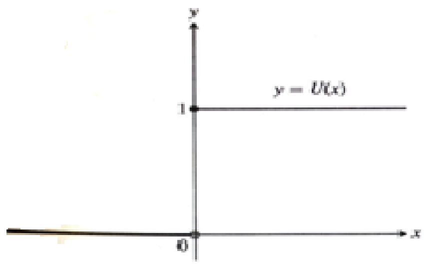
The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$.



This includes $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous.

Example 2: *The unit step function has jump discontinuity*.

The unit step function is graphed below.



It is right-continuous at $x = 0$, but is neither left-continuous there nor continuous at $x = 0$. It has a jump discontinuity at $x = 0$.

Continuity Test

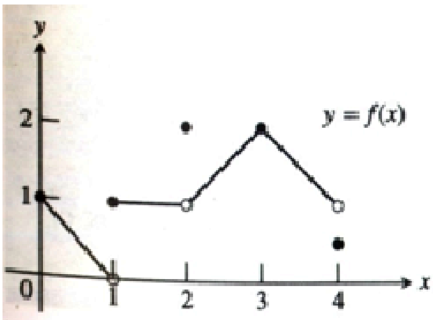
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

Example 3:

Consider the function $y = f(x)$, in the figure below, whose domain is the closed interval $[0, 4]$. Discuss the continuity of f at $x = 0, 1, 2, 3, 4$.



Solution: f is continuous at $x = 0$ because $f(0)$ exists and

$\lim_{x \to 0^+} f(x) = 1 = f(0)$.

f is discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist; f has different right- and left- hand limits at the interior point $x = 1$. However, f is right continuous at $x = 1$ because $f(1)$ exists, $\lim_{x\to 1^+} f(x) = 1$, and this equals the function value.

Note that $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1^-} f(x) = 0$. There fore $x = 1$ is a point of discontinuity and it is a jump discontinuity

f is discontinuous at $x = 2$ because $\lim_{x\to 2} f(x) \neq f(2)$.

Therefore $x = 2$ is a removable discontinuity, by setting $\lim_{x\to 2} f(x) = 1$.

f is continuous at $x = 3$ because $f(3)$ exists, $\lim_{x\to 3} f(x) = 2$, and this is equal to the function value.

f is discontinuous at the right endpoint $x = 4$ because $\lim_{x\to 4^-} f(x) \neq f(4)$.

PROBLEM SET

IP1: The function f given by

$$f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2 - 8x^{-3} & \text{if } x > 2 \end{cases}$$

Discuss the continuity at $x = 2$. Name the discontinuity if it is discontinuous at $x = 2$.

Solution: The Given function is

$$f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & if\ 0 < x < 2 \\ 0 & if\ x = 2 \\ 2 - 8x^{-3} & if\ x > 2 \end{cases}$$

Now, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2 - 8x^{-3})$

$$= \lim_{h \to 0} (2 - 8(2+h)^{-3})$$

$$= \lim_{h \to 0} \left(2 - \frac{8}{8 + 12h + 6h^2 + h^3}\right) = 1$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{2}(x^2 - 4)$$

$$= \lim_{h \to 0} \frac{1}{2}((2-h)^2 - 4)$$

$$= \lim_{h \to 0} \frac{1}{2}(-4h + h^2) = 0$$

$\Longrightarrow f(x)$ is discontinuity at $x = 2$ and $x = 2$ is a jump discontinuity of $f(x)$.

P1: Examine the continuity at $x = 0, 1, 2$ of the function f defined as

$$f(x) = \begin{cases} -x & , \quad x \le 0 \\ 5x - 4 & , \quad 0 < x \le 1 \\ 4x^2 - 3x, & \quad 1 < x < 2 \\ 3x + 4 & , \quad x \ge 2 \end{cases}$$

Solution:

The behavior of $f(x)$ at $x = 0$:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5x - 4)$$

$$= \lim_{h \to 0} (5(0+h) - 4) = \lim_{h \to 0} 5h - 4 = -4$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x)$$

$$= \lim_{h \to 0} -(0-h) = \lim_{h \to 0} h = 0$$

$\Longrightarrow f(x)$ is discontinuous at $x = 0$ and $x = 0$ is a jump discontinuity of $f(x)$.

Continuity at $x = 1$:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x^2 - 3x))$$

$$= \lim_{h\to 0}(4(1+h)^2 - 3(1+h))$$

$$= \lim_{h\to 0}(4h^2 + 5h + 1) = 1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(5x-4)$$

$$= \lim_{h\to 0} 5(1-h) - 4 = 1$$

Further, $f(1) = 1$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$\Longrightarrow f(x)$ is continuous at $x = 1$

Continuity at $x = 2$:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3x+4) = 10$$

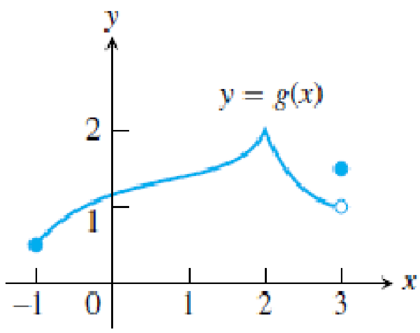
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4x^2 - 3x) \ = 10$$

Further, $f(2) = 3(2) + 4 = 10$

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)] = f(2)$$

$\Longrightarrow f(x)$ is continuous at $x = 2$

IP2: Discuss the continuity at the end points of the following:



Solution:

Notice that $g(-1)$ exists, and $g(-1) = 0.5$ and

$$\lim_{x\to -1^+} g(x) = 0.5$$

Now $\lim_{x\to -1^+} g(x) = g(-1)$. Since $x = -1$ is the left end point, $g(x)$ is continuous at $x = -1$.

Now, $\lim_{x\to 3^-} g(x) = 1$ and $g(3) = 1.5$

$$\therefore \lim_{x\to 3^-} g(x) \neq g(3)$$

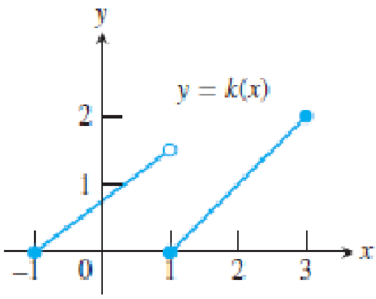
Therefore $g(x)$ is discontinuous at $x = 3$

(since $x = 3$ is left end point)

Note:

$x = 3$ is a removable discontinuity by redefining $g(3)$ as $\lim_{x\to 3^-} g(x)$ $i.e.$, 1.

P2: Discuss the continuity at $x = 1$ for the following:



Solution:

From the given graph $k(x)$ is defined at $x = 1$ and $k(1) = 0$

$$\lim_{x\to 1^-} k(x) = 1.5$$

$$\lim_{x\to 1^+} k(x) = 0$$

$$\lim_{x\to 1^-} k(x) \neq \lim_{x\to 1^+} k(x)$$

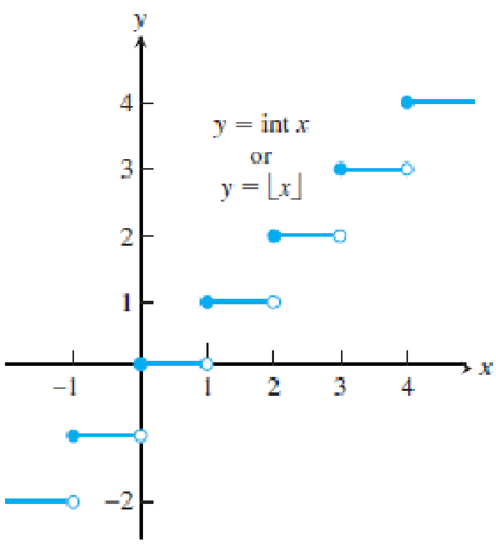
Thus, $k(x)$ is discontinuous at $x = 1$ and $x = 1$ is a jump discontinuity.

But $\lim_{x\to 1^+} k(x) = 0 = k(1)$. Therefore $k(x)$ is right continuous at $x = 1$.

IP3: Discuss the continuity of $f(x) = \lfloor x \rfloor$

Solution:

The given function is $f(x) = \lfloor x \rfloor$. $y = \lfloor x \rfloor$ is graphed below



$f(x)$ is discontinuous at every integer because the limit does not exists at an integer n

$$\lim_{x\to n^-} \lfloor x \rfloor = n - 1$$

And

$$\lim_{x\to n^+} \lfloor x \rfloor = n$$

So the left hand limit and right hand limit are not equal as $x \to n$.

Since $\lfloor n \rfloor = n$, the greatest integer function is right continuous at every integer n.

Therefore, the greatest integer function is continuous at every real number other than the integers.

P3: Examine the continuity of

$$\boldsymbol{f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}}$$ **at the origin.**

Solution: The given function is

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Now, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{|x|}{x}$

$$= \lim_{h\to 0} \frac{|0+h|}{0+h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x}$$

$$= \lim_{h\to 0} \frac{|0-h|}{0-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$\Longrightarrow f(x)$ is discontinuous at $x = 0$ and $x = 0$ is a jump discontinuity of $f(x)$.

IP4: If $\boldsymbol{f(x) = \frac{x^2-9}{x^2-2x-3}}$ for $\boldsymbol{x \neq 3}$ and $\boldsymbol{f(x)}$ is continuous at $\boldsymbol{x = 3}$. Then find the value of $\boldsymbol{f(3)}$.

Solution: The given function is

$$f(x) = \frac{x^2-9}{x^2-2x-3} = \frac{(x+3)(x-3)}{(x+1)(x-3)} = \frac{(x+3)}{(x+1)}, x \neq 3$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{(x+3)}{(x+1)} = \lim_{h\to 0} \frac{(3+h+3)}{(3+h+1)} = \frac{3}{2}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{(x+3)}{(x+1)} = \lim_{h\to 0} \frac{(3-h+3)}{(3-h+1)} = \frac{3}{2}$$

$$\therefore \lim_{x\to 3} f(x) = \frac{3}{2}$$

Given $f(x)$ is continuous at $x = 3$

$$\Longrightarrow \lim_{x\to 3} f(x) = f(3) \Longrightarrow \frac{3}{2} = f(3)$$

$$\therefore f(3) = \frac{3}{2}$$

P4: If $f(x)=\frac{x^2-10x+25}{x^2-7x+10}$ for $x\neq 5$ and f is continuous at $x=5$. Then find the value of $f(5)$.

Solution: The Given function is

$$f(x)=\frac{x^2-10x+25}{x^2-7x+10}=\frac{(x-5)(x-5)}{(x-2)(x-5)}=\frac{(x-5)}{(x-2)},x\neq 5$$

$$\lim_{x\to 5^+}f(x)=\lim_{x\to 5^+}\frac{(x-5)}{(x-2)}=\lim_{h\to 0}\frac{(5+h-5)}{(5+h-2)}=0$$

$$\lim_{x\to 5^-}f(x)=\lim_{x\to 5^-}\frac{(x-5)}{(x-2)}=\lim_{h\to 0}\frac{(5-h-5)}{(5-h-2)}=0$$

$\therefore\ \lim_{x\to 5}f(x)=0$

Given $f(x)$ is continuous at $x=5$

$\Longrightarrow \lim_{x\to 5}f(x)=f(5)\ \Longrightarrow 0=f(5)$

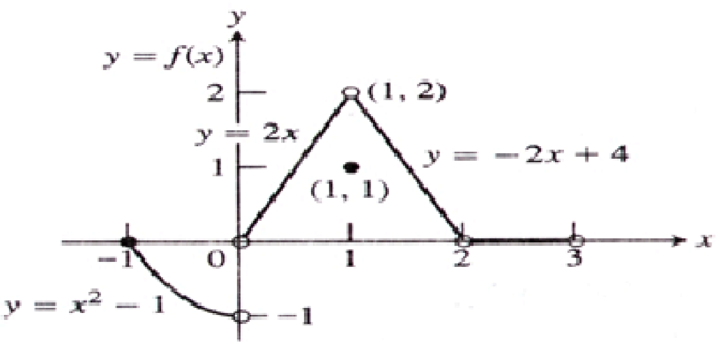
$\therefore f(5)=0$

Exercise:

1. The function

$$f(x)=\begin{cases} x^2-1 & -1\leq x<0 \\ 2x & 0<x<1 \\ 1 & x=1 \\ -2x+4 & 1<x<2 \\ 0 & 2<x<3 \end{cases}$$

is graphed below.



a. Does $f(-1)$ exist?

b. Does $\lim_{x\to 1^+}f(x)$ exist?

c. Does $\lim_{x\to 1^+}f(x)=f(-1)$?

d. Is f continuous at $x=-1$?

e. Is f defined at $x=2$?

f. Is f continuous at $x = 2$?

2. Is function f defined by $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & if \;\; x \neq 0 \\ 1 & if \;\; x = 0 \end{cases}$ continuous at $x = 0$.

3. Check the continuity of f given by $f(x) = \begin{cases} 4 - x^2 & if \;\; x \leq 0 \\ x - 5 & if \;\; 0 \leq x \leq 1 \\ 4x^2 - 9 & if \;\; 1 < x < 2 \\ 3x + 4 & if \;\; x \geq 2 \end{cases}$ at the points 0,1 and 2.

4. If f is function defined by $f(x) = \begin{cases} \dfrac{x-1}{\sqrt{x}-1} & if \; x > 1 \\ 5 - 3x & if \; -2 \leq x \leq 1 \\ \dfrac{6}{x-10} & if \; x < -2 \end{cases}$ then discuss the continuity of f.

5. If $f(x) = \frac{x^2-1}{x-1}$. Discuss the continuity at $x = 1$.

6. At what points are the functions continuous?

a. $y = \dfrac{1}{x-2} - 3x$

b. $y = \dfrac{x+1}{x^2 - 4x + 3}$

c. $y = |x-1| + \sin x$

d. $y = \dfrac{\sin x}{x}$

e. $y = \csc 2x$

f. $y = \dfrac{x \tan x}{x^2 + 1}$

g. $y = \sqrt{2x+3}$

h. $y = (2x-1)^{1/3}$

3.9. Rules of Continuity

Learning objectives:

- To state the properties of continuous functions.
- To study the continuity of polynomials, rational functions, absolute value function and trigonometric functions.
- To define the continuous extension of a function to a point.

  And

- To practice related problems.

Algebraic combinations of continuous functions are continuous wherever they are defined

Theorem: Continuity of Algebraic Combinations

If functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $f + g$ and $f - g$
2. fg
3. kf , where k is any number
4. $\frac{f}{g}$, provided $g(c) \neq 0$
5. $\left(f(x)\right)^{\frac{m}{n}}$, m and n are integers, $n \neq 0$.

As a consequence, polynomials and rational functions are continuous at every point where they are defined.

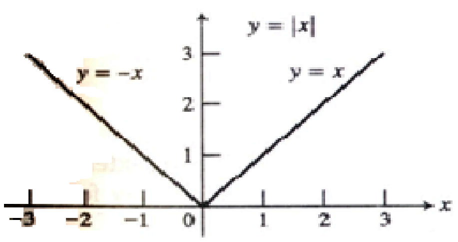
Theorem: Continuity of Polynomials and Rational Functions

Every polynomial is continuous at every point of the real line. Every rational function is continuous at every point where its denominator is different from zero.

Example 1: The functions $f(x) = x^4 + 20$ and $g(x) = 5x(x-2)$ are continuous at every value of x. The function $r(x) = \frac{x^4+20}{5x(x-2)}$ is continuous at every value of x except $x = 0$ and $x = 2$, where the denominator is 0.

Example 2: Continuity of $f(x) = |x|$

The function $f(x) = |x|$ is continuous at every value of x.



If $x > 0$, we have $f(x) = x$ is a polynomial.

If $x < 0$, we have $f(x) = -x$ is another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$.

Example 3:

We will later show that the functions $\sin x$ and $\cos x$ are continuous at every value of x. It then follows that the quotients

$$\tan x = \frac{\sin x}{\cos x} \quad , \quad \cot x = \frac{\cos x}{\sin x}$$

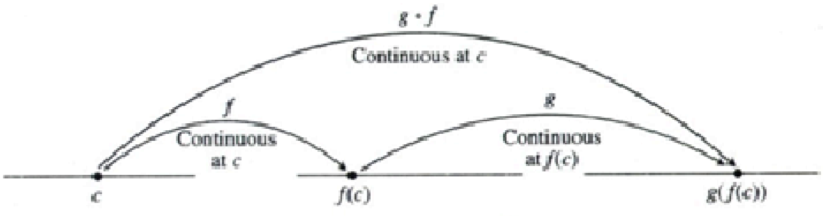
$$\sec x = \frac{1}{\cos x} \quad , \quad \csc x = \frac{1}{\sin x}$$

are continuous at every point where they are defined.

Continuity of Composites:

Theorem:

If f is continuous at c, and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied.

Example 4: The following functions are continuous everywhere on their respective domains.

a) $y = \sqrt{x}$
b) $y = \sqrt{x^2 - 2x - 5}$
c) $y = \frac{x \cos\left(x^{\frac{2}{3}}\right)}{1+x^4}$
d) $y = \left|\frac{x-2}{x^2-2}\right|$

Continuous Extension to a Point

If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } \ x = c \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension** of f to $x = c$. For rational functions f, continuous extensions are usually found by canceling common factors.

Example 5: Show that $f(x) = \frac{x^2+x-6}{x^2-4}$ has a continuous extension to $x = 2$, and find that extension.

Solution:

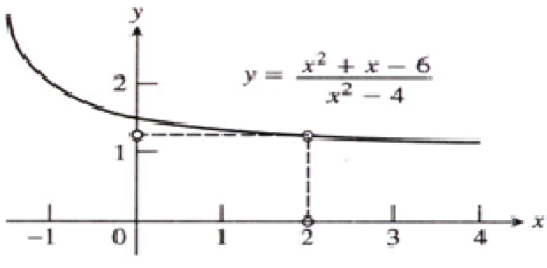
Notice that $f(2)$ is not defined. If $x \neq 2$, we have

$$f(x) = \frac{x^2+x-6}{x^2-4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

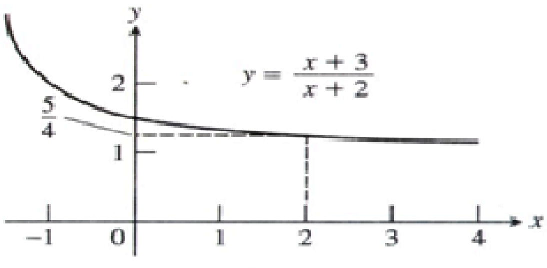
The function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$. It is continuous at $x = 2$, having the value of $5/4$. Thus F is the continuous extension of f to $x = 2$. The graphs of f and F are shown below.



(a)



(b)

The continuous extension F has the same graph except with no hole at $(2, 5/4)$.

PROBLEM SET

IP1: Every polynomial is continuous at every point c of the real line.

Proof:

Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_i \epsilon \boldsymbol{R}$,

$i = 0,1,2, \dots, n$, $a_n \neq 0$ be any polynomial. Let c be any real number. We know that $\lim_{x\to c} x = c$ and $\lim_{x\to c} k = k$.

Therefore,

$$\lim_{x\to c} p(x) = \lim_{x\to c}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$

$$= \lim_{x\to c} a_0 + a_1 \lim_{x\to c} x + a_2 \left(\lim_{x\to c} x\right)^2 + \cdots + a_n \left(\lim_{x\to c} x\right)^n$$

(By the properties of limits)

$$= a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n$$
$$= p(c)$$

Thus $p(x)$ is continuous at every $c \in \boldsymbol{R}$.

Hence the result

P1: If the functions f and g are continuous at $x = c$ then $f \pm g$ are continuous at $x = c$.

Proof: Given f and g are continuous functions at $x = c$

$i.e., \lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$

Now, $\lim_{x \to c} (f + g)(x) = \lim_{x \to c} \big(f(x) + g(x)\big)$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

(sum rule for limits)

$$= f(c) + g(c)$$

(since f, g are continuous at c)

$$= (f + g)(c)$$

Therefore, $f + g$ is continuous at $x = c$.

Similarly $f - g$ is continuous at $x = c$.

Hence the result

IP2: Prove that $\sin x$, $\cos x$ are continuous everywhere.

Proof: We first note that

$$\lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1$$

Let $f(x) = \sin x$. Let c be any real number. Then,

$$\lim_{x \to c^+} f(x) = \lim_{h \to 0} f(c + h)$$
$$= \lim_{h \to 0} \sin(c + h)$$
$$= \lim_{h \to 0} \sin c \cos h + \lim_{h \to 0} \cos c \sin h$$
$$= \sin c \lim_{h \to 0} \cos h + \cos c \lim_{h \to 0} \sin h$$
$$= \sin c$$

$$\lim_{x \to c^-} f(x) = \lim_{h \to 0} f(c - h)$$
$$= \lim_{h \to 0} \sin(c - h)$$
$$= \lim_{h \to 0} \sin c \cos h - \lim_{h \to 0} \cos c \sin h$$

$$= \sin c \lim_{h\to 0} \cos h - \cos c \lim_{h\to 0} \sin h$$

$$= \sin c$$

$$\Longrightarrow \lim_{x\to c^+} \sin x = \lim_{x\to c^-} \sin x = \sin c$$

$\Longrightarrow \lim\limits_{x\to c} \sin x = \sin c$. Therefore, $\sin x$ is continuous at c.

Thus, $\sin x$ is continuous everywhere (since c is any real number).

Similarly $\cos x$ is continuous everywhere.

P2: Every rational function is continuous wherever it is defined.

Proof: Let the rational function be $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Let c be any real number such that $q(c) \neq 0$. Then,

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{\lim_{x\to c} p(x)}{\lim_{x\to c} q(x)} \quad \text{(Quotient rule for limits)}$$

$$= \frac{p(c)}{q(c)}$$

(since polynomials are continuous everywhere)

Therefore, $\frac{p(x)}{q(x)}$ is continuous at $x = c$. Thus every rational function is continuous wherever it is defined.

Hence the result

IP3: Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Solution: The given function is $f(x) = |\cos x|$

Define g by $g(x) = \cos x$ and h by $h(x) = |x|$

$$(hog)(x) = h\big(g(x)\big)$$

$$= h(\cos x)$$

$$= |\cos x|$$

$$= f(x)$$

Notice that h and g are continuous everywhere. Since f is the composite of two functions g and h; f is continuous everywhere.

P3: Prove that the function $f(x) = \big|1 - x + |x|\big|$ is continuous everywhere.

Solution: The given function is $f(x) = \big|1 - x + |x|\big|$

Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$, for all real x.

Now ,

$$(hog)(x) = h\big(g(x)\big)$$

$$= h(1 - x + |x|)$$

$$= |1 - x + |x||$$
$$= f(x)$$

Since $|x|$ is a continuous function for all x, $h(x)$ is continuous everywhere.

Since $g(x)$ is a sum of a polynomial function and the modulus function, $g(x)$ is continuous everywhere.

Since $f(x)$ is a composite of two everywhere continuous functions $g(x)$ and $h(x)$; $f(x)$ is continuous everywhere.

IP4: Show that $g(x) = \frac{x^2-16}{x^2-3x-4}$ has a continuous extension to $x = 4$. Find that extension.

Solution: The given function is $g(x) = \frac{x^2-16}{x^2-3x-4}$

At $x = 4$, $g(x)$ is not defined. If $x \neq 4$ we have

$$g(x) = \frac{x^2-16}{x^2-3x-4} = \frac{(x+4)(x-4)}{(x-4)(x+1)} = \frac{x+4}{x+1}$$

Let, $G(x) = \frac{x+4}{x+1}$ and $G(x) = g(x)$ at $x \neq 4$

Since $G(x)$ is a rational function with denominator not vanishing at $x = 4$;

$G(x)$ is continuous at $x = 4$ and has the value $\frac{8}{5}$ at $x = 4$.

Thus, G is the continuous extension of g to $x = 4$ and

$$\lim_{x\to 4}\frac{x^2-16}{x^2-3x-4} = \lim_{x\to 4} g(x) = \frac{8}{5}$$

P4: Show that $h(t) = \frac{t^2+3t-10}{t-2}$ has a continuous extension to $t = 2$, and find that extension.

Solution:

The given function is $h(t) = \frac{t^2+3t-10}{t-2}$. Notice that $h(t)$ is not defined at $t = 2$.
If $t \neq 2$, we have

$$h(t) = \frac{t^2+3t-10}{t-2} = \frac{(t+5)(t-2)}{t-2} = t + 5$$

Let $H(t) = t + 5$ and $H(t) = h(t)$ at $t \neq 2$.

Since $H(t)$ is a polynomial; $H(t)$ is continuous at $t = 2$ and has the value 7 at $t = 2$.

Thus, $H(t)$ is the continuous extension of $h(t)$ to $t = 2$ and

$$\lim_{t\to 2} h(t) = 7$$

Exercise:

7. Find the limits. Are the functions continuous at the point being approached?

   a. $\lim_{x\to\pi} \sin(x - \sin x)$

   b. $\lim_{y\to 1} \sec(y\sec^2 y - \tan^2 y - 1)$

   c. $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$

   d. $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$

   e. $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{\frac{1}{3}}\right)\right)$

   f. $\lim_{x\to\frac{\pi}{6}} \sqrt{\cos^2 x + 5\sqrt{3}\tan x}$

8. Find the continuous extension of

   a. $g(x) = \frac{x^2-9}{x-3}$ to $x = 3$

   b. $f(s) = \frac{s^3-1}{s^2-1}$ to $s = 1$

   c. $g(x) = \frac{x^2-16}{x^2-3x-4}$ to $x = 4$

   d. $h(t) = \frac{(t^2+3t-10)}{t-2}$ to $t = 2$

   e. $f(x) = \frac{x^2-9}{x^2-x-12}$ to $x = -3$

9. For what value of a is $f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases}$ continuous at every .

10. Given that the function f defined by

$$f(x) = \begin{cases} 2x - 1 & if\ x > 2 \\ k & if\ x = 2 \\ x^2 - 1 & if\ x < 2 \end{cases}$$ is continuous at

every x. Then find the value of k.

11. For what values of b is $g(x) = \begin{cases} x & if\ x < -2 \\ bx^2 & if\ x \geq 2 \end{cases}$ continuous at every x.

12. Prove that the function $f(x) = \sqrt{1 + \sqrt{2x+1}}$ is continuous at $x = 2$.

13. Prove that the function $g(x) = \cos(3t + 4)$ is continuous at every real number.

3.10. Continuity on Intervals

Learning objectives:

- To define continuity of a function on its domain.
- To study intermediate value theorem and its application to assert the existence of a zero of a function.

  And

- To practice the related problems.

A function is called **continuous** if it is continuous every where in its domain.

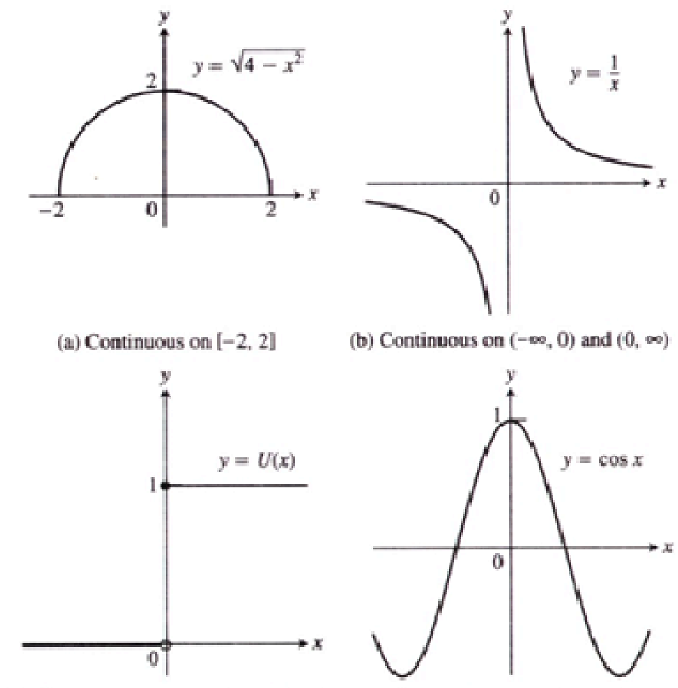
A function that is not continuous throughout its entire domain may be continuous when restricted to particular intervals within the domain.

A function f is said to be ***continuous on an interval*** I in its domain if $\lim_{x\to c} f(x) = f(c)$ at every interior point c and if the appropriate one-sided limits equal the function values at the endpoints.

A function continuous on an interval I is automatically continuous on any interval contained in I.

Polynomials are continuous on every interval, and rational functions are continuous on every interval on which they are defined.

Example 1: Functions continuous on intervals



(a) Continuous on [−2, 2] (b) Continuous on (−∞, 0) and (0, ∞)

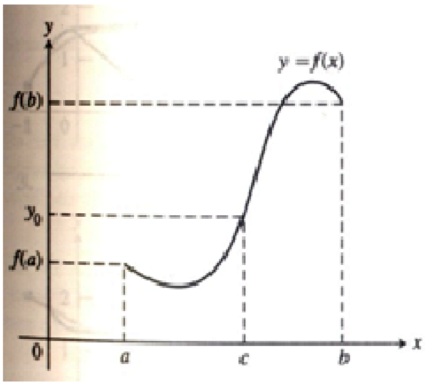
(c) Continuous on (−∞, 0) and [0, ∞) (d) Continuous on (−∞, ∞)

Functions that are continuous on intervals have properties that make them particularly useful in applications. One of these is the intermediate value property.

A function is said to have the ***intermediate value property*** if whenever it takes on two values, it also takes on all the values in between.

Theorem: The Intermediate Value Theorem

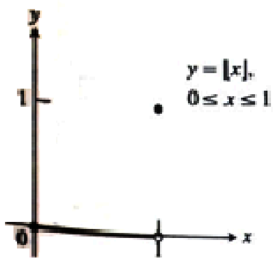
Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$.



The function f, being continuous on $[a, b]$, takes on every value between $f(a)$and $f(b)$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system.

The continuity of $f$on I is essential to the theorem. If $f$is discontinuous even at one point of f, the theorem does not apply. For example, it will not apply for the function graphed below.



The function $f(x) = [x], 0 \leq x \leq 1$, does not take on any value between $f(0) = 0$ and $f(1) = 1$.

The above theorem is the reason for the graph of a function continuous on an interval I cannot have any breaks. It will be connected, a single, unbroken curve, like the graph of $\sin x$.

It will not have jumps like the graph of the greatest integer function $[x]$ or separate branches like the graph of $\frac{1}{x}$.

We call a solution of the equation $f(x) = 0$ a **root** or **zero** of the function f. The Intermediate Value Theorem tells the following:

If f is continuous, then any interval on which f changes sign must contain a zero of the function.

Example 2: Is any real number exactly 1 less than its cube?

Solution: Any such number must satisfy the equation

$$x = x^3 - 1$$
$$i.e., \; x^3 - x - 1 = 0$$

Hence we are looking for zeros of $f(x) = x^3 - x - 1$. By trial, we find that $f(1) = -1$ and $f(2) = 5$. Then, by the Intermediate Value Theorem, there is at least one number in $[1,2]$ where f is zero. The answer to the question is then "yes".

PROBLEM SET

IP1: If $\boldsymbol{f(x) = \begin{cases} x^2 + ax + b & \text{for } 0 \le x < 2 \\ 3x + 2 & \text{for } 2 \le x \le 4 \\ 2ax + 5b & \text{for } 4 < x < 8 \end{cases}}$

is continuous on $[0, 8]$, then find the values of a and b.

Solution: Since f is continuous on $[0,8]$, it is continuous at $x = 2$ and $x = 4$

Now, for $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$
$$\Rightarrow \lim_{x\to 2^-} (x^2 + ax + b) = \lim_{x\to 2^+} (3x + 2) = 3(2) + 2$$
$$\Rightarrow 4 + 2a + b = 8$$
$$\Rightarrow 2a + b = 4 \qquad \dots(1)$$

Now for $x = 4$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$$
$$\lim_{x\to 4^-} (3x + 2) = \lim_{x\to 4^+} (2ax + 5b) = 3(4) + 2$$
$$\Rightarrow 8a + 5b = 14 \qquad \dots(2)$$

Solving (1) and (2), we get $a = 3$, $b = -2$.

P1: If $\boldsymbol{f(x) = \begin{cases} -2\sin x, & \text{for } -\pi \le x \le -\frac{\pi}{2} \\ a\sin x + b, & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases}}$

is continuous on $[-\pi, \pi]$, then find the values of a and b.

Solution: Since f is continuous on $[-\pi, \pi]$, it is continuous at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

Now, f is continuous at $x = -\frac{\pi}{2}$

$$\Longrightarrow \lim_{x \to (-\frac{\pi}{2})^-} f(x) = \lim_{x \to (-\frac{\pi}{2})^+} f(x) = f(-\frac{\pi}{2})$$

$$\Longrightarrow \lim_{x \to -\frac{\pi}{2}} (-2\sin x) = \lim_{x \to -\frac{\pi}{2}} (a \sin x + b)$$

$$\Longrightarrow -2\sin(-\frac{\pi}{2}) = a\sin(-\frac{\pi}{2}) + b$$

$$\Longrightarrow 2 = -a + b = 2$$

$$\Longrightarrow -a + b = 2 \qquad \dots(1)$$

Again f is continuous at $x = \frac{\pi}{2}$

$$\Longrightarrow \lim_{x \to (\frac{\pi}{2})^-} f(x) = \lim_{x \to (\frac{\pi}{2})^+} f(x) = f(\frac{\pi}{2})$$

$$\Longrightarrow \lim_{x \to \frac{\pi}{2}} (a \sin x + b) = \lim_{x \to \frac{\pi}{2}} \cos x = \cos\frac{\pi}{2}$$

$$\Longrightarrow a\sin\frac{\pi}{2} + b = \cos\frac{\pi}{2}$$

$$\Longrightarrow a + b = 0 \qquad \dots(2)$$

Solving equations (1) and (2), we get $a = -1, b = 1$

IP2: Show that the function

$$f(x) = \begin{cases} x^2, & if \ \ 0 \le x < 1 \\ 1, & if \ \ 1 \le x < \sqrt{2} \\ \frac{2}{x^2}, & if \ \ \sqrt{2} \le x < \infty \end{cases}$$ **is continuous on $[0, \infty)$.**

Solution:

When $0 \le x < 1$, we have $f(x) = x^2$.

Since it is a polynomial, it is continuous on $[0,1)$.

When $1 \le x < \sqrt{2}$, we have $f(x) = 1$.

Since it is a constant function, it is continuous on $(1, \sqrt{2})$.

When $\sqrt{2} \le x < \infty$, we have $f(x) = \frac{2}{x^2}$.

Since it is a rational function and $x^2 \neq 0$ in the defined domain, it is continuous on $(\sqrt{2}, \infty)$.

We have to prove $f(x)$ is continuous on $[0, \infty)$ it is enough to prove f is continuous at $x = 1$ and $x = \sqrt{2}$.

At $x = 1$,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = \lim_{h \to 0} (1 + h)^2 = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} 1 = 1$$

Further, $f(1) = 1$

$$\therefore \lim_{x\to1^+} f(x) = \lim_{x\to1^+} f(x) = f(1)$$

Thus, f is continuous at $x = 1$.

At $x = \sqrt{2}$,

$$\lim_{x\to(\sqrt{2})^+} f(x) = \lim_{x\to(\sqrt{2})^+} \left(\frac{2}{x^2}\right)$$

$$= \lim_{h\to0} \frac{2}{\left(\sqrt{2}+h\right)^2} = 1$$

$$\lim_{x\to(\sqrt{2})^-} f(x) = \lim_{x\to(\sqrt{2})^-} (1) = 1$$

Further, $f\left(\sqrt{2}\right) = 1$

$$\therefore \lim_{x\to(\sqrt{2})^-} f(x) = \lim_{x\to(\sqrt{2})^-} f(x) = f\left(\sqrt{2}\right)$$

Thus, f is continuous at $x = \sqrt{2}$.

Therefore, f is continuous on $[0, \infty)$

P2: Examine the continuity in the interval $(-\infty, \infty)$ of the function defined as follows

$$f(x) = \begin{cases} 2 & , \quad if\ x \in (-\infty, 0) \\ 1 + \cos x & , \quad if\ x \in \left[0, \frac{\pi}{2}\right) \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & , \quad if\ x \in \left[\frac{\pi}{2}, \infty\right) \end{cases}$$

Solution:

When $x \in (-\infty, 0)$, we have $f(x) = 2$. It is a constant function and so is continuous on $(-\infty, 0)$.

When $x \in \left[0, \frac{\pi}{2}\right)$, we have $f(x) = 1 + \cos x$. Clearly, it is continuous on its domain.

When $x \in \left[\frac{\pi}{2}, \infty\right)$, we have $(x) = 2 + \left(x - \frac{\pi}{2}\right)^2$. Since it is a polynomial, it is continuous on $\left(\frac{\pi}{2}, \infty\right)$.

Therefore, the function f is continuous everywhere except possibly at $= 0, \frac{\pi}{2}$.

Firstly we consider $x = 0$

Now,

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} (1 + \cos x) = \lim_{h\to0} (1 + \cos(0 + h))$$

$$= \lim_{h\to0} (1 + \cos h) = 1 + 1 = 2$$

$$(\because \lim_{h\to0} \cos h = 1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2 = 2$$

Further, $f(0) = 1 + \cos 0 = 1 + 1 = 2$

$$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

Thus, f is continuous at $x = 0$.

Now, we consider $x = \frac{\pi}{2}$, we have

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \left(2 + \left(x - \frac{\pi}{2}\right)^2\right)$$

$$= \lim_{h\to 0} \left(2 + \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^2\right)$$

$$= \lim_{h\to 0} (2 + h^2) = 2$$

$$\lim_{x\to\left(\frac{\pi}{2}\right)_-} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)_-} (1 + \cos x)$$

$$= \lim_{h\to 0} \left(1 + \cos\left(\frac{\pi}{2} - h\right)\right)$$

$$= \lim_{h\to 0} (1 + \sin h)$$

$$= 1 + 0 = 1 \qquad (\because \lim_{h\to 0} \sin h = 0)$$

$$\therefore \lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) \neq \lim_{x\to\left(\frac{\pi}{2}\right)_-} f(x)$$

Thus, f is discontinuous at $\quad = \frac{\pi}{2}$.

Therefore, f is continuous on $\left(-\infty, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \infty\right)$

IP3: Show that the function $F(x) = (x-a)^2(x-b)^2 + x$ takes on the value $\frac{(a+b)}{2}$ for some value x.

Solution: The given function is: $F(x) = (x-a)^2(x-b)^2 + x$.

Without loss of generality, assume that $a < b$.

Then, $F(a) = a$ and $F(b) = b$

By the intermediate value theorem we have: If f continuous on $[a, b]$, then it takes every value between $f(a)$ and $f(b)$. Since $a < \frac{a+b}{2} < b$, there is a number c between a and b such that $F(c) = \frac{a+b}{2}$.

P3: Show that the function $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

Solution: Let $f(x) = x^3 - 15x + 1$. Clearly $f(x)$ is continuous on $[-4,4]$.

Now, $f(-4) = (-4)^3 - 15(-4) + 1 = -3 < 0$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15 > 0$$
$$f(1) = (1)^3 - 15(1) + 1 = -13 < 0$$
$$f(4) = (4)^3 - 15(4) + 1 = 5 > 0$$

By the intermediate value theorem, $f(x) = 0$ for some x in each intervals $(-4,-1), (-1,1)$ and $(1,4)$.

$i.e.,\ \ x^3 - 15x + 1 = 0$ has three solutions in $[-4,4]$

IP4: Prove that the function $f(x) = x^3 + x^2 - 4$ has at least one zero in the interval $(1, 2)$.

Solution: The given function is: $f(x) = x^3 + x^2 - 4.$

Since all polynomial functions are continuous everywhere, $f(x)$ is continuous everywhere.

Now,

$$f(1) = 1^3 + 1^2 - 4 = -2 < 0$$
$$f(2) = 2^3 + 2^2 - 4 = 8 > 0$$

Therefore, by intermediate value theorem there exists a number c in $(1,2)$ such that $f(c) = 0$.Thus, the function $f(x)$ has at least one zero in $(1,2)$.

P4: The function f is defined by $f(x) = 2x^3 - 5x^2 - 10x + 5.$

Prove that $f(x)$ has at least one zero in $[0, 1]$.

Solution: The Given function is $f(x) = 2x^3 - 5x^2 - 10x + 5.$

Since all polynomial functions are continuous everywhere;

$f(x)$ is continuous every where.

By trial we see

$$f(0) = 2(0)^3 - 5(0)^2 - 10(0) + 5 = 5 > 0$$
$$f(1) = 2(1)^3 - 5(1)^2 - 10(1) + 5 = -8 < 0$$

By the intermediate value theorem there exists a number

c in $(0,1)$ such that $f(c) = 0$. Thus, the function $f(x)$ has at least one zero in $[0,1]$.

Exercise:

1. If the function f, defined by $f(x) = \begin{cases} kx + 1 & if\ -\infty \le x \le 1 \\ x^2 - 1 & if\ \ \ 1 \le x \le \infty \end{cases}$ is a continuous function on $[-1,2]$. Then find the value of k.
2. Find the values of a and b so that the function $f(x)$ defined by $f(x) = \begin{cases} x + a\sqrt{2}\sin x & , if\ 0 \le x < \frac{\pi}{4} \\ 2x\cos x + b & , if\ \frac{\pi}{2} \le x < \frac{\pi}{2} \\ a\cos 2x - b\sin x & , if\ \frac{\pi}{2} \le x < \pi \end{cases}$.

3. Discuss the continuity of $f(x) = \sqrt{16 - x^2}$.
4. Discuss the continuity of the function $f(x) = \sqrt{\frac{x-2}{5-x}}$ in the interval $(2,5)$.
5. Find the interval in which the function $f(x) = \sqrt{x} + \sqrt{2x-1}$ is continuous.
6. Explain why the equation $x - \cos x = 0$ has at least one solution.
7. Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a zero some where between -1 and 2.
8. Show that $x^3 - x + 1 = 0$ has a zero in the interval $[-2,0]$

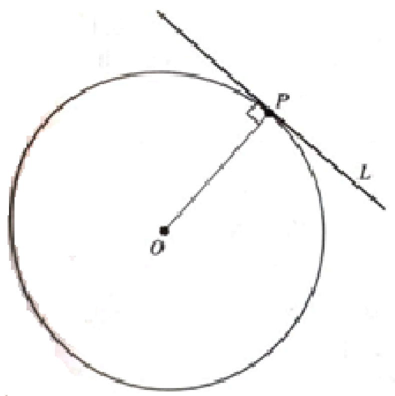
3.11. Tangent Lines

Learning objectives:

- To define the tangent to a curve at a point on the curve and to find it. And
- To practice the related problems.

Tangent to a Curve

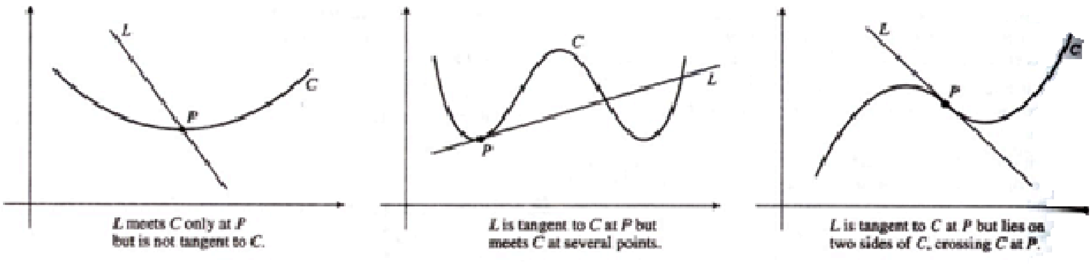
From the geometry, we know the tangents to circles. A line L is tangent to a circle at a point P if L passes through P and is perpendicular to the radius at P. Such a line just *touches* the circle.



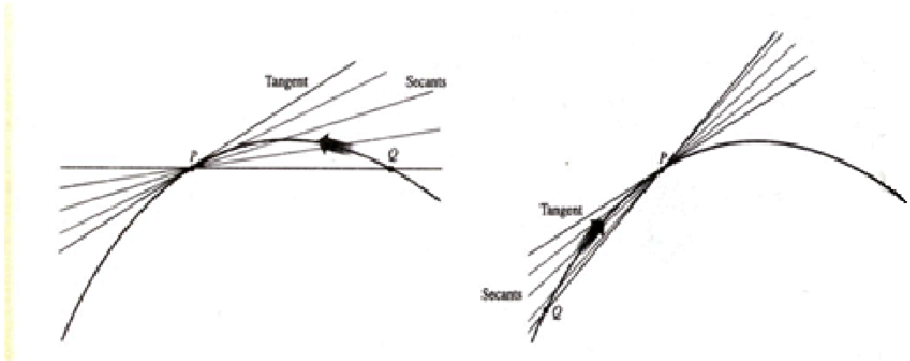
The following statements are valid.

1. L passes through P and is perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.

These statements may not apply consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at the point of tangency.



To define tangency for general curves, we take into account the behavior of the secants through P and nearby points Q (on C) as Q moves toward P along the curve.



The procedure is as follows:

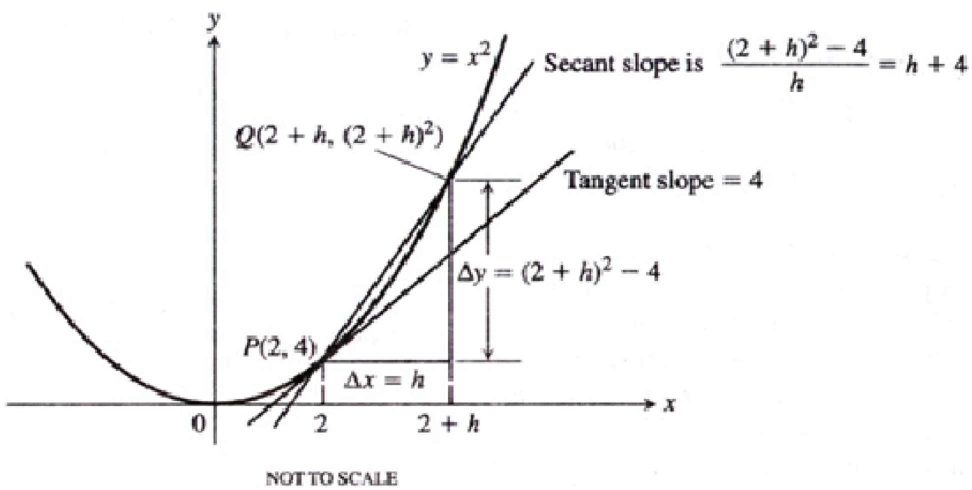
1. We calculate the slope of the secant PQ
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, we take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

Example 1: Find the slope of the parabola $y = x^2$ at the point P(2,4). Write an equation for the tangent to the parabola at this point.

Solution: Consider the secant line through $P(2,4)$ and Q$(2+h, (2+h)^2)$ nearby.

Secant slope $= \frac{\Delta y}{\Delta x} = \frac{(2+h)^2-2^2}{h} = \frac{h^2+4h+4-4}{h} = \frac{h^2+4h}{h} = h+4$

If $h > 0$, Q lies above and to the right of P, as in the figure below.



As Q approaches P along the curve, h approaches zero and the secant slope approaches 4:

$$\lim_{h\to 0} h + 4 = 4$$

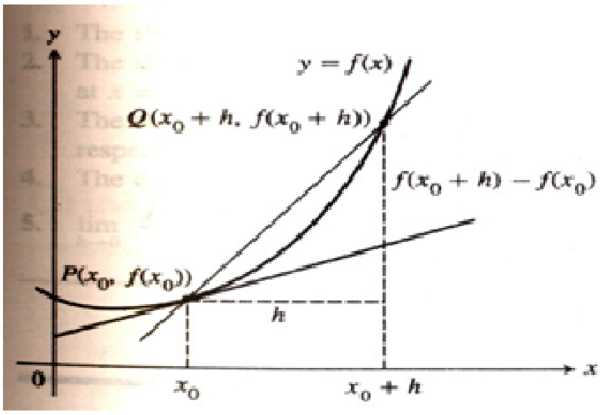
We take 4 to be the parabola's slope at P. The tangent to the parabola at P is the line through P with slope 4. The equation of the tangent to the parabola at P is,

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$

$$\Longrightarrow y = 4x - 4$$

Finding a Tangent to the Graph of a function

We use the same procedure to find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$. We calculate the slope of the secant through P and a point Q$(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$.



The tangent slope is $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$. If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

Definitions

The **slope of the curve** $y = f(x)$ at the point P$(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example 2:

a) Find the slope of the curve $y = \frac{1}{x}\ at\ x = a \neq 0$

b) Where does the slope equal $-1/4$?

c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

Solution

a) We have $f(x) = \frac{1}{x}$. The slope at $\left(a, \frac{1}{a}\right)$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

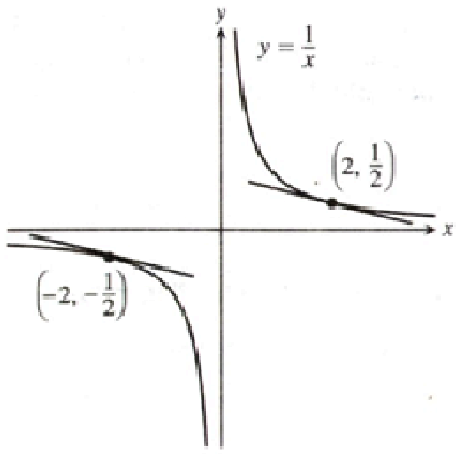
$$= \lim_{h \to 0} \frac{-h}{ha(a+h)}$$
$$= -\frac{1}{a^2}$$

b) Given the slope is $-1/4$. Therefore,

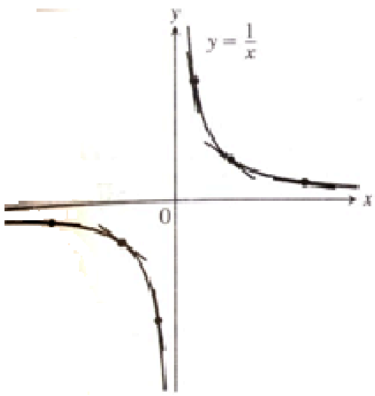
$$\frac{-1}{a^2} = \frac{-1}{4} \quad \Longrightarrow \quad a^2 = 4 \quad \Longrightarrow a = 2, -2$$

The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$.

c)



We notice that the slope $-{}^{1}/_{a^2}$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep.



We also see that the slope approaches $-\infty$ as $a \to o^-$. As a moves away from the origin, the slope approaches 0^- and the tangent levels off.

Rates of Change

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}$$

is called the **difference quotient of *f* at x_0**. If the difference quotient has a limit as *h* approaches zero, that limit is called the **derivative of *f* at x_0**.

If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where $x = x_0$. If we interpret the difference quotient as an average rate of change, the derivative gives the function's rate of change with respect to x at the point $x = x_0$.

Example 3: A rock falls from the top of a $50\ m$ cliff. Physical experiments show that a solid object dropped from the rest to fall freely near the surface of the earth will be $y = 5t^2\ m$ during the first t sec.

What is the rock's speed at $t = 1$ sec?

Solution: $f(t) = 5t^2$

The rock's speed at the instant $t = 1$ sec is

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{5(1+h)^2-5(1)^2}{h}$$

$$= \lim_{h\to 0}\frac{5(h^2+2h)}{h} = \lim_{h\to 0}5(h+2)$$

$$= 10\ m/sec$$

We note that the following statements refer to the same thing:

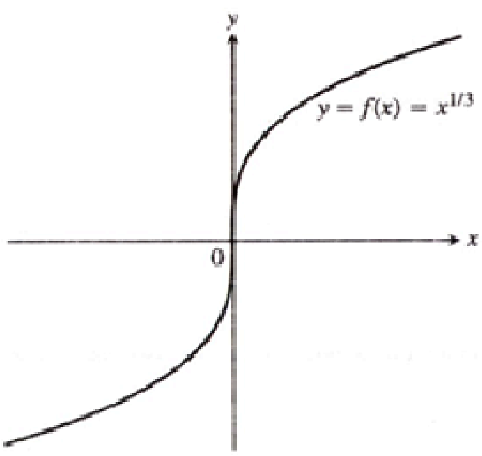
1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

Vertical tangents:

We say that the curve $y = f(x)$ has a ***vertical tangent*** at the point $x = x_0$ if

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \infty \text{ or } -\infty$$

1) Consider the function $y = f(x) = x^{\frac{1}{3}}$. Its graph is shown below.

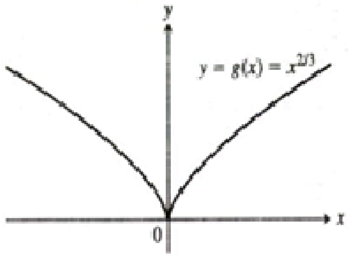


At $x = 0$:

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(h)^{\frac{1}{3}}-0}{h} = \lim_{h\to 0} \frac{1}{(h)^{\frac{2}{3}}} = \infty$$

So, there is a vertical tangent at $x = 0$.

2) Consider the function $y = g(x) = x^{\frac{2}{3}}$. Its graph is shown below.



At $x = 0$:

$$\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{(h)^{\frac{2}{3}}-0}{h} = \lim_{h\to 0} \frac{1}{(h)^{\frac{1}{3}}}$$

The limit does not exist, because the limit is ∞ from the right and $-\infty$ from the left.

So, there is no vertical tangent at $x = 0$.

PROBLEM SET

IP1: Find the slope of curve $y = f(x) = \frac{x-1}{x+1}$ at $x = x_0 = 0$.

Solution:

The given curve is: $y = \frac{x-1}{x+1}$. At $x = 0$ we have, $y = \frac{0-1}{0+1} = -1$.

Now, the point $P(0, -1)$ is on the given curve.

We know that the slope of the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$\boldsymbol{m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}}$$

$$\boldsymbol{f(x_0 + h) = f(0 + h) = f(h) = \frac{h - 1}{h + 1}}$$

$$\boldsymbol{f(x_0)} = \boldsymbol{f(0)} = \frac{0 - 1}{0 + 1} = \frac{-1}{1} = -1$$

The slope of the curve $y = \frac{x-1}{x+1}$ at the point $P(0, -1)$ is

$$m = \lim_{h\to 0} \frac{\left(\frac{h-1}{h+1}\right)-(-1)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{h-1}{h+1}+1\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{h - 1 + h + 1}{h + 1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{h+h}{h+1}\right)}{h} = \lim_{h \to 0} \frac{\left(\frac{2h}{h+1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{(2h)}{h(h+1)}$$

$$= \lim_{h \to 0} \frac{2h}{h(h+1)} = \lim_{h \to 0} \frac{2}{h+1} = 2$$

Slope of the curve given curve at $x = 0$ is 2.

P1: Find the slope of the curve $y = f(x) = 1 - x^2$ at $x = x_0 = 2$.

Solution: The given curve is: $y = 1 - x^2$. At $x = 2$ we have

$y = 1 - 2^2 = -3$. Then the point $P(2, -3)$ is on the given curve.

We know that the slope of the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

The slope of the curve $y = 1 - x^2$ at the point $P(2, -3)$ is

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(1-(2+h)^2) - (-3)}{h}$$

$$= \lim_{h \to 0} \frac{-h(4+h)}{h}$$

$$= \lim_{h \to 0} -(4+h) = -4$$

Slope of the given curve at $x = 2$ is -4.

IP2: Find the equation for the tangent line to the curve $y = \frac{1}{x^2}$ at the point $P(-1, 1)$.

Solution: The given curve is: $y = \frac{1}{x^2}$. The point $P(-1,1)$ is on the curve.

We know that the slope of the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

The slope of the tangent to the curve $y = \frac{1}{x^2}$ at the point $P(-1,1)$ is

$$m = \lim_{h \to 0} \frac{\frac{1}{(-1+h)^2} - \frac{1}{(-1)^2}}{h}$$

$$= \lim_{h \to 0} \frac{-(-2h+h^2)}{h(-1+h)^2}$$

$$= \lim_{h \to 0} \frac{2-h}{(h-1)^2} = 2$$

Equation of the tangent line to the given curve at $P(-1,1)$ with slope $m = 2$ is

$$y + 1 = 2(x - 1) \implies 2x - y + 3 = 0$$

P2: Find the equation for the tangent line to the curve

$y = (x - 1)^2 + 1$ at the point $P(1, 1)$.

Solution: The given curve is: $y = (x - 1)^2 + 1$. The point $P(1,1)$ is on the curve.

We know that the slope of the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$ is

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The slope of the tangent to the curve $y = (x - 1)^2 + 1$ at the point $P(1,1)$ is

$$m = \lim_{h \to 0} \frac{[(1+h-1)^2+1]-[(1-1)^2+1]}{h}$$

$$= \lim_{h \to 0} \frac{h^2+1-1}{h} = \lim_{h \to 0} h = 0$$

Equation of the tangent line to the given curve at $P(1,1)$

having slope $m = 0$ is

$$(y - 1) = 0(x - 1) \implies y - 1 = 0$$

IP3: Find the equation of the straight line having slope 2 that is tangent to the curve $y = x^2 - 2x + 3$.

Solution: Given that the slope of the tangent to the curve

$y = x^2 - 2x + 3$ is $m = 2$

We know that slope of the tangent to the curve $y = f(x)$ at $(x, x^2 - 2x + 3)$ is

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\implies 2 = \lim_{h \to 0} \frac{[(x+h)^2-2(x+h)+3]-[x^2-2x+3]}{h}$$

$$\implies 2 = \lim_{h \to 0} \frac{[x^2+2xh+h^2-2x-2h+3]-[x^2-2x+3]}{h}$$

$$\implies 2 = \lim_{h \to 0} \frac{2xh-2h+h^2}{h}$$

$$\implies 2 = \lim_{h \to 0} 2x + h - 2 \implies 2 = 2x + 2$$

$$\implies x = 0$$

Now, $x = 0 \implies y = 0^2 - 2(0) + 3 = 3$

The equation of the straight line at the point $(0,3)$ and having slope $m = 2$ is

$$y - 3 = 2(x - 0) \implies 2x - y + 3 = 0$$

P3: Find the equation of the straight line having slope $\frac{1}{4}$ that is tangent to the curve $y = \sqrt{x}$.

Solution: Given that the slope of the tangent to the curve $y = \sqrt{x}$ is $m = \frac{1}{4}$.

We know that slope of the tangent to the curve $y = f(x)$ is at $(x, \sqrt{x})$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Longrightarrow \frac{1}{4} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\Longrightarrow \frac{1}{4} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\Longrightarrow \frac{1}{4} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$\Longrightarrow \frac{1}{4} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \Longrightarrow \frac{1}{4} = \frac{1}{2\sqrt{x}}$$

$$\Longrightarrow \sqrt{x} = 2 \Longrightarrow x = 4$$

Now, $x = 4 \implies y = \sqrt{4} = 2$ and the equation of the tangent line at the point $(4,2)$ and having slope $\frac{1}{4}$ is

$$y - 2 = \frac{1}{4}(x - 4) \Longrightarrow x - 4y + 4 = 0$$

IP4: Verify whether the curve $y = x^{\frac{2}{3}} - (x-1)^{\frac{1}{3}}$ has a vertical tangent at $x = 1$.

Solution: The given curve is: $y = x^{\frac{2}{3}} - (x-1)^{\frac{1}{3}}$.

We know that the curve $y = f(x)$ has a vertical tangent at

$x = x_0$ if

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \infty \text{ or } -\infty$$

Vertical tangent at $x = 1$:

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^{\frac{2}{3}} - (1+h-1)^{\frac{1}{3}} - 1}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^{\frac{2}{3}} - 1}{h} - \lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h}$$

Now, $$\lim_{h \to 0} \frac{(1+h)^{\frac{2}{3}} - 1}{h} = \lim_{h \to 0} \left[\frac{(1+h)^{\frac{2}{3}} - 1}{h} \times \frac{\left((1+h)^{\frac{4}{3}} + (1+h)^{\frac{2}{3}} + 1\right)}{\left((1+h)^{\frac{2}{3}} + (1+h)^{\frac{2}{3}} + 1\right)} \right]$$

$$= \lim_{h \to 0} \left[\frac{(1+h)^2 - 1}{h\left((1+h)^{\frac{2}{3}} + (1+h)^{\frac{2}{3}} + 1\right)} \right]$$

$$= \lim_{h \to 0} \left[\frac{h(h+2)}{h\left((1+h)^{\frac{2}{3}} + (1+h)^{\frac{2}{3}} + 1\right)} \right] = \frac{2}{3}$$

$$\therefore \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\frac{2}{3}-\lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}}=\infty \quad \left(\because \lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}}=\infty\right)$$

Therefore, the given curve has a vertical tangent at $x=1$.

P4: What is the rate of change of the volume of a ball$\left(V=\frac{4}{3}\pi r^3\right)$ with respect to the radius when radius is $r=2$.

Solution: Volume of the ball $V=f(r)=\frac{4}{3}\pi r^3$

We know that the rate of change of $f(x)$ with respect to x at $x=x_0$ is

$$=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

The rate of change of $f(r)$ with respect to the radius r at $r=2$ is

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{4}{3}\pi(2+h)^3-\frac{4}{3}\pi(2)^3}{h} \quad =\lim_{h\to 0}\frac{\frac{4}{3}\pi(8+h^3+6h^2+12h-8)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{4}{3}\pi(h^3+6h^2+12h)}{h} \quad =\lim_{h\to 0}\frac{4}{3}\pi(h^2+6h+12)$$

$$=\frac{4}{3}\pi(12)=16\pi$$

The rate of change of the volume of a ball with respect to the radius when radius is $r=2$ is 16π

Exercise:

1. Find the slope of the curve at the point indicated.
   - **a.** $y=5x^2$, $\quad x=-1$
   - **b.** $y=\frac{1}{x-1}$, $\quad x=3$
   - **c.** $y=3x^4-4x$, $\quad x=4$
   - **d.** $y=x^3-x+1$, $\quad x=2$
   - **e.** $y=x^3-3x+2$, $\quad x=3$
2. Find the slope of the function’s graph at the given point. Then find an equation for the line tangent to the graph there.
   - **a.** $f(x)=x^2+1$, $\quad (2,5)$
   - **b.** $g(x)=\frac{x}{x-2}$, $\quad (3,3)$
   - **c.** $h(t)=t^3$, $\quad (2,8)$
   - **d.** $f(x)=\sqrt{x}$, $\quad (4,2)$
   - **e.** $f(x)=x-2x^2$, $\quad (1,-1)$
   - **f.** $g(x)=\frac{8}{x^2}$, $\quad (2,2)$
   - **g.** $h(t)=t^3+3t$, $\quad (1,4)$
   - **h.** $f(x)=\sqrt{x+1}$, $\quad (8,3)$

3. Find an equation for the tangent to the curve at the given point.

a. $y = 4 - x^2$ $\quad (-1,3)$
b. $y = 2\sqrt{x}$ $\quad (1,2)$
c. $y = x^3$ $\quad (-2,-8)$
d. $y = \frac{1}{x^3}$ $\quad \left(-2,-\frac{1}{8}\right)$

4. At what points do the graph of the function has horizontal tangents?

a. $f(x) = x^2 + 4x - 1$
b. $f(x) = x^3 - 3x$

5.

a. Does the graph of

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

b. Does the graph of

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

has a vertical tangent at the point $(0,1)$? Give reasons for your answer.

c. Does the graph of

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

has a vertical tangent at the origin? Give reasons for your answer.

6. Verify whether the curve has a vertical tangent at the point indicated.

a. $y = x^{\frac{2}{5}}$, $\quad x = 0$
b. $y = x^{\frac{1}{5}}$, $\quad x = 0$
c. $y = 4x^{\frac{2}{5}} - 2x$, $\quad x = 0$
d. $y = x^{\frac{1}{3}} + (x-1)^{\frac{1}{3}}$, $\quad x = 1$

7. Find equations of all lines having slope −1 that are tangent to the curve $y = 1/(x-1)$.

8. An object is dropped from the top of a 100-m-high tower. Its height aboveground after t seconds is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

9. At t sec after liftoff, the height of a rocket is $3t^2$ft. how fast the rocket climbing 10sec after liftoff?

10. What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to its radius when the radius is $r = 3$?

4.1. The Derivative of a Function

Leaning objectives:

In this module, we study

- To define the derivative of a function with respect to its independent variable.
- To compute the derivatives using the definition.
- To define differentiability on an interval.
  And
- To solve the problems related to the above concepts.

In the unit on limits, we defined the slope of a curve $y = f(x)$ at the point where $x = x_0$ to be

$$m = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

We called this limit the derivative of f at x_0. Now, we investigate the derivative as a function derived from f by considering the limit of the difference quotient at each point of the domain.

Definition:

*The **derivative** of the function f with respect to the variable x is the function f' whose value at x is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

*If $f'(x)$ exists, we say that f **has a derivative** at x or f **is differentiable** at x.*

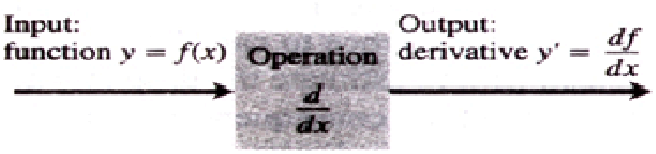
The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f.

There are many ways to denote the derivative of a function

$y = f(x)$, where x is the independent variable and y is the dependent variable.

Besides f', the other notations are $y', \frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx}f(x), D_x f$. These are pronounced as "f prime", "y prime", "dy by dx",

"df by dx", "d by dx of $f(x)$" and "dx of f" respectively. The notation $\dot{y}$ (y dot) is used for time derivatives. The notation $\frac{dy}{dx}$ is also read as "the derivative of y with respect to x," and $\frac{df}{dx}$ and $\frac{d}{dx}f(x)$ as "the derivative of f with respect to x."



The above figure gives a flow diagram for the operation of taking a derivative with respect to x.

Calculating the Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. The procedure for calculating the derivative is as follows:

1. Write expressions for $f(x)$ and $f(x+h)$
2. Expand and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$
3. Using the simplified quotient, find $f'(x)$ by evaluating the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Example 1: The derivative of $y = mx + b$ is m at any x.

Solution: We have $y = f(x)$ where $f(x) = mx + b$.

Now, $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \; = \lim_{h\to 0} \frac{m(x+h)+b-mx-b}{h} = \lim_{h\to 0} \frac{mh}{h} = m$

Example 2:

a) Differentiate $f(x) = \frac{x}{x-1}$

b) Where does the curve $y = f(x)$ have slope -1?

Solution:

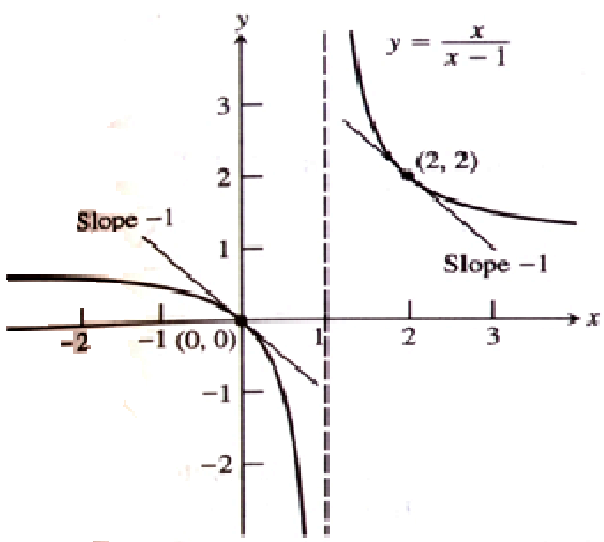
a) $$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h}$$

$$= \frac{1}{h}.\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)} = \frac{1}{h}.\frac{-h}{(x+h-1)(x-1)}$$

$$f'(x) = \lim_{h\to 0} \frac{-h}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

b) The slope of the curve $y = f(x)$ is $f'(x)$. Given that the

slope is -1, $\quad \frac{-1}{(x-1)^2} = -1 \;\Rightarrow (x-1)^2 = 1 \;\Rightarrow x = 2$ or $x = 0$



Example 3:

a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.

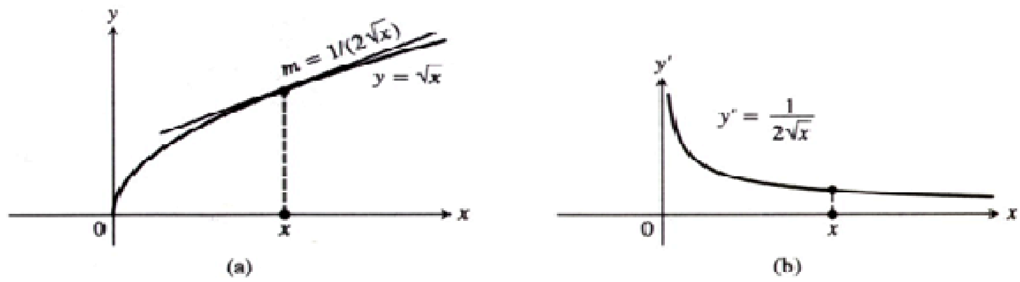
b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution:

a) . $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$ Multiply by $\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$



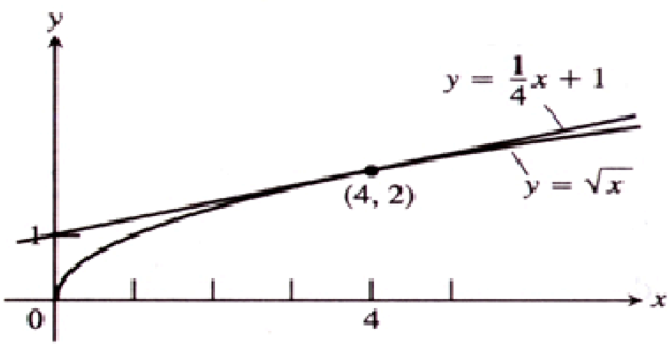
The function is defined at $x = 0$, but its derivative is not.

b). The slope of the curve at $x = 4$ is

$$\frac{dy}{dx}\Big|_{x=4} = \frac{1}{2\sqrt{x}}\Big|_{x=4} = \frac{1}{4}$$

The tangent is the line through the point $(4,2)$ with slope $1/4$.

$$y = 2 + \frac{1}{4}(x-4) \implies y = \frac{1}{4}x + 1$$



The value $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ of the derivative of $y = f(x)$ with respect to x at $x = a$ can be denoted in the following ways:

$$y'|_{x=a} = \frac{dy}{dx}\Big|_{x=a} = \frac{d}{dx} f(x)\Big|_{x=a}$$

Here, the symbol $|_{x=a}$, called an **evaluation symbol**, tells us to evaluate the expression to its left at $x = a$.

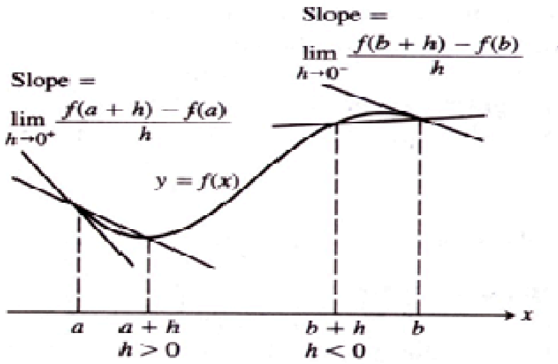
Differentiability on an Interval:

A function $y = f(x)$ is **differentiable** on an open interval if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$ ***Right-hand derivative at a***

$\lim_{h \to 0^-} \frac{f(b+h)-f(b)}{h}$ ***Light-hand derivative at b***

exist at the endpoints.



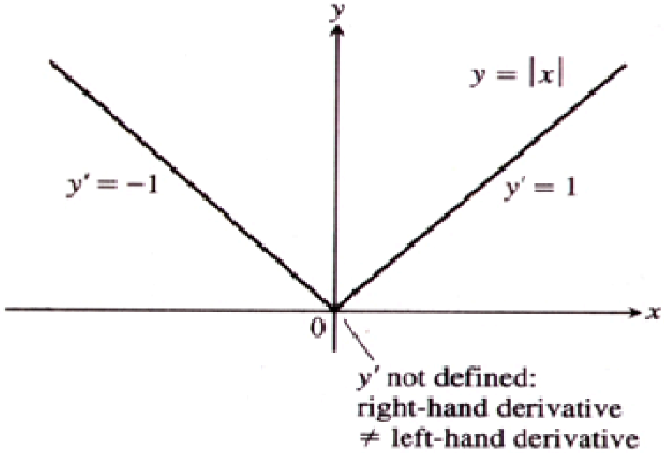
Right-hand and left-hand derivatives may be defined at any point in the domain. A function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

Example 4:

The function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

To the right of the origin, $\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = 1$

To the left, $\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = -1$



There can be no derivative at the origin because the one-sided derivatives differ there:

Right-hand derivative of $|x|$ at zero

$$\lim_{h\to 0}\frac{|0+h|-|0|}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$

Left-hand derivative of $|x|$ at zero

$$\lim_{h\to 0}\frac{|0+h|-|0|}{h}=\lim_{h\to 0}\frac{-h}{h}=\lim_{h\to 0}-1=-1$$

PROBLEM SET

IP1: Find the derivative of the function $f(x)=\frac{1-x}{2+x}$ using the definition of the derivative.

Solution: The given function is $f(x)=\frac{1-x}{2+x}$

First we calculate the difference quotient

$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{1-(x+h)}{2+(x+h)}-\frac{1-x}{2+x}}{h}$$

$$=\frac{(1-x-h)(2+x)-(1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$=\frac{(2+x-2x-x^2-2h-xh)-(2+x+h-2x-x^2-xh)}{h(2+x+h)(2+x)}$$

$$=\frac{-3h}{h(2+x+h)(2+x)}=\frac{-3}{(2+x+h)(2+x)}$$

We know that derivative of $f(x)$ is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{-3}{(2+x+h)(2+x)}$$

$$=-\frac{3}{(2+x)^2}$$

Derivative of the given function is $f'(x)=-\frac{3}{(2+x)^2}$.

P1: Find the derivative of the function $f(x)=x^3-x$ using the definition of the derivative.

Solution: The given function is: $f(x)=x^3-x$

Now we calculate the difference quotient

$$\frac{f(x+h)-f(x)}{h}=\frac{[(x+h)^3-(x+h)]-[x^3-x]}{h}$$

$$=\frac{[(x^3+3x^2h+3xh^2+h^3)-(x+h)]-[x^3-x]}{h}$$

$$=\frac{(3x^2h+3xh^2+h^3-h)}{h}$$

$$= 3x^2 + 3xh + h^2 - 1$$

We know that the derivative of $f(x)$ is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}(3x^2 + 3xh + h^2 - 1)$$

$$= 3x^2 - 1$$

Derivative of function $f(x) = x^3 - x$ is $f'(x) = 3x^2 - 1$.

IP2: If a particle moves in a straight line such that the distance travelled in time t is given by $S = 8 + 9t - 6t^2$, find the initial velocity of the particle.

Solution: Given $S = f(t) = 8 + 9t - 6t^2$.

Initial velocity of the particle $V = \frac{dS}{dt}|_{t=0}$

$i.e.$, Derivative of $f(t)$ at $t = 0$.

We know that derivative of $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Now,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(8+9h-6h^2)-(8)}{h}$$

$$= \lim_{h\to 0} \frac{9h-6h^2}{h} = 9$$

$\therefore$ Initial velocity of the particle is 9 units/unit time.

P2: The distance $f(t)$ in meters moved by a particle travelling in a straight line in t seconds is given by $f(t) = t^2 + 3t + 4$. Find the speed of the particle at the end of 2seconds.

Solution: We have $f(t) = t^2 + 3t + 4$

The speed of the particle at the end of 2 sec given by $f'(2)$

$i.e.$, Derivative of $f(t)$ at $t = 2$.

We know that derivative of $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Now,

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{[(2+h)^2+3(2+h)+4]-[2^2+3(2)+4]}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 7h}{h} = 7$$

$$f'(2) = 7$$

$\therefore$ Speed of the particle at the end of the 2 sec is 7 m/s.

IP3: Find the derivative of the function $f(x) = 2x^2 + 3x - 5$.

Also prove that $f'(0) + 3f'(-1) = 0$.

Solution: The given function is $f(x) = 2x^2 + 3x - 5$

We know that derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[2(x+h)^2 + 3(x+h) - 5] - [2x^2 + 3x - 5]}{h}$$

$$= \lim_{h \to 0} \frac{[2(x^2 + 2xh + h^2) + 3x + 3h - 5] - [2x^2 + 3x - 5]}{h}$$

$$= \lim_{h \to 0} \frac{(2h^2 + 4xh + 3h)}{h}$$

$$f'(x) = 4x + 3$$

$f'(x)$ at $x = 0$ is $f'(0) = 4(0) + 3 = 3$

$f'(x)$ at $x = -1$ is $f'(-1) = 4(-1) + 3 = -1$

$\therefore \quad f'(0) + 3f'(-1) = 3 + 3(-1) = 3 - 3 = 0$

IP4: If $w = z + \sqrt{z}$, then find $\left.\frac{dw}{dz}\right|_{z=4}$.

Solution: Given $w = f(z) = z + \sqrt{z}$

$$\frac{dw}{dz} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(z+h) + \sqrt{(z+h)}\right] - [z + \sqrt{z}]}{h}$$

$$= \lim_{h \to 0} \frac{h + \sqrt{(z+h)} - \sqrt{z}}{h}$$

$$= \lim_{h \to 0} \left(1 + \frac{\sqrt{(z+h)} - \sqrt{z}}{h}\right)$$

$$= \lim_{h \to 0} \left(1 + \frac{\sqrt{(z+h)} - \sqrt{z}}{h} \times \frac{\sqrt{(z+h)} + \sqrt{z}}{\sqrt{(z+h)} + \sqrt{z}}\right)$$

$$= \lim_{h \to 0} \left(1 + \frac{z + h - z}{h\left(\sqrt{(z+h)} + \sqrt{z}\right)}\right)$$

$$= \lim_{h \to 0} \left(1 + \frac{1}{\sqrt{(z+h)} + \sqrt{z}}\right)$$

$$= 1 + \frac{1}{2\sqrt{z}}$$

Now, $\frac{dw}{dz}|_{z=4} = 1 + \frac{1}{2\sqrt{4}} = 1 + \frac{1}{4} = \frac{5}{4}$.

P4: If $y = 1 - \frac{1}{x}$, then find $\frac{dy}{dx}|_{x=\sqrt{3}}$.

Solution: Given $y = f(x) = 1 - \frac{1}{x}$

We know that

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left(1-\frac{1}{x+h}\right)-\left(1-\frac{1}{x}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\left(\frac{1}{x}-\frac{1}{x+h}\right)}{h} = \lim_{h\to 0}\frac{x+h-x}{hx(x+h)}$$

$$= \lim_{h\to 0}\frac{1}{x(x+h)} = \frac{1}{x^2}$$

Now, $\frac{dy}{dx}|_{x=\sqrt{3}} = \frac{1}{(\sqrt{3})^2} = \frac{1}{3}$

Exercises:

1. Find the derivative of the following functions from the definition.
   a. $x^3 - 27$
   b. $x^4 + 4$
   c. $\sqrt{x+1}$
   d. $\frac{1}{x^2+1}$
   e. $\frac{2x+3}{x-2}$
   f. $ax^2 + \frac{b}{x}$

2. Using the definition, calculate the derivatives of the function and then find the values of the derivatives as specified.
   a. $f(x) = 4 - x^2$; $f'(-3), f'(0), f'(1)$
   b. $g(t) = \frac{1}{t^2}$; $g'(-1), g'(2), g'(\sqrt{3})$
   c. $p(\theta) = \sqrt{3\theta}$; $p'(1), p'(3), p'(2/3)$
   d. $f(x) = x^2 + 7x + 4$; $f'(2), f'(5)$
   e. The function f given by $f(x) = x^2 - 6x + 8$, prove that $f'(5) - 3f'(2) = f'(8)$.
   f. Show that the derivative of the function f is given by $f(x) = 2x^3 - 9x^2 + 12x + 9$ at $x = 1$ and $x = 2$ are equal.

3. Find the indicated derivatives.
   a. $\frac{dy}{dx}$ if $y = 2x^3$
   b. $\frac{ds}{dt}$ if $s = \frac{t}{2t+1}$
   c. $\frac{dp}{dq}$ if $p = \frac{1}{\sqrt{q+1}}$
   d. $\frac{dy}{dx}|_{t=-1}$ if $s = 1 - 3t^2$

e. $\frac{dr}{d\theta}|_{\theta=1}$ if $r = \frac{2}{\sqrt{4-\theta}}$

4. Differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

   a. $f(x) = x + \frac{9}{x}$, $x = -3$

   b. $s = t^3 - t^2$, $t = -1$

   c. Differentiate the function $f(x) = \frac{8}{\sqrt{x-2}}$. Find an equation of the tangent line at the point $(6, 4)$ on the graph of the function.

5. Using first principles prove that $\frac{d}{dx}\left\{\frac{1}{f(x)}\right\} = \frac{-f'(x)}{\{f(x)\}^2}$.

4.2. Derivatives and Continuity

Learning objectives:

- To investigate the reasons for a function fail to have a derivative at a point.
- To prove that a function is continuous at every point where it has a derivative.

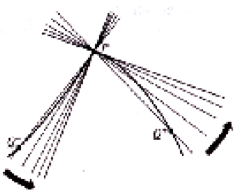
  And

- To practice related problems.

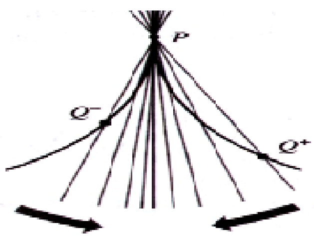
Function Not Having a Derivative at a Point

A function has a derivative at a point x_0 if the slopes of secant lines, through $P(x_0, f(x_0))$ and a nearby point Q on the graph, approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. A function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

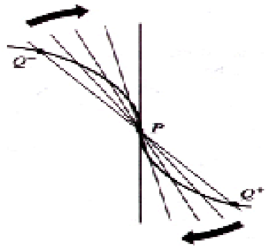
1. A ***corner,*** where the one-sided derivatives differ



2. A ***cusp,*** where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



3. A ***vertical tangent,*** where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides



4. A ***discontinuity.***

Differentiability and Continuity

A function is continuous at every point where it has a derivative.

Theorem:

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof:

Given that $f'(c)$ exists ,we must show that $\lim\limits_{h\to 0} f(x) = f(c)$,

or equivalently, that $\lim\limits_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(c+h) - f(c) = \frac{f(c+h)-f(c)}{h}.h$$

$$i.e., \quad f(c+h) = f(c) + \frac{f(c+h)-f(c)}{h}.h$$

Now, we take limits as $h \to 0$.

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \,.\, \lim_{h\to 0} h$$

$$= f(c) + f'(c).0 = f(c)$$

Similar arguments with one-sided limits show that *if f has a derivative from one side* (*right or left*) *at $x = c$, then f is continuous from that side at $x = c$.*

The above theorem says that

If a function has a discontinuity at a point (*for instance, a jump discontinuity*) *then it can not be differentiable there.*

Example 1: The greatest integer function $y = \lfloor x \rfloor = int\ x$ is not differentiable at every integer $x = n$ (since it jumps at every integer).

The converse of the above theorem is false. A function need not have a derivative at a point where it is continuous.

Example 2: The absolute value function $y = |x|$ is continuous at $x = 0$ is but it is not differentiable at $x = 0$.

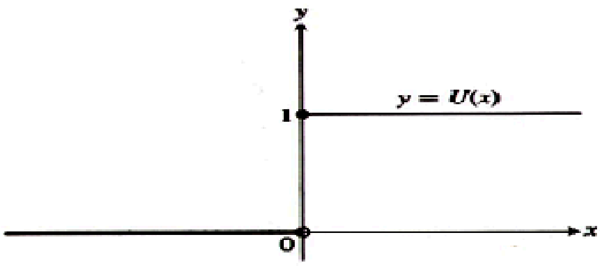
The Intermediate Value Property of Derivatives

Not every function can be some function's derivative, as we see from the following theorem.

Theorem:

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

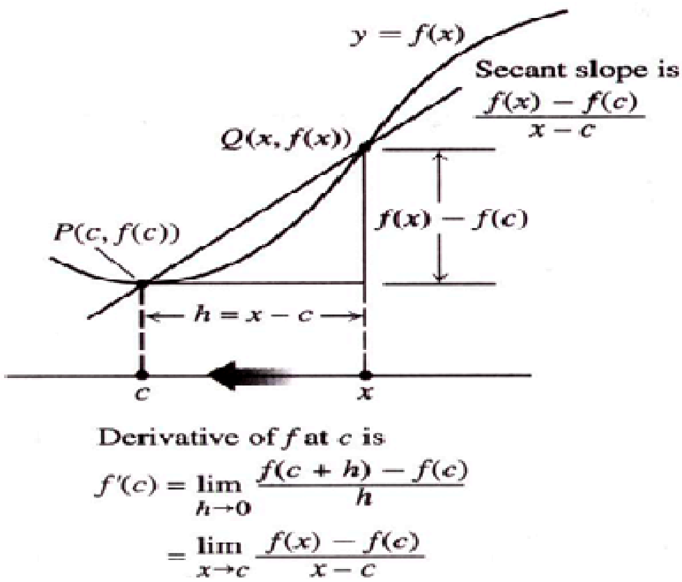
This theorem says that a function cannot be a derivative on an interval unless it has the intermediate value property there.



The unit step function does not have the intermediate value property and cannot be the derivative of a function on the real line.

An alternative formula for calculating derivatives

The formula for the secant slope whose limit leads to the derivative depends on how the points involved are labeled.



The secant slope is $\frac{f(x)-f(c)}{x-c}$. The slope of the curve at P is

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$
$$=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

The use of this formula simplifies some derivative calculations.

PROBLEM SET

IP1: Show that the function $f(x)=\begin{cases} x & for \quad x<1 \\ 3-x & for \;\; 1\le x\le 3 \\ x^2-4x+3 & for \quad x>3 \end{cases}$ is not differentiable at $x=1,3$.

Solution:

Given function is $f(x)=\begin{cases} x & for \quad x<1 \\ 3-x & for \;\; 1\le x\le 3 \\ x^2-4x+3 & for \quad x>3 \end{cases}$

Differentiability of f at $x=1$:

For $x=1, f(1)=2$

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1}(3-x)=2$$
$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1}x=1$$
$$\therefore \lim_{x\to 1^+}f(x)\neq \lim_{x\to 1^-}f(x)$$

Therefore, $f(x)$ is discontinuous at $x=1$.

$\therefore$ $f(x)$ is not differentiable at $x=1$.

Differentiability of f at $x=3$:

For $x=3, f(3)=0$

Right hand derivative at $x=3$:

$$\operatorname*{Lim}_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{(3+h)^2-4(3+h)+3-0}{h}$$
$$=\lim_{h\to 0}\frac{h^2+6h-4h}{h}=2$$

Left hand derivative at $x=3$:

$$\lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}=\lim_{h\to 0}\frac{3-(3-h)}{-h}=-1$$

The right hand derivative at $x=3$ is not equal to the left hand derivative at $x=3$ .i.e., $x=3$ is a corner. Thus, $f(x)$ is not differentiable at $x=3$.

P1: Discuss the differentiability of the function $f(x) = |x| + |x-1|$ at $x = 0, 1$.

Solution: The Given function is $f(x) = |x| + |x-1|$

$$f(x) = \begin{cases} -x-(x-1) = -2x+1, & if \ \ x < 0 \\ x-(x-1) = 1 \ , & if \ \ 0 \le x \le 1 \\ x+(x-1) = 2x-1 \ , & if \ x > 1 \end{cases}$$

Differentiability of $f(x)$ at $x = 0$:

For $x = 0$, $f(0) = 1$

Right hand derivative at $x = 0$:

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

Left hand derivative at $x = 0$:

$$\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{1-2(-h)-1}{-h} = -2$$

The right hand derivative at $x = 0$ is not equal to the left hand derivative at $x = 0$. $i.e.$, $x = 0$ is a corner. Thus, $f(x)$ is not differentiable at $x = 0$.

Differentiability of $f(x)$ at $x = 1$:

For $x = 1$, $f(1) = 1$.

Right hand derivative at $x = 1$:

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{2(1+h)-1-1}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

Left hand derivative at $x = 1$:

$$\lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{1-1}{-h} = 0$$

The right hand derivative at $x = 1$ not equal to the right hand derivative at $x = 1. i.e., x = 1$ is a corner. Thus, $f(x)$ is not differentiable at $x = 1$.

IP2: For what values of a and b is the function

$$f(x) = \begin{cases} x^2 & , x \le c \\ ax+b & , x > c \end{cases}$$ **is differentiable at $x = c$.**

Solution: Given that $f(x)$ is differentiable at $x = c$.

So, $f(x)$ is continuous at $x = c$.

$$\Rightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

$$\Rightarrow \lim_{x\to c} x^2 = \lim_{x\to c} (ax+b) = c^2$$

$$\Rightarrow c^2 = ac + b \qquad \ldots(i)$$

Again $f(x)$ is differentiable at $x = c$

$$\Rightarrow \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$$

$$\Rightarrow \lim_{x\to c} \frac{x^2-c^2}{x-c} = \lim_{x\to c} \frac{(ax+b)-c^2}{x-c}$$

$$\Rightarrow \lim_{x\to c} (x+c) = \lim_{x\to c} \frac{(ax+b)-(ac+b)}{x-c} \qquad (\because from(i))$$

$$\Rightarrow 2c = \lim_{x\to c} \frac{a(x-c)}{x-c}$$

$$\Rightarrow a = 2c$$

Now substituting value of a in (i) we get

$$c^2 = (2c)c + b \Rightarrow b = -c^2$$

Hence $a = 2c, b = -c^2$

P2: If $f(x) = \begin{cases} x^2+3x+a & for \ x \le 1 \\ bx+2 & for \ x > 1 \end{cases}$ is every where differentiable find the values of a and b.

Solution: Given that $f(x)$ is everywhere differentiable

$\therefore f(x)$ is differentiable at $x = 1$

So, $f(x)$ is continuous at $x = 1$

$$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1}(x^2+3x+a) = \lim_{x\to 1}(bx+2) = 1+3+a$$

$$\Rightarrow 1+3+a = b+2$$

$$\Rightarrow b = 2+a \qquad \dots (i)$$

Again , $f(x)$ is differentiable at $x = 1$

$$\Rightarrow \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1} \frac{(x^2+3x+a)-(4+a)}{x-1} = \lim_{x\to 1} \frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1} \frac{x^2+3x-4}{x-1} = \lim_{x\to 1} \frac{bx-(2+a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1} \frac{(x+4)(x-1)}{x-1} = \lim_{x\to 1} \frac{bx-b}{x-1} \qquad (\because from\ (i))$$

$$\Rightarrow \lim_{x\to 1}(x+4) = \lim_{x\to 1} b$$

$$\Rightarrow 5 = b$$

Substituting value of b in (i) we get

$$5 = 2+a \Rightarrow a = 3$$

Hence $a = 3, b = 5$

IP3: If $f(x)$ is differentiable at $x = a$, find $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x-a}$.

Solution: Since $f(x)$ is differentiable at $x = a$. Therefore, $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exits.

Let $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$.

Further, $f(x)$ is differentiable at $x = a \Longrightarrow f(x)$ is continuous at $x = a$.

$$\therefore \lim_{x \to a} f(x) = f(a)$$

Now, $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x-a}$

$$= \lim_{x \to a} \frac{x^2 f(a) - a^2 f(a) + a^2 f(a) - a^2 f(x)}{x-a}$$

$$= \lim_{x \to a} \frac{(x^2 - a^2) f(x) - a^2 \big(f(x) - f(a)\big)}{x-a}$$

$$= \lim_{x \to a} \frac{(x^2 - a^2) f(x)}{x-a} - a^2 \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to a} (x+a) f(x) - a^2 f'(a)$$

$$= \lim_{x \to a} (x+a) . \lim_{x \to a} f(x) - a^2 f'(a)$$

$$= 2a f(a) - a^2 f'(a)$$

Hence $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x-a} = 2a f(a) - a^2 f'(a)$.

P3: If $f(2) = 4$ and $f'(2) = 1$, then find $\lim_{x \to 2} \frac{x f(2) - 2 f(x)}{x-2}$.

Solution: We have,

$$\lim_{x \to 2} \frac{x f(2) - 2 f(x)}{x-2}$$

$$= \lim_{x \to 2} \frac{x f(2) - 2 f(2) + 2 f(2) - 2 f(x)}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-2) f(2) - 2 (f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-2) f(2)}{x-2} - 2 \lim_{x \to 2} \frac{f(x)-f(2)}{x-2}$$

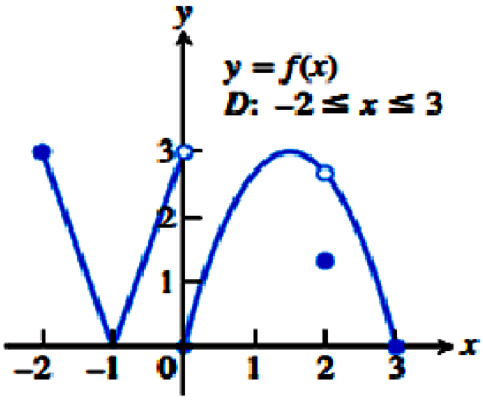
$$= \lim_{x \to 2} f(2) - 2 f'(2)$$

$$= f(2) - 2 f'(2)$$

Given that $f(2) = 4$ and $f'(2) = 1$

$$\therefore \lim_{x \to 2} \frac{x f(2) - 2 f(x)}{x-2} = 4 - 2(1) = 2$$

IP4: Figures below show the graph of a function over a closed interval D. At what domain points does the function appear to be differentiable, continuous but not differentiable, neither continuous nor differentiable.



Solution:

Differentiable:

The graph of function $f(x)$ appears differentiable on

$-2 \leq x < -1,\ -1 < x < 0,\ 0 \leq x < 2$ and $2 < x \leq 3$.

Continuous but not differentiable:

$$\lim_{x \to -1^-} f(x) = 0\ ,\ \lim_{x \to -1^+} f(x) = 0 \text{ and } f(-1) = 0$$

$$\therefore\ \lim_{x \to -1} f(x) = f(-1)$$

$\Longrightarrow f$ is continuous at $x = -1$

But f is not differentiable at $x = -1$ because there is a ***corner*** at $x = -1$.

$\Longrightarrow f$ is continuous but not differentiable at $x = -1$.

Neither continuous nor differentiable:

At $x = 0$,

$$\lim_{x \to 0^-} f(x) = 3 \text{ and } \lim_{x \to 0^+} f(x) = 0$$

$$\therefore\ \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

f is discontinuous at $x = 0$ so, f is not differentiable at $x = 0$

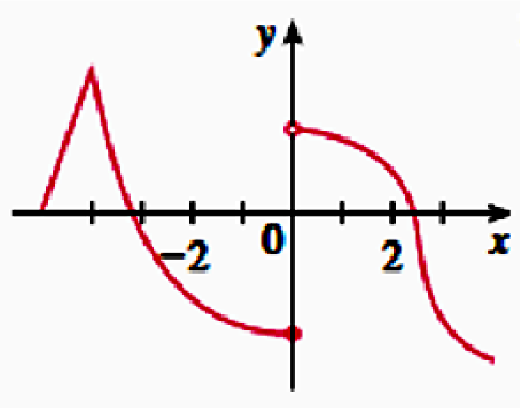
At $x = 2$, notice that

$$\lim_{x \to 2} f(x) \neq f(2)$$

$\therefore f$ is discontinuous at $x = 2$. So, f is not differentiable at $x = 2$.

Thus, f is neither continuous nor differentiable at $x = 0, x = 2$.

P4: The graph of $f(x)$ is given below. State with reason at which $f(x)$ is not differentiable.



Solution:

Observe that the graph of $f(x)$ has a ***corner*** at $x = -4$.

So, $f(x)$ is not differentiable at $x = -4$.

Now, at $x = 0$

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$$\therefore \lim_{x \to 0} f(x) \text{ is not exists.}$$

Thus, $f(x)$ is discontinuous at $x = 0$.

So, $f(x)$ is not differentiable at $x = 0$.

Hence $f(x)$ is not differentiable at $x = -4\ ,0$.

Exercises:

6. Using the alternative formula for the derivative, find the derivative of the functions at the given value of c.

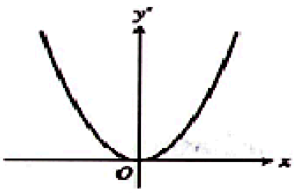
a. $f(x) = \frac{1}{x+2}\ ,\quad c = -1$

b. $g(x) = \frac{t}{x-1}\ ,\quad c = 3$

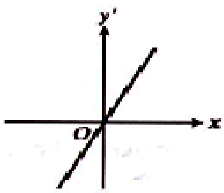
c. $f(x) = \frac{1}{(x-1)^2}\quad c = 2$

d. $g(x) = 1 + \sqrt{x}\quad c = 4$

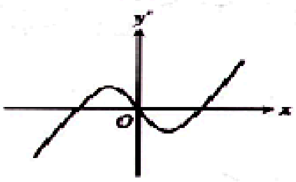
7. The following are the graphs of some derivatives



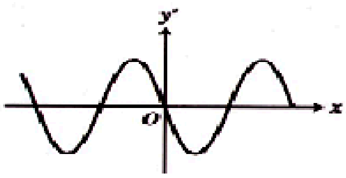
(a)



(b)



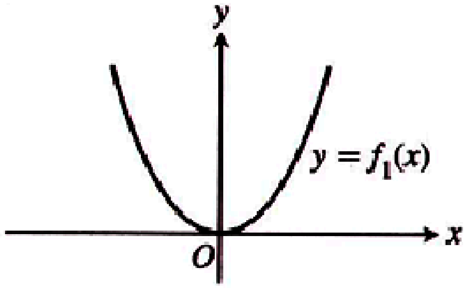
(c)



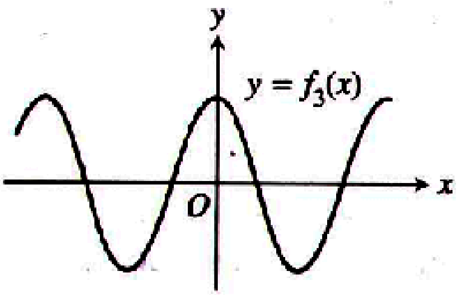
(d)

Match the functions graphed below with the derivatives graphed above.

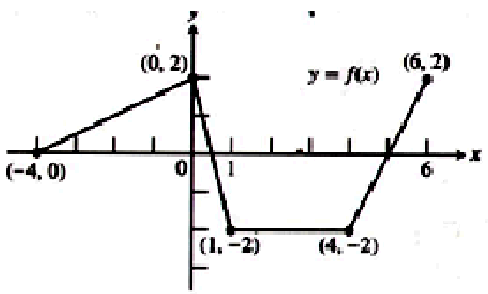
a.



b.



8. The graph below is made of line segments joined end to end.

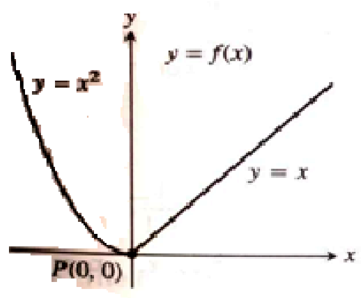


a. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.

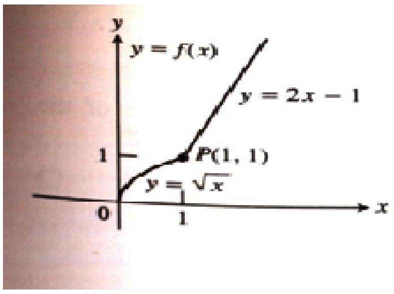
b. Graph the derivative of f. Call the vertical axis the y' axis. The graph should show a step function.

9. Compare the right-hand and left-hand derivatives to show that the functions are not differentiable at the point P.

a.

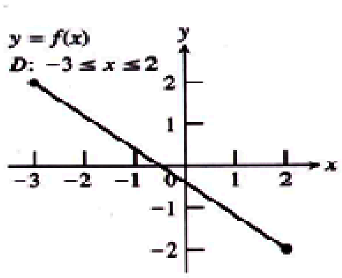


b.

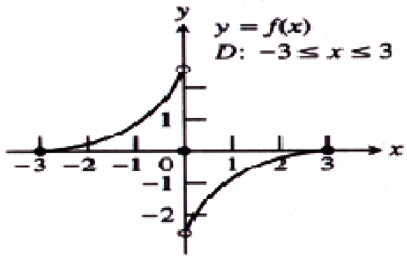


10. Figures below show the graph of a function over a closed interval D. At what domain points does the function appear to be differentiable, continuous but not differentiable, neither continuous nor differentiable.

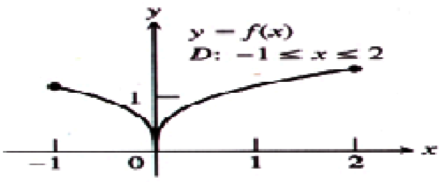
a.



b.



c.



11. Let $f(0) = 0$ and $f'(0) = 1$ for a positive integer k

Show that

$$\lim_{x\to 0}\frac{1}{x}\left[f(x) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{3}\right) + \cdots + f\left(\frac{x}{k}\right)\right] = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$

12. Discuss the differentiability of $f(x) = |x-1| + |x-2|$.

13. Discuss the continuity and differentiability of

$$f(x) = \begin{cases} 1-x & \text{for } x < 1 \\ (1-x)(2-x) & \text{for } 1 \le x \le 2 \\ 3-x & \text{for } x > 2 \end{cases}$$

4.3. Differentiation Rules – Sums and Differences

Learning objectives:

- To formulate (i) the power rule for positive integers, (ii) the constant multiple rule, (iii) the sum rule and (iv) the difference rule for differentiation.
  And
- To practice the related problems.

In this module and the next, we formulate some rules for differentiation of functions without having to apply the definition each time.

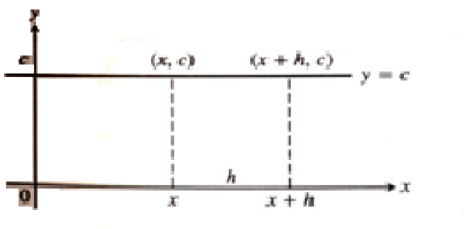
The first rule of differentiation is that *the derivative of every constant function is zero.*

Rule 1: Derivative of a Constant

If c is a constant, then $\frac{d}{dx}(c) = 0$

Proof:

We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c.



At every value of x, we find that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = 0$$

Example 1: $\frac{d}{dx}(8) = 0$, $\frac{d}{dx}\left(-\frac{1}{2}\right) = 0$, $\frac{d}{dx}(\sqrt{3}) = 0$

The next rule tells how to differentiate x^n , if n is a positive integer.

Rule 2: Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

Proof: Let $f(x) = x^n$.Since n is a positive integer, we use the fact that

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Now, $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

$= \lim_{h \to 0} \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)(x)^{n-2} + (x)^{n-1}]}{h}$

$= \lim_{h \to 0} [(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]$

$= nx^{n-1}$

Example 2: $\frac{d}{dx} x = 1 , \frac{d}{dx} x^2 = 2x , \frac{d}{dx} x^3 = 3x^2$

The next rule says that *when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.*

Rule 3: The Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(c\,u) = c\,\frac{du}{dx}$$

Proof: $\quad \frac{d}{dx}(cu) = \lim_{h \to 0} \frac{c\,u(x+h) - c\,u(x)}{h}$

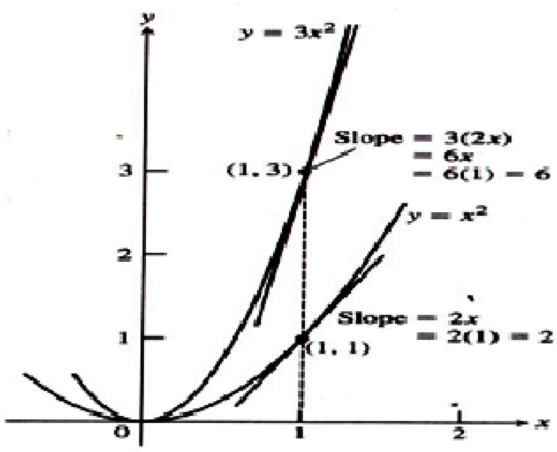
$= c \lim_{h \to 0} \frac{u(x+h)-u(x)}{h} \quad = c\frac{du}{dx}$

In particular, if n is a positive integer, then $\frac{d}{dx}(cx^n) = cnx^{n-1}$

Example 3: The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \times 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3.



A special case:

The derivative of the negative of a differentiable function is the negative of the function's derivative.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1.u) = -1.\frac{d}{dx}(u) = -\frac{du}{dx}$$

The next rule says that *the derivative of the sum of two differentiable functions is the sum of their derivatives.*

Rule 4: The Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

Proof:

$$\frac{dy}{dx}\big(u(x)+v(x)\big)=\lim_{h\to 0}\frac{[u(x+h)+v(x+h)]-[u(x)-v(x)]}{h}$$
$$=\lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\right]$$
$$=\lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right]+\lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\right]$$
$$=\frac{du}{dx}+\frac{dv}{dx}$$

Combining the Sum Rule with the Constant Multiple Rule gives the equivalent ***Difference Rule**,* which says that *the derivative of a difference of differentiable functions is the difference of their derivatives.*

$$\frac{d}{dx}(u-v)=\frac{d}{dx}(u+(-1)v)=\frac{du}{dx}+(-1)\frac{dv}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions. If $u_1, u_2, \dots\dots.., u_n$ are differentiable at x, then so is $u_1+u_2+\cdots+u_n$, and

$$\frac{d}{dx}(u_1+u_2+\cdots+u_n)=\frac{du_1}{dx}+\frac{du_2}{dx}+\cdots+\frac{du_n}{dx}$$

Proof:

We recall the principle of mathematical induction. The steps in proving a formula by induction are

1. Check that it holds for $n = n_0$.
2. Prove that if it holds for any positive integer $n = k \geq n_0$ then it holds for $n = k + 1$.

We now prove the Sum Rule for sums of more than two functions. We prove the statement

$$\frac{d}{dx}(u_1+u_2+\cdots+u_n)=\frac{du_1}{dx}+\frac{du_2}{dx}+\cdots+\frac{du_n}{dx}$$

by mathematical induction. We assume that the rule holds for $n = k$,

$i.e.,$ $\frac{d}{dx}(u_1+u_2+\cdots+u_k)=\frac{du_1}{dx}+\frac{du_2}{dx}+\cdots+\frac{du_k}{dx}$

Now,

$$\frac{d}{dx}(u_1+u_2+\cdots+u_k+u_{k+1})=\frac{d}{dx}(u_1+u_2+\cdots+u_k)+\frac{du_{k+1}}{dx}$$
$$=\frac{du_1}{dx}+\frac{du_2}{dx}+\cdots+\frac{du_k}{dx}+\frac{du_{k+1}}{dx}$$

Thus, if the statement is true for $n = k$, it is true for $n = k+1$. The statement is true for $n = 2$. Therefore, the Sum Rule is true for every integer $n \geq 2$.

Example 5:

$$a)\ y = x^4 + 12x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4\right) + \frac{d}{dx}(12x) = 4x^3 + 12$$

b) $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$
$$= 3x^2 + \frac{8}{3}x - 5$$

We can *differentiate any polynomial term by term,* the way we differentiated the polynomials in the examples above.

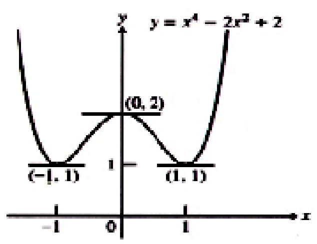
Example 6: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so , where?

Solution: The horizontal tangents occur where the slope $\frac{dy}{dx}$ is zero.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4 - 2x^2 + 2\right) = 4x^3 - 4x = 0$$

$$4x\left(x^2 - 1\right) = 0 \qquad\qquad x = 0, 1, -1$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0$, 1 and -1. The corresponding points on the curve are $(0, 2)$, $(1, 1)$, and $(-1, 1)$.



PROBLEM SET

IP1: Every polynomial is differentiable everywhere.

Proof:

Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be any polynomial. First notice that $1, x, x^2, \ldots, x^n$ are differentiable everywhere and $a_0, a_1x, a_2x^2, \ldots, a_nx^n$ are differentiable everywhere.

Now,

$$\frac{d}{dx}p(x) = \frac{d}{dx}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$

$$= \frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2) + \cdots + \frac{d}{dx}(a_nx^n)$$

(sum rule)

$$= 0 + a_1\frac{d}{dx}(x) + a_2\frac{d}{dx}(x^2) + \cdots + a_n\frac{d}{dx}(x^n)$$

(constant multiple rule)

$$= a_1 + 2a_3x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

(power rule)

Thus, every polynomial is differentiable everywhere.

P1: If $y = 6x^5 + 3x^2 + 3x + 1$, then find $\frac{dy}{dx}|_{x=1}$.

Solution: Given $y = 6x^5 + 3x^2 + 3x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^5) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(1)$$

$$= 6\left(\frac{d}{dx}(x^5)\right) + 3\left(\frac{d}{dx}(x^2)\right) + 3\left(\frac{d}{dx}(x)\right) + 0$$

$$= 6(5x^{5-1}) + 3(2x^{2-1}) + 3(1)$$

$$= 30x^4 + 6x + 3$$

Now,

$$\frac{dy}{dx}|_{x=1} = 30(1)^4 + 6(1) + 3 = 39.$$

IP2: Find the points on the curve $y = -\frac{1}{3}x^3 + 2x^2$ at which the tangent line has the slope – 5.

Solution: The given curve is: $y = -\frac{1}{3}x^3 + 2x^2$

Differentiating the curve with respect to x.

$$\frac{dy}{dx} = \frac{d}{dx}(-\frac{1}{3}x^3 + 2x^2)$$

$$= \frac{d}{dx}\left(-\frac{1}{3}x^3\right) + \frac{d}{dx}(2x^2)$$

$$= -\frac{1}{3}\frac{d}{dx}(x^3) + 2\frac{d}{dx}(x^2)$$

$$= -\frac{1}{3}(3x^2) + 2(2x)$$

$$\frac{dy}{dx} = -x^2 + 4x$$

By the hypothesis slope of the tangent line is -5 so that,

$$\frac{dy}{dx} = -5$$

$$\Rightarrow\ -x^2 + 4x = -5$$

$$\Rightarrow\ x^2 - 4x - 5 = 0$$

$$\Rightarrow\ (x+1)(x-5) = 0$$

$$\Rightarrow x = -1,5$$

When $x = -1\ \Rightarrow y = -\frac{1}{3}(-1)^3 + 2(-1)^2 = \frac{7}{3}$

$x = 5\quad \Rightarrow y = -\frac{1}{3}(5)^3 + 2(5)^2 = \frac{25}{3}$

Points on the curve at which the tangent line has the slope -5 are $\left(-1,\frac{7}{3}\right), \left(5,\frac{25}{3}\right)$.

P2: Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

Solution: The given curve is: $y = 2x^3 + 3x^2 - 12x + 1$

Differentiating the curve with respect to x.

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 + 3x^2 - 12x + 1)$$

$$= \frac{d}{dx}(2x^3) + \frac{d}{dx}(3x^2) - \frac{d}{dx}(12x) + \frac{d}{dx}(1)$$

$$= 2\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) - 12\frac{d}{dx}(x) + \frac{d}{dx}(1)$$

$$= 2(3x^2) + 3(2x) - 12(1) + 0$$

$$= 6x^2 + 6x - 12 = 6(x^2 + x - 2)$$

Curve has horizontal tangent where the slope of the tangent is zero.

$$i.e.,\ \frac{dy}{dx} = 0$$

$$\Rightarrow\ 6(x^2 + x - 2) = 0$$

$$\Rightarrow\ x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$$

$$\Rightarrow x = 1, -2$$

Given curve have horizontal tangents at $x = 1, -2$.

Corresponding points on the curve are $(1, -6), (-2, 21)$.

IP3: For the function $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1$ prove that $f'(1) = 100f'(0)$.

Solution: The given function is:

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1$$

Differentiating both sides with respect to x, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1\right)$$

$$= \frac{d}{dx}\left(\frac{x^{100}}{100}\right) + \frac{d}{dx}\left(\frac{x^{99}}{99}\right) + \cdots + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$$

$$= x^{99} + x^{98} + \cdots + x^2 + x + 1 + 0$$

$$f'(x) = 1 + x + x^2 + \cdots + x^{98} + x^{99}$$

Now,

$$f'(0) = 1 + (0)^2 + \cdots + (0)^{98} + (0)^{99} = 1$$

$$f'(1) = 1 + 1 + (1)^2 + \cdots + (1)^{98} + (1)^{99}$$

$$= 1 + 1 + \cdots + 1 \quad (100\ times)$$

$$= 100$$

$$\therefore f'(1) = 100f'(0)$$

P3: If $f(x) = 1 + x + x^2 + \cdots + x^{100}$, then find $f'(1)$.

Solution: Given that $f(x) = 1 + x + x^2 + \cdots + x^{100}$.

$$f'(x) = \frac{d}{dx}(1 + x + x^2 + \cdots + x^{100})$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \cdots + \frac{d}{dx}(x^{100})$$

$$= 0 + 1 + 2x + \cdots + 100x^{99}$$

$$= 1 + 2x + \cdots + 100x^{99}$$

Now, $f'(1) = 1 + 2(1) + \cdots + 100(1)^{99}$

$$= 1 + 2 + \cdots + 100$$

Sum of n natural numbers is $\frac{n(n+1)}{2}$.

Here $n = 100$

$$f'(1) = \frac{100(100+1)}{2} = 50(101) = 5050.$$

IP4: If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, then prove that $\frac{dy}{dx} = y$.

Solution: Given that $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\frac{dy}{dx} = \frac{d}{dx}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(1) + \frac{d}{dx}\left(\frac{x}{1!}\right) + \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \cdots$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(1) + \frac{1}{1!}\frac{d}{dx}(x) + \frac{1}{2!}\frac{d}{dx}(x^2) + \frac{1}{3!}\frac{d}{dx}(x^3) + \cdots$$

$$\Rightarrow \frac{dy}{dx} = 0 + \frac{1}{1!}(1) + \frac{1}{2!}(2x) + \frac{1}{3!}(3x^2) + \cdots$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow \frac{dy}{dx} = y$$

P4: If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$, then

Solution: Given that $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}\left(\frac{x}{1!}\right) + \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \cdots + \frac{d}{dx}\left(\frac{x^n}{n!}\right)$$

$$= \frac{d}{dx}(1) + \frac{1}{1!}\frac{d}{dx}(x) + \frac{1}{2!}\frac{d}{dx}(x^2) + \frac{1}{3!}\frac{d}{dx}(x^3) + \cdots + \frac{1}{n!}\frac{d}{dx}(x^n)$$

$$= 0 + \frac{1}{1!}(1) + \frac{1}{2!}(2x) + \frac{1}{3!}(3x^2) + \cdots + \frac{1}{n!}(nx^{n-1})$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!}$$

$$= \left\{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\right\} - \frac{x^n}{n!}$$

$$\Rightarrow \frac{dy}{dx} = y - \frac{x^n}{n!} \Rightarrow \frac{dy}{dx} - y + \frac{x^n}{n!} = 0$$

Exercise:

1. Find the derivative of the given functions.

   a. $y = -x^2 + 3$

   b. $s = 5t^3 - 3t^5$

   c. $y = \frac{4x^3}{3} - x$

   d. $y = 6x^2 - 10x$

   e. $y = 6x^{100} - x^{55} + x$

2. Find the derivative of $f(x) = 1 + x + x^2 + \cdots + x^{50}$ at $x = 1$.
3. If $y = \frac{2x^9}{3} - \frac{5}{7}x^7 + 6x^3 - x$, then find $\frac{dy}{dx}$ at $x = 1$.
4. If $f(x) = \lambda x^2 + \mu x + 12$, $f'(4) = 15$ and $f'(2) = 11$
   Then find the values of λ and μ.
5. If $f(x) = \alpha x^n$, then prove that $\alpha = \frac{f'(1)}{n}$.
6. If $f(x) = mx + c$ and $f(0) = f'(0) = 1$ then $f(2) =?$
7. For what values of x does the graph of function
   $f(x) = x^3 + 3x^2 + x + 3$ have the horizontal tangent.

8. Find an equation for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the point of tangency.
9. The curve $y = ax^2 + bx + c$ passes through the point $(1,2)$ and is tangent to the line $y = x$ at the origin. Find a, b, and c.
10. The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1,0)$. Find a, b and c.
11. Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1,0)$.
12. Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.
13.
    a. Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2,1)$.
    b. Find equation for the tangents to the above curve at the points where the slope of the curve is 8.

4.4. Differentiation Rules – Products and Quotients

Learning objectives:

- To derive the following rules of differentiation:
    - The product rule.
    - The quotient rule.
    - Power Rule for negative Integers.

And

- To practice related problems.

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not the product of their derivatives*. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$$

The derivative of a product of two functions is the sum of *two* products, as we see now from the next rule.

Rule 5: The Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In prime notation,

$$(uv)' = uv' + vu'.$$

Proof:

$$\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right] \\
&= \lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \\
&= u\frac{dv}{dx} + v\frac{du}{dx}
\end{aligned}$$

Example 1:Find the derivative of $y = (x^2+1)(x^3+3)$.

Solution:

From the product rule with $u = x^2+1$ and $v = x^2+3$, we find

$$\begin{aligned}
\frac{d}{dx}\left[(x^2+1)(x^3+3)\right] &= (x^2+1)\frac{d}{dx}(x^3+3) + (x^3+3)\frac{d}{dx}(x^2+1) \\
&= (x^2+1)(3x^2) + (x^3+3)(2x) \\
&= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x
\end{aligned}$$

The above example can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial.

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x$$

There are times, however, when the product rule must be used.

Quotients

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is *not the quotient of their derivatives.*

Rule 6: The Quotient Rule

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

Proof:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h \to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}$$

$$= \lim_{h \to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

We subtract and add $v(x)u(x)$ in the numerator. We then get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h \to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h \to 0} \frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}$$

Taking the limit in the numerator and denominator now gives the Quotient Rule.

Example 2: Find the derivative of $y = \frac{t^2-1}{t^2+1}$.

Solution:

We apply the quotient rule with $u = t^2 - 1$ and $v = t^2 + 1$

$$\frac{dy}{dt} = \frac{(t^2+1)\frac{d}{dt}(t^2-1) - (t^2-1)\frac{d}{dt}(t^2+1)}{(t^2+1)^2}$$

$$= \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

The Power Rule for negative integers is the same as the rule for positive integers.

Rule 7: Power Rule for Negative Integers

If n is a negative integer and $n \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof:

If n is a negative integer, then $n = -m$ where m is a positive integer. Hence, $x^n = x^{-m} = \frac{1}{x^m}$ and

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \frac{d}{dx}(1) - 1.\frac{d}{dx}(x^m)}{(x^m)^2}$$

$$= \frac{0 - mx^{m-1}}{x^{2m}} = -mx^{-m-1}$$

$$= nx^{n-1}$$

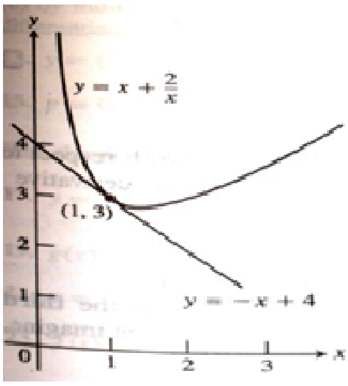
Example 3:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

Example 4:

Find an equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1,3)$.



Solution:

The slope of the curve is

$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 - \frac{2}{x^2}$$

The slope at $x = 1$ is

$$\frac{dy}{dx}\Big|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1$$

The line through $(1,3)$ with slope $m = -1$ is

$$y - 3 = (-1)(x - 1)$$

$$y = -x + 4$$

Example 5:Find the derivative of $y = \frac{(x-1)(x^2-2x)}{x^4}$.

Solution:

This problem can be solved using the Quotient Rule. But it is more easily solved by expanding the numerator, dividing by x^4, and then applying the Sum and Power Rules.

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}$$

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$$

PROBLEM SET

IP1: If $y = (\sqrt{x}+1)(x^2-4x+2)$, then find $\frac{dy}{dx}$.

Solution: $y = (\sqrt{x}+1)(x^2-4x+2)$

Differentiating y with respect to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left((\sqrt{x}+1)(x^2-4x+2)\right)$$

Applying the product rule with $u = \sqrt{x}+1$ and $v = (x^2-4x+2)$.

$$\frac{dy}{dx} = (\sqrt{x}+1)\frac{d}{dx}(x^2-4x+2) + (x^2-4x+2)\frac{d}{dx}(\sqrt{x}+1)$$

$$\Rightarrow \frac{dy}{dx} = (\sqrt{x}+1)(2x-4) + (x^2-4x+2)\left(\frac{1}{2\sqrt{x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = 2x\sqrt{x} - 4\sqrt{x} + 2x - 4 + \frac{1}{2}x\sqrt{x} - 2\sqrt{x} + \frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{2}x\sqrt{x} - 6\sqrt{x} + \frac{1}{\sqrt{x}} + 2x - 4$$

P1: If $y = (\sqrt{x}-3x)\left(x+\frac{1}{x}\right)$, then find $\frac{dy}{dx}$ at $x = 1$.

Solution: Given $y = (\sqrt{x}-3x)\left(x+\frac{1}{x}\right)$.

$$\frac{dy}{dx} = \frac{d}{dx}\left((\sqrt{x}-3x)\left(x+\frac{1}{x}\right)\right)$$

Applying the product rule with $u = \sqrt{x}-3x$ and $v = x+\frac{1}{x}$

$$\frac{dy}{dx} = (\sqrt{x}-3x)\frac{d}{dx}\left(x+\frac{1}{x}\right) + \left(x+\frac{1}{x}\right)\frac{d}{dx}(\sqrt{x}-3x)$$

$$\Rightarrow \frac{dy}{dx} = (\sqrt{x}-3x)\left(\frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{1}{x}\right)\right) + \left(x+\frac{1}{x}\right)\left(\frac{d}{dx}(\sqrt{x}) - \frac{d}{dx}(3x)\right)$$

$$\Rightarrow \frac{dy}{dx} = (\sqrt{x}-3x)\left(1-\frac{1}{x^2}\right) + \left(x+\frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}} - 3\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\sqrt{x} - \frac{1}{x\sqrt{x}} - 3x + \frac{3}{x}\right) + \left(\frac{\sqrt{x}}{2} - 3x + \frac{1}{2x\sqrt{x}} - \frac{3}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{3\sqrt{x}}{2} - \frac{1}{2x\sqrt{x}} - 6x$$

Now, $\frac{dy}{dx}$ at $x = 1$ is

$$\frac{dy}{dx}\Big|_{x=1} = \frac{3\sqrt{1}}{2} - \frac{1}{2(1)\sqrt{1}} - 6(1) = -5$$

Hence the value of $\frac{dy}{dx}$ at $x = 1$ is -5.

IP2: If $y = \frac{px^2+qx+r}{ax+b}$, then find $\frac{dy}{dx}$.

Solution: Given $y = \frac{px^2+qx+r}{ax+b}$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{px^2+qx+r}{ax+b}\right)$$

We have the quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Here $u = px^2 + qx + r$ and $v = ax + b$

$$\frac{d}{dx}\left(\frac{px^2+qx+r}{ax+b}\right) = \frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$= \frac{(ax+b)(2px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$= \frac{2pax^2+aqx+2pbx+bq-pax^2-aqx-ar}{(ax+b)^2}$$

$$= \frac{pax^2+2pbx+bq-ar}{(ax+b)^2}$$

Hence $\frac{dy}{dx} = \frac{pax^2+2pbx+bq-ar}{(ax+b)^2}$.

P2: $\frac{d}{dx}\left(\frac{x^2-x+1}{x^2+x+1}\right) =$

Solution: We have the quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Here $u = x^2 - x + 1$ and $v = x^2 + x + 1$

$$\frac{d}{dx}\left(\frac{x^2-x+1}{x^2+x+1}\right) = \frac{(x^2+x+1)\frac{d}{dx}(x^2-x+1)-(x^2-x+1)\frac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{(x^2+x+1)(2x-1)-(x^2-x+1)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{(2x^3-x^2+2x^2-x+2x-1)-(2x^3+x^2-2x^2-x+2x+1)}{(x^2+x+1)^2}$$

$$= \frac{2x^2-2}{(x^2+x+1)^2} = \frac{2(x^2-1)}{(x^2+x+1)^2}$$

IP3: If $y = (1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\left(1+x^{2^n}\right)$, then find $\frac{dy}{dx}$.

Solution: We have, $y = (1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\left(1+x^{2^n}\right)$.

Multiply and divide with $(1-x)$

$$y = \frac{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\left(1+x^{2^n}\right)}{(1-x)}$$

$$\Rightarrow \quad y = \frac{(1-x^2)(1+x^2)(1+x^4)(1+x^8)\dots\left(1+x^{2^n}\right)}{(1-x)}$$

$$\Rightarrow \quad y = \frac{(1-x^4)(1+x^4)(1+x^8)\dots\left(1+x^{2^n}\right)}{(1-x)} = \frac{\left(1-x^{2^{n+1}}\right)}{(1-x)}$$

Now differentiating with respect to x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\left(1-x^{2^{n+1}}\right)}{(1-x)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1-x)\frac{d}{dx}\left(1-x^{2^{n+1}}\right)-\left(1-x^{2^{n+1}}\right)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1-x)\left(-2^{n+1}x^{2^{n+1}-1}\right)+\left(1-x^{2^{n+1}}\right)}{(1-x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2^{n+1}x^{2^{n+1}-1}+2^{n+1}x^{2^{n+1}}+1-x^{2^{n+1}}}{(1-x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2^{n+1}x^{2^{n+1}-1}+x^{2^{n+1}}(2^{n+1}-1)+1}{(1-x)^2}$$

P3: The function $\boldsymbol{f(x)}$ is given by

$\boldsymbol{f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)}$. Then find $\boldsymbol{f'(-1)}$.

Solution:

The given function $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$.

Multiply and divide with $(1-x)$

$$f(x) = \frac{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)}{(1-x)}$$

$$\Rightarrow f(x) = \frac{(1-x^2)(1+x^2)(1+x^4)(1+x^8)}{(1-x)}$$

$$\Rightarrow f(x) = \frac{(1-x^4)(1+x^4)(1+x^8)}{(1-x)} = \frac{(1-x^8)(1+x^8)}{(1-x)} = \frac{(1-x^{16})}{(1-x)}$$

Differentiating $f(x)$ with respect to x

$$f'(x) = \frac{d}{dx}\left(\frac{(1-x^{16})}{(1-x)}\right)$$

$$\Rightarrow f'(x) = \frac{(1-x)\frac{d}{dx}(1-x^{16})-(1-x^{16})\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$\Rightarrow f'(x) = \frac{(1-x)\left(-16x^{15}\right)-(1-x^{16})(-1)}{(1-x)^2}$$

$$\Rightarrow f'(x) = \frac{(x-1)\left(16x^{15}\right)+(1-x^{16})}{(1-x)^2}$$

Now,

$$f'(-1) = \frac{(-1-1)\left(16(-1)^{15}\right)+(1-(-1)^{16})}{(1+1)^2}$$

$$= \frac{(-2)(16(-1))+(1-1)}{(2)^2} = \frac{32}{4} = 8.$$

IP4: If $u(5) = 1, u'(5) = 3, v(5) = 2$ and $v'(5) = -2$, then find the following values:

a. $\frac{d}{dx}(uv)$ at $x = 5$
b. $\frac{d}{dx}\left(\frac{u}{v}\right)$ at $x = 5$
c. $\frac{d}{dx}\left(\frac{v}{u}\right)$ at $x = 5$

Solution:

a) $\frac{d}{dx}(uv) = uv' + vu'$

$$\frac{d}{dx}(uv)|_{x=5} = u(5)v'(5) + v(5)u'(5)$$

Substituting the given values

$$\frac{d}{dx}(uv)|_{x=5} = (1)(-2) + (2)(3) = 4.$$

b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu'-uv'}{v^2}$

$$\frac{d}{dx}\left(\frac{u}{v}\right)|_{x=5} = \frac{v(5)u'(5)-u(5)v'(5)}{\left(v(5)\right)^2}$$

Substituting the given values

$$\frac{d}{dx}\left(\frac{u}{v}\right)|_{x=5} = \frac{(2)(3)-(1)(-2)}{(2)^2} = \frac{8}{4} = 2.$$

c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv'-vu'}{u^2}$

$$\frac{d}{dx}\left(\frac{u}{v}\right)|_{x=5} = \frac{u(5)v'(5)-v(5)u'(5)}{\left(u(5)\right)^2}$$

Substituting the given values

$$\frac{d}{dx}\left(\frac{v}{u}\right)|_{x=5} = \frac{(1)(-2)-(2)(3)}{(1)^2} = -8.$$

P4: Suppose u and v are functions of x that are differentiable at $x = 0$ and $u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2.$

Then find the values of $\frac{d}{dx}(uv)$ and $\frac{d}{dx}\left(\frac{u}{v}\right)$ at $x = 0.$

Solution:

The given values are

$$u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2$$

$$\frac{d}{dx}(uv) = uv' + vu'$$

$$\Longrightarrow \frac{d}{dx}(uv)|_{x=0} = u(0)v'(0) + v(0)u'(0)$$

Substituting the given values

$$\Longrightarrow \frac{d}{dx}(uv)|_{x=0} = (5)(2) + (-1)(-3) = 13$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{\left(v(0)\right)^2}$$

Substituting the given values

$$\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)|_{x=0} = \frac{(-1)(-3) - (5)(2)}{(-1)^2} = \frac{3-10}{1} = -7$$

Exercise:

14. Find y' both by applying the Product Rule and by multiplying the factors to produce a sum of simpler terms to differentiate.

a. $y = (3 - x^2)(x^3 - x + 1)$

b. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$

c. $y = (x - 1)(x^2 + x + 1)$

d. $y = \left(x + \frac{1}{x}\right)(x - \frac{1}{x} + 1)$

15. Find the derivatives of the functions

a. $y = \frac{2x + 5}{3x - 2}$

b. $g(x) = \frac{x^2 - 4}{x + 0.5}$

c. $v = (1 - t)(1 + t^2)^{-1}$

d. $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$

e. $z = \frac{2x+1}{x^2-1}$

f. $f(t) = \frac{t^2-1}{t^2+t-2}$

g. $w = (2x - 7)^{-1}(x + 5)$

h. $u = \frac{5x+1}{2\sqrt{x}}$

i. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$

16. Suppose u and v are differentiable functions of x and that $u(1) = 2$, $u'(1) = 0$, $v(1) = 5$, $v'(1) = -1$ find the values of the following derivatives at $x = 1$

a. $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **d.** $\frac{d}{dx}(7v-2u)$

17. find the equation of the tangent line to the curve at the specified point

a. $y=\frac{2x}{x^2+1}$ at $(1,1)$.

b. $y=\frac{\sqrt{x}}{x-2}$ at $(4,1)$.

4.5. Second and Higher Order Derivatives

Learning objectives:

- To find the second and higher order derivatives of a function $f(x)$.

  And

- To practice the related problems.

The derivative $y'=\frac{dy}{dx}$ is the ***first (first order) derivative*** of y with respect to x. This derivative may itself be a differentiable function of x; if so, its derivative

$$y''=\frac{dy'}{dx}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$$

is called the ***second (second order) derivative*** of y with respect to x. The symbol $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ does not mean multiplication. It means *the derivative of the derivative.*

If y'' is differentiable, its derivative $y'''=\frac{dy''}{dx}=\frac{d^3y}{dx^3}$ is the ***third (third order) derivative*** of y with respect to x. The names continue with

$$y^{(n)}=\frac{d}{dx}y^{(n-1)}$$

denoting ***the n^{th} (n^{th} order) derivative*** of y with respect to x, for any positive integer x.

We can interpret the second derivative as the rate of change of the slope of the tangent to the graph of $y=f(x)$ at each point.

Example 1: The first four derivatives of $y=x^3-3x^2+2$ are

First Derivative: $y'=3x^2-6x$

Second Derivative: $y''=6x-6$

Third Derivative: $y'''=6$

Fourth Derivative: $y^{(4)}=0$

The function has derivatives of all orders, the fifth and later derivatives all being zero.

PROBLEM SET

IP1: Find $\frac{d^4}{dx^4}[5x^6+2x^5-9x^3+32x-1]$.

Solution: Let $y=5x^6+2x^5-9x^3+32x-1$

$$\frac{dy}{dx}=\frac{d}{dx}(5x^6+2x^5-9x^3+32x-1)$$

$$=5(6x^5)+2(5x^4)-9(3x^2)+32(1)-0$$

$$=30x^5+10x^4-27x^2+32$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(30x^5+10x^4-27x^2+32)$$

$$=30(5x^4)+10(4x^3)-27(2x)+0$$

$$=150x^4+40x^3-54x$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}(150x^4+40x^3-54x)$$

$$=150(4x^3)+40(3x^2)-54$$

$$=600x^3+120x^2-54$$

$$\frac{d^4y}{dx^4}=\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)=\frac{d}{dx}(600x^3+120x^2-54)$$

$$=1800x^2+240x$$

$$\therefore \frac{d^4}{dx^4}[5x^6+2x^5-9x^3+32x-1]=1800x^2+240x.$$

P1: Find the third derivative of $y=\frac{(2x^2+6)(x^3+2x^2-4)}{(x-1)^3+(x+1)^3}$ at $x=1$.

Solution:

Given $\quad y=\frac{(2x^2+6)(x^3+2x^2-4)}{(x-1)^3+(x+1)^3}$

$$y=\frac{(2x^2+6)(x^3+2x^2-4)}{(x^3-3x^2+3x-1)+(x^3+3x^2+3x+1)}$$

$$\Rightarrow \quad y=\frac{(2x^2+6)(x^3+2x^2-4)}{2x^3+6x}$$

$$\Rightarrow \quad y=\frac{x^3+2x^2-4}{x}=x^2+2x-\frac{4}{x} \qquad \ldots(1)$$

Differentiating (1) w.r.t x

$$y'=\frac{d}{dx}\left(x^2+2x-\frac{4}{x}\right)=2x+2+\frac{4}{x^2} \qquad \ldots(2)$$

Differentiating (2) w.r.t x

$$y''=\frac{d}{dx}\left(2x+2+\frac{4}{x^2}\right)$$

$$\Rightarrow \quad y''=2+0+4\left(\frac{-2}{x^3}\right)=2-\frac{8}{x^3} \qquad \ldots(3)$$

Differentiating (3) w.r.t x

$$y''' = \frac{d}{dx}\left(2 - \frac{8}{x^3}\right) = 0 - 8(-3x^{-4}) = \frac{24}{x^4}$$

At $x = 1,\ y'''(1) = \frac{24}{(1)^4} = 24.$

IP2: If $y = \frac{x^2+4x-1}{2x+1}$, then find y''.

Solution: Given $y = \frac{x^2+4x-1}{2x+1}$

Differentiating y w.r.t x

$$y' = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+4x-1}{2x+1}\right)$$

$$= \frac{(2x+1)\frac{d}{dx}(x^2+4x-1)-(x^2+4x-1)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$= \frac{(2x+1)(2x+4)-(x^2+4x-1)(2)}{(2x+1)^2}$$

$$= \frac{(4x^2+10x+4)-(2x^2+8x-2)}{(2x+1)^2}$$

$$= \frac{2x^2+2x+6}{(2x+1)^2}$$

Differentiating y' w.r.t x

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{2x^2+2x+6}{(2x+1)^2}\right)$$

$$= \frac{(4x^2+4x+1)\frac{d}{dx}(2x^2+2x+6)-(2x^2+2x+6)\frac{d}{dx}(4x^2+4x+1)}{(2x+1)^4}$$

$$= \frac{(4x^2+4x+1)(4x+2)-(2x^2+2x+6)(8x+4)}{(2x+1)^4}$$

$$= \frac{(2x+1)[8x^2+8x+2-8x^2-8x-24]}{(2x+1)^4}$$

$$y'' = \frac{-22}{(2x+1)^3}$$

P2: If $y = \frac{x^2+x+2}{x-1}$, then find y'' at $x = 3$.

Solution: Given $y = \frac{x^2+x+2}{x-1}$

Differentiating y w.r.t x

$$y' = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+x+2}{x-1}\right)$$

$$= \frac{(x-1)\frac{d}{dx}(x^2+x+2)-(x^2+x+2)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x+1)-(x^2+x+2)(1)}{(x-1)^2}$$

$$= \frac{(2x^2-x-1)-(x^2+x+2)}{(x-1)^2}$$

$$= \frac{x^2-2x-3}{(x-1)^2}$$

Differentiating y' w.r.t x

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{x^2-2x-3}{(x-1)^2}\right)$$

$$= \frac{(x^2-2x+1)\frac{d}{dx}(x^2-2x-3)-(x^2-2x-3)\frac{d}{dx}(x^2-2x+1)}{(x-1)^4}$$

$$= \frac{(x^2-2x+1)(2x-2)-(x^2-2x-3)(2x-2)}{(x-1)^4}$$

$$= \frac{2(x-1)[x^2-2x+1-x^2+2x+3]}{(x-1)^4}$$

$$= \frac{2(4)}{(x-1)^3} = \frac{8}{(x-1)^3}$$

y'' at $x = 3$

$$y''(3) = \frac{8}{(3-1)^3} = \frac{8}{8} = 1.$$

IP3: If $y = ax + \frac{b}{x}$ then prove that $x^2y'' + xy' = y$.

Solution: Given $y = ax + \frac{b}{x}$ ...(1)

Differentiating (1) w.r.t x

$$y' = \frac{d}{dx}\left(ax + \frac{b}{x}\right)$$

$$\Rightarrow \quad y' = a - \frac{b}{x^2} \qquad \text{... (2)}$$

Again differentiating (2) w.r.t x

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(a - \frac{b}{x^2}\right) = 0 - b\left(\frac{-2}{x^3}\right) = \frac{2b}{x^3}$$

Now, $\quad x^2y'' + xy' = x^2\left(\frac{2b}{x^3}\right) + x\left(a - \frac{b}{x^2}\right)$

$$= \frac{2b}{x} + ax - \frac{b}{x} = ax + \frac{b}{x} = y$$

Thus, $x^2y'' + xy' = y$.

P3: If $y = ax^{n+1} + bx^{-n}$, then $x^2y'' =$

Solution: We have $y = ax^{n+1} + bx^{-n}$ ...(1)

Differentiating (1) w.r.t x

$$y' = \frac{d}{dx}(ax^{n+1} + bx^{-n})$$

$$= a(n+1)x^n + b(-n)x^{-n-1} \qquad \text{... (2)}$$

Again differentiating (2) w.r.t x

$$y'' = \frac{d}{dx}(a(n+1)x^n - nbx^{-n-1})$$

$$= an(n+1)x^{n-1} - nb(-n-1)x^{-n-2}$$

$$= n(n+1)[ax^{n-1} + bx^{-n-2}]$$

Multiplying by x^2 on both sides

$$x^2y'' = n(n+1)[ax^{n+1} + bx^{-n}]$$

$$\Longrightarrow \quad x^2y'' = n(n+1)y$$

IP4: If $y = \left(\frac{x^3+3}{12x}\right)\left(\frac{x^4-1}{x^3}\right)$, then find y''.

Solution: Given $y = \left(\frac{x^3+3}{12x}\right)\left(\frac{x^4-1}{x^3}\right)$

$$y = \frac{x^7-x^3+3x^4-3}{12x^4} = \frac{1}{12}\left(x^3 - \frac{1}{x} + 3 - \frac{3}{x^4}\right)$$

Now, $y' = \frac{dy}{dx} = \frac{1}{12}\frac{d}{dx}\left(x^3 - \frac{1}{x} + 3 - \frac{3}{x^4}\right)$

$$= \frac{1}{12}\left(3x^2 + \frac{2}{x^2} + \frac{12}{x^5}\right)$$

Now, $y'' = \frac{dy'}{dx} = \frac{1}{12}\frac{d}{dx}\left(3x^2 + \frac{2}{x^2} + \frac{12}{x^5}\right)$

$$= \frac{1}{12}\left(6x - \frac{4}{x^3} - \frac{60}{x^6}\right)$$

P4: If $y = \frac{1+x-4\sqrt{x}}{x}$, then find y''.

Solution: Given $y = \frac{1+x-4\sqrt{x}}{x} = \frac{1}{x} + 1 - \frac{4}{\sqrt{x}}$

Differentiating y w.r.t x

$$y' = \frac{d}{dx}\left(1 + \frac{1}{x} - \frac{4}{\sqrt{x}}\right)$$

$$= 0 - \frac{1}{x^2} - 4\left(\frac{-1}{2}x^{-3/2}\right) = -\frac{1}{x^2} + \frac{2}{x^{3/2}}$$

$$y'' = \frac{d}{dx}\left(-\frac{1}{x^2} + \frac{2}{x^{3/2}}\right)$$

$$= \left(\frac{2}{x^3} + 2\left(\frac{-3}{2}x^{-5/2}\right)\right) = \frac{2}{x^3} - \frac{3}{x^{5/2}}$$

Exercise:

1. Find the derivatives of all orders of the function
   - **a.** $y = \frac{x^4}{2} + \frac{3}{2}x^2 - x$
   - **b.** $y = \frac{x^5}{120}$
2. Find the first and second derivatives of the functions.
   - **a.** $y = \frac{x^3+7}{x}$
   - **b.** $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3}$

c. $w = \left(\frac{1+3z}{3z}\right)(3-z)$

d. $s = \frac{t^2+5t-1}{t^4}$

e. $u = \frac{(x^2+x)(x^2-x+1)}{x^4}$

f. $w = (z+1)(z-1)(z^2+1)$

g. $p = \frac{q^2+3}{(q-1)^3+(q+1)^3}$

4.6. Derivatives of Trigonometric Functions

Learning objectives:

- To derive the derivatives of sine and cosine functions.
- To study an example of simple harmonic motion.
- To define jerk in the motion of a body.
  And
- To practice related problems.

The Derivative of $y = \sin x$:

We have , $\boldsymbol{y = \sin x}$ $\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$

$= \sin x \,.\, \lim_{h\to 0}\frac{\cos h-1}{h} + \cos x \,.\, \lim_{h\to 0}\frac{\sin h}{h}$

$= \cos x \qquad \left(\because \lim_{h\to 0}\frac{\cos h-1}{h} = 0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h} = 1\right)$

The derivative of the sine is the cosine.

$$\frac{d}{dx}(\sin x) = \cos x$$

Example 1:

a) $y = x^2 - \sin x \qquad \frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) = 2x - \cos x$

b) $y = x^2 \sin x$

$\frac{dy}{dx} = x^2\frac{d}{dx}(\sin x) + sin\, x\frac{d}{dx}(x^2) = x^2\cos x + 2xsin\ x$

c) $y = \frac{\sin x}{x}$

$\frac{dy}{dx} = \frac{x.\frac{d}{dx}(\sin x)-\sin x.1}{x^2} = \frac{x\cos x-\sin x}{x^2}$

The Derivative of $y = \cos x$:

We have, $y = \cos x$

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h\to 0} \cos x \,.\, \frac{(\cos h - 1)}{h} - \lim_{h\to 0} \sin x \,.\, \frac{\sin h}{h}$$

$$= \cos x \,.\, \lim_{h\to 0} \frac{(\cos h - 1)}{h} - \sin x \,.\, \lim_{h\to 0} \frac{\sin h}{h}$$

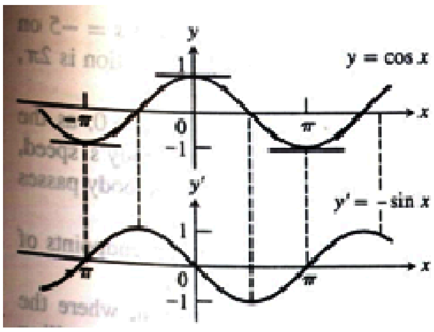
$$= \cos x \,.\, 0 - \sin x \,.\, 1$$

$$\left(\because \lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \,,\, \lim_{h\to 0} \frac{\sin h}{h} = 1\right)$$

$$= -\sin x$$

The derivative of the cosine is the negative of the sine.

$$\frac{d}{dx}(\cos x) = -\sin x$$



The curve $\boldsymbol{y' = -\sin x}$ is the graph of the tangents to the curve $\boldsymbol{y = \cos x}$.

Example 2:

a) $y = 5x + \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x$$

b) $y = \sin x \cos x$

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

$$= \sin x\,(-sinx) + \cos x\,(\cos x)$$

$$= cos^2 x - sin^2 x$$

c) $y = \frac{\cos x}{1-\sin x}$

$$\frac{dy}{dx} = \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + sin^2 x + cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1-\sin x}{(1-\sin x)^2}$$

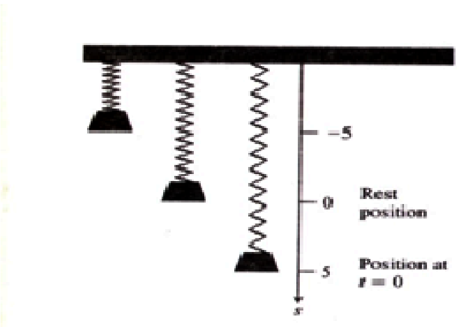
$$= \frac{1}{1-\sin x}$$

Simple harmonic motion:

The motion of a body moving up and down on the end of a spring is an example of ***simple harmonic motion***.

Example 3:

A body hanging from a spring is stretched 5 units beyond its rest position and released at time $t = 0$ to move up and down.



Its position at any later time t is $s = 5\cos t$

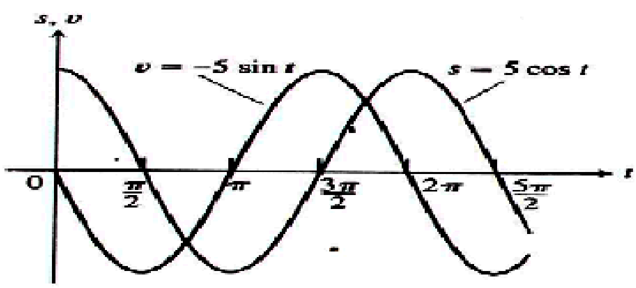
What are its velocity and acceleration at time t?

Solution: We have

Position: $s = 5\cos t$

Velocity: $v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = 5\frac{d}{dt}(\cos t) = -5\sin t$

Acceleration: $a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\frac{d}{dt}(\sin t) = -5\cos t$



From these equations, we can learn the following:

1. As time passes, the body moves up and down between $s = 5$ and $s = -5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π, the period of $\cos t$.
2. The function $\sin t$ attains its greatest magnitude 5, when $\cos t = 0$, as the graphs of the sine and cosine show. Hence, the body's speed, $|v| = |5 \sin t|$, is greatest every time $\cos t = 0$, i.e., every time the body passes its rest position.
   The body's speed is zero when $\sin t = 0$. This occurs at the endpoints of the interval of motion, when $\cos t = \pm 1$.
3. The acceleration, $a = -5 \cos t$, is zero only at the rest position, where the cosine is zero. When the body is anywhere else, the spring is either pulling on it or pushing on it. The acceleration is greatest in magnitude at the points farthest from the origin, where $\cos t = \pm 1$.

A sudden change in acceleration is called a ***"jerk"***. A ride in a bus is jerky when there are abrupt changes in acceleration, and the accelerations involved need not necessarily be large.

Jerk is the derivative of acceleration. If a body's position at time t is $s = f(t)$, the body's jerk at time t is

$$j = \frac{da}{dt} = \frac{d^3 s}{dt^3}$$

The jerk of the simple harmonic motion in the previous example is

$$j = \frac{da}{dt} = \frac{d}{dx}(-5 \cos t) = 5 \sin t$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the origin, where the acceleration changes direction and sign.

PROBLEM SET

IP1: $\boldsymbol{\frac{d}{dx}\left((ax^2 + \sin x)(p + q \cos x)\right) =}$

Solution:

$\frac{d}{dx}\left((ax^2 + \sin x)(p + q \cos x)\right)$

$= (p + q \cos x)\frac{d}{dx}(ax^2 + \sin x) + (ax^2 + \sin x)\frac{d}{dx}(p + q \cos x)$

$= (p + q \cos x)(a(2x) + \cos x) + (ax^2 + \sin x)(0 - q \sin x)$

$= (2ax + \cos x)(p + q \cos x) - q(ax^2 + \sin x) \sin x$

P1: If $\boldsymbol{y = x^2 \sin x + 2x \ \cos x - 2 \sin x}$, then find $\boldsymbol{\frac{dy}{dx}}$.

Solution:

Given $y = x^2 \sin x + 2x \ \cos x - 2 \sin x$.

Differentiating y w.r.t x.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sin x + 2x\ \cos x - 2\sin x)$$

$$= \left[x^2 \frac{d}{dx}(\sin x) + \sin x\,(2x)\right] + 2\left[x\frac{d}{dx}(\cos x) + \cos x\right] - 2\frac{d}{dx}(\sin x)$$

$$= [x^2(\cos x) + 2x\sin x] + 2[-x\sin x + \cos x] - 2\cos x$$

$$= x^2(\cos x) + 2x\sin x - 2x\sin x + 2\cos x - 2\cos x$$

$$= x^2 \cos x$$

IP2: $\frac{d}{dx}\left(\frac{\sin x - x\cos x}{x\sin x + \cos x}\right) =$

Solution:

$$\frac{d}{dx}\left(\frac{\sin x - x\cos x}{x\sin x + \cos x}\right)$$

$$= \frac{(x\sin x+\cos x)\frac{d}{dx}(\sin x - x\cos x) - (\sin x - x\cos x)\frac{d}{dx}(x\sin x+\cos x)}{(x\sin x+\cos x)^2}$$

$$= \frac{(x\sin x+\cos x)\left(\cos x - \frac{d}{dx}(x\cos x)\right) - (\sin x - x\cos x)\left(\frac{d}{dx}(x\sin x) - \sin x\right)}{(x\sin x+\cos x)^2}$$

$$= \frac{(x\sin x+\cos x)(\cos x + x\sin x - \cos x) - (\sin x - x\cos x)(\sin x + x\cos x - \sin x)}{(x\sin x+\cos x)^2}$$

$$= \frac{(x\sin x+\cos x)(x\sin x) - (\sin x - x\cos x)(x\cos x)}{(x\sin x+\cos x)^2}$$

$$= \frac{x(x\sin^2 x + \sin x\cos x - \sin x\cos x + x\cos^2 x)}{(x\sin x+\cos x)^2}$$

$$= \frac{x^2(\sin^2 x+\cos^2 x)}{(x\sin x+\cos x)^2}$$

$$= \frac{x^2}{(x\sin x+\cos x)^2}$$

P2: If $y = \frac{\sin x+\cos x}{\sin x-\cos x}$, then find $\frac{dy}{dx}$ at $x = 15^o$.

Solution: Given $y = \frac{\sin x+\cos x}{\sin x-\cos x}$

Differentiating y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$$

$$= \frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x) - (\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$= \frac{(\sin x-\cos x)(\cos x-\sin x) - (\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{-(\sin x-\cos x)^2 - (\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$= \frac{-[(\sin x-\cos x)^2 + (\sin x+\cos x)^2]}{(\sin x-\cos x)^2}$$

$$= \frac{-2(\sin^2 x+\cos^2 x)}{\sin^2 x+\cos^2 x - 2\sin x\cos x}$$

$\frac{dy}{dx} = \frac{-2}{1-\sin 2x}$

$\left(\frac{dy}{dx}\right)_{x=15°} = \frac{-2}{1-\sin 2(15°)} = \frac{-2}{1-\sin 30°} = \frac{-2}{1-\frac{1}{2}} = -4$

IP3: Find the equation of tangent line to the curve

$y = x\cos x + \sin x$ at $x = \pi$.

Solution: The given curve is: $y = x\cos x + \sin x$

Slope of the tangent $= \frac{dy}{dx} = \frac{d}{dx}(x\cos x + \sin x)$

$$= \frac{d}{dx}(x\cos x) + \cos x$$

$$= (\cos x - x\sin x) + \cos x$$

$$= 2\cos x - x\sin x$$

Slope of the tangent at $x = \pi$ is

$$\frac{dy}{dx}|_{x=\pi} = 2\cos\pi - \pi\sin\pi = -2$$

If $x = \pi$, then $y = \pi\cos\pi + \sin\pi = -\pi$

The equation of the tangent to the given curve at the point $(\pi, -\pi)$ having slope -2 is

$$(y+\pi) = -2(x-\pi)$$

$$i.e., \quad 2x + y - \pi = 0$$

P3: Find the tangent to the curve $y = \frac{1}{\sin x + \cos x}$ at the point $(0, 1)$.

Solution: The given curve is: $y = \frac{1}{\sin x + \cos x}$

Slope of the tangent $m = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\sin x + \cos x}\right)$

$$= \frac{(\sin x + \cos x)(0) - (1)(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x - \cos x}{(\sin x + \cos x)^2}$$

Slope of the tangent at the point $(0,1)$ is

$$m = \frac{\sin 0 - \cos 0}{(\sin 0 + \cos 0)^2} = \frac{0-1}{(0+1)^2} = -1$$

The equation of the tangent to the given curve at the point $(0,1)$ having slope -1 is

$$(y-1) = -1(x-0)$$

$$i.e., \quad x + y - 1 = 0$$

IP4: The equation for the position of a body moving on a coordinate line is $s = \sin t + \cos t$ where s is in meters and t is in seconds. Find the body's velocity, speed, acceleration and jerk at time $t = \frac{\pi}{6}$ seconds.

Solution: Given $s = \sin t + \cos t$.

Velocity $(v) = \frac{ds}{dt} = \frac{d}{dt}(\sin t + \cos t)$

$$v = \cos t - \sin t$$

At time $t = \frac{\pi}{6}$ sec

$$v = \cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}\ m/sec$$

Speed $= |v| = \left|\frac{\sqrt{3}-1}{2}\right| = \frac{\sqrt{3}-1}{2}\ m/sec$

Acceleration $(a) = \frac{dv}{dt} = \frac{d}{dt}(\cos t - \sin t)$

$$a = -\sin t - \cos t$$

At time $t = \frac{\pi}{6}$ sec

$$a = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{1}{2} - \frac{\sqrt{3}}{2} = -(\frac{\sqrt{3}+1}{2})$$

Jerk $(j) = \frac{da}{dt} = \frac{d}{dt}(-\sin t - \cos t)$

$$j = -\cos t + \sin t$$

At time $t = \frac{\pi}{6}$ sec

$$j = -\cos\frac{\pi}{6} + \sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1-\sqrt{3}}{2}$$

P4: The equation for the position of a body moving on a coordinate line is given by $s = 2 - 2\ \sin t$ where s is in meters and t is in seconds. Find the jerk at time $t = \frac{\pi}{3}$ seconds.

Solution:

Given $s = 2 - 2\ \sin t$.

If a body's position at time t is $s = f(t)$, then the body's jerk at time t is

$$\begin{aligned} j &= \frac{d^3s}{dt^3} \\ &= \frac{d^3}{dt^3}(2 - 2\ \sin t) \\ &= \frac{d^2}{dt^2}\left(\frac{d}{dt}(2 - 2\ \sin t)\right) \\ &= \frac{d^2}{dt^2}(0 - 2\cos t) = \frac{d^2}{dt^2}(-2\cos t) \\ &= \frac{d}{dt}\left(\frac{d}{dt}(-2\cos t)\right) \\ &= \frac{d}{dt}(2\sin t) \\ &= 2\cos t \end{aligned}$$

The body's jerk at time $t = \frac{\pi}{3}$ sec is

$$j = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1\ m/sec^3$$

Exercise:

1. Find $\frac{dy}{dx}$.

a. $y = -10x + 3\cos x$

b. $y = \csc x - 4\sqrt{x} + 7$

c. $y = (\sec x + \tan x)(\sec x - \tan x)$

d. $y = (\sin x + \cos x)\sec x$

e. $y = \frac{\cot x}{1+\cot x}$

f. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

g. $y = x^2\cos x - 2x\sin x - 2\cos x$

h. $y = \frac{1+\sin x}{x+\cos x}$

i. $y = \frac{\sin x}{x^2}$

2. Find $\frac{ds}{dt}$.

a. $s = \tan t - t$

b. $s = \frac{1+\csc t}{1-\csc t}$

3. Find $\frac{dr}{d\theta}$.

a. $r = 4 - \theta^2\sin\theta$

b. $r = \sec\theta\csc\theta$

4. Find $\frac{dp}{dq}$.

a. $p = 5 + \frac{1}{\cot q}$

b. $p = \frac{\sin q + \cos q}{\cos q}$

5. The equation for the position of a body moving on a coordinate line is $s = \sin t + \cos t$ where s is in meters and t is in seconds. Find the body's velocity, speed, acceleration and jerk at time $t = \frac{\pi}{4}$ seconds.

6. The equation for the position of a body moving on a coordinate line is $s = 2 - 2\sin t$ where s is in meters and t is in seconds. Find the body's velocity, speed, acceleration and jerk at time $t = \frac{\pi}{4}$ seconds.

7. The equation for the position of a body moving on a coordinate line is $s = 4\sin t$ where s is in meters and t is in seconds. Find the body's velocity, acceleration and jerk at time.

8. An object at the end of a vertical spring is stretched $4\ cm$ beyond its rest position and released at time $t = 0$. Its position at time t is $s = 2\sin t - 4\cos t$. Find its velocity, acceleration and jerk at time $t = \frac{\pi}{4}$ sec.
9. Find the equation of the tangent to the curve at the given point
   a. $y = x + \cos x \quad (0,1)$.
   b. $y = 2x\sin x \quad \left(\frac{\pi}{2}, \pi\right)$.

4.7. Continuity of Trigonometric Functions

Learning objectives:

- To obtain the derivatives of the trigonometric functions.
- To discuss the continuity of the trigonometric functions.
  And
- To practice the related problems.

The Derivatives of the Other Basic Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x} \quad , \quad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} \quad , \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas.

$$\frac{d}{dx}(\tan x) = sec^2 x \quad , \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -csc^2 x \, , \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

There is minus sign in the derivative formulas for the co functions. We derive the formula for $\tan x$ below.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Similarly, other formulas can be derived.

Example:

Find y'' if $y = \sec x$.

Solution:

Given that $\boldsymbol{y = \sec x}$

$$y' = \sec x \tan x$$

$$y'' = \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$
$$= \sec x\,(sec^2 x) + \tan x\,(\sec x \tan x)$$
$$= \sec^3 x + \sec x \tan^2 x$$

Example:

a. $\boldsymbol{\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \csc^2 x}$

b. $\boldsymbol{\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\csc x) = 2\frac{d}{dx}(\csc x) = -2\csc x \cot x}$

Continuity of Trigonometric Functions:

Since the six basic trigonometric functions are differentiable throughout their domains they are also continuous throughout their domains. This means that $\sin x$ and $\cos x$ are continuous for all x, that $\sec x$ and $\tan x$ are continuous except when x is a nonzero odd integer multiples of $\pi/2$, and that $\csc x$ and $\cot x$ are continuous except when x is an integer multiple of π. For each function, $\lim_{x\to c} f(x) = f(c)$ whenever $f(c)$ is defined. As a result, we can calculate the limits of many algebraic combinations and composites of trigonometric functions by direct substitution.

Example:

$$\boldsymbol{\lim_{x\to 0} \frac{\sqrt{2+sec\,x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+sec\,0}}{\cos(\pi-\tan 0)} = \frac{\sqrt{2+1}}{cos\,(\pi-0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}}$$

Other Limits

The equation $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ holds no matter how θ may be expressed

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\qquad \theta = x;$ $\qquad As\ \ x \to 0, \theta \to 0$

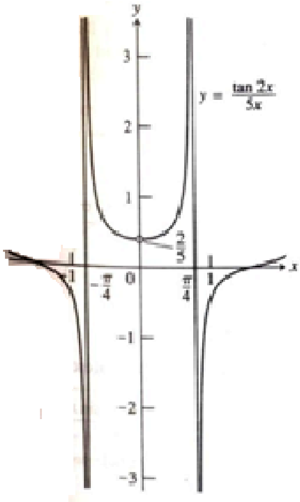
$\lim_{x\to 0} \frac{\sin 7x}{7x} = 1$ $\qquad \theta = 7x;$ $\qquad As\ \ x \to 0, \theta \to 0$

$\lim_{x\to 0} \frac{\sin(2/3)x}{(2/3)x} = 1$ $\qquad \theta = (2/3)x;$ $\qquad As\ \ x \to 0, \theta \to 0$

Example:

a. $\boldsymbol{\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{(2/5).\sin 2x}{(2/5).5x} = \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}}$
$$\boldsymbol{= \frac{2}{5}(1) = \frac{2}{5}}$$

b. $\lim_{x\to 0} \frac{\tan 2x}{5x} = \lim_{x\to 0}\left(\frac{\sin 2x}{5x} . \frac{1}{\cos 2x}\right)$
$$= \left(\lim_{x\to 0}\frac{\sin 2x}{5x}\right)\left(\lim_{x\to 0}\frac{1}{\cos 2x}\right)$$
$$= \left(\frac{2}{5}\right)\left(\frac{1}{\cos 0}\right) = \frac{2}{5}$$



The graph of $y = \frac{\tan 2x}{5x}$ steps across the y-axis at $y = 2/5$.

Example:

$\lim_{t\to\frac{\pi}{2}} \frac{\sin\left(t-\frac{\pi}{2}\right)}{t-\frac{\pi}{2}}$ set $\theta = t - \frac{\pi}{2}$. Then $\theta \to 0$ as $t \to \frac{\pi}{2}$ $= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

PROBLEM SET

IP1: Find the derivative of $y = x^2 \sec x + \sqrt{x} \csc x$ w.r.t x.

Solution:

Given that $y = x^2 \sec x + \sqrt{x}\csc x$.

Now, differentiating w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \sec x + \sqrt{x}\csc x\right)$$

$$= \frac{d}{dx}(x^2 \sec x) + \frac{d}{dx}\left(\sqrt{x}\csc x\right)$$

$$= \left(\sec x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(\sec x)\right) + \left(\csc x \frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(\csc x)\right)$$

$$= (2x \sec x + x^2 \sec x \tan x) + \left(\frac{1}{2\sqrt{x}}\csc x - \sqrt{x}\csc x \cot x\right)$$

P1: Find the derivative of $y = \sec x \tan x + \cot x$ w.r.t x.

Solution:

Given $y = \sec x \tan x + \cot x$.

Now, differentiating y w.r.t x. We get,

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x \tan x + \cot x)$$

$$= \frac{d}{dx}(\sec x \tan x) + \frac{d}{dx}(\cot x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) - \csc^2 x$$

$$= \sec x\,(sec^2 x) + \tan x\,(\sec x \tan x) - \csc^2 x$$

$$= \sec x\,(\sec^2 x + \tan^2 x) - \csc^2 x$$

$$\therefore \frac{dy}{dx} = \sec x\,(\sec^2 x + \tan^2 x) - \csc^2 x$$

IP2: If $y = \frac{\tan x}{\sec x + 1}$, then find $\frac{dy}{dx}$.

Solution: We have, $y = \frac{\tan x}{\sec x + 1}$

Differentiating y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\tan x}{\sec x + 1}\right)$$

$$= \frac{(\sec x + 1)\frac{d}{dx}(\tan x) - (\tan x)\frac{d}{dx}(\sec x + 1)}{(\sec x + 1)^2}$$

$$= \frac{(\sec x + 1)(\sec^2 x) - (\tan x)(\sec x \tan x + 0)}{(\sec x + 1)^2}$$

$$= \frac{\sec^3 x + \sec^2 x - \sec x \tan^2 x}{(\sec x + 1)^2}$$

$$= \frac{\sec x(\sec^2 x + \sec x - \tan^2 x)}{(\sec x + 1)^2}$$

$$= \frac{\sec x}{\sec x + 1} \qquad (\because \sec^2 x - \tan^2 x = 1\ )$$

P2: Find $\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right)$.

Solution:

$$\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right) = \frac{(1-\tan x)\frac{d}{dx}(1+\tan x) - (1+\tan x)\frac{d}{dx}(1-\tan x)}{(1-\tan x)^2}$$

$$= \frac{(1-\tan x)(0+\sec^2 x) - (1+\tan x)(0-\sec^2 x)}{(1-\tan x)^2}$$

$$= \frac{\sec^2 x - \tan x \sec^2 x + \sec^2 x + \tan x \sec^2 x}{(1-\tan x)^2}$$

$$= \frac{2\sec^2 x}{(1-\tan x)^2} = \frac{2}{(\cos x - \sin x)^2}$$

$$= \frac{2}{\cos^2 x + \sin^2 x - 2\sin x \cos x} = \frac{2}{1 - \sin 2x}$$

IP3: Evaluate $\lim_{x\to 0} \frac{5x + 4\sin 3x}{4\sin 2x + 7x}$.

Solution: $\lim_{x\to 0} \frac{5x + 4\sin 3x}{4\sin 2x + 7x}$

Dividing numerator and denominator by x

$$= \lim_{x\to 0} \frac{5 + 4\left(\frac{\sin 3x}{x}\right)}{4\left(\frac{\sin 2x}{x}\right) + 7}$$

$$= \lim_{x\to 0} \frac{5 + 12\left(\frac{\sin 3x}{3x}\right)}{8\left(\frac{\sin 2x}{2x}\right) + 7}$$

$$= \frac{5+12.\lim\limits_{x\to0}\left(\frac{\sin 3x}{3x}\right)}{8.\lim\limits_{x\to0}\left(\frac{\sin 3x}{3x}\right)+7}$$

$$= \frac{5+12(1)}{8(1)+7} = \frac{17}{15}$$

$$\therefore \lim_{x\to0}\frac{5x+4\sin 3x}{4\sin 2x+7x} = \frac{17}{15}$$

P3: $\lim\limits_{x\to0}\frac{\sin 2x+\sin 3x}{2x+\sin 3x} =$

Solution: $\lim\limits_{x\to0}\frac{\sin 2x+\sin 3x}{2x+\sin 3x}$

Divide numerator and denominator by x.

$$= \lim_{x\to0}\frac{\frac{\sin 2x}{x}+\frac{\sin 3x}{x}}{2+\frac{\sin 3x}{x}}$$

$$= \lim_{x\to0}\frac{2\left(\frac{\sin 2x}{2x}\right)+3\left(\frac{\sin 3x}{3x}\right)}{2+3\left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{2\lim\limits_{x\to0}\left(\frac{\sin 2x}{2x}\right)+3\lim\limits_{x\to0}\left(\frac{\sin 3x}{3x}\right)}{2+3\lim\limits_{x\to0}\left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{2\times1+3\times1}{2+3\times1} = \frac{5}{5} = 1$$

$$\therefore \lim_{x\to0}\frac{\sin 2x+\sin 3x}{2x+\sin 3x} = 1$$

IP4: Evaluate $\lim\limits_{x\to0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$.

Solution: We have,

$$\lim_{x\to0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$$

$$= \lim_{x\to0}\frac{8}{x^8}\left\{\left(1-\cos\frac{x^2}{2}\right)-\cos\frac{x^2}{4}\left(1-\cos\frac{x^2}{2}\right)\right\}$$

$$= \lim_{x\to0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)$$

$$= \lim_{x\to0} 8\left\{\frac{1-\cos\frac{x^2}{2}}{x^4}\right\}\left\{\frac{1-\cos\frac{x^2}{4}}{x^4}\right\}$$

$$= \lim_{x\to0}\left[8\times\frac{2sin^2\frac{x^2}{4}}{x^4}\times\frac{2sin^2\frac{x^2}{8}}{x^4}\right]$$

$$= \lim_{x\to0}\left[32\times\left\{\frac{\sin\frac{x^2}{4}}{x^2}\right\}^2\times\left\{\frac{\sin\frac{x^2}{8}}{x^2}\right\}^2\right]$$

$$= 32.\lim_{x\to0}\left\{\frac{\sin\frac{x^2}{4}}{4\left(\frac{x^2}{4}\right)}\right\}^2\times\lim_{x\to0}\left\{\frac{\sin\frac{x^2}{8}}{8\left(\frac{x^2}{8}\right)}\right\}^2$$

$$= 32 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^2$$

$$= 32 \times \frac{1}{16} \times \frac{1}{64} = \frac{1}{32}$$

P4: $\lim_{x\to 3} \frac{\sin(x^2-3x)}{x^2-9} =$

Solution:

$$\lim_{x\to 3} \frac{\sin(x^2-3x)}{x^2-9} = \lim_{x\to 3} \frac{\sin\big(x(x-3)\big)}{(x-3)(x+3)}$$

$$= \lim_{x\to 3} \left[\frac{\sin\big(x(x-3)\big)}{x(x-3)} \times \frac{x}{x+3}\right]$$

$$= \left[\lim_{x\to 3} \frac{\sin\big(x(x-3)\big)}{x(x-3)}\right] \times \left[\lim_{x\to 3} \frac{x}{x+3}\right]$$

$$= \left[\lim_{x\to 3} \frac{\sin\big(x(x-3)\big)}{x(x-3)}\right] \times \left[\frac{3}{3+3}\right]$$

$$= \frac{1}{2} \lim_{x\to 3} \frac{\sin\big(x(x-3)\big)}{x(x-3)}$$

Now assume $x(x-3) = \theta$ as $x \to 3$

Then $x(x-3) \to 0 \;\; i.e., \quad \theta \to 0$

$$= \frac{1}{2} \lim_{\theta\to 0} \frac{\sin\theta}{\theta}$$

$$= \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\therefore \lim_{x\to 3} \frac{\sin(x^2-3x)}{x^2-9} = \frac{1}{2}$$

Exercises:

10. Find the limits

a. $\lim_{\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

b. $\lim_{y\to 0} \frac{\sin 3y}{4y}$

c. $\lim_{x\to 0} \frac{\tan 2x}{x}$

d. $\lim_{x\to 0} \frac{x \csc 2x}{\cos 5x}$

e. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x \cos x}$

f. $\lim_{t\to 0} \frac{\sin(1-cos\, t)}{1-\cos t}$

g. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$

h. $\lim_{\theta\to 0} \frac{\tan 3\theta}{\sin 8\theta}$

11. Find $\frac{dy}{dx}$.

a. $y = x^3 \sec x$

b. $y = \sqrt{x}\tan x$

c. $y = \frac{2 - \sec x}{3 + 4\csc x}$

d. $y = \frac{1}{\sec x + \tan x}$

e. $y = \frac{\csc x}{1 + \csc x}$

12. Find y'' if

a. $y = \csc x$

b. $y = \sec x$

13. Graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

a. $y = \sin x \qquad -3\pi/2 \le x \le 2\pi \qquad x = -\pi, 0, 3\pi/2$

b. $y = \sec x \qquad -\pi/2 \le x \le \pi/2 \qquad x = -\pi/3, \pi/4$

14. Do the graphs of the functions have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so where? If not, why not?

a. $y = x + \sin x$

b. $y = x - \cot x$

5.1. The Chain Rule

Learning objectives

- To discuss the chain rule, the rule for differentiation of composite functions

  And

- To practice related problems.

We have learnt how to differentiate simple functions like $\sin x$ and (x^2-4). Now we extend the differentiation to composites like $sin\ (x^2-4)$ by using the Chain Rule.

The Chain Rule says that *the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points*. The Chain Rule is probably the most widely used differentiation rule in calculus. This module describes the rule and explains how to use it. First, we consider a few examples.

Example 1: The function $y=6x-10=2(3x-5)$ is the composite of the functions $y=2u$ and $u=3x-5$. How are the derivatives of these three functions related?

Solution: We have, $\frac{dy}{dx}=6, \frac{dy}{du}=2, \& \ \frac{du}{dx}=3$**. Then** $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

If we think of the derivative as a rate of change, this relationship seems to be reasonable. For, $y=f(u)$ and $u=g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x.

Example 2:

The function $y=9x^4+6x^2+1=\left(3x^2+1\right)^2$ is a composite of $y=u^2$ and $u=3x^2+1$.

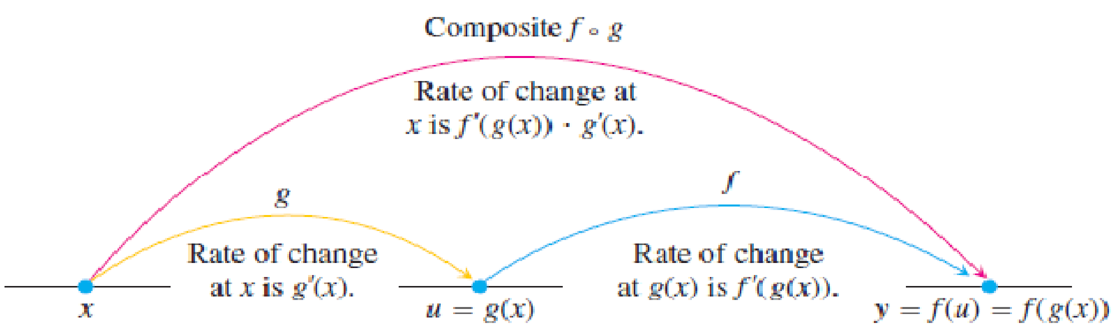
Calculating the derivatives, we see that

$$\frac{dy}{du}\cdot\frac{du}{dx}=2u\cdot 6x=2\left(3x^2+1\right)\cdot 6x=36x^3+12x$$

And $$\frac{dy}{dx}=\frac{d}{dx}\left(9x^4+6x^2+1\right)=36x^3+12x$$

Thus, $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

The derivative of the composite function $f\left(g\left(x\right)\right)$ at x is the derivative of f at $g(x)$ times the derivative g at x. This is known as the Chain Rule.



Theorem

If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \qquad \text{... (1)}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{...(2)}$$

where $\dfrac{dy}{du}$ is evaluated at $u = g(x)$.

Example 3: Find the derivative of $y = \sqrt{x^2 + 1}$.

Solution: Here $y = f(u(x))$, where $f(u) = \sqrt{u}$ and $u = x^2 + 1$.

Since the derivatives f and g are $\quad f'(u) = \dfrac{1}{2\sqrt{u}}, \quad u'(x) = 2x,$

the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x)$$

$$= \frac{1}{2\sqrt{u(x)}} \cdot u'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x) = \frac{x}{\sqrt{x^2+1}}$$

The "Outside-Inside" Rule:

It sometimes helps to think about the Chain Rule the following way:

If $y = f(g(x))$, equation (2) tells us that

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x). \qquad \text{... (3)}$$

In words, equation (3) says:

To find $\dfrac{dy}{dx}$, differentiate the "outside" function f and leave the "inside" $g(x)$ alone; then multiply by the derivative of the inside.

Example 4:

$$\frac{d}{dx}\overset{outside}{\sin}\underbrace{\left(x^2+x\right)}_{inside}=\overset{\substack{derivative\,of\\ the\,outside}}{\cos}\underbrace{\left(x^2+x\right)}_{\substack{inside\\ left\,alone}}\cdot\underbrace{\left(2x+1\right)}_{\substack{derivative\\ of\ the\,inside}}$$

Written simply without explanation, the "Outside-Inside" Rule leads to the evaluation

$$\frac{d}{dx}\sin\left(x^2+x\right)=\cos\left(x^2+x\right)\cdot\left(2x+1\right)$$

We sometimes have to use the Chain Rule two or more times to find a derivative.

Example 5: Find the derivative of $g(t)=\tan\left(5-\sin 2t\right)$**.**

Solution:

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left(\tan\left(5-\sin 2t\right)\right) \\ &= \sec^2\left(5-\sin 2t\right)\cdot\frac{d}{dt}\left(5-\sin 2t\right) \\ &= \sec^2\left(5-\sin 2t\right)\cdot\left(0-\cos 2t\cdot\frac{d}{dt}(2t)\right) \\ &= \sec^2\left(5-\sin 2t\right)\cdot\left(-\cos 2t\right)\cdot 2 \\ &= -2(\cos 2t)\sec^2\left(5-\sin 2t\right) \end{aligned}$$

PROBLEM SET

IP1: Find the derivative of the function given by , $\boldsymbol{f(x)=\sec(\tan x)}$

Solution: Given, $f(x)=\sec(tanx)$

$f(x)=secu$, where $u(x)=tanx$

$\frac{df}{du}=secu\ tanu\ and\ \frac{du}{dx}=sec^2x$

By chain rule, $\frac{df}{dx}=\frac{df}{du}.\frac{du}{dx}=secu\ tanu.sec^2x$

$=\sec(tanx)\ \ tan(tanx).sec^2x$

Thus $\frac{df}{dx}=\sec(tanx)\ \ tan(tanx).sec^2x$

P1: Find the derivative of the function given by, $\boldsymbol{f(x)=\sin(3x-5)}$

Solution: Given, $f(x)=\sin(3x-5)$,

now $f(x)=\sin u$,where $u=3x-5$

By chain rule

$$\frac{df}{dx}=\frac{df}{du}.\frac{du}{dx}$$

$$= \frac{d}{du}(sinu).\frac{d}{dx}(3x-5) = \cos\ u.3$$

Thus $\frac{df}{dx} = 3\cos\ (3x-5)$

IP2: If $f(x) = sin\big(cos(tanx)\big)$, then find $f'(x)$

Solution: $f(x) = sin\big(cos(tanx)\big)$

$f(x) = \sin\ u$, $u(v) = cosv$ and $v(x) = tanx$

$\frac{df}{du} = cosu$, $\frac{du}{dv} = -sinv$ and $\frac{dv}{dx} = sec^2x$

By chain rule, $\frac{df}{dx} = \frac{df}{du}.\frac{du}{dv}.\frac{dv}{dx} = -cosu.(sinv).sec^2x$

$$= -cos\big(cos(tanx)\big).\big(sin(tanx)\big).sec^2x$$

$$\frac{df}{dx} = -cos\big(cos(tanx)\big).\big(sin(tanx)\big).sec^2x$$

P2: Find the derivative of the function given by, $f(x) = \sin\left(cos(x^2)\right)$

Solution: Given, $f(x) = \sin(cosx^2)$

The given function is a composite of three functions.

If $f(x) = \sin v, v(w) = cos\ w\ and\ w(x) = x^2$

$\frac{df}{dv} = cosv$, $\frac{dv}{dw} = -sinw$ and $\frac{dw}{dx} = 2x.$

Hence, by chain rule

$$\frac{df}{dx} = \frac{df}{dv}.\frac{dv}{dw}.\frac{dw}{dx} = cosv.(-sinw).2x = -2xcos(x^2).sin(x^2)$$

Thus $\frac{df}{dx} = -2xcos(cosx^2).sin(x^2)$

IP3: Find the derivative of $y = cot\left(\frac{sint}{t}\right)$

Solution: $y = cot\left(\frac{sint}{t}\right)$

$y = cotu$, where $u(t) = \frac{sint}{t}$

By chain rule, $\frac{dy}{dt} = \frac{dy}{du}.\frac{du}{dt} = \frac{d}{du}(cotu).\frac{d}{dt}\left(\frac{sint}{t}\right)$

$$= -csc^2u.\left(\frac{t.cost-sint}{t^2}\right) = -\left(\frac{t.cost-sint}{t^2}\right)csc^2\left(\frac{sint}{t}\right).$$

P3: Find the derivative of $y = sin(x - cosx)$

Solution: $y = sin(x - cosx)$

$y = sinu$, where $u(x) = x - cosx$

By chain rule , $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{d}{du}(sinu).\frac{d}{dx}(x - cosx)$

$$= cosu.(1 + sinx) = (1 + sinx)cos(x - cosx).$$

IP4: Differentiate $y = \cos\left(5sin\left(\frac{t}{5}\right)\right)$:

Solution: $y = \cos\left(5\sin\left(\frac{t}{5}\right)\right)$

$y = cosu, u = 5sinv$, where $v = \frac{t}{5}$

By chain rule , $\frac{dy}{dt} = \frac{dy}{du} . \frac{du}{dv} . \frac{dv}{dt}$

$$= \frac{d}{du}(cosu).\frac{d}{dv}(5sinv).\frac{d}{dt}\left(\frac{t}{5}\right)$$

$$= -sinu.(5cosv).\frac{1}{5} = -sin\left(5\sin\left(\frac{t}{5}\right)\right)\cos\left(\frac{t}{5}\right)$$

P4: Differentiate $\boldsymbol{y = \sin(cos(2t-5))}$

Solution: $y = \sin(cos(2t-5))$

$y = sinu, u = cosv$, where $v = (2t-5)$

By chain rule, $\frac{dy}{dt} = \frac{dy}{du} . \frac{du}{dv} . \frac{dv}{dt}$

$$= \frac{d(sinu)}{du} . \frac{d(cosv)}{dv} . \frac{d(2t-5)}{dt}$$

$$= cosu.(-sinv).2 = -2cos(cos(2t-5))\sin(2t-5)$$

Exercise:

I. Given $\boldsymbol{y = f(u)\ and\ u = g(x)}$**, find** $\boldsymbol{\frac{dy}{dx} = f'(g(x))g'(x)}$

1. $y = sinu, u = 3x + 1$
2. $y = cosu, u = -\frac{x}{3}$
3. $y = cosu, u = sinx$
4. $y = sinu, u = x - cosx$
5. $y = tanu, u = 10x - 5$

II. Find the derivative of the functions below:

1. $s = \frac{4}{3\pi}sin3t + \frac{4}{5\pi}cos5t$
2. $s = sin\left(\frac{3\pi t}{2}\right) + cos\left(\frac{3\pi t}{2}\right)$
3. $y = \sin(\cos(2t-5))$
4. $y = cos\left(5sin\left(\frac{t}{3}\right)\right)$
5. $y = \sec(tanx)$
6. $y = \cot\left(\pi - \frac{1}{x}\right)$

5.2. Differentiation formulas that includes chain rule

Learning objectives:

- To study the power chain rule

  And

- To solve related problems

Differentiation formulas that includes chain rule

Many of the differentiation formulas we encounter already include the Chain Rule. We illustrate this as shown below.

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ in the Chain Rule formula $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

leads to the formula, $\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$

For example, if u is a differentiable function of x, n is an integer, and $y = u^n$, then the Chain Rule gives, $\frac{dy}{dx} = \frac{d}{du}\left(u^n\right) \cdot \frac{du}{dx} = nu^{n-1} \frac{du}{dx}$

Power Chain Rule

If $u(x)$ is a differentiable function and n is an integer, then u^n is differentiable and

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}$$

Example 1:

a) $\frac{d}{dx} \sin^5 x = 5\sin^4 x \frac{d}{dx}(\sin x) = 5\sin^4 x \cos x$

b) $\frac{d}{dx}(2x+1)^{-3} = -3(2x+1)^{-4} \frac{d}{dx}(2x+1) = -6(2x+1)^{-4}$

c) $\frac{d}{dx}\left(5x^3 - x^4\right)^7 = 7\left(5x^3 - x^4\right)^6 \frac{d}{dx}\left(5x^3 - x^4\right) = 7\left(5x^3 - x^4\right)^6 \left(15x^2 - 4x^3\right)$

d) $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} = (-1)(3x-2)^{-2}(3) = -\frac{3}{(3x-2)^2}$

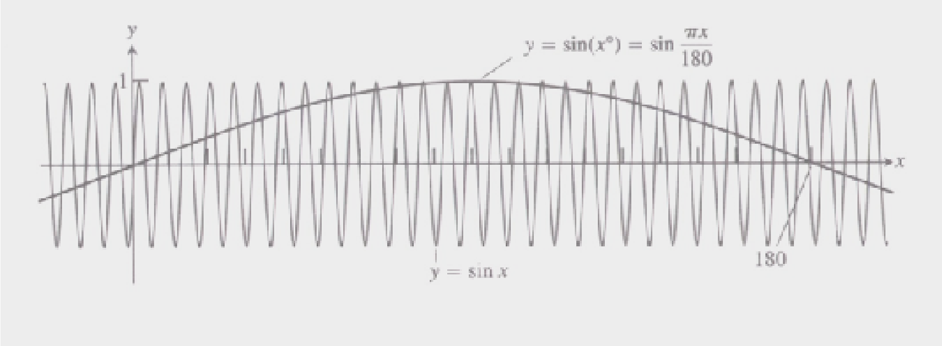
Example 2:

It is important to remember that the formulas for the derivatives of $\sin x$ and $\cos x$ were obtained under the assumption that x *is measured in radians, not in degrees*.

Since $180° = \pi$ radians, $x° = \frac{\pi x}{180}$ radians.

By the Chain Rule,

$$\frac{d}{dx}\sin\left(x^{\circ}\right)=\frac{d}{dx}\sin\left(\frac{\pi x}{180}\right)=\cos\left(\frac{\pi x}{180}\right).\frac{d}{dx}\left(\frac{\pi x}{180}\right)=\frac{\pi}{180}\cos\left(x^{\circ}\right)$$



$sin\left(x^{\circ}\right)$ oscillates only $\frac{\pi}{180}$ times as often as $\sin x$ oscillates. Its maximum slope is $\frac{\pi}{180}$.

In a similar manner, we obtain the derivative of $\cos x°$.

$$\frac{d}{dx}(\cos x°) = -\frac{\pi}{180°}\sin(x°)$$

The factor $\frac{\pi}{180}$, annoying in the first derivative, would compound with repeated differentiation. This is the compelling reason for the use of radian measure.

PROBLEM SET

IP1: Find the derivative of the function $y = sin^3 x$

Solution:

$y = sin^3 x \Longrightarrow y = u^3,$ Where $u(x) = sinx$

By chain rule,

$\frac{dy}{dx} = y^{'}(u).\frac{du}{dx}$, where $\frac{dy}{du} = 3(u)^2$ and $\frac{du}{dx} = cosx$

Now

$$\frac{dy}{dx} = 3(sinx)^2 cosx = 3sin^2 xcosx = \frac{3}{2}\,sin2x.sinx$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}\,sin2x.sinx$$

P1: Find the derivative of the function $y = \left(\frac{x}{5}+\frac{1}{5x}\right)^5$

Solution:

$$y = \left(\frac{x}{5}+\frac{1}{5x}\right)^5 \Longrightarrow y = (u)^5, where\; u(x) = \left(\frac{x}{5}+\frac{1}{5x}\right)$$

By chain rule

$\frac{dy}{dx} = y'(u).\frac{du}{dx}$, where $\frac{dy}{du} = 5.u^4$ and $\frac{du}{dx} = \left(\frac{1}{5}-\frac{1}{5x^2}\right)$

$$\therefore \frac{dy}{dx} = 5\left(\frac{x}{5}+\frac{1}{5x}\right)^4.\left(\frac{1}{5}-\frac{1}{5x^2}\right)$$

IP2: Find the derivative of the function $y = \frac{1}{6}\left(1 + cos^2(7t)\right)^3$

Solution:

$$y = \frac{1}{6}\left(1 + cos^2(7t)\right)^3$$

$$\frac{dy}{dt} = \frac{1}{6}\frac{d}{dt}\left(1 + cos^2(7t)\right)^3$$

$$= \frac{1}{6}.3.\left(1 + cos^2(7t)\right)^2 \frac{d}{dt}\left(1 + cos^2(7t)\right)$$

$$= \frac{1}{2}.\left(1 + cos^2(7t)\right)^2 \left(0 + 2cos(7t)\frac{d}{dt}\left(cos(7t)\right)\right)$$

$$= \frac{1}{2}.\left(1 + cos^2(7t)\right)^2 \left(2cos(7t).(-sin(7t))\frac{d}{dt}(7t)\right)$$

$$= -\frac{7}{2}.\left(1 + cos^2(7t)\right)^2 sin14t$$

P2: Find the derivative of the function $s = sin^2\left(\frac{3\pi t}{2}\right) - cos^2\left(\frac{3\pi t}{2}\right)$

Solution:

$$s = sin^2\left(\frac{3\pi t}{2}\right) - cos^2\left(\frac{3\pi t}{2}\right)$$

$$\frac{ds}{dt} = \frac{d}{dt}\left(sin^2\left(\frac{3\pi t}{2}\right)\right) - \frac{d}{dt}\left(cos^2\left(\frac{3\pi t}{2}\right)\right)$$

$$= 2sin\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(sin\left(\frac{3\pi t}{2}\right)\right) - 2cos\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(cos\left(\frac{3\pi t}{2}\right)\right)$$

$$= 2sin\left(\frac{3\pi t}{2}\right)cos\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(\frac{3\pi t}{2}\right) + 2cos\left(\frac{3\pi t}{2}\right)sin\left(\frac{3\pi t}{2}\right)\frac{d}{dt}\left(\frac{3\pi t}{2}\right)$$

$$= 2sin\left(\frac{3\pi t}{2}\right)cos\left(\frac{3\pi t}{2}\right)\left(\frac{3\pi}{2}\right) + 2cos\left(\frac{3\pi t}{2}\right)sin\left(\frac{3\pi t}{2}\right)\left(\frac{3\pi}{2}\right)$$

$$= \left(\frac{3\pi}{2}\right)\left[2sin\left(\frac{3\pi t}{2}\right)cos\left(\frac{3\pi t}{2}\right) + 2cos\left(\frac{3\pi t}{2}\right)sin\left(\frac{3\pi t}{2}\right)\right]$$

$$= 3\pi\left[2sin\left(\frac{3\pi t}{2}\right)cos\left(\frac{3\pi t}{2}\right)\right] = 3\pi\left[sin\left(2.\frac{3\pi t}{2}\right)\right] = 3\pi\sin(3\pi t)$$

IP3: Find the derivative of $y = 4sin^2\left(\sqrt{1+\sqrt{t}}\right)$

Solution:

$$y = 4sin^2\left(\sqrt{1+\sqrt{t}}\right)$$

$$\frac{dy}{dt} = 4\frac{d}{dt}\left(sin^2\left(\sqrt{1+\sqrt{t}}\right)\right)$$

$$= 4.2.sin\left(\sqrt{1+\sqrt{t}}\right)\frac{d}{dt}\left(sin\left(\sqrt{1+\sqrt{t}}\right)\right)$$

$$= 4.2.sin\left(\sqrt{1+\sqrt{t}}\right)cos\left(\sqrt{1+\sqrt{t}}\right)\frac{d}{dt}\left(\sqrt{1+\sqrt{t}}\right)$$

$$= 4.sin\left(2\sqrt{1+\sqrt{t}}\right)\frac{1}{2\left(\sqrt{1+\sqrt{t}}\right)}\frac{d}{dt}\left(1+\sqrt{t}\right)$$

$$= 4.sin\left(2\sqrt{1+\sqrt{t}}\right)\frac{1}{2\left(\sqrt{1+\sqrt{t}}\right)}.\frac{1}{2\sqrt{t}} \quad = \frac{sin\left(2\sqrt{1+\sqrt{t}}\right)}{\sqrt{t}\left(\sqrt{1+\sqrt{t}}\right)}$$

P3: Find the derivative of $y = \left(1+tan^4\left(\frac{t}{12}\right)\right)^3$

Solution:

Given $y = \left(1+tan^4\left(\frac{t}{12}\right)\right)^3$

$$\frac{dy}{dt} = \frac{d}{dt}\left(1+tan^4\left(\frac{t}{12}\right)\right)^3$$

$$= 3\left(1+tan^4\left(\frac{t}{12}\right)\right)^2\frac{d}{dt}\left(1+tan^4\left(\frac{t}{12}\right)\right)$$

$$= 3\left(1+tan^4\left(\frac{t}{12}\right)\right)^2\left(0+4tan^3\left(\frac{t}{12}\right)\frac{d}{dt}(tan\left(\frac{t}{12}\right))\right)$$

$$= 3\left(1+tan^4\left(\frac{t}{12}\right)\right)^2\left(4tan^3\left(\frac{t}{12}\right)sec^2\left(\frac{t}{12}\right)\frac{d}{dt}\left(\frac{t}{12}\right)\right)$$

$$= 3\left(1+tan^4\left(\frac{t}{12}\right)\right)^2\left(4tan^3\left(\frac{t}{12}\right)sec^2\left(\frac{t}{12}\right)\frac{1}{12}\right)$$

$$= \left(1+tan^4\left(\frac{t}{12}\right)\right)^2\left(tan^3\left(\frac{t}{12}\right)sec^2\left(\frac{t}{12}\right)\right)$$

IP4: Differentiate $f(y) = \left(\frac{y^2}{y+1}\right)^5$

Solution:

Given, $f(y) = \left(\frac{y^2}{y+1}\right)^5$

Let, $f(u) = u^5$, where $u(y) = \frac{y^2}{y+1}$

By chain rule,

$$f'(y) = f'(u).u'$$

where $f'(u) = 5.u^4$,

$$u'(y) = \frac{(y+1)(2y)-y^2(1)}{(y+1)^2} = \frac{2y^2+2y-y^2}{(y+1)^2} = \frac{2y-y^2}{(y+1)^2}$$

$$f'(y) = f'(u).u'(y)$$

$$= 5.\left(\frac{y^2}{y+1}\right)^4.\frac{(2y-y^2)}{(y+1)^2}$$

P4: If $y = (x^3 - 1)^{100}$ then find $\frac{dy}{dx}$.

Solution:

Given $y = (x^3 - 1)^{100}$

$y = u^{100}(x)$, where $u(x) = x^3 - 1$

By chain rule,

$\frac{dy}{dx} = y'(u).u'(x)$, where $\frac{dy}{dx} = 100.u^{99}$, $\quad u'(x) = 3x^2$

$$\frac{dy}{dx} = 100(x^3 - 1)^{99} \times 3x^2 = 300x^2(x^3 - 1)^{99}$$

Exercise

I. **Find the derivative of the following functions:**

a. $y = (2x + 1)^5$

b. $y = (4 - 3x)^9$

c. $y = \left(1 - \frac{x}{7}\right)^{-7}$

d. $y = \left(\frac{x}{2} - 1\right)^{-10}$

e. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

f. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$

g. $y = \frac{1}{21}(3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

h. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$

i. $y = (4x + 3)^4(x + 1)^{-3}$

j. $y = (2x - 5)^{-1}(x^2 - 5x)^6$

II. **Find the derivative of the following functions:**

a. $f(\theta) = \left(\frac{sin\theta}{1+cos\theta}\right)^2$

b. $g(t) = \left(\frac{1+cost}{sint}\right)^{-1}$

c. $y = sin^2(\pi t - 2)$

d. $y = sec^2\pi t$

e. $y = (1 + cos2t)^{-4}$

f. $y = \left(1 + cot\left(\frac{t}{2}\right)\right)^{-2}$

g. $y = sin^3x$

h. $y = 5\ cos^{-4}x$

i. $y = \left(1 + tan^4\left(\frac{t}{12}\right)\right)^3$

j. $y = \frac{1}{6}\left(1 + cos^2(7t)\right)^3$

k. $r = (csc\,\theta + cot\theta)^{-1}$

l. $r = -(sec\theta + tan\theta)^{-1}$

m. $y = x^2 sin^4 x + x cos^{-2} x$

n. $y = \frac{1}{x} sin^{-5} x - \frac{x}{3} cos^3 x$

5.3. Implicit Differentiation

Learning objectives:

- To discuss implicit differentiation
  And
- To solve related problems.

Implicit Differentiation

When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate in the usual way, we may still be able to find $\frac{dy}{dx}$ by ***implicit differentiation***.

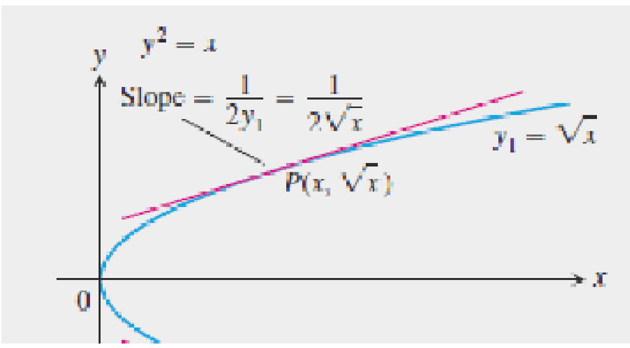
This consists of differentiation both sides of the equation with respect to x and then solve resulting equation for $\frac{dy}{dx}$.

The process by which we find $\frac{dy}{dx}$ is called ***implicit differentiation***.

Example 1: Find $\frac{dy}{dx}$ if $y^2 = x$

Solution: The equation $y^2 = x$ defines two differentiable functions of x

$$y_1 = \sqrt{x} \qquad y_2 = -\sqrt{x}$$



The derivative for each of these for $x > 0$ is

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}$$

We can find the derivatives for both by implicit differentiation. We simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x \Longrightarrow 2y\,\frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{2y}$$

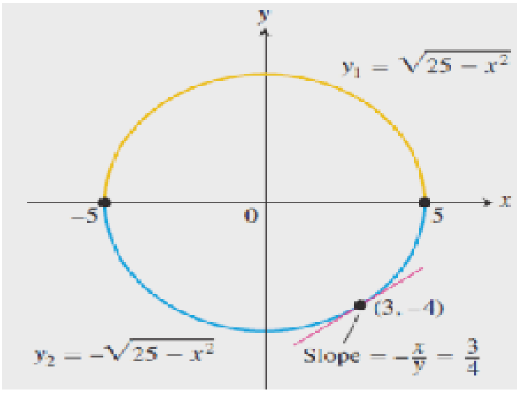
This one formula gives the derivatives we calculated for both of the explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ $\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}}$, $\frac{dy_2}{dx} = \frac{1}{2y_2} = -\frac{1}{2\sqrt{x}}$

Example 2: Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

The circle is not the graph of a single function of x. It is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$. The point (3, −4) lies on the graph of y_2, so we can find the slope by calculating explicitly:

$$\frac{dy_2}{dx}\Big|_{x=3} = \frac{2x}{2\sqrt{25-x^2}}\Big|_{x=3} = \frac{6}{2\sqrt{25-9}} = \frac{3}{4}$$



We can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$\Longrightarrow 2x + 2y\,\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

The slope at (3, −4) is $-\frac{x}{y}\Big|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$

The formula $\frac{dy}{dx} = -\frac{x}{y}$ applies *everywhere the circle has a slope*. The derivative involves both variables x and y, not just the independent variable x.

Implicit differentiation takes four steps.

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation.
3. Factor out $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$ by dividing.

Example 3: Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$.

Solution:

$$2y = x^2 + \sin y$$

Differentiate both sides with respect to x

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2) + \frac{d}{dx}(siny)$$

$$\Rightarrow 2\frac{dy}{dx} = 2x + cosy\ \frac{dy}{dx}$$

$$\Rightarrow 2\frac{dy}{dx} - cosy\ \frac{dy}{dx} = 2x$$

$$\Rightarrow (2 - cosy)\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{2-cosy}$$

PROBLEM SET:

IP1: Find the derivative of $x + \sin y = xy$

Solution:

Given, $x + \sin y = xy$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(x) + \frac{d}{dx}(\sin y) = \frac{d}{dx}(xy)$$

$$\Rightarrow 1 + \cos y\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow (\cos y - x)\frac{dy}{dx} = y - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-1}{\cos y - x}$$

P1: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3axy$

Solution:

$x^3 + y^3 = 3axy$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x,we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(3axy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\ \frac{dy}{dx} + y\right]$$

$$\Rightarrow x^2 + y^2\ \frac{dy}{dx} = a\left[x\ \frac{dy}{dx} + y\right]$$

$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$$

IP2: Find $\frac{dr}{d\theta}$ if $\cos r + \sin\theta = r\ \theta$

Solution:

Given, $cos\ r + sin\ \theta = r\ \theta$

We find $\frac{dr}{d\theta}$ through implicit differentiation

Differentiate both sides with respect to θ

$$\frac{d}{d\theta}(\cos r) + \frac{d}{d\theta}(\sin\theta) = \frac{d}{d\theta}(r\,\theta)$$

$$\Longrightarrow -\sin r\,\frac{dr}{d\theta} + \cos\theta = r + \theta\,\frac{dr}{d\theta}$$

$$\Longrightarrow \cos\theta - r = \frac{dr}{d\theta}(\theta + \sin r)$$

$$\Longrightarrow \frac{dr}{d\theta} = \frac{\cos\theta - r}{\theta + \sin r}$$

P2: Find $\frac{dr}{d\theta}$ if $\theta^{\frac{1}{2}} + r^{\frac{1}{2}} = 1$

Solution:

Given, $\theta^{\frac{1}{2}} + r^{\frac{1}{2}} = 1$

We find $\frac{dr}{d\theta}$ through implicit differentiation

Differentiate both sides with respect to θ

$$\frac{d}{d\theta}\left(\theta^{\frac{1}{2}}\right) + \frac{d}{d\theta}\left(r^{\frac{1}{2}}\right) = \frac{d}{d\theta}(1)$$

$$\Longrightarrow \frac{1}{2}\,\theta^{-\frac{1}{2}} + \frac{1}{2}\,r^{-\frac{1}{2}}\,\frac{dr}{d\theta} = 0$$

$$\Longrightarrow \theta^{-\frac{1}{2}} = -r^{-\frac{1}{2}}\,\frac{dr}{d\theta}$$

$$\Longrightarrow \frac{dr}{d\theta} = -\sqrt{\frac{r}{\theta}}$$

IP3: Find the slope of $y = 2\sin(\pi x - y)$ at $(1, 0)$

Solution:

Given, $y = 2\sin(\pi x - y)$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(y) = 2\frac{d}{dx}\left(\sin(\pi x - y)\right)$$

$$\Longrightarrow \frac{dy}{dx} = 2\cos(\pi x - y).\left[\pi - \frac{dy}{dx}\right]$$

$$\Longrightarrow \frac{dy}{dx} = 2\cos(\pi x - y)\,\pi - 2\cos(\pi x - y)\,\frac{dy}{dx}$$

$$\frac{dy}{dx}\,[1 + 2\cos(\pi x - y)] = 2\cos(\pi x - y)\pi$$

$$\frac{dy}{dx} = \frac{2\cos(\pi x - y)\pi}{1 + 2\cos(\pi x - y)}$$

$$\therefore slope = \left(\frac{dy}{dx}\right)_{(1,0)} = \frac{2\cos(\pi).\pi}{1 + 2\cos\pi} = \frac{-2\pi}{1 + 2(-1)} = \frac{-2\pi}{-1} = 2\pi$$

P3: Find the slope of $x\sin 2y = y\cos 2x$. at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Solution:

Given, $x\sin 2y = y\cos 2x$.

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(x\sin 2y) = \frac{d}{dx}(y\cos 2x)$$

$$\Rightarrow x\ \cos 2y.2.\frac{dy}{dx} + \sin 2y = y.(-sin\ 2x).2 + cos\ 2x\frac{dy}{dx}$$

$$\Rightarrow x\ \cos 2y.2.\frac{dy}{dx} + \sin 2y = (-sin2x)(2y) + cos\ 2x\,.\frac{dy}{dx}$$

$$\Rightarrow (2x\,cos\ 2y - cos\ 2x)\frac{dy}{dx} = -2y\sin 2x - \sin 2y$$

$$\Rightarrow \frac{dy}{dx} = -\frac{[\sin 2y + 2y\sin 2x]}{(2x\cos 2y - \cos 2x)}$$

$$slope = \frac{dy}{dx}\bigg|_{\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)} = -\frac{[sin\ (\pi) + \pi.(sin\ (\frac{\pi}{2}))]}{\left[\frac{\pi}{2}\,cos\ (\pi) - cos\ (\frac{\pi}{2})\right]} \quad = \frac{[0+\pi]}{\left[\frac{\pi}{2}(-1) - 0\right]} = -2$$

IP4: If $siny = x\ sin(a+y)$ then $\frac{dy}{dx} =$

Solution:

Given, $siny = x\ sin(a+y)$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(siny) = \frac{d}{dx}(x\ sin(a+y))$$

$$\Rightarrow cosy\frac{dy}{dx} = x\ \frac{d}{dx}\sin(a+y) + sin(a+y)$$

$$\Rightarrow cosy\frac{dy}{dx} = x\cos(a+y)\frac{dy}{dx} + sin(a+y)$$

$$\Rightarrow cosy\frac{dy}{dx} = \left(\frac{siny}{\sin(a+y)}\right)\ \cos(a+y)\frac{dy}{dx} + sin(a+y)$$

$$since\ x = \left(\frac{siny}{\sin(a+y)}\right)$$

$$\Rightarrow (cosy\sin(a+y) - siny\cos(a+y))\frac{dy}{dx} = sin^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = sin^2(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a+y)}{sina}$$

P4: Find $\frac{dy}{dx}$ using implicit differentiation, $6y^2 + 3xy + 2y^2 + 17y - 6 = 0$

Solution:

Given, $6y^2 + 3xy + 2y^2 + 17y - 6 = 0$

We find $\frac{dy}{dx}$ through implicit differentiation

Differentiate both sides with respect to x

$$\frac{d}{dx}(\ 6y^2) + \frac{d}{dx}(3xy) + \frac{d}{dx}(2y^2) + \frac{d}{dx}(17y) - \frac{d}{dx}(6) = 0$$

$$\Rightarrow 12y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = 0$$

$$\Rightarrow (12y + 4y + 3x + 17)\frac{dy}{dx} = -3y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3y}{(16y + 3x + 17)}$$

Exercise:

I. Use implicit differentiation to find $\frac{dy}{dx}$.

a. $x^2y + xy^2 = 6$
b. $x^3 + y^3 = 18xy$
c. $2xy + y^2 = x + y$
d. $x^3 - xy + y^3 = 1$
e. $x^2(x-y)^2 = x^2 - y^2$
f. $(3xy+7)^2 = 6y$
g. $y^2 = \frac{x-1}{x+1}$
h. $x^2 = \frac{x-y}{x+y}$

II. Use implicit differentiation to find $\frac{dy}{dx}$.

a. $x = tany$
b. $xy = \cot(xy)$
c. $x + \tan(xy) = 0$
d. $x + siny = xy$
e. $ysin\left(\frac{1}{y}\right) = 1 - xy$
f. $y^2\cos\left(\frac{1}{y}\right) = 2x + 2y$

III. Use implicit differentiation to find $\frac{dr}{d\theta}$.

a. $\sin(r\theta) = \frac{1}{2}$
b. $cosr + cot\theta = r\theta$

IV. Find the slope of the following functions:

a. $6y^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$
b. $x^2 - \sqrt{3xy} + 2y^2 = 5, \quad (\sqrt{3}, 2)$
c. $2xy + \pi\, siny = 2\pi, \quad \left(1, \frac{\pi}{2}\right)$
d. $x^2\, cos^2y - siny = 0, \quad (0, \pi)$

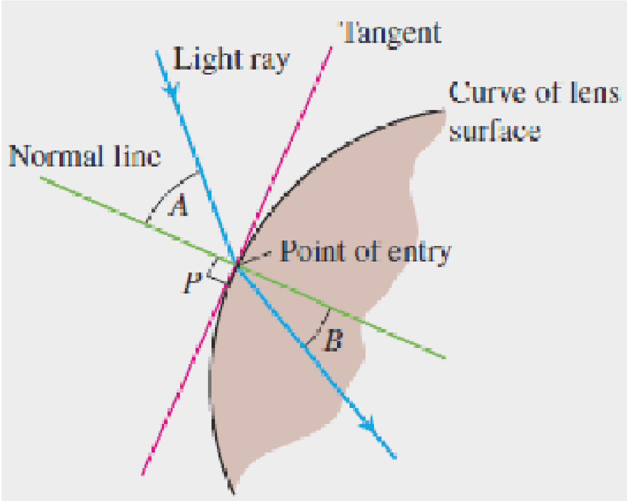
5.4. Tangents and Normal Lines

Learning objectives:

- To find the tangent and the normal lines of a given curve at a given point.
- To find higher order derivatives using implicit differentiation
  And
- To practice related problems.

Tangents and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in figure below). This line is called the *normal* to the surface at the point of entry.



The above figure is a profile view of a lens and the normal is the line perpendicular to the tangent to the profile curve at the point of entry.

Definition

A line is normal to a curve at a point if it is perpendicular to the curve's tangent there. The line is called the ***normal*** to the curve at that point.

The profiles of lenses are often described by quadratic curves. When they are, we can use implicit differentiation to find the tangents and normals.

Example 1: Find the tangent and normal to the curve $\boldsymbol{x^2 - xy + y^2 = 7}$ at the point $\boldsymbol{(-1, 2)}$.

Solution: We first use implicit differentiation to find $\boldsymbol{\frac{dy}{dx}}$:

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

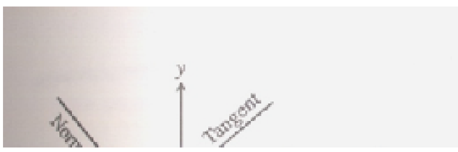
$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

We then evaluate the derivative at $(x, y) = (-1, 2)$ to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \left.\frac{y-2x}{2y-x}\right|_{(-1,2)} = \frac{2-2(-1)}{2(2)-(-1)} = \frac{4}{5}$$



The tangent to the curve at $(-1, 2)$ is the line

$$y = 2 + \frac{4}{5}(x - (-1))$$

$$y = \frac{4}{5}x + \frac{14}{5}$$

The normal to the curve at $(-1,2)$ is

$$y = 2 - \frac{5}{4}(x - (-1))$$
$$y = -\frac{5}{4}x + \frac{3}{4}$$

Using Implicit Differentiation to find derivatives of Higher Order

Implicit differentiation can also be used to calculate higher order derivatives.

Example 2: Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$.

Solution: We differentiate both sides of the equation with respect to x to find $y' = \frac{dy}{dx}$:

$$2x^3 - 3y^2 = 7$$
$$\Rightarrow \frac{d}{dx}(2x^3) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(7)$$
$$6x^2 - 6\,yy' = 0 \Rightarrow x^2 - yy' = 0 \Rightarrow y' = \frac{x^2}{y} \qquad (\text{If } y \neq 0)$$

We differentiate $x^2 - yy' = 0$ again to find y'':

$$\frac{d}{dx}(x^2 - yy') = \frac{d}{dx}(0)$$
$$\Rightarrow 2x - yy'' - y'y' = 0 \Rightarrow yy'' = 2x - (y')^2$$
$$\Rightarrow y'' = \frac{2x}{y} - \frac{(y')^2}{y} \qquad (y \neq 0)$$

Finally, we substitute $y' = \frac{x^2}{y}$ to express y'' in terms of x and y:

$$y'' = \frac{2x}{y} - \frac{\left(\frac{x^2}{y}\right)^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3}\ , \qquad (y \neq 0)$$

PROBLEM SET

IP1: Find the tangent and normal to the curve $x^2 + xy - y^2 = 1$ at $(2,3)$

Solution:

The given curve is $x^2 + xy - y^2 = 1$ and $(2,3)$ lies on it.

Differentiate with respect to x

$$2x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$$
$$\Rightarrow (x - 2y)\frac{dy}{dx} + y + 2x = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{x-2y}$$
$$\therefore \text{slope} = \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{-(4+3)}{2-6} = \frac{-7}{-4} = \frac{7}{4}$$

Tangent to the curve at $(2,3)$

$$y - 3 = \frac{7}{4}(x - 2) \Rightarrow 4y - 12 = 7x - 14$$
$$\Rightarrow 7x - 4y = 2$$

Slope of the normal to the curve at $(2,3) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(2,3)}} = -\frac{4}{7}$

Normal to the curve at $(2,3)$

$$y - 3 = \frac{-4}{7}(x - 2) \Rightarrow 4x + 7y - 29 = 0$$

P1: Find the tangent and normal to the curve $x^2 + y^2 = y^4 - 2x$ at $(-2, 1)$.

Solution:

The given curve is $x^2 + y^2 = y^4 - 2x$ and $(-2,1)$ lies on it.

Differentiate with respect to x

$2x + 2y\frac{dy}{dx} = 4y^3\frac{dy}{dx} - 2$

$\Rightarrow (2y - 4y^3)\frac{dy}{dx} = -2 - 2x$

$\Rightarrow \frac{dy}{dx} = \frac{1+x}{2y^3 - y}$

$\therefore$ slope $= \frac{dy}{dx}_{(-2,1)} = \frac{1+(-2)}{2(1)-1} = \frac{-1}{1} = -1$

Tangent to the curve at $(-2, 1)$

$y - 1 = (-1)(x + 2) \Rightarrow x + y + 1 = 0$

Slope of the normal to the curve at $(-2, 1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(-2,1)}} = 1$

Normal to the curve at $(-2, 1)$

$y - (1) = 1(x + 2) \Rightarrow x - y + 3 = 0$

IP2: Find the tangent and normal to the curve $x^2\, cos^2y -\ siny = 0$ at $(0,\ \pi)$

Solution:

The given curve is $x^2\, cos^2y -\ siny = 0$ and $(0,\ \pi)$ lies on it.

Differentiate both sides with respect to x

$2x\, cos^2y + x^2 .2\, cosy(-\, siny)\ \frac{dy}{dx} - cosy\frac{dy}{dx} = 0$

$\Rightarrow 2x\, cos^2y - x^2\ sin2y\ \frac{dy}{dx} - cosy\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2x\, cos^2y}{(x^2\, sin2y + cosy)}$

$\therefore$ slope $= \frac{dy}{dx}_{(0,\ \pi)} = \frac{2x\, .cos^2y}{x^2\, sin2y + cosy} = 0$

Tangent to the curve at $(0,\ \pi)$ $y - \pi = 0 \Rightarrow y = \pi$

Slope of the normal to the curve at $(0,\ \pi) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(0,\pi)}} = s$

where,$s \longrightarrow \infty$

the equation of the Normal to the curve at $(0,\ \pi)$

$y - \pi = s(x - 0) \Rightarrow x - 0 = \frac{1}{s}(y - \pi) = 0, s \longrightarrow \infty$

Thus, the normal to the curve at $(0,\ \pi)$ is $x = 0$ that is $y\_\ axis$

P2: Find the tangent and normal to the curve $x\, sin2y = y\, cos2x$ at $\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)$

Solution:

The given curve is $xsin2y = y\, cos2x$ and $\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)$ lies on it.

Differentiate both sides with respect to x

$sin2y + x.cos2y.2.\frac{dy}{dx} = y.(-sin2x).2 + cos2x.\frac{dy}{dx}$

$\Rightarrow (2x\, cos2y - cos2x)\frac{dy}{dx} = -2y\, sin2x - sin2y$

$\Rightarrow \frac{dy}{dx} = \frac{-[sin2y + 2y\, sin2x]}{2x\, cos2y - cos2x}$

$\therefore$ slope $= \frac{dy}{dx}_{\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)} = \frac{-[0+\pi.(1)]}{2\times\frac{\pi}{4}(-1)-0} = \frac{-\pi}{-\frac{\pi}{2}} = 2$

Tangent to the curve at $\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)$

$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right) \Longrightarrow 4x - 2y = 0 \Longrightarrow 2x - y = 0$

Slope of the normal to the curve at $\left(\frac{\pi}{4},\ \frac{\pi}{2}\right) = -\frac{1}{\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},\frac{\pi}{2}\right)}} = -\frac{1}{2}$

Normal to the curve at $\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)$

$y - \frac{\pi}{2} = \frac{-1}{2}\left(x - \frac{\pi}{4}\right) \Longrightarrow x + 2y = \frac{5\pi}{4}$

IP3: If $ax^2 + 2hxy + by^2 = 1$ prove that $\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$

Solution:

$$ax^2 + 2hxy + by^2 = 1 \qquad \ldots (1)$$

Differentiate w. r. t. x, we get

$$2ax + 2h\left[x\frac{dy}{dx} + y.1\right] + 2by.\frac{dy}{dx} = 0$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{(hy+ax)}{hx+by} \qquad \ldots (2)$$

$$\frac{d^2y}{dx^2} = -\left[\frac{(hx+by)\left(h\frac{dy}{dx}+a\right)-(hy+ax)\left(h+b\frac{dy}{dx}\right)}{(hx+by)^2}\right]$$

$$= \frac{-x(h^2-ab)\frac{dy}{dx}+(h^2-ab)y}{(hx+by)^2}$$

$$= \frac{h^2-ab}{(hx+by)^2}\left[y + x\left(\frac{hy+ax}{hx+by}\right)\right] \quad \text{using (2)}$$

$$= \frac{h^2-ab}{(hx+by)^3}(h\,xy + by^2 + h\,xy + ay^2)$$

$$= \frac{h^2-ab}{(hx+by)^3} \quad \text{using (1)}$$

$$\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$$

P3: For the conic $ax^2 + by^2 = c$, Find $\frac{d^2y}{dx^2}$.

Solution:

Given conic is $ax^2 + by^2 = c$

Differentiate with respect to x

$$2ax + 2by\frac{dy}{dx} = 0 \qquad \left(\therefore \frac{dy}{dx} = -\frac{ax}{by}\right)$$

Differentiate again with respect to x

$$a + b\left(\frac{dy}{dx}\right)^2 + by\frac{d^2y}{dx^2} = 0$$

$$\Longrightarrow \frac{d^2y}{dx^2} = \frac{a+b\left(\frac{dy}{dx}\right)^2}{by}$$

$$= -\frac{a+b\left(\frac{a^2}{b^2}\times\frac{x^2}{y^2}\right)}{by} = -\frac{a(by^2+ax^2)}{b^2y^3} = -\frac{ac}{b^2y^3}$$

IP4: $xy + y^2 = 16$, Find the value of $\frac{d^2y}{dx^2}$ at $(0,-1)$

Solution:

$xy + y^2 = 16$

Differentiate with respect to x

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0 \qquad \left(\therefore \frac{dy}{dx} = \frac{-y}{(x+2y)}\right)$$

$$\Longrightarrow (x+2y)\frac{dy}{dx} + y = 0$$

Differentiate with respect to x

$$(x+2y)\frac{d^2y}{dx^2} + \left(1 + 2\frac{dy}{dx}\right)\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\Longrightarrow (x+2y)\frac{d^2y}{dx^2} + 2\left(1 + \frac{dy}{dx}\right)\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = -\frac{2\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}}{(x+2y)} = \frac{2\left(1-\frac{y}{(x+2y)}\right)\left(\frac{y}{(x+2y)}\right)}{(x+2y)}$$

$$\frac{d^2y}{dx^2} = \frac{2y(x+y)}{(x+2y)^3}$$

The value of $\frac{d^2y}{dx^2}$ at $(0,-1) = -\frac{1}{4}$

P4: If $x^3 + y^3 = 16$, then find the value of $\frac{d^2y}{dx^2}$ at $(2,2)$

Solution:

$x^3 + y^3 = 16$

Differentiation with respect to x

$$x^2 + y^2\frac{dy}{dx} = 0 \quad \left(\frac{dy}{dx} = -\left(\frac{x}{y}\right)^2\right)$$

Differentiation with respect to x

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = 0$$

$$\Longrightarrow \frac{d^2y}{dx^2} = -\frac{\left(2x + 2y\left(-\left(\frac{x}{y}\right)^2\right)^2\right)}{y^2}$$

The value of $\frac{d^2y}{dx^2}$ at $(2,2) = -\frac{8}{4} = -2$

Exercise:

a. Find the lines that are (i) tangent and (ii) normal to the curve at the given point.

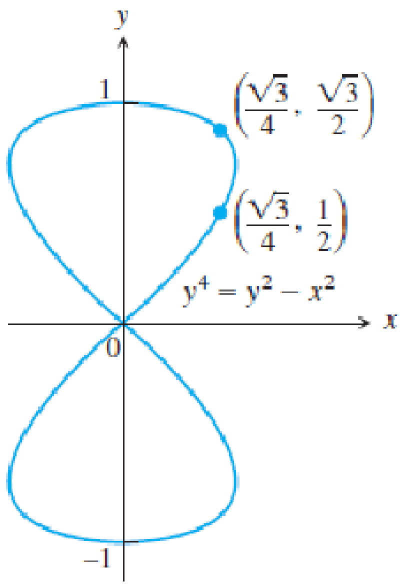
a.	$x^2 + xy - y^2 = 1$	$(2,3)$
b.	$x^2y^2 = 9$	$(-1,3)$
c.	$6x^2 + 3xy + 2y^2 + 17y - 6 = 0$	$(-1,0)$
d.	$2xy + \pi \sin y = 2\pi$	$(1,\frac{\pi}{2})$
e.	$y = 2\sin(\pi - y)$	$(1,0)$

b. Find second order differentiation to the following functions

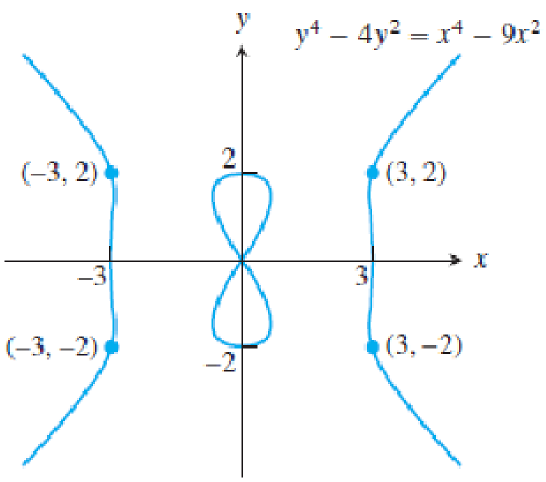
a. $x^2 + y^2 = 1$
b. $x^2 + 2x = y^2$
c. $y^2 - 2x = 1 - 2y$
d. $y = \frac{1}{9}cot(3x - 1)$
e. $y^3 + y = 2cosx$ at $(0,1)$

c. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

d. Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



e. Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



5.5. Rational Powers of Differential Functions

Learning objectives:

- To extend the power rule for differentiation to rational exponents through implicit differentiation

  And
- To practice related problems.

This module uses the technique of implicit differentiation to extend the Power Rule for differentiation to include all rational exponents.

The power rule $\frac{d}{dx}x^n = nx^{n-1}$

was proved to hold when n is an integer. We can now show that it holds when n is any rational number.

Theorem: Power rule for differentiation for rational numbers

If n is a rational number, then x^n is differentiable at every interior point x of the domain of x^{n-1}, and

$$\frac{d}{dx}x^n = nx^{n-1}$$

Proof:

Let $n = \frac{p}{q}$ where p and q are integers with $q \neq 0$ so that $y = x^n = x^{\frac{p}{q}}$. Then $y^q = x^p$

This equation is an algebraic combination of powers of x and y, and y is a differentiable function of x. Since p and q are integers, we can differentiate both sides of the equation implicitly with respect to x and obtain $q\, y^{q-1}\frac{dy}{dx} = p\, x^{p-1}$

We solve for $\frac{dy}{dx}$ and

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{\left(x^{p/q}\right)^{q-1}}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} \Longrightarrow = \frac{p}{q} \cdot x^{p-1-p+p/q}$$

$$= \frac{p}{q} \cdot x^{\left(\frac{p}{q}\right)-1}$$

$$= nx^{n-1}$$

This proves the rule.

Example 1:

a) $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}} = \frac{1}{2\sqrt{x}}$; function is defined for $x \geq 0$, derivative is defined only for $x > 0$.

b) $\frac{d}{dx}\left(x^{\frac{1}{5}}\right) = \frac{1}{5}x^{-\frac{4}{5}}$; function is defined for all x, derivative is not defined at $x = 0$

In terms of the Chain rule, the Power Rule states that if n is a rational number, u is differentiable at x, and u^{n-1} is defined, then u^n is differentiable at x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Example 2:

a) $\frac{d}{dx}(1-x^2)^{\frac{1}{4}} = \frac{1}{4}\,(1-x^2)^{-\frac{3}{4}}(-2x) = \frac{-x}{2(1-x^2)^{\frac{3}{4}}}$;

Notice that the function is defined on [−1, 1], but the derivative is defined only on (−1, 1).

b) $\frac{d}{dx}(\cos x)^{-\frac{1}{5}} = -\frac{1}{5}\,(\cos x)^{-\frac{6}{5}}\,\frac{d}{dx}\,cosx$

$= -\frac{1}{5}\,(\cos x)^{-\frac{6}{5}}(-\sin x)$

$= \frac{1}{5}\sin x\,(\cos x)^{-\frac{6}{5}}$

PROBLEM SET

IP1: Find $\frac{dy}{dx}$ if $y = \sqrt[4]{5x}$

Solution:

$y = \sqrt[4]{5x}$

$\frac{dy}{dx} = \frac{d}{dx}\left((5x)^{\frac{1}{4}}\right) = \frac{1}{4}(5x)^{\frac{1}{4}-1} = \frac{1}{4}(5x)^{-\frac{3}{4}}$

Function is defined for all $x \geq 0$, derivative is not defined at $x = 0$

P1: Find $\frac{dy}{dx}$ if $y = x^{\frac{-3}{5}}$

Solution:

$y = x^{\frac{-3}{5}}$

$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{-3}{5}}\right) = \left(-\frac{3}{5}\right)x^{-\frac{1}{5}-1} = -\frac{3}{5}x^{-\frac{6}{5}}$

Function is defined for all $x \in \boldsymbol{R}$, $x \neq 0$, derivative is not defined at $x = 0$

IP2: Find $\frac{dy}{dx}$ if $y = x(x^2+1)^{\frac{1}{2}}$

Solution:

$y = x(x^2+1)^{\frac{1}{2}}$

$\Longrightarrow \frac{dy}{dx} = x\frac{d}{dx}(x^2+1)^{\frac{1}{2}} + (x^2+1)^{\frac{1}{2}}\frac{d}{dx}(x)$

$= x\frac{1}{2}(x^2+1)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+1) + (x^2+1)^{\frac{1}{2}}.1$

$= 2x^2\,\frac{1}{2\sqrt{x^2+1}} + \sqrt{x^2+1}$

$= \frac{x^2+x^2+1}{\sqrt{(x^2+1)}} = \frac{(2x^2+1)}{\sqrt{(x^2+1)}}$

P2: Find $\frac{dy}{dx}$ if $y = (1-6x)^{\frac{2}{3}}$

Solution:

$y = (1-6x)^{\frac{2}{3}}$

$\frac{dy}{dx} = \frac{d}{dx}(1-6x)^{\frac{2}{3}} = \frac{2}{3}(1-6x)^{\frac{2}{3}-1}.\frac{d}{dx}(1-6x)$

$$= \frac{2}{3}(1-6x)^{-\frac{1}{3}}.(-6) = \frac{-4}{(1-6x)^{\frac{1}{3}}}$$

Function is defined for all x, derivative is defined for all $x \; except \; x \neq \frac{1}{6}$.

IP3: Find the derivative of $y = \cos\left((1-6\mathrm{t})^{\frac{2}{3}}\right)$

Solution:

$y = \cos\left((1-6\mathrm{t})^{\frac{2}{3}}\right)$

$y = cosu$, where $u = v^{\frac{2}{3}}$ and $v(t) = 1 - 6\mathrm{t}$

Now $\frac{dy}{du} = -sinu$, $\frac{du}{dv} = \frac{2}{3}v^{\frac{-1}{3}}$ and $\frac{dv}{dt} = -6$

By chain rule $\frac{dy}{dt} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dt}$

$$\frac{dy}{dt} = -sinu.\frac{2}{3}v^{-\frac{1}{3}}.(-6)$$

$$= 4sin\left((1-6\mathrm{t})^{\frac{2}{3}}\right).(1-6\mathrm{t})^{-\frac{1}{3}}$$

P3: Find the derivative of $y = (\sin(\theta+5))^{\frac{5}{4}}$

Solution:

$y = (\sin(\theta+5))^{\frac{5}{4}}$

$y = u^{\frac{5}{4}}$, where $u(v) = \sin v$ and $v(\theta) = \theta + 5$

Now, $\frac{dy}{du} = \frac{5}{4}u^{\frac{1}{4}}$, $\frac{du}{dv} = \cos v$ and $\frac{dv}{d\theta} = 1$

By chain rule $\frac{dy}{d\theta} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{d\theta}$

$$\frac{dy}{d\theta} = \frac{5}{4}u^{\frac{1}{4}}.\cos v\,.1$$

$$= \frac{5}{4}(\sin(\theta+5))^{\frac{1}{4}}.\cos(\theta+5)\,.1$$

$$= \frac{5}{4}(\sin(\theta+5))^{\frac{1}{4}}.\cos(\theta+5)$$

IP4: Find the derivative of the function given by $y = \sin\left[(2t+5)^{-\frac{2}{3}}\right]$

Solution:

$y = \sin\left[(2t+5)^{-\frac{2}{3}}\right]$

Now , $y = \sin u$, $u = v^{-\frac{2}{3}}$ and $v = (2t+5)$

$$\frac{dy}{du} = \cos u,\ \frac{du}{dv} = \frac{-2}{3}v^{\frac{1}{3}},\ \frac{dv}{dt} = 2$$

By chain rule

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dt}$$

$$\frac{dy}{dt} = \cos u\left[-\frac{2}{3}v^{1/3}.2\right]$$

$$= -\frac{4}{3}\cos\left((2t+5)^{-\frac{2}{3}}\right).(2t+5)^{1/3}$$

P4: Find the derivative of $y = \sqrt[3]{1 + cos2\theta}$

Solution:

$y = \sqrt[3]{1 + cos2\theta}$

Now, $y = u^{\frac{1}{3}}$ where $u = 1 + cos2\theta$

By chain rule

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta}$$

$$= \frac{d}{du}\left(u^{\frac{1}{3}}\right) \times \frac{d}{d\theta}(1 + cos2\theta) = \frac{1}{3}u^{-\frac{2}{3}}(-sin2\theta)(2)$$

$$= -\frac{2}{3}(1 + cos2\theta)^{-\frac{2}{3}} sin2\theta$$

Exercise:

f. State and prove power rule for differentiation for rational numbers.

g. Find $\frac{dy}{dx}$.

   a. $y = x^{9/4}$

   b. $x = \sqrt[3]{2x}$

   c. $y = 7\sqrt{x + 6}$

   d. $y = (2x + 5)^{-\frac{1}{2}}$

   e. $y = x(x^2 + 1)^{\frac{1}{2}}$

h. Find the first derivatives of the functions.

   a. $s = \sqrt[7]{t^2}$

   b. $f(x) = \sqrt{1 - \sqrt{x}}$

   c. $r = \frac{3}{2}\theta^{\frac{2}{3}} + \frac{4}{3}\theta^{\frac{3}{4}}$

5.6. Related Rates of Change

Learning objectives:

- To practice some related rates problems.

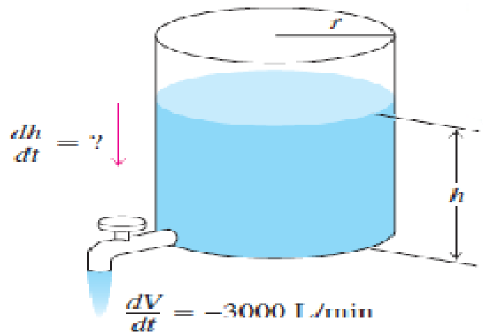
The problem of finding a rate, which can't be measured easily, from some other measurable rates, is called a *related rates problem*.
When we pump the fluid out stored in a vertical cylindrical storage tank, the fluid level inside the tank drops down. In general it is difficult to measure the rate of change in the level of fluid. However it can be determined from the rate of pumping which can be easily measured. We write an equation that relates the variables involved and differentiate to get an equation that relates the rate we seek to the rate we know.

Example 1:

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of $3000\ L/min$?

Solution

Let r be the radius of the cylinder, h the height of the fluid, and V the volume.



As time passes, the radius remains constant, but V and h change. We think of V and h as differentiable functions of time and use t to represent time. It is given that

$$\frac{dV}{dt} = -3000$$

Since we pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.

We are asked to find $\frac{dh}{dt}$: How fast will the fluid level drop?

Rates of change are represented by derivatives.

To find $\frac{dh}{dt}$, we first write an equation that relates h to V.

If V measured in litres and r, h in metres then $V = \pi r^2 h$ cubic metres. Since one cubic metre contains $1000L$, we write $V = 1000\pi r^2 h$

We differentiate both sides of the equation with respect to time:

$$\frac{dv}{dt} = 1000\pi r^2 \frac{dh}{dt}, \quad r \text{ is a constant.}$$

We substitute the known value $\frac{dv}{dt} = -3000$ and solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}$$

The fluid level will drop at the rate of $\frac{3}{\pi r^2}\ m/min$.

The rate at which the fluid level drops depends on the tank's radius. If r is small, $\frac{dh}{dt}$ will be large; if r is large, $\frac{dh}{dt}$ will be small.

If $r = 1\ m$: $\frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95\ m/min = -95\ cm/min$

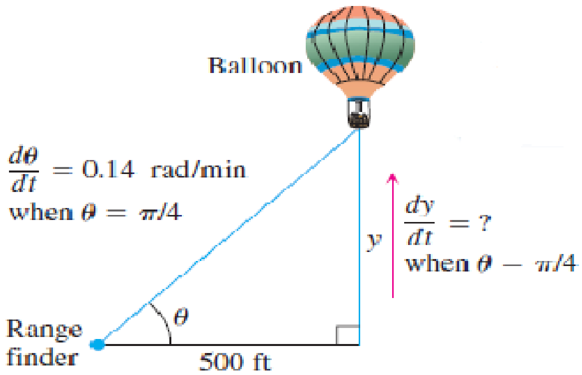
If $r = 10\ m$: $\frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095 m/min = -0.95 cm/min$

Example 2:

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution

We draw a picture and name the variables and constants.



The variables are: θ is the angle the range finder makes with the ground (radians), y is the height of the balloon

We assume θ and y to be differentiable functions of time t. *Write down the numerical information.*

$$\frac{d\theta}{dt} = 0.14\ rad/min \qquad when\ \theta = \frac{\pi}{4}$$

Write down what we are asked to find

We want $\frac{dy}{dt}$ when $\theta = \frac{\pi}{4}$.

Write an equation that relates the variables y and θ.

$$y = 500 \tan\theta$$

Differentiate with respect to t using the Chain Rule.

The result tells how $\frac{dy}{dt}$ (which we want) is related to $\frac{d\theta}{dt}$ (which we know).

$$\frac{dy}{dt} = 500\ sec^2\theta \frac{d\theta}{dt}$$

Evaluate with $\theta = \frac{\pi}{4}$ *and* $\frac{d\theta}{dt} = 0.14$ to find $\frac{dy}{dt}$.

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2(0.14) = 140$$

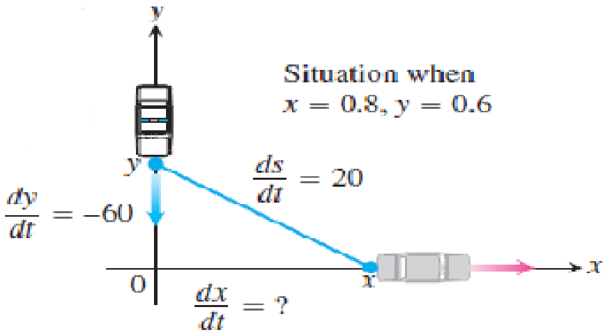
At the moment in question, the balloon is rising at the rate of $140\ ft/min$.

Example 3:

A police cruiser, approaching a right-angled intersection from the north,is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 km north of the intersection and the car is 0.8 km to the east, the police determine with radar that the distance between them and the car is increasing at 20 $km/hour$. If the cruiser is moving at $60\ km/hour$ at the instant of measurement, what is the speed of the car?

Solution

We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the northbound highway.



We let t represent time and set

x= position of car at time t

y= position of cruiser at time t

s = distance between car and cruiser at time t

We assume $x, y,$ and s are differentiable functions of t.

At the instant in question,

$$x = 0.8\ km, y = 0.6\ km, \frac{ds}{dt} = 20 km/h, \quad \frac{dy}{dt} = -60\ km/h$$

$\frac{dy}{dt}$ is negative because y is decreasing.

We need to find $\frac{dx}{dt}$.

The variables are related by $\quad s^2 = x^2 + y^2$

We differentiate with respect to t.

$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

We evaluate with

$x = 0.8\ km, y = 0.6\ km, \frac{ds}{dt} = 20 km/h, \frac{dy}{dt} = -60 km/$ and solve for $\frac{dx}{dt}$.

$$20 = \frac{1}{\sqrt{0.8^2+0.6^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right) = 0.8\frac{dx}{dt} - 36$$

$$\frac{dx}{dt} = \frac{20+36}{0.8} = 70$$

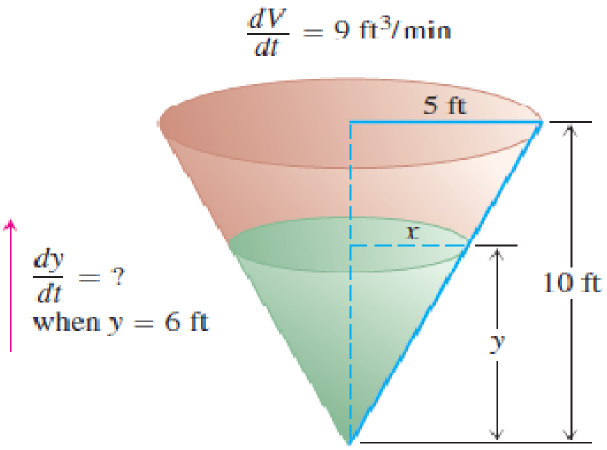
At the moment in question, the car's speed is 70 km/h.

Example 4:

Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when water is 6 ft deep?

Solution

We draw a picture of a partially filled conical tank.



The variables in the problem are

V = volume (ft^3) of water in the tank at time t (min),
x= radius (ft) of the surface of water at time t,
y = depth (ft) of water in the tank at time t.

We wish to find $\frac{dy}{dt}$.

The variables are related by $V = \frac{1}{3}\pi x^2 y$.

The equation involves x also. Because no information is given about x and $\frac{dx}{dt}$ at the time in question, we need to eliminate x. From similar triangles,

$$\frac{x}{y} = \frac{5}{10} \qquad x = \frac{y}{2}$$

Therefore, $V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$

We differentiate with respect to t.

$$\frac{dV}{dt} = \frac{\pi}{12}.3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}$$

We then solve for $\frac{dy}{dt}$: $\frac{dy}{dt} = \frac{4}{\pi y^2}\frac{dV}{dt}$

We evaluate with $y = 6\ and\ \frac{dV}{dt} = 9$..

$$\frac{dy}{dt} = \frac{4}{\pi(6)^2}.9 = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about $0.32\ ft/min$.

PROBLEM SET

IP1:

A ladder 10ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution :

Let x feet be the distance from the bottom of the ladder to the wall and y feet be the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

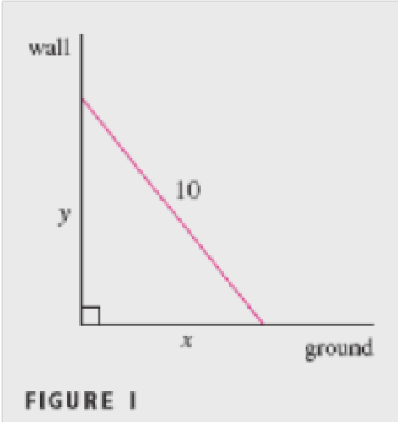


FIGURE 1

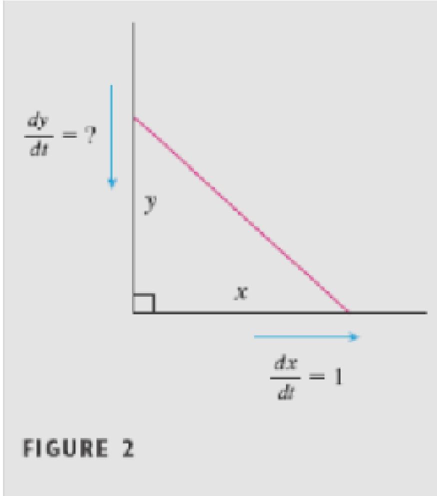


FIGURE 2

Given $\frac{dx}{dt} = 1\ ft/s$ and
Find $\frac{dy}{dt}$ when $x = 6\ ft$
The relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the chain Rule,
we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longrightarrow 2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

$$\Longrightarrow 2y\frac{dy}{dt} = -2x\frac{dx}{dt} \Longrightarrow y\frac{dy}{dt} = -x\frac{dx}{dt}$$

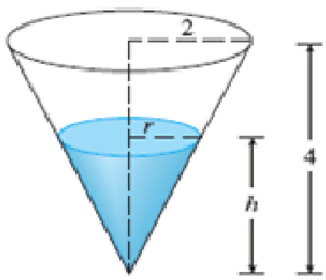
$$\Longrightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$, the Pythagorean theorem gives $y = 8$ and so, substituting these values and $\frac{dx}{dt} = 1$, we get

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}\ ft/s$$

The fact that $\frac{dy}{dt}$ is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of $\frac{3}{4} ft/s$. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4} ft/s$.

P1:
A water tank has the shape of an inverted circular cone with base radius $2m$ and height $4m$. If water is being pumped into the tank at a rate of $2m^3/min$, find the rate at which the water level is rising when the water is $3m$ deep.
Solution:
Let V, r and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.



Given, $dV/dt = 2m^3/min$

We are asked to find $\frac{dh}{dt}$ when h is $3m$.

The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

To express V as a function of h alone.

In order to eliminate r, we use the similar triangles to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

The expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

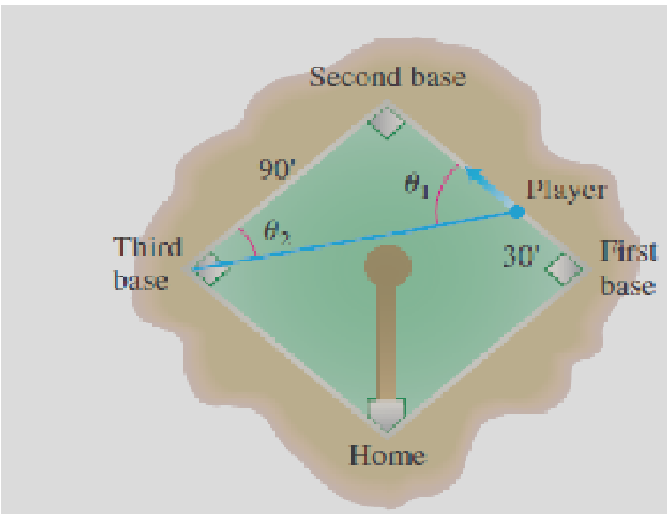
Substituting $h = 3\, m$ and $\frac{dV}{dt} = 2m^3/min$,

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}.2 = \frac{8}{9\pi}m/min$$

The water level is rising at a rate of $\frac{8}{9\pi}m/min \approx 0.28m/min$.

IP2: A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec. At what rate is the player's distance from third base changing when the player is 30 ft from first base?

Solution:



$s(t)$ = Distance of the player from third base

$x(t)$ = Distance of the player from second base

$\frac{dx}{dt} = -16\ ft/sec$

From the image, $s^2 = x^2 + 8100$

$\Longrightarrow 2s\frac{ds}{dt} = 2x\,\frac{dx}{dt}$

$\Longrightarrow s\,\frac{ds}{dt} = x\frac{dx}{dt}$ ... (1)

When the player is 30 ft from 1st base then $x = 60$

$S = \sqrt{(90)^2 + (60)^2} = 30\sqrt{13}$

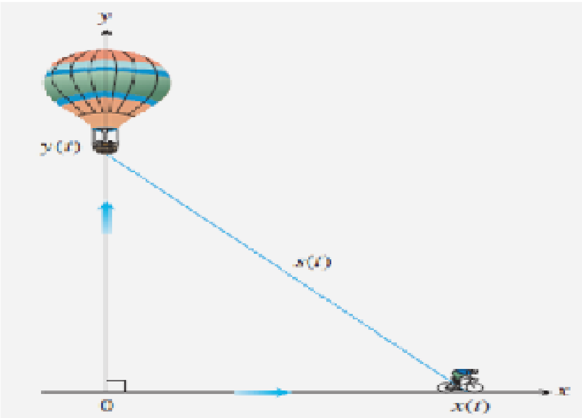
From (1)

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt} = \frac{60}{30\sqrt{3}}(-16)$$

$$= -\frac{32}{\sqrt{3}} \approx -8.875\ ft/sec$$

P2: A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?

Solution:



Here $y(t) =$ balloons height from O.

$x(t) =$ bicycle distance from O.

$S(t) =$ distance between balloon and bicycle

$t = 3$ sec

$\frac{ds}{dt} = ?$

From the given data, $s^2(t) = y^2(t) + x^2(t)$

$$\Rightarrow 2s(t)\frac{ds}{dt} = 2y(t)\frac{dy}{dt} + 2x(t)\frac{dx}{dt}$$

$$\Rightarrow s(t)\frac{ds}{dt} = y(t)\frac{dy}{dt} + x(t)\frac{dx}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{s(t)}\left[y(t)\frac{dy}{dt} + x(t)\frac{dx}{dt}\right]$$

$y(t) = 65 + 3 = 68\ ft$, $x(t) = 51\ ft$ $s(t) = \sqrt{(68)^2 + (51)^2} = 85$

$$\frac{ds}{dt} = \frac{1}{85}[68(1) + 51(17)] = 11 ft/sec$$

IP3: A man walks along a straight path at a speed of $4ft/s$. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the search light rotating when the man is 15 ft from the point on the path closest to the searchlight?

Solution:

let x be the distance from the man to the point on the path closest to the searchlight. θ be the angle between the beam of the searchlight and the perpendicular to the path.
Given , $\frac{dx}{dt} = 4ft/s$
Find $\frac{d\theta}{dt}$ when $x = 15$.
The equation that relates x and θ is
$\frac{x}{20} = tan\theta \Longrightarrow x = 20\ tan\theta$
Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = 20sec^2\theta\frac{d\theta}{dt}$$

So $\quad \frac{d\theta}{dt} = \frac{1}{20}\ cos^2\theta\frac{dx}{dt} = \frac{1}{20}cos^2\theta(4) = \frac{1}{5}cos^2\theta$

When $x = 15$, the length of the beam is 25, so $cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of $0.128\ rad/s$.

P3: Car A is traveling west at $50\ mi/h$ and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 and car B is $0.4\ mi$ from the intersection?

Solution:

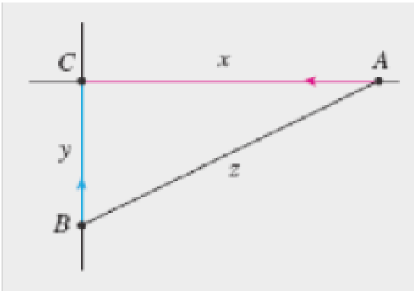


FIGURE 4

Here C is the intersection of the roads.
At a given time t, let x be the distance from car A to C, y be the distance from car B to C, and z be the distance between the cars, where x, y, and z are measured in miles.
Given, $\frac{dx}{dt} = -50mi/h$ and $\frac{dy}{dt} = -60mi/h$.
To find $\frac{dz}{dt}$
The equation that relates x, y and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating both sides with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $x = 0.3\ mi$ and $y = 0.4\ mi$, $z = 0.5\ mi$, so

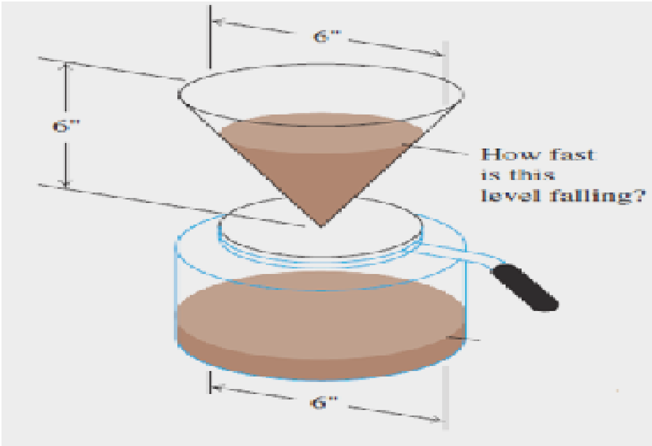
$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)]$$

$$= -78\ mi/h$$

The cars are approaching each other at a rate of $78\ mi/h$.

IP4: Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10\ inch^3/min$. How fast is the level in the cone falling, when the coffee in the cone is 5 inch. deep?

Solution:



Let h be the height of the coffee in the cone.

From the figure, the radius of the cone $r = \frac{h}{2}$

Volume of the cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$

$$V = \frac{1}{12}\pi h^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{12}\pi(3h^2)\frac{dh}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

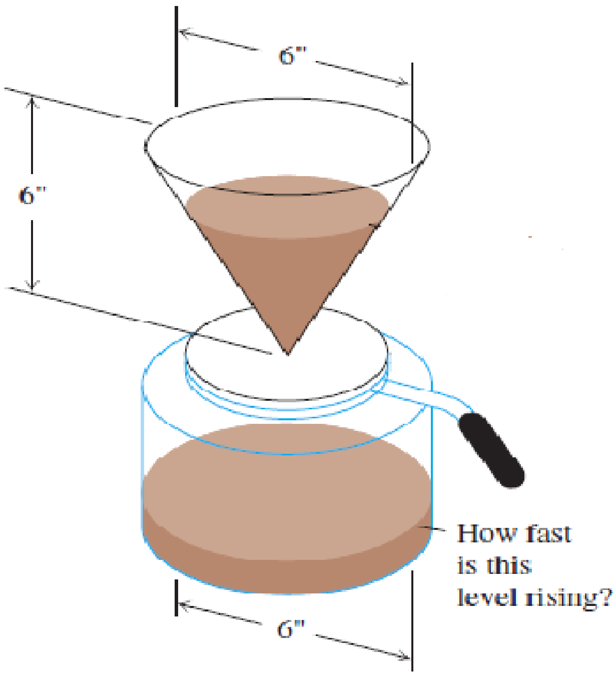
When $h = 5\ inch$, then

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi(25)}(-10) = -\frac{8}{5\pi}\ inch/min$$

The rate the coffee level *falling* in the cone is $\frac{8}{5\pi}\ inch/min$

P4:

Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10\ inch^3/min$.How fast is the level in the pot rising when the coffee in the cone is 5 inch. deep?



Solution:

Radius of cylindrical coffee pot $= 3\ inch$

Let h be the height of the coffee in the pot

Volume of the coffee $V = \pi r^2 h = 9\pi h$

$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$

The rate the coffee is rising is

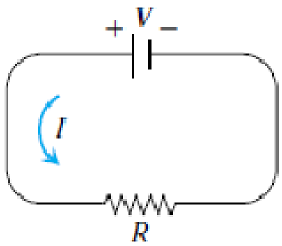
$\frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt}$

$\frac{dh}{dt} = \frac{1}{9\pi}[10]$

$= \frac{10}{9\pi}\ inch/min$

Exercise:

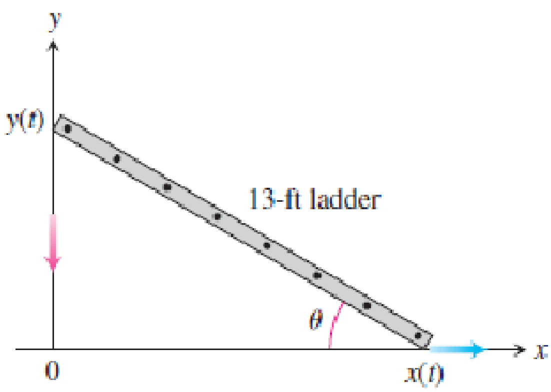
1. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates $\frac{dA}{dt}$ to $\frac{dr}{dt}$.
2. The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
   a. How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant?
   b. How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if h is constant?
   c. How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither r nor h is constant?
3. The voltage V (volts), current I (amperes) and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $\frac{1}{3}$ amp/sec. Let t denote time in seconds.



   a. What is the value of $\frac{dV}{dt}$?
   b. What is the value of $\frac{dI}{dt}$?
   c. What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?
   d. Find the rate at which R is changing when $V =$ 12 volts and $I =$ 2 amperes. Is R increasing, or decreasing?
4. Let x and $y$be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ to be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.
   a. How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?
   b. How is $\frac{\mathrm{ds}}{\mathrm{dt}}$ related to $\frac{\mathrm{dx}}{\mathrm{dt}}$ and $\frac{dy}{dt}$ if neither x nor y is constant?
   c. How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

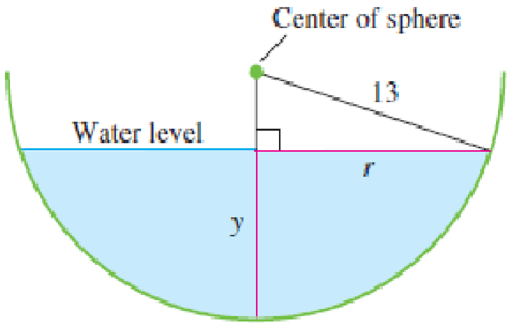
5. The area A of a triangle with sides of length a and b enclosing an angle of measure θ is
$$A = \frac{1}{2}\, ab \sin\theta$$
   a. How is $\frac{dA}{dt}$ related to $\frac{d\theta}{dt}$ if a and b are constant?
   b. How is $\frac{dA}{dt}$ related to $\frac{d\theta}{dt}$ and $\frac{da}{dt}$ if only b is constant?
   c. How is $\frac{dA}{dt}$ related to $\frac{d\theta}{dt}, \frac{da}{dt}$ and $\frac{db}{dt}$ if none of a, b, and θ are constant?
6. The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When l = 12 cm and w = 5 cm, find the rates of change of
   a. the area,
   b. the perimeter, and
   c. the lengths of the diagonals of the rectangle.

   Which of these quantities are decreasing, and which are increasing?
7. A $13-ft$ ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.



   a. How fast is the top of the ladder sliding down the wall then?
   b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
   c. At what rate is the angle θ between the ladder and the ground changing then?
8. A girl flies a kite at a height of 100 m, the wind carrying the kite horizontally away from her at a rate of 10 m/sec. How fast must she let out the string when the kite is 150 m away from her?
9. Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the
   a. height and
   b. radius changing when the pile is 4 m high?

   Answer in cm/min.
10. Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile.



Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $\mathrm{V} = \frac{\pi}{3}y^2(3\mathrm{R} - y)$ when the water is y units deep.

a. At what rate is the water level changing when the water is 8 m deep?
b. What is the radius r of the water's surface when the water is y m deep?

At what rate is the radius r changing when the water is 8 m deep?

6.1. Extreme Values of Functions

Learning objectives:

- To define absolute maximum, absolute minimum, local maximum and local minimum of a function.
- To study the Max-Min Theorem (The Extreme Value Theorem)
  AND
- To practice related problems.

Global Extrema

Absolute Extreme Values

Let f be a function with domain D. Then f has an ***absolute maximum*** value on D at a point c if, $\quad f(x) \leq f(c) \quad$ for all x in D

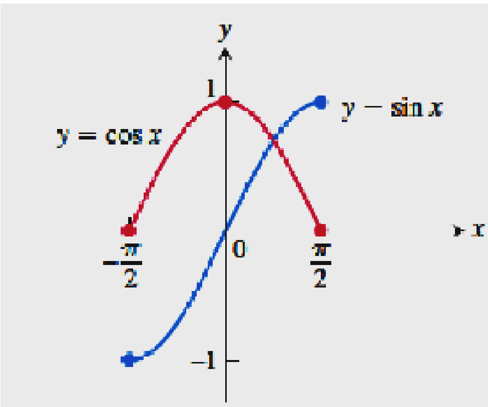
and an ***absolute minimum*** value on D at c if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D$$

Absolute maximum and minimum values are called absolute ***extrema***.

Absolute extrema are also called ***global*** extrema.

Example 1:



On $[-\pi/2, \pi/2], f(x) = \cos x$ takes on an absolute maximum value of 1(once) and an absolute minimum value of 0 (twice). The function $g(x) = \sin x$ takes on an absolute maximum value of 1 and an absolute minimum value of -1.

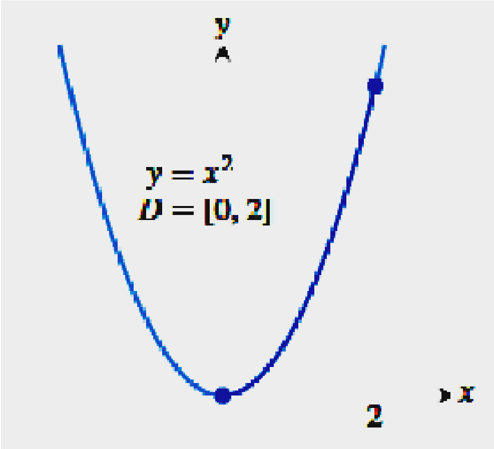
Functions with the same defining rule can have different extrema, depending on the domain.

Example 2:

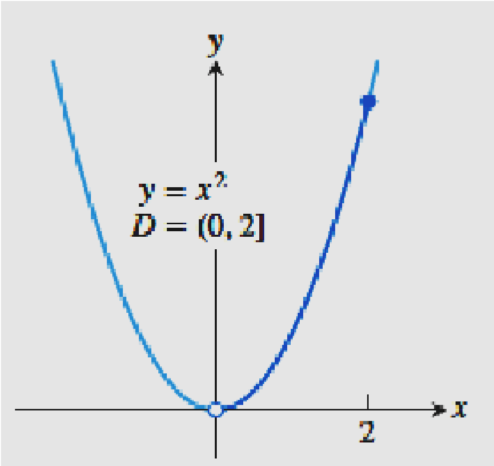
We consider the function rule $y = x^2$ with various domains:

a) $\quad (-\infty, \infty)$; No absolute maximum. Absolute minimum of 0 at $x = 0$.

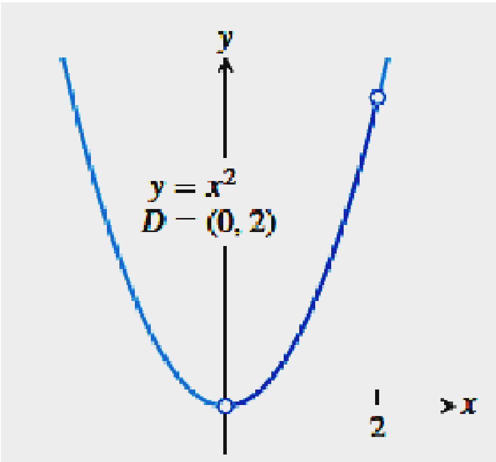
b) $[0, 2]$; Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.



c) $(0, 2]$; Absolute maximum of 4 at $x = 2$. No absolute minimum.



d) $(0, 2)$; No absolute extrema.



The following theorem asserts that a function that is continuous at every point of a closed interval has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function. They play a role in problem solving and in the development of integral calculus.

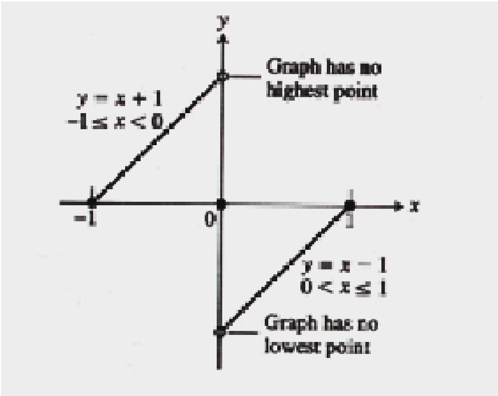
The Max-Min Theorem: (**The Extreme Value Theorem**)

If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in I.

On an open interval, a continuous function need not have either a maximum or a minimum value. The function $f(x) = x$ has neither a largest nor a smallest value on $(0,1)$.
Even a single point of discontinuity can keep a function from having either a maximum or a minimum value on a closed interval. The function

$$y = \begin{cases} x+1 & -1 \le x < 0 \\ 0 & x = 0 \\ x-1 & 0 < x \le 1 \end{cases}$$

is continuous at every point of $[-1,1]$ except $x = 0$, yet its graph over $[-1,1]$ has neither a highest nor a lowest point.



As the above discussions show, the requirements that the interval be closed and the function continuous are key ingredients of the Max-Min Theorem. Without them, the conclusion of the theorem need not hold. Further example 2 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite.

Local Extreme Values

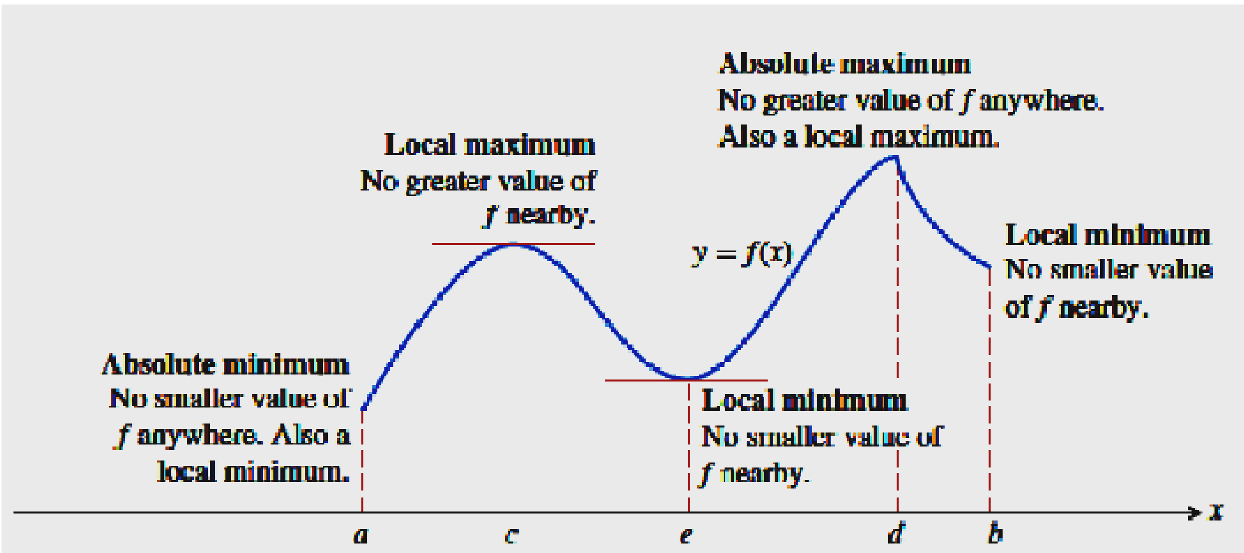
A function f has a ***local maximum*** value at an interior point c of its domain if

$f(x) \le f(c)$ for all x in some open interval containing c.

A function f has a ***local minimum*** value at an interior point c of its domain if

$f(x) \ge f(c)$ for all x in some open interval containing c.

We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a ***local maximum*** or ***local minimum*** value *at an endpoint* c if the appropriate inequality holds for all x in some half-open interval in its domain containing c.

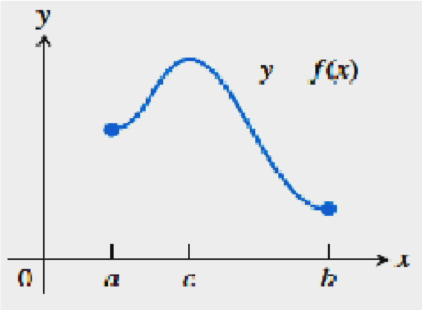


The figure above shows a graph with five extreme points. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point nearby.

The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.
The function f has local maxima at c and d and local minima at a, e and b.
An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, a list of all local maxima will automatically include the absolute maximum if there is one. Similarly, a list of all local minima will include the absolute minimum if there is one.

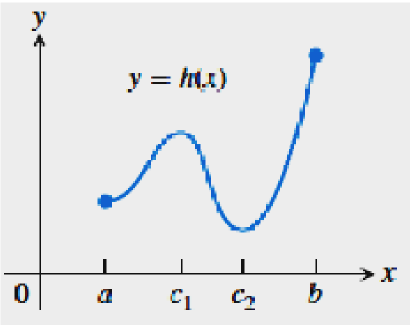
PROBLEM SET

IP1: Determine from the graphs whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Max-Min theorem.



Solution: From the graph, we notice that the function has an absolute minimum at $x = b$ and an absolute maximum at $x = c$.
Max-Min theorem guarantees the existence of such extreme values because f is continuous on $[a, b]$.

P1: Determine from the graphs whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Max-Min theorem.

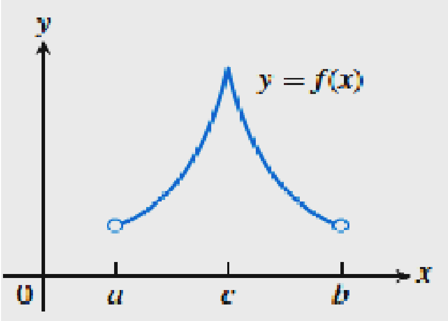


Solution: From the graph, we notice that the function has an absolute minimum at $x = c_2$ and an absolute maximum at $x = b$.
Min-max theorem guarantees the existence of such extreme values because h is continuous on $[a, b]$.

IP2: Determine from the graphs whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Max-Min theorem.

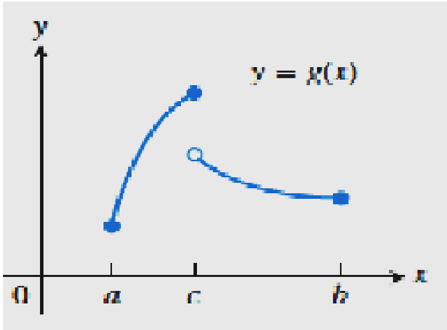
Solution: From the graph, we notice that the function has no absolute extrema.
The function is neither continuous nor defined on a closed interval, so it need not full fill the conclusions of Max-Min theorem.

P2: Determine from the graphs whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Max-Min theorem.



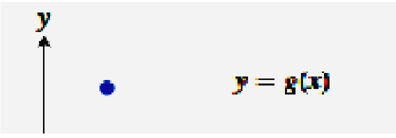
Solution: From the graph, we notice that the function has an absolute maximum at $x = c$. There is no absolute minimum.
Since the function's domain is an open interval, the function does not satisfy the hypotheses of Max-Min theorem and need not have absolute extreme values.

IP3: Determine from the graphs whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Max-Min theorem.



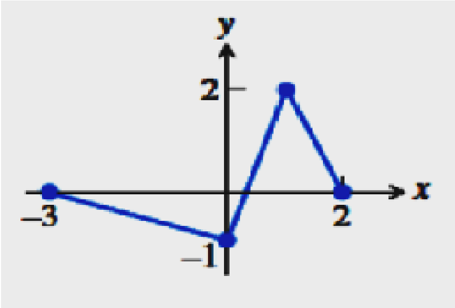
Solution: From the graph, we notice that the function has an absolute minimum at $x = a$ and an absolute maximum at $x = c$.
Note that $y = g(x)$ is not continuous but still has absolute extrema. When the hypothesis of the Max-Min Theorem is satisfied then extrema are guaranteed, but when hypothesis is not satisfied, absolute extrema may or may not occur.

P3: Determine from the graphs whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Max-Min theorem.



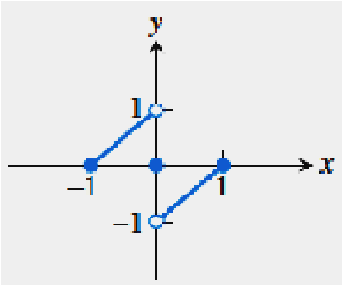
Solution: From the graph, we notice that the function has an absolute maximum at $x = a$ and an absolute minimum at $x = c$.
Note that $y = g(x)$ is not continuous but still has absolute extrema. When the hypotheses of the Max-Min Theorem is satisfied then extrema are guaranteed, but when hypothesis is not satisfied, absolute extrema may or may not occur.

IP4: Find the extreme values and where they occur.



Solution: From the graph, we notice that the function has local maximum at $(-3,0)$, local minimum at $(2,0)$, an absolute maximum at $(1,2)$ and an absolute minimum at $(0,-1)$.

P4: Find the extreme values and where they occur.

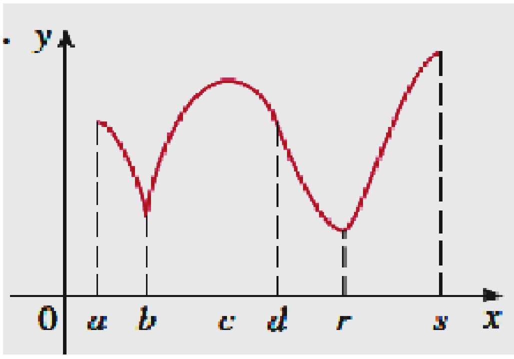


Solution: From the graph, we notice that the function has local minimum at $(-1,0)$ and local maximum at $(1,0)$.

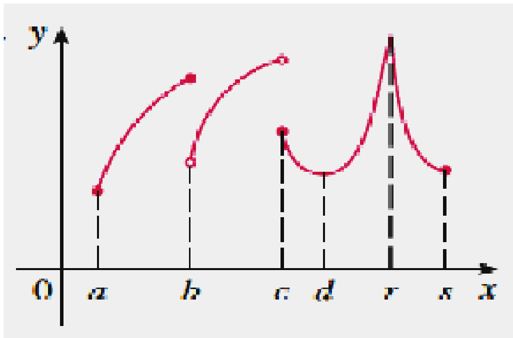
Exercises:

1. For each of the numbers $a, b, c, d, r,$ and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

   a.

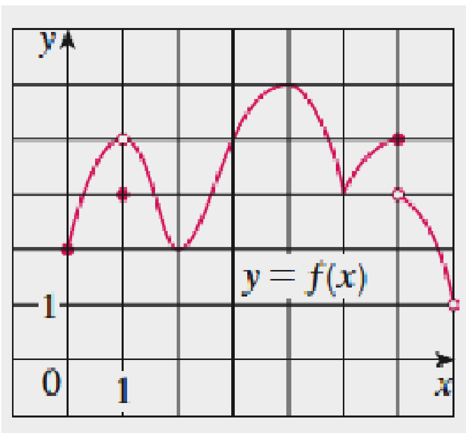


   b.

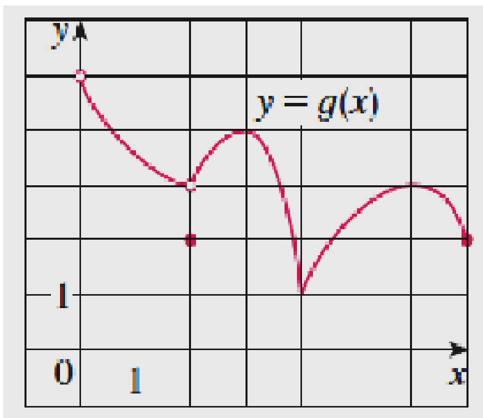


2. Use the graph to state the absolute and local maximum and minimum values of the function.

   a.

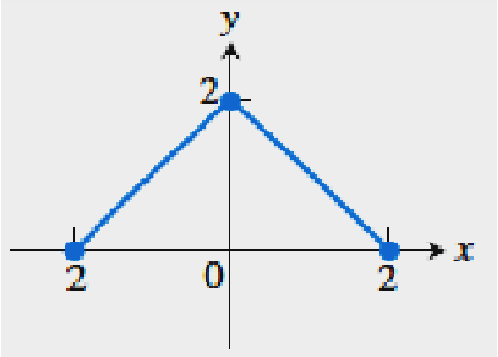


   b.

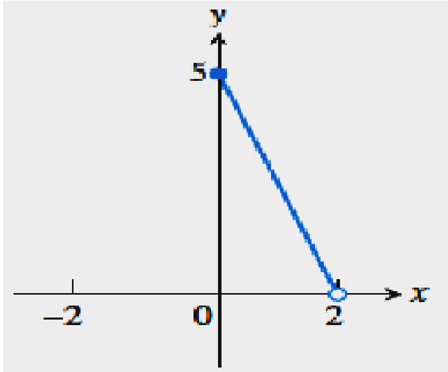


3. Find the extreme values and where they occur.

a.



b.



6.2. Finding Extrema

Learning objectives:

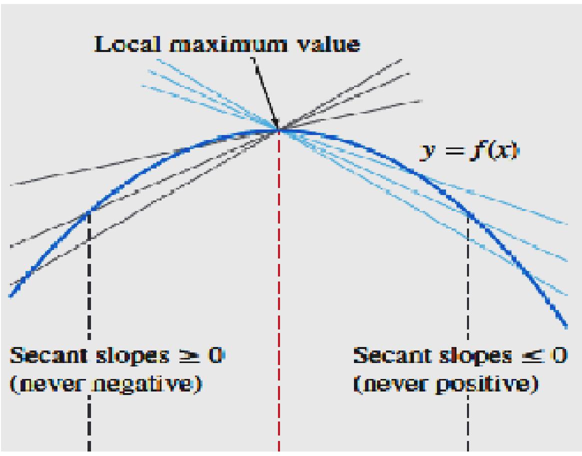
- To prove the First Derivative Theorem for local extrema.
- To define a critical points of a function.
  And
- To practice the related problems.

We usually need to investigate only a few values to find a function's extrema.

The First Derivative Theorem for local extrema

If f has a local maximum or minimum value at an interior point c of its domain, and if $f^{'}$ is defined at c, then $f^{'}(c) = 0$

Proof:



Suppose that f has a local maximum value at $x = c$.

So that $f(x) \leq f(c)$ for all values of x near enough to c.

$\Longrightarrow f(x) - f(c) \leq 0.$

Since c is an interior point of the domain of f say D, $f^{'}(c)$ is defined by the two-sided limit

$$f^{'}(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f^{'}(c)$.

From the right-hand limit, we find that

$f^{'}(c) = \lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0 \Longrightarrow f^{'}(c) \leq 0$ -------(1)

(because $(x-c) > 0$ and $f(x) \leq f(c)$)

Similarly, from the left-hand limit

$f^{'}(c) = \lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0 \Longrightarrow f^{'}(c) \geq 0$ --------(2)

(because $(x-c) < 0$ and $f(x) \leq f(c)$)

From (1) and (2) we get,

$$f^{'}(c) = 0$$

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$ which reverses the inequalities in (1) and (2).

The First Derivative Theorem says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

1. *interior points where $f^{'} = 0$,*
2. *interior points where $f^{'}$ is undefined,*
3. *endpoints of the domain of f.*

Definition:

An interior point of the domain of a function f where $f^{'}$ is zero or undefined is a ***critical point*** of f.

The only domain points where a function can assume extreme values are critical points and endpoints.

Most quests for extreme values call for finding the absolute extrema of a continuous function on a closed interval. The max-Min Theorem assures us that such values exist; The First Derivative Theorem tells us that they are taken on only at critical points and endpoints. We can simply list these points and calculate the corresponding function values to see what the largest and smallest are.

Example 1: Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution:

The function is differentiable over its entire domain, so the only critical point is where

$$f^{'}(x) = 2x = 0, \qquad i.e., x = 0$$

We need to check the function's values at the critical point $x = 0$ and at the endpoints $x = -2$ and $x = 1$.

$f(0) = 0, f(-2) = 4$ and $f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

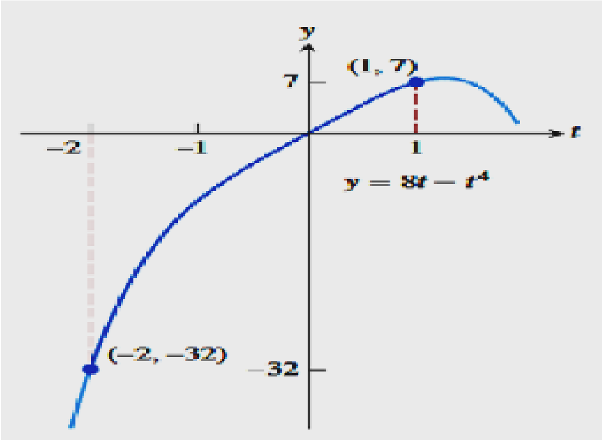
Example 2: Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution:

The function is differentiable on its entire domain, so the only critical points occur where $g^{'}(t) = 0$. Solving this equation gives $8 - 4t^3 = 0 \implies t^3 = 2 \implies t = 2^{1/3}$ a point not in the given domain. The function's local extrema therefore occur at the endpoints, where we find

$g(-2) = -32$ (Absolute minimum)

$g(1) = 7$ (Absolute maximum)



Example 3: Find the absolute extrema of $h(x) = x^{2/3}$ on $[-2, 3]$.

Solution: The first derivative, $h'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

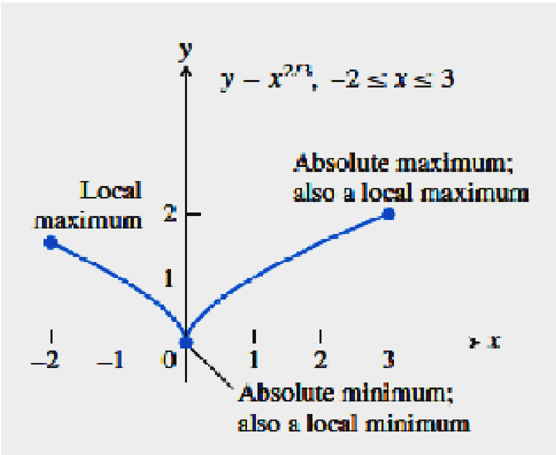
has no zeros but is undefined at $x = 0$.

The values of h at this one critical point and at the endpoints $x = -2$ and $x = 3$ are

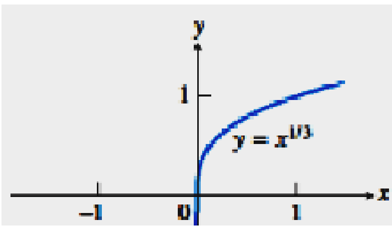
$h(0) = 0, h(-2) = (-2)^{2/3} = 4^{1/3}, h(3) = (3)^{2/3} = 9^{1/3}$

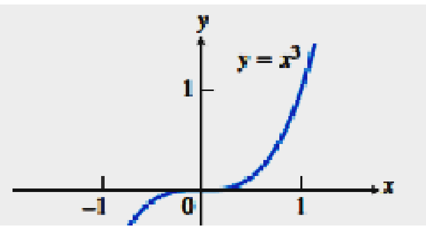
The absolute maximum value is $9^{1/3}$, assumed at $x = 3$;

the absolute minimum is zero, assumed at $x = 0$.



While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figures below illustrate this for interior points.





$f(x) = x^{\frac{1}{3}}$ has no extremum at $x = 0$, even though

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{1/3}}$ is undefined at $x = 0$.

$g(x) = x^3$ has no extremum at $x = 0$ even though $g'(x) = 3x^2$ is zero at $x = 0$.

PROBLEM SET

IP1: Find the critical points of the function $g(x) = \sqrt{x^2 - 2x - 3}$.

Solution: $x^2 - 2x - 3 \geq 0$

Domain is $(-\infty, -1] \cup [3, \infty)$

Given function is $g(x) = \sqrt{x^2 - 2x - 3}$

Derivative of $g(x)$ is $g'(x) = \frac{1}{2\sqrt{x^2-2x-3}}(2x-2) = \frac{x-1}{\sqrt{x^2-2x-3}}$.

We know that the critical points occur when $g'(x) = 0$ and $g'(x)$ is undefined.

Therefore, $g'(x) = 0$ if $x - 1 = 0$ $i.e.,$ $x = 1$

and $g'(x)$ is undefined when $x^2 - 2x - 3 = 0$

$$\Rightarrow (x-3)(x+1) = 0 \quad \Rightarrow x = -1, 3$$

Thus, the critical points are $x = -1, 3$.

P1: Find the critical points of function $f(x) = x^{\frac{3}{5}}(4 - x)$.

Solution: The given function is $\boldsymbol{f(x) = x^{\frac{3}{5}}(4 - x)}$

$$= 4x^{\frac{3}{5}} - x^{\frac{8}{5}}.$$

Derivative of $f(x)$ is $f'(x) = 4\frac{d}{dx}\left(x^{\frac{3}{5}}\right) - \frac{d}{dx}\left(x^{\frac{8}{5}}\right)$

$$f'(x) = 4\left(\frac{3}{5}\right)x^{\frac{3}{5}-1} - \frac{8}{5}x^{\frac{8}{5}-1}$$

$$f'(x) = \frac{12}{5}x^{\frac{-2}{5}} - \frac{8}{5}x^{\frac{3}{5}} = \frac{12}{5x^{\frac{2}{5}}} - \frac{8x^{\frac{3}{5}}}{5}$$

$$= \frac{12-8x^{\frac{3}{5}}x^{\frac{2}{5}}}{5x^{\frac{2}{5}}} = \frac{12-8x^{\frac{3}{5}+\frac{2}{5}}}{5x^{\frac{2}{5}}} = \frac{12-8x}{5x^{\frac{2}{5}}}$$

We know that the critical points occur when $f'(x) = 0$ and $f'(x)$ is undefined.

Therefore, $f'(x) = 0$ if $12-8x = 0$ $i.e.,$ $x = \frac{3}{2}$

and $f'(x)$ is undefined when $x = 0$.

Thus, the critical points are $x = 0$ and $= \frac{3}{2}$.

IP2: Find the absolute maximum and minimum values of $f(x) = x^{\frac{3}{5}}$ on $[-32, 1]$.

Solution: The given function is $f(x) = x^{\frac{3}{5}}$

Now, $f'(x) = \frac{3}{5}x^{\frac{3}{5}-1} = \frac{3}{5}x^{\frac{-2}{5}} = \frac{3}{5x^{\frac{2}{5}}}$

Therefore, $f'(x)$ is undefined at $x = 0$.

$x = 0$ is the only critical point of $f(x)$.

The values of f at this critical point $x = 0$ and at the endpoints $x = -32$ and $x = 1$ are $f(0) = 0, f(-32) = -8$ and $f(1) = 1$

∴The function has the absolute maximum value of 1 assumed at $x = 1$ and the absolute minimum value of -8 assumed at $x = -32$.

P2: Find the absolute maximum and minimum values of $f(x) = x^{\frac{4}{3}}$ on $[-1, 8]$.

Solution: The given function is $f(x) = x^{\frac{4}{3}}$.

Now, $f'(x) = \frac{4}{3}x^{\frac{4}{3}-1} = \frac{4}{3}x^{\frac{1}{3}} = \frac{4x^{\frac{1}{3}}}{3}$

Therefore, $f'(x) = 0$ at $x = 0$.
$x = 0$ is the only critical point of $f(x)$.
The values of f at this critical point and at the endpoints $x = -1$ and $x = 8$ are
$f(0) = 0, f(-1) = 1$ and $f(8) = 16$
∴The function has the absolute maximum value of 16 assumed at $x = 8$ and the absolute minimum value of 0 assumed at $x = 0$.

IP3: Find the absolute maximum and minimum values of the function $f(x) = \frac{-1}{x^2}$ on $[0.5, 2]$.

Solution: $f(x) = \frac{-1}{x^2} = -x^{-2} \to f'(x) = -(-2)x^{-2-1} = 2x^{-3}$
The given function is differentiable on its domain entire domain.
The derivative of $f(x)$ is $f'(x) = \frac{2}{x^3}$.
$f'(x)$ is undefined at $x = 0$ but $x = 0$ is not a critical point since 0 is not in domain.
There are no critical points.
We need to check the function values at the end points $x = 0.5$ and $x = 2$.

$$f(0.5) = -4 \text{ and } f(2) = -0.25$$

∴ The function has an absolute maximum value of -0.25 at $x = 2$ and an absolute minimum value of -4 at $x = 0.5$.

P3: Find the absolute maximum and minimum values of the function $f(x) = \frac{2}{3}x - 5$ on $[-2, 3]$.

Solution: The given function is differentiable on its domain entire domain.
The derivative of $f(x)$ is $f'(x) = \frac{2}{3}$.
There are no critical points.
We need to check the function values at the end points $x = -2$ and $x = 3$.

$$f(-2) = \frac{-19}{3} \text{ and } f(3) = -3$$

∴ The function has an absolute maximum value of -3 at $x = 3$ and an absolute minimum value of $\frac{-19}{3}$ at $x = -2$.

IP4: Find the absolute maximum and minimum values of $f(x) = \sec x$ in $\left[\frac{-\pi}{3}, \frac{\pi}{6}\right]$.

Solution: The given function is differentiable on its domain.
So the only critical points occur where $f'(x) = 0$.
$i.e.,\ f'(x) = \sec x \tan x = 0$
∴ $x = 0$ is the only critical point of $f(x)$.
We need to check the function values at the critical point $x = 0$ and at the end points $x = \frac{-\pi}{3}, x = \frac{\pi}{6}$.
$f(0) = \sec 0 = 1, f\left(\frac{-\pi}{3}\right) = \sec\left(-\frac{\pi}{3}\right) = 2$, and
$f\left(\frac{\pi}{6}\right) = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$.
Therefore, the function has the absolute maximum value of 2 at $x = -\frac{\pi}{3}$ and absolute minimum value of 1 at $x = 0$.

P4: Find the absolute maximum and minimum values of $f(\theta) = \sin\theta$ in $\left[\frac{-\pi}{2}, \frac{5\pi}{6}\right]$.

Solution: The given function is differentiable on its domain.

So the only critical points occur where $f'(\theta) = 0$.

$i.e.,\ f'(\theta) = \cos\theta = 0$

The solution of $\cos\theta = 0$ in $\left[\frac{-\pi}{2}, \frac{5\pi}{6}\right]$ is $\theta = \frac{\pi}{2}$.

$\therefore \theta = \frac{\pi}{2}$ is the only critical point of $f(\theta)$.

We need to check the function values at the critical point $\theta = \frac{\pi}{2}$ and at the end points $\theta = \frac{-\pi}{2}, \theta = \frac{5\pi}{6}$.

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$, $f\left(\frac{-\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$, and

$f\left(\frac{5\pi}{6}\right) = \sin\frac{5\pi}{6} = \frac{1}{2}$.

Therefore, the function has the absolute maximum value of 1 at $\theta = \frac{\pi}{2}$ and absolute minimum value of -1 at $\theta = -\frac{\pi}{2}$.

Exercises:

1. **Find the absolute maximum and the absolute minimum values of the functions in the following intervals**
    a. $f(x) = x^2 - 1 \qquad -1 \leq x \leq 2$
    b. $f(x) = 4 - x^2 \qquad -3 \leq x \leq 1$
    c. $f(x) = -\frac{1}{x} \qquad -2 \leq x \leq -1$
    d. $h(x) = \sqrt[3]{x} \qquad -1 \leq x \leq 8$
    e. $g(x) = \sqrt{4 - x^2} \qquad -2 \leq x \leq -1$
    f. $f(x) = -\sqrt{5 - x^2} \qquad -\sqrt{5} \leq x \leq 0$
    g. $f(\theta) = \tan x \qquad -\frac{\pi}{3} \leq x \leq \frac{\pi}{4}$
    h. $g(x) = \csc x \qquad \frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$
    i. $f(t) = 2 - |t| \qquad -1 \leq t \leq 3$
2. **Find the function's absolute maximum and minimum values and say where they are assumed**.
    a) $f(x) = x^{5/3} \qquad -1 \leq x \leq 8$
    b) $g(\theta) = 3\theta^{2/3} \qquad -27 \leq \theta \leq 8$

3. **Find the values of any local maxima and minima the functions may have on the given domains, and say where they are assumed. Which extrema, if any, are absolute for the given domain?**

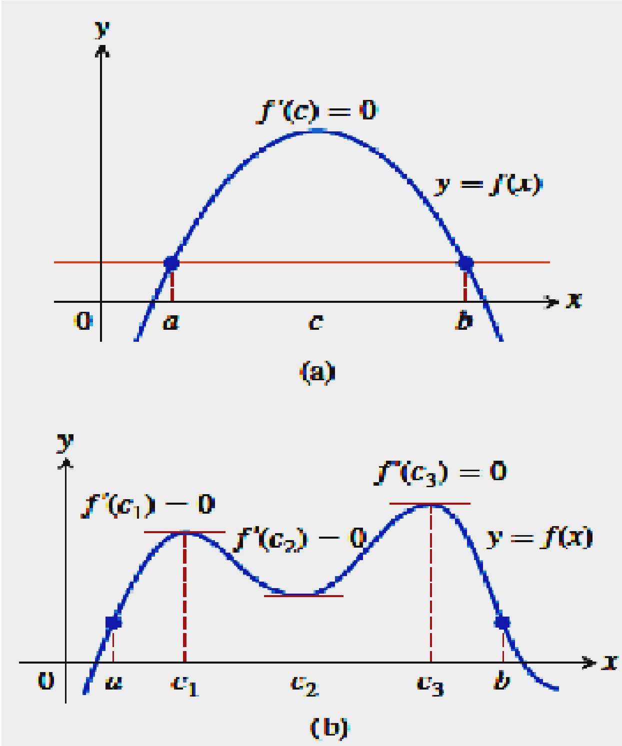
    a. $f(x) = x^2 - 4 \qquad -2 \leq x \leq 2$
    b. $g(x) = x^2 - 4 \qquad -2 \leq x < 2$
    c. $h(x) = x^2 - 4 \qquad -2 < x < 2$
    d. $k(x) = x^2 - 4 \qquad -2 < x < \infty$
    e. $l(x) = x^2 - 4 \qquad 0 < x < \infty$

6.3. Rolle's Theorem

Learning objectives:

- To state and prove Rolle's Theorem.
  And
- To practice the related problems.

We may visualize geometrically that between any two points where a differentiable curve crosses a horizontal line there is at least one point on the curve where the tangent is horizontal. Rolle's Theorem asserts that this is indeed the case.



Rolle's Theorem:

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$ then there is at least one number c in (a, b) at which $f'(c) = 0$.

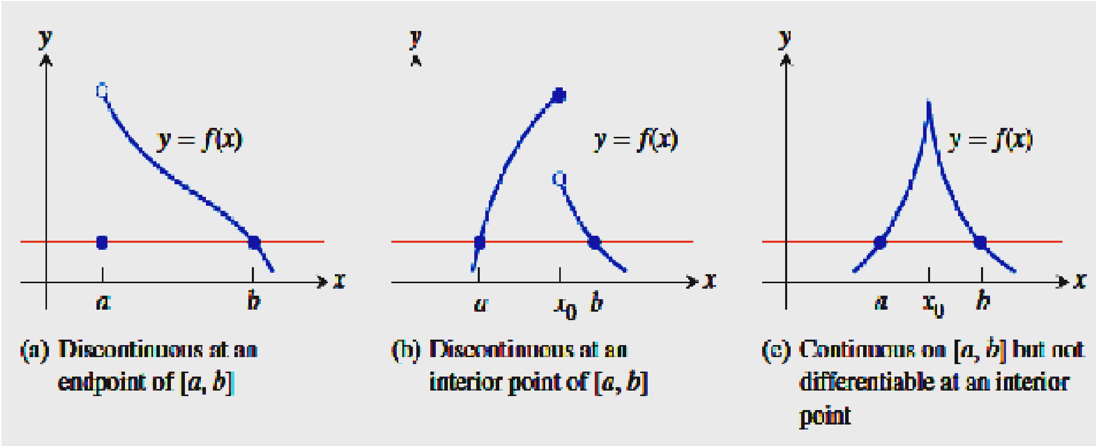
Proof:

Being continuous on $[a, b]$, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at the endpoints of the function's domain, a and b.

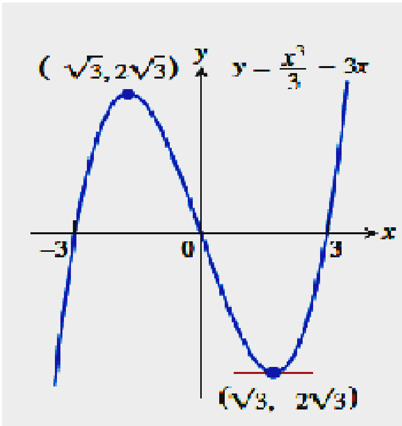
We rule out (2) since f has a derivative at every interior point.
If either the maximum or the minimum occurs at a point c inside the interval, then $f'(c) = 0$ by the First Derivative Theorem and we have a point for Rolle's Theorem.
Suppose that the absolute maximum and absolute minimum occur at the end points.
Since $f(a) = f(b)$, the absolute maximum is equal to absolute minimum. This forces f to be a constant function with $f(x) = f(a) = f(b)$ for all $x \in [a, b]$.
Therefore $f'(x) = 0$ for all $x \in [a, b]$ and the point c can be taken any where in the interior (a, b).

Hence the theorem.
The hypotheses of Rolle's Theorem are essential. If they fail at even one point, the graph may not have a horizontal tangent.



Example 1:

The polynomial function $f(x) = \frac{x^3}{3} - 3x$ graphed below is continuous at every point of $[-3,3]$ and is differentiable at every point of $(-3,3)$.



Since $f(-3) = f(3) = 0$, Rolle's Theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$. In fact $f'(x) = x^2 - 3$ is zero twice in this interval, once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$.

PROBLEM SET

IP1: Discuss the applicability of the Rolle's Theorem on the function

$$\boldsymbol{f(x) = \begin{cases} x^2 + 1, & if\ \ 0 \leq x < 1 \\ 3 - x, & if\ \ 1 < x \leq 2 \end{cases}}$$

Solution:

The given function $f(x)$ is continuous on $[0,2]$ except possibly at $x = 1$. Notice that

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - x) = 3 - 1 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2$$

Therefore, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$, $\lim_{x \to 1} f(x) = 2$ and $x = 1$ is a removable discontinuity.

Now, define $f(1) = \lim_{x \to 1} f(x) = 2$. Therefore $f(x)$ is continuous on $[0,2]$. Further $f(0) = f(2) = 1$ and $f(x)$ is differentiable on $(0,2)$ except possibly at $x = 1$.

Now, left hand derivative at $x = 1$ is

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=2$$
and right hand derivative at $x=1$ is
$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=-1$$
Thus, $f(x)$ is not differentiable at $x=1$ and the condition of derivability on $(0,2)$ is not satisfied.
Therefore, Rolle's Theorem is not applicable for $f(x)$ on $[0,2]$.

P1: Discuss the applicability of Rolle's Theorem for the function $f(x)=|x|$ on the interval $[-1,1]$.

Solution: We have $f(x)=|x|, x\,\epsilon\,[-1,1]$ and $f(1)=f(-1)=1$.
It is known that $f(x)$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$ except at $x=0$.
Thus the condition of derivability on $(-1,1)$ is not satisfied. Therefore, Rolle's Theorem is not applicable for $f(x)=|x|$ on $[-1,1]$.

IP2: Verify Rolle's Theorem for the function $f(x)=x^3-6x^2+9x$ on the interval $[0,3]$.

Solution: Since $f(x)$ is a polynomial, it is continuous on $[0,3]$ and differentiable on $(0,3)$.
Further $f(0)=f(3)=0$.
Thus all the conditions of Rolle's Theorem are satisfied.
By Rolle's Theorem there is a number $c\in(0,3)$ such that $f'(c)=0$
$$\Rightarrow\ 3c^2-12c-9=0\ \Rightarrow c=1,3\ .$$
Therefore, there exist a number $c=1\in(0,3)$ such that $f'(c)=0$
Thus, Rolle's Theorem is verified.

P2: Verify Rolle's Theorem for the function $f(x)=x^2-5x+6$ on the interval $[2,3]$.

Solution: Since $f(x)$ is a polynomial, it is continuous on $[2,3]$ and differentiable on $(2,3)$.
Further $f(2)=f(3)=0$.
Thus all the conditions of Rolle's Theorem are satisfied.
By Rolle's Theorem there is a number $c\in(2,3)$ such that $f'(c)=0$
$$\Rightarrow\ 2c-5=0\ \Rightarrow c=\frac{5}{2}.$$
Therefore, there exist a number $c=\frac{5}{2}\in(2,3)$ such that $f'(c)=0$
Thus, Rolle's Theorem is verified.

IP3: Show that the function $f(x)=x^4+3x+1$ has exactly one zero in $[-2,-1]$.

Solution: Since $f(x)$ is a polynomial, it is continuous and differentiable on $[-2,-1]$.
Now, $f(-2)=11>0$ and $f(-1)=-1<0$.
By intermediate value theorem $f(x)$ has at least one zero in $(-2,-1)$, say α.
Required to show that $f(x)$ has only one zero in $[-2,-1]$.
Notice that $f'(x)=4x^3+3<0$ for all $x\,\epsilon\,(-2,-1)$.
Assume that $f(x)$ has a root $\beta\neq\alpha$ in $(-2,-1)$,
with out loss of generality we may assume $\alpha<\beta$.
Then $[\alpha,\beta]\subset[-2,-1]$. Therefore $f(x)$ is continuous on $[\alpha,\beta]$, derivable on (α,β) and $f(\alpha)=f(\beta)=0$.
Thus $f(x)$ satisfies the condition of Rolle's Theorem.
By Rolle's Theorem $\exists\,\gamma\,\epsilon\,(\alpha,\beta)$ such that $f'(\gamma)=0$.
This contradicts the fact that $f'(x)<0,\forall\,x\,\epsilon\,(-2,-1)$.
Thus our assumption is wrong. Therefore $f(x)$ has only one zero in $[-2,-1]$.

P3: Prove that the equation $x^5 + x^4 - 1 = 0$ has exactly one root in $[0,\ 1]$.
Solution: let $f(x) = x^5 + x^4 - 1$
Since $f(x)$ is a polynomial, it is continuous and differentiable on $[0,1]$. Now, $f(0) = -1 < 0$ and $f(1) = 1 > 0$.
By intermediate value theorem $f(x)$ has at least one zero in $(0,1)$, say α. Required to show that $f(x)$ has only one zero in $[0,1]$.
Notice that $f^{'}(x) = 5x^4 + 4x^3 > 0$ for all $x \in (0,1)$.
Assume that $f(x)$ has a root $\beta \neq \alpha$ in $(0,1)$.
Without loss of generality we may assume $\alpha < \beta$.
Then $[\alpha, \beta] \subset [0,1]$. Therefore $f(x)$ is continuous on $[\alpha, \beta]$, derivable on (α, β) and $f(\alpha) = f(\beta) = 0$.
Thus $f(x)$ satisfies the condition of Rolle's Theorem.
By Rolle's Theorem $\exists\ \gamma \in (\alpha, \beta)$ such that $f^{'}(\gamma) = 0$.
This contradicts the fact that $f^{'}(x) > 0, \forall\ x \in (0,1)$.
Thus our assumption is wrong. Therefore $f(x)$ has only one zero in $[0,1]$.
IP4: It is given that Rolle's Theorem holds for the function $f(x) = x^3 + bx^2 + ax$ on $[1,3]$ with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b.
Solution: It is given that the Rolle's Theorem holds for $f(x)$ on $[1,3]$ with $c = 2 + \frac{1}{\sqrt{3}}$.
$\therefore \quad f(1) = f(3)$ and $f^{'}(c) = 0$
First $f(1) = f(3)$
$\Longrightarrow\ 1 + b + a = 27 + 9b + 3a$
$\Longrightarrow\ 2a + 8b + 26 = 0 \Longrightarrow\ a + 4b + 13 = 0$ ….(1)
Next $f^{'}(c) = 0$
and $3c^2 + 2bc + a = 0$
and $3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$

$$3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) + 4b + \frac{2b}{\sqrt{3}} + a = 0$$

$$12 + 1 + \frac{12}{\sqrt{3}} + 4b + \frac{2b}{\sqrt{3}} + a = 0$$

$$a + 4b + 13 + \frac{2b}{\sqrt{3}} + \frac{12}{\sqrt{3}} = 0$$

$$(a + 4b + 13) + \frac{2}{\sqrt{3}}(b + 6) = 0$$

$\Longrightarrow\ (a + 4b + 13) + \frac{2}{\sqrt{3}}(b + 6) = 0$
$\Longrightarrow\ 0 + \frac{2}{\sqrt{3}}(b + 6) = 0$
$\Longrightarrow\ \frac{2}{\sqrt{3}}(b + 6) = 0 \Longrightarrow b = -6$
Now from (1)
$\Longrightarrow\ a = 11$ and $b = -6$
P4: Verify Rolle's Theorem for the function $f(x) = (x - a)^m (x - b)^n$ on the interval $[a, b]$, where m, n are positive integers.
Solution: We have, $f(x) = (x - a)^m (x - b)^n$ where $m, n \in \boldsymbol{N}$
Notice that $f(x)$ is a polynomial of degree $m + n$.
Therefore, $f(x)$ is continuous on $[a, b]$ and $f(x)$ is derivable on (a, b).

Further, $f(a) = f(b) = 0$.
Thus the conditions of Rolle's Theorem are satisfied.
Now, we have to show that there exists $c \in (a, b)$ such that $f'(c) = 0$.
We have,

$$f'(x) = \frac{d}{dx}(x-a)^m \cdot (x-b)^n + (x-a)^m \cdot \frac{d}{dx}(x-b)^n$$

$$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$$

$$f'(x) = m(x-a)^{m-1}(x-b)(x-b)^{n-1} + n(x-a)(x-a)^{m-1}(x-b)^{n-1}$$

$$f'(x) = m(x-b)(x-a)^{m-1}(x-b)^{n-1} + n(x-a)(x-a)^{m-1}(x-b)^{n-1}$$

$\Longrightarrow \quad f'(x) = (x-a)^{m-1}(x-b)^{n-1}\{m(x-b) + n(x-a)\}$

$\Longrightarrow \quad f'(x) = (x-a)^{m-1}(x-b)^{n-1}\{x(m+n) - mb - na\}$

$\therefore \quad f'(c) = 0$

$\Longrightarrow \quad (c-a)^{m-1}(c-b)^{n-1}\{c(m+n) - mb - na\} = 0$

$\Longrightarrow (c-a) = 0$ or $(c-b) = 0$ or $c(m+n) - (mb+na) = 0$

$\Longrightarrow \quad c = \frac{mb+na}{m+n} \qquad (\because c \in (a,b))$

Since $c = \frac{mb+na}{m+n}$ divides (a, b) internally in the ratio $m{:}n$, $c \in (a, b)$.
Thus, there exists $c = \frac{mb+na}{m+n} \in (a, b)$ such that $f'(c) = 0$.
Hence, Rolle's Theorem is verified.

Exercises:

1. State and prove Rolle's Theorem.
2. Verify the Rolle's Theorem for the function on the indicated intervals.
   a. $f(x) = \begin{cases} -4x + 5 & 0 \le x \le 1 \\ 2x - 3 & 1 \le x \le 2 \end{cases}$
   b. $f(x) = (x^2 - 1)(x - 2)$ on $[-1, 2]$
   c. $f(x) = \sin^2 x$ on $[0, \pi]$
   d. $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$

3. Show that the function have exactly one zero in the given interval.
   a. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$
   b. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad \left[0, \frac{\pi}{2}\right]$

6.4. The Mean Value Theorem

Learning objectives:

- To prove the Mean Value Theorem and to discuss two of its important consequences. And
- To practice the related problems.

The main use of Rolle's Theorem is to prove the Mean Value Theorem.
The Mean value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem.

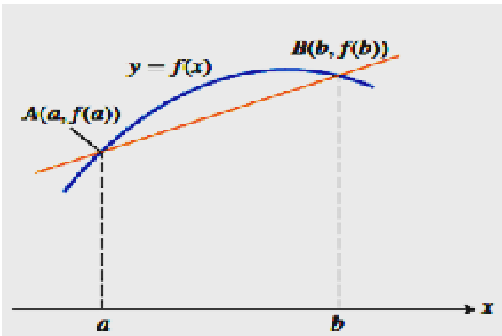
The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ --------(1)}$$

Proof:

We draw the graph of $y = f(x)$ as a curve in the plane and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$.



The line is the graph of the function $g(x)$, where

$$g(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x - a) \qquad \text{(Point-slope equation)}$$

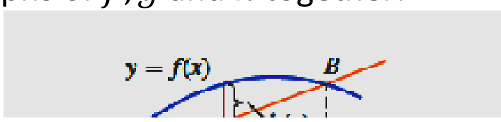
$$i.e., \quad g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a) \quad \text{--------(2)}$$

The vertical difference between the graphs of f and g at x is

$$h(x) = f(x) - g(x)$$

$$= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a) \qquad \text{----(3)}$$

The figure below shows the graphs of f, g and h together.



The function h is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Thus,

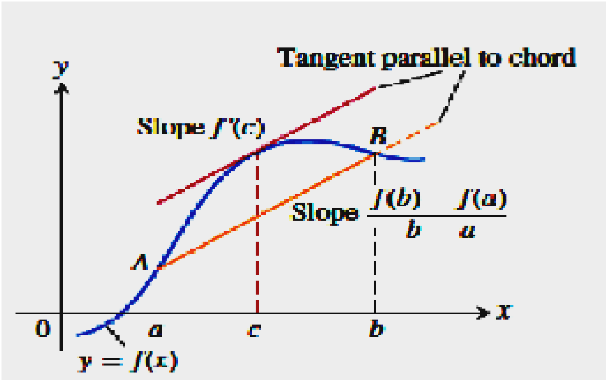
the function h satisfies the hypotheses of Rolle's theorem on $[a, b]$.Therefore, $h'(c) = 0$ for some c in (a, b). This is the point we want for equation (1).
We differentiate both sides of equation (3) with respect to x and then set $x = c$.

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$
$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$
$$0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

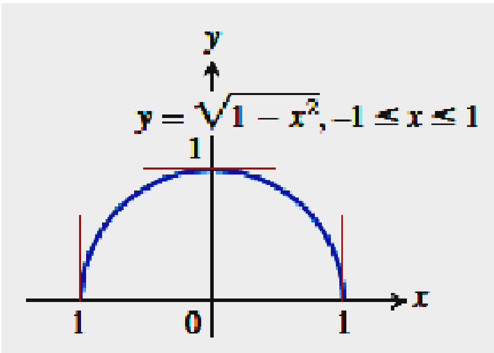
Rearranging,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

This proves the theorem.
Geometrically, the Mean Value Theorem says that some where between A and B the curve has at least one tangent parallel to the chord AB.



The hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. Continuity at a and b is enough.



In the graph above, the function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

Example 1:

The function $f(x) = x^2$ is continuous for $0 \leq x \leq 2$ and differentiable for $0 < x < 2$.

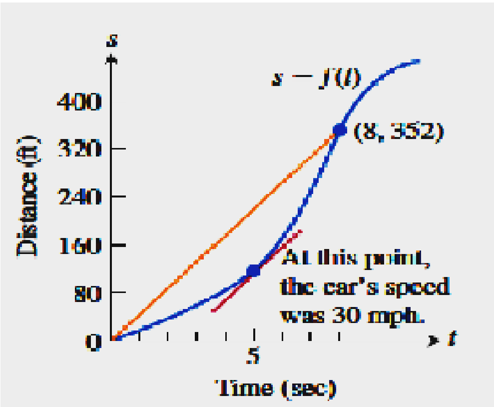
Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. We can identify c by solving the equation $2c = 2$ to get $c = 1$.

In general, we are not able to identify the location of c; the above example is an exceptional case. The theorem simply tells that c exists, and does not give an indication where it is located.

If we think of the number $\big(f(b) - f(a)\big)/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Example 2:

If a car accelerating from zero takes 8 seconds to go $352\ ft$, its average velocity for the $8\ sec$ interval is $352/8 = 44\ ft/s$ or 30 mph.



At some point during the acceleration, the Mean Value Theorem says, the speedometer must read exactly 30 mph.

We know that if a function f has a constant value on an interval I, then f is differentiable on I and $f'(x) = 0$ for all x in I. The converse of this is corollary 1 of the Mean Value Theorem.

Corollary 1:

Functions with zero Derivatives are Constant

If $f'(x) = 0$ at each point of an interval I, then $f(x) = c$ for all x in I, where c is a constant.

Proof: Suppose that $x_1$and x_2 are two points in I, numbered from left to right so that $x_1 < x_2$.Then f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$, and hence continuous at every point as well. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point c between x_1 and x_2. Since $f' = 0$ throughout I, this equation translates into

$$f(x_2) - f(x_1) = 0 \quad \Rightarrow \quad f(x_2) = f(x_1)$$

Since $x_1$and x_2 are any two points in $I$and $f(x_2) = f(x_1)$, it follows that f has a constant value in I. This completes the proof of the corollary.

Corollary 2:

Functions with the Same Derivative Differ by a Constant:

If $f'(x) = g'(x)$ at each point of an interval I, then there exists a constant C such that $f(x) = g(x) + C$ for all x in I.

Proof:

At each point x in I the derivative of the difference function $h = f - g$ is

$h'(x) = f'(x) - g'(x)$

$h'(x) = g'(x) - g'(x) = 0$

$h'(x) = 0$

Thus, $h(x) = C$ on I (Corollary 1).

That is, $f(x) - g(x) = C$ on I,

so $f(x) = g(x) + C$. Hence the result.

Corollary 2 says that the functions with the same derivative differ by a constant.

It says that functions can have identical derivatives on an interval only if their values on the interval have a constant difference. We know, for instance, that the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$. Any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C.

Example 3:

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0,2)$.

Solution: Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that

$$f(x) = -\cos x + C$$

for some constant C. The value of C can be determined from the condition that

$$f(0) = -\cos(0) + C = 2 \quad \Longrightarrow C = 3$$

The function is $f(x) = -\cos x + 3$.

PROBLEM SET

IP1: Find the value of c that satisfy the equation $\frac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of the mean value theorem for the function $f(x) = x + \frac{1}{x}$ in the interval $\left[\frac{1}{2}, 2\right]$.

Solution: The given function is: $f(x) = x + \frac{1}{x}$

Notice that $f(x)$ is continuous in $\left[\frac{1}{2}, 2\right]$ and derivable in $\left(\frac{1}{2}, 2\right)$. By Lagrange Mean Value Theorem there exists a $c \in \left(\frac{1}{2}, 2\right)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

We have $f'(x) = 1 - \frac{1}{x^2}$. Therefore,

$$1 - \frac{1}{c^2} = \frac{f(2)-f(\frac{1}{2})}{2-\frac{1}{2}} = \frac{\left(2+\frac{1}{2}\right)-\left(\frac{1}{2}+2\right)}{3/2} = 0 \implies c = \pm 1$$

$$\therefore c = 1 \in \left(\frac{1}{2}, 2\right)$$

Notice that $c = -1 \notin \left(\frac{1}{2}, 2\right)$.

P1: Find the value of c that satisfy the equation $\frac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of the mean value theorem for the function $f(x) = x^{2/3}$ in the interval $[0,1]$.

Solution: The given function is: $f(x) = x^{2/3}$

Notice that $f(x)$ is continuous in $[0,1]$ and derivable in $(0,1)$. By Lagrange Mean Value Theorem there exists a $c \in (0,1)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

We have $f'(x) = \frac{2}{3x^{1/3}}$. Therefore,

$$\frac{2}{3c^{1/3}} = \frac{f(1)-f(0)}{1-0} = \frac{1-0}{1} = 1$$

$$\Rightarrow \quad c = \frac{8}{27} \in (0,1).$$

IP2: Verify the mean value theorem for the function $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$ on the interval $[-\pi, 0]$.

Solution: The given function is $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$.

First we check the continuity of f on $[-\pi, 0]$.

Continuity of $f(x)$ at $x = 0$,

Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$ and $f(0) = 0$

$\therefore \lim_{x\to 0^-} f(x) \ne f(0)$

Therefore, $f(x)$ is discontinuous at $x = 0$.

So, $f(x)$ is discontinuous on $[-\pi, 0]$.

Therefore, $f(x)$ does not satisfy the hypotheses of the mean value theorem.

P2: For what values of a, m and b does the function

$$f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Solution: Given that $f(x)$ satisfy the hypotheses of the Mean Value Theorem on the interval $[0,2]$.

$\therefore f(x)$ must be continuous on $[0,2]$ and differentiable on $(0,2)$.

Since $f(x)$ is continuous at $x = 0$

$$\therefore \quad \lim_{x\to 0^+} f(x) = f(0)$$

$$\lim_{x\to 0}(-x^2 + 3x + a) = 3 \quad \Rightarrow \quad a = 3$$

Since $f(x)$ is continuous at $x = 1$

$$\therefore \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad \lim_{x\to 1}(-x^2 + 3x + a) = \lim_{x\to 1}(mx + b) = m + b$$

$$\Rightarrow \quad a + 2 = m + b \quad \Rightarrow \quad m + b = 5\text{---------}(i)$$

Since $f(x)$ must be differentiable at $x = 1$.

$$\therefore \quad \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \quad \lim_{x\to 1} \frac{(-x^2+3x+a)-(2+a)}{x-1} = \lim_{x\to 1} \frac{(mx+b)-(m+b)}{x-1}$$

$\Rightarrow \quad \lim_{x\to1}\frac{-x^2+3x-2}{x-1}=\lim_{x\to1}\frac{m(x-1)}{x-1}$

$\Rightarrow \quad \lim_{x\to1}\frac{-(x-2)(x-1)}{x-1}=\lim_{x\to1} m \quad \Rightarrow \quad m=1$

From (i) $\quad 1+b=5 \quad \Rightarrow \quad b=4$

Therefore, $a=3, m=1$ and $b=4$.

IP3: Verify the function $f(x)=x-2\sin x$ satisfies the hypotheses of the mean value theorem on the interval $[-\pi,\pi]$. Then find all numbers c that satisfy the conclusions of the mean value theorem.

Solution: Since x and $\sin x$ are everywhere continuous and differentiable, $f(x)$ is continuous on $[-\pi,\pi]$ and differentiable on $(-\pi,\pi)$.

Thus, both the conditions of Mean Value Theorem are satisfied.

So, by Mean Value Theorem there exists at least one real number $c\,\epsilon\,(-\pi,\pi)$ such that

$$f'(c)=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}$$

We have, $f(x)=x-2\sin x$

$\Rightarrow\ f'(x)=1-2\cos x$, $f(b)=\ f(\pi)=\pi$ and

$$f(a)=f(-\pi)=-\pi$$

$$\therefore\ f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}$$

$\Rightarrow 1-2\cos c=\frac{\pi-(-\pi)}{\pi-(-\pi)} \Rightarrow 1-2\cos c=1$

$\Rightarrow -2\cos c=0$

$\Rightarrow \cos c=0 \quad \Rightarrow c=\pm\frac{\pi}{2}$

Now, there exists $c=\pm\frac{\pi}{2}\,\epsilon\,(-\pi,\pi)$ such that $\quad f'(c)=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}$.

Thus, Mean Value Theorem is verified.

P3: Verify the function $f(x)=3x^2+2x+5$ satisfies the hypotheses of the mean value theorem on the interval $[-1,1]$. Then find all numbers c that satisfy the conclusions of the mean value theorem.

Solution: Given function is $f(x)=3x^2+2x+5$

Since $f(x)$ is a polynomial function; it is every where continuous and differentiable.

Therefore, $f(x)$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$.

Thus both conditions of mean value theorem are satisfied. So, by Mean Value Theorem there exists at least one real number $c\,\epsilon\,(-1,1)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{f(1)-f(-1)}{1-(-1)}$$

where $f'(x)=6x+2$, $f(-1)=6$ and $f(1)=10$

$$\therefore 6c+2=\frac{10-6}{2} \Rightarrow 6c+2=\frac{4}{2}\Rightarrow 6c+2=2$$

$$\Rightarrow 6c=2-2\Rightarrow 6c=0 \Rightarrow c=0$$

Since $c=0\,\epsilon\,(-1,1)$ such that $f'(c)=\frac{f(1)-f(-1)}{1-(-1)}$.

Thus, Mean Value Theorem is verified.

IP4: Find the function whose derivative is $8-csc^2x$ and whose graph passing through the point $\left(\frac{\pi}{4},0\right)$.

Solution: Since $f(x)$ has the same derivative as $g(x)=8x+\cot x+;$ $f(x)$ and $g(x)$ differ by a constant. Therefore,

$f(x) = 8x + \cot x + C$ for some constant C.

The value C can be determined from the condition that $f\left(\frac{\pi}{4}\right) = 0$.

$\therefore\ 8\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right) + C = 0 \implies 2\pi + 1 + C = 0$

$\implies C = -2\pi - 1$

The required function is $f(x) = 8x + \cot x - 2\pi - 1$.

P4: Find the function $f(x)$ whose derivative is $2x - 1$ and whose graph passes through the point $(0, 0)$.

Solution: Since $f(x)$ has the same derivative as $g(x) = x^2 - x$; $f(x)$ and $g(x)$ differ by a constant. Therefore,

$f(x) = x^2 - x + C$ for some constant C.

The value C can be determined from the condition that $f(0) = 0$.

$\therefore f(0) = 0 - 0 + C \implies C = 0$

The required function is $f(x) = x^2 - x$.

Exercises:

- State and prove the Mean Value Theorem.
- Prove that functions with zero derivatives are constant.
- Prove that functions with the same derivative differ by a constant.

1. Find the value or values of c that satisfy the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$ in the conclusion of the Mean Value Theorem for the functions and intervals in the problems below.
   a. $f(x) = x^2 + x - 1$ $[0,1]$
   b. $f(x) = \sqrt{x-1}$ $[1,3]$
   c. $f(x) = (x-1)(x-2)(x-3)$ $[0,4]$
   d. $f(x) = \sqrt{x^2 - 4}$ $[2,3]$

2. Which of the functions in the problems below satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.
   a. $f(x) = x^{2/3}$ $[-1,8]$
   b. $f(x) = x^{4/5}$ $[0,1]$
   c. $f(x) = \sqrt{x(1-x)}$ $[0,1]$
   d. $f(x) = \frac{x}{x+2}$ $[1,4]$
   e. $f(x) = \sqrt{x-2}$ $[2,3]$
   f. $f(x) = 2\sin x + \sin 2x$ $[0,\pi]$

3. Find the function with the given derivative whose graph is passes through the point P.
   a. $g'(x) = \frac{1}{x^2} + 2x$, $P(-1,1)$
   b. $r'(t) = \sec t \tan t - 1$, $P(0,0)$
4. If the velocity $v = \frac{ds}{dt}$ and initial position of a body moving along a coordinate line is given then find the body's position at any time t.
   a. $v = 9.8t + 5$, $s(0) = 10$

b. $v = 32t - 2, \quad s(0.5) = 4$
c. $v = \sin \pi t, \quad s(0) = 0$
d. $v = \frac{2}{\pi} \cos \frac{2t}{\pi}, \quad s(\pi^2) = 1$

6.5. Increasing and Decreasing Functions

Learning objectives:

- To define a monotonic function.
- To state and prove First Derivative Test for monotonic functions.
  And
- To practice the related problems.

We inquire into what kinds of functions have positive derivatives or negative derivatives. The answer is provided by the corollary 3 (given below) of the Mean Value Theorem: The only functions with positive derivatives are increasing functions; the only functions with negative derivatives are decreasing functions.

Definitions

Let f be a function defined on an interval I.
(i) f is said to be ***increasing*** on I if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$ such that $x_1 < x_2$.
(ii) f is said to be ***decreasing*** on I if $f(x_2) < f(x_1)$ for all $x_1, x_2 \in I$ such that $x_1 < x_2$.
A function is said to be monotonic on I if it is either increasing or decreasing on I.

Corollary 3:

The First derivative Test for monotonic functions:

Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).
If $f'(x) > 0$ at each point $x \in (a,b)$, then f is increasing on $[a,b]$.
If $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$.

Proof:

Let x_1 and x_2 be two points in $[a,b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side is the same as the sign of $f'(c)$ because $(x_2 - x_1)$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a,b), and $f(x_2) < f(x_1)$ if f' is negative on (a,b).

Example 1:

The function $f(x) = x^2$ decreases on $(-\infty, 0)$, where $f'(x) = 2x < 0$. It increases on $(0, \infty)$, where $f'(x) = 2x > 0$.

Application of the first derivative test to find where a function is increasing and decreasing

If $a < b$ are two critical points for a function f and if f' exists but not zero on (a, b), then $f'(x)$ must be positive or negative for $x \in (a, b)$. (By intermediate value property of derivatives: *If a and b are any two points in an interval on which f is differentiable then f' takes on every value between $f'(a)$ and $f'(b)$*). One way we can determine the sign of f' on the interval is simply by evaluating f' at some point in (a, b). Then we apply corollary 3.

Note: The corollary 3 is applicable for infinite intervals also.

PROBLEM SET

IP1: Find the intervals in which $f(x) = -3 + 12x - 9x^2 + 2x^3$ increasing and decreasing.

Solution: The function f is everywhere differentiable. The first derivative

$$f'(x) = 12 - 18x + 6x^2 = 6(x^2 - 3x + 2)$$
$$= 6(x - 1)(x - 2)$$

$f'(x)$ is zero at $x = 1$ and $x = 2$. So the critical points are $x = 1$ and $x = 2$.

These critical points subdivide the domain of f into intervals $(-\infty, 1), (1,2)$ and $(2, \infty)$.

To determine the sign of f', we evaluate f' at a convenient point in each sub interval.

We apply corollary 3 to determine the behavior of f on each sub interval.

See the following table:

Interval	$(-\infty, 1)$	$(1,2)$	$(2, \infty)$
f' evaluated	$f'(0) = 12$	$f'\left(\frac{3}{2}\right) = -\frac{3}{2}$	$f'(3) = 12$
Sign of f'	$+ve$	$-ve$	$+ve$
Behavior of f	increasing	decreasing	increasing

$\therefore f$ is increasing on $(-\infty, 1), (2, \infty)$ and decreasing on $(1,2)$.

P1: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify intervals on which f is increasing and decreasing.

Solution:

The function f is everywhere differentiable.

The first derivative $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$

$f'(x)$ is zero at $x = -2$ and $x = 2$. So the critical points are $x = -2$ and $x = 2$.

These critical points subdivide the domain of f into intervals $(-\infty, -2), (-2, 2)$ and $(2, \infty)$ on which f' is either positive or negative.

To determine the sign of f', we evaluate f' at a convenient point in each sub interval.

We apply corollary 3 to determine the behavior of f on each sub interval. See the following table:

Interval	$(-\infty, -2)$	$(-2,2)$	$(2, \infty)$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+ve$	$-ve$	$+ve$
Behavior of f	increasing	decreasing	increasing

$\therefore f$ is increasing on $(-\infty, -2), (2, \infty)$ and decreasing on $(-2,2)$.

IP2: Find the intervals on which the function $\boldsymbol{f(x) = 4x^3 + 3x^2 - 6x + 1}$ is increasing and decreasing.

Solution: Given function is: $f(x) = 4x^3 + 3x^2 - 6x + 1$.

It is differentiable on $(-\infty, \infty)$. The first derivative

$$f'(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$$

The critical points are $x = -1, \frac{1}{2}$.

These critical points subdivide the domain of f into intervals $(-\infty, -1), \left(-1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$.

To determine the sign of f', we evaluate f' at a convenient point in each sub interval.

We apply corollary 3 to determine the behavior of f on each sub interval. See the following table:

Interval	$(-\infty, -1)$	$\left(-1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \infty\right)$
f' evaluated	$f'(-2) = 30$	$f'(0) = -6$	$f'(2) = 54$
Sign of f'	$+ve$	$-ve$	$+ve$
Behavior of f	increasing	decreasing	increasing

$\therefore f$ is increasing on $(-\infty, -1), \left(\frac{1}{2}, \infty\right)$ and decreasing on $\left(-1, \frac{1}{2}\right)$.

P2: Find the intervals in which the function $\boldsymbol{f(x) = x^3(x - 2)^2}$ is increasing and decreasing.

Solution: Given function is: $f(x) = x^3(x - 2)^2$.

It is differentiable on $(-\infty, \infty)$. The first derivative,

$$\begin{aligned} f'(x) &= x^3\big(2(x - 2)\big) + (3x^2)(x - 2)^2 \\ &= 2x^3(x - 2) + 3x^2(x - 2)^2 \\ &= 2x^2x(x - 2) + 3x^2(x - 2)(x - 2) \\ &= x^2(x - 2)\big(2x + 3(x - 2)\big) \\ &= x^2(x - 2)(2x + 3x - 6) \\ &= x^2(x - 2)(5x - 6) \end{aligned}$$

Critical point are $x = 0, 2$ and $\frac{6}{5}$. These critical points subdivide the domain of f into intervals $(-\infty, 0), \left(0, \frac{6}{5}\right), \left(\frac{6}{5}, 2\right)$ and $(2, \infty)$.

To determine the sign of f', we evaluate f' at a convenient point in each sub interval.

We apply corollary 3 to determine the behavior of f on each sub interval. See the following table:

Interval	$(-\infty, 0)$	$\left(0, \frac{6}{5}\right)$	$\left(\frac{6}{5}, 2\right)$	$(2, \infty)$
f' evaluated	$f'(-1) = 33$	$f'(1) = 1$	$f'\left(\frac{7}{5}\right) = -\frac{147}{125}$	$f'(3) = 81$
Sign of f'	$+ve$	$+ve$	$-ve$	$+ve$
Behavior of f	increasing	increasing	decreasing	increasing

$\therefore f$ is increasing on $(-\infty, 0), \left(0, \frac{6}{5}\right), (2, \infty)$ and decreasing on $\left(\frac{6}{5}, 2\right)$.

IP3: Find the intervals on which the function $\boldsymbol{f(x) = \sec^2 x - 2\tan x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}}$ is increasing or decreasing.

Solution: Given function is: $f(x) = \sec^2 x - 2\tan x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
It is differentiable on its domain. The first derivative,

$$f'(x) = 2(\sec x)(\sec x \tan x) - 2\sec^2 x$$
$$= 2\sec^2 x\,(\tan x - 1)$$

$f'(x) = 0 \implies 2\sec^2 x\,(\tan x - 1) = 0 \implies \tan x - 1 = 0$
$\implies x = \frac{\pi}{4}$

$f'(x)$ is zero at $x = \frac{\pi}{4}$. The critical point of $f(x)$ is $x = \frac{\pi}{4}$.
This critical point subdivide the domain of f into intervals $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$.
To determine the sign of f', we evaluate f' at a convenient point in each sub interval.
We apply corollary 3 to determine the behavior of f on each sub interval.
See the following table:

Interval	$\left[-\frac{\pi}{2}, \frac{\pi}{4}\right)$	$\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$
$\boldsymbol{f'}$ **evaluated**	$f'(0) = -2$	$f'\left(\frac{\pi}{3}\right) = 8(\sqrt{3} - 1)$
Sign of $\boldsymbol{f'}$	$-ve$	$+ve$
Behavior of $\boldsymbol{f}$	decreasing	increasing

$\therefore f$ is increasing on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ and decreasing on $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right)$.

P3: Find the intervals in which $\boldsymbol{f(x) = x^2\sqrt{5 - x}}$ is increasing and decreasing.

Solution: Given function is: $(x) = x^2\sqrt{5 - x}$. It is defined on $(-\infty, 5)$ and differentiable on it. The first derivative,

$$f'(x) = 2x.\sqrt{5 - x} - x^2.\frac{1}{2\sqrt{5-x}} = \frac{4x(5-x) - x^2}{2\sqrt{5-x}} = \frac{5x\,(4-x)}{2\,\sqrt{5-x}}.$$

Critical points of $f(x)$ are $x = 0, 4$.
These critical points subdivide the domain of f into intervals $(-\infty, 0), (0, 4)$ and $(4, 5)$.
To determine the sign of f', we evaluate f' at a convenient point in each sub interval.
We apply corollary 3 to determine the behavior of f on each sub interval. See the following table:

Interval	$(-\infty, 0)$	$(0,4)$	$(4,5)$
f' evaluated	$f'(-1) = \frac{-25}{2\sqrt{6}}$	$f'(1) = \frac{15}{4}$	$f'\left(\frac{9}{2}\right) = \frac{-45}{4\sqrt{2}}$
Sign f'	$-ve$	$+ve$	$-ve$
Behavior of f	decreasing	increasing	decreasing

$\therefore f$ is increasing on $(0,4)$ and decreasing on $(-\infty, 0), (4,5)$.

IP4: Find the critical points of $\boldsymbol{f(r) = (r + 7)^3}$ and identify intervals on which $\boldsymbol{f}$ is increasing and decreasing.

Solution: Given function is: $f(r) = (r + 7)^3$. It is differentiable on $(-\infty, \infty)$. The first derivative, $f'(r) = 3(r + 7)^2$
$f'(r)$ is zero at $r = -7$. So the critical points is $r = -7$.
These critical points subdivide the domain of f into intervals $(-\infty, -7)$ and $(-7, \infty)$ on which f' is either positive or negative.
To determine the sign of f', we evaluate f' at a convenient point in each sub interval.
We apply corollary 3 to determine the behavior of f on each sub interval. See the following table:

Interval	$(-\infty,-7)$	$(-7,\infty)$
f' evaluated	$f'(-8)=3$	$f'(-6)=3$
Sign of f'	$+ve$	$+ve$
Behavior of f	increasing	increasing

$\therefore f$ is increasing on $(-\infty,-7)\cup(-7,\infty)$.

P4: Find the critical points of $f(t)=-3t^2+9t+5$ and identify intervals on which f is increasing and decreasing.

Solution: Given function is: $f(t)=-3t^2+9t+5$. It is differentiable on $(-\infty,\infty)$.

The first derivative, $f'(t)=-6t+9$

$f'(t)$ is zero at $t=\frac{3}{2}$. So the critical points is $t=\frac{3}{2}$.

These critical points subdivide the domain of f into intervals $\left(-\infty,\frac{3}{2}\right)$ and $\left(\frac{3}{2},\infty\right)$ on which f' is either positive or negative.

To determine the sign of f', we evaluate f' at a convenient point in each sub interval.

We apply corollary 3 to determine the behavior of f on each sub interval.

See the following table:

Interval	$\left(-\infty,\frac{3}{2}\right)$	$\left(\frac{3}{2},\infty\right)$
f' evaluated	$f'(0)=9$	$f'(2)=-3$
Sign of f'	$+ve$	$-ve$
Behavior of f	increasing	decreasing

$\therefore f$ is increasing on $\left(-\infty,\frac{3}{2}\right)$ and decreasing on $\left(\frac{3}{2},\infty\right)$.

Exercises:

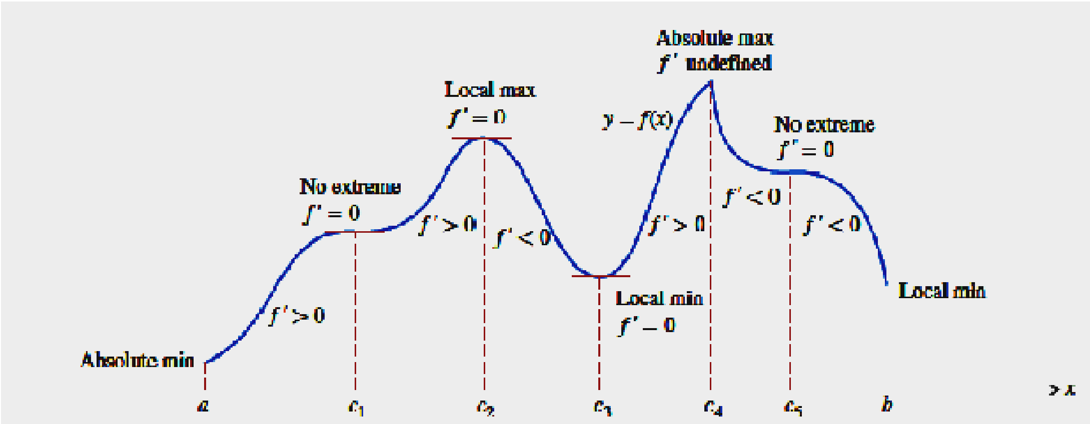
1. Show that $g(x)=1/x$ decreases on every interval in its domain.
2. Find the intervals on which the functions are increasing and decreasing.
   a. $f(x)=x^2-4x+6$
   b. $f(x)=4x^3-6x^2-72x+36$
   c. $f(x)=x^2+2x-5$
   d. $f(x)=-2x^3-9x^2-12x$
   e. $f(x)=(x+1)^3(x-3)^2$
   f. $f(x)=\sqrt{25-4x^2}$
   g. $f(x)=\sin 3x$, $\quad x\in\left(0,\frac{\pi}{2}\right)$
   h. $f(x)=\sin x+\cos x$, $\quad x\in[0,2\pi]$
3. For what values of a such that the function f given by $f(x)=x^2+ax+1$ is strictly increasing on $(1,2)$.

6.6. The First Derivative Test

Learning objectives:

- To discuss the First Derivative Test for local extrema.
  And
- To practice the related problems.

We use the first derivative to test a function's critical points for the presence of local extreme values.

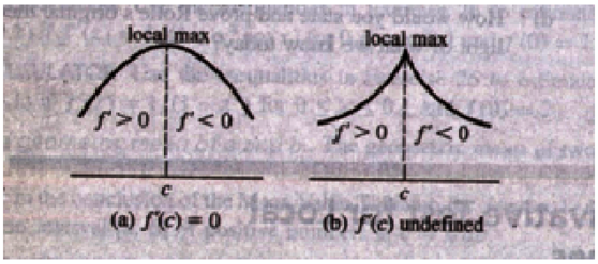


A function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in the point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.
At the points where f has a minimum, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. This means that the curve is falling on the left of the minimum value and rising on its right.
Similarly, at the points where f has a maximum, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising on the left of the maximum value and falling on its right.

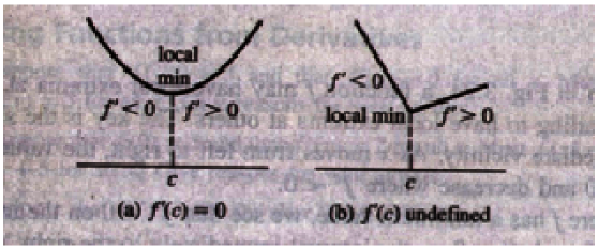
The First Derivative Test for Local Extrema

Let $f(x)$ be a continuous function with domain $[a, b]$ and c be a critical point in its domain. Let f be differentiable at every point in some interval containing c except possibly at c it self. Moving across c from left to right

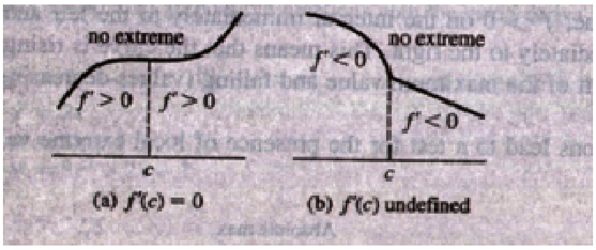
1) If f' changes from positive to negative at c, then f has a local maximum at c.



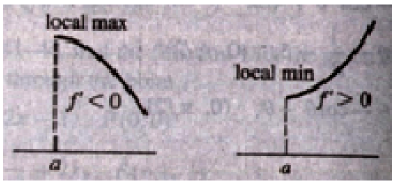
2) If f' changes from negative to positive at c, then f has a local minimum at c.



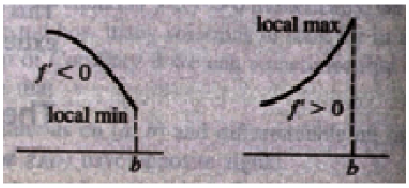
3) If f' does not change sign at c, then f has no local extreme at c.



At a left endpoint a: If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) at a.



At a right endpoint b: If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) at b.



Example 1:

Find the critical points of $f(x) = x^{\frac{1}{3}}(x-4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution:

The function f is continuous on $(-\infty, \infty)$.

The first derivative,

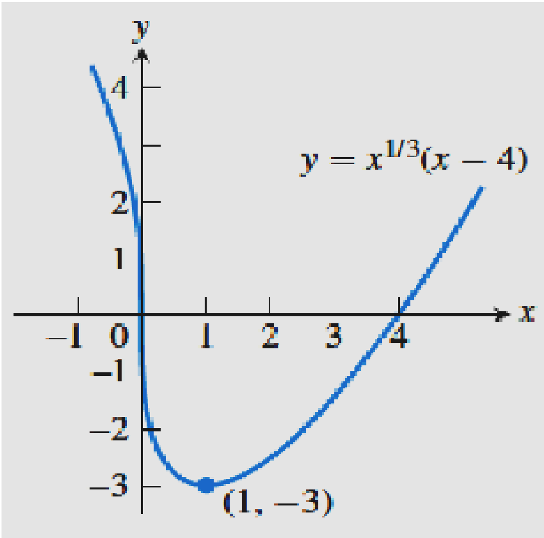
$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$
$$= \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain of f, so the critical points, $x = 0$ and $x = 1$, are the only points where f might have an extreme value of any kind.

These critical points divide $(-\infty, \infty)$ into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points. The information is displayed in the following table.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
***f′* evaluated**	$f'(-1) = -\frac{8}{3}$	$f'\left(\frac{1}{2}\right) = -\frac{2}{3\left(\frac{1}{4}\right)^{\frac{1}{3}}}$	$f'(2) = \frac{4}{3(4)^{\frac{1}{3}}}$
Sign of *f′*	$-ve$	$-ve$	$+ve$
Behavior of *f*	decreasing	decreasing	increasing

Notice that f' does not change the sign at $x = 0$. Therefore, f has no extremum at $x = 0$.
Notice that f' changes from negative to positive at $x = 1$. Therefore, f has a local minimum at $x = 1$.
The value of the local minimum is $f(1) = 1^{1/3}(1-4) = -3$. This is also an absolute minimum.
Note that $\lim_{x\to 0} f'(x) = -\infty$. Therefore, the curve $y = f(x)$ has a vertical tangent at the origin.

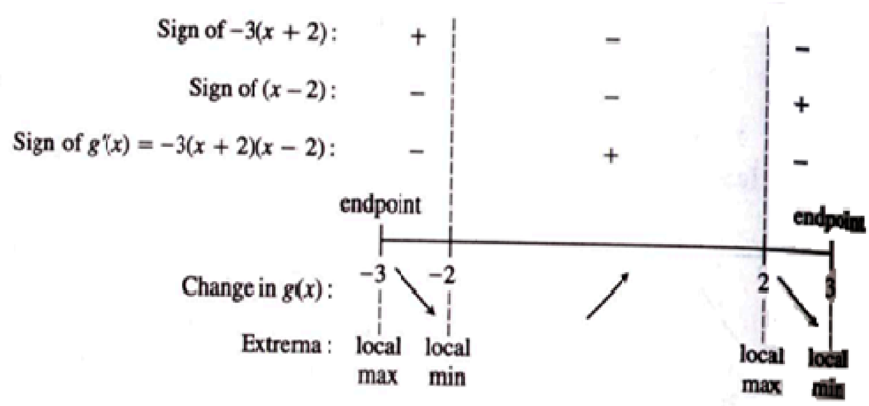


Example 2: Find the intervals on which, $g(x) = -x^3 + 12x + 5 \quad , \quad -3 \le x \le 3$ is increasing and decreasing. Where does the function assume extreme values and what are the values?

Solution: The function f is continuous on its domain, $[-3, 3]$.
The first derivative

$g'(x) = -3x^2 + 12 = -3(x^2 - 4) = -3(x-2)(x+2)$

defined at all points of $[-3, 3]$, is zero at $x = -2$ and $x = 2$.
These critical points divide the domain of g into intervals on which g' is either positive or negative.

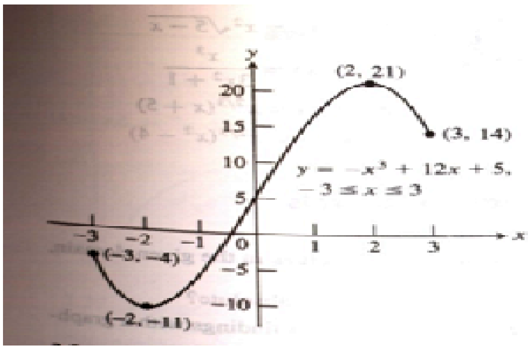


Thus, conclude that g has local maxima at $x = -3$ and $x = 2$ and local minima at $x = -2$ and $x = 3$. The corresponding values of $g(x) = -x^3 + 12x + 5$ are

Local maxima: $g(-3) = -4 \qquad g(2) = 21$

Local minima: $g(-2) = 11 \qquad g(3) = 14$

Since g is defined on a closed interval, $g(-2)$ is the absolute minimum and $g(2)$ is the absolute maximum.



PROBLEM SET

IP1: Find the intervals on which $f(x) = (x+7)^3$ is increasing and decreasing. Where does the function assume extreme values and what are the values?

Solution: The function f is continuous on $(-\infty, \infty)$.
The first derivative is $f'(x) = 3(x+7)^2$.
$f'(x)$ is zero at $x = -7$. So the critical point $x = -7$ subdivides the domain of f into intervals $(-\infty, -7)$ and $(-7, \infty)$ on which f' is either positive or negative.
Notice that

$$f'(x) = 3(x+7)^2 > 0, \forall\, x \in (-\infty, -7)$$
$$f'(x) = 3(x+7)^2 > 0, \forall\, x \in (-7, \infty)$$

f is increasing on $(-\infty, -7) \cup (-7, \infty)$, by first derivative test for monotonic functions.
Notice that f' does not change sign at $x = -7$. Therefore, f has no local extrema at $x = -7$. Thus f has no local extrema.

P1: Find the intervals on which $f(x) = -3x^2 + 9x + 5$ is increasing and decreasing. Where does the function assume extreme values and what are the values?

Solution: The function f is continuous on $(-\infty, \infty)$.
The first derivative,

$$f'(x) = -6x + 9 = -3(2x - 3)$$

$f'(x)$ is zero at $x = \frac{3}{2}$. So there is only one critical point $x = \frac{3}{2}$ and it divides the domain of f into intervals on which f' is either positive or negative.
The sign pattern of f' reveals the behavior of f between and at the critical point. The behavior of f is analysed below.

Interval	$\left(-\infty, \frac{3}{2}\right)$	$\left(\frac{3}{2}, \infty\right)$
f' evaluated	$f'(0) = 9$	$f'(2) = -3$
Sign of f'	$+ve$	$-ve$
Behavior of f	increasing	decreasing

Thus, f is increasing in $\left(-\infty, \frac{3}{2}\right)$ and decreasing in $\left(\frac{3}{2}, \infty\right)$.
Notice that f' changes from $+ve$ to $-ve$ at $x = \frac{3}{2}$, f has a local maximum at $x = \frac{3}{2}$. Further f has absolute maximum at $x = \frac{3}{2}$ and absolute maximum value is $f\left(\frac{3}{2}\right) = \frac{47}{4}$.

IP2: Answer the following questions about whose derivative is $f'(x) = x^{-1/3}(x+2)$.

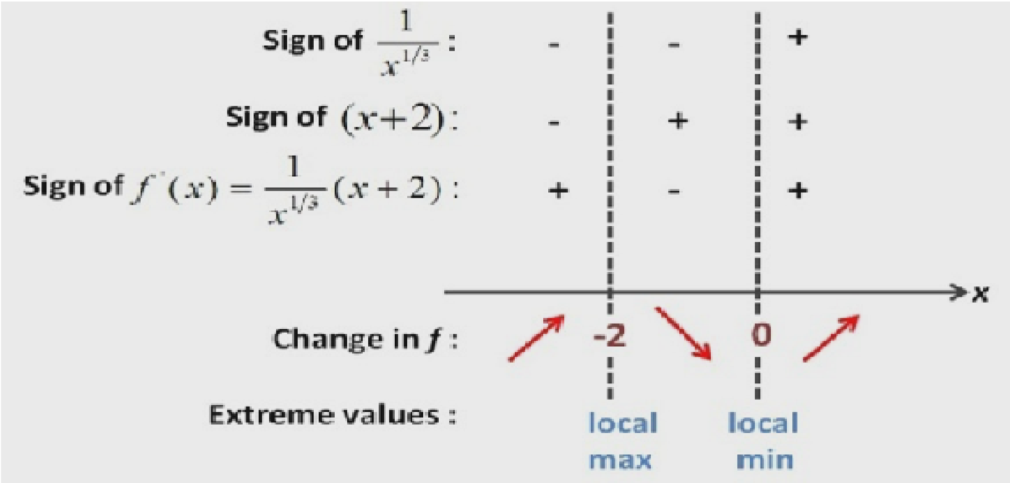
a) **What are the critical points of f ?**

b) **On what intervals is f increasing or decreasing?**

c) **At what points, if any, does f assume local maximum and minimum values?**

Solution: Given $f'(x) = x^{-1/3}(x+2)$.

$f'(x)$ is zero at $x = -2$ and undefined at $x = 0$. So the only critical points of f are $x = -2$ and $x = 0$.



f is increasing on $(-\infty,-2)$ and $(0,\infty)$, decreasing on $(-2,0)$.

We conclude that f has local maximum at $x = -2$ and local minimum at $x = 0$.

P2: Answer the following questions about whose derivative is $f'(x) = (x-1)(x+2)$.

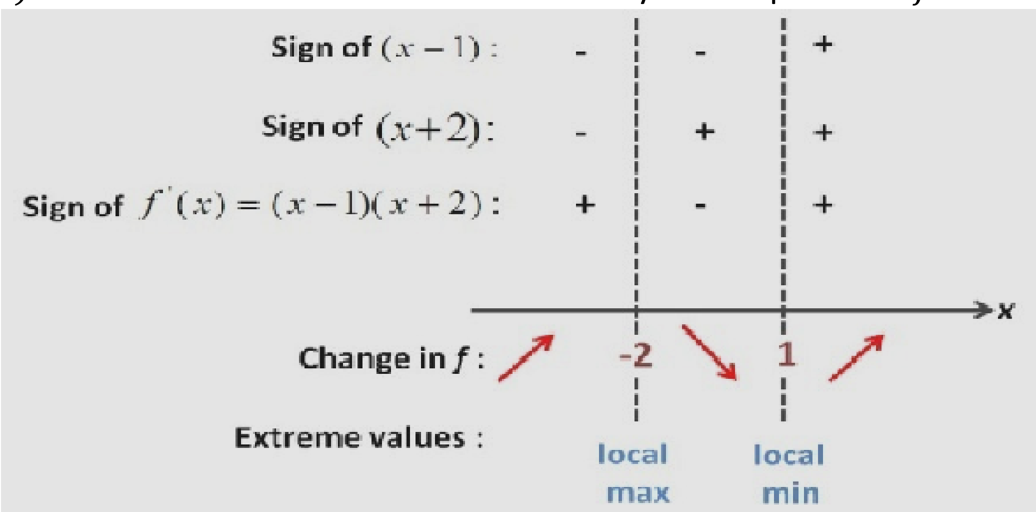
d) **What are the critical points of f ?**

e) **On what intervals is f increasing or decreasing?**

f) **At what points, if any, does f assume local maximum and minimum values?**

Solution: Given $f'(x) = (x-1)(x+2)$.

$f'(x)$ is zero at $x = -2$ and $x = 1$. So the only critical points of f are $x = -2$ and $x = 1$.



f is increasing on $(-\infty,-2)$ and $(1,\infty)$, decreasing on $(-2,1)$.

We conclude that f has local maximum at $x = -2$ and local minimum at $x = 1$.

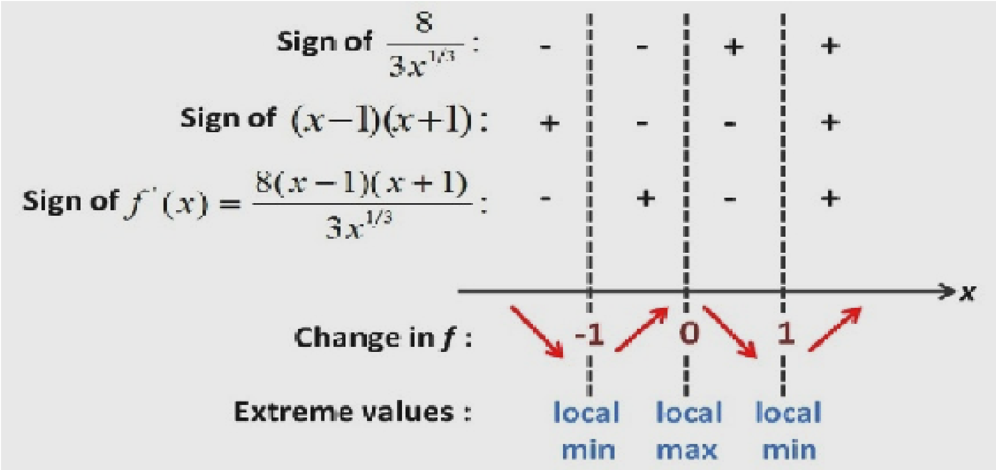
IP3: Find the critical points of $f(x) = x^{2/3}(x^2-4)$. Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution: The function f is defined for all real numbers and continuous. The first derivative,

$$f'(x) = \frac{d}{dx}\left(x^{8/3} - 4x^{2/3}\right) = \frac{8}{3}x^{5/3} - \frac{8}{3}x^{-1/3}$$

$$= \frac{8}{3}x^{-1/3}(x^2-1) = \frac{8(x+1)(x-1)}{3x^{1/3}}$$

$f'(x)$ is zero at $x = \pm 1$ and undefined at $x = 0$. There are no end points in the domain of f. So the points $x = \pm 1$ and $x = 0$, are the only critical points of f where f might have an extreme value of any kind.
These critical points divide the domain of f into intervals on which f' is either positive or negative.



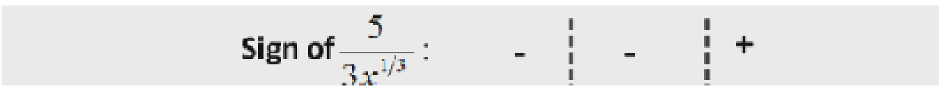
f is increasing on $(-1,0)$ and $(1,\infty)$, decreasing on $(-\infty,-1)$ and $(0,1)$.
We conclude that the function f has local maximum at $x = 0$ and local minimum at $x = -1, 1$.
The corresponding values of $f(x) = x^{2/3}(x^2 - 4)$ are
Local minimum: $f(\pm 1) = (\pm 1)^{2/3}(1 - 4) = -3$
Local maximum: $f(0) = 0$.
Further notice that f has absolute minimum at $x = \pm 1$.

P3: Find the critical points of $f(x) = x^{2/3}(x + 5)$. Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution: The function f is defined for all real numbers and continuous. The first derivative,

$$f'(x) = \frac{d}{dx}\left(x^{5/3} + 5x^{2/3}\right) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$
$$= \frac{5}{3}x^{-1/3}(x + 2) = \frac{5(x+2)}{3x^{1/3}}$$

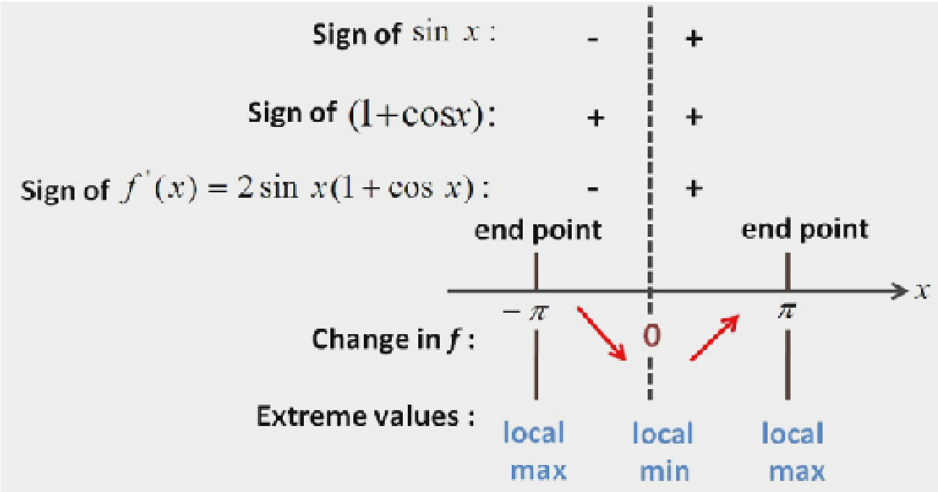
$f'(x)$ is zero at $x = -2$ and undefined at $x = 0$. There are no end points in the domain of f. So the points $x = -2$ and $x = 0$, are the only critical points where f might have an extreme value of any kind.
These critical points divide the domain of f into intervals on which f' is either positive or negative.



f is increasing on $(-\infty,-2)$ and $(0,\infty)$, decreasing on $(-2,0)$.
We conclude that the function f has local maximum at $x=-2$ and local minimum at $x=0$.
The corresponding values of $f(x)=x^{2/3}(x+5)$ are
Local maximum: $f(-2)=(-2)^{2/3}(-2+5)=3\sqrt[3]{4}$
Local minimum: $f(0)=0$.

IP4: Find the local extrema of the function $f(x)=-2\cos x-\cos^2 x$, on the interval $-\pi\le x\le\pi$.

Solution: The function f is continuous on its domain $[-\pi,\pi]$.
The first derivative, $f'(x)=2\sin x-2\cos x\,(-\sin x)=2\sin x\,(1+\cos x)$
$f'(x)$ is defined at all points of $[-\pi,\pi]$, is zero at $x=\pm\pi$ and $x=0$. These critical points divide the domain of f into intervals $[-\pi,0)$ and $(0,\pi]$ on which f' is either positive or negative.



We conclude that f has local maxima at $x=-\pi$ and $x=\pi$, local minima at $x=0$.
The corresponding values of $f(x)=-2\cos x-\cos^2 x$ are

Local maxima: $f(-\pi)=1$ $\quad f(\pi)=1$
Local minima: $f(0)=-3$

Further notice that f has absolute maximum at $x=-\pi,\pi$ and absolute minimum at $x=0$.

P4: Find the local extrema of the function $f(x)=x^3-3x^2$ on the interval $-\infty<x\le 3$.

Solution: The function f is continuous on its domain $(-\infty,3]$.
The first derivative, $f'(x)=3x^2-6x=3x(x-2)$
$f'(x)$ is defined at all points of $(-\infty,3]$, is zero at $x=0$ and $x=2$. These critical points divide the domain of f into intervals $(-\infty,0),(0,2)$ and $(2,3)$ on which f' is either positive or negative.

We conclude that f has local maxima at $x = 0$ and $x = 3$, local minima at $x = 2$.
The corresponding values of $f(x) = x^3 - 3x^2$ are

Local maxima: $f(0) = 0$, $f(3) = 0$

Local minima: $f(2) = -4$

Further, note that f has absolute maximum at $x = 0$ and $x = 3$.

Exercises:

1. Answer the following questions in problems i to iv about the functions whose derivatives are given below.

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

- **i.** $f'(x) = x(x-1)$
- **ii.** $f'(x) = (x+2)(x-1)^2$
- **iii.** $f'(x) = (x+2)(x-1)(x-3)$
- **iv.** $f'(x) = x^{-1/3}(x+2)$

2. In the problems i - x

- **a.** Find the intervals on which the function is increasing and decreasing.
- **b.** Then identify the function's local extreme values, if any, saying where they are taken on.
- **c.** Which, if any, of the extreme values are absolute?
  - **i.** $g(t) = -t^3 - 3t + 3$
  - **ii.** $h(x) = -x^3 + 2x^2$
  - **iii.** $f(x) = 3\theta^2 - 4\theta^3$
  - **iv.** $f(r) = 3r^3 + 16r$
  - **v.** $f(x) = x^4 - 8x^2 + 16$
  - **vi.** $H(t) = \frac{3}{2}t^4 - t^6$
  - **vii.** $g(x) = x\sqrt{8 - x^2}$
  - **viii.** $f(x) = \frac{x^2-3}{x-2}, x \neq 2$
  - **ix.** $f(x) = x^{1/3}(x+8)$
  - **x.** $h(x) = x^{1/3}(x^4 - 4)$

3. In problems $i - iv$:

- **a.** Identify the function's local extreme values in the given domain, and say where they are assumed.
- **b.** Which of the extreme values, if any, are absolute?
  - **i.** $f(x) = 2x - x^2 \qquad -\infty < x \leq 2$
  - **ii.** $g(x) = x^2 - 4x + 4 \qquad 1 \leq x < \infty$
  - **iii.** $f(t) = 12t - t^3 \qquad -3 \leq t < \infty$
  - **iv.** $h(x) = \frac{x^3}{3} - 2x^2 + 4x \qquad 0 \leq x < \infty$

6.7. Curve Sketching (concavity)

Learning objectives:

- To define concavity for a differentiable function.
- To study the second derivative test for the concavity for the graphs of twice differentiable functions.
- To define a point of inflection for the graph of a function.

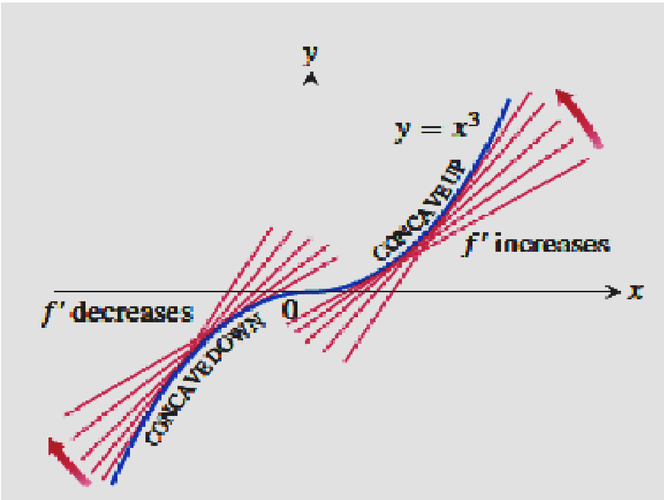
And

- To practice the related problems.

We have seen that the first derivative played a role in locating a function's extreme values. A function can have extreme values only at the endpoints of its domain and at its critical points. We also have seen that critical points do not necessarily yield extreme values. The first derivative was used to ascertain whether there is really an extreme value there or whether the graph just continues to rise or fall.

Concavity:

As seen from the figure below, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways.



As we come in from the left toward the origin along the curve, the curve turns to our right and falls below its tangents. As we leave the origin, the curve turns to our left and rises above its tangents.

To put it another way, the slopes of the tangents decrease as the curve approaches the origin from the left and increase as the curves moves from the origin into the first quadrant. The graph of a differential function $y = f(x)$ is ***concave up*** on an interval I if y' is increasing on I and ***concave down*** on an interval I if y' is decreasing on I.

If $y = f(x)$ has a second derivative, the Corollary 3 of the Mean Value Theorem says that y' increases if $y'' > 0$ and decreases if $y'' < 0$.

The Second Derivative Test for Concavity

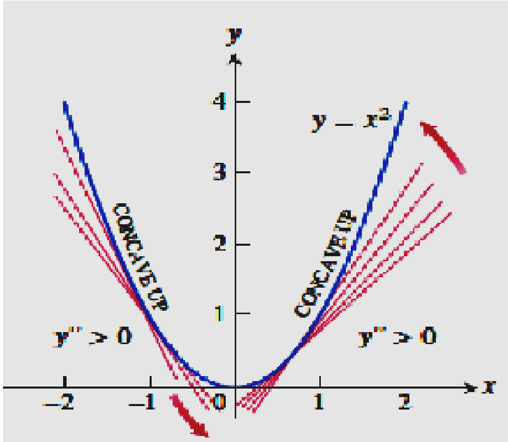
Let $y = f(x)$ be twice differentiable on an interval I.

If $y'' > 0$ on I, the graph of f over I is concave up.

If $y'' < 0$ on I, the graph of f over I is concave down.

Example 1:

a) The curve $y = x^3$ is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$. (See the previous figure)
b) The parabola $y = x^2$ is concave up on every interval because $y'' = 2 > 0$.



Points of Inflection

To study the motion of a body along a line, we graph the body's position as a function of time. The graph reveals where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points where the concavity changes.

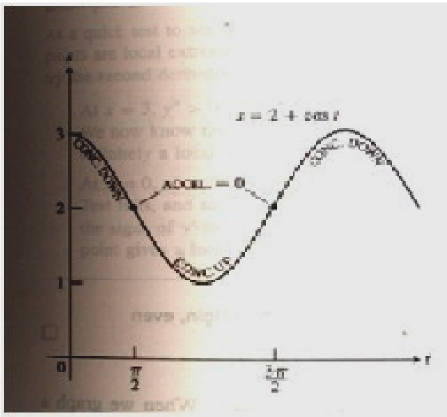
*A point where the graph of a function has a tangent line and where the concavity change is called a **point of inflection**.*

Thus a point of inflection on a curve is a point where y'' is positive on one side and negative on the other. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined.

On the graph of a twice-differentiable function, $y'' = 0$ at a point of inflection.

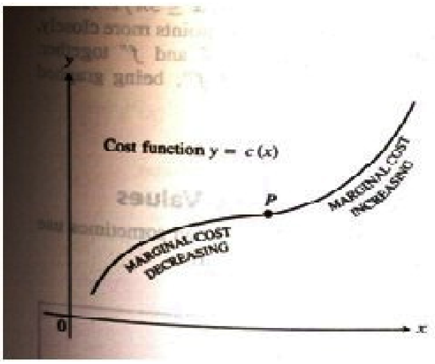
Example 2:

The graph of $s = 2 + \cos t \geq 0$, changes concavity at $t = \pi/2\,, 3\pi/2\,, \ldots$, where the acceleration $s'' = -\cos t$ is zero.



Example 3:

Inflection points have applications in some areas of economics. Suppose that $y = c(x)$ is the total cost of producing x units of something.

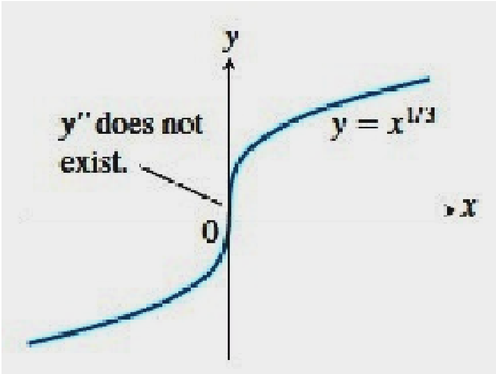


The point of inflection at P is then the point at which the marginal cost (the approximate cost of producing one more unit) changes from decreasing to increasing.

Example 4:

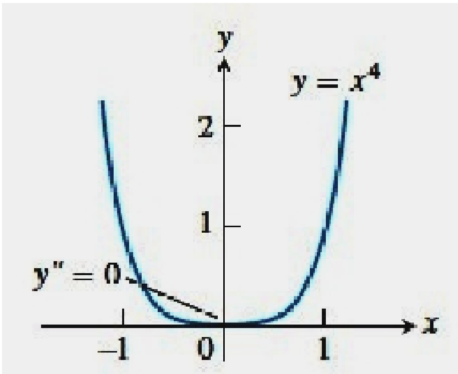
The curve $y = x^{1/3}$ has a point of inflection at $x = 0$, but y'' does not exist there.

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}$$



Example 5:

The curve $y = x^4$ has no inflection point at $x = 0$. Even though $y'' = 12x^2$ is zero there, it does not change sign.



PROBLEM SET

IP1: Find the inflection points and the intervals in which the function $f(x) = x^4 - 4x^3$ is concave up and concave down.

Solution: Given function is: $f(x) = x^4 - 4x^3$

Now, $f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

$f''(x) = 0$ when $x = 0$ and $x = 2$.

If $-\infty < x < 0$, $f''(x) = 12x(x-2) > 0$

If $0 < x < 2$, $f''(x) = 12x(x-2) < 0$

If $2 < x < \infty$, $f''(x) = 12x(x-2) > 0$

Therefore, $f(x)$ is concave down on $(0,2)$, concave up on $(-\infty, 0), (2, \infty)$ and $x = 0, x = 2$ are the inflection points.

P1: Find the inflection points and intervals in which the function $f(x) = 2x^3 + 3x^2 - 36x$ is concave up and concave down.

Solution: Given function is: $f(x) = 2x^3 + 3x^2 - 36x$

Now, $f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6$

$f''(x) = 0 \implies x = -\frac{1}{2}$.

For $-\infty < x < -\frac{1}{2}$ we have $f''(x) = 12x + 6 < 0$

For $-\frac{1}{2} < x < \infty$ we have $f''(x) = 12x + 6 > 0$

Therefore, $f(x)$ is concave down on $\left(-\infty, -\frac{1}{2}\right)$, concave up on $\left(-\frac{1}{2}, \infty\right)$ and there is a point of inflection at $x = -\frac{1}{2}$.

IP2: Find the inflection points and intervals in which the function $f(x) = \frac{x^2}{x^2+3}$ is concave up and concave down.

Solution: Given function is: $f(x) = \frac{x^2}{x^2+3}$

Now, $f'(x) = \frac{2x(x^2+3)-x^2(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$

$f''(x) = \frac{6(x^2+3)^2-6x[2(x^2+3)(2x)]}{(x^2+3)^4} = \frac{18(1-x^2)}{(x^2+3)^3}$

$f''(x) = 0 \implies \frac{18(1-x^2)}{(x^2+3)^3} = 0$

$\implies x = \pm 1$

Possible points of inflection are at $x = -1$ and $x = 1$.

For $-\infty < x < -1$, we have $f''(x) = \frac{18(1-x^2)}{(x^2+3)^3} < 0$

For $-1 < x < 1$, we have $f''(x) = \frac{18(1-x^2)}{(x^2+3)^3} > 0$

For $1 < x < \infty$, we have $f''(x) = \frac{18(1-x^2)}{(x^2+3)^3} < 0$

Therefore, $f(x)$ is concave down on $(-\infty, 1)$ and $(1, \infty)$, concave up on $(-1,1)$ and there is a point of inflection at $x = -1$ and $x = 1$.

P2: Find the inflection points and intervals in which the function $f(x) = 3 + \sin x$ is concave up and concave down in the interval $[0, 2\pi]$.

Solution: Given function is: $f(x) = 3 + \sin x$

Now, $f'(x) = \cos x$

$f''(x) = -\sin x$

$f''(x) = 0 \implies -\sin x = 0 \implies x = \pi$

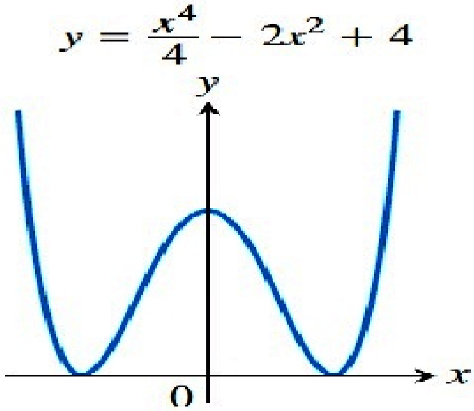
Possible point of inflection at $x = \pi$.

For $0 < x < \pi$, we have $f''(x) = -\sin x < 0$

For $\pi < x < 2\pi$, we have $f''(x) = -\sin x > 0$

Therefore, $f(x)$ is concave down on $(0, \pi)$, concave up on $(\pi, 2\pi)$ and there is a point of inflection at $x = \pi$.

IP3: Identify the inflection points and local maxima and minima of the function graphed below. Identify the intervals on which the function is concave up and concave down.

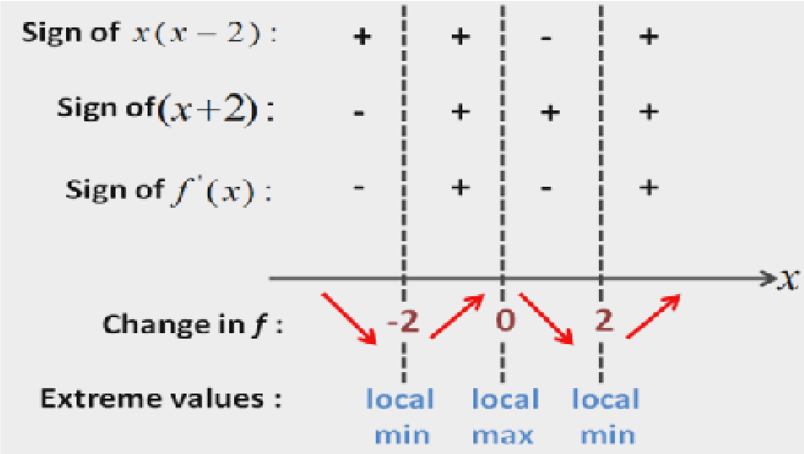


Solution: Given $y = \frac{x^4}{4} - 2x^2 + 4$

Now, $y' = x^3 - 4x = x(x+2)(x-2)$

$y' = 0 \Longrightarrow x(x+2)(x-2) = 0 \Longrightarrow x = 0, \pm 2.$

$\therefore$ Critical points are $x = -2$, $x = 0$ and $x = 2$

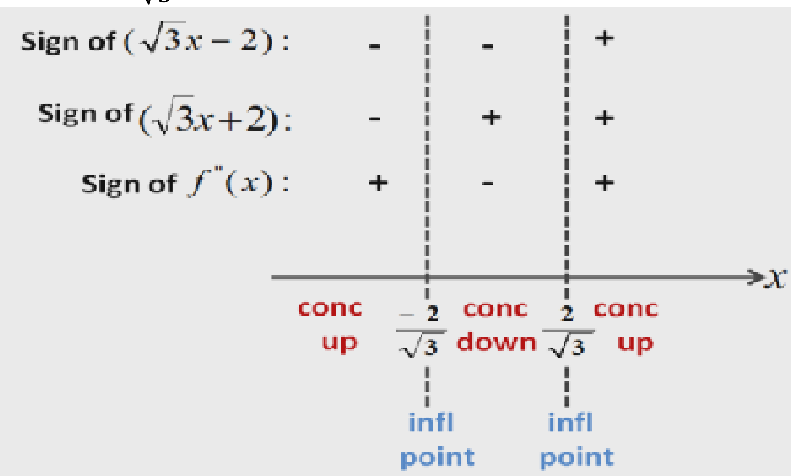


The graph is rising on $(-2,0)$ and $(2,\infty)$, falling on $(-\infty,-2)$ and $(0,2)$.

Local maximum is 4 at $x = 0$ and local minima are 0 at $x = \pm 2$.

Now, $y'' = 3x^2 - 4 = \left(\sqrt{3}x + 2\right)\left(\sqrt{3}x - 2\right)$

$y'' = 0 \;\Longrightarrow\; x = \pm\frac{2}{\sqrt{3}}.$

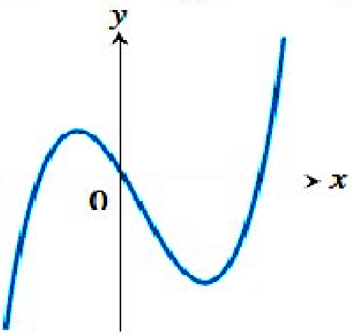


Therefore, the graph is concave up on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$, concave down on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

$\therefore$ $\left(-\frac{2}{\sqrt{3}}, \frac{16}{9}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{16}{9}\right)$ are the inflection points.

P3: Identify the inflection points and local maxima and minima of the function graphed below. Identify the intervals on which the function is concave up and concave down.

$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



Solution: Given $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 4$

Now, $y' = x^2 - x - 2 = (x+1)(x-2)$

$y' = 0 \Longrightarrow (x+1)(x-2) = 0 \Longrightarrow x = -1,\ 2.$

$\therefore$ Critical points are $x = -1$ and $x = 2$

For $-\infty < x < -1$, we have $y' = (x+1)(x-2) > 0$

For $-1 < x < 2$, we have $y' = (x+1)(x-2) < 0$

For $2 < x < \infty$, we have $y' = (x+1)(x-2) > 0$

The graph is rising on $(-\infty, -1)$ and $(2, \infty)$, falling on $(-1, 2)$.

Local maximum is $\frac{3}{2}$ at $x = -1$ and local minimum is -3 at $x = 2$.

Now, $y'' = 2x - 1$

$y'' = 0 \Longrightarrow x = \frac{1}{2}.$

For $-\infty < x < \frac{1}{2}$, we have $y'' = 2x - 1 < 0$

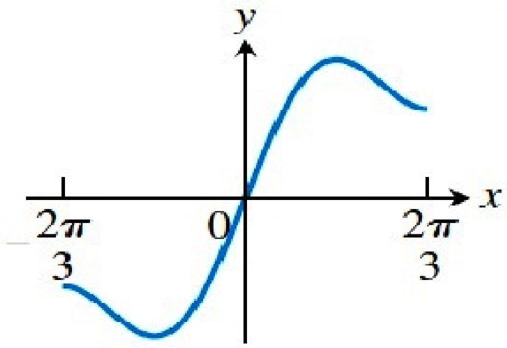
For $\frac{1}{2} < x < \infty$, we have $y'' = 2x - 1 > 0$

Therefore, the graph is concave down on $\left(-\infty, \frac{1}{2}\right)$, concave up on $\left(\frac{1}{2}, \infty\right)$.

Therefore, $\left(\frac{1}{2}, -\frac{3}{4}\right)$ is the inflection point.

IP4: Identify the inflection points and local maxima and minima of the function graphed below. Identify the intervals on which the function is concave up and concave down.

$$y = x + \sin 2x, -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$$



Solution: Given $y = x + \sin 2x$, $\frac{-2\pi}{3} \le x \le \frac{2\pi}{3}$

Now, $y' = 1 + 2\cos 2x$

$y' = 0 \Longrightarrow 1 + 2\cos 2x = 0 \Longrightarrow \cos 2x = -\frac{1}{2}$

$\Longrightarrow 2x = \pm\frac{2\pi}{3} \Longrightarrow x = \pm\frac{\pi}{3}$

Critical points are $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$

For $\frac{-2\pi}{3} < x < \frac{-\pi}{3}$ we have $y' = 1 + 2\cos 2x < 0$

For $\frac{-\pi}{3} < x < \frac{\pi}{3}$ we have $y' = 1 + 2\cos 2x > 0$

For $\frac{\pi}{3} < x < \frac{2\pi}{3}$ we have $y' = 1 + 2\cos 2x < 0$

Therefore, the graph is rising on $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, falling on $\left(\frac{-2\pi}{3}, \frac{-\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$.

Local maxima are $\frac{-2\pi}{3} + \frac{\sqrt{3}}{2}$ at $x = \frac{-2\pi}{3}$ and $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$ at $x = \frac{\pi}{3}$, local minima are $\frac{-\pi}{3} - \frac{\sqrt{3}}{2}$ at $x = \frac{-\pi}{3}$ and $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ at $x = \frac{2\pi}{3}$.

Now, $y'' = -4\sin 2x$

$y'' = 0 \Longrightarrow -4\sin 2x = 0 \Longrightarrow \sin 2x = 0$

$\Longrightarrow 2x = 0, \pm\pi \;\; \Longrightarrow x = 0, \pm\frac{\pi}{2}.$

For $-\frac{2\pi}{3} < x < -\frac{\pi}{2}$, we have $y'' = -4\sin 2x < 0$

For $-\frac{\pi}{2} < x < 0$, we have $y'' = -4\sin 2x > 0$

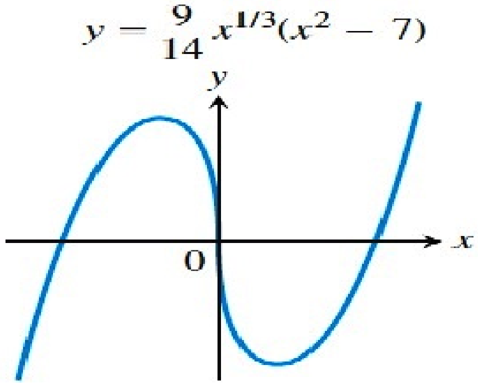
For $0 < x < \frac{\pi}{2}$, we have $y'' = -4\sin 2x < 0$

For $\frac{\pi}{2} < x < \frac{2\pi}{3}$, we have $y'' = -4\sin 2x > 0$

Therefore, the graph is concave down on $\left(-\frac{2\pi}{3}, -\frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$,
concave up on $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

Point of inflections are $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, $(0,0)$ and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

P4: Identify the inflection points and local maxima and minima of the function graphed below. Identify the intervals on which the function is concave up and concave down.



Solution:

Given $y = \frac{9}{14}x^{1/3}(x^2 - 7)$

Now, $y' = \frac{3}{14}x^{-2/3}(x^2 - 7) + \frac{9}{14}x^{1/3}(2x) = \frac{3}{2}x^{-2/3}(x^2 - 1)$

$y' = 0$ when $x = 0, \pm 1$. The critical points are $x = 0, \pm 1$.

Sign of $\frac{3}{2x^{2/3}}$:	+	+	+	+
Sign of $(x-1)(x+1)$:	+	-	-	+
Sign of $f'(x)$:	+	-	-	+

The graph is rising on $(-\infty, -1)$ and $(1, \infty)$, falling on $(-1,1)$.

Local maximum is $\frac{27}{7}$ at $x = -1$ and local minima is $-\frac{27}{7}$ at $x = 1$.

Now, $y'' = -x^{-5/3}(x^2 - 1) + \frac{3}{2}x^{-2/3}(2x) = x^{-5/3}(2x^2 + 1)$

y'' is undefined when $x = 0$. The possible inflection point is at $x = 0$.

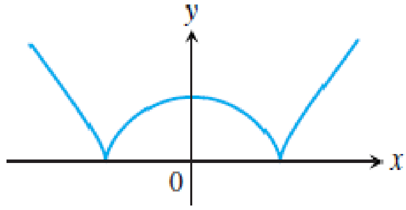
For $-\infty < x < 0$, we have $y'' = x^{-5/3}(2x^2 + 1) < 0$

For $0 < x < \infty$, we have $y'' = x^{-5/3}(2x^2 + 1) > 0$
Therefore, the graph is concave up on $(0, \infty)$, concave down on $(-\infty, 0)$.
The point of inflection is $(0,0)$.

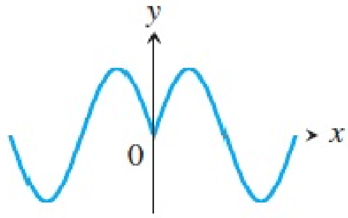
Exercises:

1. Identify the inflection points and local maxima and minima of the functions graphed below. Identify the intervals on which the functions are concave up and concave down.

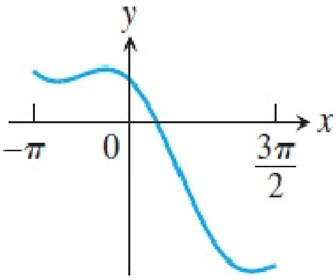
$y = \frac{3}{4}(x^2 - 1)^{2/3}$



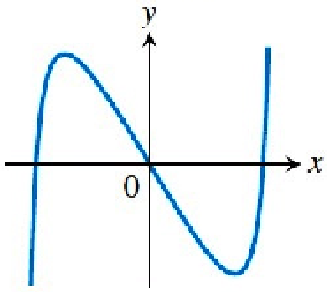
$y = \sin|x|, -2\pi \le x \le 2\pi$



$y = 2\cos x - \sqrt{2}x, \; -\pi \le x \le \frac{3\pi}{2}$



$y = \tan x - 4x, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$



6.8. Second Derivative Test

Learning objectives:

- To study the second derivative test for local extrema.

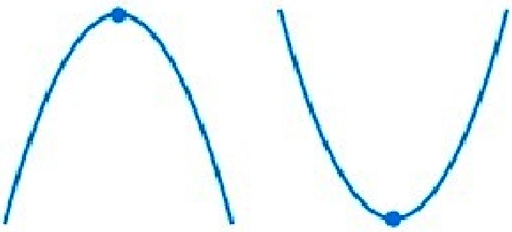
  And

- To practice the related problems.

Instead of examining y' for sign changes at a critical point, we can use the second derivative y'' to determine the presence of a local extremum.

The Second Derivative Test for Local Extrema

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.



$f' = 0, f'' < 0 \Rightarrow$ local max $\qquad$ $f' = 0, f'' > 0 \Rightarrow$ local min

We notice that the test requires us to know y'' only at c itself, and not in an interval about c.

The test is inconclusive if $y'' = 0$ or if y'' does not exist. When this happens, we use the first derivative test for local extreme values.

Example 1: Discuss the local extrema for the function $y = x^4 - 4x^3 + 10$.
Solution: We calculate y' and y''.
$y = x^4 - 4x^3 + 10$
$y' = 4x^3 - 12x^2 = 4x^2(x - 3)$
$y' = 0$ at $x = 0, x = 3$.
$y'' = 12x^2 - 24x = 12x(x - 2)$
Now, $y''(3) = 36 > 0$. By second derivative test for local extrema y has a local minimum at $x = 3$.
Now, $y''(0) = 0$. The second derivative test for local extrema fails and we use first derivative test for local extrema.
Notice that, $y' = 4x^2(x - 3) < 0$ for $x < 0$ and $0 < x < 3$.
y' does not change its sign at $x = 0$.
Therefore, y has no local extrema at $x = 0$.

PROBLEM SET

IP1: Find all the points of local maxima and minima of the function $f(x) = x^3 - 12x$.
Solution: Given function is $f(x) = x^3 - 12x$.
Now, $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$
$f'(x) = 0$ at $x = 2$ and $x = -2$.
$f''(x) = 6x$
Hence $f''(2) = 12 > 0$, $f''(-2) = -12 < 0$
By second derivative test for local extrema, $f(x)$ has a local maximum at $x = -2$ and local minimum at $x = 2$.
P1: Find all the points of local maxima and local minima of the function $f(x) = x^4 - 8x^2$.
Solution: We have $f(x) = x^4 - 8x^2$.
Now, $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$
$f'(x) = 0$ at $x = 0, x = 2$ and $x = -2$.
$f''(x) = 12x^2 - 16$
Hence $f''(0) = -16 < 0$, $f''(2) = f''(-2) = 32 > 0$
By second derivative test for local extrema, $f(x)$ has a local maximum at $x = 0$ and local minimum at $x = \pm 2$.
IP2: Find all the points of local minima and local maxima of the function $f(x) = x^5 - 5x^4$.
Solution: We have $f(x) = x^5 - 5x^4$.
Now, $f'(x) = 5x^4 - 20x^3 = 5x^3(x - 4)$
$f'(x) = 0$ at $x = 0$ and $x = 4$.
$f''(x) = 20x^3 - 60x^2 = 20x^2(x - 3)$
At $x = 4, f''(4) = 320 > 0$:
By second derivative test for local extrema, $f(x)$ has a local minimum at $x = 4$.
At $x = 0, f''(0) = 0$: The second derivative test fails and so we need to check the signs of $f'(x)$ to know whether this point gives a local extreme value.
Notice that, $f'(x) = 5x^3(x - 4) > 0$ for $x < 0$ and
$f'(x) = 5x^3(x - 4) < 0$ for $0 < x < 4$.
Thus $f'(x)$ changes sign at $x = 0$. Therefore, $f(x)$ has a local maximum at $x = 0$.

P2: Find all the points of local maxima and local minima of the function $y = x^{\frac{4}{3}}$.

Solution: We have, $y = x^{\frac{4}{3}}$. Now, $y' = \frac{4}{3}x^{\frac{1}{3}}$. $y' = 0$ at $x = 0$.

Now, $y'' = \frac{4}{9}x^{\frac{-2}{3}} = \frac{4}{9x^{\frac{2}{3}}}$. At $x = 0$, y'' does not exists.

Therefore, the second derivative test fails and so we need to check the signs of y' to know whether this point gives a local extreme value.

Notice that

$$y' < 0 \ , \ \forall\, x < 0 \quad \text{and} \quad y' > 0 \ , \forall\, x > 0$$

Notice that y' changes from negative to positive at $x = 0$.

Therefore, y has a local minimum at $x = 0$.

IP3: Find all the points of local maxima and local minima of the function

$$\boldsymbol{f(x) = 2\cos x + x} \text{ , where } < \boldsymbol{x} < \boldsymbol{\pi} \text{ .}$$

Solution: We have, $f(x) = 2\cos x + x$, $0 < x < \pi$.

Now, $f'(x) = -2\sin x + 1$

For local maximum or minimum, we have

$$f'(x) = 0 \quad \Longrightarrow \quad -2\sin x + 1 = 0 \quad \Longrightarrow \quad \sin x = \frac{1}{2}$$

$$\Longrightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad (\because\ 0 < x < \pi)$$

Now, $f''(x) = -2\cos x$ and

$$f''\left(\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0 \ ,$$

$$f''\left(\frac{5\pi}{6}\right) = -2\cos\frac{5\pi}{6} = -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3} > 0 \ .$$

By second derivative test for local extrema, $f(x)$ has a local maximum at $x = \frac{\pi}{6}$ and local minimum at $x = \frac{5\pi}{6}$.

Therefore, $x = \frac{\pi}{6}$ is a point of local maximum and local maximum value is

$$f\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} + \frac{\pi}{6} = \sqrt{3} + \frac{\pi}{6}.$$

$x = \frac{5\pi}{6}$ is a point of local minimum and local minimum value is

$$f\left(\frac{5\pi}{6}\right) = 2\cos\frac{5\pi}{6} + \frac{5\pi}{6} = -\sqrt{3} + \frac{5\pi}{6}.$$

P3: Find all the points of local maxima and local minima of the function

$$\boldsymbol{f(x) = \sin^4 x + \cos^4 x} \text{ where } \boldsymbol{0} < x < \frac{\boldsymbol{\pi}}{\boldsymbol{2}} \text{ .}$$

Solution: We have, $f(x) = \sin^4 x + \cos^4 x$, $0 < x < \frac{\pi}{2}$.

Now,

$$\begin{aligned} f'(x) &= 4\sin^3 x(\cos x) + 4\cos^3 x(-\sin x) \\ &= -4\sin x\cos x\,(\cos^2 x - \sin^2 x) \\ &= -2\sin 2x\cos 2x = -\sin 4x \end{aligned}$$

For local maximum or minimum, we have

$$f'(x) = 0 \quad \Longrightarrow \quad -\sin 4x = 0 \quad \Longrightarrow \quad 4x = \pi \quad \Longrightarrow \quad x = \frac{\pi}{4} \qquad (\because\ 0 < 4x < 2\pi)$$

Now, $f''(x) = -4\cos 4x$ and $f''\left(\frac{\pi}{4}\right) = 4 > 0$.

By second derivative test for local extrema, $f(x)$ has a local minimum at $x = \frac{\pi}{4}$ and the local minimum value is

$f\left(\frac{\pi}{4}\right) = \sin^4\left(\frac{\pi}{4}\right) + \cos^4\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$

IP4: Show that $\boldsymbol{f(x) = \left(k - \frac{1}{k} - x\right)(4 - 3x^2)}$, where $\boldsymbol{k}$ is a positive constant, has one and only one maximum value and one and only minimum value and their difference is $\boldsymbol{\frac{4}{9}\left(k + \frac{1}{k}\right)^3}$.

Solution:

We have, $f(x) = \left(k - \frac{1}{k} - x\right)(4 - 3x^2)$.

$$f'(x) = \left(k - \frac{1}{k} - x\right)(-6x) - (4 - 3x^2)$$
$$= 9x^2 - 6\left(k - \frac{1}{k}\right)x - 4$$
$$f''(x) = 18x - 6\left(k - \frac{1}{k}\right)$$

Now, $f'(x) = 0 \implies 9x^2 - 6\left(k - \frac{1}{k}\right)x - 4 = 0$

$$\implies (3x - 2k)\left(3x + \frac{2}{k}\right) = 0 \implies x = \frac{2k}{3}, \frac{-2}{3k}$$

Critical points are $= \frac{2k}{3}, \frac{-2}{3k}$.

$(i)\ f''\left(\frac{2k}{3}\right) = 18\left(\frac{2k}{3}\right) - 6\left(k - \frac{1}{k}\right) = 6k + \frac{6}{k} > 0$

$\therefore$ At $x = \frac{2k}{3}$, $f(x)$ has minimum.

Minimum value $= f\left(\frac{2k}{3}\right) = \left(k - \frac{1}{k} - \frac{2k}{3}\right)\left(4 - 3\left(\frac{2k}{3}\right)^2\right)$

$$= \left(\frac{k}{3} - \frac{1}{k}\right)\left(4 - \frac{4k^2}{3}\right)$$

$(ii)\ f''\left(-\frac{2}{3k}\right) = 18\left(-\frac{2}{3k}\right) - 6\left(k - \frac{1}{k}\right) = -6\left(k + \frac{1}{k}\right) < 0$

$\therefore$ At $= -\frac{2}{3k}$, $f(x)$ has maximum.

Maximum value $= f\left(-\frac{2}{3k}\right) = \left(k - \frac{1}{k} + \frac{2}{3k}\right)\left(4 - 3\left(-\frac{2}{3k}\right)^2\right)$

$$= \left(k - \frac{1}{3k}\right)\left(4 - \frac{4}{3k^2}\right)$$

Difference between maximum and minimum values is

$$= \left(k - \frac{1}{3k}\right)\left(4 - \frac{4}{3k^2}\right) - \left(\frac{k}{3} - \frac{1}{k}\right)\left(4 - \frac{4k^2}{3}\right)$$
$$= 4\left[\left(k - \frac{1}{3k}\right)\left(1 - \frac{1}{3k^2}\right) - \left(\frac{k}{3} - \frac{1}{k}\right)\left(1 - \frac{k^2}{3}\right)\right]$$
$$= 4\left[k - \frac{1}{3k} - \frac{1}{3k} + \frac{1}{9k^3} - \frac{k}{3} + \frac{k^3}{9} + \frac{1}{k} - \frac{k}{3}\right]$$
$$= \frac{4}{9}\left(k^3 + 3k + \frac{3}{k} + \frac{4}{k^3}\right)$$
$$= \frac{4}{9}\left(k + \frac{1}{k}\right)^3$$

P4: Show that the function $\boldsymbol{f}$ defined by $\boldsymbol{f(x) = x^p(1 - x)^q}$, $\forall\, \boldsymbol{x} \in \boldsymbol{R}$ where $\boldsymbol{p, q}$ are positive integers has a maximum value for $\boldsymbol{x = \frac{p}{p+q}}$ for all $\boldsymbol{p, q}$.

Solution: We have, $f(x) = x^p(1 - x)^q \quad \forall\, x \in \boldsymbol{R}$

$$f'(x) = px^{p-1}(1 - x)^q - qx^p(1 - x)^{q-1}$$
$$= x^{p-1}(1 - x)^{q-1}\big(p - x(p + q)\big)$$

$$f'(x) = 0 \quad \Rightarrow x = 0, 1, \frac{p}{p+q}$$

Again,

$$f''(x) = p(p-1)x^{p-2}(1-x)^q - pqx^{p-1}(1-x)^{q-1} - \ x^{p-1}(1-x)^{q-1}$$
$$+q(q-1)x^p(1-x)^{q-2}$$
$$\Rightarrow f''(x) = (p-1)x^{p-2}(1-x)^q - 2pqx^{p-1}(1-x)^{q-1}$$
$$+\ q(q-1)x^p(1-x)^{q-2}$$
$$\Rightarrow \ f''(x) = x^{p-1}(1-x)^{q-1}\left[p(p-1)\left(\frac{1-x}{x}\right) - 2pq + \ q(q-1)\left(\frac{x}{1-x}\right)\right]$$

At $x = \frac{p}{p+q}$, we have $1 - x = \frac{q}{p+q}$ and $\frac{1-x}{x} = \frac{q}{p}$.

$$f''\left(\frac{p}{p+q}\right) = \left(\frac{p}{p+q}\right)^{p-1}\left(\frac{q}{p+q}\right)^{q-1}\left[p(p-1)\left(\frac{q}{p}\right) - 2pq + q(q-1)\left(\frac{p}{q}\right)\right]$$
$$= \left(\frac{p}{p+q}\right)^{p-1}\left(\frac{q}{p+q}\right)^{q-1}[q(p-1) - 2pq + p(q-1)]$$
$$= -(p+q)\left(\frac{p}{p+q}\right)^{p-1}\left(\frac{q}{p+q}\right)^{q-1} < 0 \qquad (\because p, q > 0)$$

Thus, the given function is maximum at $x = \frac{p}{p+q}$ for all p, q and the maximum value is $\left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q = \frac{p^p . q^q}{(p+q)^{p+q}}$.

Exercises:

1. Find the points at which the local maxima or local minima are attained for the following functions. Also find the local maximum or local minimum values, as the case may be.
   a. $x^3 - 3x$
   b. $x^3 - 6x^2 + 9x + 15$
   c. $(x-1)(x+2)^2$
   d. $\frac{x}{2} + \frac{2}{x}$, $(x > 0)$
   e. $\frac{1}{x^2+2}$
   f. $x\sqrt{1-x}$ $(0 < x < 1)$
   g. $-(x-1)^3(x+1)^2$
2. Find all the local maxima and local minima of the sine function.

6.9. Strategy for Graphing

Learning objectives:

- To study the cusps of a continuous function.
- To present a general strategy for sketching the graph of a function.

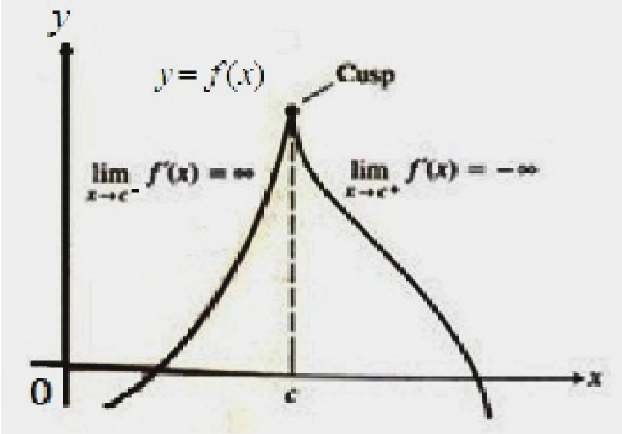
  And
- To sketch the graphs of certain functions using graphing strategy.

In this module we study a general procedure to sketch the graph of a function. We first study the possible cusps of the graph of a continuous function.

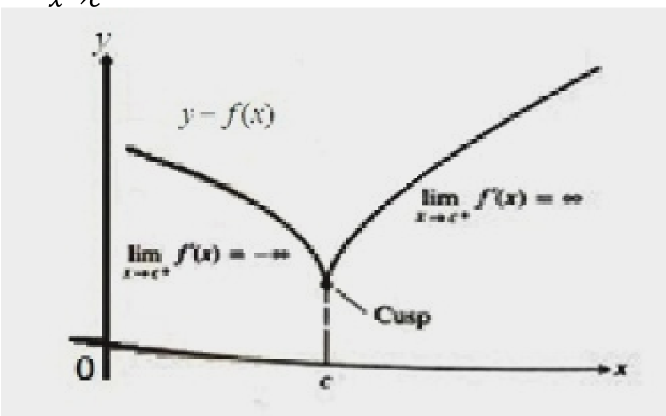
Cusps

The graph of a continuous function $y = f(x)$ has a cusp at a point $x = c$ if the concavity is the same on both sides of c and either

1. $\lim_{x \to c^-} f'(x) = \infty$ and $\lim_{x \to c^+} f'(x) = -\infty$



2. $\lim_{x \to c^-} f'(x) = -\infty$ and $\lim_{x \to c^+} f'(x) = \infty$



A cusp can be either a local maximum as in (1) or a local minimum as in (2).
To study the shape of the graph of $y = f(x)$, we compute f' and f''. These two together reveal the shape of the curve $= f(x)$, $i.e.$, the presence of critical points, increasing and decreasing nature of the curve and its concavity. We use this information to sketch the graph of $y = f(x)$.
A general strategy for sketching the graph of $y = f(x)$ is given below:

1. Find y' and y''.
2. Find the rise and fall of the curve.
3. Determine the concavity of the curve.
4. Make a summary and show the curve's general shape.
5. Plot specific points and sketch the curve.

Example 1: Sketch the graph of $y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$.
Solution: The domain of $\boldsymbol{f}$ is $(-\infty, \infty)$ and it is continuous on it. There are no symmetries about either axes or the origin.
Step 1: Find y' and y''

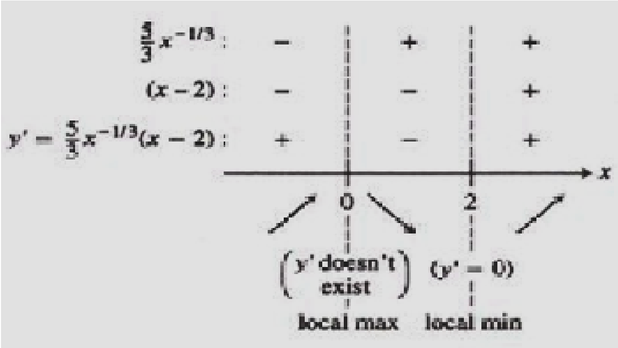
$y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}} = x^{\frac{2}{3}}(x - 5)$

$y' = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{\frac{-1}{3}} = \frac{5}{3}x^{\frac{-1}{3}}(x - 2)$

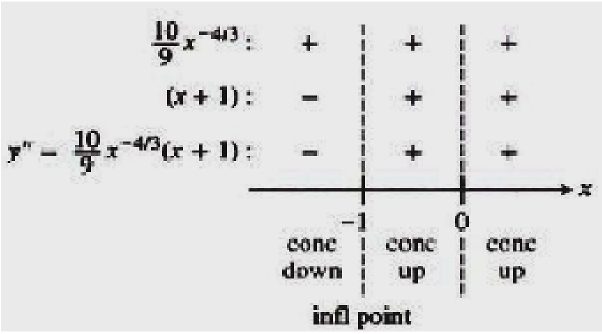
critical points: $x = 0$ (y' undefined), $x = 2$.

$y'' = \frac{10}{9}x^{\frac{-1}{3}} + \frac{10}{9}x^{\frac{-4}{3}} = \frac{10}{9}x^{\frac{-4}{3}}(x + 1)$

Possible inflection points: $x = 0$ (y'' undefined), $x = -1$.
Step 2: *Rise and fall*



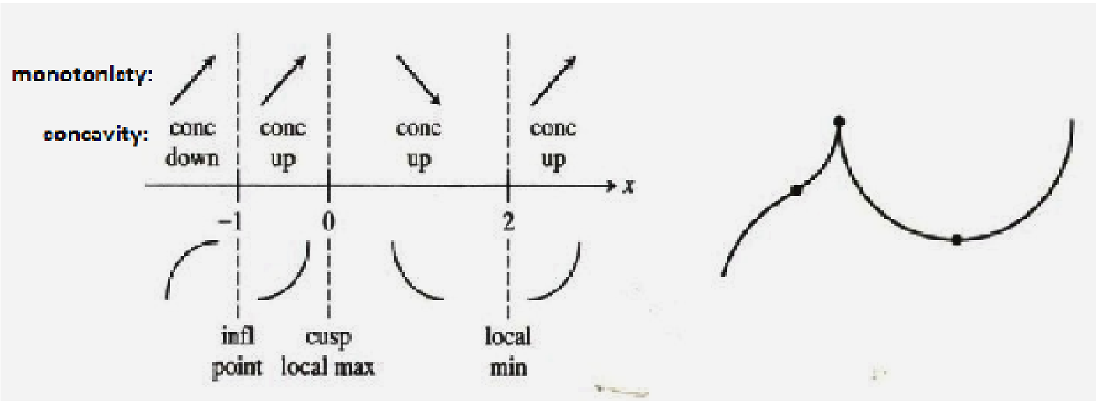
Step 3: *Concavity*



From the sign pattern for y'', we see that there is an inflection point at $x = -1$, but not at $x = 0$. However, knowing that

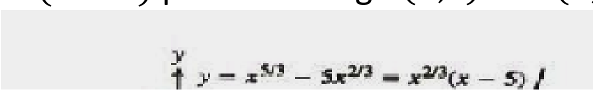
1. the function $y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$ is continuous,
2. $y' \to \infty$ as $x \to 0^-$ and $y' \to -\infty$ as $x \to 0^+$, and
3. the concavity does not change at $x = 0$ tells us that the graph has a *cusp* at $x = 0$.

Step 4: *Summary*



Step 5: *Specific points and sketching the curve*

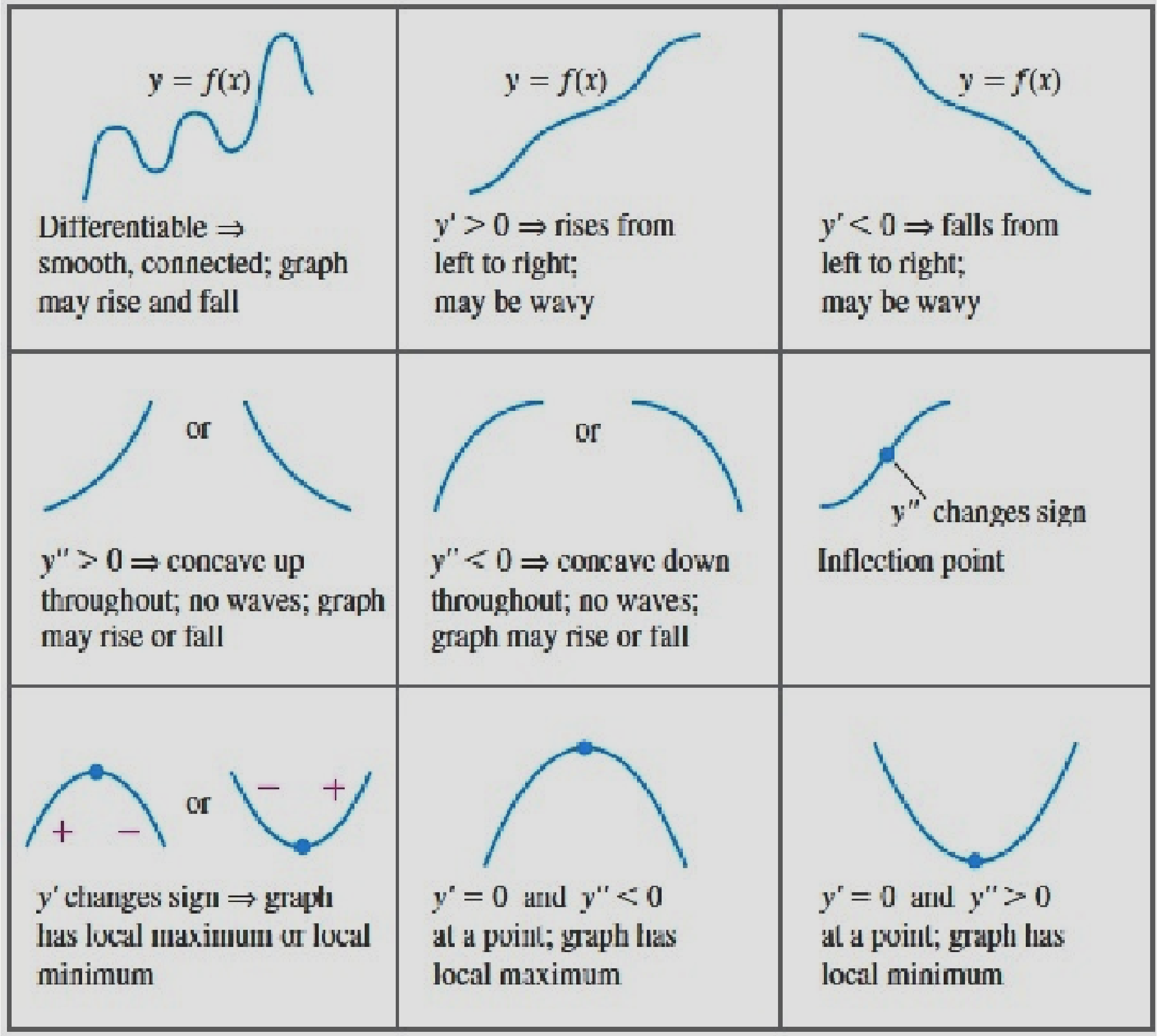
The curve $y = x^{\frac{2}{3}}(x-5)$ passes through $(0,0)$ and $(5,0)$.



Learning About Functions from Derivatives

From the examples, we see that we have been able to recover almost everything we need to know about a differentiable function $y = f(x)$ by examining y'. We can find where the graph rises and falls and where the local extremes are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. We have also learnt that if we know the value of f at a single point in addition to y', we know how to place the graph in the xy-plane.

The following is a summary of what derivatives tell us about graphs of functions.



PROBLEM SET

IP1: Graph the equation $y = x^5 - 5x^4$.

Solution:

The domain of f is $(-\infty, \infty)$ and it is continuous on it. There are no symmetries about either axes or the origin.

Step1:

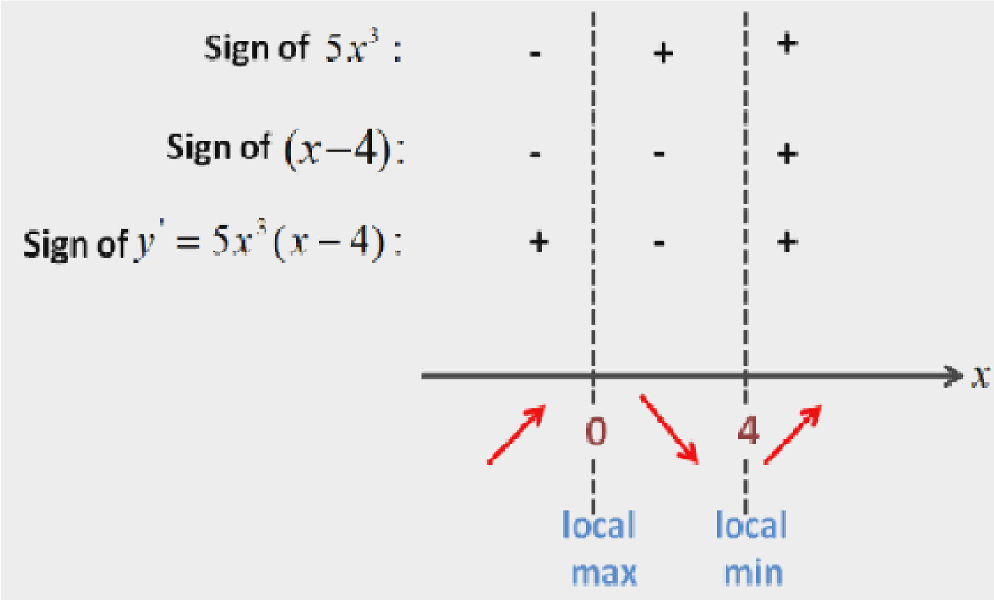
$y' = 5x^4 - 20x^3 = 5x^3(x - 4)$

Critical points are $x = 0, 4$.

$y'' = 20x^3 - 60x^2 = 20x^2(x - 3)$

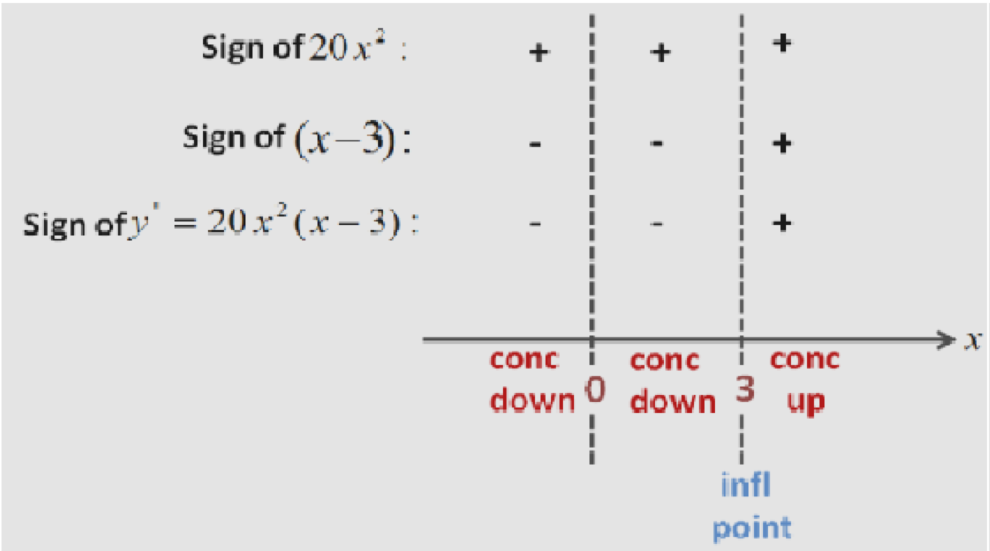
Possible inflection points are $x = 0, 3$.

Step2: *Rise and fall*



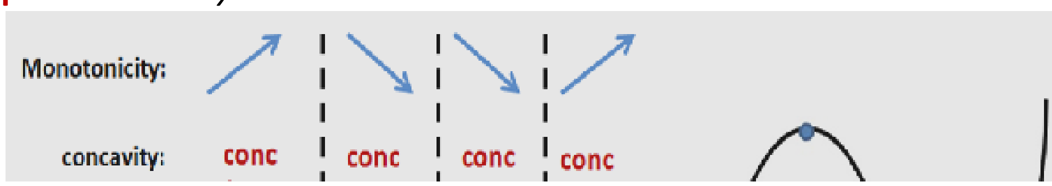
The graph rises on $(-\infty, 0)$ and $(4, \infty)$, falls on $(0,4)$. There is a local maximum at $x = 0$ and there is a local minimum at $x = 4$.

Step3: *concavity*



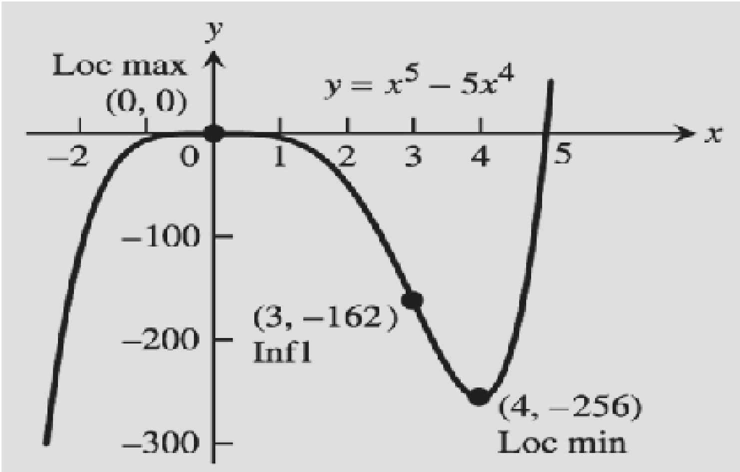
The graph is concave up on$(-\infty, 0)$ and $(0,3)$, concave down on $(3, \infty)$. At $x = 3$ there is a point of inflection.

Step4: *summary*



Step5: *Specific points and sketching the curve*

The curve $y = x^5 - 5x^4$ passes through $(0,0)$ and $(5,0)$.



P1: Graph the function $y = 4x^3 - x^4$.

Solution: The domain of f is $(-\infty, \infty)$ and it is continuous on it. There are no symmetries about either axes or the origin.

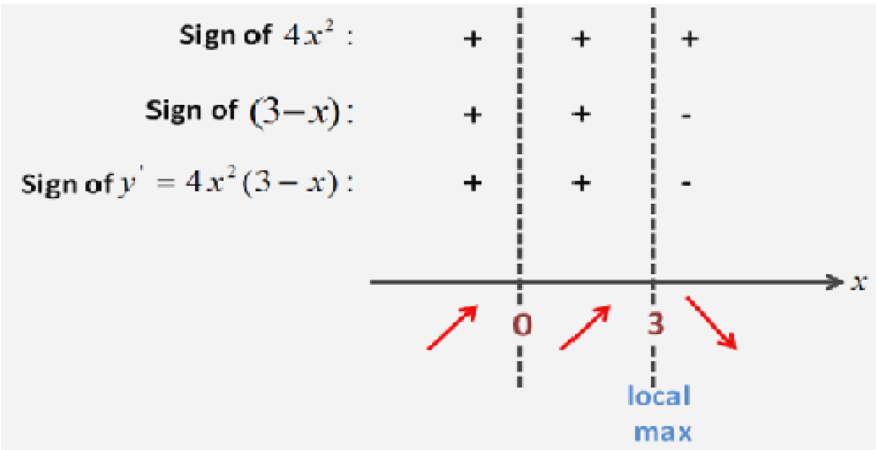
Step1: $y' = 12x^2 - 4x^3 = 4x^2(3 - x)$:

Critical points are $x = 0, 3$.

$y'' = 24x - 12x^2 = 12x(2 - x)$:

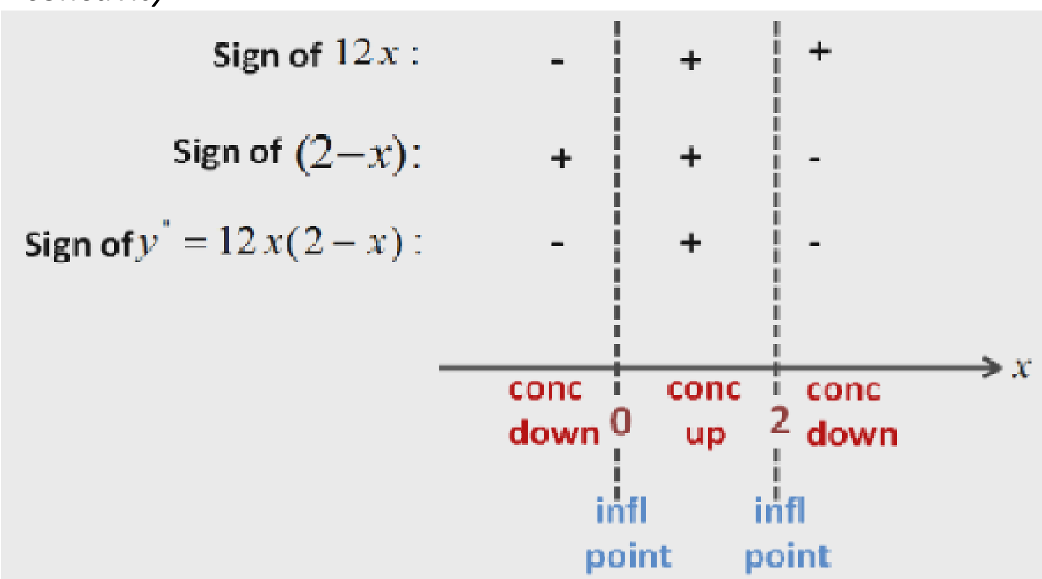
Possible inflection points are at $x = 0, 2$.

Step2: *Rise and fall*



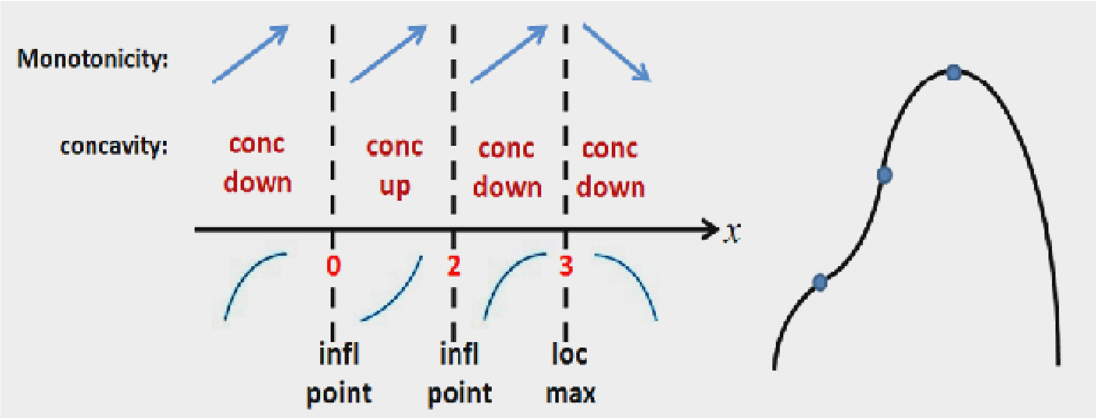
The curve rises on $(-\infty, 0)$ and $(0,3)$, falls on $(3, \infty)$. At $x = 3$ there is a local maximum, but there is no local minimum.

Step3: *concavity*



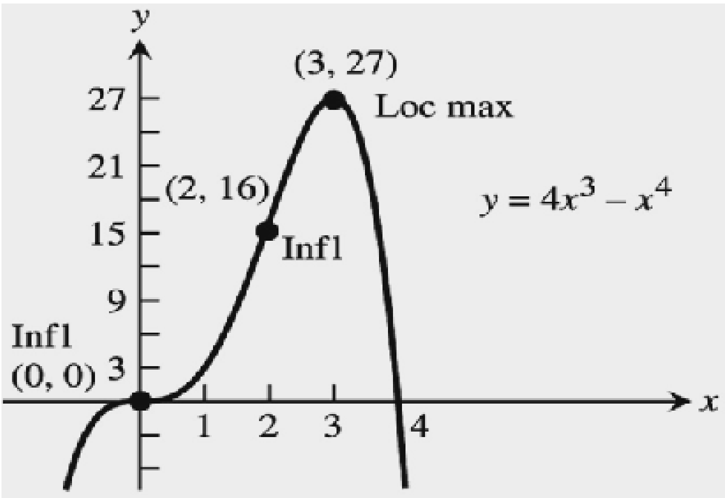
The graph is concave up on $(0,2)$, concave down on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $x = 0$ and $x = 2$.

Step4: *summary*



Step5: *Specific points and sketching the curve*

The curve $y = 4x^3 - x^4$ is passes through $(0,0)$ and $(4,0)$.



IP2: Graph the function $y = x^{2/5}$.

Solution:

Given $y = f(x) = x^{2/5}$.

The domain of f is $(-\infty, \infty)$ and it is continuous on it. Since f is even function of x, its graph is symmetric with respect to the y-axis.

Step1:

$y' = \frac{2}{5}x^{-3/5}$:

Critical point is $x = 0$ (y' is undefined).

$y'' = \frac{2}{5}\left(\frac{-3}{5}x^{-8/5}\right) = -\frac{6}{25}x^{-8/5}$

Possible inflection point at $x = 0$ (y'' is undefined).

Step2: *Rise and fall*

For $-\infty < x < 0$, we have $y' = \frac{2}{5}x^{-3/5} < 0$.

For $0 < x < \infty$, we have $y'' = \frac{2}{5}x^{-3/5} > 0$.

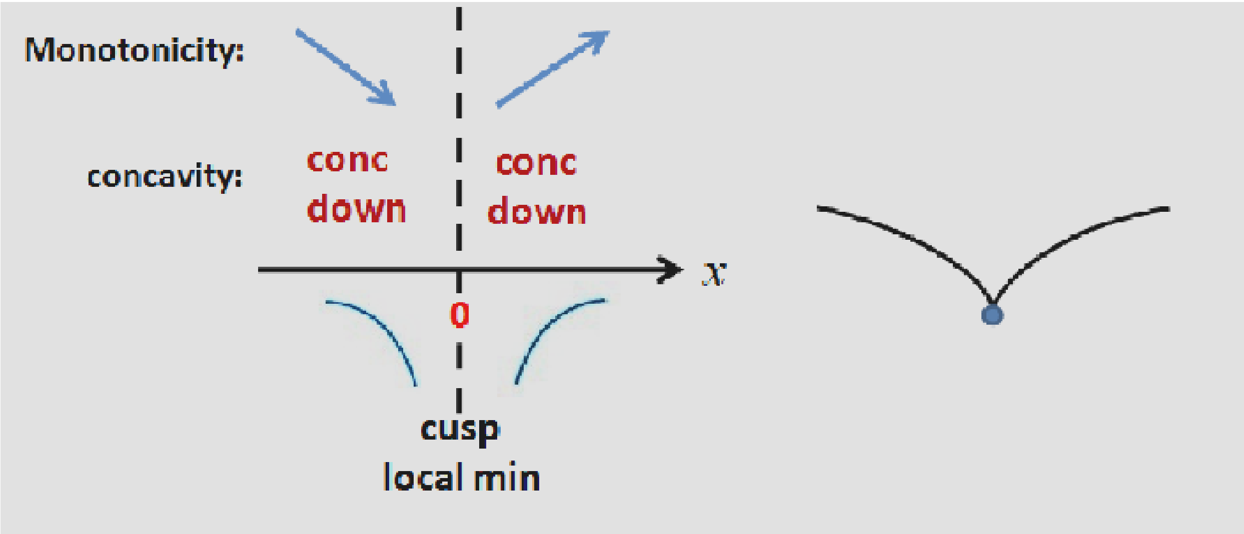
The graph is rises on $(0, \infty)$ and falls on $(-\infty, 0)$.

There is a local minimum at $x = 0$ but there is no local maximum.

Step2: *concavity*

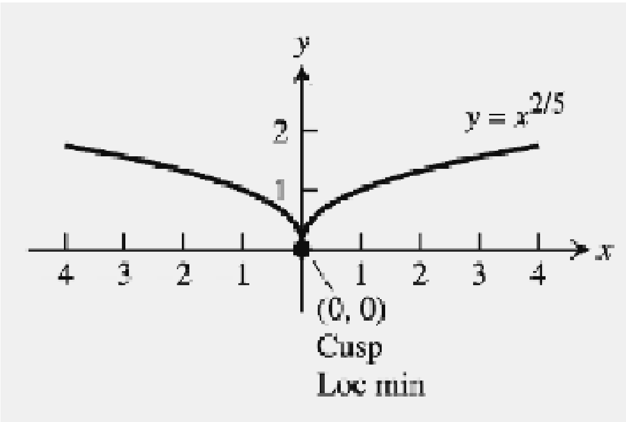
Notice that $y' = -\frac{6}{25x^{8/5}} < 0$ for $x \in (-\infty, 0)$ and $x \in (0, \infty)$.
The graph is concave down on $(-\infty, 0)$ and $(0, \infty)$.
The concavity does not changes at $x = 0$ and $y' \to -\infty$ as $\to 0^-$, $y' \to \infty$ as $x \to 0^+$ tells us that the graph has a cusp at $x = 0$.

Step3: *summary*



Step4: *specific points and sketching the curve*

The curve passes through $(0,0)$.



P2: Graph the function $y = x^{3/5}$.

Solution:

Given function is: $y = f(x) = x^{3/5}$.
The domain of f is $(-\infty, \infty)$ and it is continuous on it. Since f is odd function of x, its graph is symmetric with respect to the origin.

Step1:

$y' = \frac{3}{5}x^{-2/5}$

Critical point is $x = 0$ (y' is undefined).

$y'' = \frac{3}{5}\left(\frac{-2}{5}x^{-7/5}\right) = -\frac{6}{25}x^{-7/5}$

Possible inflection point at $x = 0$ (y'' is undefined).

Step2: *Rise and fall*

Notice that $y' = \frac{3}{5x^{2/5}} > 0$ for $x \in (-\infty, 0)$ and $x \in (0, \infty)$

The graph is rises on $(-\infty, 0) \cup (0, \infty)$.

There is no local maximum and local minimum.

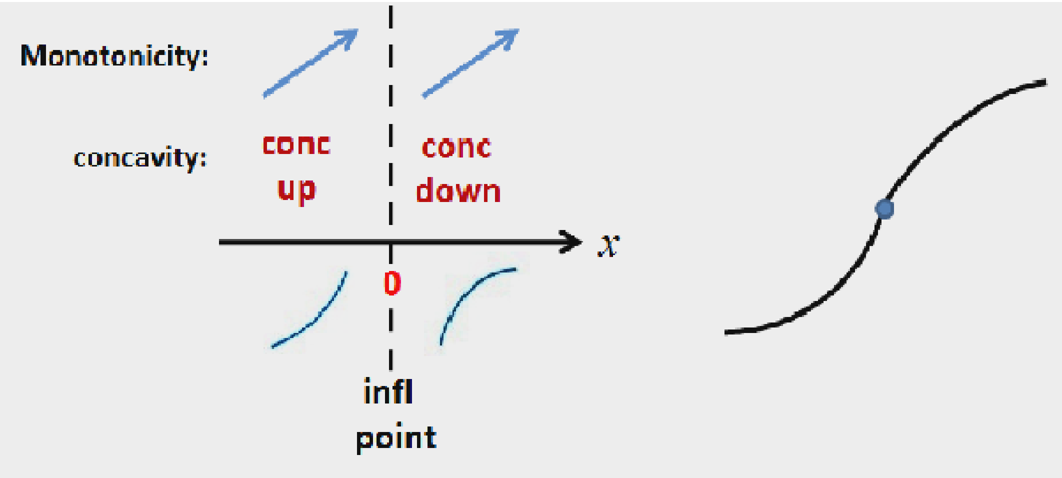
Step3: *concavity*

For $-\infty < x < 0$, we have $y'' = -\frac{6}{25}\, x^{-7/5} > 0$.

For $0 < x < \infty$, we have $y'' = -\frac{6}{25}\, x^{-7/5} < 0$.

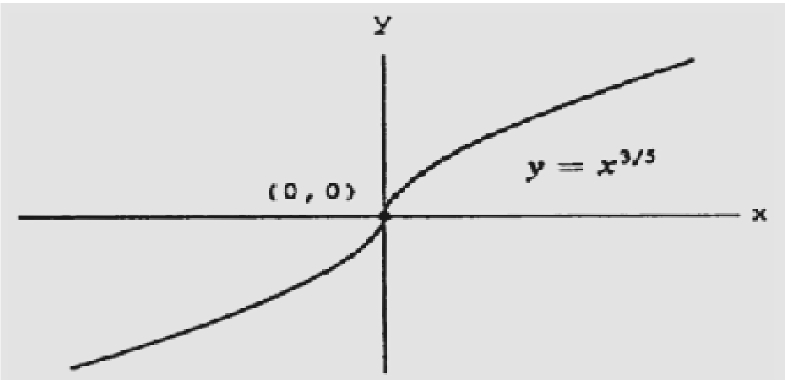
The graph is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. At $x = 0$ there is a point of inflection.

Step4: *summary*



Step5: *Specific points and sketching the curve*

The curve passes through $(0,0)$.



IP3: Graph the function $y = x^{2/3}\left(\frac{5}{2} - x\right)$.

Solution: Given function is: $y = x^{2/3}\left(\frac{5}{2} - x\right) = \frac{5}{2}x^{2/3} - x^{5/3}$

The domain of f is $(-\infty, \infty)$ and it is continuous on it. There are no symmetries about either axes or the origin.

Step1:

$y' = \frac{5}{2}\left(\frac{2}{3}x^{-1/3}\right) - \frac{5}{3}x^{2/3} = \frac{5}{3}x^{-1/3}(1 - x)$:

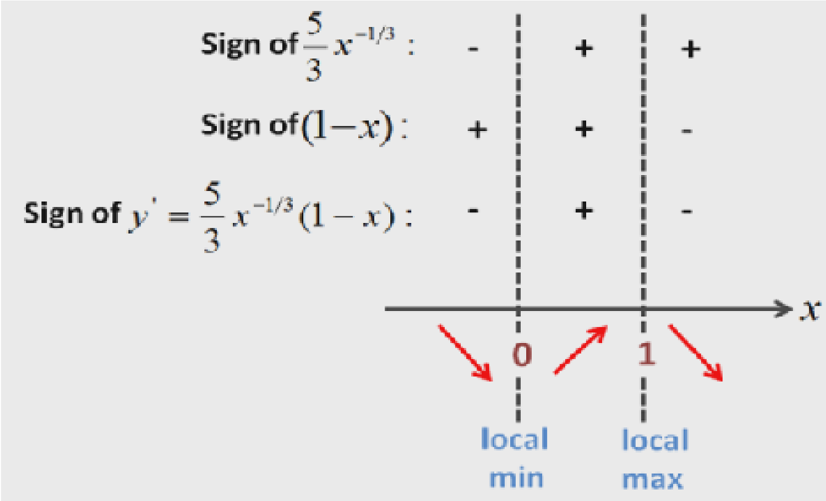
Critical points are $x = 0$ (y'undifend), $x = 1$.

$y'' = \frac{5}{3}\left(-\frac{1}{3}x^{-4/3}\right) - \frac{5}{3}\left(\frac{2}{3}x^{-1/3}\right) = -\frac{5}{9}x^{-4/3} - \frac{10}{9}x^{-1/3}$

$= -\frac{5}{9}x^{-4/3}(1 + 2x)$

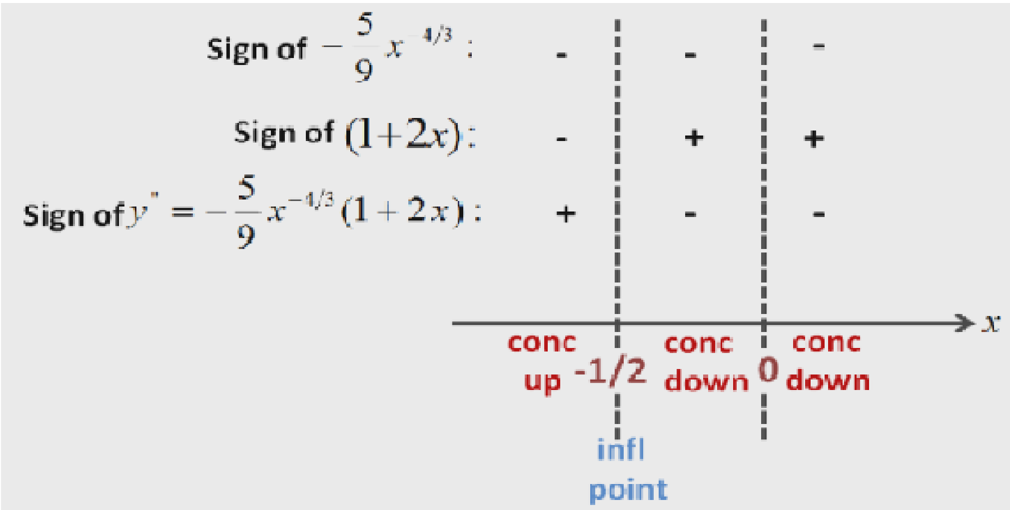
Possible inflection points at $x = 0$ (y''undifend), $x = -\frac{1}{2}$.

Step2: *Rise and fall*



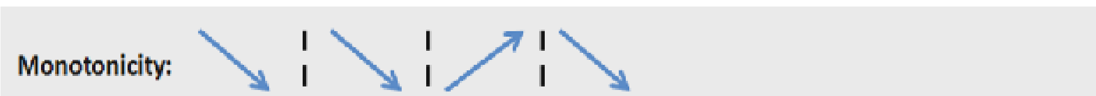
The graph is rises on $(0,1)$, falls on $(-\infty,0)$ and $(1,\infty)$.
There is a local minimum at $x = 0$ and local maximum at $x = 1$.

Step3: *concavity*



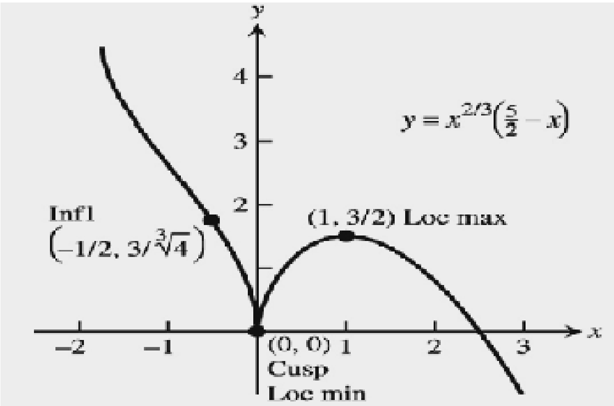
The graph is concave up on $\left(-\infty,-\frac{1}{2}\right)$, concave down on $(-\frac{1}{2},0)$ and $(0,\infty)$.
At $x = -\frac{1}{2}$ there is a point of inflection.
The concavity does not changes at $x = 0$ and $y' \to \infty$ as $x \to 0^-$, $y' \to -\infty$ as $x \to 0^+$ tells us that the graph has a cusp at $x = 0$.

Step4: *summary*



Step5: *specific points and sketching the curve*

The curve $y = x^{2/3}\left(\frac{5}{2} - x\right)$ passes through $(0,0)$ and $\left(\frac{5}{2}, 0\right)$.



P3: Graph the function $y = 2x - 3x^{2/3}$.

Solution:

The domain of f is $(-\infty, \infty)$ and it is continuous on it. There are no symmetries about either axes or the origin.

Step1:

$y' = 2 - 3\left(\frac{2}{3}x^{-1/3}\right) = 2 - 2x^{-1/3} = \frac{2}{x^{1/3}}\left(x^{1/3} - 1\right)$

Critical points are $x = 0$ (y' undefined), $x = 1$.

$y'' = -2\left(\frac{-1}{3}x^{-4/3}\right) = \frac{2}{3}x^{-4/3}$

Possible inflection point is at $x = 0$ (y'' undefined).

Step2: *Rise and fall*

When $-\infty < x < 0$, we have $y' = \frac{2}{x^{1/3}}\left(x^{1/3} - 1\right) > 0$

When $0 < x < 1$, we have $y' = \frac{2}{x^{1/3}}\left(x^{1/3} - 1\right) < 0$

When $1 < x < \infty$, we have $y' = \frac{2}{x^{1/3}}\left(x^{1/3} - 1\right) > 0$

The graph is rises on $(-\infty, 0)$ and $(1, \infty)$, falls on $(0,1)$.

There is a local maximum at $x = 0$ and local minimum at $x = 1$.

Step3: *concavity*

Notice that $y'' = \frac{2}{3}x^{-4/3} > 0$ for $x \in (-\infty, 0)$ and $x \in (0, \infty)$.

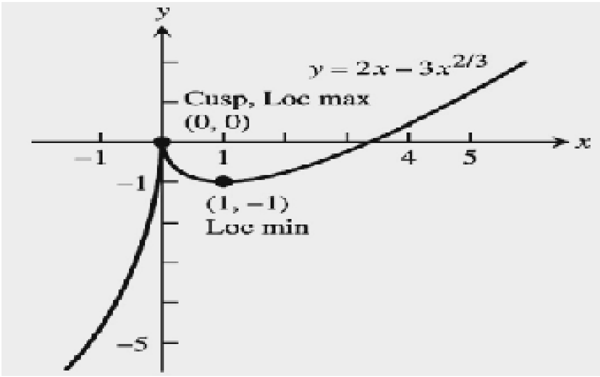
Therefore, the graph is concave up on $(-\infty, 0)$ and $(0, \infty)$.

The concavity does not changes at $x = 0$ and $y' \to \infty$ as $x \to 0^-$, $y' \to -\infty$ as $x \to 0^+$ tells us that the graph has a cusp at $x = 0$.

Step4: *summary*

Step5: *specific points and sketching the curve*

The curve $y = 2x - 3x^{2/3}$ passes through $(0,0)$ and $\left(\frac{27}{8}, 0\right)$.



IP4: Graph the function $y = (2 - x^2)^{3/2}$.

Solution: Given function is: $y = f(x) = (2 - x^2)^{3/2}$.

The domain of f is $(-\sqrt{2}, \sqrt{2})$ and it is continuous on it. Since f is even function of x, its graph is symmetric with respect to the y-axis.

Step1:

$$y' = \frac{3}{2}(2 - x^2)^{1/2}(-2x) = -3x\sqrt{(\sqrt{2} + x)(\sqrt{2} - x)}$$

Critical point is at $x = 0$.

$$y'' = -3(2 - x^2)^{1/2} + (-3x)\left(\frac{1}{2}(2 - x^2)^{-1/2}(-2x)\right)$$

$$= (2 - x^2)^{-1/2}(-3(2 - x^2) + 3x^2) = \frac{6(x+1)(x-1)}{\sqrt{(\sqrt{2}+x)(\sqrt{2}-x)}}$$

Possible points of inflection are at $x = \pm 1$.

Step2: *Rise and fall*

When $-\sqrt{2} < x < 0$, we have $y' > 0$.

When $0 < x < \sqrt{2}$, we have $y' < 0$.

The curve is rises on $(-\sqrt{2}, 0)$, falls on $(0, \sqrt{2})$.

At $x = 0$, there is a local maximum.

Step3: *concavity*

When $-\sqrt{2} < x < -1$, we have $y'' > 0$.

When $-1 < x < 1$, we have $y'' < 0$.

When $1 < x < \sqrt{2}$, we have $y'' > 0$.

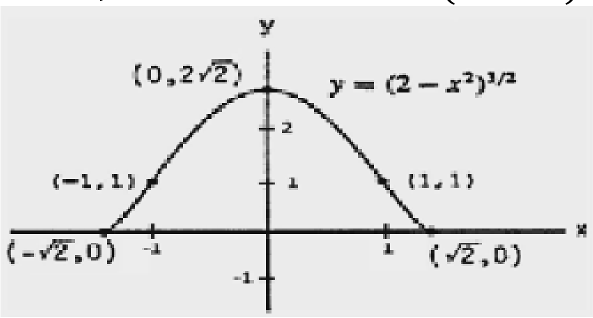
The curve is concave up on $(-\sqrt{2}, -1)$ and $(1, \sqrt{2})$, concave down on $(-1,1)$.

There are inflection points at $x = \pm 1$.

Step4: *summery*

Step5: *specific points and sketching the curve*

The curve $y = (2 - x^2)^{3/2}$ is passes through $(-\sqrt{2}, 0), (\sqrt{2}, 0)$ and $(0, 2\sqrt{2})$.



P4: Graph the function $y = \frac{x^3}{3x^2+1}$.

Solution:

Given function is: $y = f(x) = \frac{x^3}{3x^2+1}$.

The domain of f is $(-\infty, \infty)$ and it is continuous on it. Since f is odd function of x, its graph is symmetric with respect to the origin.

Step1:

$$y' = \frac{3x^2(3x^2+1) - x^3(6x)}{(3x^2+1)^2} = \frac{3x^2(x^2+1)}{(3x^2+1)^2}$$

Critical point at $x = 0$.

$$y'' = \frac{(12x^3+6x)(3x^2+1)^2 - 2(3x^2+1)(6x)(3x^4+3x^2)}{(3x^2+1)^4}$$

$$= \frac{(3x^2+1)\left(36x^5+12x^3+18x^3+6x-36x^5-36x^3\right)}{(3x^2+1)^4}$$

$$= \frac{-6x^3+6x}{(3x^2+1)^3} = \frac{6x(1-x)(1+x)}{(3x^2+1)^3}$$

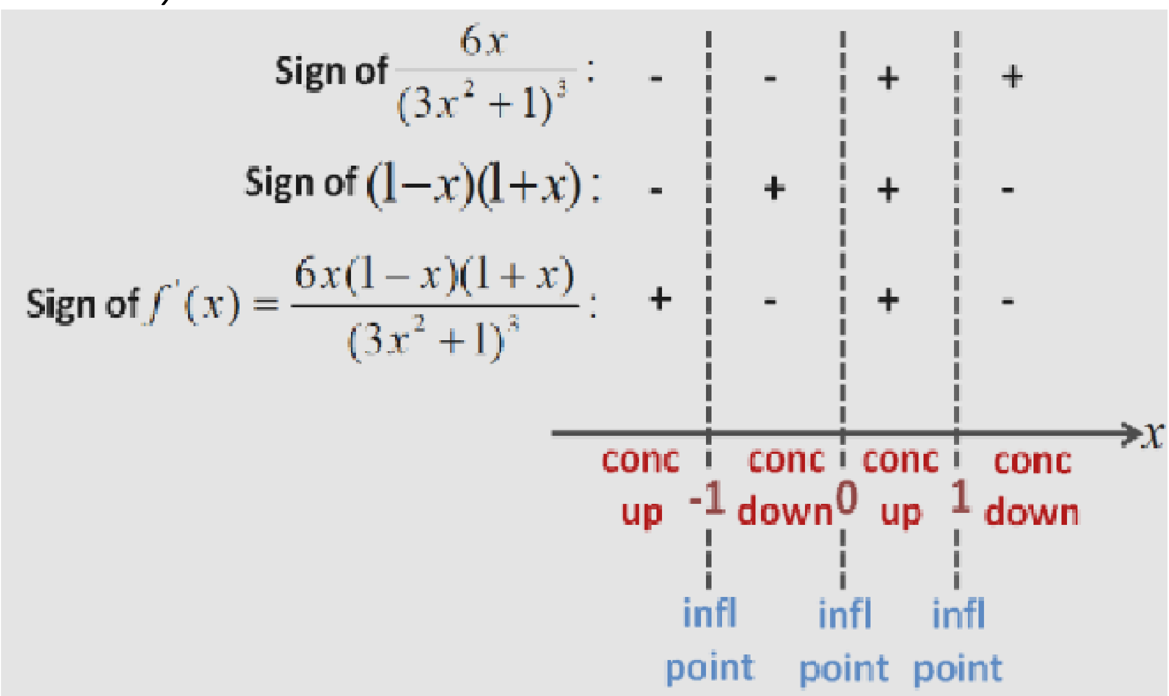
Possible inflection points are at $x = 0, \pm 1$.

Step2: *Rise and fall*

Notice that $y' = \frac{3x^2(x^2+1)}{(3x^2+1)^2} > 0$ for $x \,\epsilon\, (-\infty, 0)$ and $x \,\epsilon\, (0, \infty)$.

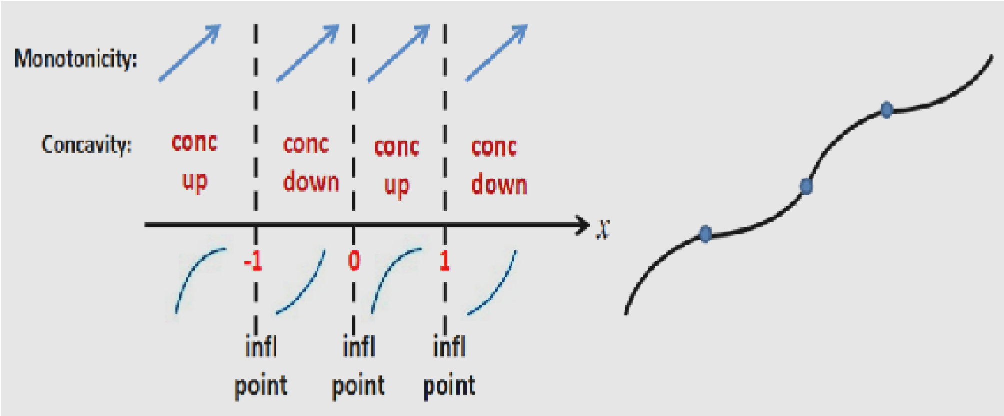
∴ The graph is rises on $(-\infty, 0) \cup (0, \infty)$ so there are no local extrema.

Step3: *concavity*



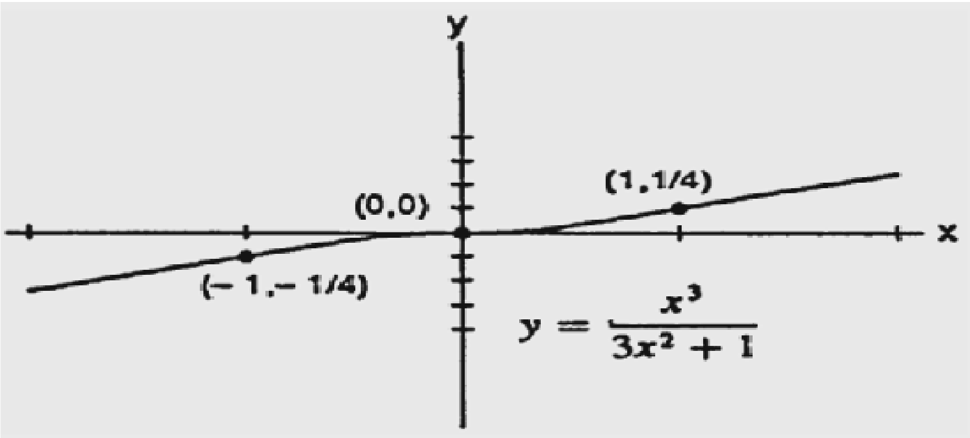
The graph is concave up on $(-\infty,-1)$ and $(0,1)$, concave down on $(-1,0)$ and $(1,\infty)$. The point of inflections are at $x = 0, \pm 1$.

Step4: *summary*



Step5: *specific points and sketching the curve*

The curve $y = \frac{x^3}{3x^2+1}$ is passes through $(0,0)$.



Exercises:

Graph the equations in problems $1-15$. Include the coordinates of any local extreme points and inflection points.

1. $y = x^2 - 4x + 3$
2. $y = x^3 - 3x + 3$
3. $y = -2x^3 + 6x^2 - 3$
4. $y = x^4 - 2x^2 = x^2(x^2 - 2)$
5. $y = (x-2)^3 + 1$
6. $y = 4x^3 - x^4 = x^3(4 - x)$
7. $y = x^5 - 5x^4 = x^4(x - 5)$
8. $y = x + \sin x \qquad 0 \le x \le 2\pi$
9. $y = x^{1/5}$
10. $y = x^{2/5}$
11. $y = 5x^{2/5} - 2x$
12. $y = x\sqrt{8 - x^2}$
13. $y = \frac{x^3-3}{x-2} \qquad x \ne 2$
14. $y = |x^2 - 1|$
15. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x} & x \le 0 \\ \sqrt{x} & x > 0 \end{cases}$

6.10. Asymptotes and Dominant terms

Learning objectives:

- To define the horizontal and vertical asymptotes of the graph of a function and to find them.
- To express a rational function $f(x)$ in dominant terms and to find its oblique (linear) asymptote when the degree of the numerator of $f(x)$ is one greater than the degree of the denominator of $f(x)$.

And

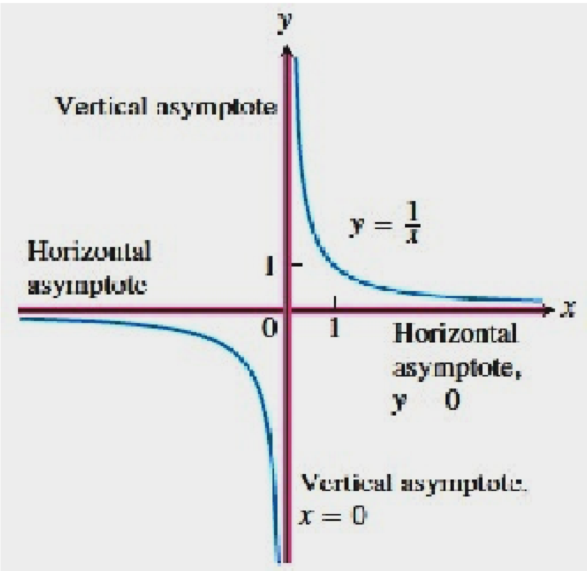
- To practice the related problems.

Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an ***asymptote*** of the graph.

Example 1:

The coordinate axes are asymptotes of the curve $y = \frac{1}{x}$.



The x −axis is an asymptote of the curve on the right because

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

The y −axis is an asymptote of the curve both above and below because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The function is undefined at $x = 0$ since the denominator is zero there.

Definitions

A line $y = b$ is a ***horizontal asymptote*** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b$$

A line $x = a$ is a ***vertical asymptote*** of the graph if either

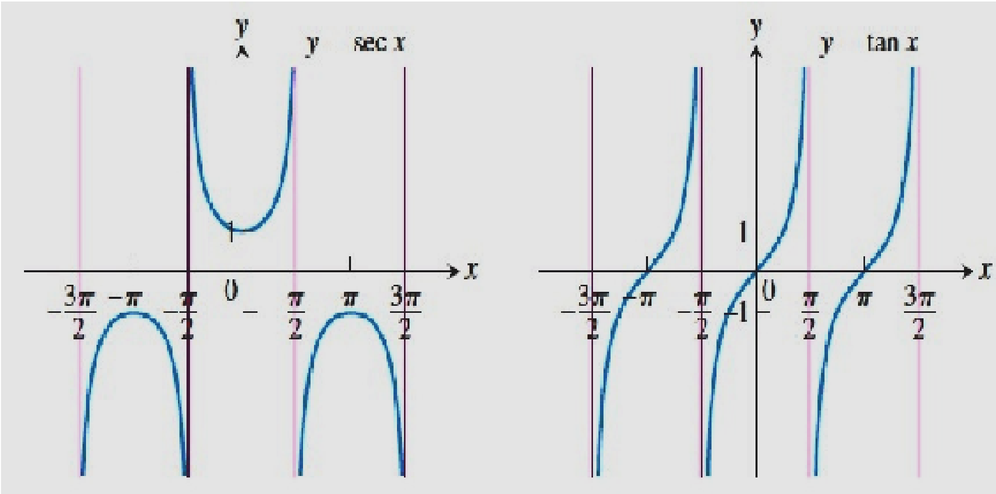
$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Example 2:

The curves

$$y = \sec x = \frac{1}{\cos x} \quad \text{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

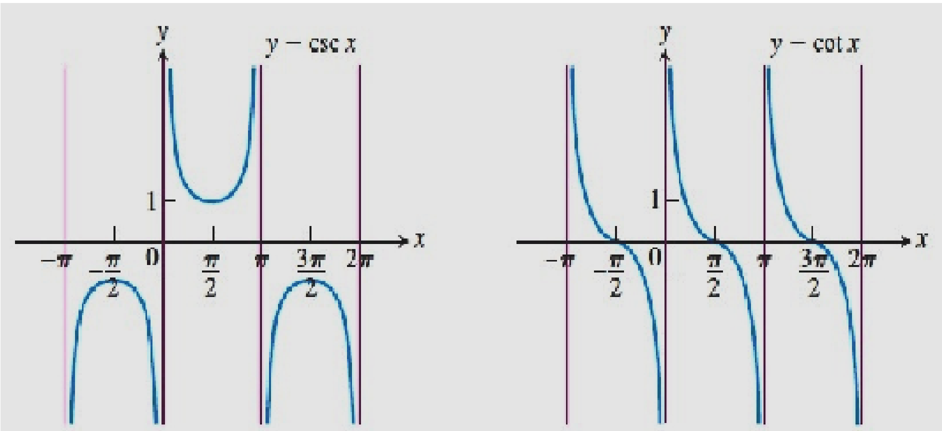
both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$.



The graphs of

$$y = \csc x = \frac{1}{\sin x} \quad \text{and} \quad y = \cot x = \frac{\cos x}{\sin x}$$

both have vertical asymptotes at integer multiples of π, where $\sin x = 0$.



Example 3:

Find the asymptotes of the curve $y = \frac{x+3}{x+2}$.

Solution:

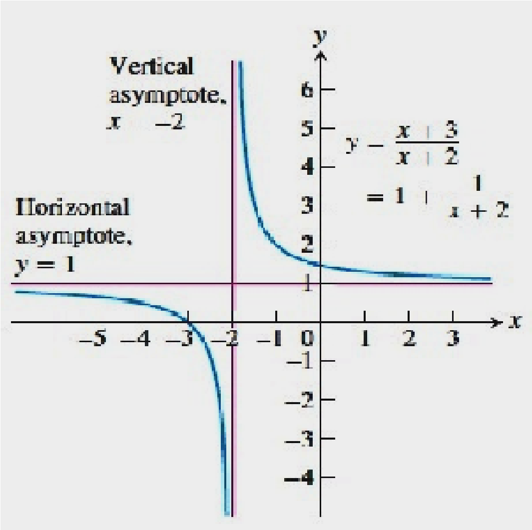
We are interested in the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -2$ where the denominator is zero.

$y = \lim_{x \to \pm\infty} \left(\frac{x+3}{x+2}\right) \Longrightarrow y = 1$

The asymptotes are revealed if we recast the rational function as a polynomial with a remainder, by dividing $x + 3$ by $x + 2$.

This enables us to rewrite y: $\quad y = 1 + \frac{1}{x+2}$

From this we see that the curve in question is the graph of $y = \frac{1}{x}$ shifted 1 unit up and 2 units left.



The asymptotes are the lines $y = 1$ and $x = -2$.

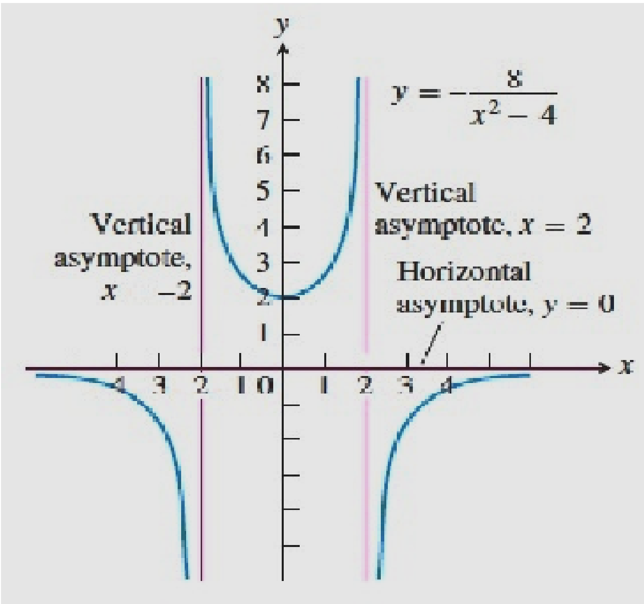
Example 4:

Find the asymptotes of the graph of $f(x) = -\frac{8}{x^2-4}$.

Solution:

We are interested in the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Since f is an even function of x, its graph is symmetric with respect to the y −axis.

Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is an asymptote of the graph to the right. By symmetry it is an asymptote to the left as well.



Since $\lim_{x\to 2^+} f(x) = -\infty$ and $\lim_{x\to 2^-} f(x) = \infty$

the line $x = 2$ is an asymptote both from the right and from the left. By symmetry, the same holds for the line $x = -2$.

There are no other asymptotes because f has a finite limit at every other point.

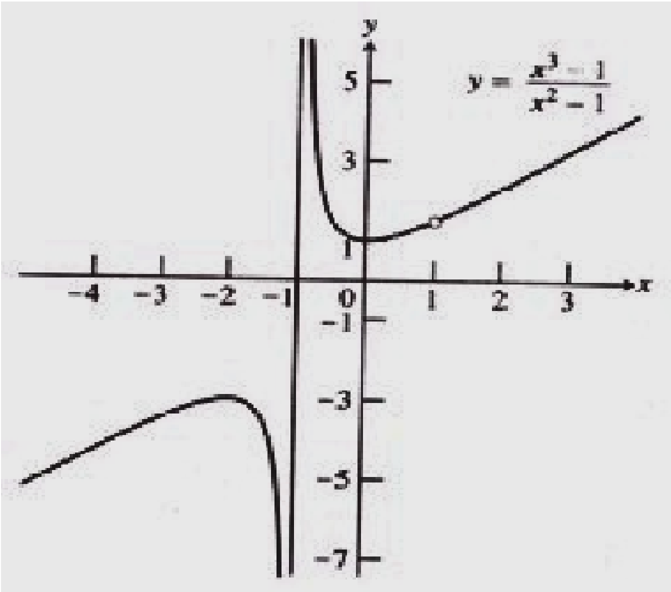
It looks like that rational functions have vertical asymptotes where their denominators are zero. This is not that true. **Rational functions *reduced to lowest terms* have vertical asymptotes where their denominators are zero.**

Example 5:

The graph of $f(x) = \frac{x^3-1}{x^2-1}$ has a vertical asymptote at $x = -1$ but not at $x = 1$. Since

$$\frac{x^3-1}{x^3+1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}$$

the function has a finite limit $(3/2)$ as $x \to 1$ and the discontinuity is removable.



This is an example of a removable discontinuity at a zero of the denominator.
The sandwich theorem also holds for limits as $x \to \pm\infty$. The following is an example of its application.

Example 6:

Find the asymptotes of the curve $y = 2 + \frac{\sin x}{x}$.

Solution:

We are interested in the behavior as $x \to \pm\infty$ and as $x \to 0$, where the denominator is zero.

We know that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, so there is no asymptote at the origin.

Since $0 \leq \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|$ and $\lim\limits_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0$, we have $\lim\limits_{x\to\pm\infty} \frac{\sin x}{x} = 0$, by the Sandwich theorem. Hence,

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2$$

and the line $y = 2$ is an asymptote of the curve on both left and right.

Oblique Asymptotes

If the degree of the numerator of a rational function is *one greater than* the degree of the denominator, then the graph has an ***oblique asymptote***, that is, a *linear asymptote* that is neither vertical nor horizontal.

Example 1:

Find the asymptotes of the graph of $f(x) = \frac{x^2-3}{2x-4}$.

Solution:

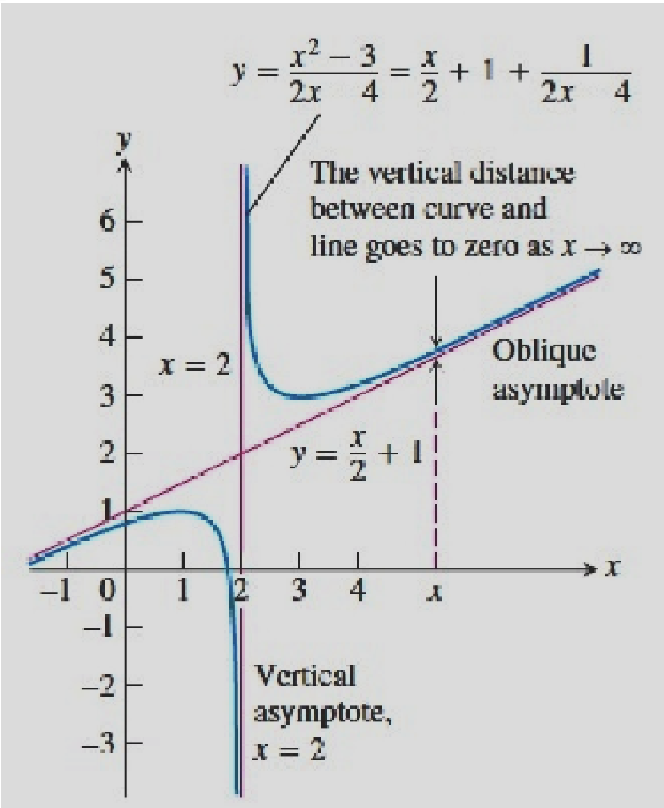
Notice that $f(x)$ is a rational function and the degree of the numerator of $f(x)$ is one greater than the degree of the denominator of $f(x)$.

We are interested in the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to 2$, where the denominator is zero. We divide $x^2 - 3$ by $2x - 4$ and we find

, $f(x) = \frac{x^2-3}{2x-4} = \frac{x}{2} + 1 + \frac{1}{2x-4}$.

Since $\lim_{x\to 2^+} f(x) = \infty$ and $\lim_{x\to 2^-} f(x) = -\infty$, the line $x = 2$ is a two-sided vertical asymptote. As $x \to \pm\infty$, the remainder approaches 0 and $f(x) = \frac{x}{2} + 1$.

The line $y = \frac{x}{2} + 1$ is an oblique asymptote both to the right and to the left.



Dominant Terms

From the previous example $f(x) = \frac{x}{2} + 1 + \frac{1}{2x-4}$, we infer that

$f(x) \approx \frac{x}{2} + 1$ for x numerically large

$f(x) \approx \frac{1}{2x-4}$ for x near 2

The function f behaves like $y = \frac{x}{2} + 1$ when x is numerically large and the contribution of $1/(2x-4)$ to the total value of f is insignificant. It behaves like $1/(2x-4)$ when x is so close to 2 that $1/(2x-4)$ makes the dominant contribution.

We say that $(x/2)+1$ ***dominates*** when x is numerically large, and we say that $1/(2x-4)$ ***dominates*** when x is near 2. ***Dominant terms*** like these are the key to predicting a function's behavior.

PROBLEM SET

IP1: Find the asymptotes of the curve $y=\frac{3x+3}{x-3}$.

Solution:

Let $y=f(x)=\frac{3x+3}{x-3}$. $f(x)$ is a rational function and the denominator is zero at $x=3$.

To find the asymptotes of the curve, we have to study the behavior of $f(x)$ as $x\to\pm\infty$ and as $x\to 3$.

Notice that $\lim_{x\to 3^-} f(x)=-\infty$ and $\lim_{x\to 3^+} f(x)=\infty$.

Therefore, the line $x=3$ is a vertical asymptote of the curve.

Notice that $\lim_{x\to\pm\infty} f(x)=3$.

Therefore, the line $y=3$ is a horizontal asymptote of the curve.

There are no other asymptotes because f has a finite limit at every point.

P1: Find the asymptotes of the curve $y=\frac{2x}{x+1}$.

Solution:

Let $y=f(x)=\frac{2x}{x+1}$. $f(x)$ is a rational function and the denominator is zero at $x=-1$.

To find the asymptotes of the curve, we have to study the behavior of $f(x)$ as $x\to\pm\infty$ and as $x\to -1$.

Notice that $\lim_{x\to -1^-} f(x)=\infty$ and $\lim_{x\to -1^+} f(x)=-\infty$.

Therefore, the line $x=-1$ is a vertical asymptote of the curve.

Notice that $\lim_{x\to\pm\infty} f(x)=2$.

Therefore, the line $y=2$ is a horizontal asymptote of the curve.

There are no other asymptotes because f has a finite limit at every point.

IP2: Find the asymptotes of the curve $y=\frac{x+1}{2x^2+3x-27}$.

Solution: Given $y=f(x)=\frac{x+1}{2x^2+3x-27}=\frac{x+1}{(2x+9)(x-3)}$.

$f(x)$ is a rational function and the denominator is zero at $x=-\frac{9}{2},\ x=3$.

To find the asymptotes of the curve, we have to study the behavior of $f(x)$ as $x\to\pm\infty$ and as $x\to-\frac{9}{2}$, $x\to 3$.

Notice that $\lim_{x\to -\left(\frac{9}{2}\right)^-} f(x)=-\infty$ and $\lim_{x\to -\left(\frac{9}{2}\right)^+} f(x)=\infty$. Therefore, the line $x=-\frac{9}{2}$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to 3^-} f(x)=-\infty$ and $\lim_{x\to 3^+} f(x)=\infty$.

Therefore, the line $x=3$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to\pm\infty} f(x)=0$.

Therefore, the line $y=0$ is a horizontal asymptote of the graph.

There are no other asymptotes because f has a finite limit at every point.

P2: Find the asymptotes of the curve $y = \frac{3x^2+12x-15}{x^2-2x-8}$.

Solution: Given $y = f(x) = \frac{3x^2+12x-15}{x^2-2x-8} = \frac{3(x+4)(x-1)}{(x+2)(x-4)}$.

$f(x)$ is a rational function and the denominator is zero at $x = -2, x = 4$.

To find the asymptotes of the curve, we have to study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -2$, $x \to 4$.

Notice that $\lim_{x\to -2^-} f(x) = -\infty$ and $\lim_{x\to -2^+} f(x) = \infty$.

Therefore, the line $x = -2$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = \infty$.

Therefore, the line $x = 4$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to \pm\infty} f(x) = 3$.

Therefore, the line $y = 3$ is a horizontal asymptote of the graph.

There are no other asymptotes because f has a finite limit at every point.

IP3: Find the asymptotes of the graph of the function $f(x) = \frac{1+x^4}{x^2-x^4}$.

Solution: Given function is $f(x) = \frac{1+x^4}{x^2-x^4} = \frac{1+x^4}{x^2(1-x^2)} = \frac{1+x^4}{x^2(1-x)(1+x)}$.

$f(x)$ is a rational function and the denominator is zero at $x = 0,\ x = \pm 1$.

To find the asymptotes of the curve,

we have to study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to 0,\ x \to \pm 1$.

Since f is an even function of x, its graph is symmetric with respect to the y-axis.

Notice that $\lim_{x\to 0^-} f(x) = \infty$ and $\lim_{x\to 0^+} f(x) = \infty$.

Therefore, the line $x = 0$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to -1^-} f(x) = -\infty$ and $\lim_{x\to -1^+} f(x) = \infty$.

Therefore, the line $x = -1$ is a vertical asymptote of the graph from the right and from the left. By the symmetry, the same holds for the line $x = 1$.

Notice that $\lim_{x\to \infty} f(x) = -1$. Therefore, the line $y = -1$ is a horizontal asymptote of the graph to the right. By the symmetry it is an asymptote to the left as well.

There are no other asymptotes because f has a finite limit at every point.

P3: Find the asymptotes of the graph of the function $f(x) = \frac{2x^2+x-1}{x^2+x-2}$.

Solution: Given function is $f(x) = \frac{2x^2+x-1}{x^2+x-2} = \frac{(2x-1)(x+1)}{(x+2)(x-1)}$.

$f(x)$ is a rational function and the denominator is zero at $x = -2, x = 1$.

To find the asymptotes of the curve, we have to study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -2$, $x \to 1$.

Notice that $\lim_{x\to -2^-} f(x) = \infty$ and $\lim_{x\to -2^+} f(x) = -\infty$.

Therefore, the line $x = -2$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.

Therefore, the line $x = 1$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to \pm\infty} f(x) = 2$. Therefore, the line $y = 2$ is a horizontal asymptote of the graph.

There are no other asymptotes because f has a finite limit at every point.

IP4: Find the asymptotes of the function $f(x) = \frac{2x^2}{x^2-1}$.

Solution:

Given $(x) = \frac{2x^2}{x^2-1}$. $f(x)$ is a rational function and the denominator is zero at $x = \pm 1$.

To find the asymptotes of the curve, we have to study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to \pm 1$.

Since f is an even function of x, its graph is symmetric with respect to the y −axis.

Notice that $\lim_{x\to\infty} f(x) = 2$. Therefore, the line $y = 2$ is an asymptote of the graph to the right. By symmetry it is an asymptote to the left as well.

Notice that $\lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = -\infty$.

Therefore, the line $x = 1$ is an asymptote both from the right and from the left.

By symmetry, the same holds for the line $x = -1$.

There are no other asymptotes because f has a finite limit at every other point.

P4: Find the asymptotes of graph of the $f(x) = \frac{x^2-16}{x^2-2x-8}$.

Solution: We have, $f(x) = \frac{x^2-16}{x^2-2x-8} = \frac{(x+4)(x-4)}{(x+2)(x-4)}$.

Now, $f(x) = \frac{x+4}{x+2}$ (when reduced to lowest terms)

$f(x)$ is a rational function and the denominator is zero at $x = -2$.

To find the asymptotes of the curve, we have to study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -2$.

$x = 4$ is not a vertical asymptote because $f(x)$ has a finite limit $\left(\frac{4}{3}\right)$ as $x \to 4$ and the discontinuity is removable.

Notice that $\lim_{x\to -2^-} f(x) = -\infty$ and $\lim_{x\to -2^+} f(x) = \infty$.

Therefore, the line $x = -2$ is a vertical asymptote of the graph.

Notice that $\lim_{x\to\pm\infty} f(x) = 1$. Therefore, the line $y = 1$ is a horizontal asymptote of the graph.

There are no other asymptotes because f has a finite limit at every point.

IP5: Find the asymptotes of the graph of $f(x) = \frac{x^2+1}{x+1}$.

Solution: Notice that $f(x)$ is a rational function and the degree of the numerator is one greater than the degree of the denominator.

We write the rational function (in dominant terms) as a polynomial plus remainder.

$$f(x) = \frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1}$$

To find the asymptotes, we study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -1$ where the denominator of $f(x)$ is zero.

Since $\lim_{x\to -1^+} f(x) = \infty$ and $\lim_{x\to -1^-} f(x) = -\infty$, the line $x = -1$ is a two sided vertical asymptote.

As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to x - 1$.

Therefore, the line $y = x - 1$ is an oblique asymptote both to the right and to the left.

P5: Find the asymptotes of the graph of $f(x) = \frac{x^2+2x-1}{x}$.

Solution: Notice that $f(x)$ is a rational function and the degree of the numerator is one greater than the degree of the denominator.
We write the rational function (in dominant terms) as a polynomial plus remainder.

$$f(x) = \frac{x^2+2x-1}{x} = (x+2) - \frac{1}{x}$$

To find the asymptotes, we study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to 0$ where the denominator of $f(x)$ is zero.
Since $\lim_{x \to 0^+} f(x) = -\infty$ and $\lim_{x \to 0^-} f(x) = \infty$, the line $x = 0$ is a two sided vertical asymptote.
As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to x + 2$.
Therefore, the line $\boldsymbol{y = x + 2}$ is an oblique asymptote both to the right and to the left.

IP6: Find the asymptotes of the graph of $\boldsymbol{f(x) = \frac{2x^2+x+1}{x+1}}$.

Solution: Notice that $f(x)$ is a rational function and the degree of the numerator is one greater than the degree of the denominator.
We write the rational function (in dominant terms) as a polynomial plus remainder.

$$f(x) = \frac{2x^2+x+1}{x+1} = 2x - 1 + \frac{2}{x+1}$$

To find the asymptotes, we study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -1$ where the denominator of $f(x)$ is zero.
Since $\lim_{x \to -1^+} f(x) = \infty$ and $\lim_{x \to -1^-} f(x) = -\infty$, the line $x = -1$ is a two sided vertical asymptote.
As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to 2x - 1$.
Therefore, the line $y = 2x - 1$ is an oblique asymptote both to the right and to the left.

P6: Find the asymptotes of the graph of $\boldsymbol{f(x) = \frac{x^3}{2(x+1)^2}}$.

Solution: Notice that $f(x)$ is a rational function and the degree of the numerator is one greater than the degree of the denominator.
We write the rational function (in dominant terms) as a polynomial plus remainder.

$$f(x) = \frac{x^3}{2x^2+4x+2} = \frac{1}{2}x - 1 + \frac{3x+2}{2x^2+4x+2}$$

To find the asymptotes, we study the behavior of $f(x)$ as $x \to \pm\infty$ and as $x \to -1$ where the denominator of $f(x)$ is zero.
Since $\lim_{x \to -1^+} f(x) = -\infty$ and $\lim_{x \to -1^-} f(x) = -\infty$, the line $x = -1$ is a vertical asymptote.
As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to \frac{1}{2}x - 1$. Therefore, the line $y = \frac{1}{2}x - 1$ is an oblique asymptote both to the right and left.

Exercises:

Find the asymptotes of the following curves.

1. $y = \frac{1}{x-1}$
2. $y = \frac{1}{2x+4}$
3. $y = \frac{2x^2+x-1}{x^2-1}$
4. $y = \frac{1}{x^2-1}$

6.11. Optimization

Learning objectives:

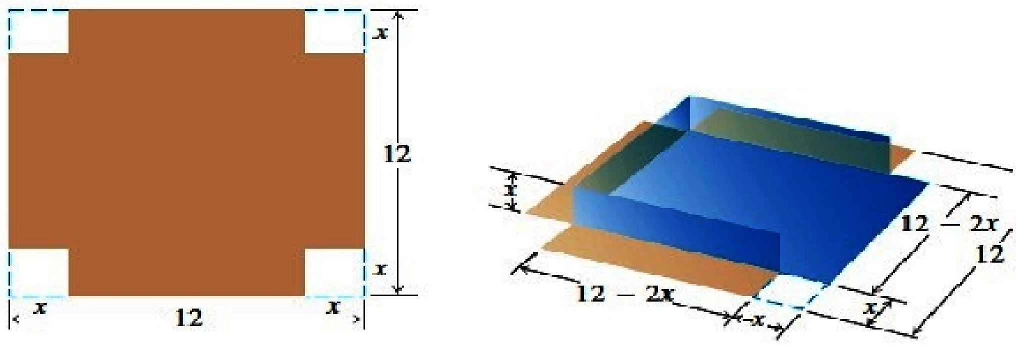
- To discuss and solve a variety of optimization problems.

Optimization is a term used to mean maximize or minimize some aspect of a thing such as the profit of a production run. In the mathematical models, optimization aims at finding the maxima and minima of a differentiable function.

Example 1:

An open box is to be made by cutting small congruent squares from the corners of 12-by-12-cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution:



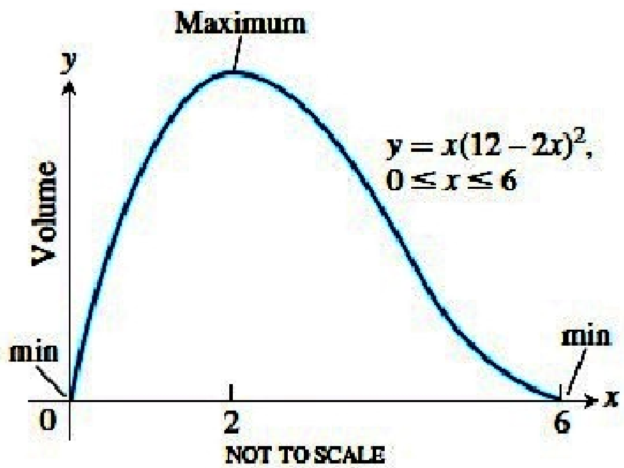
(a) (b)

The corner squares are $x\ cm$ on a side. The volume of the box is a function of this variable:

$$V(x) = x(12-2x)^2 = 144x - 48x^2 + 4x^3$$

Since the sides of the tin sheet are 12 cm long, $x \le 6$.

Thus, the domain of V is the interval $0 \le x \le 6$.



A graph of V suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. We examine the first derivate of V with respect to x:

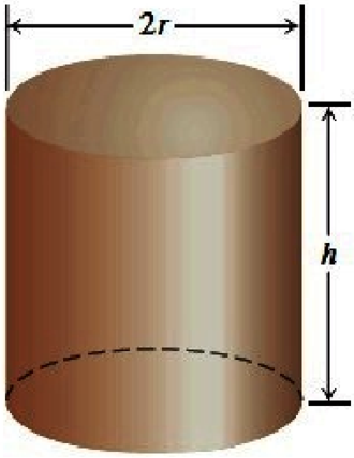
$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2)$$
$$= 12(x^2 - 8x + 12) = 12(x-2)(x-6)$$

Only $x = 2$ lies in the interior of the domain and constitute the critical point. The values of Vat this one critical point and two endpoints are

$V(2) = 128\,, V(0) = 0$ and $V(6) = 0$

The maximum value is $128\ cm^3$. The cut-out squares should be 2 cm on a side.

Example 2:
In the design a $1-L$ oil can, shaped like a right circular cylinder, determine the dimensions so that the can uses the least material.
Solution:



Since 1 liter is equal to $1000\ cm^3$,

$$\pi r^2 h = 1000 \qquad \text{----------}\ (1)$$

For the least material, we wish to determine the dimensions r and h that make the total surface area

$$A = 2\pi r^2 + 2\pi rh \qquad \text{----------}\ (2)$$

as small as possible while satisfying the constraint $\pi r^2 h = 1000$. We solve the equation (1) for h:

$$h = \frac{1000}{\pi r^2}$$

This changes the formula for A to

$$A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r\frac{1000}{\pi r^2}$$

$$A(r) = 2\pi r^2 + \frac{2000}{r} \text{ where } r > 0$$

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2}$$

$$A'(r) = 0 \Longrightarrow \frac{4\pi r^3 - 2000}{r^2} = 0$$

$$\Longrightarrow 4\pi r^3 - 2000 = 0$$

$$\Longrightarrow 4\pi r^3 = 2000$$

$$\Longrightarrow r^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$$

Critical point: $r = \sqrt[3]{\frac{500}{\pi}}$

We investigate the second derivative $\frac{d^2A}{dr^2}$:

$$A'(r) = 4\pi r - \frac{2000}{r^2}$$

$$A''(r) = 4\pi(1) - 2000\left(\frac{-2}{r^3}\right)$$

$$A''(r) = 4\pi + \frac{4000}{r^3}$$

$$A''\left(r = \sqrt[3]{\frac{500}{\pi}}\right) = 4\pi + \frac{4000}{\left(\sqrt[3]{\frac{500}{\pi}}\right)^3}$$

$$= 4\pi + \frac{4000}{\frac{500}{\pi}} = 4\pi + 8\pi = 12\pi > 0$$

The second derivative is positive throughout the domain of A.

The value of A at $r = \sqrt[3]{\frac{500}{\pi}}$ is therefore an absolute minimum because the graph of A is concave up.

When $r = \sqrt[3]{\frac{500}{\pi}}$ $\quad h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{2(500)}{\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}}$

$$h = 2\frac{(500)}{\pi\cdot\left(\frac{500^{\frac{2}{3}}}{\pi^{\frac{2}{3}}}\right)} = 2\frac{(500)500^{\frac{-2}{3}}}{\pi\cdot\pi^{\frac{-2}{3}}} = 2\frac{500^{1-\frac{2}{3}}}{\pi^{1-\frac{2}{3}}} = 2\frac{500^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}$$

$$= 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}} = 2\left(\sqrt[3]{\frac{500}{\pi}}\right) = 2r$$

$\therefore h = 2r$

The 1-L can that uses the least material has height equal to the diameter.

Example 3:

Find two positive numbers whose sum is 20 and whose product is as large as possible.

Solution:

If one number is x, the other is $(20 - x)$. Their product is

$$f(x) = x(20 - x) = 20x - x^2$$

We want the value or values of x that make $f(x)$ as large as possible. The domain of f is the closed interval $0 \leq x \leq 20$.

We evaluate f at the critical points and endpoints. The first derivative,

$$f'(x) = 20 - 2x$$

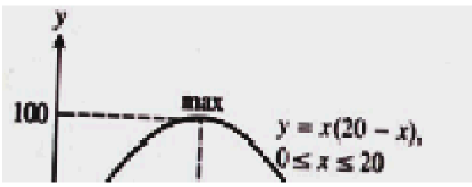
is defined at every point of the interval $0 \leq x \leq 20$.

and is zero only at $x = 10$. We list the values of f at this one critical point and the endpoints.

$$f(10) = 20(10) - 10^2 = 100$$

$$f(0) = 0 \qquad f(20) = 0$$

We conclude that the maximum value is 100. The corresponding numbers are 10 and 10.

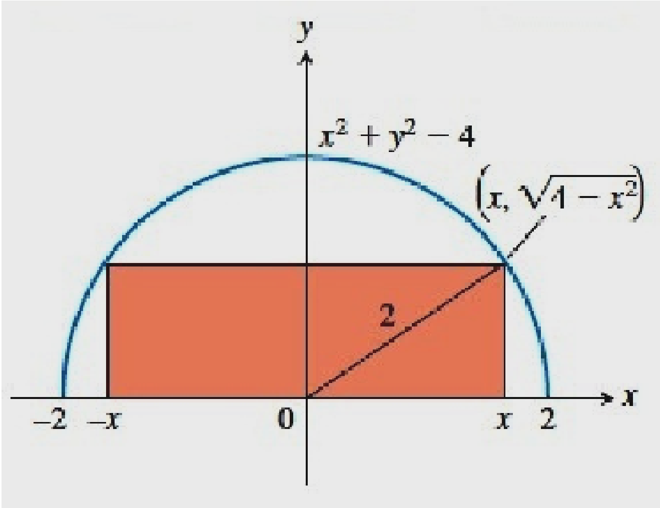


Example 4:

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution:

We place the circle and rectangle in the coordinate plane. We express the length, height, and area of the rectangle in terms of the position x of the lower right-hand corner.



Length $= 2x$
Height $= \sqrt{4 - x^2}$
The area is $A(x) = 2x\sqrt{4 - x^2}$
on the domain $[0, 2]$. We examine the values of A at the critical points and endpoints.
The derivative

$$\begin{aligned}
\frac{dA}{dx} &= \frac{d}{dx}\left(2x\sqrt{4 - x^2}\right) \\
&= 2x\frac{d}{dx}\left(\sqrt{4 - x^2}\right) + \sqrt{4 - x^2}\frac{d}{dx}(2x) \\
&= (2x)\frac{1}{2\sqrt{4-x^2}}\frac{d}{dx}(4 - x^2) + \sqrt{4 - x^2}(2) \\
&= (x)\frac{1}{\sqrt{4-x^2}}(0 - 2x) + 2\sqrt{4 - x^2} \\
&= \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4 - x^2} \\
&= \frac{-2x^2 + 2\sqrt{4-x^2}\sqrt{4-x^2}}{\sqrt{4-x^2}} = \frac{-2x^2 + 2(4-x^2)}{\sqrt{4-x^2}} \\
&= \frac{-2x^2 + 8 - 2x^2}{\sqrt{4-x^2}} = \frac{8-4x^2}{\sqrt{4-x^2}}
\end{aligned}$$

$\therefore \frac{dA}{dx} = \frac{8-4x^2}{\sqrt{4-x^2}}$

$\frac{dA}{dx}$ is not defined when$4 - x^2 = 0 \Longrightarrow x = 2$ and

$\frac{dA}{dx}$ is equal to zero when $8 - 4x^2 = 0 \Longrightarrow x = \pm\sqrt{2}$

$x = -\sqrt{2}$ is not admissible as it lies outside the domain; $x = \sqrt{2}$ is the critical point.
The values of A at the endpoints and at this one critical point are

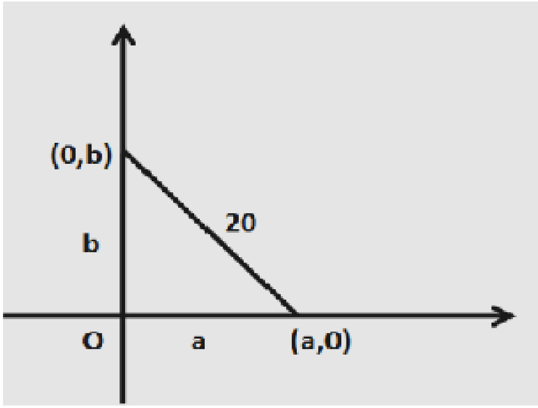
$$A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4 - 2} = 4$$

$$A(0) = 0 \qquad A(2) = 0$$

The area has a maximum value of 4 when $x = \sqrt{2}$
the rectangle length is $2x = 2\sqrt{2}$
and width is $y = \sqrt{4 - x^2} = \sqrt{4 - \sqrt{2}^2} = \sqrt{4 - 2} = \sqrt{2}$ units length $y = \sqrt{2}$ units long.

PROBLEM SET

IP1: You are planning to close off a corner of the first quadrant with a line segment $20\ units$ long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.



Solution:

The area of the triangle is $A = \frac{1}{2}ab$.

By Pythagorean theorem $a^2 + b^2 = 400$.

$\Longrightarrow a = \sqrt{400 - b^2}$

$$\therefore\ A(b) = \frac{b}{2}\sqrt{400 - b^2}\ ,\ \text{where}\ \ 0 \le b \le 20.$$

The derivative $\frac{dA}{db} = \frac{b}{2}\frac{d}{db}\left(\sqrt{400 - b^2}\right) + \sqrt{400 - b^2}\,\frac{d}{db}\left(\frac{b}{2}\right)$

$$\frac{dA}{db} = \frac{b}{2}\frac{1}{2\sqrt{400-b^2}}\frac{d}{db}(400 - b^2) + \sqrt{400 - b^2}\left(\frac{1}{2}\right)$$

$$\frac{dA}{db} = \frac{b}{2}\frac{1}{2\sqrt{400-b^2}}(-2b) + \sqrt{400 - b^2}\left(\frac{1}{2}\right)$$

$$\frac{dA}{db} = \frac{-b^2}{2\sqrt{400-b^2}} + \frac{\sqrt{400-b^2}}{2}$$

$$\frac{dA}{db} = \frac{-b^2+\left(\sqrt{400-b^2}\right)\left(\sqrt{400-b^2}\right)}{2\sqrt{400-b^2}} = \frac{-b^2+400-b^2}{2\sqrt{400-b^2}} = \frac{400-2b^2}{2\sqrt{400-b^2}}$$

$$\frac{dA}{db} = \frac{2(200-b^2)}{2\sqrt{400-b^2}} = \frac{200-b^2}{\sqrt{400-b^2}}.$$

$$\frac{dA}{db} = 0 \implies \frac{200-b^2}{\sqrt{400-b^2}} = 0 \implies 200 - b^2 = 0 \implies b = \pm 10\sqrt{2}$$

$b = -10\sqrt{2}$ is not in the domain. The only critical point is at $b = 10\sqrt{2}$.

The values of A at the endpoints and at this one critical point are

$$A(b) = \frac{b}{2}\sqrt{400 - b^2}$$

$$A\left(10\sqrt{2}\right) = \frac{10\sqrt{2}}{2}\sqrt{400 - 200} = \frac{10\sqrt{2}}{2}\sqrt{200} = \frac{10\sqrt{2}}{2}10\sqrt{2} = 100$$

$$A(0) = 0 \quad , \quad A(20) = 0$$

Area of the triangle is maximum when $b = 10\sqrt{2}$ and

$a = \sqrt{400 - b^2} = \sqrt{400 - 200} = \sqrt{200} = 10\sqrt{2}.$

Hence the maximum area occurs when $a = b$.

P1: What is the smallest perimeter possible for a rectangle whose area is 16 cm^2. and what are its dimensions.

Solution: Let l and w be the length and width of the rectangle respectively.

The perimeter of rectangle is

$$P = 2l + 2w$$

Given that, area of rectangle is $16\ cm^2$.

We have, $lw = 16 \implies w = \frac{16}{l}$.
The perimeter of rectangle is

$$P = 2l + 2w = 2l + 2\left(\frac{16}{l}\right) = 2l + \frac{32}{l} \text{ where } l > 0.$$

$$P(l) = 2l + \frac{32}{l} \text{ where } l > 0.$$

Now, $P'(l) = 2(1) + 32\left(\frac{-1}{l^2}\right) = 2 - \frac{32}{l^2}$

and $P''(l) = 0 - 32\left(\frac{-2}{l^3}\right) = \frac{64}{l^3}$.

$P'(l) = 0 \implies 2 - \frac{32}{l^2} = 0 \implies -\frac{32}{l^2} = -2 \implies \frac{l^2}{32} = \frac{1}{2}$

$$\implies l^2 = \frac{32}{2} \implies l^2 = 16 \implies l = \pm 4$$

Now $l = 4$ (since $l > 0$).
Notice that $P''(l) = \frac{64}{l^3}$

$\therefore P''(4) = \frac{64}{4^3} = \frac{64}{64} = 1 > 0$

Thus P is minimum when $l = 4$.
Therefore, $l = 4, w = \frac{16}{l} = \frac{16}{4} = 4$ and the smallest perimeter is $P = 2l + 2w = 16\ cm$.

IP2: Show that among all rectangles with an $8\ m$ perimeter the one with largest area is a square.

Solution: Let x and y be length and width of the rectangle.
Area of the rectangle $A = xy$
Perimeter of the rectangle $P = 2x + 2y$.
By the hypotheses $P = 8\ m$

$$\therefore 2x + 2y = 8 \implies x + y = 4 \implies y = 4 - x$$

Now,
the Area of the rectangle $A(x) = xy = x(4 - x) = 4x - x^2$

$$A(x) = 4x - x^2 \text{where } 0 \leq x \leq 4.$$

The derivative is $A'(x) = 4 - 2x$.
The critical point occurs at$A'(x) = 0 \implies 4 - 2x = 0 \implies x = 2$.
Now, $A''(x) = \frac{d}{dx}(4 - 2x) = 0 - 2(1) = -2$

$$\therefore A''(2) = -2 < 0$$

Therefore, the area of the rectangle is maximum at $x = 2$ since $A''(2) < 0$.
The dimensions of rectangle are length $x = 2\ m$ and width $y = 4 - x = 4 - 2 = 2\ m$.
So, it is a square.

P2: A rectangle plot of farmland will be bounded on one side by a river and on the three sides by a single stand electric fence with $800\ m$ of wire at disposal, what is the largest area you can enclose and what are its dimensions .

Solution:
Let x and y be the width and length of the rectangle respectively.
Area of the rectangle $A = xy$.

By the hypotheses the total length of the fencing is $800\, m$.

Thus, $2x + y = 800 \Longrightarrow y = 800 - 2x$

$\therefore A(x) = x(800 - 2x) = 800x - 2x^2$, where $0 \le x \le 400$.

Differentiation w.r.t x, $A'(x) = 800 - 4x$.

$$A'(x) = 800 - 4x = 0 \rightarrow 800 - 4x = 0 \rightarrow x = 200$$

The critical point is at $x = 200$.

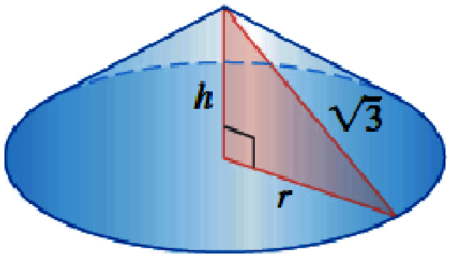
Now $A''(x) = -4$ and $A''(200) = -4 < 0$.

$\therefore A(x)$ is maximum when $x = 200$.

The dimensions are $x = 200\, m$, $y = 400\, m$.

The largest area that can be enclosed $= x.y = 80{,}000\, sq.m$.

IP3: A right triangle whose hypotenuse is $\sqrt{3}\, m$ long is revolved about one of its legs to generate a right circular cone. Find the radius, height and volume of the cone of greatest volume that can be made this way.



Solution:

From the figure $h^2 + r^2 = 3 \quad \Longrightarrow \quad r = \sqrt{3 - h^2}$.

Then the volume of cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3 - h^2)h$.

$$V(h) = \frac{\pi}{3}(3h - h^3) = \frac{\pi}{3}(3h) - \frac{\pi}{3}(h^3).$$

$$V(h) = \pi h - \frac{\pi h^3}{3}, \text{ where } 0 \le h \le \sqrt{3}$$

The derivative of V w.r.t h is

$$\frac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2)$$

The critical point occurs at $h = \pm 1$.

But $h > 0$, so $h = 1$ is the only critical point.

Now, $\frac{d^2V}{dh^2} = -2\pi h$

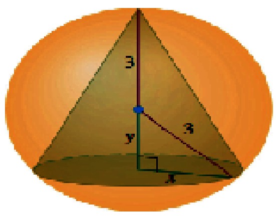
Since $\frac{d^2V}{dh^2} < 0$ at $h = 1$. Therefore, the critical point corresponds to maximum volume.

The cone of greatest volume has radius $r = \sqrt{2}\, m$, height $h = 1\, m$.

Greatest Volume $V = \frac{1}{3}\pi r^2 h = \frac{2\pi}{3}\, m^3$.

P3:

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



Solution:

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$

From the figure $r = x = \sqrt{9 - y^2}$ and $h = y + 3$.

Thus, $V(y) = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}(9 - y^2)(y + 3)$

$= \frac{\pi}{3}(27 + 9y - 3y^2 - y^3)$ where $0 \leq y \leq 3$.

The derivative of V w.r.t y is

$$V'(y) = \frac{\pi}{3}(9 - 6y - 3y^2) = \frac{\pi}{3}3(3 - 2y - y^2)$$
$$= \pi(3 - 2y - y^2)$$
$$= \pi(1 - y)(3 + y)$$

The critical points are $y = -3$ and $\boldsymbol{y = 1}$,

but $y = -3$ is not in the domain.

Now, $V''(y) = \frac{d}{dy}\left(\frac{\pi}{3}(9 - 6y - 3y^2)\right)$

$= \frac{\pi}{3}(-6 - 6y)$

Thus, $V''(1) = \frac{\pi}{3}(-6 - 6) = -4\pi < 0$.

If $y = 1 \Longrightarrow x = \sqrt{9 - y^2} = \sqrt{9 - 1^2} = \sqrt{8}$

$$\therefore r = x = \sqrt{8}$$

And $h = y + 3 = 1 + 3 = 4$

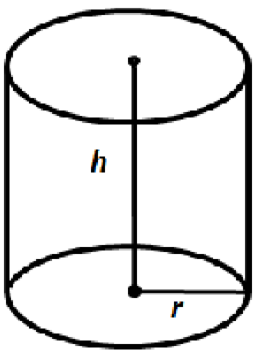
Therefore, the largest volume of the right circular cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\sqrt{8}\right)^2(4) = \frac{\pi}{3}(8)(4) = \frac{32\pi}{3}\ cubic\ units.$$

IP4: The sum of the height and diameter of the base of a right circular cylinder is given as $\boldsymbol{3\ units}$. Find the radius of the base and height of the cylinder so that the volume is maximum.

Solution:

Step1:



Let h be height and r be the radius of the base of the right circular cylinder.

Given that $h + 2r = 3 \Longrightarrow h = 3 - 2r$.

Volume of the cylinder

$$V = \pi r^2 h = \pi r^2(3 - 2r)$$
$$V(r) = 3\pi r^2 - 2\pi r^3\ ;\ 0 \leq r \leq 3$$

Now V is a function of r. Differentiating V w.r.t r,

$$\frac{dV}{dr} = \frac{d}{dr}(3\pi r^2 - 2\pi r^3) = 3\pi(2r) - 2\pi(3r^2)$$
$$= 6\pi r - 6\pi r^2$$
$$\frac{d^2V}{dr^2} = \frac{d}{dr}(6\pi r - 6\pi r^2) = 6\pi(1) - 6\pi(2r) = 6\pi - 12\pi r$$

If $\frac{dV}{dr} = 0$, then$6\pi r - 6\pi r^2 = 0 \rightarrow 6\pi r(1-r) = 0$

$i.e., r = 0$ or 1

Since $r > 0$, $r = 1$ is the only positive value at which $\frac{dV}{dr}$ vanishes.

$$\frac{d^2V}{dr^2}|_{r=1} = 6\pi - 12\pi(1) = -6\pi < 0$$

Hence V has the maximum value when $r = 1$.

If $r = 1$, then $h = 3 - 2(1) = 3 - 2 = 1$.

Therefore, when the volume is maximum the dimensions are $r = 1\ unit$ and $h = 1\ unit$.

P4: Find two positive numbers whose sum is 24 and whose product is as large as possible.

Solution: Let x and y be the two positive numbers such that $x + y = 24$.

Their product is $f(x) = xy = x(24 - x) = 24x - x^2$

We want the value x that make $f(x)$ as large as possible.

The domain of f is the closed interval $0 \leq x \leq 24$.

We evaluate f at the critical points and endpoints. The first derivative,

$$f'(x) = 24 - 2x$$

is defined at every point of the interval $0 \leq x \leq 24$ and is zero only at $x = 12$.

$$f''(x) = 0 - 2(1) = -2$$

Now , $f''(12) = -2 < 0$

We list the values of f at this one critical point and the endpoints.

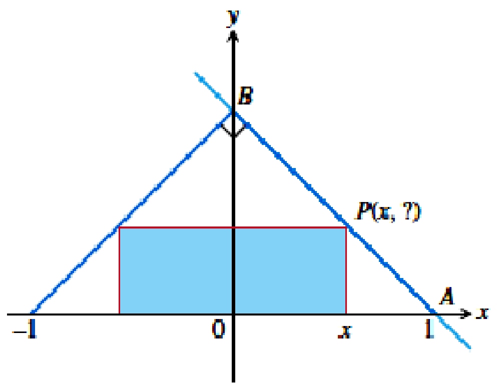
$$f(12) = 24(12) - 12^2 = 144$$

$$f(0) = 0 \qquad f(24) = 0$$

We conclude that the maximum value is 144. The corresponding numbers are 12 and 12.

Exercises:

1. The figure shown here shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



   a. Express the $y-$coordinate of P in terms of x.
   b. Express the area of the rectangle in terms of x.
   c. What is the largest area the rectangle can have?

2. You are planning to make an open rectangular box from an 8-by-15-cm piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way?
3. Two sides of triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area?
4. What are the dimensions of the lightest (least material) open-top right circular cylindrical can that will hold a volume of $1000\ cm^3$?

5. The sum of two non negative numbers is 36. Find the numbers if
   i) the difference of their square roots is to be as large as possible.
   ii) the sum of their square roots to be as large as possible.
6. The sum of two non negative numbers is 20. Find the numbers
   i) if the product of one number and the square root of the other is to be as large as possible.
   ii) if one number plus square root of the other is to be as large as possible.

6.12. Differentials

Learning objectives:

- To introduce the concept of differentials.
- To estimate the change of a function with differentials.
- To calculate the error in the differential approximation of a differentiable function.

And

- To practice related problems.

We sometimes use the Leibnitz notation $\frac{dy}{dx}$ to represent the derivative of y w.r.t x.
Contrary to its appearance; it is not a ratio. In this module, we introduce two new variables dx and dy with the property that if their ratio exists, it will be equal to its derivative.

Definition
Let $y = f(x)$ be a differentiable function. The ***differential*** $\boldsymbol{dx}$ is an independent variable.
The ***differential*** $\boldsymbol{dy}$ is

$$dy = f'(x)dx \qquad \text{-------------}(1)$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx.
If dx is given a specific value and x is a particular number in the domain of the function f, then the numerical value of dy is determined.
If $dx \neq 0$, the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because
from (1) $\quad dy \div dx = f'(x) = \frac{dy}{dx}$
This equation says that when $dx \neq 0$, we can regard the derivative $\frac{dy}{dx}$ as a quotient of differentials.
We sometimes write

$$df = f'(x)dx$$

in place of $dy = f'(x)dx$, and call df the differential of f.
Example 1:
Find dy if (a) $y = x^5 + 37x$ (b) $y = \sin 3x$
Solution: **We have,** $y = f(x) \implies dy = f'(x)dx$**.**
(a) $dy = (5x^4 + 37)dx$
(b) $dy = (3\cos 3x)dx$
Example 2:
Find df if $f(x) = 3x^2 - 6$.
Solution: $\quad df = d(3x^2 - 6) = 6xdx$

Every differentiation formula like

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

has a corresponding differential form like

$$d(u+v) = du + dv$$

Several formulas for the differentials are given below.

$$dc = 0$$
$$d(cu) = c\,du$$
$$d(u+v) = du + dv$$
$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$$
$$d(u^n) = nu^{n-1}du$$
$$d(\sin u) = \cos u\,du$$
$$d(\cos u) = -\sin u\,du$$
$$d(\tan u) = sec^2u\,du$$
$$d(\cot u) = -csc^2u\,du$$
$$d(\sec u) = \sec u \tan u\;\,du$$
$$d(\csc u) = -\csc u \cot u\;\,du$$

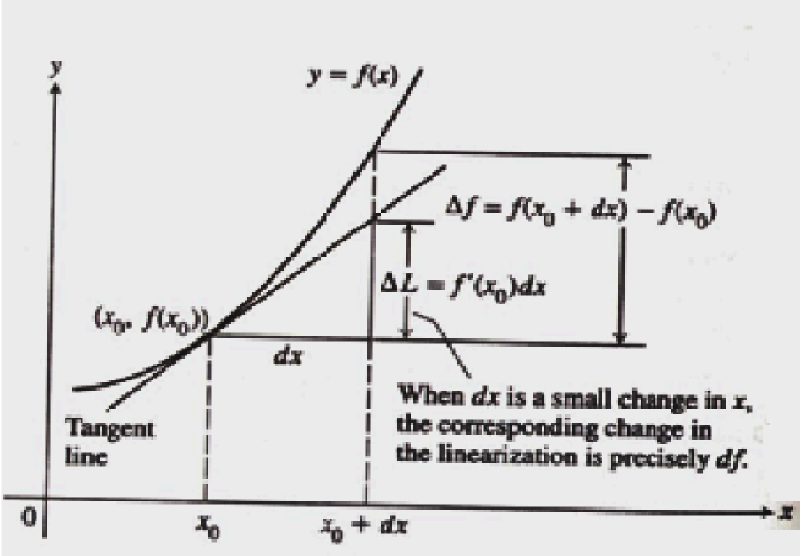
Example 3:

$$d(\tan 2x) = sec^2(2x)d(2x) = 2sec^2(2x)dx$$
$$d\left(\frac{x}{x+1}\right) = \frac{(x+1)dx - xd(x+1)}{(x+1)^2} = \frac{xdx + dx - xdx}{(x+1)^2} = \frac{dx}{(x+1)^2}$$

Estimating change with Differentials:

Suppose we know the value of a differentiable function $f(x)$ at a point x_0 and we want to predict how much this value will change if we move to a nearby point $x_0 + dx$ (where $dx = \Delta x$).

$$\Delta f = f(x_0 + dx) - f(x_0).$$



The change in f is $\Delta f = f(x_0 + dx) - f(x_0)$ ---------(2)

If dx is small, f and its linearization L at x_0 will change by nearly the same amount. Therefore, we approximate Δf by ΔL.

$$\begin{aligned}\Delta f = \Delta L &= L(x_0 + dx) - L(x_0)\\ &= f(x_0) + f'(x_0)[(x_0 + dx) - x_0] - f(x_0)\\ &= f(x_0) + f'(x_0)[x_0 + dx - x_0] - f(x_0)\\ &= f'(x_0)dx \qquad \text{----------------(3)}\end{aligned}$$

We have from (2) $f(x_0 + dx) = f(x_0) + \Delta f$
The differential approximation gives

$$f(x_0 + dx) \approx f(x_0) + df, \text{ where } dx = \Delta x.$$

Thus, $f(x_0 + dx) - f(x_0) \approx df$,

$$i.e., \ \Delta f \approx df.$$

Therefore, from (3) $df = f'(x_0)dx$.
Now, $f(x_0 + dx)$ can be predicted when $f(x_0)$ is known and dx is small.
We summarize the above and state the following:
Let $f(x)$ be differentiable at $x = x_0$. The approximate change in the value of f when x changes from x_0 to $x_0 + dx$ is

$$df = f'(x_0)dx$$

The Error in differential Approximation
Let $f(x)$ be differentiable at $x = x_0$ and suppose that Δx is an increment of x. The true change in f as x changes from x_0 to $x_0 + \Delta x$ is

$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$

while the differential estimate of the change is

$$df = f'(x_0)\Delta x$$

We measure the approximation error by subtracting df from Δf:

$$\begin{aligned}\text{Approximation error} &= \Delta f - df \\ &= \Delta f - f'(x_0)\Delta x \\ &= f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x \\ &= \left(\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - f'(x_0)\right)\Delta x\end{aligned}$$

As $\Delta x \to 0$, the difference quotient $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ approaches $f'(x_0)$, so the quantity in parentheses becomes a very small number, which we call ε.

In fact $\varepsilon \to 0$ as $\Delta x \to 0$. When Δx is small, the approximation error $\varepsilon \Delta x$ is smaller still.

$$\Delta f - df = \varepsilon\, \Delta x$$
$$\Delta f = df + \varepsilon\, \Delta x$$

Therefore,

$$\text{True change} = \text{estimated change} + \text{error}$$
$$\Delta f = f'(x_0)\Delta x + \varepsilon\, \Delta x$$

Change in $\boldsymbol{y = f(x)}$ near $\boldsymbol{x = x_0}$
If $y = f(x)$ is differentiable at $x = x_0$, and x changes from x_0 to $x_0 + \Delta x$, the change Δy in f is given by an equation of the form

$$\Delta y = f'(x_0)\Delta x + \varepsilon\, \Delta x$$

in which $\varepsilon \to 0$ as $\Delta x \to 0$.
Sensitivity
The equation $df = f'(x)dx$ tells how sensitive the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater is the effect of a given change dx. As we move from x_0 to a nearby point $x_0 + dx$, we can describe the change in f as follows:
(i) The absolute change is $\Delta f = f(x_0 + dx) - f(x_0)$.
It is estimated by $df = f'(x_0)dx$.

(ii) The relative change is $\frac{\Delta f}{f(x_0)}$. It is estimated by $\frac{df}{f(x_0)}$.

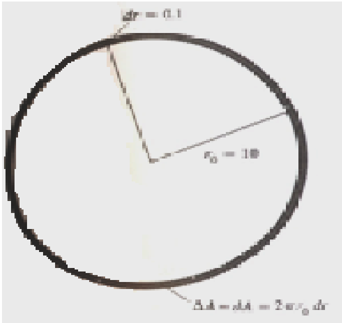
(iii) The Percentage change is $\frac{\Delta f}{f(x_0)} \times 100$. It is estimated by $\frac{df}{f(x_0)} \times 100$.

Example 4:

The radius r of a circle increases from $r_0 = 10\ m$ to $10.1\ m$. Estimate the increase in the circle's area A by calculating dA. Compare this with the true change ΔA.

Calculate the estimated percentage change in the area of the circle.

Solution:



$$A(r) = \pi r^2 \Longrightarrow dA = d(\pi r^2) = 2\pi r dr$$

$$dA_{r_0=10} = 2\pi r_0 dr = 2\pi(10)(0.1) = 2\pi$$

The true change is

$$\Delta A = A(r_0 + dr) - A(r_0)$$
$$= A(10.1) - A(10)$$
$$\Delta A = \pi(10.1)^2 - \pi(10)^2$$
$$= (102.01 - 100)\pi = 2\pi + 0.01\pi$$

The error $= \Delta A - dA = 2\pi + 0.01\pi - 2\pi = 0.01\pi$.

The estimated percentage change in the area of the circle is

$$\frac{dA}{A(r_0)} \times 100 = \frac{2\pi}{100\pi} \times 100 = 2\%$$

Example 5:

About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

Solution: We need to satisfy the inequality

$$|\Delta S| \leq 1\% os\ S = \frac{1}{100} S = \frac{4\pi r^2}{100}$$

$$|\Delta S| \leq \frac{4\pi r^2}{100}$$

We replace $\Delta S = ds$ in this inequality with

$$dS = d(4\pi r^2) = 4\pi d(r^2) = 4\pi(2r dr) = 8\pi r\ dr$$

$$dS = 8\pi r\ dr$$

This gives, $|8\pi r\ dr| \leq \frac{4\pi r^2}{100}$

$$|dr| \leq \frac{1}{8\pi r} . \frac{4\pi r^2}{100} = \frac{r}{200} = \frac{\left(\frac{1}{2}\right)}{100} r = (0.5\%)\ r$$

$$|dr| \leq \frac{\left(\frac{1}{2}\right)}{100} r = \frac{(0.5)}{100} r = (0.5\%)\ r$$

$$|dr| \leq (0.5\%)\ r$$

We should measure r with an error dr that is no more than 0.5% of the true value.

Example 6: You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculation be to a $0.1\ sec$ error in measuring the time?

Solution: The size of ds in the equation $ds = 32t\, dt$ depends on how big t is.

If $t = 2\ s$, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4\ m$$

Three seconds later, at $t = 5\ s$, the error caused by the same dt is

$$ds = 32(5)(0.1) = 16\ m$$

The estimated depth of the well differs from its true depth by a greater distance the longer the time it takes the stone to splash into the water below, for a given error in measuring the time.

PROBLEM SET

IP1: If $y = 3\csc(1 - 2\sqrt{x})$, then find dy.

Solution: Given $y = 3\csc(1 - 2\sqrt{x})$.

$$dy = d\left(3\csc(1 - 2\sqrt{x})\right)$$
$$= 3\left(-\csc(1 - 2\sqrt{x})\cot(1 - 2\sqrt{x})\right)\left(d(1 - 2\sqrt{x})\right)$$
$$= 3\left(-\csc(1 - 2\sqrt{x})\cot(1 - 2\sqrt{x})\right)\left(0 - 2.\frac{1}{2\sqrt{x}}\right)dx$$
$$= 3\left(-\csc(1 - 2\sqrt{x})\cot(1 - 2\sqrt{x})\right)\left(-\frac{1}{\sqrt{x}}\right)dx$$
$$= \frac{3}{\sqrt{x}}\csc(1 - 2\sqrt{x})\cot(1 - 2\sqrt{x})dx$$

P1: If $y = \frac{2\sqrt{x}}{3(1+\sqrt{x})}$, then find dy.

Solution:

Given $y = \frac{2\sqrt{x}}{3(1+\sqrt{x})}$.

$$dy = d\left(\frac{2\sqrt{x}}{3(1+\sqrt{x})}\right) = \frac{3(1+\sqrt{x})d[2\sqrt{x}] - (2\sqrt{x})d[3(1+\sqrt{x})]}{[3(1+\sqrt{x})]^2}$$
$$= \frac{3(1+\sqrt{x})\left[2\left(\frac{1}{2\sqrt{x}}\right)dx\right] - (2\sqrt{x})\left[3\left(0+\frac{1}{2\sqrt{x}}\right)dx\right]}{3^2[(1+\sqrt{x})]^2}$$
$$= \frac{(3+3\sqrt{x})\left[2\left(\frac{1}{2\sqrt{x}}\right)dx\right] - (2\sqrt{x})\left[\frac{3}{2\sqrt{x}}dx\right]}{9[(1+\sqrt{x})]^2}$$
$$= \frac{(3+3\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)dx - 3dx}{9[(1+\sqrt{x})]^2} = \frac{3\left(\frac{1}{\sqrt{x}}\right)dx + 3dx - 3dx}{9[(1+\sqrt{x})]^2}$$
$$= \frac{3\left(\frac{1}{\sqrt{x}}\right)dx}{9[(1+\sqrt{x})]^2} = \frac{1}{3\sqrt{x}(1+\sqrt{x})^2}dx$$

IP2: The function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a) **the change $\Delta f = f(x_0 + dx) - f(x_0)$,**
b) **the value of the estimate $df = f'(x_0)dx$; and**
c) **the approximate error $|\Delta f - df|$.**

$$\boldsymbol{f(x) = x^3 - 2x + 3, \quad x_0 = 2, dx = 0.1}$$

Solution:

Given $f(x) = x^3 - 2x + 3, \quad x_0 = 2, dx = 0.1$

Now, $f'(x) = 3x^2 - 2$

a) $\Delta f = f(x_0 + dx) - f(x_0) = f(2 + 0.1) - f(2)$

$\Delta f = f(2.1) - f(2)$

$= ((2.1)^3 - 2(2.1) + 3) - 7$

$= 9.261 - 4.2 - 4 = 1.061$.

b) $df = f'(x_0)dx$

$df = [3(2)^2 - 2](0.1) = 1$

c) The approximate error $|\Delta f - df| = |1.061 - 1| = 0.061$.

P2: The function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

d) the change $\Delta f = f(x_0 + dx) - f(x_0)$,

e) the value of the estimate $df = f'(x_0)dx$; and

f) the approximate error $|\Delta f - df|$.

$\boldsymbol{f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, dx = 0.1}$

Solution: Given $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, dx = 0.1$

Now, $f'(x) = 4x + 4$

d) $\Delta f = f(x_0 + dx) - f(x_0) = f(-1 + 0.1) - f(-1)$

$\Delta f = f(-0.9) - f(-1)$

$= (2(-0.9)^2 + 4(-0.9) - 3) - (-5)$

$= 2(0.81) - 3.6 - 3 + 5 = 6.62 - 6.6 = 0.02$

e) $df = f'(x_0)dx = \big(f'(-1)\big)(0.1)$

$df = [4(-1) + 4](0.1) = 0$

f) The approximate error= $|\Delta f - df| = |0.02 - 0| = 0.02$.

IP3: Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly with in 3% of its true value.

Solution: Volume of the sphere $V = \frac{\pi D^3}{6}$.

Now, $dV = \frac{\pi}{6}(3D^2 dD) = \frac{\pi D^2}{2} dD$.

If the volume is to be calculated correctly with in 3% of its true value, then

$$|\Delta V| \leq (3\%)V = \left(\frac{3}{100}\right)\left(\frac{\pi D^3}{6}\right) = \frac{\pi D^3}{200}$$

$$|\Delta V| \leq \frac{\pi D^3}{200}$$

We know that $\Delta V \approx dV$.

$$|dV| \leq \frac{\pi D^3}{200}$$

$$\Rightarrow \quad \left|\frac{\pi D^2}{2} dD\right| \leq \frac{\pi D^3}{200}$$

$$\Rightarrow \quad |dD| \leq \frac{\pi D^3}{200} \times \frac{2}{\pi D^2}$$

$$\Rightarrow \quad |dD| \leq \frac{D}{100} = (1\%)D$$

The allowable percentage error in measuring the diameter is 1%.

P3: The diameter of a sphere is measured as $100 \pm 1\ cm$ and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

Solution: Let D and V be the diameter and volume of the sphere respectively.

By the hypotheses $D = 100\ cm$ and $dD = 1\ cm$.

Volume of the sphere $V = \frac{4}{3}\pi(r)^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{4}{3}\pi\frac{D^3}{8} = \frac{\pi D^3}{6}$

Now, $dV = \frac{\pi}{6}d(D^3) = \frac{\pi}{6}(3D^2 dD) = \frac{\pi D^2}{2}dD.$

The percentage error in the volume is $\frac{dV}{V} \times 100.$

$$\begin{aligned}\frac{dV}{V} \times 100 &= \frac{\frac{\pi D^2}{2}dD}{\frac{\pi D^3}{6}} \times 100 \\ &= \frac{\pi D^2}{2}dD \times \frac{6}{\pi D^3} \times 100 \\ &= \frac{3}{D}dD \times 100 = \frac{3}{100} \times 100 = 3\end{aligned}$$

The percentage error in the volume is 3% .

IP4: If an error $0.01\ cm$ occurs in measuring the perimeter of a circle as $42\ cm$, then find the error and the relative error in the area.

Solution: Let the radius, perimeter and area of the circle be r, P and A respectively.

By the hypotheses $P = 42\ cm$, $dP = 0.01\ cm$.

Perimeter $P = 2\pi r \quad \Rightarrow \quad r = \frac{P}{2\pi}$

Area of the circle $A = \pi r^2 = \pi\left(\frac{P}{2\pi}\right)^2 = \frac{P^2}{4\pi}$

The error in the area is

$$dA = \frac{1}{4\pi}(2P\ dP) = \frac{P\ dP}{2\pi} = \frac{42(0.01)}{2\pi} = \frac{0.21}{\pi}$$

Relative error in the area is

$$\frac{dA}{A} = \frac{\frac{P\ dP}{2\pi}}{\frac{P^2}{4\pi}} = \frac{P\ dP}{2\pi} \times \frac{4\pi}{P^2} = \frac{2dP}{P} = \frac{2(0.01)}{42} = \frac{1}{2100}.$$

P4: The radius of a circle is increased from $2\ m$ to $2.02\ m$

a) **Estimate the resulting change in area.**
b) **Express the estimate as percentage of the circle's original area.**

Solution: Let r be the radius of the circle.

By the hypotheses $r = 2\ m$ and $dr = 2.02 - 2 = 0.02\ m$.

a) Area of the circle $A = \pi r^2$

   Now, $dA = 2\pi r\ dr$

   $\Rightarrow \quad dA = 2\pi(2)(0.02) = 0.08\pi\ m^2$.

   The resulting change in area of the circle if radius is increased from $2\ m$ to $2.02\ m$ is $0.08\pi\ m^2$.

b) The estimated percentage change in the area of the circle is

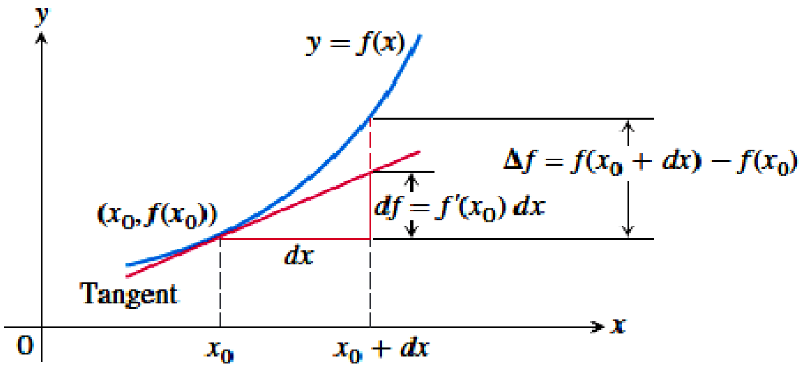
$$\begin{aligned}\frac{dA}{A} \times 100 &= \frac{2\pi r\ dr}{\pi r^2} \times 100 = \frac{2dr}{r} \times 100 \\ &= \frac{2(0.02)}{2} \times 100 = 2\%.\end{aligned}$$

Exercises:

1. Find dy.
   1. $y = x^3 - 3\sqrt{x}$
   2. $y = x\sqrt{1 - x^2}$
   3. $y = \frac{2x}{1+x^2}$
   4. $2y^{3/2} + xy - x = 0$

5. $xy^2 - 4x^{\frac{3}{2}} - y = 0$
6. $y = \sin(5\sqrt{x})$
7. $y = \cos(x^2)$
8. $y = \sec(x^2 - 1)$
9. $y = 4\tan(x^3/3)$
10. $y = 2\cot\left(\frac{1}{\sqrt{x}}\right)$

2. Each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find
   (a) the change $\Delta f = f(x_0 + dx) - f(x_0)$
   (b) the value of the estimate $df = f'(x_0)dx$
   (c) the approximation error $|\Delta f - df|$



a. $f(x) = x^2 + 2x$ $\quad x_0 = 0 \quad dx = 0.1$
b. $f(x) = x^3 - x$ $\quad x_0 = 1 \quad dx = 0.1$
c. $f(x) = x^4$ $\quad x_0 = 1 \quad dx = 0.1$
d. $f(x) = x^{-1}$ $\quad x_0 = 0.5 \quad dx = 0.1$

3. Write a differential formula that estimates the given change in volume or surface area.
   a. The change in volume$V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$
   b. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$
   c. The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from r_0 to $r_0 + dr$ and the height does not change.

Applications:

1. The diameter of a tree was 10 $in.$ During the following year, the circumference increased 2 $in.$ About how much did the tree's diameter increase? The tree's cross section area.
2. Estimate the volume of material in a cylindrical shell with height 30 $in.$, radius 6 $in.$ and thickness 0.5 $in.$
3. The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.
4. About how much accurately should you measure the side of a square to be sure of calculating t he area with in 2% of its true value.