Chapter 2

MONTE CARLO METHOD

2.1 Monte Carlo Method

The technique of making calculations by random sampling was though already used for solving statistical problems, the term 'Monte Carlo Method' seems to have come into existence in 1949,* when a paper, "The Monte Carlo Method" was published in the *Journal of American Statistical Association*. John Von Neumann and Stanislav Ulam are thought to have developed this method. In USSR, the first papers on the Monte Carlo Method were published in 1955-56.**

The technique 'Monte Carlo' derives its name from the city of Monte Carlo in the principality of Monaco, France, famous for gambling and its casino. Gambling is the game of probability and random sampling and so is the Monte Carlo Method. The roulette is a simple mechanical device of generating random numbers and is commonly used in gambling. Though the birth of the term Monte Carlo Method is connected with gambling games, yet it cannot help to win in roulette. It is a technique of experimental sampling with random numbers or the method of trials, which can be used to solve many problems, which are otherwise difficult or impossible to solve. J. Von Neumann and S. Ulam, who coined this term 'Monte Carlo' were involved in the design of nuclear shields at the Los Alamos Scientific Laboratory. They needed to know how far neutrons would travel through various materials. The problem was too difficult to solve analytically, and it was highly time consuming and hazardous to conduct experiments. They, simulated the experiment on a computer, using random numbers, and gave it the name Monte Carlo Method.

The term "Monte Carlo Method" is a very general term and the Monte Carlo Methods are used widely varying from economics to nuclear physics to waiting lines and to regulating the flow of traffic etc. Monte Carlo Methods are stochastic techniques and make use of random numbers and probability statistics to solve the problems. The way this technique is applied varies from field to field and problem to problem. Monte Carlo Method is applied to solve both deterministic as well as stochastic problems. There are many deterministic problems also which are solved by using random numbers and interactive procedure of calculations. In such a case we convert the deterministic model into a stochastic model, and the results obtained are not exact values, but only estimates. The application of Monte Carlo Method for evaluation of Pi (π) is converting a deterministic model into a stochastic model. Thus to call something a "Monte Carlo" experiment, all you need to do is to use random numbers to analyze the problem. Some simple examples will help to illustrate the use of random sampling in problem solving.

Example 2.1. Area of an Irregular Figure

Let us consider a very simple case of determining the area of a plane Fig. 2.1. This problem can easily be solved by simple calculations and has been taken only to demonstrate the application of Monte Carlo Method.

* Metropolis N., Ulam S., 'The Monte Carlo Method', J.Amer. Statistical Assoc., 1949, 44, No. 247.

** Papers by V.V.Chaveha nidze, Yu. A.Shreider and V.S.Vladiminov.

Enclose this figure in a regular figure of known area, say square of area A. Inside the square mark, at random, N number of points. This may be done by asking a man with closed eyes, to mark the points on the paper. Count the points inside the irregular figure, let these be M, while the points inside the square are N. Now, it is geometrically obvious that the ratio of area F of the irregular figure to the known area A, would approximately be the same as the ratio M to N.

i.e., $$\frac{F}{A} = \frac{M}{N} \quad \text{or} \quad F = \frac{M}{N}A$$

In Fig. 2.1 (*a*), $N = 50$ and $M = 17$ which gives the ratio $\frac{M}{N}$ as 0.34, while the true ratio in this case is 0.3126. If the number of random points N is increased, the accuracy of the result will also increase. In the second experiment, Fig. 2.1 (*b*), $N = 84$, and count of M comes to 26, giving the ratio $\frac{M}{N} = 0.3095$ which is more nearer to the actual ratio of 0.3126. Thus if $A = 9 \times 9 = 81$ cm^2, area F $= .3095 \times 81 = 25.07$ cm^2.

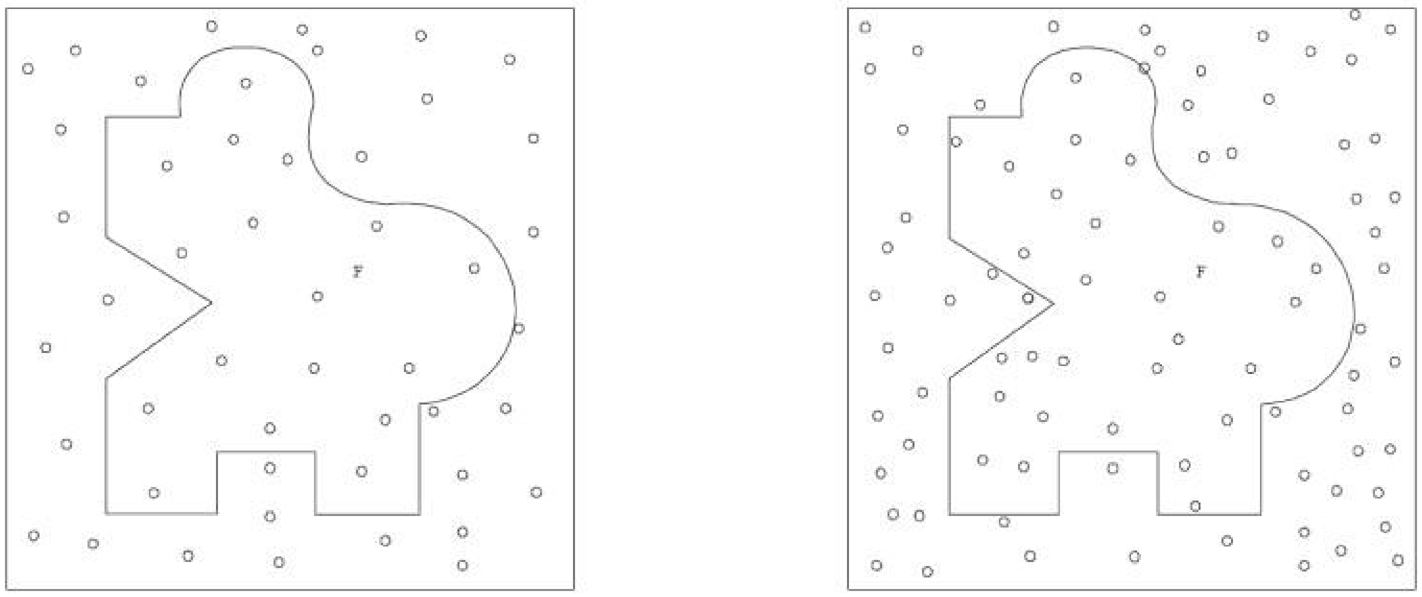
In this experiment, the randomness of the points is very important. Various methods can be employed for marking the points, and one such method is the use of random number tables. For each random point, two random numbers will be used to generate the two coordinates of the point.

In the present case our square is 9×9 cm^2. That is each side is 9 cm long. If we take the random numbers between 0 and 1, then the co-ordinates of a random point are obtained by multiplying the random number by 9.

Let .96, .42, .71, .10, .03, and .32, ..., be the random numbers, then we have

$$\begin{aligned} x1 &= .96 \times 9 = 8.64, & y1 &= .42 \times 9 = 3.78 \\ x2 &= .71 \times 9 = 6.39, & y2 &= .10 \times 9 = 0.9 \\ x3 &= .03 \times 9 = 0.27, & y3 &= .30 \times 9 = 2.7 \end{aligned}$$

Larger the number of points, more accurate will be the result. The error of calculation is proportional to $\frac{1}{N^2}$. Thus to reduce the error to half, value of N will have to be increased four times, and to have one more accurate decimal digit in the result a 100 fold increase in N will be required. It is thus clear that the Monte Carlo Method cannot be used to attain high accuracy and hence the Method is suitable for those problems only which do not require a very high degree of accuracy.



(*a*) $M/N = 17/50$ (*b*) $M/N = 26/84$

Fig. 2.1

Example 2.2. A Gambling Game

To illustrate the case of a random process, let us consider a game in which an unbiased coin is repeatedly flipped. For each flip you have to pay Re. 1. and when the difference between heads tossed and tails tossed becomes 3, you get Rs. 8. Thus if the required difference is obtained in less than 8 flips, you win some money and if it is in more than 8 flips you lose. How to decide, whether to play this game or not ?

The easiest way is to play the game at home and check, that is what the Monte Carlo Method is, to simulate the process by carrying trials. A coin can be used for this purpose or a random number table can be made use of. Larger the number of trials we carry, more accurate will be our estimate.

When a coin is flipped, the probability of head or tail is same that is 50%, but there are 10 possible one digit random numbers. Thus half of these (0, 1, 2, 3, 4) can be taken for head and the remaining half (5, 6, 7, 8, 9) for the tail.

Following is a string of random numbers noted down from a random number table.

5, 9, 3, 6, 4, 8, 6, 8, 1 ————

If H denotes head and T denotes tail, the above random numbers simulate the flips as,

T T H T H T T , T H ———.

We notice that after 7 trails the difference between heads and tails becomes 3 (2 heads 5 tails). Thus in the first game you win Re. 1. Some more games are simulated in Table 2.1. For the first seven games, we find that total money paid is Rs. $7 + 9 + 5 + 11 + 11 + 9 + 7 = 59$, while the return is Rs. 56. Thus each game on the average requires $\frac{59}{7} = 8.43$ flips.

This small simulation shows that you will lose some money in the long run. But can this estimate be relied upon ? No, since the number of trials conducted is very small, the accuracy of result would be very poor. If this simulation is continued over a sufficiently long period that is for a large number of trials the estimate close to the true mean can be obtained, which in this case is 9 flips for each game. Thus in the long run, you will lose Re. 1 per game.

Table 2.1. Simulation of a Gambling Game

Game No.	*Sl. No.*	*Random Number*	*Head or Tail*	*Cumulative Heads*	*Cumulative Tails*	*Difference*
1	1	5	T	0	1	1
	2	9	T	0	2	2
	3	3	H	1	2	1
	4	6	T	1	3	2
	5	4	H	2	3	1
	6	8	T	2	4	2
	7	6	T	2	5	3
						Win Re. 1
2	1	8	T	0	1	1
	2	1	H	1	1	0
	3	5	T	1	2	1
	4	2	H	2	2	0
	5	4	H	3	2	1
	6	0	H	4	2	2
	7	6	T	4	3	1
	8	3	H	5	3	2
	9	1	H	6	3	3
						Lose Re. 1

3	1	2	H	1	0	1
	2	9	T	1	1	0
	3	4	H	2	1	1
	4	2	H	3	1	2
	5	3	H	4	1	3
						Win Rs. 3
4	1	3	H	1	0	1
	2	3	H	2	0	2
	3	7	T	2	1	1
	4	1	H	3	1	2
	5	8	T	3	2	1
	6	0	H	4	2	2
	7	6	T	4	3	1
	8	2	H	5	3	2
	9	9	T	5	4	1
	10	3	H	6	4	2
	11	4	H	7	4	3
						Lose Rs. 3
5	1	3	H	1	0	1
	2	2	H	2	0	2
	3	9	T	2	1	1
	4	6	T	2	2	0
	5	1	H	3	2	1
	6	8	T	3	3	0
	7	7	T	3	4	1
	8	0	H	4	4	0
	9	8	T	4	5	1
	10	6	T	4	6	2
	11	7	T	4	7	3
						Lose Rs. 3
6	1	4	H	1	0	1
	2	1	H	2	0	2
	3	8	T	2	1	1
	4	2	H	3	1	2
	5	6	T	3	2	1
	6	0	H	4	2	2
	7	9	T	4	3	1
	8	2	H	5	3	2
	9	4	H	6	3	3
						Lose Re. 1
7	1	9	T	0	1	1
	2	8	T	0	2	2
	3	3	H	1	2	1
	4	9	T	1	3	2
	5	3	H	2	3	2
	6	7	T	2	4	2
	7	5	T	2	5	3
						Win Re. 1

Example 2.3. Numerical Integration by Monte Carlo Method

The application of Monte Carlo Method for the integration of a function, can best be illustrated with the help of an example. Let us consider a simple case of a single variable function.

$$I = \int_2^5 x^3 dx = \int_a^b f(x) dx$$

The exact value of the integral, $I = \left[\frac{x^4}{4}\right]_2^5 = 152.25$

This function $f(x) = x^3$ is plotted in Fig. 2.2.

The area under the curve between the limits $x = 2$ and $x = 5$, shown shaded is the value of the integral. The value of the area can be obtained in the same way as of irregular figure in Example 2.1. Let this area be enclosed in a rectangle $ABCD$, whose area is known ($140 \times 3 = 420$). The darts are thrown at random on to the figure. Let N darts fall in the rectangle $ABCD$ and out of these let M darts fall in the shaded area. In other words, out of N random marks made on $ABCD$, M marks lie in the shaded area.

Then, the ratio of the shaded area I to the area A of rectangle $ABCD$ is M to N.

or $$I = \frac{M}{N} A$$

The correctness of the result will depend upon the number of trials that is darts thrown. Larger the value of N, more accurate will be the result.

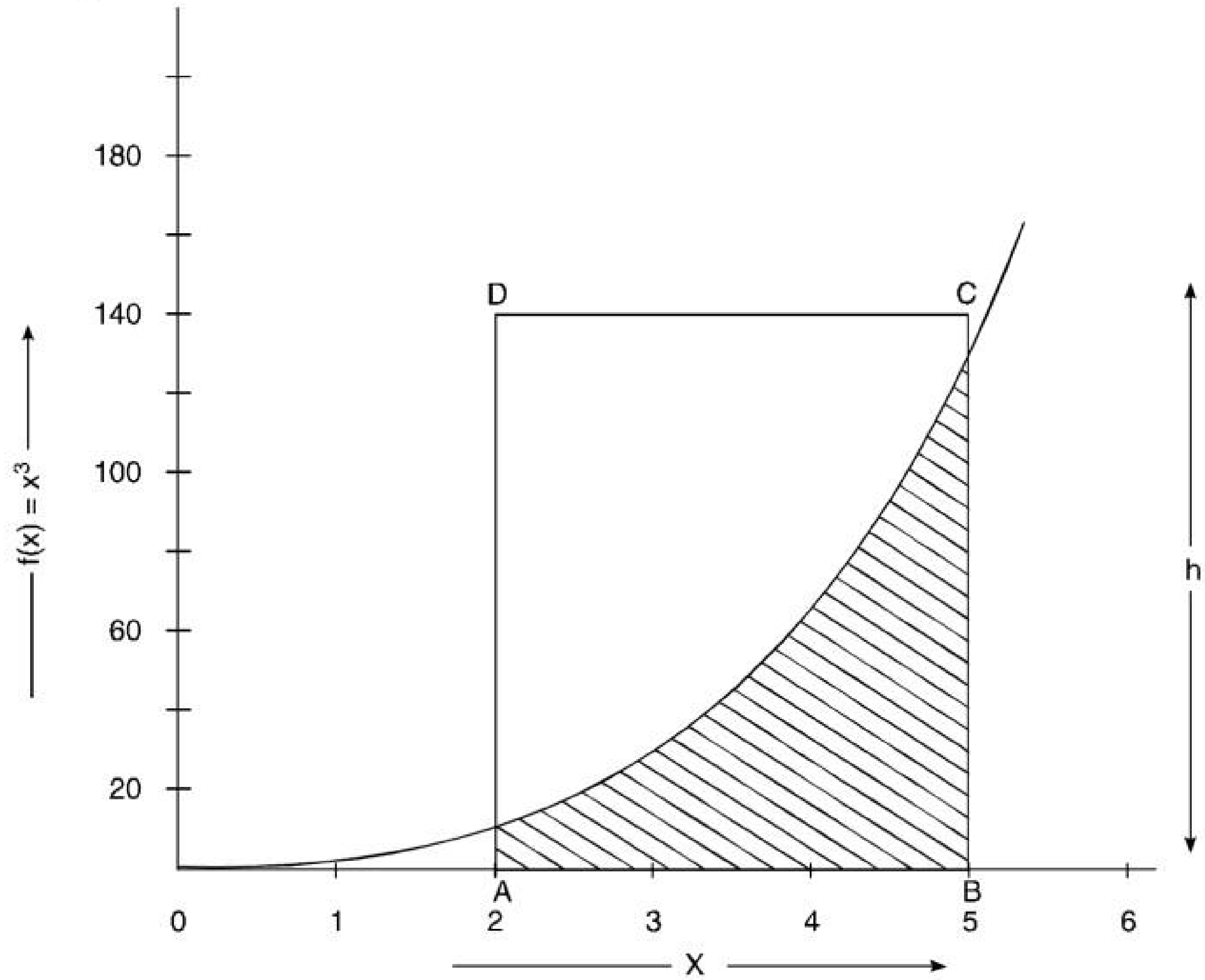


Fig. 2.2

Instead of throwing darts on the area, we can make use of random numbers. To mark a random point we need two random numbers to generate the two co-ordinates x and y. We can make use of a random number generator or of a random number table.

Let us take two digit random numbers between 20 and 50 to represent the x co-ordinate. If r is a random number between 20 and 50, then $x = 0.1\ (r)$. To generate y co-ordinate, take random number between 0 and 1, and multiply it by the height of the rectangle $ABCD$. Now check if $y \le x^3$. If so, the point lies below the curve, otherwise it is above the curve. Simulation procedure is demonstrated in Table 2.2. The point which lies below the curve *i.e.*, have $y \le x^3$ is added to M, while N is the number of trials (points generated). For twenty points simulated in the table, 8 points lie in the shaded area ($M = 8$; $N = 20$). Thus, the value of integral is $I = \frac{8}{20} \times A = 0.4 \times 420 = 168.00$. Actual value of the integral is 152.25. By increasing the number of points, value of I quite close to the actual value can be obtained, and this large number of points can be generated by a computer. Statistical techniques can be employed to determine the sample size, so as to obtain the result with reasonable accuracy. A flow diagram of the computer program, for evaluating an integral by Monte Carlo Method is given in Fig. 2.3.

Table 2.2. Simulation of $I = \int_2^5 x^3 dx$

Random Number	*x 0.1 (r)*	*Random Number*	*y 140 (r)*	x^3	*M*	*N*
22	2.2	.57	79.8	10.65	0	1
25	2.5	.18	25.2	15.63	0	2
18	1.8	.00	0.0	5.83	1	2
45	4.5	.90	126.0	91.13	1	4
25	2.5	.05	7.00	15.63	2	5
27	2.7	.77	107.8	19.68	2	6
48	4.8	.66	92.4	110.60	3	7
43	4.3	.10	14.0	79.51	4	8
40	4.0	.76	106.4	64.00	4	9
47	4.7	.42	58.8	103.82	5	10
38	3.8	.78	109.2	54.87	5	11
33	3.3	.88	123.2	35.94	5	12
24	2.4	.03	4.2	13.82	6	13
47	4.7	.09	12.6	103.80	7	14
42	4.2	.77	107.8	74.09	7	15
25	2.5	.61	85.4	15.63	7	16
33	3.3	.27	37.8	35.94	7	17
50	5.0	.60	84.0	125.00	8	18
34	3.4	.29	40.6	39.30	8	19
21	2.1	.40	56.0	9.26	8	20

$$I = \left(\frac{8}{20}\right) \times (140 \times 3) = 168.00$$

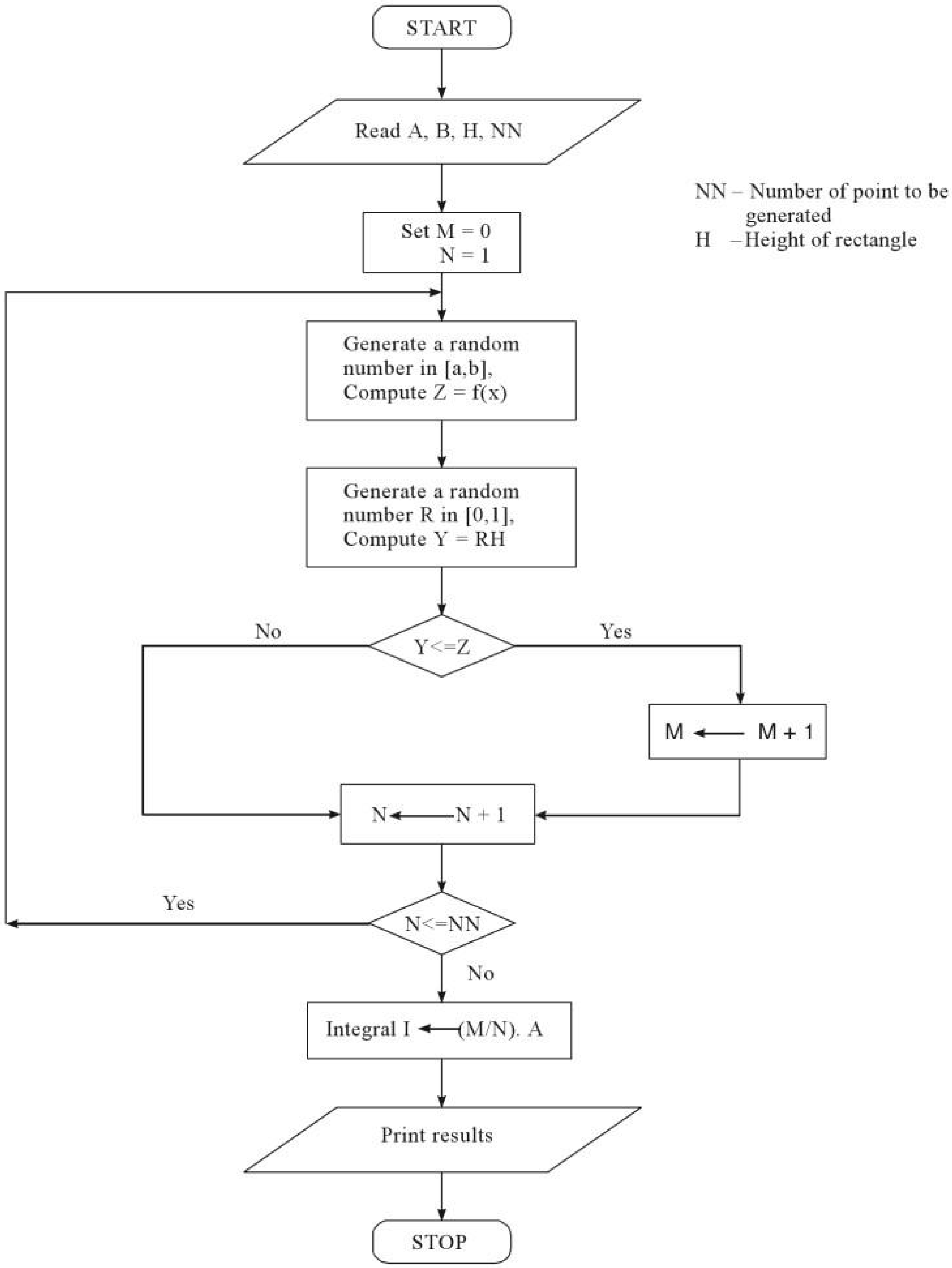


Fig. 2.3

A Note: It is not computationally economical to use the Monte Carlo Method for evaluating an integral of a single variable, especially when more efficient numerical methods are available. The method however is often used for solving multivariable integrals.

The problems of numerical integration and the area of an irregular figure, have been discussed to illustrate the application of Monte Carlo technique to problems of deterministic type, while Example 2.2, a gambling game, demonstrated the application of the method to probabilistic problems. Some, more problems are discussed in the following sections to illustrate the use of Monte Carlo Method, in handling various types of situation.

Example 2.4. Determination of value of π by Monte Carlo Method

The value of π can be determined by using the relation for area of a circle.

$$\text{Area of circle} = \pi r^2$$

Let us consider a quadrant of a unit circle as shown in Fig. 2.4.

Area of the quadrant is $\frac{\pi r^2}{4} = \frac{\pi}{4}$.

All points satisfying the equations $x^2 + y^2 \le 1;\ x, y \ge 0$ lie in this quadrant. Now if we have a pair of random numbers R_1 and R_2 in the range (0,1), then the point R_1 and R_2 may lie within the quadrant or outside the quadrant but within the square enclosing the quadrant. If we generate a large number of such points (say N) by taking pairs of random numbers, and out of them M lie within the quadrant, then the ration M/N will approach the area under the curve, which is $\frac{\pi}{4}$.

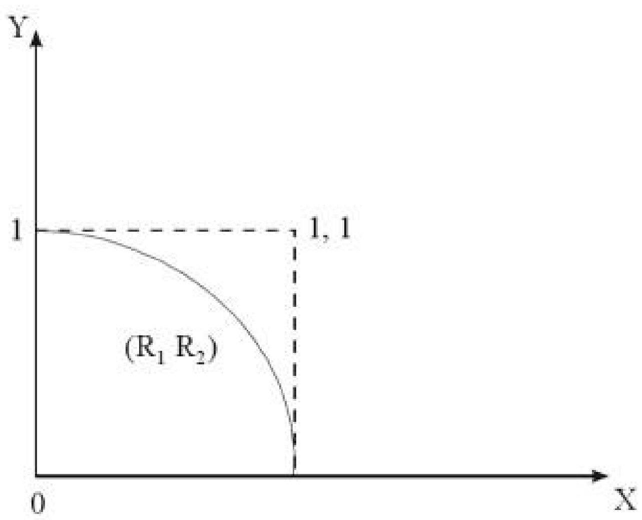


Fig. 2.4

The computations for 40 points are given in Table 2.3. For each pair of random numbers, the point is within the quadrant, when $R_1^2 + R_2^2 \le 1$

or $$R_2 \le \sqrt{1 - R_1^2}\ .$$

Table 2.3. Value of π by Monte Carlo Method

R_1	R_2	$\sqrt{1-R_1^2}$	*In/Out*	R_1	R_2	$\sqrt{1-R_1^2}$	*In/Out*
.82	.95	.5724	Out	.48	.37	.8773	In
.34	.14	.9404	In	.51	.72	.8602	In
.20	.84	.9797	In	.06	.33	.9982	In
.58	.14	.8146	In	.22	.90	.9755	In
.52	.03	.8542	In	.80	.76	.6000	Out
.51	.60	.8602	In	.56	.29	.8285	In
.34	.38	.9404	In	.06	.66	.9982	In
.72	.11	.6940	In	.92	.40	.3919	Out
.50	.55	.8760	In	.51	.11	.8602	In
.08	.21	.9967	In	.13	.44	.9915	In
.76	.18	.6499	In	.65	.74	.7599	In
.33	.81	.9440	In	.60	.27	.8000	In
.35	.96	.9368	Out	.51	.16	.8602	In
.29	.57	.9570	In	.50	.41	.8660	In
.95	.47	.3123	Out	.13	.20	.9915	In
.84	.51	.5426	In	.94	.68	.3412	Out
.34	.68	.9404	In	.51	.95	.8216	Out
.85	.11	.5268	In	.26	.28	.9656	In
.58	.92	.8146	Out	.78	.75	.6258	Out
.69	.59	.7238	In	.33	.16	.9440	In

Out of 40 points, 31 lie within the quadrant, giving the value of π_i as

$$\pi = \frac{31}{40} \times 4 = 3.10$$

The accuracy of the result can be increased by increasing the number of observations.

Example 2.5. Random Walk Problem

A drunkard moves from a point to a destination and takes steps in the forward direction, to his left and to his right, at random. The length of the step is almost constant. The probability of taking a step in the forward direction is 50%, while the probability of taking a step to the left 30% and to the right is 20%. If the co-ordinates of the starting point are taken as (0, 0), and the forward is due Y-direction, find the position of the drunkard after he takes 50 steps.

This type of problem, where a person walks at random in different direction is called a random walk. In this simple case, the position of the drunkard can easily be determined by using the probability theory. But we will simulate the walk using random numbers.

Using single digit random numbers, the random numbers can be allocated to steps in different directions, as under:

Direction		*Probability*	*Random Numbers*
Forward (F)	:	0.5	0, 1, 2, 3, 4
Left (L)	:	0.3	5, 6, 7
Right(R)	:	0.2	8, 9

Table 2.4. Random Walk Simulation

Step	*Random Number*	*Direction*	*Position*	
			x	y
1	6	L	– 1	0
2	2	F	–1	1
3	0	F	–1	2
4	6	L	–2	2
5	8	R	1	2
6	5	L	–2	2
7	7	L	–3	2
8	7	L	–4	2
9	9	R	–3	2
10	8	R	–2	2
11	4	F	–2	3
12	8	R	–1	3
13	2	F	–1	4
14	6	L	–2	4
15	2	F	–2	5
16	1	F	–2	6
17	3	F	–2	7
18	0	F	–2	8
19	8	R	–1	8
20	4	F	–1	9

When the drunkard takes steps in forward direction, the value of y is incremented by one. If the drunkard moves to the right value of x is incremented by one, and if he moves towards his left, value of x is decremented by one. The simulation for 20 steps is given in Table 2.4. The trace of the movement of the drunkard for the 20 steps is given in Fig. 2.5. In the first 20 steps, the drunkard moves to a position (– 1, 9).

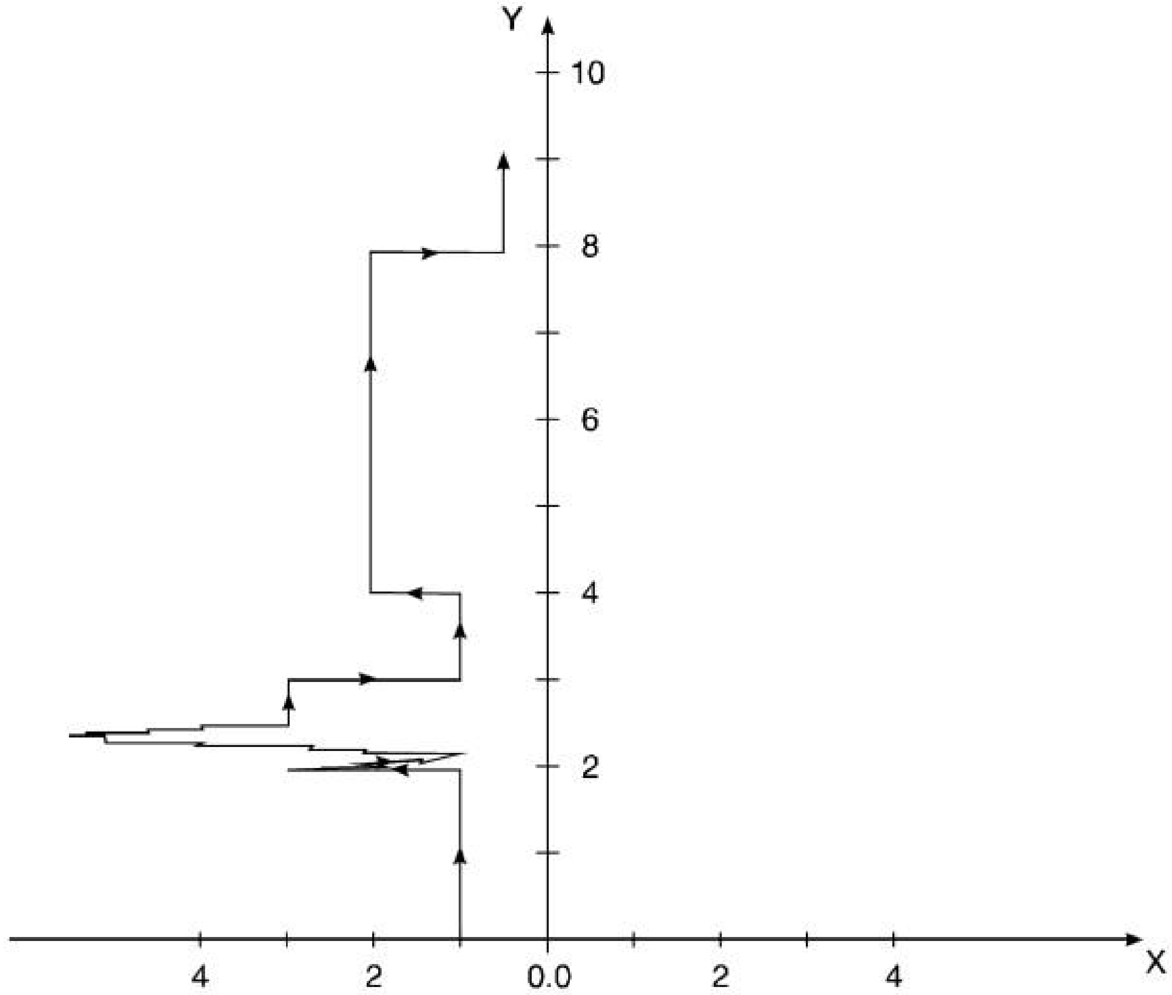


Fig. 2.5

Example 2.6. Reliability Problem

A machining center has three identical bearings, which fail according to the following probability distribution.

Bearing Life (Hours)	*Probability*
1000	0.10
1100	0.14
1200	0.24
1300	0.14
1400	0.12
1500	0.10
1600	0.06
1700	0.05
1800	0.03
1900	0.02

The present maintenance policy is to change a bearing as and when it fails. When a bearing fails, machining center stops, a repairman is called to replace the failed bearing with a new one. The time between the failure of the bearing and reporting of the repairman (delay time) is random and is distributed as under.

Delay Time (Minutes)	*Probability*
4	0.3
6	0.6
8	0.1

It takes 20 minutes to change one bearing, 30 minutes to change two bearings and 40 minutes to change all the three bearings. Downtime of the machining center costs Rs. 5 per minute, direct on job cost of the repairman is Rs. 25 per hour and the cost of a bearing is Rs. 20.

The maintenance department is interested in evaluating an alternative policy of replacing all the three bearings, whenever a bearing fails.

Solution: The reliability problem having random bearing lives and random repairman delay times can best be handled by employing simulation technique.

Table 2.5 shows the bearing life probability distribution, cumulative probability distribution and the two digit random numbers assigned to estimate the bearing lives. Table 2.6 shows the probability distribution and single digit random numbers assigned to determine the delay times of the repairman.

Table 2.5

Bearing Life (Hours)	*Probability*	*Cumulative Probability*	*Random Digits Assignment*
1000	0.10	.10	01 – 10
1100	0.14	.24	11 – 24
1200	0.24	.48	25 – 48
1300	0.14	.62	49 – 62
1400	0.12	.74	63 – 74
1500	0.10	.84	75 – 84
1600	0.06	.90	85 – 90
1700	0.05	.95	91 – 95
1800	0.03	.98	96 – 98
1900	0.02	1.00	99 – 00

Table 2.6

Delay Time (Minutes)	*Probability*	*Commutative Probability*	*Random Digits Assignment*
4	0.3	0.3	1 – 3
6	0.6	0.9	4 – 9
8	0.1	1.0	0

Table 2.7. Simulation of Reliability Problem – Present Policy

Bearing 1					Bearing 2					Bearing 3				
RD	Life	Clock	RD	Delay	RD	Life	Clock	RD	Delay	RD	Life	Clock	RD	Delay
25	1200	1200	9	6	65	1400	1400	5	6	94	1700	1700	7	6
82	1500	2700	8	6	91	1700	3100	0	8	87	1600	3300	0	6
16	1100	3800	0	8	30	1200	4300	5	6	63	1400	4700	6	6
14	1100	4900	5	6	66	1400	5700	6	6	66	1400	6100	2	4
82	1500	6400	2	4	32	1200	6900	5	6	30	1200	7300	2	4
45	1200	7600	3	4	29	1200	8100	3	4	69	1400	8700	7	6
81	1500	9100	7	6	11	1100	9200	2	4	37	1200	9900	2	4
80	1500	10600	3	4	43	1200	10400	8	6	01	1000	10900	8	6
97	1800	12400	7	6	40	1200	11600	0	8	66	1400	12300	8	6
62	1300	13700	4	6	65	1400	13000	8	6	51	1300	13600	3	4
30	1200	14900	9	6	82	1500	14500	2	4	92	1700	15300	8	6
31	1200	16100	5	6	73	1400	15900	4	6	36	1200	16500	3	4
80	1500	17600	6	6	15	1100	17000	4	6	47	1200	17700	7	6
72	1400	19000	7	6	70	1400	18400	6	6	80	1500	19200	0	8
63	1400	20400	6	6	65	1400	19800	9	6	94	1700	20900	4	6
59	1300	21700	0	8	33	1200	21000	9	6	31	1200	22100	7	6
09	1000	22700	2	4	54	1300	22300	5	6	07	1000	23100	1	4
25	1200	23900	3	4	87	1600	23900	7	6	09	1000	24100	6	6
70	1400	25300	9	6	27	1200	25100	7	6	19	1100	25200	2	4
64	1400	26700	8	6	37	1200	26300	4	6	29	1200	26400	3	4
61	1300	28000	3	4	99	1900	28200	8	6	29	1200	27600	2	4
92	1700	29700	3	4	94	1700	29900	4	6	94	1700	29300	5	6
12	1100	30800	2	4	12	1100	31000	4	6	87	1600	30900	6	6
		Total Delay		126			Total Delay		136			Total Delay		122

According to the existing repair policy, each bearing is replaced as and when it fails. It is highly unlikely that more than one bearing will fail at the same time, hence, it can be safely assumed that only one bearing is changed at any breakdown. Simulation of the system for 30,000 hours of operation, carried out manually, is given in Table 2.7. In case of each of the bearings, 23 changes have occurred during the simulation run. Thus in all 69 bearings have been replaced. The delay times in case of bearings numbered 1, 2 and 3 are 126, 136 and 122 minutes respectively, making a total of 384 minutes of repairman delay time. Since, it takes 20 minutes to change one bearing, the time spent in bearings replacement is 1380 minutes. Thus total downtime of the machine is $384 + 1380 = 1764$ minutes. On job time spent by repairman is $69 \times 20 = 1380$ minutes. The estimate cost of the existing replacement policy is as below.

$$
\begin{aligned}
\text{Cost of bearings} &= 69 \times 20 &&= 1380.00\\
\text{Cost of downtime} &= 1764 \times 5 &&= 8820.00\\
\text{Cost of repairman} &= 1380 \times \frac{25}{60} &&= \underline{575.00}\\
& && 10{,}775.00
\end{aligned}
$$

The simulation of the system for the proposed repair policy is given in Table 2.8. Simulation has been carried for the same simulation run of 30,000 hours, and using the same life times as in the simulation for the existing policy. Since, the number of replacements in this case is more, additional bearing lives have been generated by using the next random numbers in the related series. The advantages of common random numbers are discussed in Chapter 9 of the book.

Table 2.8. Simulation of Reliability Problem — Proposed Policy

Bearing 1 Life (Hrs.)	*Bearing 2 Life (Hrs.)*	*Bearing 3 Life (Hrs.)*	*First Failure (Hrs.)*	*Cumulated Life (Hrs.)*	*RD*	*Delay (Minutes)*
1200	1400	1700	1200	1200	4	6
1500	1700	1600	1500	2700	6	6
1100	1200	1400	1100	3800	7	6
1100	1400	1400	1100	4900	3	4
1500	1200	1200	1200	6100	4	6
1200	2400	4700	1200	7300	6	6
1500	1100	2200	1100	8400	7	6
1500	1200	1000	1000	9400	7	6
1800	1200	1400	1200	10600	4	6
1300	1400	1300	1300	11900	7	6
1200	1500	1700	1200	13100	1	4
1200	1400	1200	1200	14300	0	8
1500	1100	1200	1100	15400	0	8
1400	1400	1500	1400	16800	1	4
1400	1400	1700	1400	18200	9	6
1300	1200	1200	1200	19400	5	6
1000	1300	1000	1000	20400	0	8
1200	1600	1000	1000	21400	6	6
1400	1200	1100	1100	22500	4	6
1400	1200	1200	1200	23700	1	4
1300	1900	1200	1200	24900	4	6
1700	1700	1700	1700	26600	9	6
1100	1100	1600	1100	27700	4	6
80/1500	56/1300	85/1600	1300	29000	4	6
21/1100	51/1300	66/1400	1100	30100	6	6
					Total Delay	148

The simulation of proposed policy reveals that for the same operation time 25 replacements of bearings will occur, requiring a total of 75 bearings. The delay time is 148 minutes, while the time spent in changing the bearings will be $25 \times 40 = 1000$ minutes, giving total downtime of 1148 minutes.

$$\begin{array}{llr} \text{Cost of bearings} & = 75 \times 20 & = 1500.00 \\ \text{Cost of downtime} & = 1148 \times 5 & = 5740.00 \\ \text{Cost of repairman} & = 1000 \times \dfrac{25}{60} & = \underline{416.67} \\ & & \underline{7656.67} \end{array}$$

The proposed repair policy is thus much more economical and will result in a saving of Rs. 3118.33 over a period of 30,000 hours of machine operation and hence, can be recommended for implementation.

2.2 Normally Distributed Random Numbers

There are a large number of real life situations, where the behavior of the system is described by normal probability distribution. The marks obtained by students in a class, number of defective parts produced, dimensions of parts made on a machine, number of bombs hitting a target area etc. are generally described by the normal distribution. One method of generating the variates of such a distribution, by using the random numbers is described in Chapter 5. The value of variate y is calculated as,

$$y = \mu + \sigma \left(\sum_{i=1}^{12} r_i - 6 \right), \qquad i = 1, 2, \ldots\ldots\ldots, 12.$$

In this method, 12 random numbers are used to generate one value.

The alternative method is the use of random numbers, which themselves are normally distributed about a mean of zero.

$$y = \mu + \sigma z$$

where z is a random normal number. In small simulations, where computations involving normal distribution are to be carried out, the tables of random normal numbers can be used. A sample of such numbers is given in Appendix Table A-3.

Example 2.7. Application of Random Normal Numbers

An ammunition depot, of rectangular shape measuring 1000 m along X-direction and 600 m along Y-direction is under attack from a squadron of bombers. In each sortie 10 bombers drop bombs, one each, on the ammunition depot. If the bomb lands anywhere in the marked area, it is a hit, otherwise it is a miss. All the bombers aim at the center of the area. The point of strike is assumed to be normally distributed around the aiming point with a standard deviation of 500 meters in the X-direction and 300 meters in the Y-direction. The operation of bombing is to be simulated to determine the percentage number of strikes, which are on the target.

Since, the strike points are randomly distributed about the point of aim, and the distribution is normal, the random normal numbers (normally distributed random numbers) can be used to generate the strikes.

A normal random variable X is given by:

$$X = \mu + \sigma_z$$

where μ is the true mean of the distribution, σ is the standard deviation of x and z is the normal random number.

If x and y are the co-ordinates of a point, then

$$x = \mu_x + \sigma_x z_x$$
$$y = \mu_y + \sigma_y z_y$$

z_x and z_y are normal random numbers. Taking the co-ordinates of the center point, of the area as 0, 0,

$$\mu_x = \mu_y = 0$$

The given values of standard deviations in X- and Y-directions are 500 and 300 meters respectively. Therefore, the co-ordinates of a point are,

$$x_1 = 500\ Z_x$$
$$y_1 = 300\ Z_y$$

The subscripts x and y added to the normal random number Z are only to indicate that different random numbers are to be taken. In this example, since manual simulation is to be done, the normal random number table will be used. To find some reasonable answer to the problem, a well-designed simulation experiment of 40 to 50 bombing sorties may be required. However, a simulation of only two sorties that is 20 bombs has been done to demonstrate the simulation procedure. Table 2.9 shows the results for this simulation. In the table the mnemonic RNN stands for 'random normal number'. The co-ordinates of each point have been generated. Those who lie within the marked area are designated as 'hit', others as 'miss'. This type of simulation is called Monte Carlo or Static Simulation, as process is not time dependent.

Table 2.9. Simulation of Bombing Problem

Bomb strike	*RNN*	*x*	*RNN*	*y*	*Result*
1	.23	115	.24	72	Hit
2	–1.16	–580	–0.02	–6	Miss
3	0.39	195	0.64	192	Hit
4	–1.90	–950	–1.04	–312	Miss
5	–0.78	–390	0.68	204	Hit
6	–0.02	–10	–0.47	–141	Hit
7	–0.40	–200	–0.75	–225	Hit
8	–0.66	–330	–0.44	–132	Hit
9	1.41	705	1.21	363	Miss
10	0.07	35	–0.08	–24	Hit
11	0.53	265	–1.56	–468	Miss
12	0.03	15	1.49	447	Miss
13	–1.19	–595	–1.19	–357	Miss
14	0.11	55	–1.86	–558	Miss
15	0.16	80	1.21	363	Miss
16	–0.82	– 410	0.75	225	Hit
17	0.42	210	–1.50	–450	Miss
18	–0.89	–445	0.19	57	Hit
19	0.49	249	–1.44	–432	Miss
20	–0.21	–105	0.65	195	Hit

Number of hits = 10
Number of misses = 10
% Number of strikes on target = 50 %

2.3 Monte Carlo Method Vs. Stochastic Simulation

The term 'Monte Carlo Method' is very general and as discussed earlier is based on gambling like principles. It makes use of random numbers for finding the solution to the problems. Some authors differentiate between Monte Carlo Method and Stochastic Simulation, but that difference has only academic value.

Monte Carlo Method is considered to be a method of solving deterministic problems by employing random numbers, like finding the area of irregular figure, numerical integration, evaluation of π, trajectory computations etc. The stochastic simulation is used for finding the solution to a problem by employing random numbers, when the problems are stochastic. The gambling problem, the bombing problem, the reliability problem are the examples of stochastic simulation.

In practice both the Monte Carlo Method and stochastic simulation are employed to analyze models having random variables and both make use of random numbers and probability distributors to investigate the problem. Monte Carlo Methods has been employed in a wide range of situations varying from economics to nuclear physics, from regulating traffic to balancing of assembly lines, most of which are stochastic processes. The term "stochastic" is a synonym for random while the stochastic process is a particular type of model that represents the uncertainty in a dynamic system using the language of probability. Monte Carlo method and stochastic simulation have become almost synonyms and are used alternatively to describe the same method of stochastic modeling.

Example 2.8. Example of Stochastic Model – A Hinge Assembly

Fig. 2.6 shows a hinge assembly which comprise of four parts *A*, *B*, *C* and D, with a pin or bolt though the centre of the parts. The dimensions of the parts along the axis are critical, since if $a + b + c$ is greater than d, it will not be possible to put the parts together. The dimensions a, b, c and d are 2 ± 0.05, 2 ± 0.05, 30 ± 0.5 and 34.5 ± 0.5 respectively. Each of a, b, c and d is uniformly distributed over the given range. There is a huge stock of each part and the parts are randomly selected for making the hinge assemblies. What is the probability that the selected parts will not assemble ?

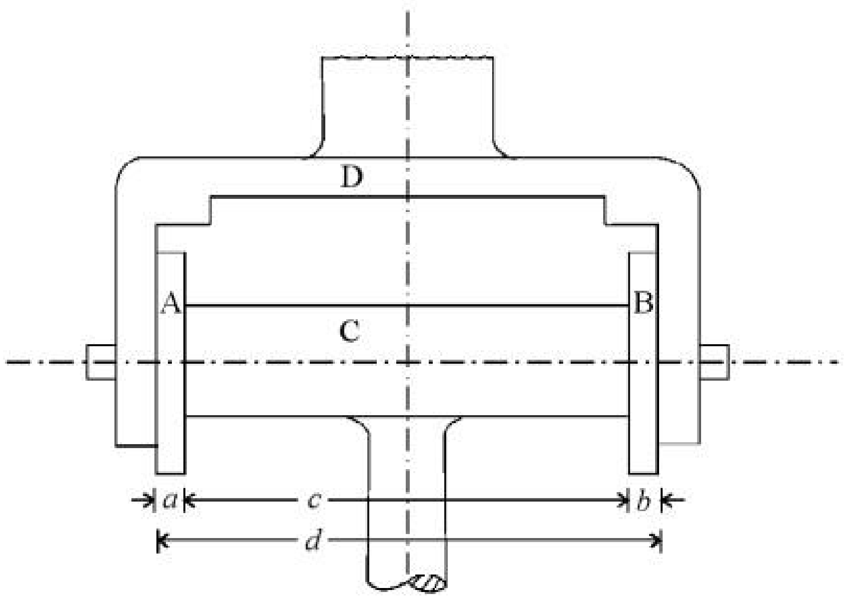


Fig. 2.6. Hinge assembly

Solution: In this example, we have to simulate the assembly process to obtain a good number of hinge assemblies. For each assembly the four parts are to be selected or we can say the dimensions a, b, c and d are to be generated. We will generate the critical dimensions by employing the two digit random numbers between 0 and 1. The range over which the dimensions are uniformly distributed is as given below:

Part	*Minimum*	*Maximum*
a	1.95	2.05
b	1.95	2.05
c	29.50	30.50
d	34.00	35.00

If R is a random number between 0 and 1, then $a = 1.95 + 0.1\ (R)$ where 1.95 is the minimum dimension.

Similarly,

$$b = 1.95 + 0.1\,(R)$$
$$c = 29.50 + 1\,(R)$$
$$d = 34.00 + 1\,(R)$$

After generating the dimension a, b, c and d we have to check the value of clearance, $x = d - (a + b + c)$. If the clearance is zero or positive parts will assemble with zero or positive clearance. If it is negative the parts will not assemble. Then we have to repeat the process of generating assemblies for different sets of parts. More the number of assemblies, better the result. Normally more than a 1000 iterations will be required to reach a reasonable answer to the problem.

Here we do it for a few iterations manually to demonstrate the simulation. The progress of simulation is shown in Table 2.10.

Table 2.10

Sr.	*Random number*	*a = 1.95 + 0.1(R)*	*Random number*	*b = 1.95 + .1R*	*Random number*	*c = 29.5 + R*	*a + b + c*	*Random number*	*d = 34 + R*	*Clearance*
1	.44	1.994	.56	2.006	.32	29.82	33.820	.12	34.12	+
2	.86	2.036	.69	2.019	.41	29.91	34.064	.83	34.83	+
3	.72	2.022	.52	2.002	.76	30.26	34.334	.34	34.34	+
4	.87	2.037	.84	2.034	.90	30.40	34.471	.42	34.42	–
5	.47	1.997	.43	2.043	.42	29.92	33.960	.97	34.37	+
6	.48	1.998	.51	2.001	.06	29.56	33.559	.22	34.22	+
7	.80	2.030	.56	2.006	.13	29.63	33.666	.92	34.92	+
8	.51	2.001	.17	1.967	.65	30.13	34.098	.60	34.60	+
9	.51	2.001	.52	2.002	.27	29.77	33.770	.94	34.94	+
10	.57	2.007	.26	1.976	.78	30.28	34.263	.33	34.33	+
11	.60	2.010	.31	1.981	.15	29.65	33.640	.64	34.64	+
12	.89	2.039	.74	2.024	.99	30.49	34.553	.63	34.63	+
13	.58	2.008	.83	2.033	.44	29.94	33.980	.64	34.64	+
14	.59	2.009	.03	1.953	.62	30.12	34.082	.30	34.30	+
15	.16	1.966	.57	2.007	.87	30.37	34.343	.21	34.21	–
16	.36	1.986	.60	2.010	.82	30.32	34.316	.83	34.83	+
17	.70	2.020	.06	1.956	.12	29.62	33.596	.59	34.59	+
18	.46	1.996	.54	2.004	.04	29.54	33.540	.51	34.51	+
19	.99	2.049	.84	2.034	.81	30.31	34.393	.15	34.15	–
20	.36	1.986	.12	1.962	.54	30.04	33.988	.97	34.97	+

Out of 20 iterations of generated samples in 3 cases the clearance is negative. Thus the probability of parts not assembling is $3/20 \times 100 = 15\%$ or 0.15.

Example 2.9

A piece of equipment contains four identical tubes and can function only if all the four are in working order. The lives of tubes has approximately uniform distribution from 1000 to 2000 hours. The current maintenance practice is to replace a tube when it fails. Equipment has to be shut down for 1 hour for replacing a tube, the cost of one tube is Rs. 100, while the shut down time costs Rs. 200 per hour. Simulate the system for about 6000 hours of run and find the maintenance cost.

P.U.M.E. (Mech.), 1991

Solution: The piece of equipment has four identical tubes and comes to halt when any one fails. It is then replaced which takes one hour time and incurs Rs. 300 as cost (cost of tube and cost of down time). The lives of tube are uniformly distributed form 1000 to 2000. To keep the computations simple, one digit random numbers can be used to generate the tube lives. If R is the random number, then the life of a tube L is given by

$$L = 1000 + (2000 - 1000)\,R/10 = 1000 + 100 \times R$$

The random numbers can be taken from a random number table. Start with any one digit in the table and move in row or column continuously. One string of random numbers is,

8, 2, 6, 3, 1, 0, 8, 9, 2, 8, 3, 7, 4, 8, 5, 6, 0, 4, 9

The corresponding tube lives are

1800, 1200, 1600, 1300, 1100, 1000, 1800, 1900, 1200, 1800,
1300, 1700, 1400, 1800, 1500, 1600, 1000, 1400, 1900.

The simulation of the system is given in Table 2.11. The variable '*clock*' indicates the time elapsed, while *T*1, *T*2, *T*3 and *T*4 represent the tube lives, that is the time after which the tube will fail. The simulation starts at zero time, with all the four tubes new having lives 1800, 1200, 1600 and 1300. The shortest life is of Tube *T*2, which will fail after 1200 hours. The clock jumps to 1200 and time to failure of tubes *T*1, *T*3 and *T*4 is computed. *T*2 is replaced with a new tube with life of 1100 hours. Now out of *T*1, *T*2, *T*3, and *T*4 the next tube to foil is *T*4 with life of 100 hours. Advance the clock by 100 hours, update the time to failure of all the tubes, and replace *T*4 by a new tube, corresponding to next random number. If two tubes fail at the same time, both are replaced but the downtime remains same. In the stipulated simulation period of 6000 hours the system failed 14 times, and 15 tubes were replaced.

Maintenance cost = 15 × 100 + 14 × 200 = Rs. 3300/- only.

Table 2.11

Clock	*T1*	*T2*	*T3*	*T4*	*No. of tubes replaced*
0	1800	**1200**	1600	1300	1
1200	600	1100	400	**100**	1
1300	500	1000	**300**	1300	1
1600	**200**	700	1000	1000	1
1800	1800	**500**	800	800	1
2300	1300	1900	**300**	**300**	2
2600	**1000**	1600	1200	1800	1
3600	1300	600	**200**	800	1
3800	1100	**400**	1700	600	1
4200	700	1400	1300	**200**	1
4400	**500**	1200	1100	1800	1
4900	1500	700	**600**	1300	1
5500	900	**100**	1600	700	1
5600	800	1000	1500	**600**	1
6200	200	400	900	–	–

2.4 Exercises

1. Make an irregular figure of known area, then determine its area by the Monte Carlo Method. Use the random number table for generating the points. Compare the accuracy of your result, when the total number of points is,

   (*i*) 50 (*ii*) 100 (*iii*) 200.

2. Solve the following integrals by Monte Carlo Method.

   (*i*) $I = \int_0^1 x dx$ (*ii*) $I = \int_1^5 \frac{x^4}{3} dx$

3. Determine the value of π by employing the Monte Carlo Method, and using the relation,

   Area of Circle = $\frac{\pi D^2}{4}$. Determine the values for 20, 40 and 100 pairs of random numbers.

4. Develop a computer program for determining the value of π and run the simulation for 1000 observations.

5. In the random walk problem of Example 2.5, the drunkard can take steps in four directions, forward, backward, to left and to right. The probabilities associated with these are 40%, 10%, 25% and 25%. The distances covered in the forward, backward, to left and to right steps are 75 cm , 45 cm, 60 cm and 60 cm respectively. Simulate the walk for 50 steps, and find the location at the end of 50 steps, while the starting point is (0, 0) on the x-y scales.
6. In the bombing problem, the ammunition depot area is a circle of 500 m radius. The point of impact is normally distributed around the aiming point with standard deviation of 500 m in X-direction and 300 m in the Y-direction. Simulate the bombing operation for 40 strikes and find the percentage of strikes on target.
7. The random variables x, y and z are distributed as under,

$$x \sim N\ (\ \mu = 50,\ \ \sigma^2 = 65\ )$$
$$y \sim N\ (\ \mu = 100,\ \sigma^2 = 100\ )$$
$$z \sim N\ (\ \mu = 150,\ \ \sigma^2 = 150\ )$$

Simulate 50 values of the random variate,

$$W = (\ x + y\ )/z.$$

8. Solve the following integral by Monte Carlo Method.

$$\int_2^5 x^3\, dx$$ **[PTU B. Tech (Prod.) May 2006]**

9. The random variables X and Y are distributed as follows.

$$Y \sim 10 \pm 10 \text{ (uniform)}$$
$$Y \sim 10 \pm 8 \text{ (uniform)}$$

Simulate 20 values of the random variables

(*i*) $z = xy$ (*ii*) $k = \dfrac{x}{y}$

10. In the reliability problem of Example 2.6, the bearing life is distributed as follows:

Hours	*Probability*
1200	.10
1400	.25
1600	.30
1800	.25
2000	.10

The remaining data is same. Simulate the system for the present and proposed policies of maintenance, and find the best policy.

11. Write a computer program in any language for simulating the reliability problem of Example 2.6 and run the program for 10,000,00 hours of operation.
12. Write a computer program in C language for simulating the Random walk problem of Example 2.5. Simulate the walk 10 times, each time for 500 steps. Use different stream of random numbers each time. Find the mean value of distance traveled.
13. Write a computer program for the simulation of hinge assembly problem of Example 2.8. Run the simulation for 1000 iterations and determine the probability of parts not assembling.