Outline

Innehåll

1 Improper Integrals and a Mean-Value Theorem

1.1 Improper Integrals of Positive Functions

Example 1 (Improper Integrals over Unbounded Domains)**.** Compute

$$\iint_D e^{-x^2}\,dA, \quad \text{where } D = \{(x,y) : -x \le y \le x,\ x \ge 0\}$$

Solution:
Let $D_R = D \cap \{0 < x < R\}$ and calculate

$$\begin{aligned}\iint_{D_R} e^{-x^2}\,dA &= \int_0^R dx \int_{-x}^{x} e^{-x^2}\,dy \\ &= 2\int_0^R xe^{-x^2} dx = -Re^{-R^2} + 1.\end{aligned}$$

Hence

$$\iint_D e^{-x^2}\,dA = \lim_{R\to\infty} \iint_{D_R} e^{-x^2}\,dA = 1.$$

Generalized double integrals

Example 2 (Improper Integrals of Unbounded Functions)**.** Let $D = \{(x,y) : x > 0, y > 0, x^2 + y^2 < 1\}$. Compute

$$\iint_D \frac{1}{(x^2+y^2)^{3/4}}\,dA.$$

(Hint: $\iint_D f(x,y)\,dA = \iint_E f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$.)

Solution: Let $D_s = D \cap \{x^2 + y^2 > s^2\}$. By the polar coordinates,

$$\begin{aligned}\int_{D_s} \frac{1}{(x^2+y^2)^{3/4}}\,dA &= \int_s^1 dr \int_0^{\pi/2} \frac{r\,d\theta}{r^{3/2}} = \frac{\pi}{2}\int_s^1 \frac{dr}{r^{1/2}} \\ &= \pi(1 - s^{1/2}) \to \pi \quad \text{as } s \to 0.\end{aligned}$$

Hence the answer is π.

1.2 A Mean-Value Theorem for Double Integrals

Theorem 1. *If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then there is a point $(x_0, y_0) \in D$ such that*

$$\iint_D f(x,y)\,dA = f(x_0,y_0) \cdot (\text{the area of } D).$$

The **mean value** *of f over D is given by*

$$\bar{f} = \frac{1}{(\text{the area of } D)} \iint_D f(x,y)\,dA$$

Example 3. Calculate the mean value of

$$x^2 + y^2$$

over the unit disk.
(Hint: $\iint_D f(x,y)\,dA = \iint_E f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$.)

Solution: Let D be the unit disk, $\{(x,y) : x^2 + y^2 \le 1\}$. Since the area of D is π, we have

$$\begin{aligned}\bar{f} &= \frac{1}{\pi}\iint_D (x^2+y^2)\,dA = \frac{1}{2\pi}\int_0^1 r\,dr \int_0^{2\pi} r^2\,d\theta \\ &= 2\int_0^1 r^3\,dr = \frac{1}{2}.\end{aligned}$$

2 Double Integrals in Polar Coordinates

2.1 Change of Variables in Double Integrals

In previous examples, we used polar coordinates to compute some integrals, without justifying the formula

$$\iint_D f(x,y)\,dA = \iint_E f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Formally, this means

$$dx\,dy = dA = r\,dr\,d\theta.$$

Note the Jacobian of map $(x,y) = (r\cos\theta, r\sin\theta)$ is given by

$$\det\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Hence, the formula can be rewritten as

$$dx\,dy = \left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta.$$

This is true in general, as the Jacobian measures the rate of change of area for a bijective map.

Theorem 2. *Let $x = x(u,v)$ and $y = y(u,v)$ be C^1 bijective mappings from E on uv-plane to D on xy-plane. If $f(x,y)$ is integrable on D and $g(u,v) = f(x(u,v), y(u,v))$ is integrable on E, then* [1]

$$\iint_D f(x,y)\,dxdy = \iint_E g(u,v)\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where

$$\frac{\partial(x,y)}{\partial(u,v)}$$

is the Jacobian matrix.

Example 4. Let $D = \{(x,y) : x^2 + y^2 \leq 1\}$. Use polar coordinates to calculate

$$\iint_D (1 - x^2 - y^2)\,dxdy.$$

Solution: As for the polar coordinates (r,θ), we have already observed that

$$\det \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Therefore, by the change of variables,

$$\iint_D (1 - x^2 - y^2)\,dx\,dy = \int_0^{2\pi} d\theta \int_0^1 (1 - r^2) r\,dr\,d\theta = \frac{\pi}{2}.$$

Example 5. Compute the improper integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

Solution: Let a be the value of the integral. Then

$$\begin{aligned} a^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\ &= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy \\ &= \lim_{R\to\infty} \iint_{D_R} e^{-x^2-y^2}\,dx\,dy, \end{aligned}$$

where $D_R = \{(x,y) : x^2 + y^2 \leq R^2\}$.

Solution continued:

However, using the polar coordinates

$$\begin{aligned} \iint_{D_R} e^{-x^2-y^2}\,dx\,dy &= \int_0^{2\pi} d\theta \int_0^R re^{-r^2}\,dr \\ &= 2\pi \left[-\frac{e^{-r^2}}{2}\right]_0^R \\ &= \pi(1 - e^{-R^2}) \to 2\pi \quad \text{as } R \to \infty. \end{aligned}$$

Thus, $a^2 = \pi$, or

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

[1] http://demonstrations.wolfram.com/2DJacobian/

2.2 More Exercises

Remark
Note that

$$\det \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\det \frac{\partial(u,v)}{\partial(x,y)}}.$$

Example 6. Determine the Jacobian $\det \frac{\partial(x,y)}{\partial(u,v)}$ where

$$(u, v) = (xy, x - y).$$

Solution:

$$\det \frac{\partial(x, y)}{\partial(u, v)} = \left(\det \frac{\partial(u, v)}{\partial(x, y)}\right)^{-1} = -\frac{1}{x + y}.$$

Example 7. Calculate the double integral

$$\iint_D xy(x^2 - y^2)dxdy,$$

where

$$D = \{(x, y) : 1 < xy < 2, 1 < x - y < 2\}.$$

Solution: Set $(u, v) = (xy, x - y)$. Then

$$\det \frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{x + y}.$$

Solution continued:
Thus, by the change of variables, we have

$$(x + y)\, dx\, dy = du\, dv.$$

Hence, setting $E = \{(u, v) : 1 < u < 2, 1 < v < 2\}$, we obtain

$$\begin{aligned} \iint_D xy(x^2 - y^2)\, dx\, dy &= \iint_D xy(x - y)(x + y)\, dx\, dy \\ &= \iint_E uv\, du\, dv = \int_1^2 dv \int_1^2 uv\, du \\ &= \frac{9}{4}. \end{aligned}$$

Example 8.

1. Show by using the polar coordinates that

$$\iint_D x\, dxdy = 8/3,$$

   if D is given by $x \geq 0,\ y \geq 0$ and $x^2 + y^2 \leq 4$.

2. Show with the change of variables that

$$\iint_D (2y - x)\, dxdy = \frac{5}{6}$$

   if D is given by $0 \leq x + y \leq 1$ and $2 \leq 2y - x \leq 3$.

3. Determine the centre of mass for D given by $x^2 \leq y \leq x$.