Answers to questions in
Lab 1: Filtering operations

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Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Good luck!

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Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?
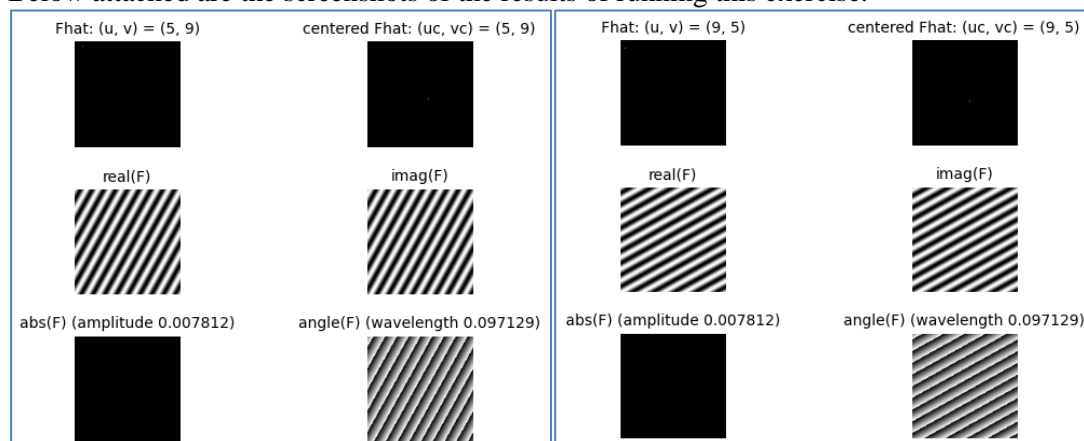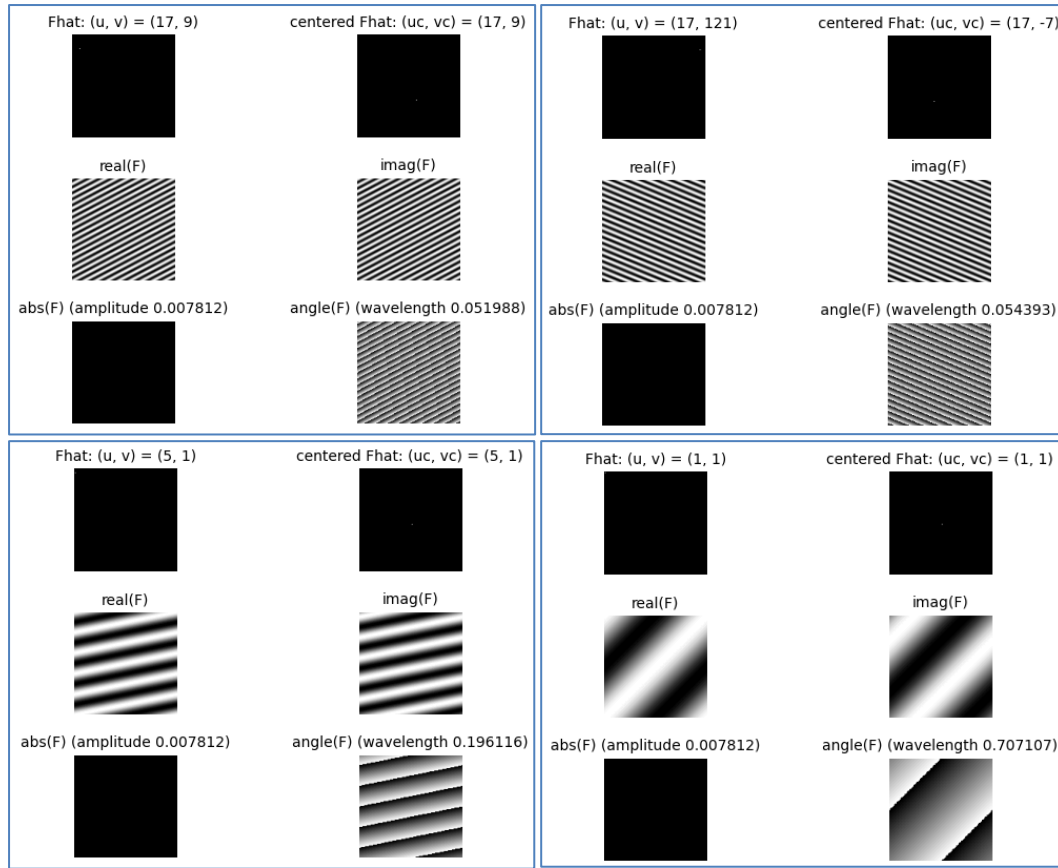
Answers:
As per my understanding, the points all correspond to the frequency component at a point in the image. The p value stands for the x-component of the frequency while the q value stands for the y-component of the frequency.
- Fhat: This shows where the frequency component lies on the frequency plane given that the top left corner denotes zero frequency.
- Centered Fhat: This shows where the frequency component lies on the frequency plane given that the centre denotes zero frequency.
- Real: This corresponds to the cosine component of the Fourier transform and denotes the magnitude of the signal at that point.
- Imaginary: This corresponds to the sine component of the Fourier transform and denotes the phase shift of the signal.
- Absolute: This shows the overall amplitude of the Fourier transform.
- Angle: This shows the overall wavelength of the Fourier transform.

As the points move further away from the zero frequency component, it can be seen that the wavelength decreases; hence showing that the frequency increases as the point moves further away from the origin.
Below attached are the screenshots of the results of running this exercise.

Fhat: (u, v) = (17, 9)	centered Fhat: (uc, vc) = (17, 9)	Fhat: (u, v) = (17, 121)	centered Fhat: (uc, vc) = (17, -7)
real(F)	imag(F)	real(F)	imag(F)
abs(F) (amplitude 0.007812)	angle(F) (wavelength 0.051988)	abs(F) (amplitude 0.007812)	angle(F) (wavelength 0.054393)

Fhat: (u, v) = (5, 1)	centered Fhat: (uc, vc) = (5, 1)	Fhat: (u, v) = (1, 1)	centered Fhat: (uc, vc) = (1, 1)
real(F)	imag(F)	real(F)	imag(F)
abs(F) (amplitude 0.007812)	angle(F) (wavelength 0.196116)	abs(F) (amplitude 0.007812)	angle(F) (wavelength 0.707107)

Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a figure.

Answers:
The position represents the frequency and orientation of the sinusoidal waves. The below equation represents the inverse Fourier transform in the discrete domain.

$$F(x) = \frac{1}{N} \sum_{u \in [0:N-1]^2} \hat{F}(u) e^{\left(\frac{2(\pi) i u^T x}{N}\right)}$$

Applying the inverse Fourier Transform at the point (p,q), we get,

$$F(x) = \frac{1}{N} e^{\left(\frac{2(\pi) i (p,q)^T x}{N}\right)}$$

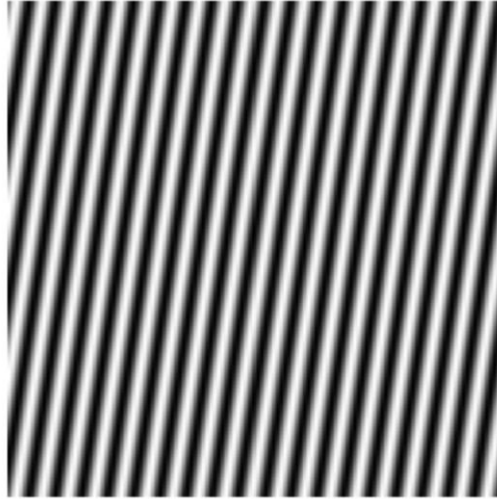The below equation shows the Euler Identity formula.

$$e^{\left(i\omega^T x\right)} = cos(\omega^T x) + i sin(\omega^T x)$$

By applying the Euler identity to the inverse Fourier transform, we get,

$$F(x) = \frac{1}{N} cos\left(\frac{2\pi(p,q)^T x}{N}\right) + i \frac{1}{N} sin\left(\frac{2\pi(p,q)^T x}{N}\right)$$

These terms within the inverse Fourier transform show us that the resultant is a sinusoid. The below image shows the sine wave from a point (p,q) = (19,3).

2D Sine Wave from Fourier Domain Point (19, 3)

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Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:
The below equation represents the inverse Fourier transform in the discrete domain.

$$F(x) = \frac{1}{N} \sum_{u \in [0:N-1]^2} \hat{F}(u) e^{\left(\frac{2\pi i u^T x}{N}\right)}$$

Applying the inverse Fourier transform at the point (p,q), we get,

$$F(x) = \frac{1}{N} e^{\left(\frac{2\pi i (p,q)^T x}{N}\right)}$$

By applying the Euler Identity to the inverse Fourier transform, we get,

$$F(x) = \frac{1}{N} \cos\left(\frac{2\pi (p,q)^T x}{N}\right) + i \frac{1}{N} \sin\left(\frac{2\pi (p,q)^T x}{N}\right)$$

The amplitude of the inverse Fourier Transform is,

$$|F(x)| = \frac{1}{N}$$

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Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answers:
To find the phase of the sine wave (direction), we use,

$$\phi = tan^{-1}\left(\frac{Im[Fhat(u,v)]}{Re[Fhat(u,v)]}\right)$$

The wavelength of the sine wave is given by,

$$\lambda = \frac{1}{\sqrt{u^2 + v^2}}$$

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Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically!

Answers:
When we center Fourier Transform, we can take advantage of the symmetry property of the frequency components. This means that the component can be remapped in the Fourier domain.

Fhat: (u, v) = (17, 121)   centered Fhat: (uc, vc) = (17, -7)

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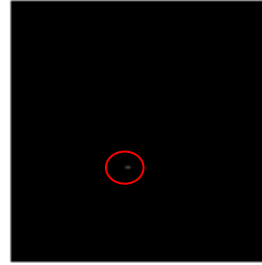
Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

Answers:
These instructions provide the coordinates for the x and y frequency components after centering the zero-frequency component on the graph.

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Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answers:
For the discrete Fourier transform in two dimensions, we have,

$$\hat{F}(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2\pi i \left(\frac{xu}{M} + \frac{yv}{N}\right)}$$

For the above signal, we get,

$$\hat{F}(u,v) = \frac{1}{128} \sum_{y=57}^{72} \sum_{x=0}^{127} e^{-2\pi i \left(\frac{xu}{128} + \frac{yv}{128}\right)}$$

$$\hat{F}(u,v) = \frac{1}{128} \sum_{y=57}^{72} \sum_{x=0}^{127} e^{-2\pi i \left(\frac{xu}{128}\right)} e^{-2\pi i \left(\frac{yv}{128}\right)}$$

$$\hat{F}(u,v) = \frac{1}{128} \sum_{y=57}^{72} e^{-2\pi i \left(\frac{yv}{128}\right)} \sum_{x=0}^{127} e^{-2\pi i \left(\frac{xu}{128}\right)}$$

$$\hat{F}(u,v) = \sum_{y=57}^{72} e^{-2\pi i \left(\frac{yv}{128}\right)} \delta_u$$

The Dirac function is defined such that the peak is only present at x = 0 hence the Fourier transform is concentrated at the border. This is the reference to the proof used to understand the presence of the Dirac function. https://math.stackexchange.com/questions/911959/the-result-of-exponential-sum-formula

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Question 8: Why is the logarithm function applied?

Answers:
The logarithm function is used to make more details visible by accounting for a large dynamic range and allowing lower ranges to become more visible.

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Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers:
In the Fourier domain, the linearity of the signal is preserved. The Fourier transforms of F and G can be seen within the first two graphs. It can be observed that adding the Fourier transforms of the individual signals is equivalent to evaluating the Fourier transform of the combined signal. We can write the general mathematical form as,
$$\mathcal{F}\big(f(t) + g(t)\big) = F(\omega) + G(\omega)$$

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Question 10: Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

Answers:
When multiplying in the Fourier domain, it is an equivalent of convolution in the Spatial domain. Another way to compute the same image is to convolve the two signals, then perform a Fourier transform on it to obtain the image above.
$$\mathcal{F}[f(x) * g(x)] = F(u)G(u)$$
The first image is the Fourier transform of f(x), while the second image is the Fourier Transform of g(x). The third image shows the convolution of the two.

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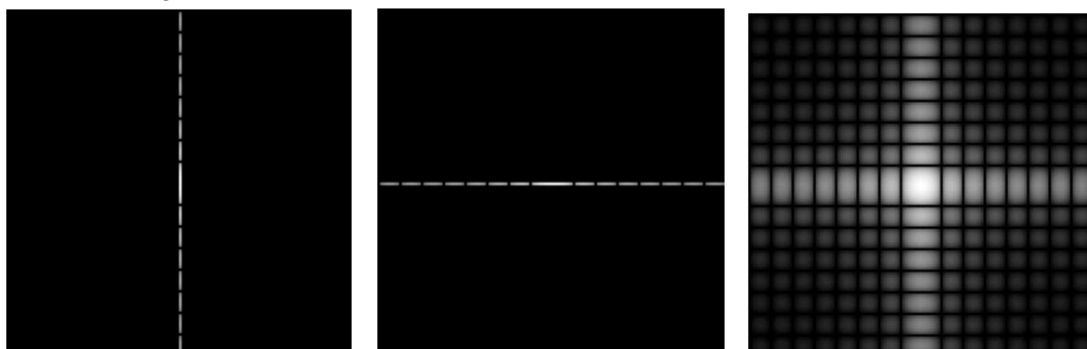
Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

Answers:
The image, in spatial domain, has been shrunk in the y-axis and expanded in the x-axis. This corresponds to expanding in the y-axis of the Fourier domain and the shrinking in the x-axis in the Fourier domain.

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Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.
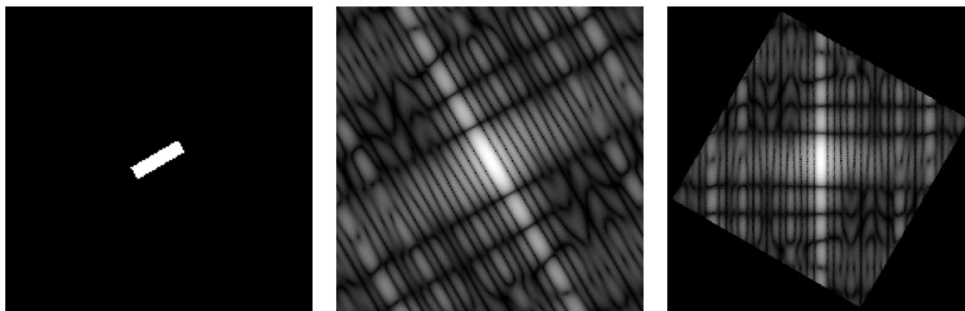
Answers:

When an image is rotated in the spatial domain, its equivalent representation in the frequency domain is also rotated. The value of the frequencies at each point remain the same except that they are rotated with the spatial image. When rotating an image at non-orthogonal angles, aliasing artifacts can occur. These can be seen when alpha has the value of 30 and 60 degrees. The frequency components are not an exact rotation of the signal when alpha is 0 degrees.

Below some screenshots are attached of the results of the exercise with the changing values of alpha. The first image denotes the image in the spatial domain. The second image denotes the image in the frequency domain. The third image denotes the frequency domain after being rotated back by alpha degrees.
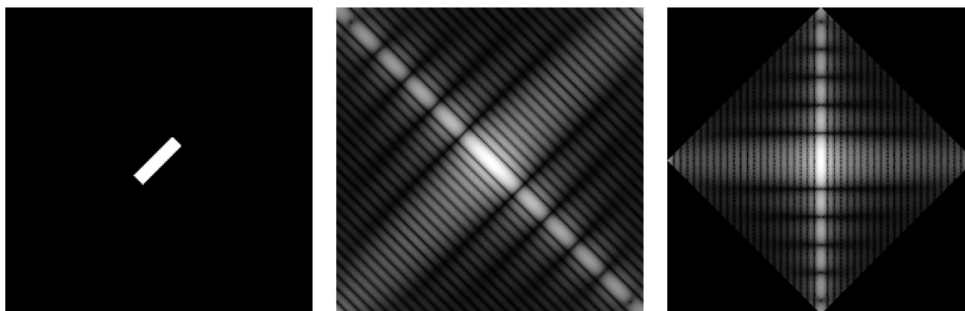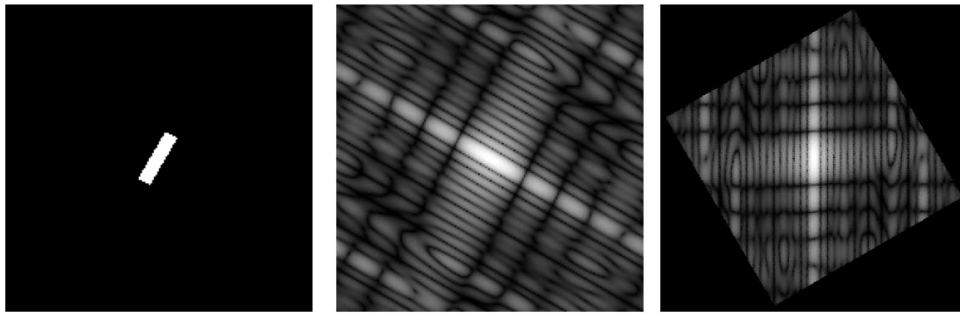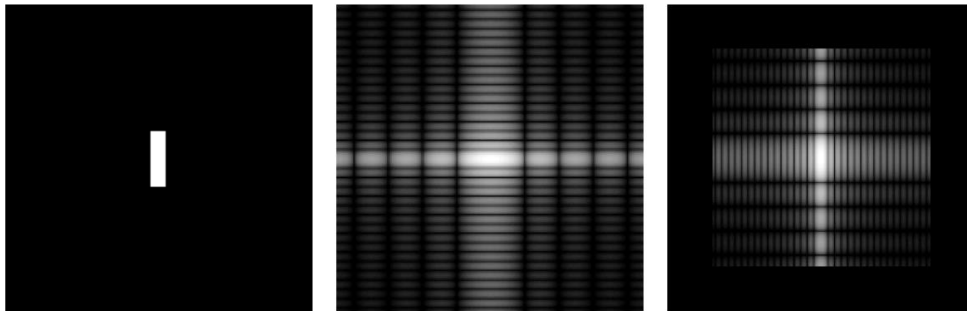
For alpha = 0,

For alpha = 30,

For alpha = 45,

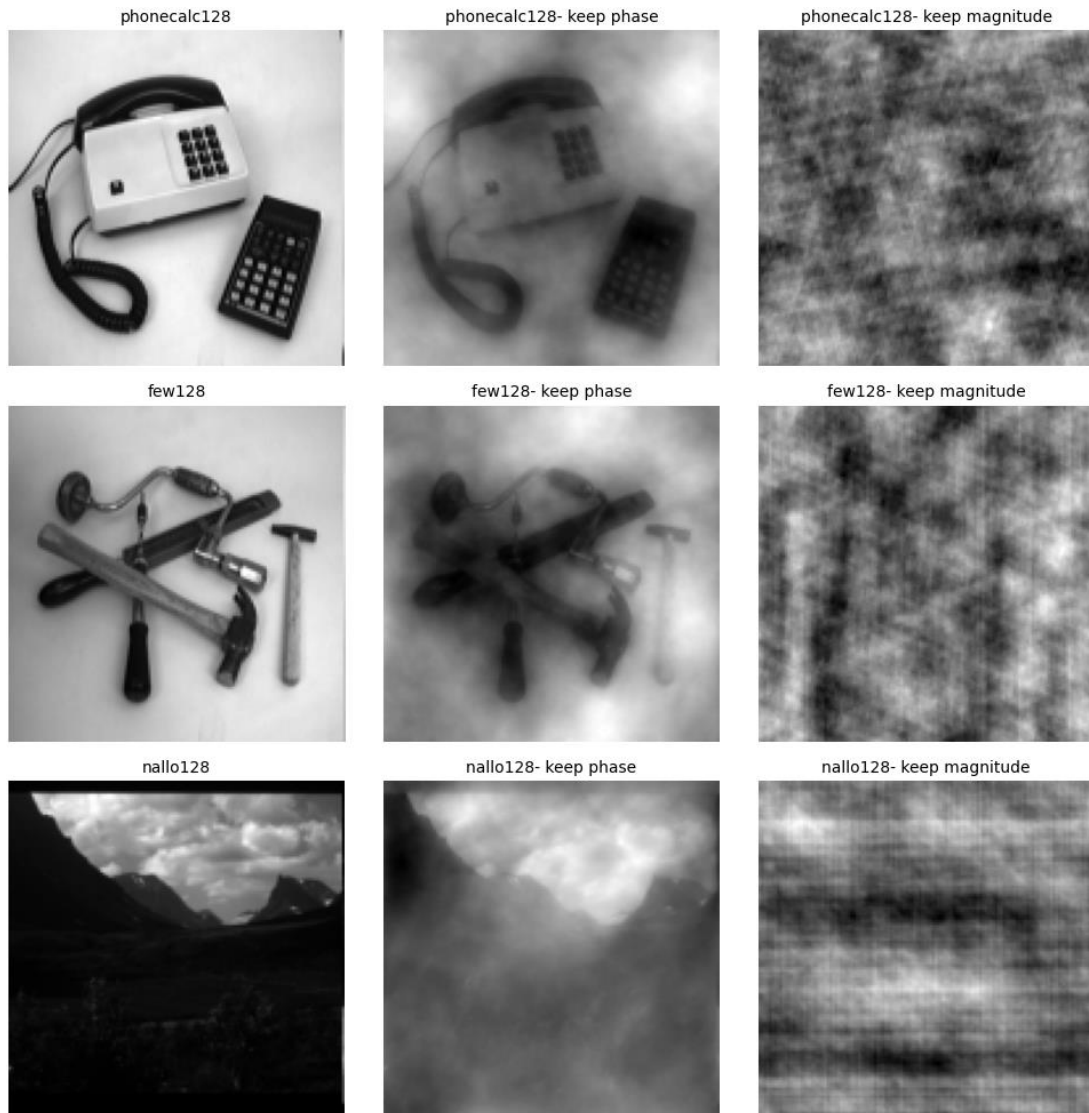For alpha = 60,
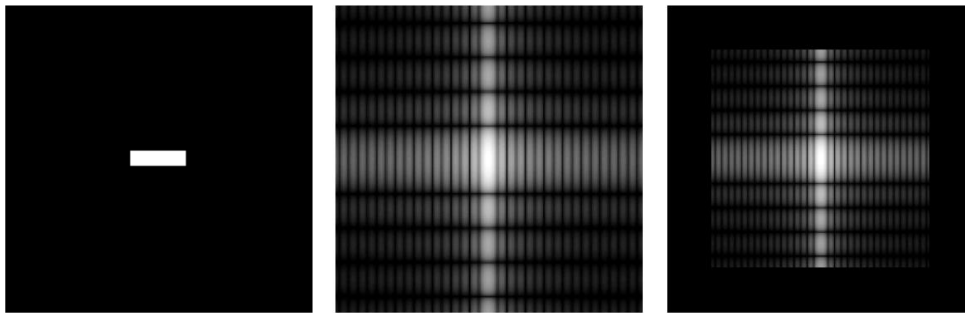
For alpha = 90,

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Question 13: What information is contained in the phase and in the magnitude of the Fourier transform?

Answers:
The magnitude of the Fourier transform represents the amplitude of each frequency component in the image. The higher the magnitude component, the more dominant the frequency component within the signal. The magnitude is a real and non-negative number.

The phase represents the phase shift or time delay of each frequency component and is important in the reconstruction of the original image as it contains information about the position and alignment of the edges and contours in the spatial domain of the image. The phase of an image is generally unique. This means that different images with the same magnitude spectrum can have significantly different phase spectra.

When using pow2image, we see that the position of the edges of the image are preserved but the magnitude has changed. When using randphaseimage, we see that the images cannot be recognized as the positions of the edges have not been preserved within the image. This means that the image can be recognized if the phase is preserved even if the magnitude is lost.

phonecalc128	phonecalc128- keep phase	phonecalc128- keep magnitude
few128	few128- keep phase	few128- keep magnitude
nallo128	nallo128- keep phase	nallo128- keep magnitude

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Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?
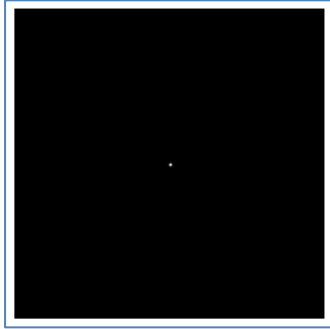
Answers:
In the below images, both **gaussfft** and **discgaussfft** have been applied to the impulse image. For t = 0.1,

      When applying **gaussfft**, we get the variance to be
      [[1.06513806e+02 2.36468622e-11]
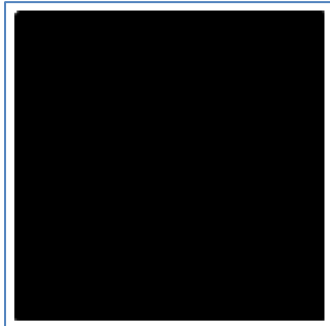   [2.36468622e-11 1.06513806e+02]]
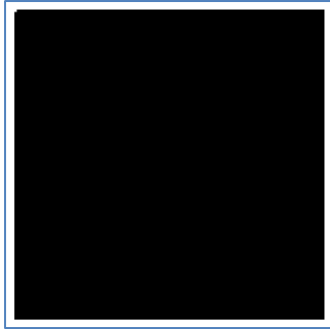
When applying **discgaussfft**, we get the variance to be
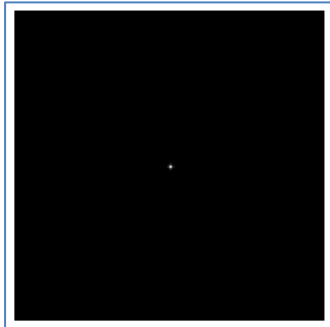[[1.00000000e-01 9.90874049e-15]
 [9.90874049e-15 1.00000000e-01]]

For t = 0.3,

When applying **gaussfft**, we get the variance to be
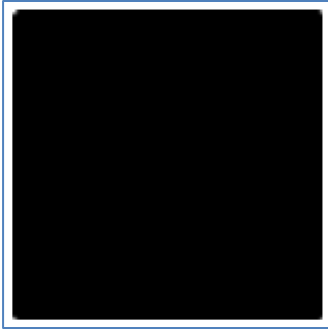[[1.91089566e+03 1.50066626e-11]
 [1.50066626e-11 1.91089566e+03]]

When applying **discgaussfft**, we get the variance to be
[[3.00000000e-01 -8.54039062e-14]
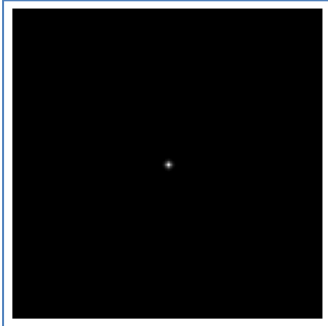 [-8.54039062e-14 3.00000000e-01]]

For t = 1.0,

When applying **gaussfft**, we get the variance to be
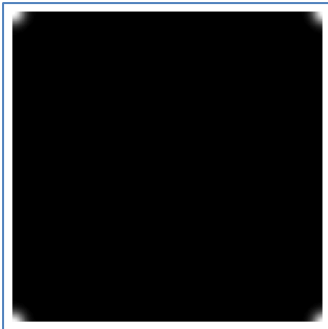[[3.35197087e+03 2.27373675e-12]
 [2.27373675e-12 3.35197087e+03]]

When applying **discgaussfft**, we get the variance to be
[[1.00000000e+00 1.80133686e-14]
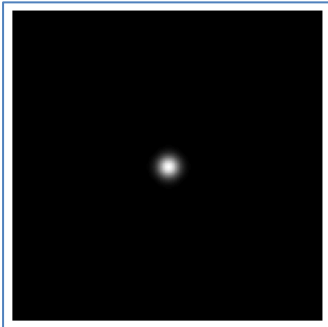 [1.80133686e-14 1.00000000e+00]]

For t = 10.0,

When applying **gaussfft**, we get the variance to be
[[3.72055413e+03 -7.10542736e-14]
 [-7.10542736e-14 3.72055413e+03]]
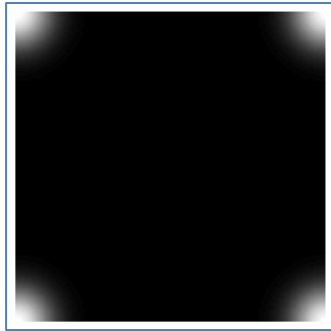

When applying **discgaussfft**, we get the variance to be
[[1.00000000e+01 1.48214774e-14]
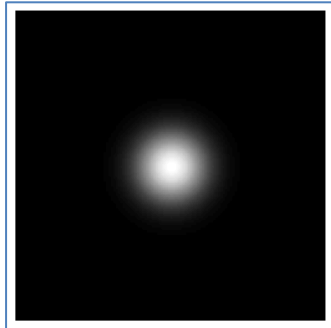 [1.48214774e-14 1.00000000e+01]]

For t = 100.0,

      When applying **gaussfft**, we get the variance to be

      [[3.16904028e+03 -9.76996262e-15]

      [-9.76996262e-15 3.16904028e+03]]

      When applying **discgaussfft**, we get the variance to be

      [[9.99999999e+01 -9.50073353e-14]

      [-9.50073353e-14 9.99999999e+01]]

As the value of t increases, the variance increases, and the impulse response becomes larger.

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Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.
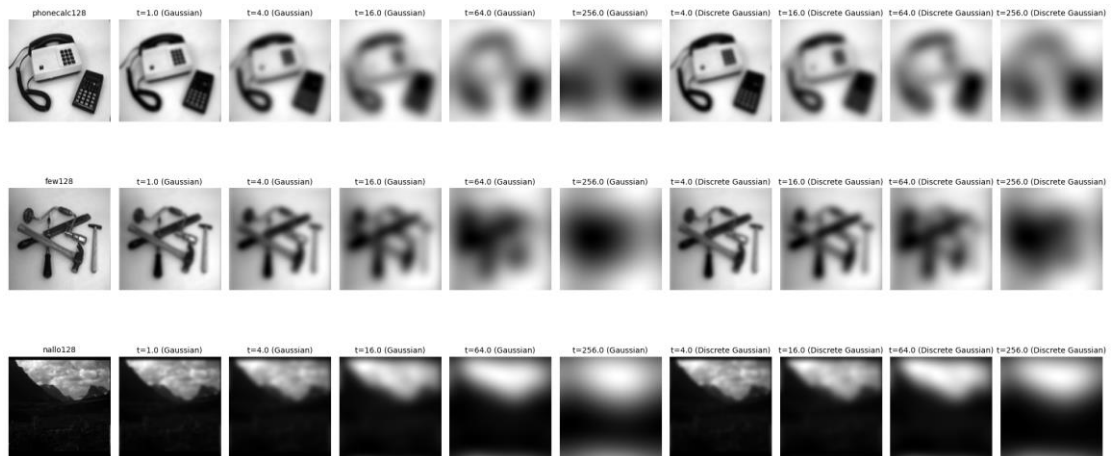
Answers:

In the ideal case, the variance is the product of t and the identity matrix. When t < 1, the variance differs from the ideal case due to incorrect Gaussian approximation and sampling. The discretized Gaussian does not estimate the continuous Gaussian. When t > 1, the variance matches the ideal case.

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Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

Answers:

As the value of t increases, the recreated image becomes more and more blur. This is because the variance of the Gaussian filter increases, and the higher frequency components are removed from the image. High frequency components correspond to the edges and contours within the image. Below attached is the result from convolving the images with the discrete and continuous filters.

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Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters? Illustrate with figure(s).

Answers:

These images show the original image, and the image with gaussian and salt and pepper noise.

Results of Gaussian Filtering:

Gaussian filtering results in the most blurring for the image. Most of the Gaussian noise generated is smoothed off. As the value of t increases, the image becomes more blurred and the edges are lost. When applied to the salt and pepper noise, the noise is blurred out along with the edges. Noise not removed from the image, making the image look more noisy. As the value of t increases, the image is more blurred.

The first image in each row corresponds to the continuous Gaussian filtering while the second image corresponds to discrete Gaussian filtering.
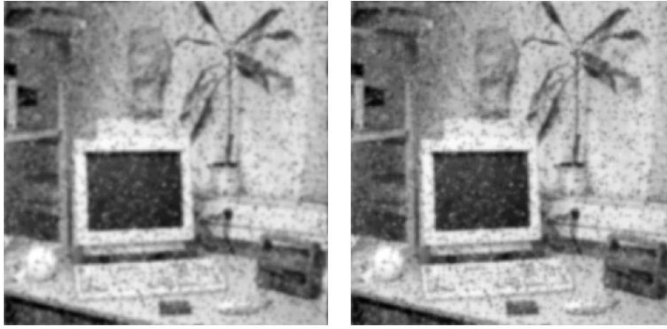
Results of Median Filtering:
This filter seems most effective and removes all the salt and pepper noise. The edges of all the objects are also preserved in both images for small window sizes. The "add" image seems noisier than the "sap" image. Some of the contours are lost as the image looks more like a painting (this applies to the "sap" image). As the window size increases, more edges and contours are lost.

The first image in each row corresponds to the median filtering of the "add" image while the second image corresponds to the median filtering of the "sap" image.

Results of Low Pass Filtering:
This filter seems more effective for the gaussian noise than the salt and pepper noise. It is used to remove the frequencies about the cuttoff frequency which means that high frequency noise can be removed. In the case of the salt and pepper noise, the noise is smoothed but not removed from the image. Both images are smoothed. As the cutoff frequency increases, fewer high frequency components are removed from the image.

The first image in each row corresponds to the low pass filtering of the "add" image while the second image corresponds to the low pass filtering of the "sap" image.

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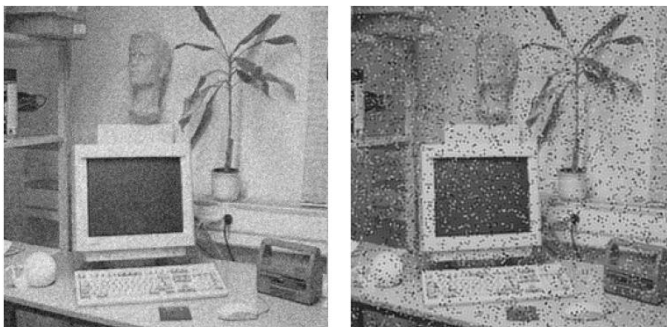
Question 18: What conclusions can you draw from comparing the results of the respective methods?

Answers:
Gaussian filtering is most effective when applied to images where only Gaussian noise is added. Median filtering is most effective when applied to the images where the Salt and Pepper noise is added. Applying the low pass filter can result in the ringing effect.
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Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

Answers:
In the fourth iteration (i = 4), the image becomes more blurred and the definition of the edges is lost. The Gaussian filter smoothens the image more and is more appealing. Low pass filter causes the image to still have a lot of noise in the fourth iteration when compared to the Gaussian. Subsampling alone leads to the presence of aliasing artifacts while applying the smoothing filters remove most of the artifacts and give a less noisy image.

This is the result of applying the Gaussian filter.

This is the result of applying the Low-pass filtering.

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Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers:
Smoothing the image first before subsampling is done to avoid the aliasing effect. When applying a smoothing filter, the high frequency components of the image are removed. Upon subsampling, every alternate pixel is retained. This reduces the overall frequency of the image and avoids artifacts due to the mirroring of high frequencies from being present in the image. This results in lowered information loss. Although some information is lost during smoothing, more is lost due to the aliasing effect if smoothing is not applied.

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