Интегрирование, часть 5, практическая часть

8.5.2. $\int \frac{dx}{\sin x} =$

1 сп.

$R(-\sin x\,;\cos x) = -R(\sin x\,;\cos x)$, случай 3

$$= \left[t = \cos x \to x = \arccos t\,; dx = -\frac{dt}{\sqrt{1-t^2}}; \sin x = \sqrt{1-t^2}\right] = \int \frac{-\frac{dt}{\sqrt{1-t^2}}}{\sqrt{1-t^2}}$$

$$= -\int \frac{dt}{\sqrt{1-t^2}\sqrt{1-t^2}} = -\int \frac{dt}{1-t^2} = \int \frac{dt}{t^2-1} = \left[\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{2*1} * \ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C = \ln\left(-\operatorname{tg}^2\frac{x}{2}\right)^{\frac{1}{2}} + C$$

$$= \ln\left|\left(\operatorname{tg}^2\frac{x}{2}\right)^{\frac{1}{2}}\right| + C = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

2 сп.

$$\int \frac{dx}{a\sin x + b\cos x + c}; a \neq 0, b \neq 0$$

$$\int \frac{dx}{\sin x} = \left[a \neq 0; t = \operatorname{tg}\frac{x}{2} \to x = 2\operatorname{arctg} t\,; dx = \frac{2td}{1+t^2}; \sin x = \frac{2t}{1+t^2}\right] = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}}$$

$$= \int \frac{2(1+t^2)}{2t(1+t^2)}dt = \int \frac{dt}{t} = \ln|t| + C = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

8.5.3. $\int \frac{dx}{5\cos x+3} = \left[t = \operatorname{tg}\frac{x}{2} \to x = 2\operatorname{arctg} t\,; dx = \frac{2dt}{1+t^2}; \cos x = \frac{1-t^2}{1+t^2}\right] = \int \frac{\frac{2dt}{1+t^2}}{5*\frac{1-t^2}{1+t^2}+3} =$

$\int \frac{\frac{2dt}{1+t^2}}{\frac{5(1-t^2)+3(1+t^2)}{t+t^2}} = \int \frac{2dt}{5-5t^2+3+3t^2} = \int \frac{2dt}{8-2t^2} = \int \frac{2dt}{2(4-t^2)} = \int \frac{dt}{4-t^2} = -\int \frac{dt}{t^2-4} = -\frac{1}{2*2}\ln\left|\frac{t-2}{t+2}\right| + C =$

$-\frac{1}{4}\ln\left|\frac{\operatorname{tg}\frac{x}{2}-2}{\operatorname{tg}\frac{x}{2}+2}\right| + C = \frac{1}{4}\ln\left|\frac{\operatorname{tg}\frac{x}{2}+2}{\operatorname{tg}\frac{x}{2}-2}\right| + C$

8.5.5. $\int \frac{dx}{3\sin^2 x+5\cos^2 x} = \left[t = \operatorname{tg} x \to x = \operatorname{arctg} t\,; dx = \frac{dt}{1+t^2}; \sin x = \frac{t}{\sqrt{1+t^2}}; \cos x = \frac{1}{\sqrt{1+t^2}}\right] =$

$\int \frac{\frac{dt}{1+t^2}}{3\left(\frac{t}{\sqrt{1+t^2}}\right)^2+5\left(\frac{1}{\sqrt{1+t^2}}\right)^2} = \int \frac{\frac{dt}{1+t^2}}{\frac{3t^2+5}{1+t^2}} = \int \frac{dt}{3t^2+5} = \int \frac{dt}{3\left(t^2+\frac{5}{3}\right)} = \frac{1}{3}\int \frac{dt}{t^2+\left(\sqrt{\frac{5}{3}}\right)^2} = \left[\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C\right] = \frac{1}{3} *$

$\frac{1}{\sqrt{\frac{5}{3}}} * \operatorname{arctg}\frac{t}{\sqrt{\frac{5}{3}}} = \frac{\sqrt{3}}{3\sqrt{5}}\operatorname{arctg}\frac{t\sqrt{3}}{\sqrt{5}} + C = \frac{1}{\sqrt{15}}\operatorname{arctg}\frac{\sqrt{3}\operatorname{tg} x}{\sqrt{5}} + C$

8.5.6. $\int \frac{dx}{\sin^5 x*\cos x} = \left[\frac{1}{\sin^5 x*\cos x}, \sin x = -\sin x\,, \cos x = -\cos x \to \frac{1}{(-\sin x)^5*(-\cos x)} =\right.$

$\left.\frac{1}{-(\sin x)^5*(-\cos x)} = \frac{1}{\sin^5 x*\cos x}; t = \operatorname{tg} x \to x = \operatorname{arctg} t\,, \sin x = \frac{t}{\sqrt{1+t^2}}; \cos x = \frac{1}{\sqrt{1+t^2}}; dx = \frac{dt}{1+t^2}\right] =$

$\int \frac{\frac{dt}{1+t^2}}{\left(\frac{t}{\sqrt{1+t^2}}\right)^5*\frac{1}{\sqrt{1+t^2}}} = \int \frac{\frac{dt}{1+t^2}}{\frac{t^5*1}{\left(\sqrt{1+t^2}\right)^5*\sqrt{1+t^2}}} = \int \frac{\left(\sqrt{1+t^2}\right)^6 dt}{\left(\sqrt{1+t^2}\right)t^5} = \int \frac{\left(\sqrt{1+t^2}\right)^3 dt}{\left(\sqrt{1+t^2}\right)t^5} = \int \frac{\left(\sqrt{1+t^2}\right)^2 dt}{t^5} = \int \frac{1+2t^2+t^4}{t^5}dt = \int \frac{dt}{t^5} +$

$\int \frac{2t^2}{t^5}dt + \int \frac{t^4}{t^5}dt = \int t^{-5}dt + 2\int t^{-3}dt + \int \frac{dt}{t} = \frac{t^{-5+1}}{-5+1} + 2\frac{t^{-3+1}}{-3+1} + \ln|t| + C = -\frac{1}{4}t^{-4} - t^{-2} + \ln|t| + C == -\frac{1}{4\,\mathrm{tg}^4 x} - \frac{1}{\mathrm{tg}^2 x} + \ln|\mathrm{tg}\, x| + C$

8.5.8. $\int \sin^3 x\, dx = \Big[R(-\sin x; \cos x) = -R(\sin x; \cos x); t = \cos x \to x = \arccos t; dx = -\frac{dt}{\sqrt{1+t^2}}; \sin x = \sqrt{1+t^2}\Big] = \int \left(\sqrt{1+t^2}\right)^3 * \left(-\frac{dt}{\sqrt{1+t^2}}\right) = -\int \frac{\left(\sqrt{1+t^2}\right)^3}{\sqrt{1+t^2}}dt = \int \left(\sqrt{1+t^2}\right)^2 dt = -\int \left(\sqrt{1+t^2}\right)dt = -\int dt + \int t^2 dt = -t + \frac{t^3}{3} + C = \frac{t^3}{3} - t + C = \frac{1}{3}\cos^3 x - \cos x + C$

8.5.9. $\int \frac{\cos^3 x dx}{\sin^4 x} = \Big[R(\sin x; -\cos x) = -R; t = \sin x \to x = \arcsin t; dx = \frac{dt}{\sqrt{1+t^2}}; \cos x = \sqrt{1+t^2}\Big] = \int \frac{\left(\sqrt{1+t^2}\right)^3}{t^4} * \frac{dt}{\sqrt{1+t^2}} = \int \frac{\left(\sqrt{1+t^2}\right)^2}{t^4}dt = \int \frac{1+t^2}{t^4}dt = \int \frac{dt}{t^4} - \int \frac{t^2}{t^4}dt = \int t^{-4}dt - \int t^{-2}dt = \frac{t^{-4+1}}{-4+1} - \frac{t^{-2+1}}{-2+1} + C = -\frac{1}{3t^3} + \frac{1}{t} + C = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C$

8.5.11. $\int \cos^4 x\, dx = \int (\cos^2 x)^2 dx = \left[\cos^2 x = \frac{1+\cos 2x}{2}\right] = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \int \frac{1}{4}(1+\cos 2x)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)dx = \frac{1}{4}\int dx + \frac{1}{4}\int 2\cos 2x\, dx + \frac{1}{4}\int \cos^2 2x\, dx = \frac{1}{4}\int dx + \frac{1}{4}\int \cos(2x)\, d(2x) + \frac{1}{4}\int \frac{1+\cos 4x}{2}dx = \frac{1}{4}\int dx + \frac{1}{4}\int \cos(2x)\, d(2x) + \frac{1}{8}(\int dx + \int \cos 4x\, dx) = \frac{1}{4}\int dx + \frac{1}{4}\int \cos(2x)\, d(2x) + \frac{1}{8}\int dx + \frac{1}{32}\int \cos(4x)\, d(4x) = \frac{3}{8}\int dx + \frac{1}{4}\int \cos(2x)\, d(2x) + \frac{1}{32}\int \cos(4x)\, d(4x) = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$