Моисеенко Павел, ИВТ, 2 курс

Интегрирование, часть 4

8.4.12. $\int\frac{dx}{x+\sqrt[3]{x^2}}=\left[\begin{array}{l}x=t^3\to t=\sqrt[3]{x}\\ dx=3t^2dt\end{array}\right]=\int\frac{3t^2dt}{t^3+t^2}=3\int\frac{t^2dt}{t^2(t+1)}=3\int\frac{d(t+1)}{t+1}=3\ln|t+1|+C=3\ln|\sqrt[3]{x}+1|+C$

8.4.13. $\int\frac{\sqrt{x}dx}{1+\sqrt[4]{x^3}}=\left[\begin{array}{l}x=t^4\to t=\sqrt[4]{x}\\ dx=4t^3dt\end{array}\right]=\int\frac{t^2\,4t^3dt}{1+t^3}=\left[\begin{array}{l}t^3=z;\ dz=3t^2dt\to\\ \to t^2dt=dz/3\end{array}\right]=\frac13\int\frac{4z\,dz}{1+z}=\frac43\int\frac{z+1-1}{z+1}dz$
$=\frac43\left(\int dz-\int\frac{dz}{z+1}\right)=\frac43(z-\ln|z+1|)+C=\frac43(t^3-\ln|t^3-\ln|t^3+1|)+C=\frac43\left(\sqrt[4]{x^3}-\ln|\sqrt[4]{x^3}+1|\right)+C$

8.4.15. $\int\frac{\sqrt{x}dx}{x-\sqrt[3]{x^2}}=\left[\begin{array}{l}x=t^6\to t=\sqrt[6]{x}\\ dx=6t^5dt\end{array}\right]=\int\frac{t^3\cdot6t^5dt}{t^6-t^4}=\int\frac{t^4\cdot6t^4dt}{t^4(t^2-1)}=6\int\frac{t^4dt}{t^2-1}=6\int\frac{t^4-1+1}{t^2-1}dt=6\cdot$
$\int\frac{(t^2-1)(t^2+1)}{t^2-1}dt+6\int\frac{dt}{t^2-1}=6\int(t^2+1)dt+6\int\frac{dt}{t^2-1}=6\left(\frac{t^3}{3}+t\right)+6\cdot\frac12\ln\left|\frac{t-1}{t+1}\right|+C=2t^3+6t+$
$+3\ln\left|\frac{t-1}{t+1}\right|+C=2\sqrt{x}+6\sqrt[6]{x}+3\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|+C$

8.4.16. $\int\frac{\sqrt{x}\,dx}{1+\sqrt{x}}=\left[\begin{array}{l}x=t^2\to t=\sqrt{x}\\ dx=2tdt\end{array}\right]=\int\frac{t\cdot2tdt}{1+t}=2\int\frac{t^2dt}{1+t}=2\int\frac{t^2-1+1}{1+t}dt=2\left(\int\frac{(t-1)(t+1)}{t+1}dt+\right.$
$\left.+\int\frac{dt}{t+1}\right)=2\left(\int(t-1)dt+\int\frac{dt}{t+1}\right)=2\left(\frac{t^2}{2}-t+\ln|t+1|\right)+C=t^2-2t+2\ln|t+1|+C=x-2\sqrt{x}+2\ln|\sqrt{x}+1|+C$

8.4.17. $\int\frac{\sqrt{x}\,dx}{1-\sqrt[3]{x}}=\left[\begin{array}{l}x=t^6\to t=\sqrt[6]{x}\\ dx=6t^5dt\end{array}\right]=\int\frac{t^3\cdot6t^5dt}{1-t^2}=6\int\frac{t^8dt}{1-t^2}=-6\int\frac{t^8-1+1}{t^2-1}dt=-6\left(\int\frac{(t^4-1)(t^4+1)}{t^2-1}dt\right.$
$\left.+\int\frac{dt}{t^2-1}\right)=-6\left(\int\frac{(t^2-1)(t^2+1)(t^4+1)}{t^2-1}dt+\int\frac{dt}{t^2-1}\right)=-6\left(\int(t^6+t^2+t^4+1)dt+\int\frac{dt}{t^2-1}\right)=-6\left(\frac{t^7}{7}+\frac{t^5}{5}+\right.$
$\left.+\frac{t^3}{3}+t+\frac12\ln\left|\frac{t-1}{t+1}\right|\right)+C=-\frac67\sqrt[6]{x^7}-\frac65\sqrt[6]{x^5}-2\sqrt{x}-3\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|+C$

8.4.18. $\int\frac{\sqrt{x+2}}{x}dx=\left[\begin{array}{l}x+2=t^2\to t=\sqrt{x+2}\\ x=t^2-2;\ dx=2tdt\end{array}\right]=\int\frac{t\cdot2tdt}{t^2-2}=2\int\frac{t^2dt}{t^2-2}=2\int\frac{t^2-2+2}{t^2-2}dt=$
$=2\left(\int dt+2\int\frac{dt}{t^2-2}\right)=2\left(t+2\cdot\frac{1}{2\sqrt2}\ln\left|\frac{t-\sqrt2}{t+\sqrt2}\right|\right)+C=2\sqrt{x+2}+\sqrt2\ln\left|\frac{\sqrt{x+2}-\sqrt2}{\sqrt{x+2}+\sqrt2}\right|+C$

8.4.20. $\int\frac{dx}{(x+1)^{3/2}+(x+1)^{1/2}}=\left[\begin{array}{l}t^2=x+1\to t=\sqrt{x+1}\\ x=t^2-1\to dx=2tdt\end{array}\right]=\int\frac{2tdt}{t^3+t}=2\int\frac{tdt}{t(t^2+1)}=2\int\frac{dt}{t^2+1}=2\,\mathrm{arctg}\,t+C=$
$=2\,\mathrm{arctg}(\sqrt{x+1})+C=2\,\mathrm{arctg}\sqrt{x+1}+C$

8.4.22. $\int\frac{x-1}{\sqrt{2x-1}}dx=\left[\begin{array}{l}t^2=2x-1\to t=\sqrt{2x-1}\\ x=\frac{t^2-1}{2}\to dx=tdt\end{array}\right]=\int\frac{\frac{t^2-1}{2}-1}{t}tdt=\int\frac{t^2-3}{2}dt=\frac12(t^2-3)dt=\frac12\left(\frac{t^3}{3}-3t\right)+C=$
$=\frac{\sqrt{(2x-1)^3}}{6}-\frac32\sqrt{2x-1}+C$

8.4.23. $\int\frac{dx}{\sqrt{1-2x}-\sqrt[4]{1-2x}}=\left[\begin{array}{l}t^4=1-2x\to t=\sqrt[4]{1-2x}\\ x=\frac{1-t^4}{2}\to dx=-2t^3\end{array}\right]=\int\frac{-2t^3}{t^2-t}dt=-2\int\frac{t^3}{t(t-1)}dt=-2\int\frac{t^2}{t-1}dt=$
$=-2\int\frac{t^2-1+1}{t-1}dt=-2\left(\int\frac{(t-1)(t+1)}{t-1}dt+\int\frac{dt}{t-1}\right)=-2\left(\int(t+1)dt+\int\frac{dt}{t-1}\right)=-2\left(\frac{t^2}{2}+t+\ln|t-1|\right)+C=$
$=-t^2-2t-2\ln|t-1|+C=-\sqrt{1-2x}-2\sqrt[4]{1-2x}-2\ln|\sqrt[4]{1-2x}-1|+C$

8.4.24. $\int\frac{1}{(2-x)^2}\sqrt{\frac{2-x}{2+x}}dx=\left[\begin{array}{l}t^2=\frac{2-x}{2+x}\to2t^2+t^2x=2-x\to t^2x+x=2-2t^2\to x(t^2+1)=2-2t^2\\ x=\frac{2-2t^2}{t^2+1}\to dx=\frac{-4t(t^2+1)-2t(2-2t^2)}{(t^2+1)^2}dt=\frac{-4t^3-4t-4t+4t^3}{(t^2+1)^2}dt=\frac{-8t}{(t^2+1)^2}dt\end{array}\right]=$
$=\int\frac{1}{\left(2-\frac{2-2t^2}{t^2+1}\right)^2}\cdot t\cdot\left(\frac{-8t}{(t^2+1)^2}\right)dt=-8\int\frac{t^2dt}{\left(\frac{2t^2+2-2+2t^2}{t^2+1}\right)^2(t^2+1)^2}=-8\int\frac{t^2dt}{(4t^2)^2}=-8\int\frac{t^2}{16t^4}dt=$
$=-\frac12\int\frac{1}{t^2}dt=-\frac12\int\frac{dt}{t^2}=-\frac12\cdot\left(-\frac1t\right)+C=\frac12\sqrt{\frac{2+x}{2-x}}+C$

8.4.25. $\int\frac{dx}{\sqrt{(x-1)^3(x-2)}}=\int\sqrt{\frac{x-2}{x-1}}\cdot\frac{dx}{(x-1)(x-2)}=\left[\begin{array}{l}t^2=\frac{x-2}{x-1}\to xt^2-t^2=x-2\to x(t^2-1)=t^2-2\\ x=\frac{t^2-2}{t^2-1}\to dx=\frac{2t(t^2-1)-2t(t^2-2)}{(t^2-1)^2}dt=\frac{2t^3-2t-2t^3+4t}{(t^2-1)^2}=\frac{2t}{(t^2-1)^2}dt\end{array}\right]=$

$$=\int t\cdot\frac{1}{\frac{t^2-2}{t^2-1}-1}\cdot\frac{1}{\frac{t^2-2}{t^2-1}-2}\cdot\frac{2t}{(t^2-1)^2}dt=2\int\frac{t^2}{(t^2-1)^2}\cdot\frac{1}{\frac{t^2-2-t^2+1}{t^2-1}}\cdot\frac{1}{\frac{t^2-2-2t^2+2}{t^2-1}}dt=$$

$$=2\int\frac{t^2}{-1(-t^2)}dt=2\int\frac{t^2}{t^2}dt=2\int dt=2t+C=2\sqrt{\frac{x-2}{x-1}}+C$$

8.4.27. $$\int\frac{dx}{(1-x)\sqrt{1-x^2}}=\int\frac{dx}{(1-x)\sqrt{(1-x)(1+x)}}=\int\frac{dx}{(1-x)(1+x)\sqrt{\frac{1-x}{1+x}}}=\left[\begin{matrix}t^2=\frac{1-x}{1+x}\Rightarrow t^2+t^2x=1-x\Rightarrow t^2x+x=t^2-1\\ x=\frac{t^2-1}{t^2+1}\to dx=\frac{2t(t^2+1)-2t(t^2-1)}{(t^2+1)^2}dt=\end{matrix}\right.$$

$$=\frac{2t^3+2t-2t^3+2t}{(t^2+1)^2}dt=\frac{4t}{(t^2+1)^2}dt\Bigg]=\int\frac{4t}{(t^2+1)^2}\cdot\frac{1}{\left(1-\frac{t^2-1}{t^2+1}\right)\left(1+\frac{t^2-1}{t^2+1}\right)}dt=\int\frac{4t\,dt}{(t^2+1)^2\left(1-\frac{t^2-2t+1}{(t^2+1)^2}\right)}=$$

$$=\int\frac{4t\,dt}{(t^2+1)^2\left(\frac{t^2+2t+2-t^2+2t-1}{(t^2+1)^2}\right)}=\int\frac{4t\,dt}{4t}=\int dt=t+C=\sqrt{\frac{1-x}{1+x}}+C$$

8.4.29. $$\int x^3\sqrt{1+x^2}dx=\left[\begin{matrix}t^2=1+x^2\to x=\sqrt{t^2-1}\\ dx=\frac{2t}{2\sqrt{t^2-1}}dt=\frac{t}{\sqrt{t^2-1}}dt\end{matrix}\right]=\int(\sqrt{t^2-1})^3\cdot t^2\cdot\frac{t}{\sqrt{t^2-1}}dt=\int(t^2-1)\cdot t^3dt=\int t^5dt-$$

$$-\int t^3dt=\frac{t^6}{6}-\frac{t^4}{4}+C=\frac{(1+x^2)^3}{6}-\frac{(1+x^2)^2}{4}+C$$

8.4.30. $$\int\frac{dx}{x^{11}\sqrt{x^4+1}}=\left[\begin{matrix}t^2=x^4+1\to x=\sqrt[4]{t^2-1}\\ dx=\frac14\cdot\frac{2t\,dt}{\sqrt[4]{(t^2-1)^3}}=\frac{t\,dt}{2\sqrt[4]{(t^2-1)^3}}\end{matrix}\right]=\frac12\int\frac{t}{\sqrt[4]{(t^2-1)^3}}\cdot\frac{1}{(\sqrt[4]{t^2-1})^{11}\cdot t}dt=\frac12\int\frac{dt}{\sqrt[4]{(t^2-1)^{14}}}=\frac12\int\frac{dt}{\sqrt{(t^2-1)^7}}=$$

$$=\frac{1}{10t\sqrt{(t^2-1)^5}}+\int\frac{dt}{10t^2\sqrt{(t^2-1)^5}}=\frac{\sqrt{x^4+1}\,(8x^8-4x^4+3)}{30x^{10}}+C$$

8.4.31. $$\int\frac{dx}{\sqrt{x}(1-\sqrt{x})^2}=\left[\begin{matrix}t^2=x\to t=\sqrt{x}\\ dx=2t\,dt\end{matrix}\right]=\int\frac{2t\,dt}{t(1-t)^2}=2\int\frac{dt}{(1-t)^2}=-2\int\frac{d(1-t)}{(1-t)^2}=-2\,\frac{(1-t)^{-1}}{-1}+C=$$

$$=\frac{2}{1-t}+C=\frac{2}{1-\sqrt{x}}+C$$

8.4.32. $$\int x^5\sqrt[3]{(1+x^3)^2}dx=\left[\begin{matrix}t^3=1+x^3\to x=\sqrt[3]{t^3-1}\\ dx=\frac13\cdot\frac{3t^2}{\sqrt[3]{(t^3-1)^2}}dt\end{matrix}\right]=\int\sqrt[3]{(t^3-1)^5}\,t^2\cdot\frac{t^2}{\sqrt[3]{(t^3-1)^2}}dt=\int(t^3-1)\cdot t^4dt=$$

$$=\int t^7dt-\int t^4dt=\frac{t^8}{8}-\frac{t^5}{5}+C=\frac{\sqrt[3]{(t^3-1)^8}}{8}-\frac{\sqrt[3]{(t^3-1)^5}}{5}+C$$

8.4.34. $$\int\sqrt{x}(1+\sqrt{x})^3dx=\left[\begin{matrix}t^2=x\to t=\sqrt{x}\\ dx=2t\,dt\end{matrix}\right]=\int t(1+t)^3\,2t\,dt=2\int t^2(1+3t+3t^2+t^3)dt=2\int(t^2+$$

$$+3t^3+3t^4+t^5)dt=2\left(\frac{t^3}{3}+\frac{3t^4}{4}+\frac{3t^5}{5}+\frac{t^6}{6}\right)+C=\frac23\sqrt{x^3}+\frac32x^2+\frac65\sqrt{x^5}+3x^3+C$$

8.4.35. $$\int\sqrt[3]{x^3-4}\,x^2dx=\left[\begin{matrix}t^3=x^3-4\to x=\sqrt[3]{t^3+4}\\ dx=\frac13\cdot\frac{3t^2}{\sqrt[3]{(t^3+4)^2}}dt\end{matrix}\right]=\int t\sqrt[3]{(t^2+4)^2}\cdot\frac{t^2}{\sqrt[3]{(t^2+4)^2}}dt=\int t^3dt=\frac{t^4}{4}+C=$$

$$=\frac14\sqrt[3]{(t^3+4)^4}+C$$