Экзамен по математике

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2 вариант

Найти производные

1.1. $y' = \left(10 \ln^4 \operatorname{arctg}\frac{\sqrt{7-x}}{x+2}\right)' = 10\cdot 4\cdot \ln^3 \operatorname{arctg}\frac{\sqrt{7-x}}{x+2}\cdot \frac{1}{\operatorname{arctg}\frac{\sqrt{7-x}}{x+2}}\cdot\left(-\frac{1}{1+\left(\frac{\sqrt{7+x}}{x+2}\right)^2}\right)\cdot\left(\frac{(\sqrt{7-x})'(x+2)-}{(x+2)^2}\right.$

$\underline{-(\sqrt{7-x})(x+2)'} = \dfrac{-40\ln^3\left(\operatorname{arctg}\frac{\sqrt{7-x}}{x+2}\right)\cdot\left(\left(-\frac{1}{2\sqrt{7-x}}\right)\cdot(x+2)-\sqrt{7-x}\right)}{\operatorname{arctg}\frac{\sqrt{7-x}}{x+2}\cdot\left(1+\frac{7-x}{(x+2)^2}\right)\cdot(x+2)^2} = -\dfrac{40\ln^3\left(\operatorname{arctg}\frac{\sqrt{7-x}}{x+2}\right)\left(\frac{x+2}{2\sqrt{7-x}}\right.}{\operatorname{arctg}\frac{\sqrt{7-x}}{x+2}\left((x+2)^2+7-x\right)}$

$\underline{-\sqrt{7-x}}$

1.2. $y = (\sqrt{x}+x^2)^{\arccos(x-\frac{3}{x})}$

$\ln y = \ln(\sqrt{x}+x^2)^{\arccos(x-\frac{3}{x})}$

$\ln y = \arccos(x-\frac{3}{x})\cdot\ln(\sqrt{x}+x^2)$

$(\ln y)' = (\arccos(x-\frac{3}{x})\cdot\ln(\sqrt{x}+x^2))'$

$\frac{y'}{y} = -\frac{1\cdot(1+\frac{3}{x^2})}{\sqrt{1-(x-3/x)^2}}\cdot\ln(\sqrt{x}+x^2)+\arccos\left(x-\frac{3}{x}\right)\cdot\frac{\frac{1}{3\sqrt{x}}+2x}{\sqrt{x}+x^2}$

$y' = y\cdot\left(-\frac{(1+3/x^2)\cdot\ln(\sqrt{x}+x^2)}{\sqrt{1-(x-3/x)^2}}+\frac{\left(\frac{1}{2\sqrt{x}}+2x\right)\cdot\arccos(x-3/x)}{\sqrt{x}+x^2}\right.$

$y' = (\sqrt{x}+x^2)^{\arccos(x-\frac{3}{x})}\cdot\left(-\frac{(1+\frac{3}{x^2})\ln(\sqrt{x}+x^2)}{\sqrt{1-(x-\frac{3}{x})^2}}+\frac{\left(\frac{1}{2\sqrt{x}}+2x\right)\cdot\arccos\left(x-\frac{3}{x}\right)}{\sqrt{x}+x^2}\right)$

1.3. $\cos(14x-3y-5x^2y)+\frac{x^4+xy-y}{y+2} = 13x+xy^2$

$\cos(14x-3y-5x^2y)+\frac{x^4+xy-y}{y+2} = 13x+xy^2$

$F_x(x;y) = -\sin(14x-3y-5x^2y)\cdot(14-10y)+\frac{(4x^3+y)(y+2)-(x^4+xy-y)\cdot 0}{(y+2)^2}-13-y^2 =$

$= -\sin(14x-3y-5x^2y)(14-10y)+\frac{4x^3+y}{y+2}-13-y^2$

$F_y(x;y) = -\sin(14x-3y-5x^2y)(-3-5x^2)+\frac{(x-1)(y+2)-(x^4+xy-y)\cdot 1}{(y+2)^2}+2xy$

$y' = \dfrac{\sin(14x-3y-5x^2y)(14-10y)+\frac{4x^3+y}{y+2}-13-y^2}{\sin(14x-3y-5x^2y)(5x^2+3)+\frac{(x-1)(y+2)-(x^4+xy-y)}{(y+2)^2}+2xy}$

Найти интегралы

2.1. $\int\frac{x\,dx}{(5-3x^2)^7} = \left[\begin{matrix}t=5-3x^2\\ dt=(5-3x^2)'dx=-6x\,dx\to x\,dx=\frac{dt}{-6}\end{matrix}\right] = \int\frac{dt}{(-6)t^7} = -\frac{1}{6}\int t^{-7}dt = -\frac{1}{6}\cdot\frac{t^{-6}}{-6}+C=$

$=\frac{1}{36t^6}+C = \frac{1}{36(5-3x^2)^6}+C$

2.2. $\int\frac{dx}{2\cos^2x+3\sin^2x} = \left[\begin{matrix}R(\sin x;\cos x)=R(-\sin x;-\cos x);\ \sin x=\frac{t}{\sqrt{1+t^2}};\ \cos=\frac{1}{\sqrt{1+t^2}}\\ t=\operatorname{tg}x;\ x=\operatorname{arctg}t;\ dx=dt/(1+t^2)\end{matrix}\right]=$

$=\int\frac{\frac{dt}{1+t^2}}{2\cdot\left(\frac{1}{\sqrt{1+t^2}}\right)^2+3\left(\frac{t}{\sqrt{1+t^2}}\right)^2} = \int\frac{dt/(1+t^2)}{(2\cdot 1+3\cdot t^2)/(1+t^2)} = \int\frac{dt}{3t^2+2} = \frac{1}{3}\int\frac{dt}{t^2+\frac{2}{3}} = \frac{1}{3}\cdot\sqrt{\frac{3}{2}}\cdot\operatorname{arctg}\left(\sqrt{\frac{3}{2}}\cdot t\right)+C$

$=\frac{1}{6}\operatorname{arctg}\left(\sqrt{\frac{3}{2}}\operatorname{tg}x\right)+C$

2.3. $\int (x^3+4x)\ln x\,dx = \left[\begin{array}{l} u=\ln x \to du=\frac{1}{x} \\ v'=x^3+4x \to v=\int(x^3+4x)dx=\frac{x^4}{4}+\frac{4x^2}{2}\end{array}\right] = \ln x\cdot\left(\frac{x^4}{4}+\frac{4x^2}{2}\right)-\int\frac{1}{x}\left(\frac{x^4}{4}+2x^2\right)dx =$

$= \ln x\left(\frac{x^4}{4}+2x^2\right)-\int\frac{x^3}{4}dx-\int 2x\,dx = \left(\frac{x^4}{4}+2x^2\right)\ln x-\frac{x^4}{4\cdot 4}-\frac{2x^2}{2}+C = \left(\frac{x^4}{4}+2x^2\right)\ln x-\frac{x^4}{16}-x^2+C$

2.4. $\int x\arcsin(2x)dx = \left[\begin{array}{l} u=\arcsin 2x \to u'=\frac{2}{\sqrt{1-4x^2}} \\ v'=x \to \int v=\frac{x^2}{2}\end{array}\right] = \frac{x^2\arcsin 2x}{2}-\int\frac{x^2}{\sqrt{1-4x^2}}dx = \left[\begin{array}{l} t=\arcsin 2x, \\ x=\frac{\sin t}{2}\end{array}\right.$

$dx=\frac{\cos t}{2}dt\Big] = \frac{x^2\arcsin 2x}{2}-\int\frac{\cos t\cdot\sin^2 t}{8\sqrt{1-\sin^2 t}}dt = \frac{x^2\arcsin 2x}{2}-\frac{1}{8}\int\sin^2 t\,dt = \Big[\int f^{n}(x)dx =$

$= \frac{n-1}{n}\int f^{n-2}(x)dx-\frac{f'(x)\cdot f^{n-1}(x)}{n}\Big] = \frac{x^2\arcsin 2x}{2}-\frac{1}{8}\cdot\frac{1}{2}\int dt-\frac{\cos t\cdot\sin t}{2} = \frac{x^2\arcsin 2x}{2}-$

$-\frac{t}{16}-\frac{\cos t\sin t}{16}+c = \frac{x^2\arcsin 2x}{2}-\frac{\arcsin 2x}{16}+\frac{x\sqrt{1-4x^2}}{8}+C$

2.5. $\int\frac{2x+5}{x^3-x^2+2x-2}dx = \int\frac{2x+5}{x^2(x-1)+2(x-1)}dx = \int\frac{2x+5}{(x-1)(x^2+2)}dx = \left[\begin{array}{l}(x-1)(x^2+2)=0 \\ \text{есть решение}\end{array}\right. \to \frac{2x+5}{(x-1)(x^2+2)} =$

$= \frac{A}{x-1}+\frac{Bx+C}{x^2+2}$ $\qquad 2x+5 = A(x^2+2)+(Bx+C)(x-1)$

Т.к. $2x+5 = Ax^2+2A+Bx^2-Bx+Cx-C$; $2x+5=(A+B)x^2+(C-B)x+(2A-C)$

$\begin{cases}A+B=0\\C-B=2\\2A-C=5\end{cases} \begin{cases}A=-B\\C-B=2\\-C-2B=5\end{cases} \begin{cases}A=-B\\C=B+2\\-3B=7\end{cases} \begin{cases}B=-\frac{7}{3}\\A=\frac{7}{3}\\C=-\frac{1}{3}\end{cases}\Bigg] = \int\left(\frac{\frac{7}{3}}{x-1}+\frac{-\frac{7}{3}x-\frac{1}{3}}{x^2+2}\right)dx = \frac{7}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{7x+1}{x^2+2}dx =$

$= \frac{7}{3}\int\frac{dx}{x-1}-\frac{1}{3}\left(\int\frac{7x}{x^2+2}dx+\int\frac{dx}{x^2+2}\right) = \frac{7}{3}\int\frac{dx}{x-1}-\frac{7}{3}\int\frac{x\,dx}{x^2+2}-\frac{1}{3}\int\frac{dx}{x^2+2} = \left[\begin{array}{l}2.\ t=x^2+2\to dt=\\ =2x\,dx \to x\,dx=\frac{dt}{2}\end{array}\right] =$

$= \frac{7}{3}\int\frac{d(x-1)}{x-1}-\frac{7}{3\cdot 2}\int\frac{dt}{t}-\frac{1}{3}\int\frac{dx}{x^2+(\sqrt{2})^2} = \frac{7}{3}\ln|x-1|-\frac{7}{6}\ln|t|-\frac{1}{3}\cdot\frac{1}{\sqrt{2}}\,\mathrm{arctg}\frac{x}{\sqrt{2}}+C = \frac{7}{3}\ln|x-1|-$

$-\frac{7}{6}\ln|x^2+2|-\frac{1}{3\sqrt{2}}\,\mathrm{arctg}\frac{x}{\sqrt{2}}+C = \frac{7}{3}\ln|x-1|-\frac{7}{6}\ln|x^2+2|-\frac{1}{3\sqrt{2}}\,\mathrm{arctg}\frac{x}{\sqrt{2}}+C$

2.6. $\int_{-3}^{-2}\frac{dx}{\sqrt{5-4x-x^2}} = \int_{-3}^{-2}\frac{dx}{\sqrt{9-4-4x-x^2}} = \int_{-3}^{-2}\frac{dx}{\sqrt{3^2-(x^2+4x+4)}} = \int_{-3}^{-2}\frac{dx}{\sqrt{3^2-(x+2)^2}} = \int_{-3}^{-2}\frac{d(x+2)}{\sqrt{3^2-(x+2)^2}} = \arcsin\frac{x+2}{3}\Big|_{-3}^{-2} =$

$= \arcsin\frac{-2+2}{3}-\arcsin\frac{-3+2}{3} = \arcsin\frac{1}{3}$

2.7. $\int\frac{dx}{1+\sqrt{2x+1}} = \left[\begin{array}{l}t^2=2x+1 \to x=\frac{t^2-1}{2}\\ dx=\frac{2t}{2}dt=t\,dt\end{array}\right] = \int\frac{t\,dt}{1+\sqrt{t^2}} = \int\frac{t\,dt}{1+t} = \int\frac{t+1-1}{1+t}dt = \int dt-\int\frac{d(t+1)}{t+1} =$

$= t-\ln|t+1|+C = \sqrt{2x+1}-\ln|\sqrt{2x+1}+1|+C$

Решить дифференциальное уравнение (ДУ)

3.1. $y'=-2y$, $y(0)=3$

$\frac{dy}{dx}=-2y$, $\frac{dy}{y}=-2dx$

$\int\frac{dy}{y}=\int(-2dx)=-2\int dx$; $\ln|y|=-2x+c$; $y=e^{-2x+c}$; $y=e^c\cdot e^{-2x}$; $y=c\cdot e^{-2x}\cdot$ $3=c\cdot e^{-2\cdot 0}$;
$3=c\cdot e^0$; $c=3$;

Частное решение: $y=3e^{-2x}$

3.2. $xy'=2\sqrt{3x^2+y^2}+y$; $y'=\frac{2\sqrt{3x^2+y^2}+y}{x}$; $y=tx$, тогда $y'=(tx)'=t'x+x't=t'x+t$; $t'x+t=$
$=\frac{2\sqrt{3x^2+tx}+tx}{x}$; $t'=\frac{dt}{dx}$, тогда $\frac{dt}{dx}x+t=\frac{2\sqrt{3+t}+t}{1}$; $\frac{dt}{dx}x=2\sqrt{3+t}+t-t$; $\frac{dt}{dx}x=2\sqrt{3+t}$;

$\frac{x}{dx}=\frac{2\sqrt{3+t}}{dt}=\frac{dt}{2\sqrt{3+t}}=\frac{dx}{x}$; интегрируем $\int\frac{dt}{2\sqrt{3+t}}=\int\frac{dx}{x}$; $\frac{1}{2}\int\frac{d(t+3)}{\sqrt{t+3}}=\int\frac{dx}{x}$;

$\frac{1}{2}\cdot\frac{1}{2}\sqrt{t+3}=\ln|x|+c$; $\frac{1}{4}\sqrt{\frac{y}{x}+3}=\ln|x|+c$; $\frac{1}{4}\sqrt{\frac{y}{x}+3}-\ln|x|=c$ — общий вид