Index Tracking and Enhanced Index Tracking in Portfolio Selection

A Report Submitted in Partial Fulfillment for the award of the Degree of

Master of Science

in

Mathematics

Submitted by:

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May, 2017

Certificate

This is to certify that the report of the major project entitled **Index Tracking and Enhanced Index Tracking**, has been submitted by **Ms. Minakshi Punam Mandal** and **Ms. Raksha Agarwal**, in the Department of Mathematics of Indian Institute of Technology, Delhi, in partial fulfillment of the requirement for the award of the degree of Master of Science (M.Sc.) in Mathematics. The report is a record of bonafide work carried out by them under my guidance and supervision and fulfilled all the requirements for the submission of the report, which has the required standard.

The matter embodied in this dissertation has not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Dr. Aparna Mehra
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Acknowledgements

We would like to take this opportunity to express our profound gratitude and deep regards to **Dr. Aparna Mehra** for her unconditional support, expert guidance and constant supervision. We thank her for regularly taking out her time for clarifying our doubts and her help in completing the project. We would also like to thank **Dr. Abha Aggarwal** for guiding us in dealing with Fuzzy Index Tracking.

We would also like to thank our M.Sc. Coordinator, **Dr. Shravan Kumar** for allotting us this project and keeping us informed about the schedule of presentations and project reports from time to time.

Special thanks to **Anubha Goyal** and **Pooja Bansal**, and all other Ph.D. scholars of the Department of Mathematics, IIT Delhi. They have been an immense support and helped us to the brim whenever we approached them.

Last, but not the least, we would thank our respective families for being our driving force and helping us get to wherever we are today, and our friends for their co-operation and encouragement.

Minakshi Punam Mandal **Raksha Agarwal**

Abstract

In this project we have studied some basic concepts of the Financial Market. We have then studied implemented (on Nifty50 Index) and analysed two models on Index Tracking and Enhanced Index Tracking, one proposed by Canakgoz and Beasley, and the other proposed by Paulo, Oliviera and Costa. We have modified and extended the models to a certain extent, to incorporate both short-selling and without short-selling and given a hybrid of the two models. We have also tried to give a comparision between the two models. Next we have given a brief introduction on Fuzzy Variables and Credibility Theory, and proposed a model for Index Tracking, by extending Markowitz's mean-variance model, but in a fuzzy setup. We have also implemented the model on the Nifty50 Index.

Contents

1 Introduction

Market Index: A market index is a measure that represents the performance of a group of stocks. It measures the value of a hypothetical portfolio of stocks. eg:- Nifty50, Russell2000

Index Tracking: Index tracking is a passive portfolio management technique where we try to replicate the stock market performance.

The easiest way to do that would be to purchase all the stocks that are present in the market index, their proportions being the same. But this method has certain disadvantages, like the proportion of certain stocks may be very small as compared to others, so in case of revision of stocks that make up the market index, the transaction costs might be high. Also, if the index has a large number of stocks, the stocks from smaller companies are relatively illiquid and hence the transaction cost becomes high. So, it is desirable to hold fewer stocks than that of the market index.

Enhanced Index Tracking: Enhanced index tracking also tries to reproduce the stock market performance, but it generates an excess return, over the return from the market index.

1.1 Keywords

- **Asset** :

  Resources or things of value owned by a company or a person.

- **Asset Return** :

  In the financial market, this indicates profit or loss in investment.

- **Asset Risk** :

  It is defined as the degree of uncertainty of return on an asset. It indicates the possibility of loss in investment in the financial market.

- **Portfolio** :

  It is the collection of assets-stocks and bonds primarily, and is represented by an ordered n-tuple, $\theta = (x_1,\, x_2,\, \ldots,\, x_n)$; $x_i \in \mathbb{R}$, is the number of units of asset a_i owned by the investor.

- **Asset Weight**:

  The weight w_i of an asset a_i is the proportionate value of a_i in the portfolio at time t

  $$w_i = \frac{x_i V_i}{\Sigma_{j=1}^{n} x_j V_j}, i = 1, 2, \ldots, n$$

  $\Sigma_{i=1}^{n} w_i = 1$.

- **Portfolio Return** :

  It is expected gain from the portfolio,

  $$\mu = E(\Sigma_{i=1}^{n} w_i r_i) = \Sigma_{i=1}^{n} w_i E(r_i) = \Sigma_{i=1}^{n} w_i \mu_i$$

  where, r_i is the return of the i^{th} asset and μ_i is the expected return of the i^{th} asset.

- **Portfolio Risk** :

  It is also known as Varience of the portfolio.

  $$\sigma = Var(\Sigma_{i=1}^{n} w_i r_i) = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} w_i w_j \sigma_{ij}$$

  $$\sigma ij = Cov(r_i, r_j) = \rho_{ij} \sigma_i \sigma_j$$

  where, σ_i = Var(r_i), σ_j = Var(r_j) & ρ_{ij} = Correlation coefficient between r_i and r_j.

- **Tracking Portfolio** :

  The set of assets or stocks which is used to track the market index.

1.2 Certain Assumptions

We make certain assumptions regarding the financial market:

Randomness: The price of stock at time T is a random variable with atleast two different values.

Positivity of prices: The prices of all the assets are strictly positive.

Divisibility: An investor may hold any unit of asset negative, positive or fractional.

Liquidity: An investor can buy or sell any amount of the asset, no matter how small or large .

Short selling: Short selling is allowed.

Solvency: The wealth of an investor is always non-negative.

Discrete unit prices: The price of an asset at any time is a random variable and can take only a finite number of values.

No Arbitrage Principle: It is not possible to make a profit without both investment and risk, i.e. , $V(0) = 0$, $V(t) \geq 0$ with probability 1, $P[V(t) > 0] > 0$; $V(t)$ is the price of the asset at time t.

Law of One Price or Comparison Principle: If two financial investments have exactly same payoffs, then they have the same price.

2 Mixed Integer Programming Approach

Canakgoz & Beasley [1] considered the problem of index tracking and enhanced index tracking using a mixed integer programming model. The model formulation comprises of transaction costs and a limit on the number of stocks purchased. In their paper they have linearised (in approximate sense) the non linear objective function and constraints.

2.1 Notation

The values of index as well as the N stocks, that make up the index, over time $0, 1, \ldots, T$ is observed. The objective is to select the optimal set of K(<N) stocks to be held along with their weights.

Let

ε_i the minimum proportion of the i^{th} stock in tracking portfolio (TP), if a particular stock i is to be held.

δ_i the maximum proportion of the i^{th} stock in tracking portfolio (TP), if a particular stock i is to be held.

X_i the number of units of i^{th} stock that is held in the current TP.

x_i the number of units of the i^{th} stock that is to be held in the new TP.

V_{it} the price of one unit of the i^{th} stock at time t.

I_t the price of the market index at time t.

$R_t = \ln \frac{I_t}{I_{t-1}}$: Single period continuous time return for the market index at time t.

$r_{it} = \ln \frac{V_{it}}{V_{i(t-1)}}$: Single period continuous time return for the i^{th} stock at time t.

$\mathcal{C}$ the value of the current portfolio, at time T, $\Sigma_{i=1}^{n} X_i V_{iT}$, plus cash change (i.e. if extra cash is invested or investment is reduced).

f_i^s the fractional cost of selling one unit of i^{th} stock at time T,.

f_i^b the fractional cost of buying one unit of i^{th} stock at time T.

γ (0$\leq \gamma \leq$1), a limit on the relative proportion of the total cash $\mathcal{C}$, that is allowed to be used as transaction cost.

G_i (≥ 0), the transaction cost of buying or selling the i^{th} stock.

z_i 1 if i^{th} stock is held in the new TP, 0 otherwise.

X_i and x_i are allowed to take fractional values. And, x_i, G_i and z_i are decision variables.

2.2 Constraints

(1) $\Sigma_{i=1}^{N} z_i =$ K, this ensures that exactly K out of N stocks are held.

(2) $\varepsilon_i z_i \leq \frac{x_i V_{iT}}{\mathcal{C}} \leq \delta_i z_i, i = 1, 2, \ldots, N.$
to guarantees that if z_i=0 then x_i=0. Also, if a stock is to be held, then the amount satisfies the limits of proportion.

(3) $G_i \geq f_i^s (X_i - x_i) V_{iT}, i = 1, 2, \ldots, N.$

(4) $G_i \leq f_i^b (x_i - X_i) V_{iT}, i = 1, 2, \ldots, N$
(3) and (4) define the limits on transaction cost.

(5) $\Sigma_{i=1}^{N} G_i \leq \gamma \mathcal{C}$, this defines the limit on total transaction cost.

(6) $\Sigma_{i=1}^{N} x_i V_{iT} = \mathcal{C} - \Sigma_{i=1}^{N} G_i$, this is the balance constraint.

(7) $x_i, G_i \geq 0, i = 1, 2, \ldots, N$

(8) $z_i \in \{0,1\}$, $i = 1, 2, \ldots, N$

The single period continuous time return $\ln \dfrac{\Sigma_{i=1}^{N} x_i V_{it}}{\Sigma_{i=1}^{N} x_i V_{i(t-1)}}$ is non linear, so it has to be linearised. We make an assumption here that this return can be written linearly as the weighted sum of returns from the individual stocks, the weights here represent the proportion that has been invested in the i^{th} stock.
Hence $W_{it} = \dfrac{x_i V_{it}}{\Sigma_{j=1}^{N} x_j V_{jt}} \rightarrow$ weight of investment in the i^{th} stock at time t.
Clearly, $\Sigma_{i=1}^{N} W_{it} = 1$, for all t. Thus, at time t the return of the portfolio can be given by $\Sigma_{i=1}^{N} W_{it} r_{it}$

Now, W_{it}'s are non linear and time dependent, we replace each W_{it} by w_i as $w_i = \dfrac{x_i V_{iT}}{\Sigma_{j=1}^{N} x_j V_{jT}}$,which is independent of time. Thus, return at time t is equal to $\Sigma_{i=1}^{N} w_i r_{it}$
Now, $\Sigma_{i=1}^{N} x_i V_{iT} = \mathcal{C} - \Sigma_{i=1}^{N} G_i$ and $\Sigma_{i=1}^{N} G_i \leq \gamma\mathcal{C}$.
Therefore w_i can be written approximately as,

$$w_i = \frac{x_i V_{iT}}{\mathcal{C} - \gamma\mathcal{C}}, \quad i = 1, 2, \ldots, N \tag{9}$$

This is a linear, time independent expression for w_i.

2.3 Index Tracking Objective

If the returns from TP are plotted against the returns from the index, $t = 1, 2, \ldots, T$, then, the intercept $\hat{\alpha}$ and slope $\hat{\beta}$ are given by,

$$\hat{\alpha} = \Sigma_{i=1}^{N} w_i \hat{\alpha}_i \tag{10}$$

$$\hat{\beta} = \Sigma_{i=1}^{N} w_i \hat{\beta}_i \tag{11}$$

where $\hat{\alpha_i}$ and $\hat{\beta_i}$ are the intercepts and slopes obtained respectively, when the return from the i^{th} stock is plotted against the return from the market.
Ideally, for index tracking, we should have $\hat{\alpha} = 0$ & $\hat{\beta} = 1$. In order to achieve this, a three stage formulation has been adopted. The primary objective is achieving an intercept of 0 and secondary objective is achieving a slope of 1 i.e. min $\mid \hat{\alpha} - 0 \mid$ followed by min $\mid \hat{\beta} - 1 \mid$ followed by min $\Sigma_{i=1}^{N} G_i$. The non linear modulus objective is transformed into linear objective by introducing variables D & E such that, $\mid \hat{\alpha} - 0 \mid \leq D$ & $\mid \hat{\beta} - 1 \mid \leq E$. Hence

$$D \geq \hat{\alpha} \tag{12}$$

$$D \geq -\hat{\alpha} \tag{13}$$

$$E \geq \hat{\beta} - 1 \tag{14}$$

$$E \geq -(\hat{\beta} - 1) \tag{15}$$

$$D, E \geq 0 \tag{16}$$

Thus for the **first stage**, the problem is

Minimize D

subject to: (1)-(16)

Suppose α_{opt} is the numeric value for $\hat{\alpha}$.

For the **second stage**, the problem is

Minimize E

subject to: (1)-(16)

$$\hat{\alpha} = \alpha_{opt}$$

Suppose β_{opt} is the numeric value for $\hat{\beta}$.

For the **third stage**,the problem is

Minimize $\Sigma_{i=1}^{N} G_i$

subject to: (1)-(16)

$$\hat{\alpha} = \alpha_{opt}$$

$$\hat{\beta} = \beta_{opt}$$

The third stage makes use of any flexibility in the problem by fixing the values of $\hat{\alpha}$ & $\hat{\beta}$ achieved in first and second stage, but minimizing the transaction cost involved in doing so.

2.4 Enhanced Index Tracking Objective

Here, a two - stage approach has been adopted, the **first stage** being:

$$\begin{aligned} &\textbf{Minimize } \mathrm{E} \\ &\textbf{subject to: } (1)\text{-}(16) \\ &\qquad\qquad\quad \hat{\alpha} = \alpha^* \end{aligned}$$

Here, α^* is the desired intercept which represents the desired excess return.

Suppose β_{opt} is the optimum value for $\hat{\beta}$.

For the **second stage**, we have

$$\begin{aligned} &\textbf{Minimize } \Sigma_{i=1}^{N} G_i \\ &\textbf{subject to: } (1)\text{-}(16) \\ &\qquad\qquad\quad \hat{\alpha} = \alpha^* \\ &\qquad\qquad\quad \hat{\beta} = \beta_{opt} \end{aligned}$$

Note: In practicality it is not always possible to retain the achieved values of $\hat{\alpha}$ & $\hat{\beta}$, so we can instead use

$$\alpha_{opt} - \epsilon \leq \hat{\alpha} \leq \alpha_{opt} + \epsilon \tag{a}$$

$$\beta_{opt} - \epsilon \leq \hat{\beta} \leq \beta_{opt} + \epsilon \tag{b}$$

where ϵ is a small positive constant.

3 Analytical Solution of Enhanced Index Tracking Problem

Paulo, Oliveira and Costa [2] approach for enhanced indexation involves a predecided set of assets that is included in the tracking portfolio. The objective function is a trade-off between the tracking error and excess return.

3.1 Notation

Suppose the market index consists of n assets. Without loss of generality first p out of n assets have been selected in the tracking portfolio (TP).

ω_B =$(\omega_{B1}, \omega_{B2}, .., \omega_{Bn})$', where ω_{Bi} represents the weight of the i^{th} asset in the benchmark portfolio (market index).

P_B return of the market index.

$\hat{\omega}$ =$(\omega_1, \omega_2, .., \omega_p)$', where ω_i represents the weight of the i^{th} asset in the tracking portfolio.

$\mathcal{R}$ =$(R_1, R_2, ..., R_p)$', where R_i represents the return from the i^{th} asset.

R = $(R_1, R_2, ..., R_n)$', vector representing the return of n assets.

Γ = E$((\mathcal{R} - r)(\mathcal{R} - r)')_{p\times p}$ denotes the co-variance matrix of $\mathcal{R}$, where r= E($\mathcal{R}$).

$\tilde{\Sigma}$ = E$((\mathcal{R} - r)(R - E(R))')_{p\times n}$ is the co-variance matrix.

P_e = $\hat{\omega}'\mathcal{R} - \omega_B' R$

μ_e = E(P_e)= $\hat{\omega}' r - \mu_B$, where μ_B= E(P_B). It is the excess return.

σ_e^2 = $\hat{\omega}'\Gamma\hat{\omega} - 2\hat{\omega}'\tilde{\Sigma}\omega_B + \sigma_B^2$, where σ_B^2 represents the variance of market index. σ_e^2 is the tracking error.

3.2 Problem Formulation

The objective function is given by J=$\varrho\sigma_e^2 - \xi\mu_e$, where ϱ & ξ are positive real numbers. ϱ & ξ determines the amount of trade off between the tracking error and excess return. Thus, the enhanced index tracking optimization problem is

$$\begin{aligned}&\textbf{Minimize } \varrho(\hat{\omega}'\Gamma\hat{\omega} - 2\hat{\omega}'\tilde{\Sigma}\omega_B + \sigma_B^2) - \xi(\hat{\omega}'r - \mu_B)\\ &\textbf{subject to}: \hat{\omega}'e = 1,\\ &\qquad\qquad\quad \hat{\omega} \in \mathbb{R}^p\end{aligned}$$

where, e is a vector of ones in $\mathbb{R}^p$

Optimal Solution

Using the method of Lagrange multipliers, the optimal tracking portfolio composition is given by

$$\hat{\omega}^* = \Gamma^{-1}(\tilde{\Sigma}\omega_B + \frac{\xi}{2\varrho}r + \frac{\tau}{\alpha}e)$$

where, $\tau = 1 - e'\Gamma^{-1}\tilde{\Sigma}\omega_B - \xi\frac{e'\Gamma^{-1}r}{2\varrho}$ and $\alpha = e'\Gamma^{-1}e$.

Note: The ratio $\frac{\xi}{\varrho}$ determines the optimal solution. By choosing this ratio appropriately, one of the three investment strategies can be defined.

- **Active Management**:

  A high value of ratio can be chosen to achieve an average return higher than a reference index.

- **Enhanced Index Tracking**:

  An intermediate value of ratio can be chosen to track a reference index with a positive excess return.

- **Index Tracking**:

  A small value(close to zero) can be chosen to replicate the return of a market.

4 Index Tracking

In this section, we analyze our model and compare the performance of the optimal portfolio with the market portfolio. We consider an index tracking strategy, where the tracking portfolio has been constructed from the stocks mentioned in the National Stock Exchange. As the Benchmark Index, we have chosen Nifty50.

We have collected the daily data from 2/1/2013 to 6/9/2016, spanning a time period of more than three and a half years, and considered the closing price of the assets. Stocks with missing values were dropped and we had 900 daily data points. To choose the assets to be held in our portfolio, we have primarily considered two approaches- one based on the market capital of the companies constituting the Benchmark Index, and another based on the correlation of the individual asset returns with the market return over the testing time period. The basic logic behind the approach based on the market capital is that companies with a higher market capital are likely to function better in the long run, and that behind the approach based on the correlation is that assets that are comparitively highly correlated with the market should track the market better.

In our analysis, we have divided our entire data into three time periods, consisting of 600 data points each, over which we have analysed the sample, and then tested the sample over the immediate next set of 100 points. We have chosen portfolios consisting of p= 5, 8, 10, 12 and 15 assets out of the entire set of stocks, where p denotes the number of assets we wish to hold in our portfolio. Our aim has been to choose the minimum number of assets that tracks the market best, in order to have low transaction costs, that arises from buying and selling of stocks.

Once we obtained satisfactory results from the assets with highest market capital and the ones that were most correlated with the market and we were pleased with our model, we tested it by selecting assets from companies with least market capital, assets that were least correlated with the market and randomly selected assets to check if the model achieved its objective no matter what set of assets we chose.

The parameters that we have considered in deciding the best set of stocks to hold are the absolute tracking error, correlation between the market return and the portfolio return over the testing period, alpha and beta of the portfolio and asset return.

4.1 Graphs

In this section, we present the graphs we obtained from the various set of assets over different time periods.

No.of assets (P): 8, chosen with Highest Market Capital

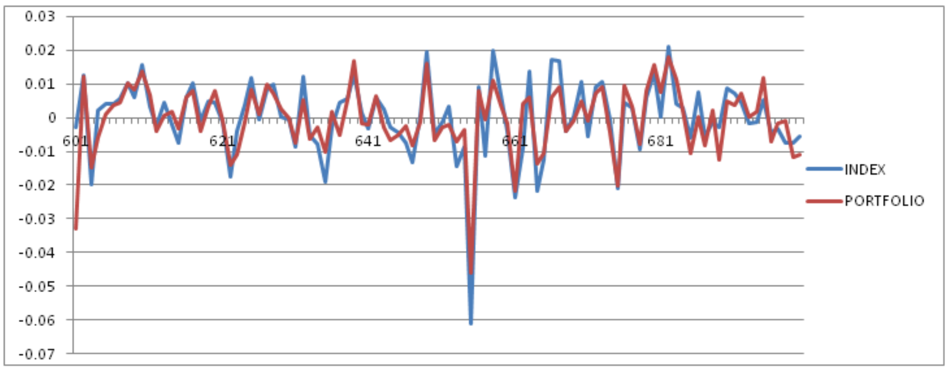

Figure 1: Testing period: 601-700

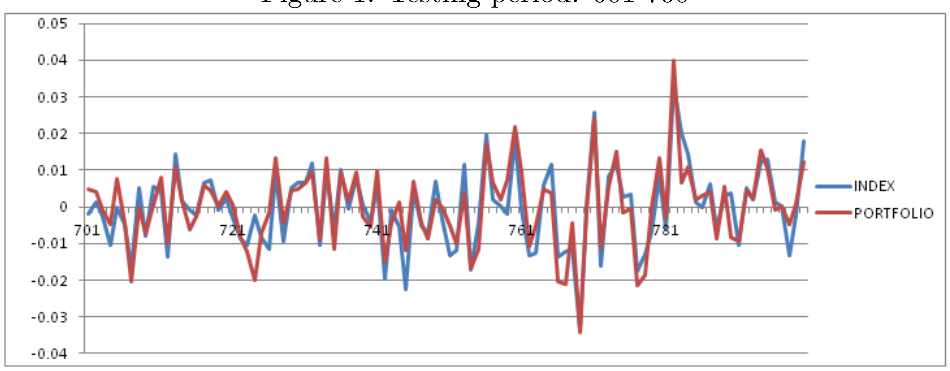

Figure 2: Testing period: 701-800

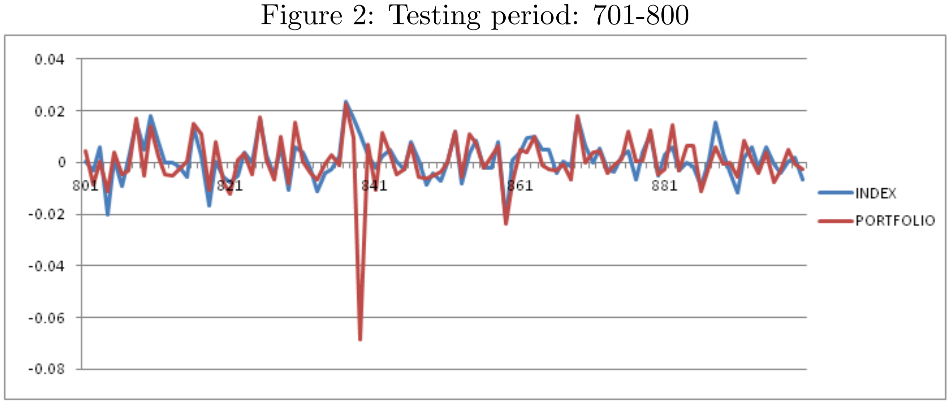

Figure 3: Testing period: 801-900

No.of assets (P): 8, chosen with Highest Correlation with Market Index

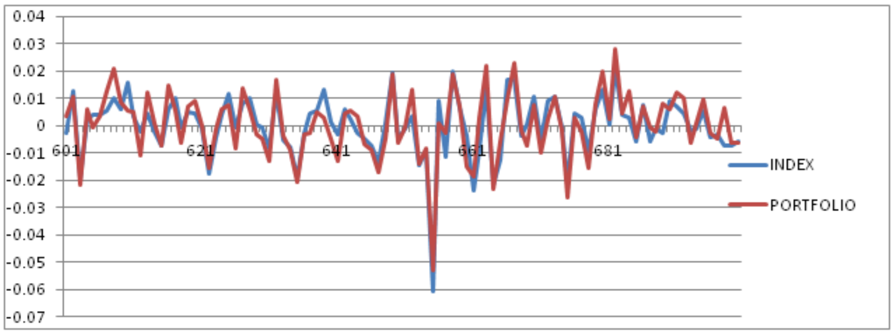

Figure 4: Testing period: 601-700

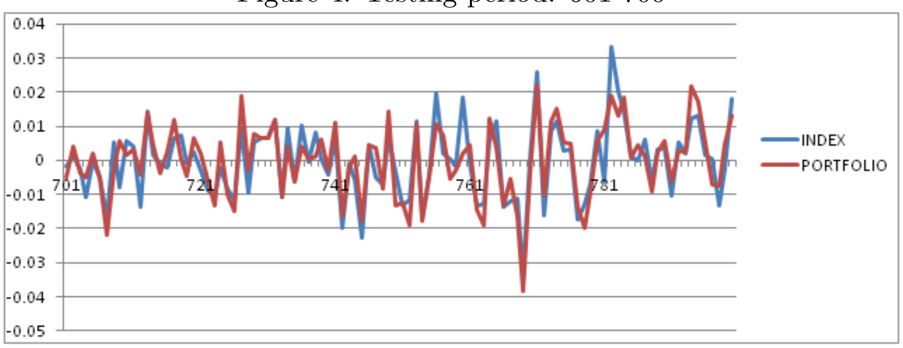

Figure 5: Testing period: 701-800

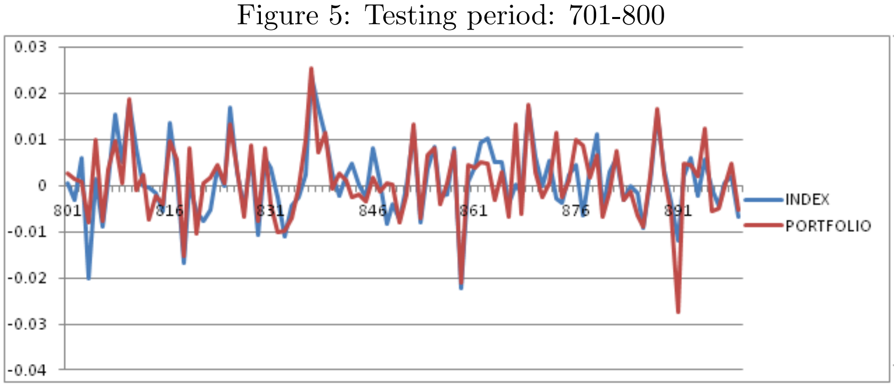

Figure 6: Testing period: 801-900

No.of assets (P): 10, chosen with Highest Market Capital

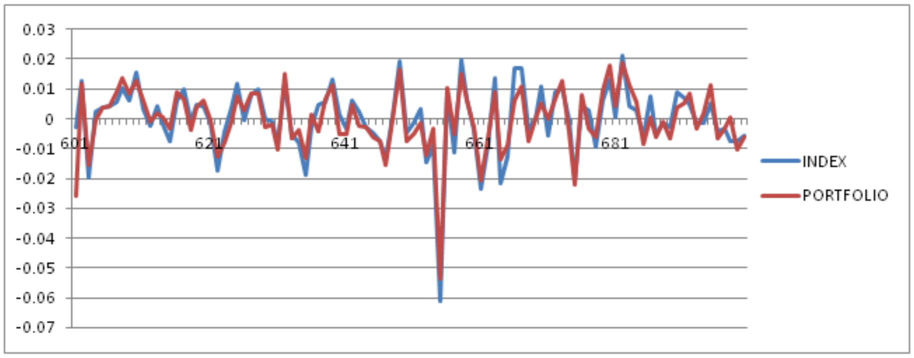

Figure 7: Testing period: 601-700

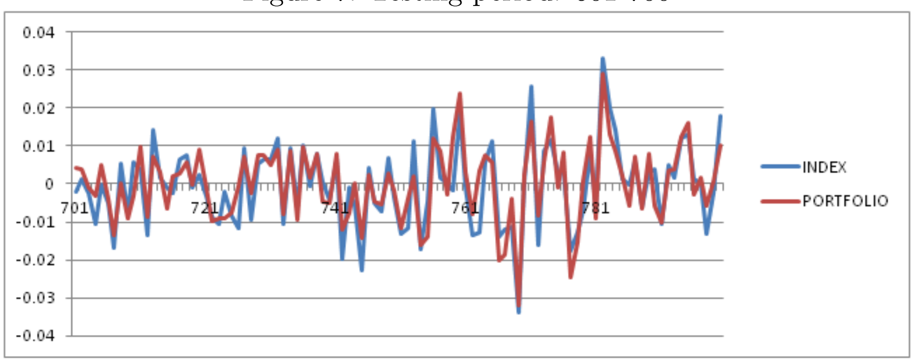

Figure 8: Testing period: 701-800

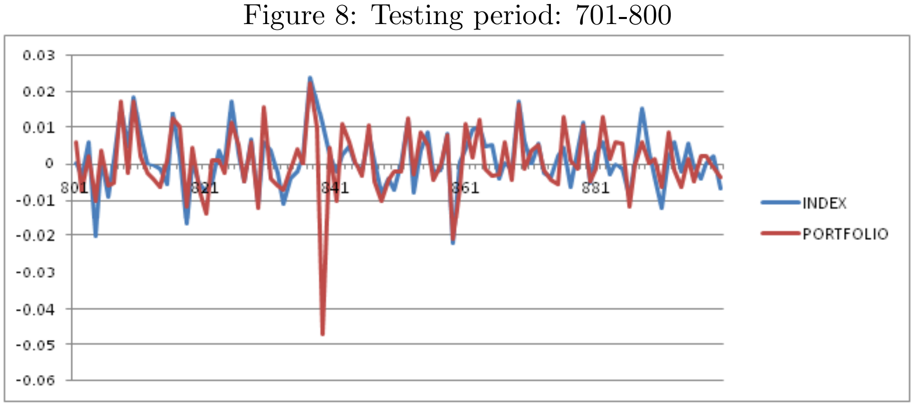

Figure 9: Testing period: 801-900

No.of assets (P): 10, chosen with Highest Correlation with Market Index

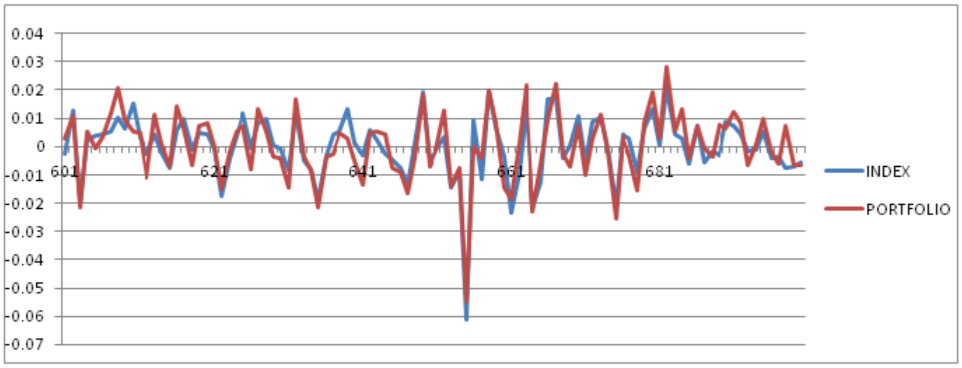

Figure 10: Testing period: 601-700

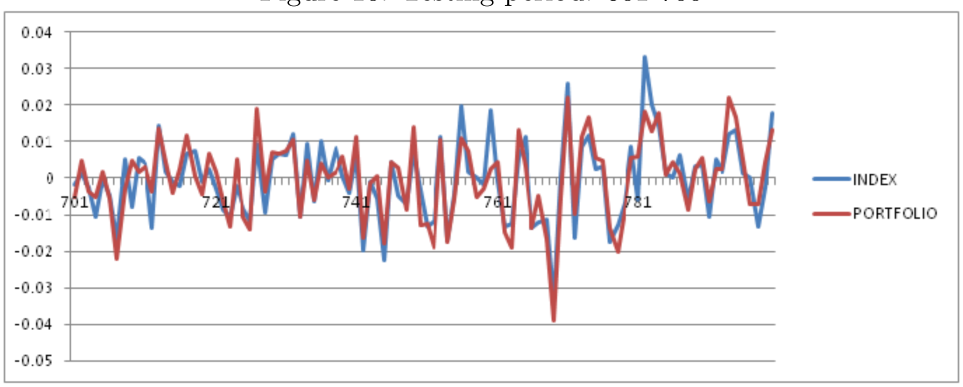

Figure 11: Testing period: 701-800

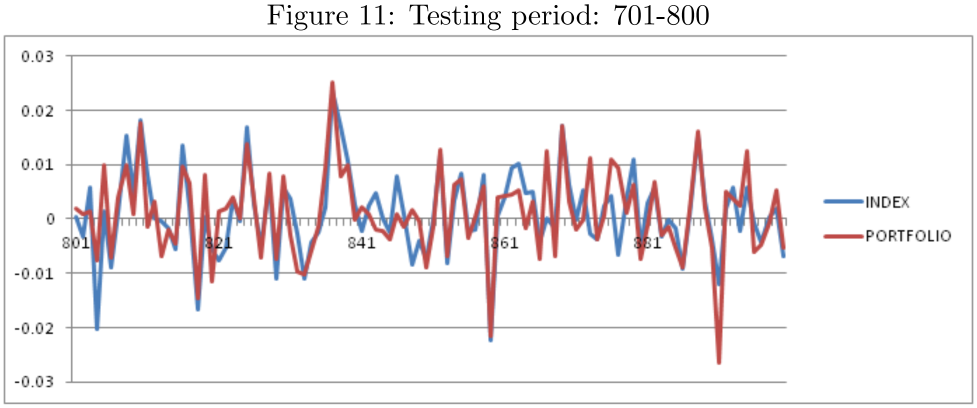

Figure 12: Testing period: 801-900

No.of assets (P): 12, chosen with Highest Market Capital

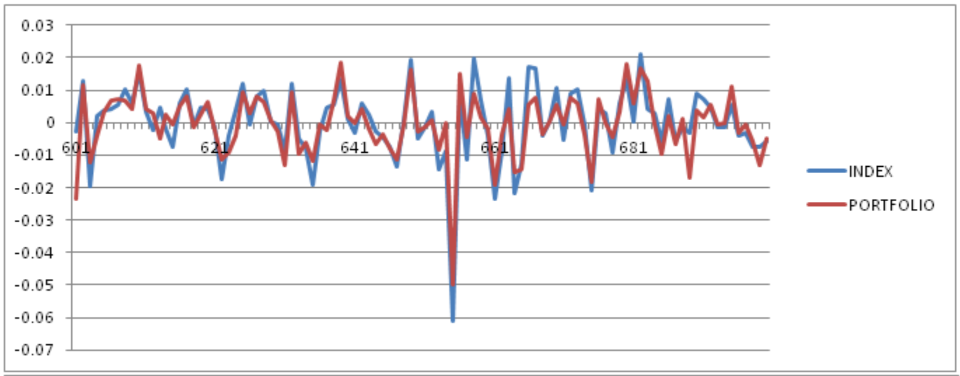

Figure 13: Testing period: 601-700

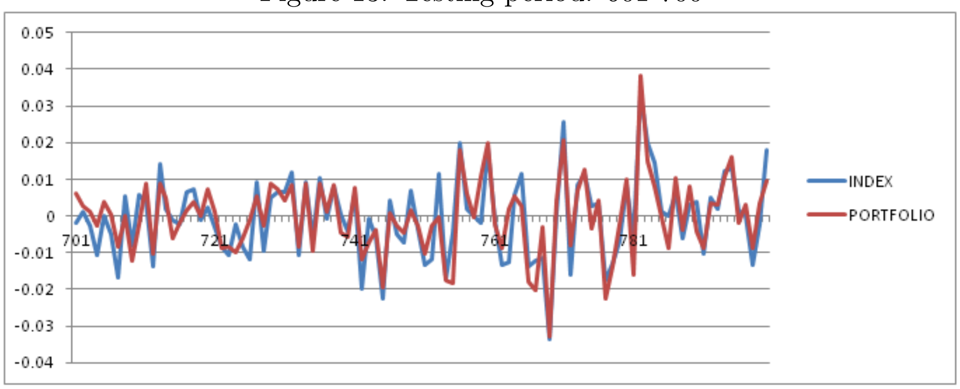

Figure 14: Testing period: 701-800

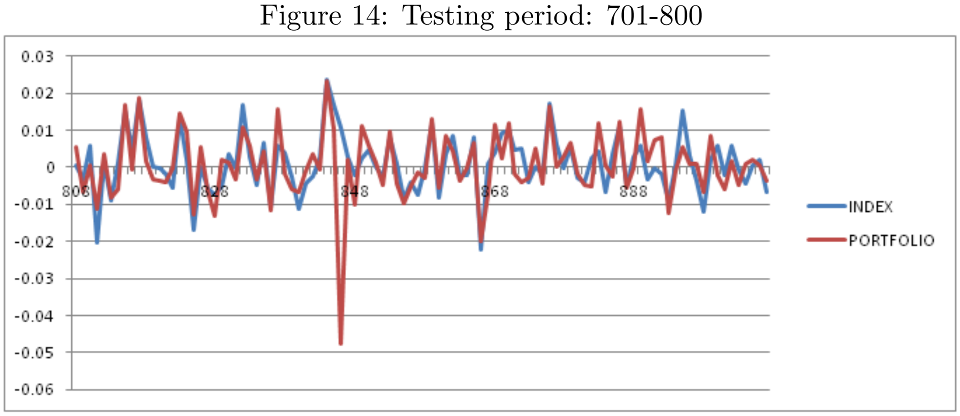

Figure 15: Testing period: 801-900

No.of assets (P): 12, chosen with Highest Correlation with Market Index

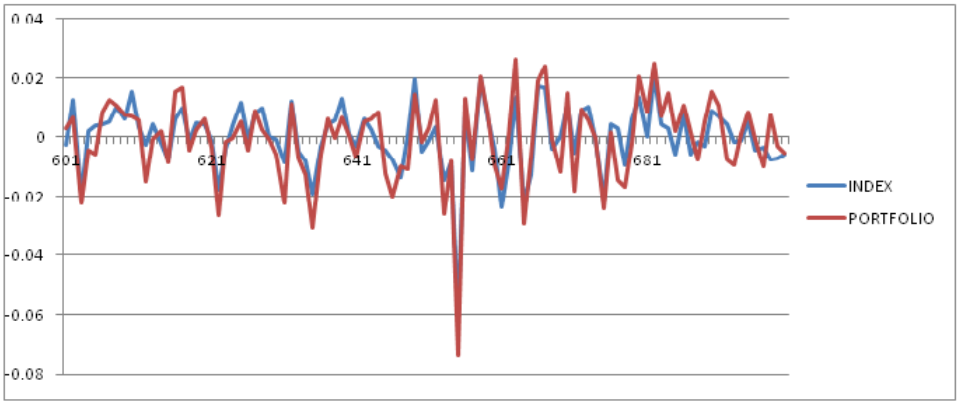

Figure 16: Testing period: 601-700

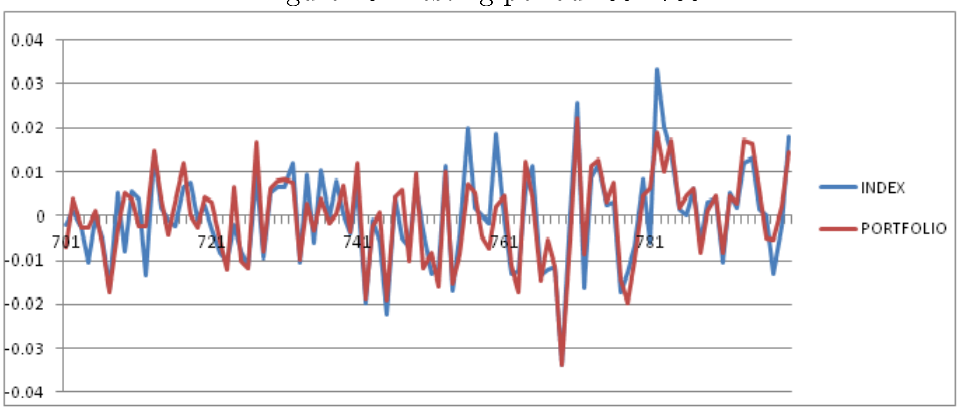

Figure 17: Testing period: 701-800

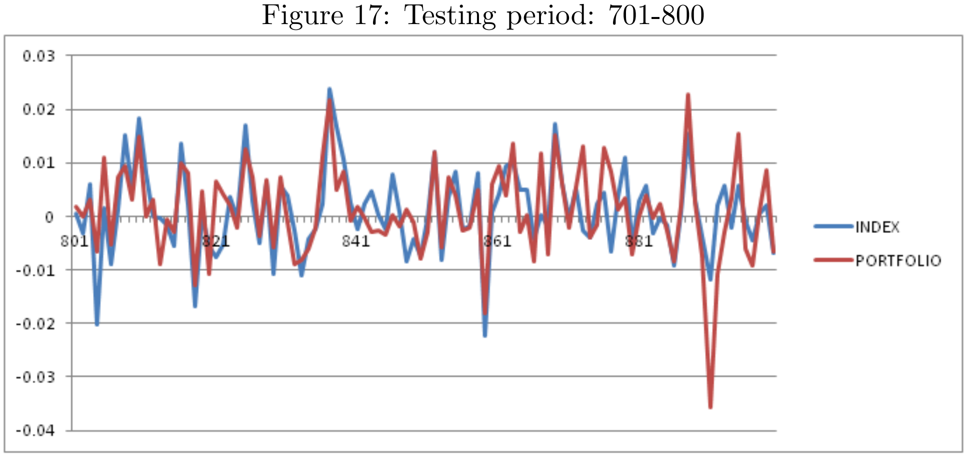

Figure 18: Testing period: 801-900

No.of assets (P): 10

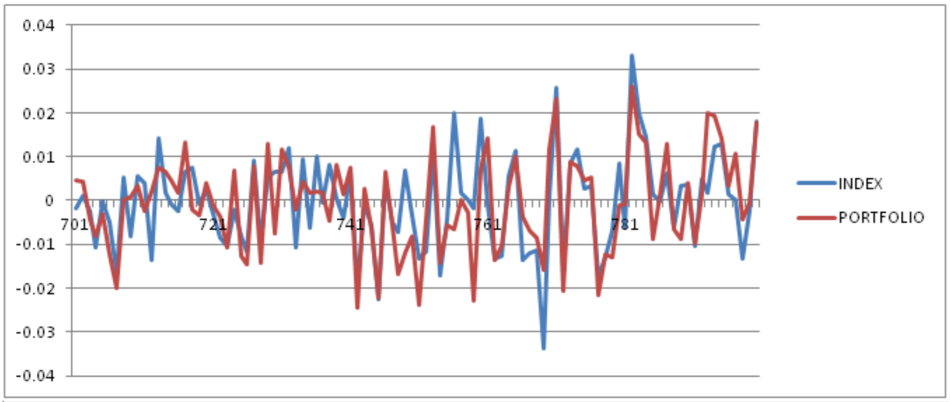

Figure 19: Selecting Assets with Least Market Capital

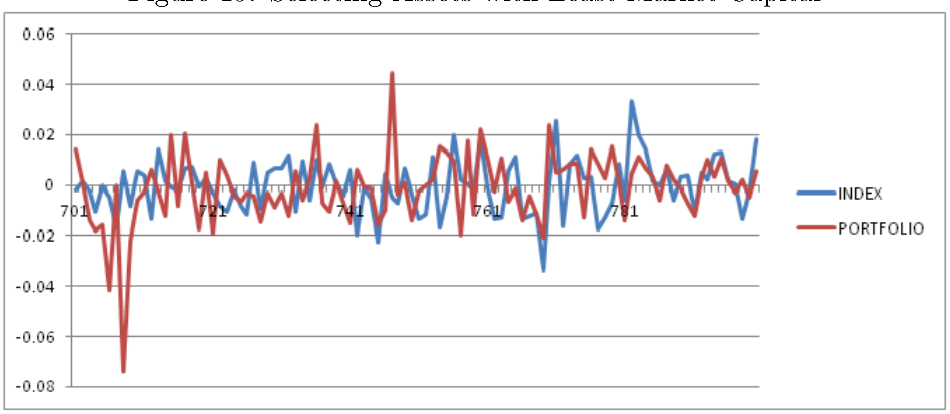

Figure 20: Selecting Assets that are least correlated with Market Index

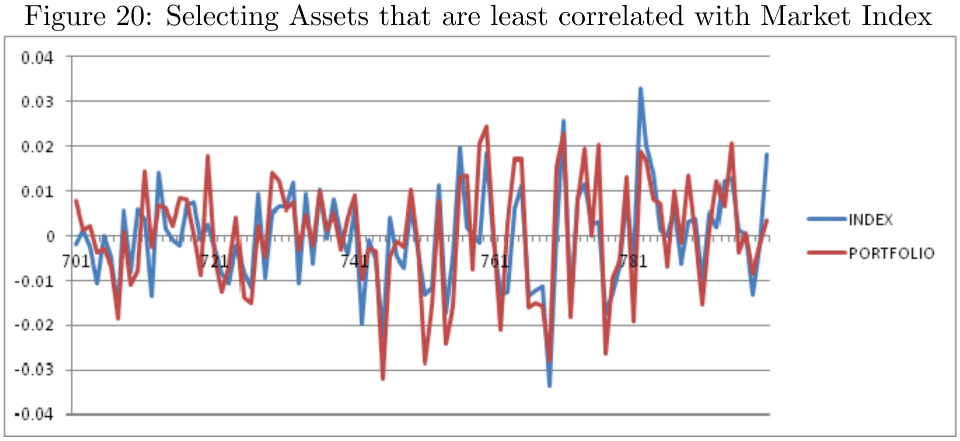

Figure 21: Selecting Assets randomly

4.2 Comparision of the results:

We have integrated the tracking error and correlation co-efficient obtained for different values of p over the various testing periods.

Table 1 shows the average tracking error (absolute difference between the market return and portfolio return) for various time periods and different set of assets whereas Table 2 shows the correlation between the market return and the portfolio return for the same. Table 3 shows the Alpha, Beta and return of the corresponding portfolios. Here, the assets were chosen based on the market capital of the companies.

P	**1-600**	**101-700**	**201-800**	**301-900**	**AVERAGE**
5	0.000878	0.0001617	0.0002256	0.0007489	0.00050356
8	0.000323	0.0002761	0.000765	0.00072	0.000521
10	0.000309	0.0004334	0.0006851	0.0007227	0.00053767
12	0.000398	0.0001916	0.0005562	0.000572	0.00042954
15	0.000413	0.0001844	0.0006193	0.0006768	0.00047331

Table 1

P	**1-600**	**101-700**	**201-800**	**301-900**	**AVERAGE**
5	0.250111	0.1049705	-0.025952	0.721693	0.26270557
8	0.858089	0.8895977	0.5207688	0.7049471	0.74335067
10	0.916734	0.8700407	0.6050772	0.7470336	0.78472131
12	0.884305	0.8733839	0.613418	0.7467659	0.77946806
15	0.827747	0.8749708	0.6277053	0.7334412	0.76596609

Table 2

P	**ALPHA**	**BETA**	**RETURN**
8	1.63E-05	0.781538	-5.1E-05
10	3.8E-05	0.751525	3.26E-05
12	0.000193	0.72253	-3.5E-05

Table 3

Table 4 shows the average tracking error for various time periods and different set of assets whereas Table 5 shows the correlation between the market return and the portfolio return for the same. Table 6 shows the Alpha, Beta and return of the corresponding portfolios. Here, the assets were chosen based on their correlation with the Market Index.

p	1-600	101-700	201-800	AVERAGE
8	0.000331	0.000273	1.26E-07	0.000201171
10	0.000253	0.000208	1.8E-05	0.000159827
12	0.000796	0.000194	6.53E-05	0.000351751

Table 4

p	1-600	101-700	201-800	AVERAGE
8	0.891213	0.864848	0.796201	0.850754034
10	0.895235	0.869889	0.795721	0.853615019
12	0.883015	0.860894	0.709798	0.817902243

Table 5

P	ALPHA	BETA	RETURN
8	-4.2E-05	0.945127	0.000421
10	-5.6E-05	0.946435	0.00038
12	-7.8E-05	0.961973	4.09E-05

Table 6

We also chose 10 assets that were least correlated with the market, assets having least Market Capital, and also randomly selected two sets of 10 assets to check if the model worked fine with any set of 10 assets. The values of the deciding parameters are provided in the table below:

PORTFOLIO	TRACKING ERROR	CORRELATION	ALPHA	BETA	RETURN
HIGHEST MARKET CAPITAL	0.000475988	0.797283869	0.000038	0.751525	3.26E-05
LOWEST MARKET CAPITAL	0.000190714	0.75577266	-0.00029	0.858969	-0.00061
MOST CORRELATED ASSETS	0.000159827	0.853615019	-5.6E-05	0.946435	0.00038
LEAST CORRELATED ASSETS	0.000889477	0.146008623	-0.00016	0.56252	-0.00131
RANDOM ASSETS 1	0.000288556	0.856766304	0.000335	0.708676	-0.00013
RANDOM ASSETS 2	0.000533116	0.80914749	3.71E-05	0.784859	0.000115

Table 7

4.3 Conclusion

So far, we have studied two models, based on Index Tracking and Enhanced Index Tracking. We have studied the first model theoritically and implemented the second model.

- The first model deals with selecting an optimal set of assets from the total set. However, we need to provide the number of assets that we wish to hold.
- In the second model, the portfolio manager has the freedom of choosing not only the number, but also the assets,in particular, i.e., if someone wants assets from a particular sector, he may do so.
  - However, any random set of assets may not yield the desired results, and one has to meticulously choose the set of assets.
  - Ideally, a portfolio manager desires to hold 10-30% of the assets of the index, inorder to reduce transaction costs, that arise with buying and selling of stocks. So, the smaller the proportion of the assets to be held, greater caution should be taken in choosing the assets.
  - The model tracked the index fairly well when 10 to 12 assets were chosen out of 50.
  - But when we chose 10 assets which were least correlated with the market, the results were not satisfactory, as can be seen from the Figure 20.

5 Enhanced Index Tracking

So far in this project, we found that the optimal portfolio for Index Tracking should consist of atleast 10 assets for desirable results. Next, we proceeded for Enhanced Index Tracking with 10 assets, inorder to keep the transaction costs at check.
The parameters we used to quantify enhancing were Sharpe Ratio, Information Ratio(IR), Alpha, Beta, Average Excess Return(AER). Tracking Error is the Standard Deviation of the excess returns.

Information Ratio measures the efficiency with which excess returns are generated. Higher IR's are desirable as they imply a greater probability of positive excess returns over an investment horizon.

$$\text{Information Ratio} = \frac{\text{Average Excess Return}}{\text{Tracking Error}}$$

The Sharpe Ratio is the average return earned in excess of the Index per unit of volatility or total risk.

$$\text{Sharpe Ratio} = \frac{\text{Average Excess Return}}{\text{Standard Deviation of Portfolio Return}}$$

5.1 Enhancing by Model 2

As we have seen previously, the assets with highest market capital and the ones that were most correlated with the market gave best Index Tracking results. As mentioned in the paper, for enhancing we should adjust the trade off between the tracking error and excess return by giving an appropriate value of the ratio $\frac{\rho}{\xi}$. We set $\rho = 0.8$ and $\xi = 0.15$.
So, we proceeded for enhancing with these assets. The graphs are given below

Assets chosen with Highest Market Capital

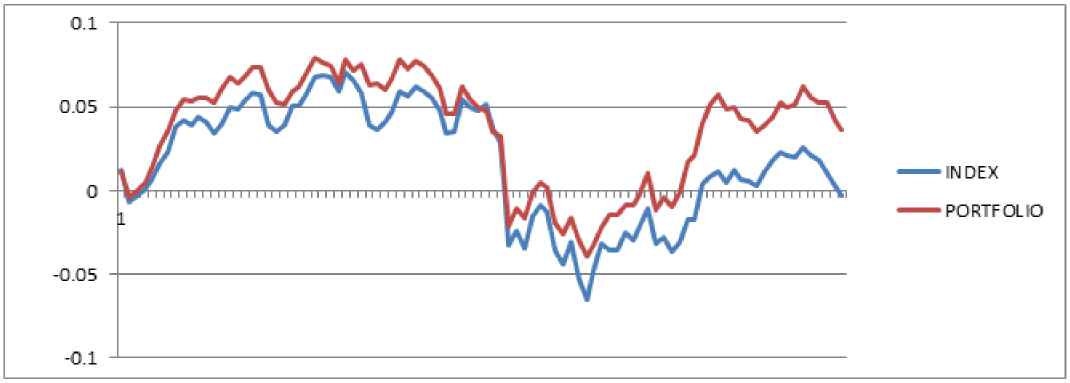

Figure 22: Testing period: 601-700

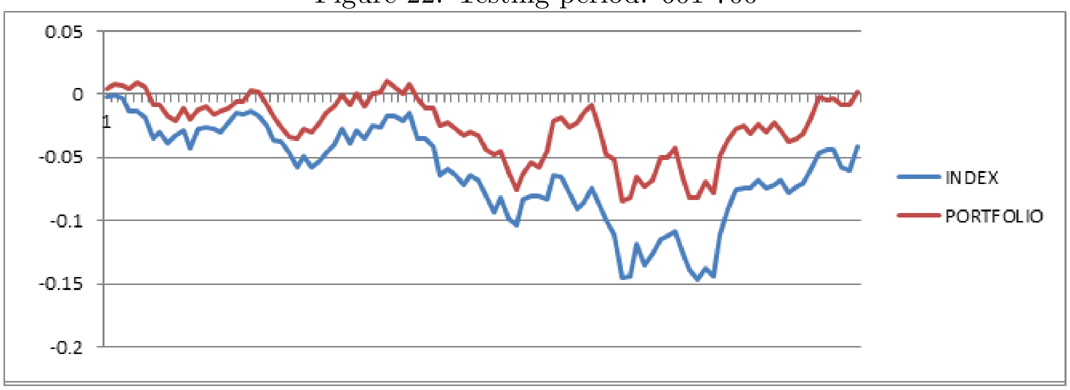

Figure 23: Testing period: 701-800

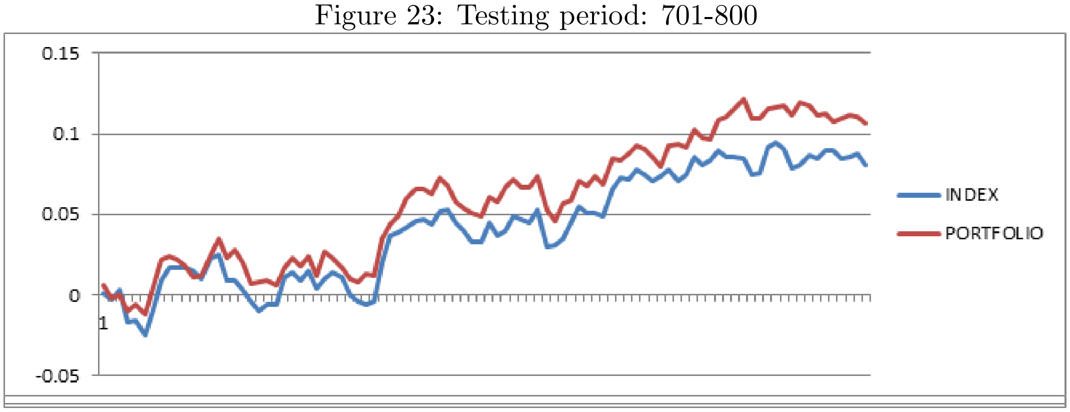

Figure 24: Testing period: 801-900

Assets that were most correlated with the market

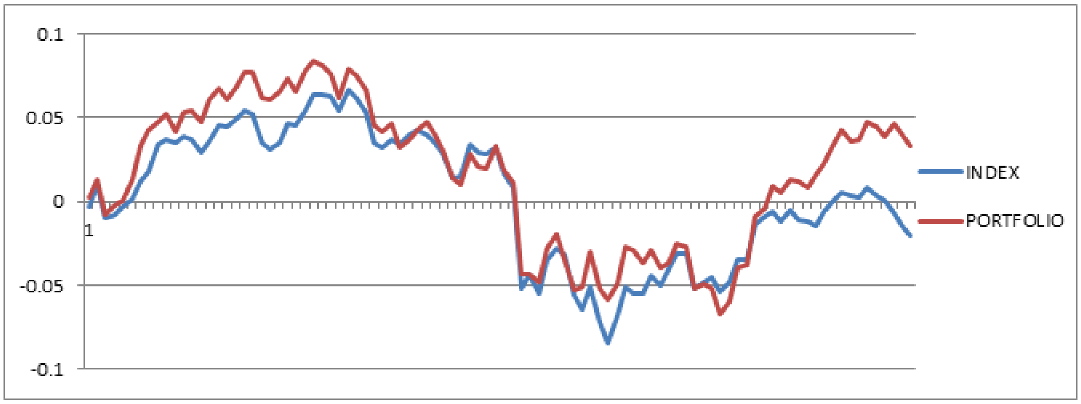

Figure 25: Testing period: 601-700

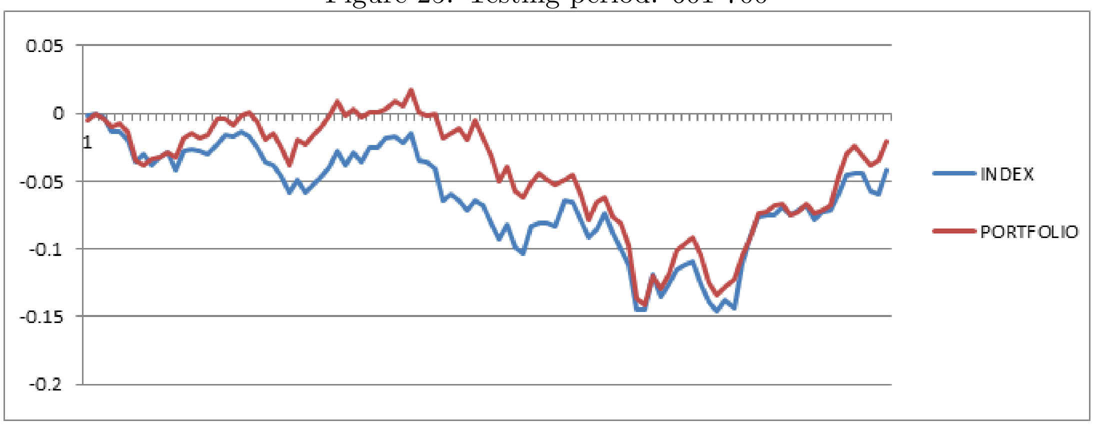

Figure 26: Testing period: 701-800

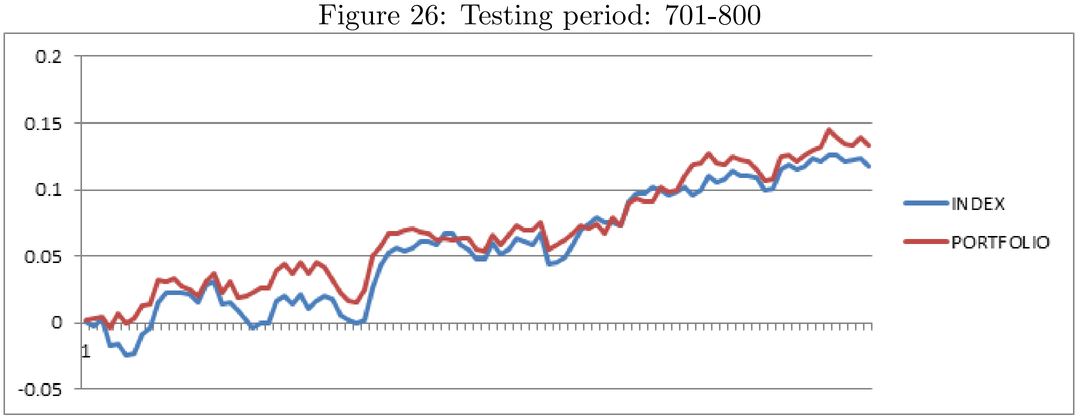

Figure 27: Testing period: 801-900

As we can see, for the assets with the highest market capital, our objective of enhancing was achieved pretty well. The assets that were most correlated with the market, also were able to generate excess return but to a lesser extent. These assets are more suitable for Index Tracking, which is also intuitively obvious. Alpha, AER, IR and Sharpe Ratio are higher for assets with highest market capital.

Tables 8 and 9, showing the various values of the parameters, are provided below.

Assets with the highest Market Capital					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	-4.34564E-05	0.84855	0.00041	0.10124	0.03989
701-800	0.00028	0.79595	0.00043	0.08092	0.04375
801-900	0.00167	0.69971	0.00028	0.05839	0.03575

Table 8

Assets that were most correlated with the Market					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	0.00017	0.96552	0.00055	0.10529	0.04514
701-800	0.00019	0.86951	0.00021	0.03777	0.01927
801-900	0.00025	0.80689	0.00017	0.03411	0.02208

Table 9

5.2 Enhancing by Model 1

But, in general, for a common investor, it is not always possible to invest in assets that have the highest market capital. So, we wanted to somehow select the stocks based on some other criteria. As mentioned in the paper by Canakgoz and Beasley, the alpha and beta of a portfolio are quite appropriate parameters for selecting the assets.

We used LINDO for coding the model, and made appropriate modifications to suit our problem. For example, we removed the constraints involving transaction costs, and modified the other constraints appropriately.

5.2.1 Enhancing by Maximizing Alpha

In mixed-integer model of Canakgoz et al.[1] the first objective is to Minimize $\mid \hat{\beta} - 1 \mid$, then we advanced in two ways.

1 We set $\hat{\beta}$=β_{opt},which is obtained in the previous stage and then maximized α.

2 We selected 20 out of 50 stocks from the first stage and maximized α on these 20 stocks, and finally selected 10 stocks.

The second approach gave better results, so we include those here.

Assets that Maximised the Alpha of the Portfolio

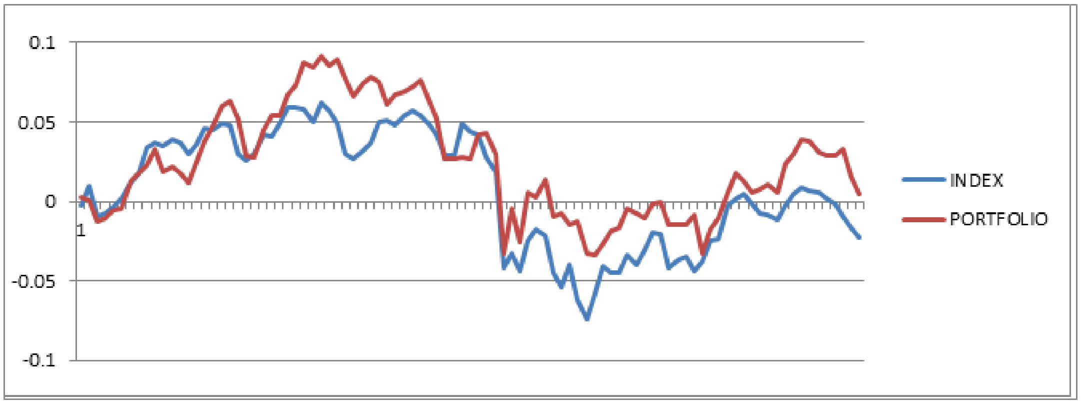

Figure 28: Testing period: 601-700

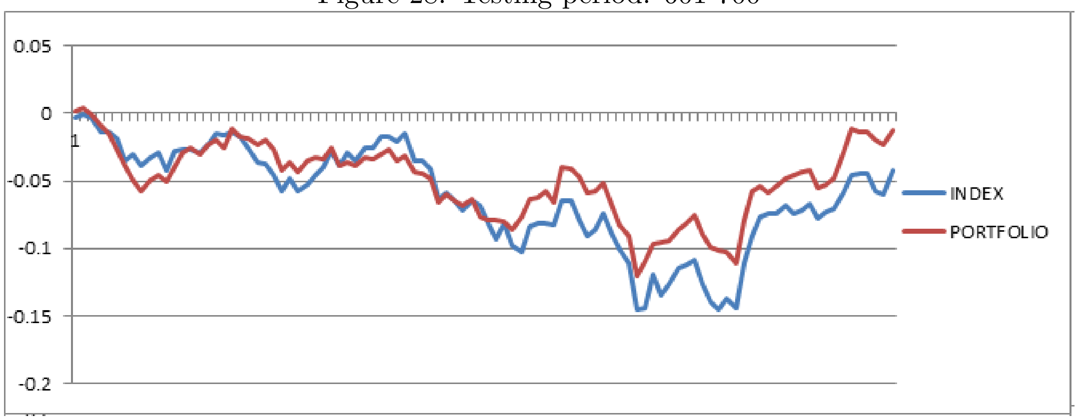

Figure 29: Testing period: 701-800

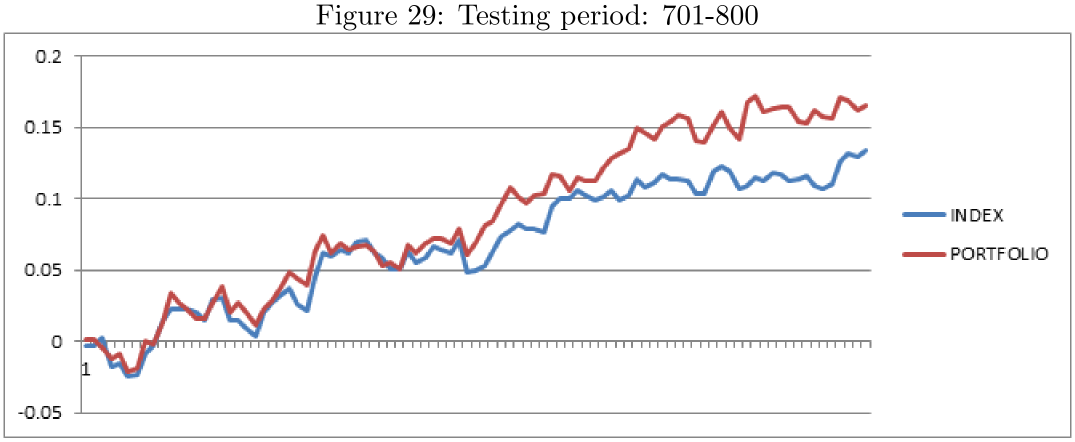

Figure 30: Testing period: 801-900

Assets Selected by Maximising Alpha					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	0.000561	0.824731	0.000285	0.02979	0.022353
701-800	0.000168	0.741028	0.000288	0.047968	0.030026
801-900	0.000365	0.628034	0.000239	0.038019	0.026361

Table 10

5.2.2 A Different Approach for Enhancing

Enhancing basically means to generate returns over and above the market return, so we considered a hypothetical benchmark portfolio whose returns were 5% higher than the index and then employed Model 1 to track this benchmark by setting the objective function to minimize $\mid \beta - 1 \mid$ (which basically means minimising the risk). Then, we plotted the original index returns and the portfolio returns.

Assets Selected according to an Optimal value of Beta					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	0.0006853	0.8961888	0.000192	0.025281	0.0161
701-800	0.000514	0.9609994	0.000104	0.016986	0.09506
801-900	0.002054	0.784734	0.00124	0.161926	0.126266

Table 11

The assets selected by both approaches were able to generate excess returns. Among the two, when assets were chosen by second approach better results were obtained, which can also be inferred from IR, AER, Alpha and Sharpe Ratio.

Assets chosen according to the Beta of the Portfolio

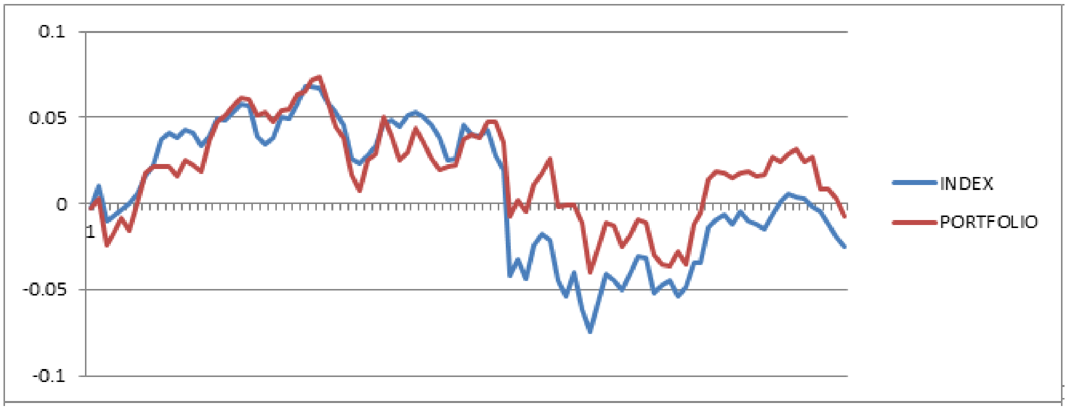

Figure 31: Testing period: 601-700

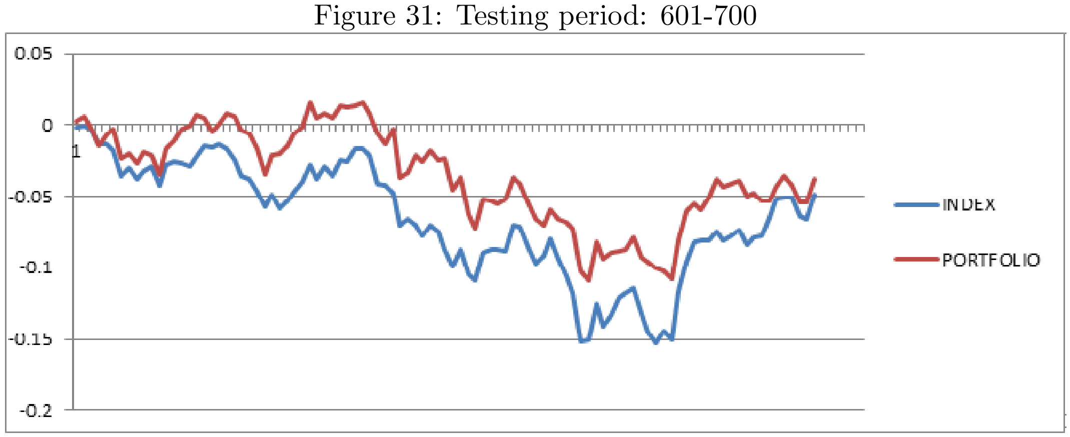

Figure 32: Testing period: 701-800

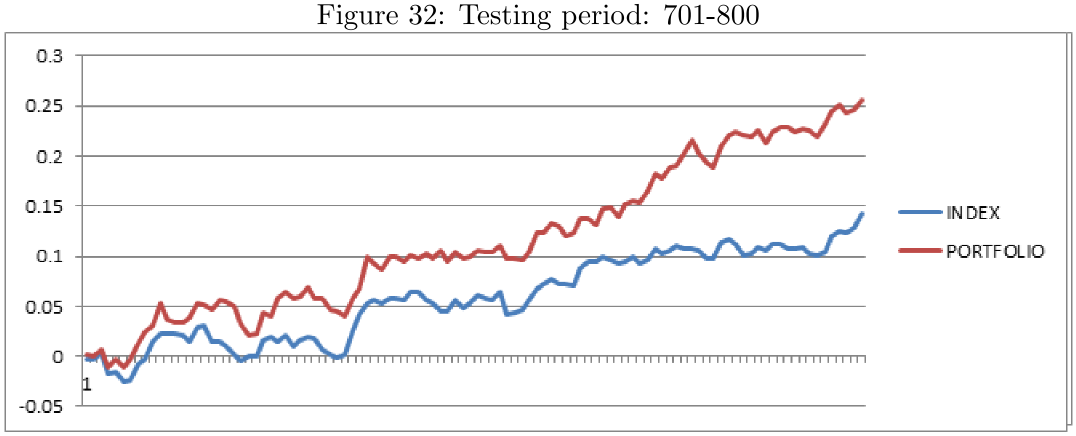

Figure 33: Testing period: 801-900

5.3 Enhancing by a Hybrid of Both Models

In Model 2, we had to provide a pre-determined set of assets, and the optimal weights for tracking the index or enhancing it were determined accordingly. For a common man, it is not always possible to always invest in the assets with the highest Market Capital. We also observed that selecting any random set of assets would not give us the desired results, and hence one has to meticulously choose the set of assets in the portfolio. We decided to select the assets according to the alpha and beta of the portfolio using Model 1 and then use this Model to find the suitable weights for tracking or enhancing the index. The respective tables and graphs for both are provided here.

Hybrid Alpha (With Short-Selling)					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	1.49E-05	0.85827	7.35E-05	0.12480944	0.007305031
701-800	-0.000883387	0.420221	0.000107078	0.010693977	0.11565387
801-900	0.000173781	0.758152	0.000351592	0.047063294	0.038787279

Table 12

Hybrid Beta (With Short-Selling)					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	0.000876	0.855866	0.000373	0.04951	0.03323
701-800	0.000302	0.851354	0.000445	0.05904	0.04648
801-900	0.000088573	0.699244	0.00072251	0.1046067	0.075952

Table 13

Assets chosen according to the Alpha of the Portfolio

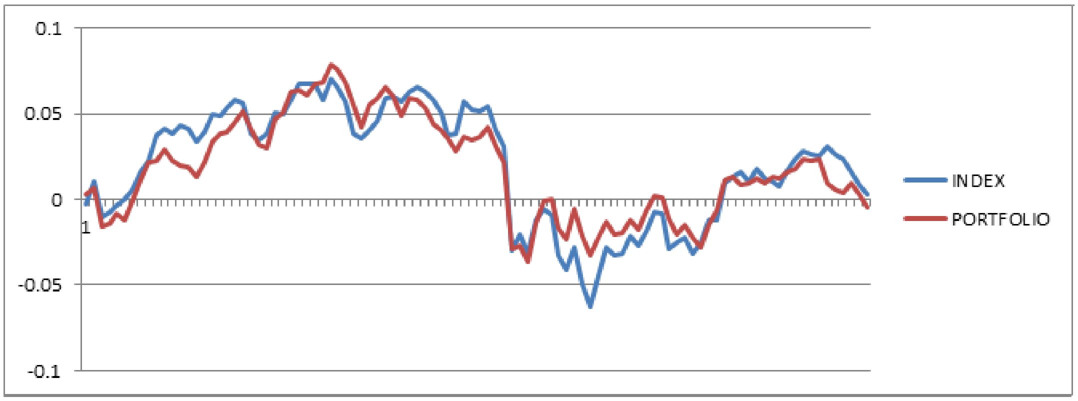

Figure 34: Testing period: 601-700

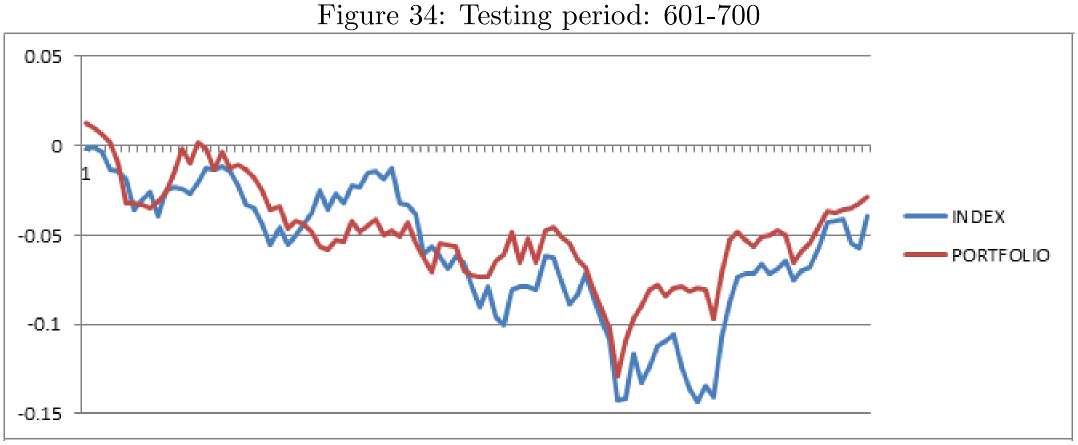

Figure 35: Testing period: 701-800

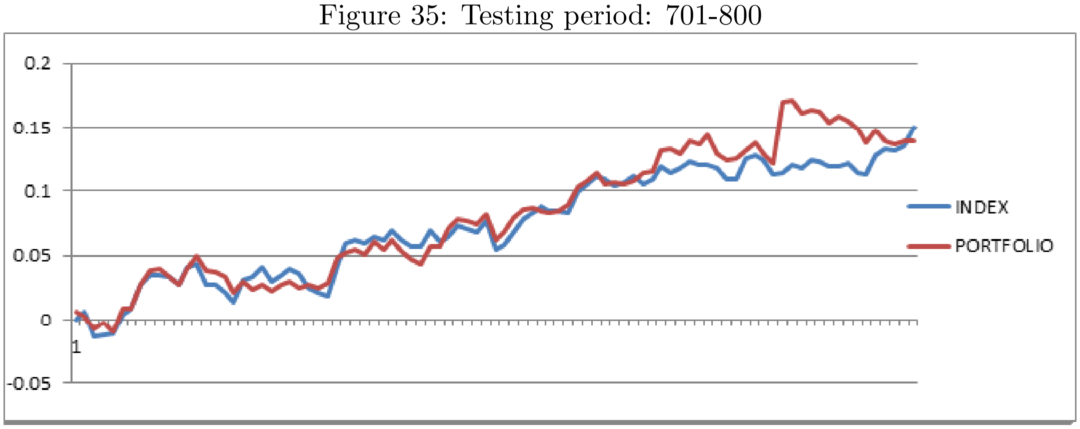

Figure 36: Testing period: 701-800

Assets chosen according to the Beta of the Portfolio

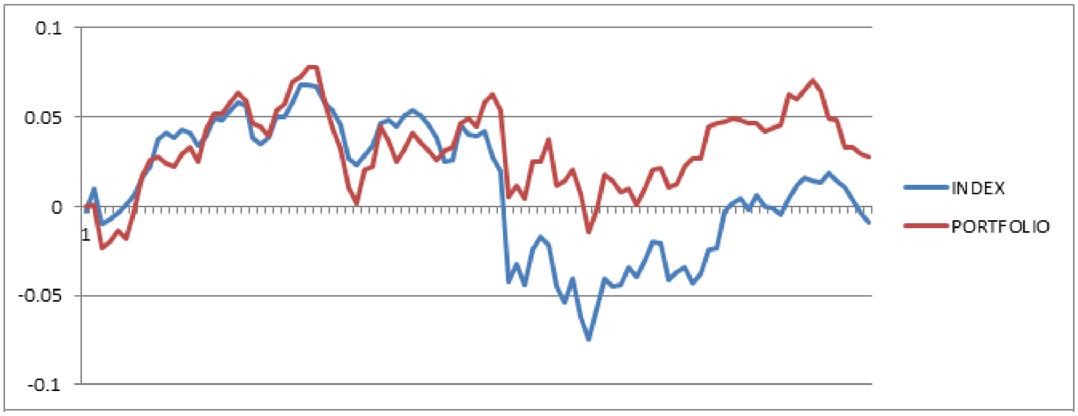

Figure 37: Testing period: 601-700

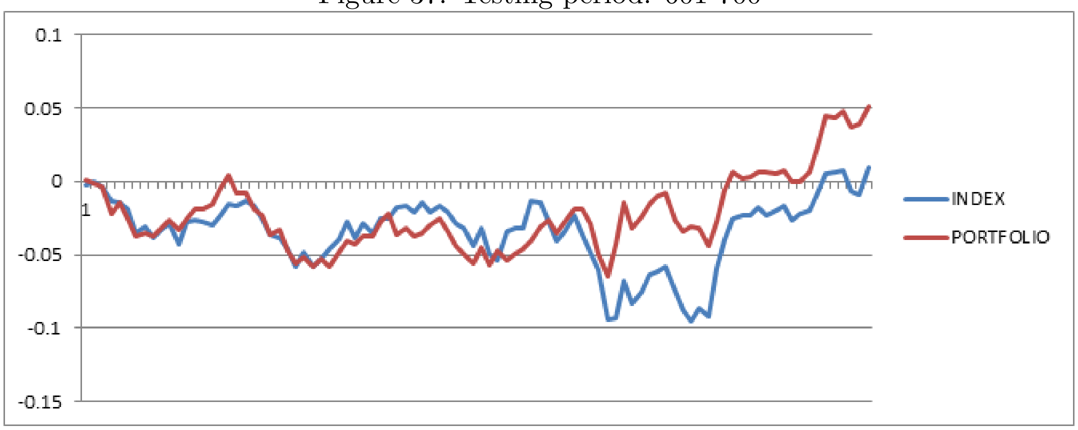

Figure 38: Testing period: 701-800

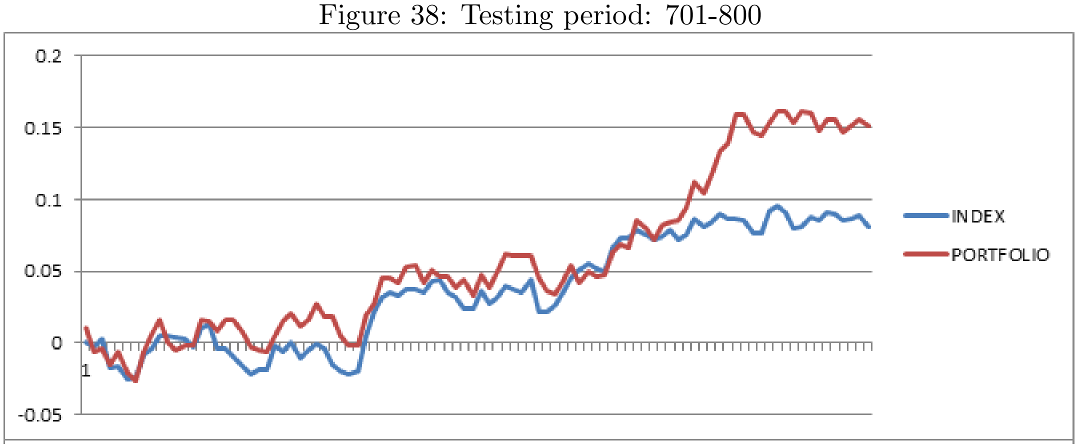

Figure 39: Testing period: 701-800

As we can see, the results for enhanced index tracking are not as good compared to when we used only the alpha or beta criteria without using the other model. Though comparing these two is not entirely justified, because in one we are allowing short selling, whereas in the other we are not. Also, once we find our weights using both the models, we have minimised the risk using two criteria, viz, Variance-Covariance matrix of the market and the portfolio and the Beta of the portfolio. Thus, we have applied two risk hedging factors, which will obviously be compromised with the excess return.

5.4 Enhancing by Model 2 without Short Selling

In order to give a justified comparision between the two models, we implemented the model by Paulo et al., but without short selling, i.e., we did not allow negative weights in our portfolio. In this too, we first selected 10 assets with highest market capital and 10 assets that were most correlated with the market.

Assets with the highest Market Capital (Without Short Selling)					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	0.000316	0.774878	0.000403	0.06853	0.04481
701-800	0.000327	0.817993	0.000411	0.076571	0.0428
801-900	0.000277	0.893371	0.00021	0.041081	0.026695

Table 14

Assets that were most correlated with the Market (Without Short Selling)					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	0.000202	1.167188	0.000321	0.064671	0.024376
701-800	0.000212	1.124584	0.000416	0.085437	0.035436
801-900	0.000394	1.262742	0.00132	0.2165	0.147712

Table 15

Though enhancing was attained by this approach, it was comparitively less than when short selling was allowed, which is obvious because the feasible set has been reduced due to the addition of an extra constraint.

Asstes chosen with Highest Market Capital

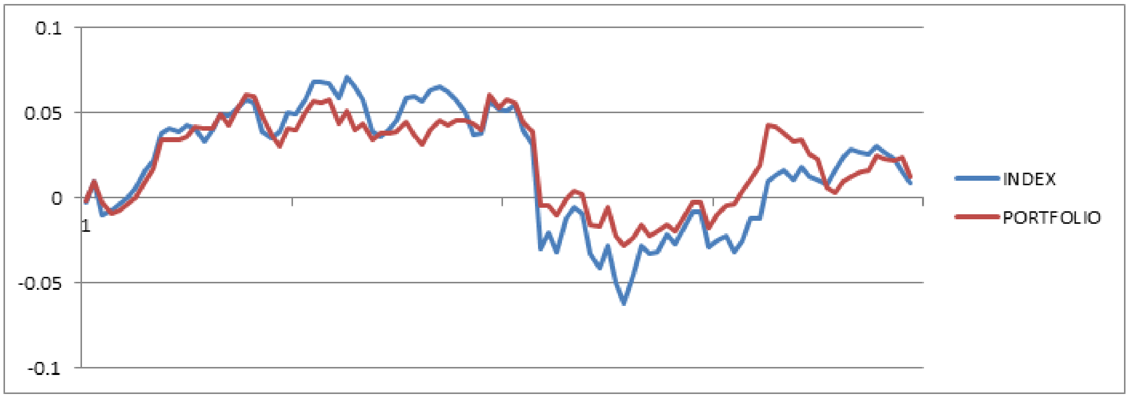

Figure 40: Testing period: 601-700

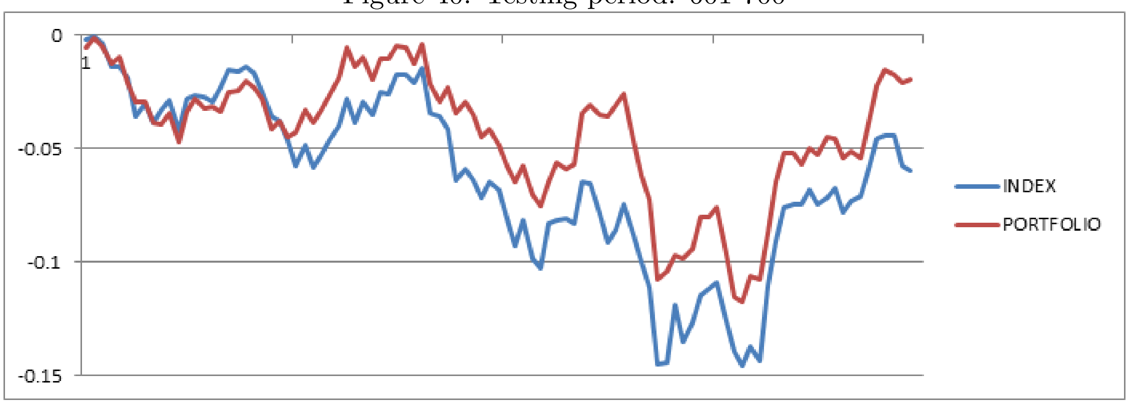

Figure 41: Testing period: 701-800

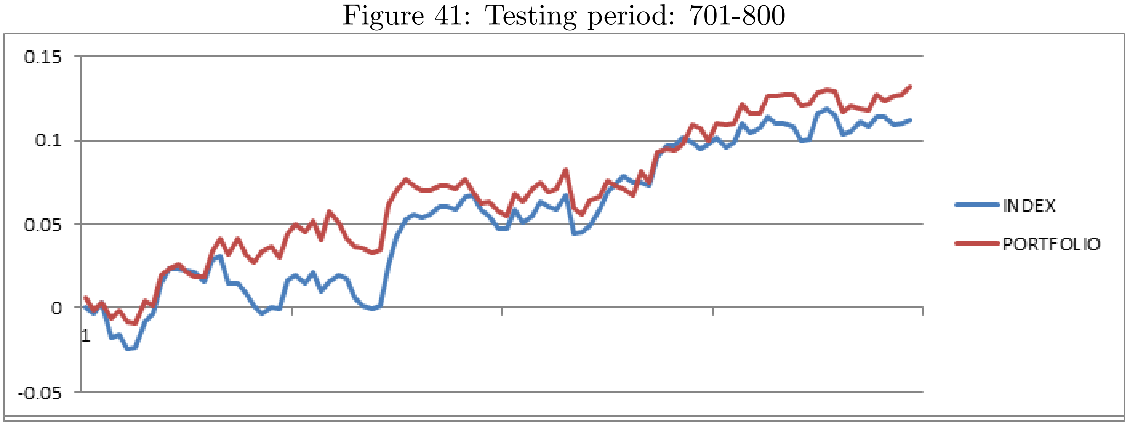

Figure 42: Testing period: 801-900

Asstes that are most are Correlated with the market

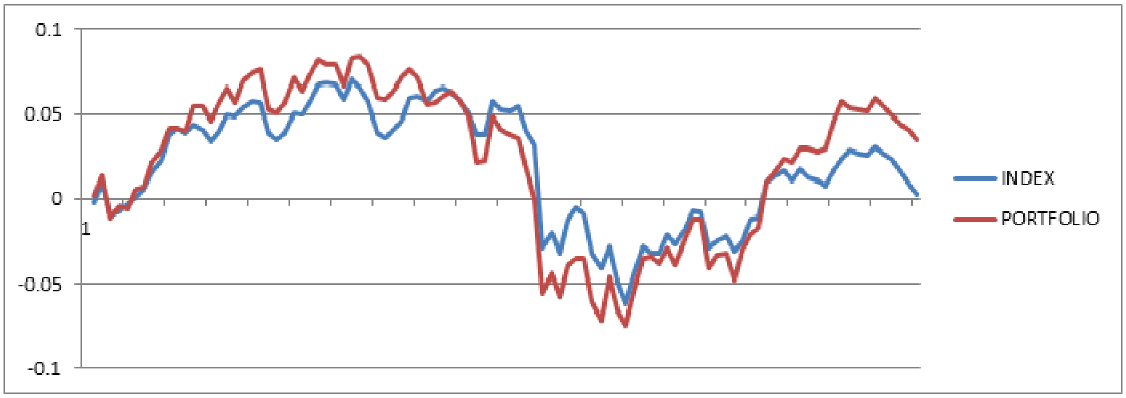

Figure 43: Testing period: 601-700

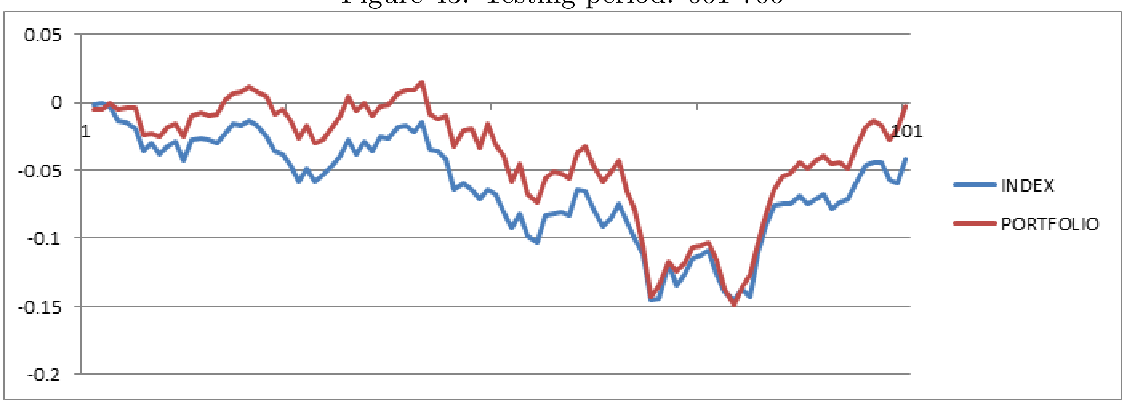

Figure 44: Testing period: 701-800

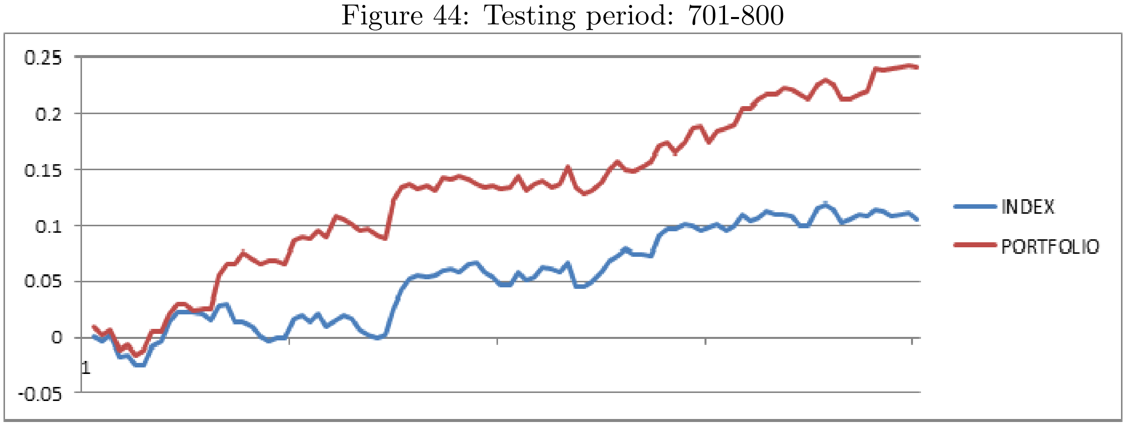

Figure 45: Testing period: 801-900

5.5 Enhancing by a Hybrid of both Models without Short Selling

Next we implemented Model 2 on the set of 10 assets picked using the model by Canakgoz and Beasley. Those 10 assets were selected once by maximising the alpha of the portfolio and once by pushing the beta of the portfolio towards 1, just like the previous hybrid models, the only difference being that now we did not allow short selling. We give the tables and graphs here.

Hybrid Alpha (Without Short Selling)					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	6.29E-04	0.882457	4.55E-05	0.00666	0.00382
701-800	0.00065	0.880194	0.00023	0.027201	0.023186
801-900	0.001914	0.780751	0.000838	0.104367	0.08151

Table 16

Hybrid Beta (Without Short Selling)					
Out of Sample	**Alpha**	**Beta**	**AER**	**IR**	**Sharpe Ratio**
601-700	0.001184	0.812842	0.000803	0.082881	0.061055
701-800	0.000049	0.889608	0.00135	0.15002	0.10981
801-900	0.002701	0.983084	0.00202	0.286956	0.192476

Table 17

This is a more justified hybrid of the two models, as in both the cases short selling was not allowed. Out of the two enhancing was better when the assets were picked by the Beta criteria.

Asstes that Maximised the Alpha of the Portfolio (Without Short-selling)

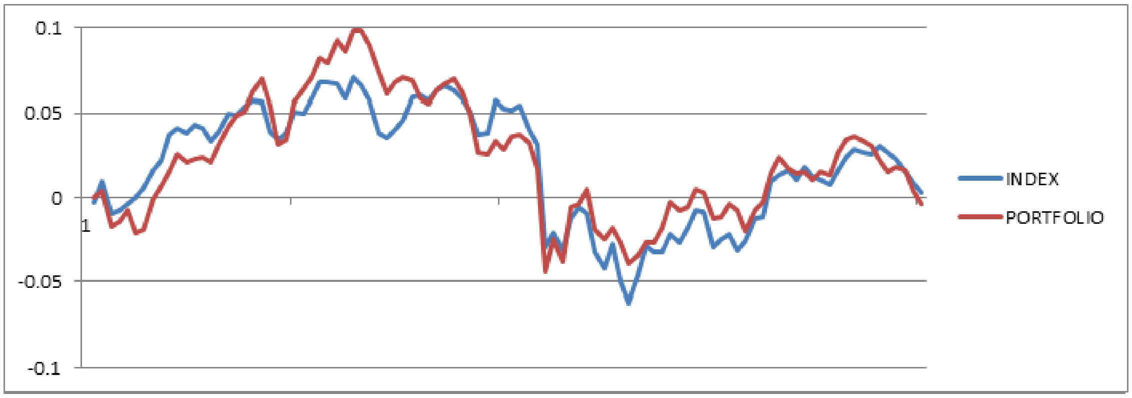

Figure 46: Testing period: 601-700

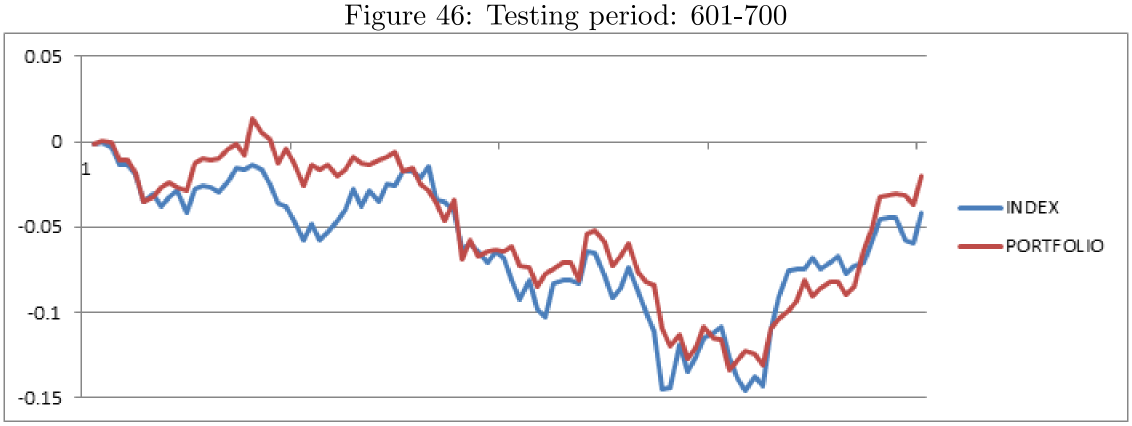

Figure 47: Testing period: 701-800

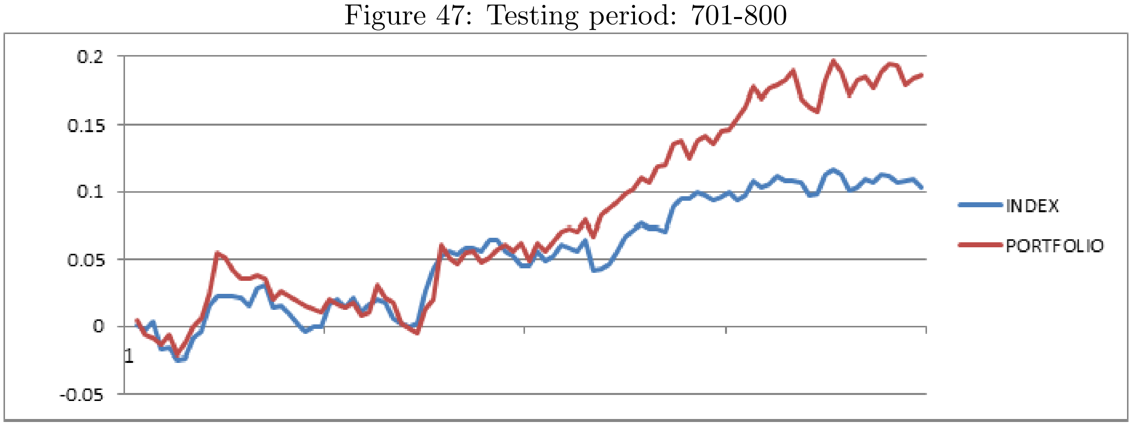

Figure 48: Testing period: 801-900

Asstes chosen according to the Beta of the Portfolio (Without Short-selling)

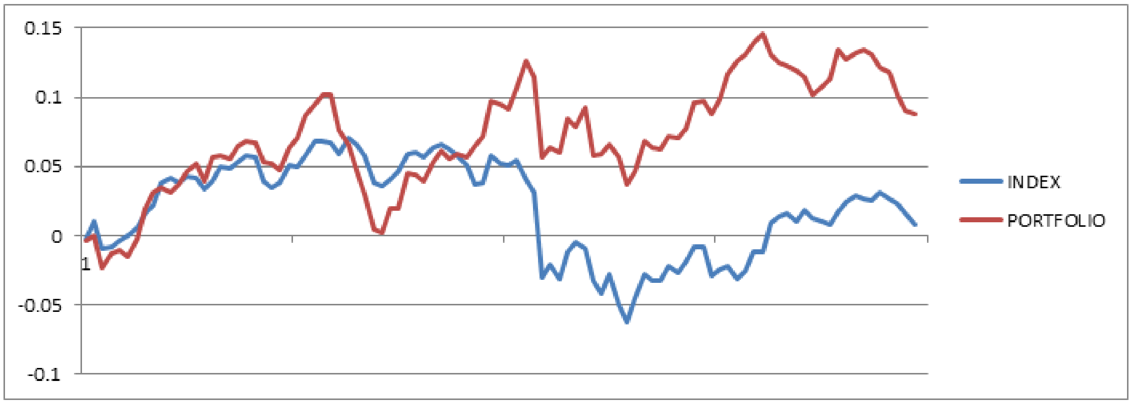

Figure 49: Testing period: 601-700

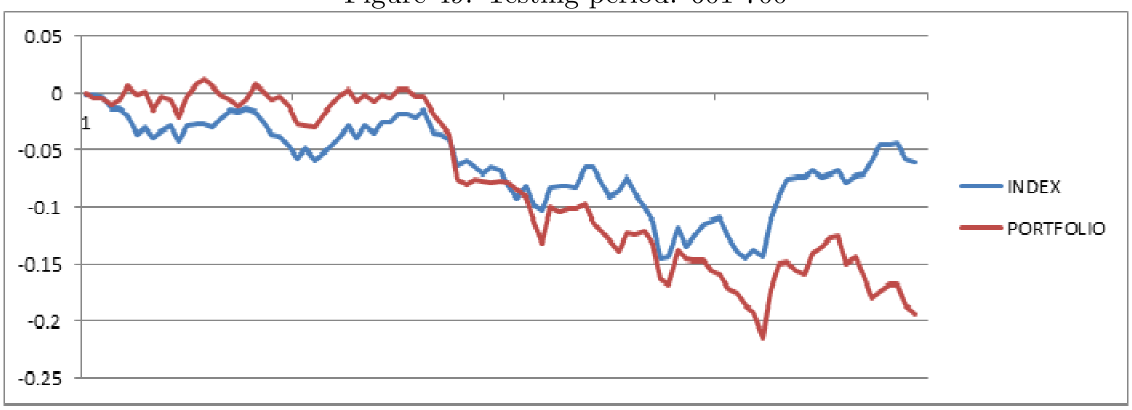

Figure 50: Testing period: 701-800

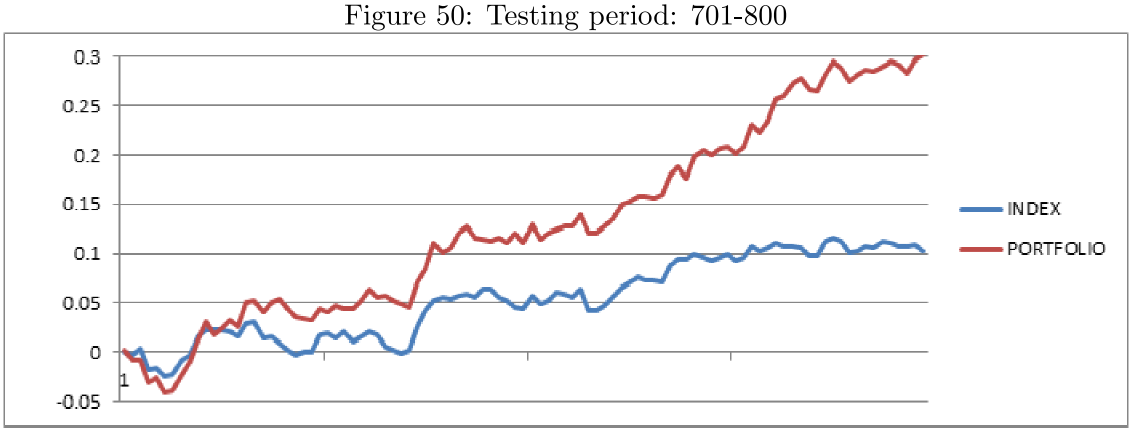

Figure 51: Testing period: 801-900

5.6 Conclusion

We have studied two models, based on Enhanced Index Tracking. We have tracked the Market Index quite satisfactorily by the model proposed by Paulo et al. We also performed enhanced index tracking for the two models, both with and without short selling. We carried out a hybrid of both the models as well.

- The model proposed by Canakgoz and Beasley was able to enhance the market returns the best, when the assets were selected according to the Alpha or Beta of the portfolio.
- The model by Paulo et al., also was able to generate significant excess returns, even when the criteria of short selling was removed.
- When the weights were determined based on both these models, the returns were moderate, when short-selling was allowed. This is also intuitively obvious as model 1 selected the assets without short-selling, whereas model 2 determined the weights with short-selling.
- When a hybrid of both the models was used and negative weights were not allowed, enhancing was better than the previous case.
- Comparitively, better results were obtained when the assets were selected according to the beta of the portfolio.
- For the model by Paulo et al., the question still remains open as to how to select the assets, because the selecton criteria used to define the subset of assets has a significant impact on the performance of the portfolio.

6 Fuzzy Index Tracking

6.1 Introduction

Credibilistic portfolio selection deals with fuzzy portfolio selection using credibility theory. Fuzzy portfolio selection has been a topic of research since 1990s. Earlier researchers used the possibility as a basic measure of the occurrence of a fuzzy event, and mostly they tried to generalize Markowitz's mean-variance selection idea. But, the problem with possibility theory is that it is not self-dual. If an investor uses possibility, when he knows the possibility level of a portfolio reaching a desired return, he is not being able to know the possibility of the opposite event, i.e., the event of the portfolio not being able to achieve the desired return! This in itself is confusing to the decision maker, and a matter that requires some pondering upon.

To get rid of this confusion, Huang proposed that the self-dual credibility should be used as the basic measure of the occurrence of a fuzzy event and then study the fuzzy portfolio selection problems. To provide an instinct and observable information about loss amount and to accurately evaluate the degree of loss, Huang proposed that each likely loss level and the chance of occurrence of loss should be evaluated, instead of just focusing on the average information of loss. Looking at loss from a broad perspective, Huang provided a general definition of risk, i.e., the risk curve, and proposed a mean risk model based on this new definition. In addition, Huang also gave a spectrum of simplified versions of credibilistic portfolio selection models, including mean-risk model, β-return-risk model, credibility minimization model, mean-variance model, mean-semi-variance model, and entropy optimization model.

Here, we will first introduce some necessary knowledge about credibility theory, the definitions of risk, and the credibilistic mean-variance portfolio selection model. Then we will extend this model for Index Tracking, where we will optimize the return and risk of the error portfolio. This idea was generalized from the paper by Paulo et al. And then we will implement this model on the Nifty50 Index of the Indian market.

6.2 Credibility Theory

6.2.1 Credibility Measure

Let Θ be a nonempty set,and $P(\Theta)$ the power set of Θ. Each element in $P(\Theta)$ is called an event. A set function $Cr\{.\}$ is called a credibility measure if it satisfies:

(1) (Normality) $Cr\{\Theta\}$=1;

(2) (Monotonicity) $Cr\{A\} \leq Cr\{B\}$, whenever $A \subset B$;

(3) (Self-Duality) $Cr\{A\} + Cr\{A^c\} = 1$ for any event A;

(4) (Maximality) $Cr(\cup_i A_i)$= $\sup_i Cr(A_i)$ for any event $\{A_i\}$with $\sup_i Cr(\cup_i A_i)$¡ 0.5.

The triplet $(\Theta, P(\Theta), Cr)$ is called a credibility space. The law of contradiction tells us that a proposition cannot be both true and false at the same time, and the law of excluded middle tells us that a proposition is either true or false. Self-duality is in fact a generalization of the law of contradiction and law of excluded middle. In other words, a mathematical system without self-duality assumption will be inconsistent with the laws. This is the main reason why self-duality axiom is assumed.

6.2.2 Fuzzy Variable

Definition 1: A fuzzy variable is a function from a credibility space $(\Theta, P(\Theta), Cr)$ to the set of real numbers.

Definition 2: Let ξ be a fuzzy variable defined on the credibility space $(\Theta, P(\Theta), Cr)$. Then its membership function is derived from the credibility measure by $\mu(t) = (2Cr\{\xi = t\}) \wedge 1$, $t \epsilon \mathbb{R}$.

In practice, a fuzzy variable may be specified by a membership function. In this case, we need a formula to calculate the credibility value of some fuzzy event. The **Credibility Inversion Theorem** provides this.

Theorem(*Credibility Inversion Theorem*) Let ξ be a fuzzy variable with membership function μ. Then for any set A of real numbers, we have

$$Cr\{\xi\} \in A = \frac{1}{2}\Big(\sup_{t\in A}\mu(t) + 1 - \sup_{t\in A^c}\mu(t)\Big)$$

Proof *(Case I)* If $Cr\{\xi \in A\} \leq 0.5$, we know from monotonicity property of credibility measure that $Cr\{\xi = t\} \leq 0.5$ for each $t \in A$. According to maximality we have

$$Cr\{\xi \in A\} = \frac{1}{2}\Big(\sup_{t\in A}(2Cr\{\xi = t\} \wedge 1)\Big) = \frac{1}{2}\sup_{t\in A}\mu(t)$$

Since the credibility measure is self dual, we have $Cr\{\xi \in A^c\} \geq 0.5$, and $\sup_{t\in A^c} Cr\{\xi = t\} \geq 0.5$. Therefore,

$$\sup_{t\in A^c}\mu(t) = \sup_{t\in A^c}\big(2Cr\{\xi = t\} \wedge 1\big) = 1.$$

(Case II) If $Cr\{\xi \in A\} \geq 0.5$, we have $Cr\{\xi \in A^c\} \leq 0.5$ because the credibility measure is self dual. From the result of the first case we have

$$Cr\{\xi \in A\} = 1 - Cr\{\xi \in A^c\} = 1 - \frac{1}{2}\Big(\sup_{t\in A^c}\mu(t) + 1 - \sup_{t\in A}\mu(t)\Big)$$

$$= \frac{1}{2}\Big(\sup_{t\in A}\mu(t) + 1 - \sup_{t\in A^c}\mu(t)\Big).$$

Example 1: A fuzzy variable is called a *triangular fuzzy variable* if it has a triangular membership function

$$\mu(t) = \begin{cases} \frac{t-r_1}{r_2-r_1} & r_1 \leq t \leq r_2 \\ \frac{t-r_3}{r_2-r_3} & r_2 \leq t \leq r_3 \\ 0 & otherwise \end{cases}$$

We denote it by $\xi = (r_1, r_2, r_3)$ with $r_1 < r_2 < r_3$.

A fuzzy variable is called a *trapezoidal fuzzy variable* if it has a trapezoidal membership function

$$\mu(t) = \begin{cases} \frac{t-r_1}{r_2-r_1} & r_1 \leq t \leq r_2 \\ 1 & r_2 \leq t \leq r_3 \\ \frac{t-r_4}{r_3-r_4} & r_3 \leq t \leq r_4 \\ 0 & otherwise \end{cases}$$

We denote it by $\xi = (r_1, r_2, r_3, r_4)$ with $r_1 < r_2 < r_3 < r_4$.

A fuzzy variable is called an *equipossible fuzzy variable on [r_1, r_2]* if it has the following membership function

$$\mu(t) = \begin{cases} 1 & r_1 \le t \le r_2 \\ 0 & otherwise \end{cases}$$

We denote it by $\xi = (r_1, r_2)$ with $r_1 < r_2$.

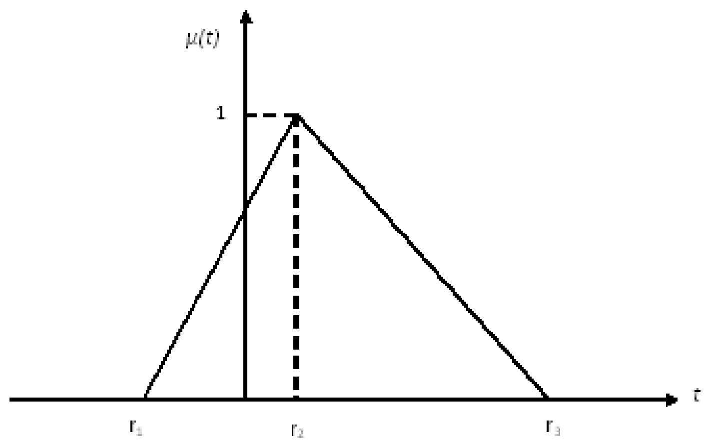

Figure 52: Triangular Membership Function

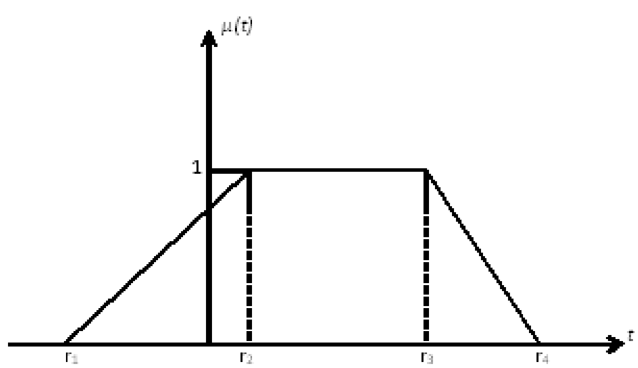

Figure 53: Trapezoidal Membership Function

Figure 54: Equipossible Membership Function

6.2.3 Expected value of fuzzy variables:

Definition 4: Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq t\}dt \; - \int_{-\infty}^{0} Cr\{\xi \leq t\}dt$$

provided that at least one of the two integrals is finite.

Example: Let $\xi = (r_1, r2, r3)$ be the triangular fuzzy variable. We know from the credibility inversion theorem the

$$Cr\{\xi \leq t\} = \begin{cases} 1 & r_3 \leq t \\ \frac{r_3-2r_2+t}{2(r_3-r_2)} & r_2 \leq t \leq r_3 \\ \frac{t-r_1}{2(r_2-r_1)} & r_1 \leq t \leq r_2 \\ 0 & otherwise \end{cases}$$

and

$$Cr\{\xi \geq t\} = \begin{cases} 0 & r_3 \leq t \\ \frac{r_3-t}{2(r_3-r_2)} & r_2 \leq t \leq r_3 \\ \frac{2r_2-r1-t}{2(r_2-r_1)} & r_1 \leq t \leq r_2 \\ 1 & otherwise \end{cases}$$

Thus if $0 \leq r_1 < r_2 < r_3$, we have $Cr\{\xi \leq t\} \equiv 0$ when $t < 0$. Then

$$\begin{aligned} E[\xi] &= \Big(\int_0^{r_1} 1dt + \int_{r_1}^{r_2} \frac{2r_2 - r_1 - t}{2(r_2 - r_1)}dt + \int_{r_2}^{r_3} \frac{r_3 - t}{2(r_3 - r_2)}dt + \int_{r_3}^{+\infty} 0dt\Big) - \int_{-\infty}^{0} 0dt \\ &= \frac{(r_1 + 2r_2 + r_3)}{4} \end{aligned}$$

If $r_1 < 0 \leq r_2$, then

$$\begin{aligned} E[\xi] &= \Big(\int_0^{r_2} \frac{2r_2 - r_1 - t}{2(r_2 - r_1)}dt + \int_{r_2}^{r_3} \frac{r_3 - t}{2(r_3 - r_2)}dt + \int_{r_3}^{+\infty} 0dt\Big) - \Big(\int_{-\infty}^{r_1} 0dt + \int_{r_1}^{0} \frac{t - r_1}{2(r_2 - r_1)}dt\Big) \\ &= \frac{(r_1 + 2r_2 + r_3)}{4} \end{aligned}$$

If $r_1 < r_2 < 0 < r_3$, then

$$\begin{aligned} E[\xi] &= \Big(\int_{r_2}^{r_3} \frac{r_3 - t}{2(r_3 - r_2)}dt + \int_{r_3}^{+\infty} 0dt\Big) - \Big(\int_{-\infty}^{r_1} 0dt + \int_{r_1}^{r_2} \frac{t - r_1}{2(r_2 - r_1)}dt \int_{r_2}^{0} \frac{r_3 - 2r_2 + t}{2(r_3 - r_2)}dt\Big) \\ &= \frac{(r_1 + 2r_2 + r_3)}{4} \end{aligned}$$

If $r_1 < r_2 < r_3 \leq 0$, then

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} 0dt - \Big(\int_{-\infty}^{r_1} 0dt + \int_{r_1}^{r_2} \frac{t - r_1}{2(r_2 - r_1)} dt + \int_{r_2}^{r_3} \frac{r_3 - 2r_2 + t}{2(r_3 - r_2)} dt + \int_{r_3}^{0} 1dt \Big) \\ &= \frac{\big(r_1 + 2r_2 + r_3\big)}{4} \end{aligned}$$

Therefore, the expected value of the triangular fuzzy variable $\xi = (r_1, r_2, r_3)$ is always $\dfrac{r_1 + 2r_2 + r_3}{4}$.

The equi-possible fuzzy variable $\xi = [r_1, r_2]$ has an expected value $\dfrac{r_1 + r_2}{2}$. The trapezoidal fuzzy variable $\xi = (r_1, r_2, r_3, r_4)$ has an expected value $\dfrac{r_1 + r_2 + r_3 + r_4}{4}$.

Let ξ be a continuous non-negative fuzzy variable with membership function μ. If μ is decreasing on $[0,\infty)$, then $Cr\{\xi \geq x\} = \frac{\mu(x)}{2}$ for any $x > 0$, and $E[\xi] = \dfrac{1}{2} \int_0^\infty \mu(x)dx$.

6.2.4 Variance of a Fuzzy Variable

The variance of a fuzzy variable provides a measure of the spread of the distribution around its expected value. A small value of variance indicates that the fuzzy variable is tightly concentrated around its expected value; and a large value of variance indicates that the fuzzy variable has a wide spread around its expected value.

Definition 5 Let ξ be a fuzzy variable with finite expected value e .Then the variance of ξ is defined by $V[\xi] = E[(\xi - e)^2]$.

Theorem Let ξ be a triangular fuzzy variable (a, b, c), then its variance

$$V[\xi] = \begin{cases} \dfrac{33\alpha^3 + 11\alpha\beta^2 + 21\alpha^2\beta - \beta^3}{384\alpha} & \alpha > \beta \\ \frac{\alpha^2}{6} & \alpha = \beta \\ \frac{33\beta^3 + 11\beta\alpha^2 + 21\beta^2\alpha - \alpha^3}{384\beta} & \beta > \alpha \end{cases}$$

where $\alpha = b - a, \ \beta = c - b$.

Proof Let $m = E[\xi]$, when $\alpha = \beta$, then $m = b$ and

$$Cr\{(\xi - m)^2 \geq r\} = \begin{cases} \frac{\alpha - \sqrt{r}}{2\alpha} & 0 \leq r \leq \alpha^2 \\ 0 & r \geq \alpha^2 \end{cases}$$

then

$$V[\xi] = E[(\xi - m)^2] = \int_0^\infty Cr\{(\xi - m)^2 \geq 2\}dr = \frac{\alpha^2}{6}$$

when $\alpha > \beta,\ m < \beta$

$$Cr\{(\xi - m)^2 \geq r\} = Cr\{(\xi - m) \geq \sqrt{r}\} \vee Cr\{(\xi - m) \leq -\sqrt{r}\}$$

(i) $0 \leq r \leq (b - m)^2$, then

$$\begin{aligned} Cr\{(\xi - m)^2 \geq r\} &= Cr\{(\xi - m) \geq \sqrt{r}\} \\ &= \frac{1}{2}\Big[1 + 1 - \frac{\sqrt{r} + m - a}{2\alpha}\Big] \\ &= 1 - \frac{\sqrt{r} + m - b + \alpha}{2\alpha} \end{aligned}$$

(ii) when $(b - m)^2 \leq r \leq r_s$, where $r_s = \dfrac{(\alpha + \beta)^2}{16}$ then

$$\begin{aligned} Cr\{(\xi - m)^2 \geq r\} &= Cr\{(\xi - m) \geq \sqrt{r}\} \\ &= \frac{1}{2}\Big[\frac{m + \sqrt{r} - b}{-\beta}\Big] \\ &= \frac{-m - \sqrt{r} + b + \beta}{2\alpha} \end{aligned}$$

(iii) when $r_s \leq r \leq (b - \alpha - m)^2$, then

$$\begin{aligned} Cr\{(\xi - m)^2 \geq r\} &= Cr\{(\xi - m) \leq -\sqrt{r}\} \\ &= \frac{1}{2}\Big[\frac{m - \sqrt{r} - a}{\alpha}\Big] \\ &= \frac{-\sqrt{r} + m - b + \alpha}{2\alpha} \end{aligned}$$

(iv) when $r \geq (b - \alpha - m)^2$, $Cr\{(\xi - m)^2 \geq r\}$ then

$$\begin{aligned} V[\xi] = E[(\xi - m)^2] &= \int_0^\infty Cr\{(\xi - m)^2 \geq r\}dr \\ &= \frac{33\alpha^3 + 11\alpha\beta^2 + 21\alpha^2\beta - \beta^3}{384\alpha} \end{aligned}$$

when $\alpha \geq \beta$. Similarly, we can prove the other parts of the theorem.

6.3 Model

We have built a credibilistic mean-variance portfolio selection model for Index Tracking here. The objective function is a trade-off between the return and the risk. As a measure of the risk, variance of a fuzzy number has been used. We have minimized the expectation and the variance of the error portfolio, which has been constructed by considering the difference between the returns of the index and the respective assets.

6.3.1 Assumptions

The following assumptions are made to build the Fuzzy Index Tracking Model:

- There are no transaction costs involved in trading.
- Short sales are not permitted.
- The returns of two different assets are assumed to be uncorrelated to each other.
- The investor is rational.

6.3.2 Notation

Suppose that the market consists of n assets. Let TP denote the tracking portfolio. Let

I $= (u, v, w)$ the return of the market index.

R_i $= (a_i, b_i, c_i)$ the return of the ith asset, $i = 1, 2, \ldots, n$.

x_i the weight of the i_{th} in the tracking portfolio.

R_p the return of the TP $R_p = \Sigma_{i=1}^{n} x_i R_i$

R_e the error portfolio representing the difference between the returns of the index and TP $R_e = I - R_p = (u - \Sigma_{i=1}^{n} x_i c_i,\ v - \Sigma_{i=1}^{n} x_i b_i,\ w - \Sigma_{i=1}^{n} x_i a_i) = (r_1, r_2, r_3)$.

μ_e the expectation of R_e given by $\dfrac{(r_1 + 2r_2 + r_3)}{4}$.

$Var(R_e)$ the variance of R_e given by $\dfrac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}$,where $\alpha = \max\ (r_2 - r_1, r_3 - r_2)$ and $\beta = \min\ (r_2 - r_1, r_3 - r_2)$.

6.3.3 Problem Formulation

The model for Index Tracking is

$$\begin{aligned} \textbf{Minimize } & \varrho r + \xi \mu_e \\ \textbf{subject to}: & \ Var(R_e) \leq r \\ & \Sigma_{i-1}^{n} x_i = 1 \\ & x_i \geq 0 \end{aligned}$$

This is a non-linear programming problem of cubic order. Here we have used risk measure as Variance of error. ρ and ξ determines the amount of trade off between the risk and return. $Var(R_e) \leq r$ reflects an average deviation level from the expected value, which is limited by the given value.

6.4 Implementation and Analysis

Fuzzification of Returns

The historical data of the stocks, including the opening prices, the highest prices, the lowest prices and the closing prices of historical trading days (from January 2015 to December 2016). The returns have been characterized by triangular possibility distributions $(a_i, b_i, c_i), i = 1, 2, \ldots, 10$, where subscript i denotes the stocks. Based on the data, we have fuzzified the returns systematically, as mentioned below.

For stock i, for a particular period, the logarithmic ratio of the lowest price of the period to the opening price on the first trading day was obtained, and this was denoted by a_i. Similarly, b_i was obtained by the logarithmic ratio of the closing price on the last trading day to the opening price of the first trading day. And, c_i is the logarithmic ratio of the highest price of the period to the opening price of the period. This method captures the entire variation of the returns over a given period, i.e., if an investor enters the market on the first day of the period and exists when the price of the asset is the lowest in the entire period, his return is denoted by a_i. If the investor continues in the market throughout the period and exists on the last day, his return would be b_i. And c_i represents the return of an investor who exits the market when the stock price is the highest. Thus there cannot be any possible return of an investor, who enters the market on the first trading day of the period, outside of this window. The entire time period has been divided into 17 time periods. Each time period consists of six months over which we have tested the model and then implemented the model over the immediate next two months.

The table showing the various values of the parameters have been provided here:

In-sample	**Out-sample**	**No. of Assets**	**Tracking Error**	**Alpha**	**Beta**	**Correlation**
Jan'15 - June'15	July'15 - Aug'15	12	0.0039	0.0003	1.0711	0.8727
Feb'15 - July'15	Aug'15 - Sep'15	11	0.0059	-0.0014	1.0493	0.8228
Mar'15 - Aug'15	Sep'15 - Oct'15	9	0.0039	-0.0068	1.0271	0.8454
Apr'15 - Sep'15	Oct'15 - Nov'15	13	0.0035	-0.0008	0.7312	0.7482
May'15 - Oct'15	Nov'15 - Dec'15	11	0.0048	-0.0009	0.519	0.5628
June'15 - Nov'15	Dec'15 - Jan'16	12	0.0037	-0.0062	0.9556	0.8187
July'15 -Dec'15	Jan'16 - Feb'16	11	0.0044	-0.0008	0.8612	0.8507
Aug'15- Jan'16	Feb'16 - Mar'16	11	0.0052	-0.0007	0.8396	0.8114
Sep'15 - Feb'16	Mar'16 - Apr'16	11	0.0048	0.001	0.6944	0.7423
Oct'15 - Mar'16	Apr'16 - May'16	11	0.0041	-0.0006	0.8121	0.7859
Nov'15 - Apr'16	May'16 - June'16	12	0.0026	-0.0005	0.7832	0.8546
Dec'15 - May'16	June'16 - July'16	8	0.0044	-0.0014	0.9624	0.6776
Jan'16 - June'16	July'16 - Aug'16	11	0.0045	0.0001	0.9064	0.6677
Feb'16 - July'16	Aug'16 - Sep'16	12	0.0029	-0.0007	0.8861	0.8306
Mar'16 - Aug'16	Sep'16 - Oct'16	11	0.0033	-0.0012	0.8406	0.7897
Apr'16 - Sep'16	Oct'16 - Nov'16	10	0.0044	-0.0012	0.6838	0.8297
May'16 - Oct'16	Nov'16 - Dec'16	9	0.0048	-0.0011	0.6973	0.8063

Table 18

6.4.1 Graphs

The out-sample tracking are shown in the following graphs.

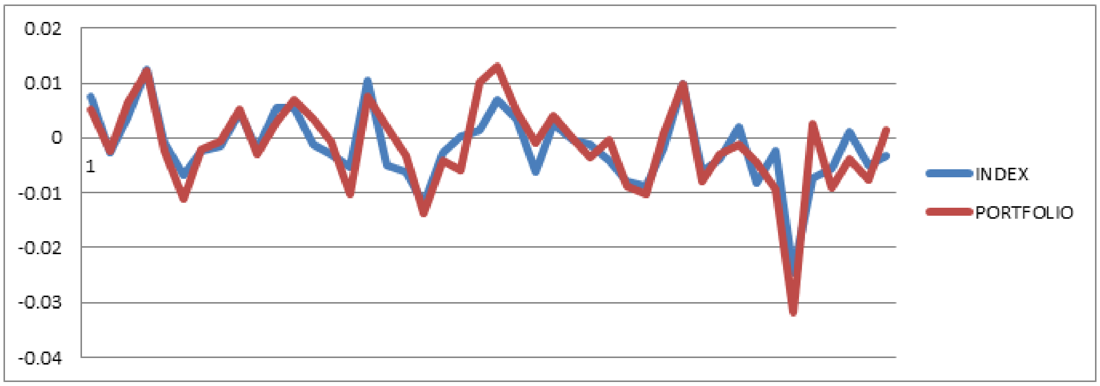

Figure 55: July 2015 - Aug 2015

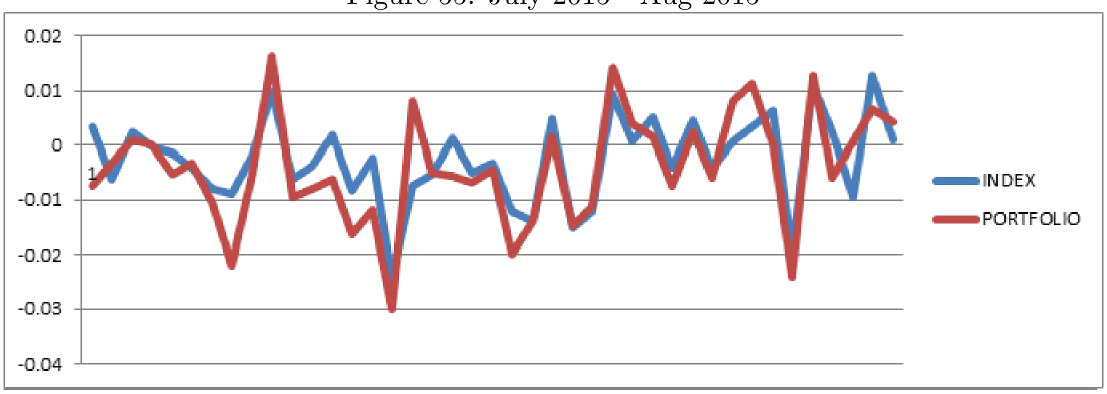

Figure 56: Aug 2015 - Sep 2015

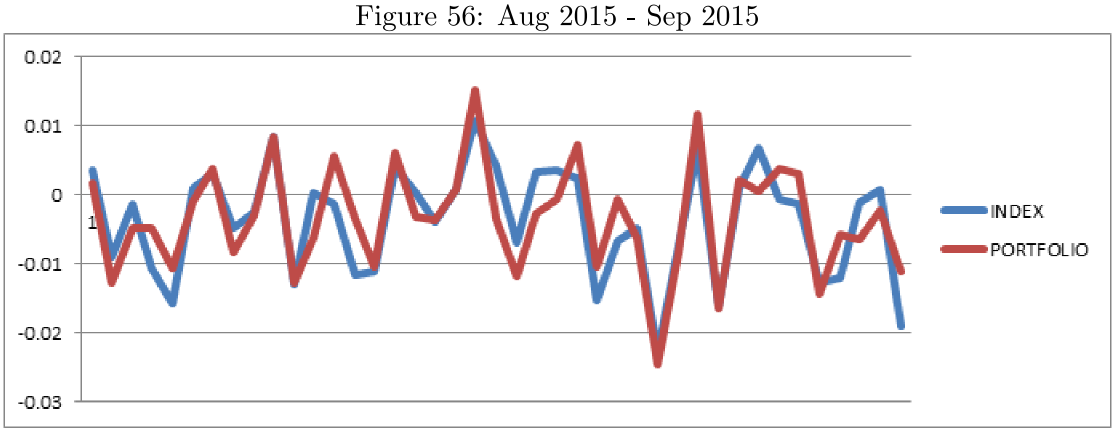

Figure 57: Jan 2016 - Feb 2016

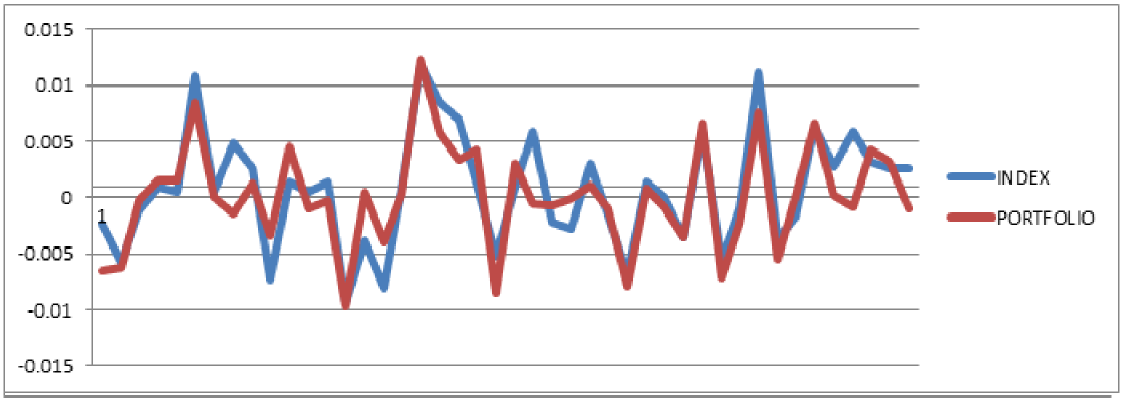

Figure 58: May 2016 - June 2016

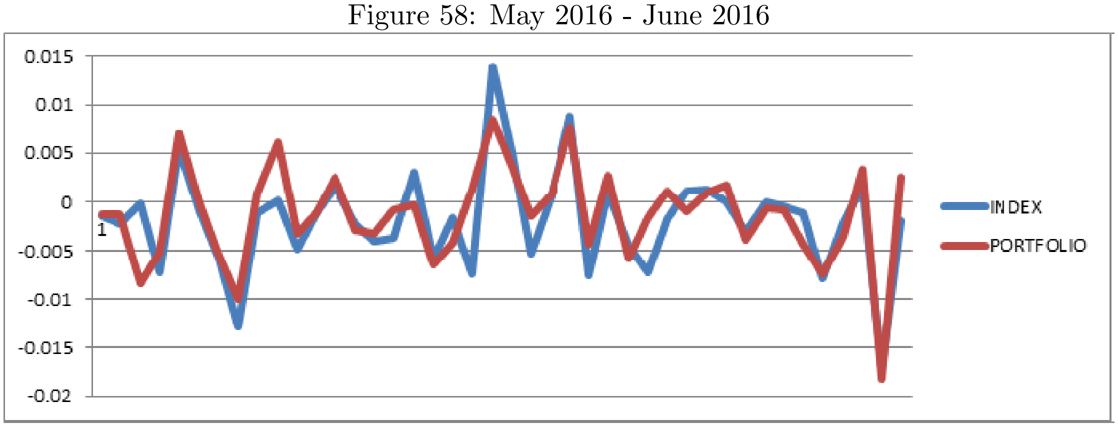

Figure 59: Aug 2016 - Sep 2016

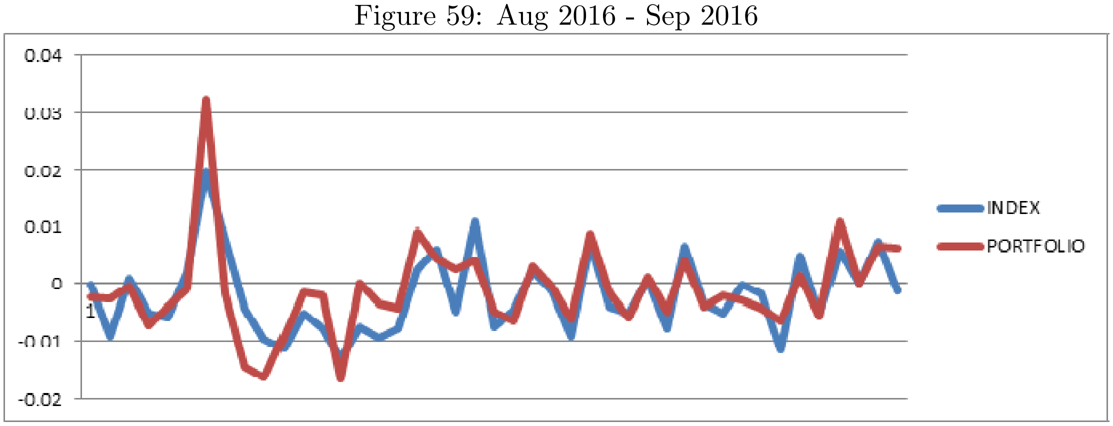

Figure 60: Nov 2016 - Dec 2016

6.5 Conclusion

Here we have constructed a fuzzy index tracking model by minimizing the mean and variance of the error portfolio.

- The number of assets in the portfolio having non-zero weights lies between 8 to 13, with 11 assets being selected 8 out of 17 times.
- We were able to push the alpha of the portfolio close to zero.
- The average tracking error was 0.0042.
- The mean beta of the portfolio was 0.8424.

This model can be modified to imcorporate the co-variance between the returns of the assets. Also, a different risk measure like entropy, CVar or semi-variance can be used instead of variance. The results we obtained were by solving the model on a Local solver in LINDO. Better results can be obtained if a global solver is used instead.

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