Drone Dynamics: Quadrotor (X Configuration)

1 State Definition

The quadrotor state vector is defined as:

$$x = [x,\ y,\ z,\ \dot{x},\ \dot{y},\ \dot{z},\ \phi,\ \theta,\ \psi,\ p,\ q,\ r]^T \tag{1}$$

2 Motor Inputs and Thrust

Each motor M_i $(i = 1, 2, 3, 4)$ produces thrust proportional to the square of its control input u_i, where:

$$M_i \in [1000, 2000], \quad \forall i = 1, 2, 3, 4 \tag{2}$$

$$u_i = \text{clip}\left(\frac{M_i - 1000}{1000},\ 0,\ 1\right) \tag{3}$$

The thrust generated by each motor is given by:

$$T_i = k_T u_i^2 \tag{4}$$

where k_T is the thrust coefficient. Here $k_T = T_{max}$.

3 Quadrotor Configuration Diagram

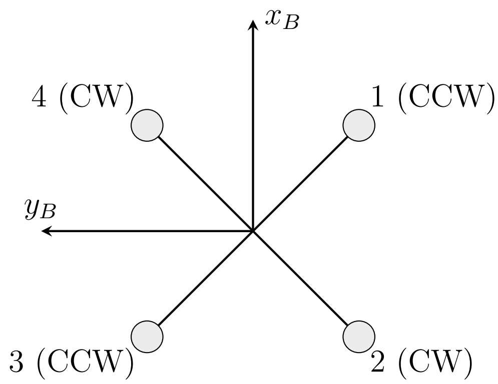

Figure 1: Quadrotor X-configuration with motor numbering and spin directions.

4 Total Thrust and Torques

Let T_i denote the thrust from each motor. The total thrust and torques about the body axes are:

(a) Total Thrust

$$T = T_1 + T_2 + T_3 + T_4 \tag{5}$$

(b) Roll Torque (about body x-axis)

$$\tau_\phi = r(-T_1 - T_2 + T_3 + T_4) \tag{6}$$

(c) Pitch Torque (about body y-axis)

$$\tau_\theta = r(-T_1 + T_2 + T_3 - T_4) \tag{7}$$

(d) Yaw Torque (about body z-axis)

$$\tau_\psi = k_q(T_1 - T_2 + T_3 - T_4) \tag{8}$$

Here, $r = \frac{L}{\sqrt{2}}$ where $L$is the arm length, and k_q is the yaw (moment) coefficient.

5 Translational Dynamics (Inertial Frame)

The translational motion is governed by Newton's second law:

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}_B + m\boldsymbol{g} \tag{9}$$

where $\boldsymbol{F}_B = [0,\, 0,\, T]^T$ is the thrust force in the body frame, and $\boldsymbol{g} = [0,\, 0,\, -g]^T$ is the gravity vector.

Transforming to the inertial frame:

$$\ddot{\boldsymbol{r}} = \frac{1}{m} R_B^N(\Theta)\boldsymbol{F}_B + \boldsymbol{g} \tag{10}$$

The rotation matrix from body to inertial frame is:

$$R_B^N(\phi, \theta, \psi) = \begin{bmatrix} c_\phi c_\psi & c_\phi s_\psi & -s_\phi \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\theta s_\psi & -c_\theta c_\psi & c_\theta c_\phi \end{bmatrix} \tag{11}$$

Thus,

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{T}{m} \begin{bmatrix} s_\phi s_\psi + c_\phi c_\psi s_\theta \\ -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ c_\phi c_\theta \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \tag{12}$$

6 Rotational Dynamics (Body Frame)

$$I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) = \boldsymbol{\tau} \tag{13}$$

where $\boldsymbol{\omega} = [p,\, q,\, r]^T$ and $I = \mathrm{diag}(I_x, I_y, I_z)$.

Expanding:

$$I_x\dot{p} = (I_y - I_z)qr + \tau_\phi \tag{14a}$$
$$I_y\dot{q} = (I_z - I_x)rp + \tau_\theta \tag{14b}$$
$$I_z\dot{r} = (I_x - I_y)pq + \tau_\psi \tag{14c}$$

7 Angular Kinematics

The Euler angle rates relate to the body angular rates via:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = T_\Theta(\Theta) \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{15}$$

where

$$T_\Theta(\Theta) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \tag{16}$$

8 State-Space Representation

Combining Eqs. (12), (14), and (15), we obtain the nonlinear state-space form:

$$\dot{x} = f(x, u) \tag{17}$$

The input vector is:

$$u = [T_1,\, T_2,\, T_3,\, T_4]^T \tag{18}$$

The control torques are expressed as:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = M_r \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \tag{19}$$

where

$$M_r = \begin{bmatrix} -1 & -1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + k_q \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} \tag{20}$$

9 Simulink Implementation

The nonlinear dynamics can be implemented in Simulink as follows:

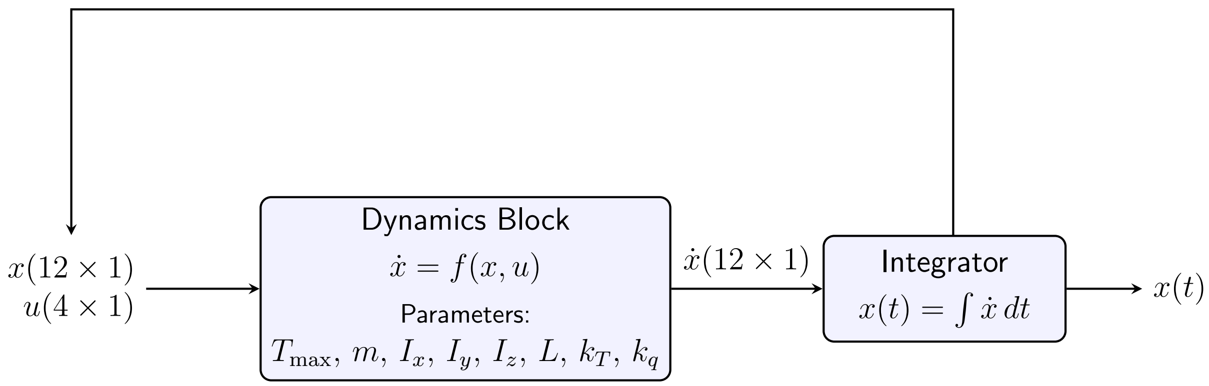

Figure 2: Simulink-style representation of the nonlinear quadrotor dynamics.