# Solution

March 12, 2023

**Python Setting**

[1]:
```
import pandas as pd
import numpy as np
import seaborn as sns
import matplotlib.pyplot as plt
import matplotlib.ticker as mtick
import statsmodels.api as sm
import scipy as sp
import itertools

import warnings
warnings.filterwarnings('ignore')

plt.style.use('seaborn-darkgrid')
MyColors = ["#0078D7", "#E74856", "#FFB900", "#10893E", "#B146C2",
            "#00B7C3", "#E3008C", "#FF8C00", "#00CC6A", "#6B69D6",
            "#0099BC", "#C30052", "#F7630C", "#00B294", "#8764B8"]
sns.set_palette(MyColors)
```

```
/usr/local/anaconda3/lib/python3.8/site-
packages/statsmodels/tsa/base/tsa_model.py:7: FutureWarning: pandas.Int64Index
is deprecated and will be removed from pandas in a future version. Use
pandas.Index with the appropriate dtype instead.
  from pandas import (to_datetime, Int64Index, DatetimeIndex, Period,
/usr/local/anaconda3/lib/python3.8/site-
packages/statsmodels/tsa/base/tsa_model.py:7: FutureWarning: pandas.Float64Index
is deprecated and will be removed from pandas in a future version. Use
pandas.Index with the appropriate dtype instead.
  from pandas import (to_datetime, Int64Index, DatetimeIndex, Period,
```

## 1 Question 1-3

Following the lecture slide 6, we implement the string model as follows.

[2]:
```
# Read discount factor data
df_init = pd.read_csv("pfilea.csv", header=None)[0]
df_init.index = np.arange(0.5, 15+0.5, 0.5)
```

```
df_init = df_init[:10]
```

[3]:
```
# Read volatility data
vol = pd.read_csv("voldat.csv", header=None)[0]
vol.index = np.arange(1, 15+0.5, 0.5)
vol[0.5] = 0
vol = vol.sort_index()[:10]
```

[4]:
```
# Read cholesky matrix
mat_chol = pd.read_csv("Homework 7 corchol.csv", header=None).values
```

[5]:
```
# Function to compute short rate based on string model
def string(df, vol, mat_chol, seed):

    # Generate dz
    np.random.seed(seed)
    dz = pd.DataFrame(mat_chol @ np.random.normal(size=[20, 20]),
                      index=np.arange(0.5, 10+0.5, 0.5),
                      columns=np.arange(0.5, 10+0.5, 0.5))

    # Repeatedly compute short rate & DF curve t=0.5 to 9.5
    # Note: index: maturity, columns: time
    B = pd.DataFrame(None,
                     index=np.arange(0.5, 10+0.5, 0.5),
                     columns=np.arange(0, 10+0.5, 0.5))
    r_save = pd.Series(index=np.arange(0, 9.5+0.5, 0.5))
    # Repeat process
    B[0] = df
    vol_t = vol.copy()
    for t in np.arange(0.5, 10+0.5, 0.5):
        r = (1/B[t-0.5][t]-1)*2
        B[t] = B[t-0.5]+r*B[t-0.5]/2 + vol_t*B[t-0.5]*dz[t]
        vol_t = vol_t.shift()
        r_save[t-0.5] = r

    return r_save
```

[6]:
```
# Monte Carlo simulation: String model
r_sim = pd.DataFrame({n: string(df_init, vol, mat_chol, n)
                      for n in range(0, 10000)})
```

[7]:
```
# Compute discount factor path
df_sim = (1/(1+r_sim/2)).cumprod()
df_sim.index = np.arange(0.5, 10+0.5, 0.5)
```

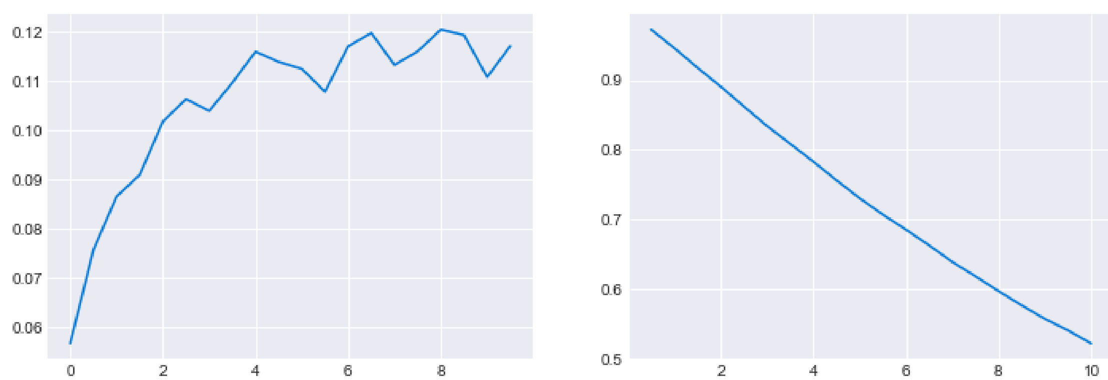
[8]:
```
r_sim.T[0:6].T
```

[8]:

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0.0 | 0.056605 | 0.056605 | 0.056605 | 0.056605 | 0.056605 | 0.056605 |
| 0.5 | -0.159872 | -0.183652 | 0.182708 | -0.205862 | 0.088557 | 0.012449 |
| 1.0 | 0.017215 | 0.155968 | 0.086947 | -0.029021 | 0.010023 | 0.193878 |
| 1.5 | -0.031646 | 0.094839 | 0.717376 | -0.140293 | 0.157849 | -0.274650 |
| 2.0 | -0.277459 | 0.194031 | -0.115602 | 0.534981 | 0.310749 | -0.084222 |
| 2.5 | -0.400526 | -0.164426 | 0.604425 | 0.184900 | -0.043334 | -0.128193 |
| 3.0 | 0.117645 | 0.576069 | 0.283137 | 0.372412 | 0.245961 | -0.281151 |
| 3.5 | -0.177417 | -0.235587 | -0.009296 | 0.101913 | 0.386836 | 0.029237 |
| 4.0 | -0.135820 | 0.167433 | 0.462785 | 0.434517 | 0.304456 | 0.176883 |
| 4.5 | 0.568107 | -0.083666 | 0.021297 | 0.402359 | 0.050822 | 0.529535 |
| 5.0 | -0.011210 | 0.342530 | 0.180708 | -0.023240 | 0.309619 | 0.379569 |
| 5.5 | 0.248220 | -0.394006 | -0.223518 | 0.364246 | -0.264376 | 0.244260 |
| 6.0 | -0.030187 | 1.038802 | -0.211351 | -0.669220 | -0.011714 | 0.641319 |
| 6.5 | 0.161459 | -0.061549 | 0.778428 | -0.173176 | -0.234246 | -0.016487 |
| 7.0 | -0.165695 | 0.037727 | 0.461555 | -0.545334 | -0.321733 | 0.548129 |
| 7.5 | 0.369705 | -0.211262 | 0.452205 | 0.012352 | -0.194104 | -0.088079 |
| 8.0 | 0.199593 | 0.302625 | -0.273338 | 0.301672 | -0.186440 | 0.335690 |
| 8.5 | 0.152603 | 0.039640 | -0.212062 | 0.259461 | 0.037316 | -0.213057 |
| 9.0 | 0.193286 | 0.329925 | -0.305849 | 0.385937 | -0.290004 | -0.426362 |
| 9.5 | 0.682637 | 0.267895 | -0.528038 | 0.499755 | 0.131577 | 0.108385 |

Also, the average short rate $E[r_T]$ and the discount factor (the price of Zero-coupon Bond) $E[B(0,T)]$ is illustrated in the below figures.

[9]:
```
# Plot average short rate curve and DF curve
fig, ax = plt.subplots(1, 2, figsize=(6*2, 4))
r_sim.T.mean().plot(ax=ax[0])
df_sim.T.mean().plot(ax=ax[1]);
```



## 2 Question 4

Based on the simulated path of discount factor, we compute the forward par rates for $M (M = 1, 2, \ldots, 5)$ year semiannual coupon bonds at $t = 5$.

$$c_M = 2\left[\frac{B(0,5) - B(0,5+M)}{\sum_{i=1}^{2M} B(0,5+i/2)}\right] \tag{1}$$

[10]:
```
# Compute forward par rate based on the simulated path
fwd_par_sim = pd.DataFrame({
    m: 2*(df_sim.loc[5,:]-df_sim.loc[5+m,:])/df_sim[(5+0.5):(5+m)].sum()
    for m in range(1, 5+1)
})
```

Taking the average, we can obtain the expected forward par rate.

[11]:
```
# Compute the expected value of forward par rate
fwd_par = fwd_par_sim.mean()
fwd_par
```

[11]:
```
1    0.101279
2    0.098716
3    0.096025
4    0.094867
5    0.092854
dtype: float64
```

[12]:
```
# Reference: Compute the forward par rate based on the initial curve
fwd_par_init = pd.Series({
    m: 2*(df_init[5]-df_init[5+m])/df_init[(5+0.5):(5+m)].sum()
    for m in range(1, 5+1)
})
fwd_par_init
```

[12]:
```
1    0.068886
2    0.069452
3    0.069903
4    0.070248
5    0.070489
dtype: float64
```

## 3 Question 5

For simplisity, we assume that the CMS rates are equal to the forward swap rates (forward par rates) and caplets are semiannual. Based on the simulated path of discount factor, we compute the par rates for $M(M = 2, 3, 4, 5, 7, 10)$ year semiannual coupon bonds.

$$c_M = 2\left[\frac{1 - DF(M)}{\sum_{i=1}^{2M} DF(i/2)}\right] \tag{2}$$

[13]:
```
# Compute par rate based on the simulated path
par_sim = pd.DataFrame({
    m: 2*(1-df_sim.loc[m,:])/df_sim[0.5:m].sum()
    for m in [2,3,5,10]
})
```

[14]:
```
par = par_sim.mean()
par
```

[14]:
```
2     0.068331
3     0.074518
5     0.080044
10    0.082530
dtype: float64
```

We implement the pricing function for cap as follows.

[15]:
```
def calc_cap(K, r_sim, df_sim):
    cf_sim = (r_sim-K)*0.5
    cf_sim = cf_sim.mask(cf_sim<0,0)
    cf_sim.index = df_sim.index
    pv = (cf_sim*df_sim).sum().mean()
    return pv
```

The price of each cap is computed as follows.

[16]:
```
pv_cap = par.apply(lambda x: calc_cap(x, r_sim, df_sim))
pv_cap
```

[16]:
```
2     0.726934
3     0.707610
5     0.690687
10    0.683179
dtype: float64
```

# 4 Question 6

We implement the pricing function for $m$ year resettable cap as follows.

[17]:
```
def calc_reset_cap(m, r_sim, df_sim):
    cf = 0.5*r_sim.diff()
    cf.index = df_sim.index
    pv = (cf.mask(cf<0,0)*df_sim).loc[:m].sum().mean()
```

```
    return pv
```

[18]:
```
calc_reset_cap(5, r_sim, df_sim)
```

[18]: 0.4483009965197659

[19]:
```
m = 5
cms_sim = pd.DataFrame({
    t: 2*(df_sim.loc[t,:]-df_sim.loc[t+m,:])/df_sim[(t+0.5):(t+m)].sum()
    for t in np.arange(0.5, 5+0.5, 0.5)}).T
```

[20]:
```
cms_sim.loc[:,0:4]
```

[20]:

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0.5 | -0.055701 | 0.067464 | 0.232266 | 0.123540 | 0.162603 |
| 1.0 | -0.020736 | 0.052849 | 0.206981 | 0.184982 | 0.141428 |
| 1.5 | -0.024471 | 0.109123 | 0.191092 | 0.144382 | 0.149492 |
| 2.0 | -0.008744 | 0.097955 | 0.154721 | 0.164366 | 0.111103 |
| 2.5 | 0.001395 | 0.081903 | 0.221743 | 0.008159 | 0.033618 |
| 3.0 | 0.076974 | 0.088572 | 0.178841 | -0.014898 | 0.014462 |
| 3.5 | 0.081370 | 0.053464 | 0.132715 | -0.019683 | -0.041748 |
| 4.0 | 0.121047 | 0.089100 | 0.124586 | -0.002599 | -0.072818 |
| 4.5 | 0.166018 | 0.097381 | 0.040279 | -0.005713 | -0.134135 |
| 5.0 | 0.143704 | 0.134150 | -0.034018 | 0.001070 | -0.113246 |

[21]:
```
cf_sim = cms_sim-0.05
cf_sim = 0.5*cf_sim.mask(cf_sim<0,0)
pv = (cf_sim*df_sim[:5]).sum().mean()
pv
```

[21]: 0.28351136587184683

[ ]:
```

```