6) En el caso 3D tenemos que si $\{e_i\}$ define un sistema de coordenadas (dextrógiro) no necesariamente ortogonal, entonces demuestre que:

a) $e^i = \dfrac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}$, $i, j, k = 1, 2, 3$ y sus permutaciones cíclicas.

Por definición tenemos $e_i \cdot e^j = \delta_i^j \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

e^i es perpendicular a e_j y e_k : $e^i = \alpha(e_j \times e_k)$
$e^i \cdot e_i = 1$

$$(e_i) \cdot e^i = \alpha(e_j \times e_k)(e_i)$$

$$1 = \alpha e_i (e_j \times e_k)_i$$

$$\frac{1 (e_j \times e_k)_i}{e_i \cdot (e_j \times e_k)_i} = \alpha(e_j \times e_k)$$

$$\boxed{\frac{(e_j \times e_k)}{e_i \cdot (e_j \times e_k)_i} = e^i} \quad i, j, k = 1, 2, 3.$$

(b) Si los volumenes $V = e_1 \cdot (e_2 \times e_3)$ y $\tilde{V} = e^1 \cdot (e^2 \times e^3)$ entonces $V\tilde{V} = 1$

$$(e_1 \cdot (e_2 \times e_3)) \cdot (e^1 \cdot (e^2 \times e^3)) = 1$$

$$(e_1 \cdot (\tfrac{1}{\alpha}) e^1) \cdot (e^1 \cdot (\tfrac{1}{\alpha}) e_1) = 1 \qquad \alpha = 1$$

$$(e_1 \cdot e^1) \cdot (e^1 \cdot e_1) = 1$$

$$1 \cdot 1 = 1$$

(c) Que vector satisface $a \cdot e^i = 1$? Demuestre que a es único.

$$a \cdot e^i = 1$$

$$a^j (e_j \cdot e^i) = 1$$

$$a^j \delta_j^i = 1 \qquad \delta_j^i \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$a^i = 1$$

$$a^j e_j \cdot e^i = 1$$

$$i = j, \; a^j = 1$$

$$e_i \cdot e^i = 1$$

(d) Encuentre el producto vectorial de dos vectores a y b que estan representados en un sistema de coordenadas oblicuo: Dada la base

$w_1 = 4\hat{\imath} + 2\hat{\jmath} + \hat{k}$, $w_2 = 3\hat{\imath} + 3\hat{\jmath}$, $w_3 = 2\hat{k}$

Entonces encuentre:

I. Las bases recíprocas $\{e^i\}$

$w_1 \cdot w_2 = 18$, $w_1 \cdot w_3 = 2$, $w_2 \cdot w_3 = 0$

No es base ortogonal

$$\boxed{w^i = \frac{w_j \times w_k}{w_i \cdot (w_j \times w_k)}} \qquad i, j, k = 1, 2, 3.$$

$$w^1 = \frac{w_2 \times w_3}{w_1 \cdot (w_2 \times w_3)} \Rightarrow w^1 = \frac{(3,3,0) \times (0,0,2)}{(4,2,1) \cdot ((3,3,0) \times (0,0,2))}$$

$$w^1 = \frac{(6,6,0)}{(4,2,1) \cdot (6,6,0)}$$

$$w^1 = \frac{1}{36}(6,6,0) \Rightarrow \boxed{w_1 = \left(\tfrac{1}{6}, \tfrac{1}{6}, 0\right)}$$

$$w^2 = \frac{w_3 \times w_1}{w_2 (w_3 \times w_1)} \Rightarrow w^2 = \frac{(0,0,2) \times (4,2,1)}{(3,3,0) \cdot ((0,0,2) \times (4,2,1))}$$

$$w^2 = \frac{(-4,8,0)}{(3,3,0) \cdot (-4,8,0)}$$

$$w^2 = \frac{1}{12} (-4,8,0) \Rightarrow \boxed{w^2 = \left(-\frac{1}{3}, \frac{2}{3}, 0\right)}$$

$$w^3 = \frac{w_1 \times w_2}{w_3 \cdot (w_1 \times w_2)} \Rightarrow w^3 = \frac{(4,2,1) \times (3,3,0)}{(0,0,2) \cdot ((4,2,1) \times (3,3,0))}$$

$$w^3 = \frac{(-3,3,6)}{(0,0,2) \cdot (-3,3,6)}$$

$$w^3 = \frac{1}{12} (-3,3,6 \Rightarrow \boxed{w^3 = \left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right)}$$

$$W_1 = (4,2,1) \text{ — } W^1 = \left(\frac{1}{6}, \frac{1}{6}, 0\right)$$

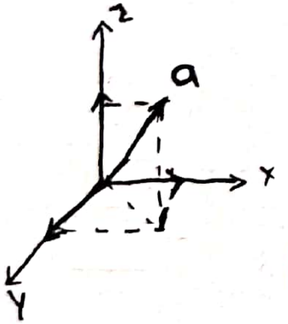
$$W_2 = (3,3,0) \text{ — } W^2 = \left(-\frac{1}{3}, \frac{2}{3}, 0\right)$$

$$W_3 = (0,0,2) \text{ — } W^3 = \left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right)$$

$\{e_i\}$ covariante $\qquad\qquad$ $\{e^i\}$ contravariante

II Componentes covariantes y contravariantes de

$$\mathbf{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$



$\mathbf{a} = a^i |w_i\rangle \rightarrow$ comp. covariante

$\mathbf{a} = a_i \langle w^i| \rightarrow$ comp. contravariante

$$\boxed{\text{Proy}_{w_i}\mathbf{a} = \left(\frac{\mathbf{a}\cdot w_i}{|w_i|^2}\right) w_i}$$

$$\text{Proy}_{w_1}\mathbf{a} = \left(\frac{(1,2,3)\cdot(4,2,1)}{|(4,2,1)|^2}\right)(4,2,1)$$

$$= \left(\frac{4+4+3}{\cancel{(\sqrt{4^2+2^2+1^2})^2}}\right)(4,2,1)$$

$$= \boxed{\frac{11}{21}(4,2,1)}$$

$$\text{Proy}_{w_2}\mathbf{a} = \left(\frac{(1,2,3)\cdot(3,3,0)}{|(3,3,0)|^2}\right)(3,3,0)$$

$$= \left(\frac{3+6}{\cancel{(\sqrt{3^2+3^2})^2}}\right)(3,3,0)$$

$$= \frac{9}{18}(3,3,0) = \boxed{\frac{1}{2}(3,3,0)}$$

$$\text{Proy}_{w_3}\mathbf{a} = \left(\frac{(1,2,3)\cdot(0,0,2)}{|(0,0,2)|^2}\right)(0,0,2)$$

$$= \left(\frac{\cancel{6}^{3}}{\cancel{4}_{2}}\right)(0,0,2)$$

$$= \boxed{\frac{3}{2}(0,0,2)}$$

$$\operatorname{Proy}_{w^1} a = \left(\frac{a \cdot w^1}{|w^1|^2}\right) w^1$$

$$= \left(\frac{(1,2,3)\cdot\left(\frac{1}{6}, \frac{1}{6}, 0\right)}{\left|\left(\frac{1}{6}, \frac{1}{6}, 0\right)\right|^2}\right)\left(\frac{1}{6}, \frac{1}{6}, 0\right)$$

$$= \left(\frac{\frac{1}{6} + \frac{2}{6} + 0}{\left(\cancel{\sqrt{}}\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2\right)^{\cancel{2}}}\right)\left(\frac{1}{6}, \frac{1}{6}, 0\right)$$

$$= \frac{\left(\frac{3}{6}\right)}{\frac{2}{36}}\left(\frac{1}{6}, \frac{1}{6}, 0\right) = \boxed{9\left(\frac{1}{6}, \frac{1}{6}, 0\right)}$$

$$\operatorname{Proy}_{w^2} a = \left(\frac{(1,2,3)\cdot\left(-\frac{1}{3}, \frac{2}{3}, 0\right)}{\left|\left(-\frac{1}{3}, \frac{2}{3}, 0\right)\right|^2}\right)\left(-\frac{1}{3}, \frac{2}{3}, 0\right)$$

$$= \frac{-\frac{1}{3} + \frac{4}{3}}{\left(\cancel{\sqrt{}}\left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right)^{\cancel{2}}}\left(-\frac{1}{3}, \frac{2}{3}, 0\right)$$

$$= \frac{\cancel{\frac{3}{3}}\,1}{\frac{5}{9}}\left(-\frac{1}{3}, \frac{2}{3}, 0\right) = \boxed{\frac{9}{5}\left(-\frac{1}{3}, \frac{2}{3}, 0\right)}$$

$$\operatorname{Proy}_{w^3} a = \left(\frac{(1,2,3)\cdot\left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right)}{\left|\left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right)\right|^2}\right)\left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right)$$

$$= \frac{-\frac{3}{12} + \frac{6}{12} + \frac{3}{2}}{\left(\cancel{\sqrt{}}\left(-\frac{3}{12}\right)^2 + \left(\frac{3}{12}\right)^2 + \left(\frac{1}{2}\right)^2\right)^{\cancel{2}}}\left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right)$$

$$= \frac{\frac{7}{4}}{\frac{3}{8}}\left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right) = \boxed{\frac{14}{3}\left(-\frac{3}{12}, \frac{3}{12}, \frac{1}{2}\right)}$$

⑦ Encuentre la base dual asociada a la base de Pauli y dado un vector genérico en el espacio de la matrices hermíticas 2x2 con la definición de producto interno definida en la sección 2.2.4, encuentre también sú 1-forma asociada.

Matrices de Pauli:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Para hallar su base dual usamos la definición de $w_i \cdot w^i = 1$

$$\sigma_1 \cdot \sigma^1 = 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_2 \cdot \sigma^2 = 1$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_3 \cdot \sigma^3 = 1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_0 \cdot \sigma^0 = 1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$