Determinants

A $n \times n$ matrix $\rightsquigarrow$ $\det A$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$ad - bc$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

$$\begin{aligned} & a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) \\ & - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) \\ & + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$\downarrow$

Use induction

<u>Thm:</u> A is invertible $\iff$ $\det(A) \neq 0$

<u>Pf:</u> Excercise!

Set Theory

"Natural definition"

A set is a well defined collection of elements.

For example

$\mathbb{N}$: The set of natural numbers

$\mathbb{Q}$: ________ rational ________

$\mathbb{R}$: ________ real ________

$\mathbb{C}$: ________ complex ________

Set operations : $A \cup B$, $A \cap B$, $A \setminus B$

We do land into a problem here.

Russell's Paradox (1901)

$$\mathcal{C} = \left\{ S \;\middle|\; \begin{array}{l} S \text{ is a set such that} \\ S \notin S \end{array} \right\}$$

for e.g $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \in \mathcal{C}$

Does $\mathcal{C} \in \mathcal{C}$?

If $\mathcal{C} \notin \mathcal{C}$, then by defn of $\mathcal{C}$
$\mathcal{C} \in \mathcal{C}$.

Logical inconsistency

So to fix Russell's paradox, we need more axioms in set theory.

Zermelo - Fraenkel Set Theory

0. Empty set axiom $\phi = \{ \}$ is a set

1. Axiom of extensionality

$A = B$ if $\forall\, x \in A \Rightarrow x \in B$ and
$\forall\, x \in B \Rightarrow x \in A$

2. Power set axiom

if A is a set, $\mathcal{P}(A)$ (Power set of A) is a set

3. Pairing axiom If A, B are sets then $\{A, B\}$ is a set

$0 := \phi$ $\quad$ $2 := \{\{\phi\}, \{\phi\}\}$

$1 := \{\phi\}$ $\quad$ $= \{\{\phi\}\}$

$3 := \{\{\{\phi\}\}\}$

(Zermelo's construction of natural numbers)

4. <u>Union Axiom</u>

If A and B are sets, $A \cup B$ is a set

$\{0, 1, \ldots n\}$ is a set.

5. <u>Axiom</u> of <u>infinity</u>

Call $\{x\}$ as $x+1$

$\exists \mathbb{N}$ s.t $0 \in \mathbb{N}$ and $\forall x \in \mathbb{N}$
$x+1 \in \mathbb{N}$

$\{0, 1, 2, 3, 4, \ldots\}$ Zermelo construction

6. <u>Axiom</u> of <u>restricted</u> <u>comprehension</u>

If A is a set then you can construct a subset using any rule you like.

Consider $S = \{\mathbb{N}, 0, 1, 2, 3, \ldots\}$

$$T = \{x \in S \mid x \notin x\} = S$$

Qun: Is $T \in T$? If yes, then $T \notin T$.

So, under this axiom we cannot think of S as a set.

(There is no valid universe containing all the elements of S.)

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7. Axiom of regularity

Every non-empty set S contains a member S' such that S & S' are disjoint sets.

Example: $S = \{1, 2, 3, 4, 5\}$

Think of 1 as a set $1 = \{0\}$

$$1 \cap S = \emptyset$$

8. Axiom schema of replacement
(Skipping for now)

9. <u>Axiom</u> of <u>choice</u>

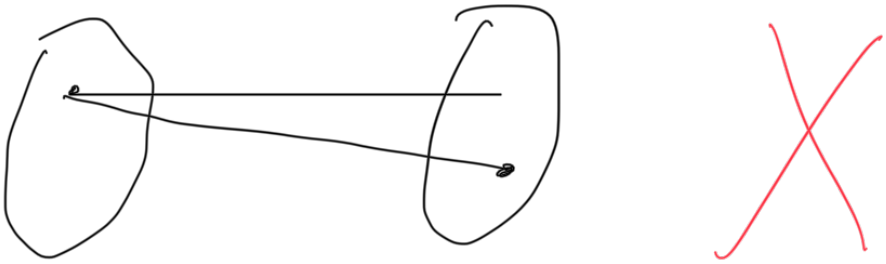
We will restate it as Zorn's lemma.
More useful version!

Functions

<u>Defn:</u> Let A, B be sets. A function f from A to B denoted as

$$f : A \to B \text{ is a set}$$

$S \subseteq A \times B$ in which every $a \in A$ appears as the first component of exactly one ordered pair.



Key words: domain, codomain, range (image, injective / one-one, surjective (onto, bijective, composition

Permutations of a finite Set

Defn: A permutation P of a finite set $\{1, 2, \ldots, n\}$ is a bijective map on $\{1, 2, \ldots, n\}$.

We can write every permutation in cycle notation.

Example: 1) $P : \{1, 2, 3, 4\} \longrightarrow \{1, 2, 3, 4\}$

$1 \mapsto 2 \qquad 3 \mapsto 4$

$2 \mapsto 1 \qquad 4 \mapsto 3$

$P = (12)(34)$

2) $P : \{1, 2, 3, 4\} \longrightarrow \{1, 2, 3, 4\}$

$1 \mapsto 2 \qquad 4 \mapsto 4$

$2 \mapsto 3$

$3 \mapsto 1$

$P = (123)$

(Missing points are fixed points.)

→ We will call set of all permutations of $\{1, 2, \ldots, n\}$ by S_n.

→ It has a natural binary operation associated to it, Composition.

(If f, g are bijective, then $f \circ g$ is bijective.)

Qun: Is the converse true?

→ Composition in cycle notation

$$P_1 = (1\,2\,3)$$

$$P_2 = (1\,3\,2)$$

$$P_1 P_2 = (1\,2\,3)(1\,3\,2) = id = P_2 P_1$$

$$P_1 = (1\,2) \qquad P_2 = (1\,2\,3)$$

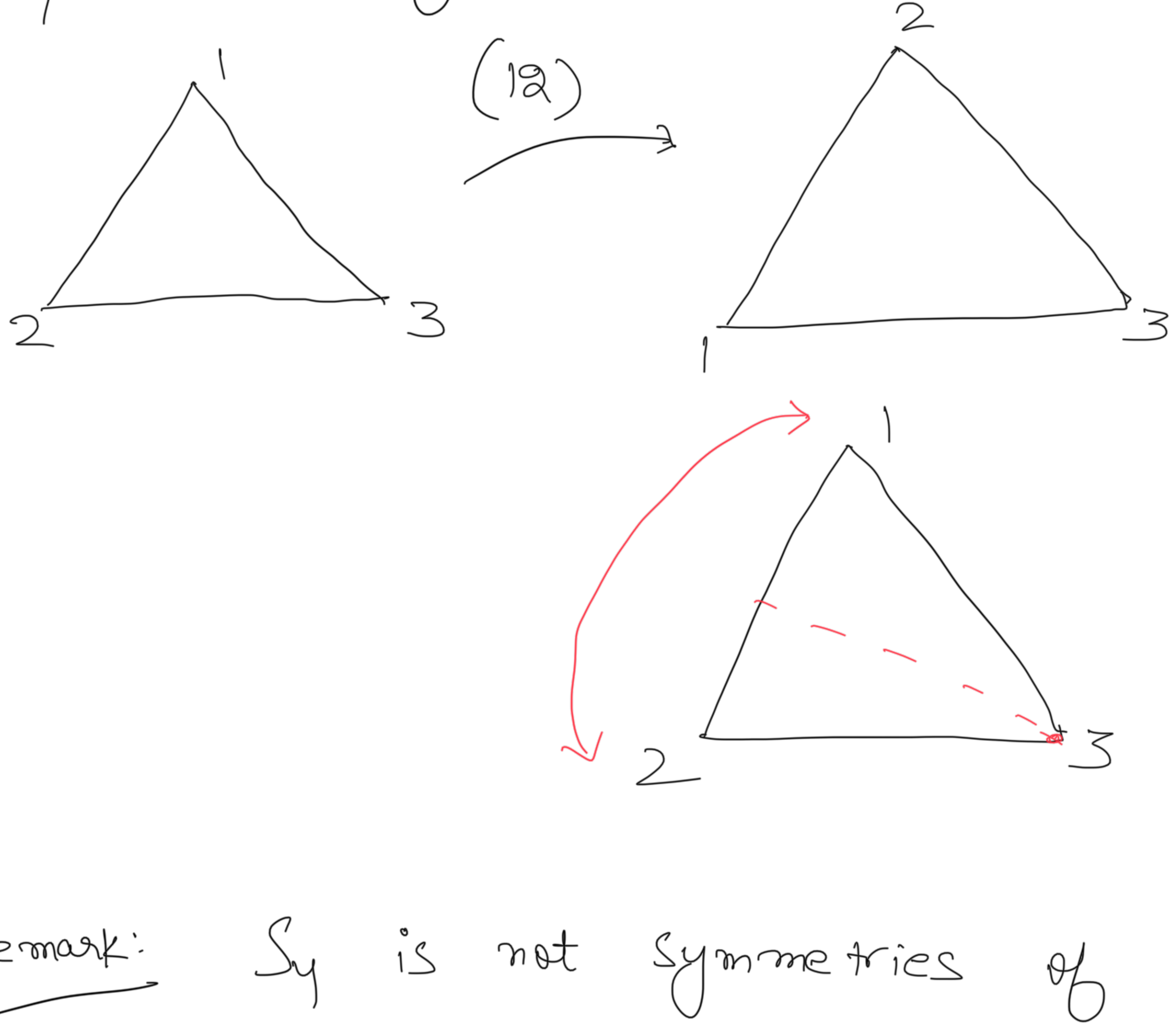
$$P_1 P_2 = (1\,2)(1\,2\,3) = (2\,3) \neq$$

$$P_2 P_1 = (1\,2\,3)(1\,2) = (1\,3)$$

Revisit S_3

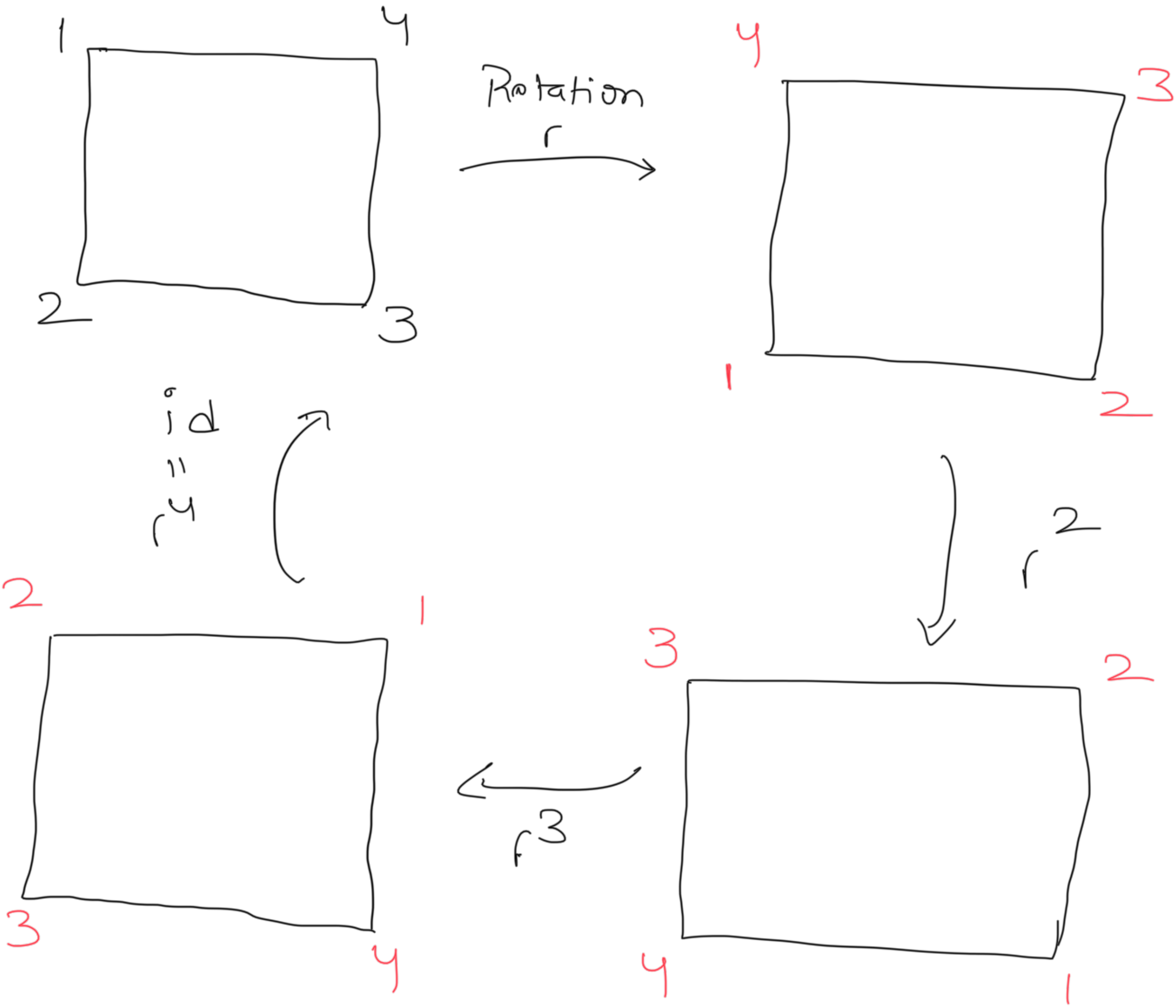
$$S_3 = \{ id, (12), (13), (23), (123), (132) \}$$

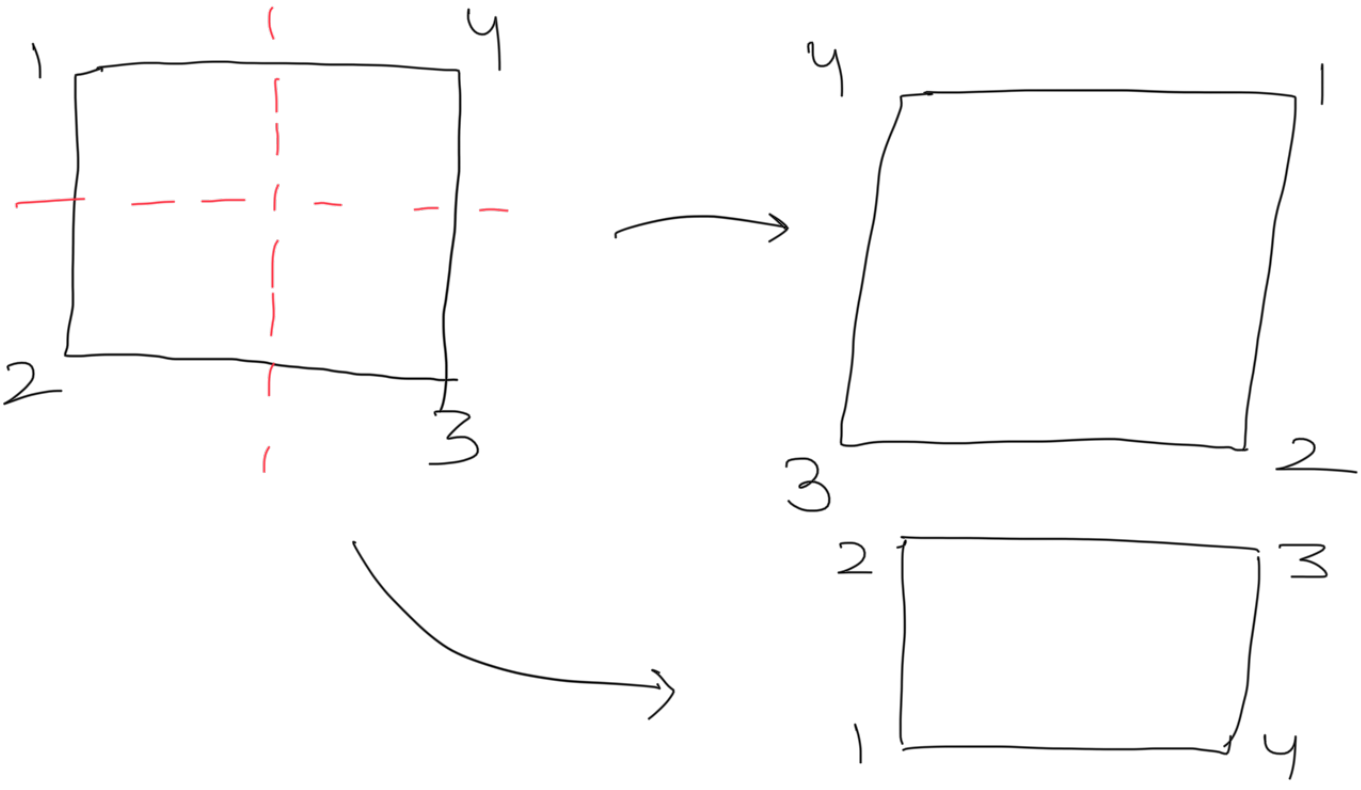
Symmetries of triangle

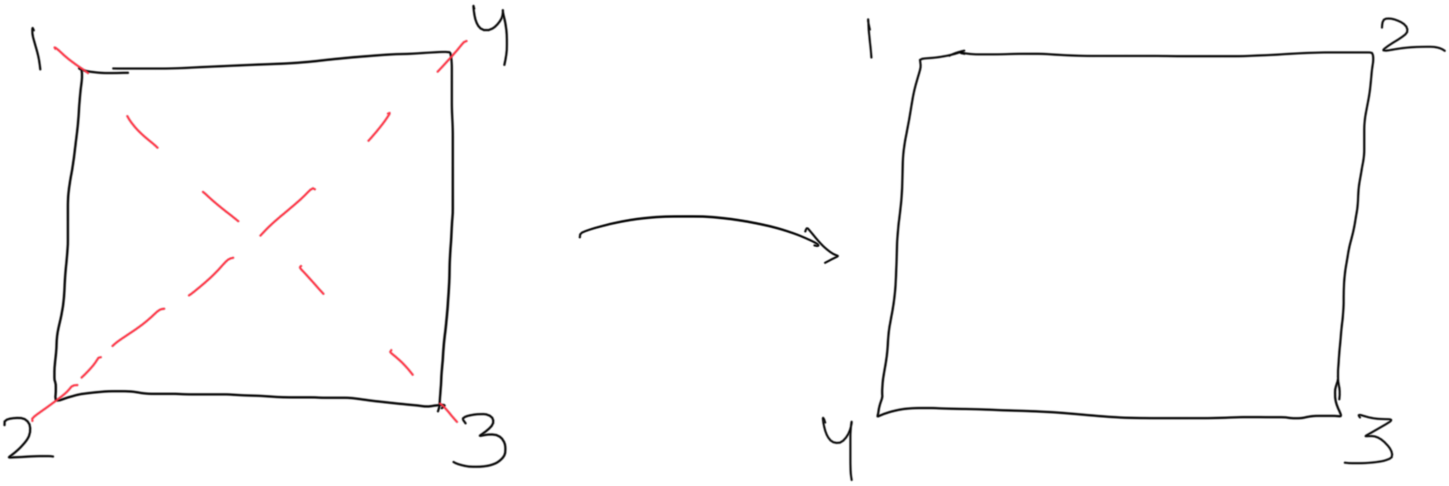


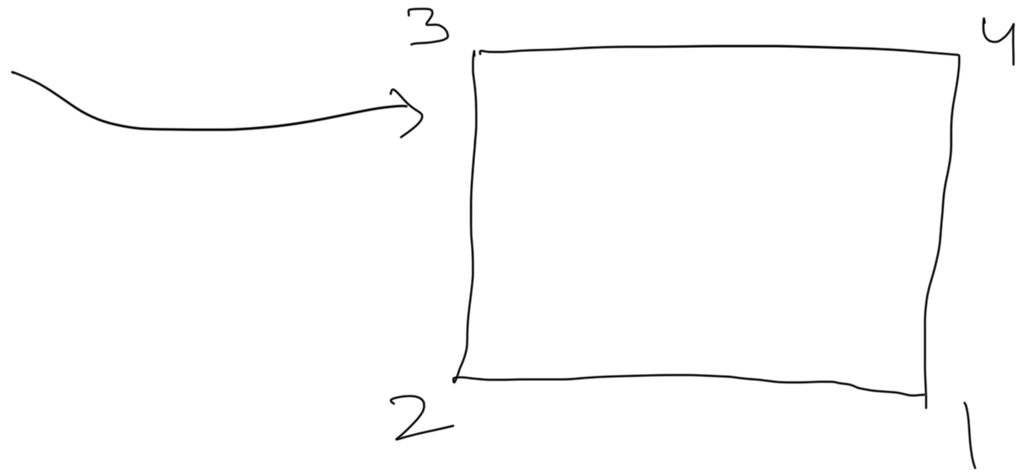
Remark: S_4 is not symmetries of square, instead S_4 is symmetry group of tetrahedron.

Symmetries of square









$$D_8 = \{ id, (1432), (13)(24), (1234), (14)(23), (12)(34), (24), (13) \}$$

For a regular n-gon, we denote its symmetry group by D_{2n}.

A cycle of length two is called transposition.

Thm: Every cycle can be written as product of transpositions.

Pf: Via induction on length of cycle, n.

Base Case: If $n=1$ or 2, nothing to show.

Inductive step: Assume this is true for all cycles of length $n-1$. Suppose P is a cycle of length n.

$$P = (a_1\, a_2\, a_3 \, \cdots \, a_n)$$

$$P = \underbrace{(a_1\, a_2 \cdots a_{n-1})}_{\text{Apply induction}}(a_{n-1}\, a_n)$$

Hence, proved.

Qn: How many transpositions do we need to write a cycle?

<u>Corollary</u>: Every permutation can be written as product of transpositions.

Signature of a permutation

If P is a cycle of length n,

$$\varepsilon(P) = (-1)^{n-1}$$

If $P = P_1 P_2 \dots P_K$ $\quad P_i$ cycle

$$\varepsilon(P) = \varepsilon(P_1)\,\varepsilon(P_2)\dots\varepsilon(P_K)$$

$$\varepsilon : S_n \longrightarrow \{\pm 1\}$$

<u>Qun</u>: How is $\varepsilon(\sigma\tau)$ related to $\varepsilon(\sigma)$ and $\varepsilon(\tau)$?

$$\{\sigma \in S_n \mid \varepsilon(\sigma) = 1\} := A_n$$

S_n Symmetric group on n-letters

A_n alternating group on —

$A_3 = \{id, (123), (132)\}$

A_4 = Set of all orientation preserving symmetries

Towards a definition of groups

Binary operations

Defn: A <u>binary operation</u> on a set S is a function

$$S \times S \xrightarrow{*} S$$
$$(a,b) \mapsto a*b$$

We will denote $(S,*)$ to denote the set S along with binary operation $*$.

Examples: $(\mathbb{N}, \max)$ $\quad \max(a,b)$

$(\mathbb{N}, \cdot)$, $(\mathbb{N}, +)$

$(\mathbb{R}, +)$, $(S_n, \cdot)$

Non-examples: $(\mathbb{N}, -)$

$$(\mathbb{R}-\{0\}, +)$$

$$(M_{m\times n}(\mathbb{R}), \cdot)$$

Restriction of binary operation:

$$(\mathbb{Q},+) \subseteq (\mathbb{R},+) \subseteq (\mathbb{C},+)$$

$$GL_n(\mathbb{R}) \subseteq (M_{n\times n}(\mathbb{R}), \cdot)$$

$$\cup\!|$$

$$SL_n(\mathbb{R})$$

$$(A_n, \cdot) \subseteq (S_n, \cdot)$$