MATH 370 ALGEBRA, SPRING 2024, HOMEWORK 7

Problem 1 Let R be a set with two laws of composition satisfying all ring axioms except the commutative law for addition. Use the distributive law to prove that the commutative law for addition holds, so that R is a ring.

Problem 2 The derivative of a polynomial f with coefficients in a field F is defined as follows, $(a_n x^n + \ldots a_1 x + a_0)' = n a_n x^{n-1} + \ldots + 1 a_1$. The integer coefficients are interpreted in F using the unique homomorphism $\mathbb{Z} \to F$. Let α be an element of F. Prove tht α is a multiple root of a polynomial f if and only if it is a common root of f and of its derivative f'.

Problem 3 The sum of two ideals I, J denoted by $I + J$ is $\{i + j \mid i \in I, j \in J\}$. The product of two ideals IJ is $\{i_1 j_1 + \ldots + i_n j_n \mid i_k \in I, j_l \in J\}$. Let I and J be two ideals of a commutative ring R such that $I + J = R$. Show that $IJ = I \cap J$. Prove the Chinese Remainder Theorem, i.e., for any two elements $a, b \in R$ there is an element $x \in R$ such that $x \equiv a(mod\ I)$ and $x \equiv b(mod\ J)$. The notation $x \equiv a(mod\ I)$ means that $x - a \in I$.

Problem 4 Find a generator for the ideal of $\mathbb{Z}[i]$ generated by $3 + 4i$ and $4 + 7i$.

Date: Friday 29[th] March, 2024.