MATH 370 ALGEBRA, SPRING 2024, MIDTERM 2

Problem 1 [10 points] Find all values of a in $\mathbb{Z}/5\mathbb{Z}$ such that the quotient ring $\mathbb{Z}/5\mathbb{Z}[x]/(x^3 + 2x^2 + ax + 3)$ is a field. Justify your answer. (You can assume that one can divide in the polynomial ring $\mathbb{Z}/5\mathbb{Z}[x]$, i.e., for any two polynomials f and g in $\mathbb{Z}/5\mathbb{Z}[x]$, there exists quotient $q(x)$ and remainder $r(x)$ in $\mathbb{Z}/5\mathbb{Z}[x]$ such that $f = gq + r$ where $r = 0$ or $deg(r) < deg(g)$.)

Problem 2 [10 points]
Let R be a commutative ring and assume there is some fixed positive integer n such that $nr = 0$ for all $r \in R$. Give an example of a ring S such that the following hold:

- There is an injective group homomorphism ϕ from R to S.
- $\phi(R)$ is an ideal of S.
- $S/\phi(R)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Problem 3 [10 points] Determine the number of elements of order 3 in a group of order 87.

Problem 4 [10+10 points]

- Let G be the group $\mathbb{Z}/17\mathbb{Z}$. Compute the automorphism group $\mathrm{Aut}(G)$ of G.
- Let G be a finite group of order $357 = 3.7.17$. Prove that every Sylow 17-subgroup of G is contained in the center $Z(G)$.

Date: Tuesday 4^{th} June, 2024.