MATH 370 ALGEBRA, SPRING 2024, HOMEWORK 2

Problem 1 [8 points] Let G be a group. Let $g \in G$ such that $\text{ord}(g) = 20$. Compute :

- $\text{ord}(g^2)$
- $\text{ord}(g^8)$
- $\text{ord}(g^5)$
- $\text{ord}(g^3)$

Problem 2 [5 points] Show that $A_{n+1} \cap S_n = A_n$, where we regard S_n as a subgroup of S_{n+1} in the following way, think of permutations in S_n as permutations of S_{n+1} that fix $n+1$. For example, we can think of the permutation $(123) \in S_3$ as an element of S_4 that fixes 4 and sends 1 to 2, 2 to 3 and 3 to 1.

Problem 3 [8 points] Let σ be a cycle of length $n \geq 2$.

- Show that for any $\tau \in S_n$, we have that $\tau\sigma\tau^{-1}$ is also a cycle of length n.
- If $n = 2k$ for some integer k, find the factorization of σ^2 into disjoint cycles.
- If $n = mq$ with $m \geq 3$ and $q \geq 2$, show that σ^m is a product of m disjoint cycles, each of length q.
- If p is a prime, show that σ^m is a cycle of length p for each $m = 1, 2, \ldots, p-1$.

Problem 4 [5 points] In each case a binary operation $*$ is given on a set S. Decide whether it is commutative and/or associative. If it is commutative and/or associative do not provide a proof of it but provide a counterexample when it is not. Also give the identity element.

- S=Z; $a * b = a - b$.
- S=Q; $a * b = ab/2$.
- S=R; $a * b = a + b - ab$.
- S is any set of cardinality at least 2, $a * b = b$.
- S=N-$\{0\}$ $a * b = gcd(a, b)$.

Problem 5 [6 points] Let g be an element of group G.

- Show that $g^2 = 1$ if and only if $g = g^{-1}$.
- If $|G|$ is finite and even, show that there exists a $g \in G$, $g \neq e$ such that $g^2 = e$.

Problem 6 [9 points] Let H and K be subgroups of a group G.

- Show that $H \cap K$ is a subgroup of G.
- Show that $H \cap K$ is the largest subgroup of G contained in both H and K, i.e., show that it contains every subgroup contained in both H and K.
- Show that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

Date: Friday 2[nd] February, 2024.