MATH 370 ALGEBRA, SPRING 2024, HOMEWORK 5

Problem 1 Fixed Point Theorem Let G be a p-group and let S be a finite set on which G acts. If the cardinality of S is not divisible by p, there is a fixed point for the operation of G on S, i.e., there exists an element s whose stabilizer is the whole group. Prove the Fixed Point Theorem.

Problem 2 Let Z be the center of a group G. Prove that if G/Z is a cyclic group, then G is abelian, and therefore $G = Z$.

Problem 3 A group G of order 12 contains a conjugacy class of order 4. Prove that the center of G is trivial.

Problem 4 Let $\phi\colon G \to G'$ be a surjective group homomorphism. Let C denote the conjugacy class of an element x of G and let C' denote the conjugacy class in G' of its image $\phi(x)$. Prove that ϕ maps C surjectively to C', and that $|C'|$ divides $|C|$.

Problem 5 Use the class equation to show that a group of order pq, with p and q prime (not necessarily distinct), contains an element of order p.

Problem 6 Prove that A_n is the only subgroup of S_n of index 2.

Problem 7 Determine the class equations of S_6 and A_6.

Date: Saturday 9^{th} March, 2024.