MATH 370 ALGEBRA, SPRING 2024, HOMEWORK 3

Problem 1 [16 points] In each case determine and justify whether G is cyclic.

- $G = (\mathbb{Z}/7\mathbb{Z})^\times$
- $G = (\mathbb{Z}/12\mathbb{Z})^\times$
- $G = (\mathbb{Z}/16\mathbb{Z})^\times$
- $G = (\mathbb{Z}/11\mathbb{Z})^\times$
- G is the group of all non-zero rational numbers with multiplication as the operation
- $G = (\mathbb{Q}, +)$
- $G = (\mathbb{R}, +)$
- G is the group of all non-zero real numbers with multiplication as the operation

Problem 2 [5 points] Let G be a group. Let X be a non-empty subset of G and define $< X >= \cap_{X \subset H} H$, the intersection of all subgroups of G that contain X as a subset. (Note that G contains X so the intersection is non-empty.) Prove that $< X >$ is the smallest subgroup of G that contains X, i.e., prove that $< X >$ is a subgroup of G and $< X >$ is a subgroup of any other subgroup H of G that contains X. The subgroup $< X >$ is called the subgroup of G generated by X.

Problem 3 [5 points] Show that $(\mathbb{Z}/10\mathbb{Z})^\times$ is not isomorphic to $(\mathbb{Z}/12\mathbb{Z})^\times$. Show that $(\mathbb{Z}/14\mathbb{Z})^\times$ is isomorphic to $(\mathbb{Z}/18\mathbb{Z})^\times$.

Problem 4 [5 points] If G is a group of order pq, where p and q are primes, show that every proper subgroup of G is cyclic.

Date: Friday 9^{th} February, 2024.