G_1, G_2 groups

$\varphi : G_1 \longrightarrow G_2$ group homomorphism

$$\boxed{G_1 / \ker \varphi \cong \text{Image}(\varphi)}$$

$H \leq G_1, \quad N \trianglelefteq G_1$

$$\boxed{HN/N \cong H / H \cap N}$$

$N \trianglelefteq K \trianglelefteq G_1$

$$\boxed{(G_1/N) / (K/N) \cong G_1/K}$$

Example (Second Isomorphism theorem)

$G = GL_n(\mathbb{C})$

$N = SL_n(\mathbb{C})$

$N \trianglelefteq G$

$H = D_n(\mathbb{C})$ = Subgroup of diagonal matrices in G

$HN = G$

$$HN/N = GL_n(\mathbb{C})/SL_n(\mathbb{C}) \cong \mathbb{C}^\times \quad \left(\begin{matrix}\text{via}\\ \text{det}\end{matrix}\right)$$

$$H \cap N = \left\{ \begin{pmatrix} \alpha_1 & & & & 0 \\ & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_{n-1} & \\ 0 & & & & \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \end{pmatrix} \middle| \; \alpha_i \in \mathbb{C}^\times \right\}$$

$$H/H \cap N \cong \mathbb{C}^\times$$

GROUP ACTIONS

Let G be a group. Let S be a set.

Definition: A group action is a map

$$\varphi : G \times S \longrightarrow S \quad \text{satisfying}$$

1) $\varphi((e, s)) = s \quad \forall \, s \in S$

2) $\varphi\big(g_1, \varphi((g_2, s))\big) = \varphi(g_1 g_2, s)$

Examples:

1) $G = S_n \qquad S = \{1, 2, \ldots, n\}$

$$\varphi : G \times S \rightarrow S$$
$$(\sigma, s) \mapsto \sigma(s)$$

2) $G = (\mathbb{R}, +)$, $S = \mathbb{R}$

$$\varphi : G \times S \longrightarrow S$$
$$(g, s) \longmapsto g + s$$

Orbit and Stabilizers

$\varphi : G \times S \to S$ group action

Fix $a \in S$.

(1) $\text{Stab}_a = \{ g \in G \mid \varphi((g,a)) = a \}$
(Stabilizer of a)

$\text{Stab}_a \subseteq G$

Is it a subgroup ?

(2) $\text{Orbit}_a = \{ \varphi((g,a)) \mid g \in G \} \subseteq X$

Example (1) (Permutations)

$G = S_n \qquad S = \{1, 2, \ldots, n\}$

$\text{Stab}_n = \{ \sigma \in G \mid \sigma(n) = n \}$

$= S_{n-1}$

$\text{Orbit}(a) = \{ \sigma(n) \mid \sigma \in S_n \} = S$

(2) $G = GL_2(\mathbb{R})$

$S = \mathbb{R}^2$ (think Column vectors)

$\varphi : G \times S \to S$

$\varphi((A, v)) = Av$

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\text{Stab}_v = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \;\middle|\; \begin{matrix} b \neq 0 \in \mathbb{R} \\ a \in \mathbb{R} \end{matrix} \right\}$

Orbit_v = All vectors in $\mathbb{R}^2$ except $(0,0)$

(3) Consider dihedral group D_8, symmetries of square.

$\varphi : D_8 \times \{1,2,3,4\} \to \{1,2,3,4\}$

$\varphi((\sigma, i)) = \sigma(i)$

What is Stab_4 and Orbit_4 ?

(4) $G = \mathbb{Z} \times \mathbb{Z}$

$$\varphi : G \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\varphi\big((m,n),(a,b)\big) = (m+a,\ n+b)$$

what is $\mathrm{Stab}_{(a,b)}$?

$\mathrm{Orbit}\,(a,b)$?

Orbit – Stabilizer Theorem

Let $\varphi : G \times X \to X$ be a group action.

Fix $a \in X$. Then

$$\frac{|G|}{|\mathrm{Stab}\, a|} = |\mathrm{Orbit}\ a|$$

Proof: $f : G \longrightarrow \mathrm{Orbit}(a) \subseteq X$

$$g \mapsto \varphi((g, a))$$

f is surjective.

Suppose $f(g_1) = f(g_2)$

$$\Leftrightarrow \varphi((g_1, a)) = \varphi((g_2, a))$$

$$\Leftrightarrow g_2^{-1} g_1 \in \mathrm{Stab}\, a$$

$$\Leftrightarrow g_1 \in g_2\, \mathrm{Stab}\, a$$

Define a new function

$$\bar{f} : \{\text{Set of cosets}\} \longrightarrow \mathrm{Orbit}(a)$$

$\bar{f}$ is both surjective & injective.

$$|\text{Orbit}(a)| = |\text{Set of cosets}|$$

$$= \frac{|G|}{|\text{Stab}_a|}$$

Application of Orbit-Stabilizer Theorem

<u>Burnside's Lemma</u> Let G be a finite group that acts on a set X. For each $g \in G$,

$$X^g = \{ x \in X \mid \phi(g,x) = x \}$$

Then,

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

<u>Pf:</u> Let's rewrite the sum on right hand side

$$\sum_{g \in G} |X^g| = \# \{ (g,x) \in G \times X \mid \phi(g,x) = x \}$$

$$= \sum_{x \in X} |\text{Stab}_x|$$

$$|\mathrm{Stab}_x| = \frac{|G|}{|\mathrm{Orbit}(x)|}$$

$$\sum_{x\in X} |\mathrm{Stab}_x| = \sum_{x \in X} \frac{|G|}{|\mathrm{Orbit}(x)|}$$

$$= |G| \sum_{x\in X} \frac{1}{|\mathrm{Orbit}(x)|}$$

Write X as disjoint union of orbits,

$$\sum_{x\in X} \frac{1}{|\mathrm{Orbit}(x)|} = \sum_{A \in \{\text{Set of orbits}\}} \sum_{x\in A} \frac{1}{|A|}$$

$$= \sum_{A \in \{\text{Set of orbits}\}} 1 = \#\text{ of orbits}$$

$$\frac{1}{|G|}\left(\sum_{g\in G} |X^g|\right) = \frac{1}{|G|}\left(|G|\,(\#\text{ of orbits})\right)$$

$$= \#\text{ of orbits}$$

Necklace

Suppose I want to construct strings of length 3, using only 0 & 1. There are 8 such possibilities

000	100
001	101
010	110
011	111

Suppose I tie the ends together,

$\{000\}$

$\{001, 010, 100\}$ ← identical upto rotation

$\{011, 101, 110\}$ ← identical upto rotation

$\{111\}$

In terms of group actions

$G = \{$group of rotations on three elements$\}$

$G = 3$

No. of orbits $= \frac{1}{3}\left(8 + 2 + 2 \right)$

everything is fixed under identity rotation

$= 4$

Cayley's Theorem

Thm: If G is a finite group, G is isomorphic to a subgroup of S_n for some value of n that depends on G.

Pf: Let us define a map

$$\psi : G \to S_n \qquad n = |G|$$
$$g \mapsto \sigma_g$$

What is σ_g here?

Let's order $G = \{g_1, g_2, \ldots, g_n\}$

& think of elements in S_n given by indices of g_i

i.e $S_n = \{1, 2, \ldots, n\}$

$1 \downarrow$ Think g_1 $\qquad$ $n \downarrow$ Think g_n

$$\sigma_g : S_n \to S_n$$
$$i \mapsto \text{Index of } g g_i$$

<u>Claim:</u> ψ is injective group homomorphism.

<u>Pf of Claim:</u> $\psi(g_1 g_2) = \sigma_{g_1 g_2}$

we want to show $= \sigma_{g_1} \circ \sigma_{g_2}$

$$\sigma_{g_1 g_2}(g_i) = g_1 g_2 (g_i)$$
$$= g_1 (g_2 g_i)$$
$$= \sigma_{g_1}(\sigma_{g_2}(g_i))$$

So, $\sigma_{g_1 g_2} = \sigma_{g_1} \circ \sigma_{g_2}$

$$\ker \psi = \{ g \in G \mid \sigma_g = \text{identity permutation} \}$$
$$= \{ g \in G \mid \sigma_g(g_i) = g_i \ \forall i \}$$
$$= \{ g \in G \mid g g_i = g_i \ \forall i \}$$
$$= \{ e \}$$

Examples:

$$G = (\mathbb{Z}/4\mathbb{Z}, +)$$
$$= \{0, 1, 2, 3\}$$

$$G \hookrightarrow S_4$$

Let's look at subgroup of rotations $\{e, (1234), (13)(24), (1432)\}$

$$\cong$$
$$(\mathbb{Z}/4\mathbb{Z}, +)$$