G_1, G_2 groups

$\varphi : G_1 \longrightarrow G_2$ group homomorphism

$$\boxed{G_1 / \ker\varphi \ \cong \ \text{Image}(\varphi)}$$

isomorphic as groups

Let's see this in a more general setting.

Quotient maps

G group $\quad H$ normal subgroup of G

Define $\eta : G \longrightarrow G/H$ as follows

$$a \longmapsto aH$$

Lemma: η is a group homomorphism.

Pf: $\eta(ab) = (ab)H$

$= (aH)(bH) = \eta(a)\,\eta(b)$

η is called quotient map.

Lemma: η is surjective.

Pf: If $aH \in G/H$, then $aH = \eta(a)$.

UNIVERSAL PROPERTY OF QUOTIENT GROUPS

Setup: $\varphi : G \longrightarrow G'$ group homomorphism

H is a normal subgroup of G that is contained in $\ker \varphi$

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/H & \end{tikzcd}$$

Punchline: Then, there exists a unique homomorphism $\overline{\varphi} : G/H \to G'$ satisfying

$$\overline{\varphi} \circ \eta = \varphi$$

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\circlearrowleft"] & G' \\ G/H \arrow[ur, dashed, "\overline{\varphi}"'] & \end{tikzcd}$$

Pf:

Given
$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/H & \end{tikzcd}$$
$H \subseteq \ker \varphi$

Want to construct red dotted arrow in the diagram
$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/H \arrow[ur, dashed, "\overline{\varphi}"'] & \end{tikzcd}$$
$H \subseteq \ker \varphi$

We want $\overline{\varphi}$ to satisfy $\overline{\varphi} \circ \eta = \varphi$.

<u>Existence</u>

Define $\overline{\varphi}(aH) = \varphi(a)$

(1) <u>Well-defined:</u> if $aH = bH$, then $b^{-1}a \in H$

but $H \subseteq \ker \varphi \implies \varphi(b^{-1}a) = 1$

$\implies \varphi(b^{-1})\,\varphi(a) = 1$

$\implies \varphi(a) = \varphi(b)$

So, $\overline{\varphi}(aH) = \overline{\varphi}(bH)$

(2) $\quad \bar{\varphi} : G/H \longrightarrow G'$

$$aH \longmapsto \varphi(a)$$

It is a group homomorphism.

Pf: $\bar{\varphi}\big((aH)(bH)\big) = \bar{\varphi}\big((ab)H\big)$

$$= \varphi(ab)$$

$$= \varphi(a)\,\varphi(b)$$

$$= \bar{\varphi}(aH)\,\bar{\varphi}(bH)$$

Uniqueness:

Suppose there are two maps $\bar{\varphi}_1$ & $\bar{\varphi}_2$

Such that

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/H \arrow[ur, dashed, "\bar{\varphi}_1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/H \arrow[ur, dashed, "\bar{\varphi}_2"'] & \end{tikzcd}$$

then we want to show that $\bar{\varphi}_1 = \bar{\varphi}_2$

$$\bar{\varphi}_1(aH) = \bar{\varphi}_1(\eta(a)) = \varphi(a) = \bar{\varphi}_2(\eta(a)) = \bar{\varphi}_2(aH).$$

So, if
$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/H \end{tikzcd}, \quad \text{then} \quad \begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/H \arrow[ur, dashed, "\bar{\varphi}"'] \end{tikzcd}$$

$H \subseteq \ker \varphi$

$$\boxed{\bar{\varphi} \circ \eta = \varphi}$$

First isomorphism theorem is a consequence of this universal property.

Second and third isomorphism theorems are corollaries of this universal property

<u>Ingredient for Second isomorphism theorem</u>:

HK $\quad$ H & K are subgroups of G

Define $HK = \{ hk \mid h \in H, k \in K \}$

Lemma: HK is a subgroup of G

$\iff HK = KH$

$\iff KH$ is a subgroup of G.

Proof: Excercise (for HW 4)

Second isomorphism theorem:

Let G be a group.

$H \subseteq G$ subgroup of G

$N \trianglelefteq G$ normal subgroup of G

Then the following statements hold:

i) $N \trianglelefteq \underbrace{HN \leq G}_{\text{subgroup}}$

ii) $H \cap N \trianglelefteq H \leq G$

iii) $HN/N \cong H/H \cap N$

Pf: i) & ii) are excercise.

iii) Want to show that

$$HN/N \cong H/H\cap N$$

Define $\psi: H \longrightarrow HN$ $\quad$ $\eta: HN \longrightarrow HN/N$

$h \longmapsto h(1)$ $\quad$ $a \longmapsto aN$

ψ is a group homomorphism.

Define $\varphi: H \longrightarrow HN/N$ as $\eta \circ \psi$

Since η & ψ are group homomorphisms, so is φ.

$$\text{Ker}\,\varphi = \{h \in H \mid \varphi(h) = 1\}$$
$$= \{h \in H \mid \varphi(h) \in N\}$$
$$= H \cap N$$

$$\text{image}(\varphi) = HN/N$$

$$\varphi : H \to HN/N$$

$$\text{image}(\varphi) = HN/N$$

$$\ker \varphi = H \cap N$$

So, first isomorphism theorem implies

$$\boxed{H/H\cap N \cong HN/N}$$

Example:

$$G = (\mathbb{Z}, +)$$

$$H = a\mathbb{Z} \quad , \quad N = b\mathbb{Z}$$

$$H \cap N = \text{lcm}(a,b)\,\mathbb{Z}$$

$$HN = \gcd(a,b)\,\mathbb{Z}$$

$$\boxed{a\mathbb{Z}/\text{lcm}(a,b)\mathbb{Z} \cong \gcd(a,b)\mathbb{Z}/b\mathbb{Z}}$$

<u>Question:</u> G group

H is a normal subgroup

So, G/H is a group

What are all the subgroups of G/H?

Answer: Correspondence or Lattice Thm

$$\left\{ \text{Subgroups of } G/H \right\} \xleftrightarrow[\text{Correspondence}]{1-1} \left\{ \begin{array}{c} \text{Subgroups} \\ \text{of } G \text{ that} \\ \text{contain } H \end{array} \right\}$$

Let $\bar{K}$ be a subgroup of G/H.

Take $K = \eta^{-1}(\bar{K})$ $\quad \eta : G \to G/H$

ie $K = \{ g \in G \mid \eta(g) \in \bar{K} \}$

It is a subgroup because

(1) $1 \in K$

(2) If $a, b \in K \Rightarrow \eta(a), \eta(b) \in \bar{K}$

$\Rightarrow \eta(a)\eta(b) \in \bar{K}$

$\Rightarrow \eta(ab) \in \bar{K}$

$\Rightarrow ab \in K$

(3) If $b \in K \Rightarrow \eta(b) \in \bar{K} \Rightarrow \eta(b^{-1}) \in \bar{K}$

$\Rightarrow b^{-1} \in K$

Similarly if one starts with a subgroup of G that contains H, say K

then $\eta(K)$ is a subgroup of G/H.

$$\left\{ \text{Subgroups of } G/H \right\} \underset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Subgroups of} \\ G \text{ that contain} \\ H \end{array} \right\}$$

$$\begin{array}{ccc} \cup\!\mid & & \cup\!\mid \end{array}$$

$$\left\{ \begin{array}{c} \text{Normal subgroups of} \\ G/H \end{array} \right\} \underset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Normal subgroups} \\ \text{of } G \text{ that} \\ \text{contain } H \end{array} \right\}$$

Pf: If $H \subseteq K \trianglelefteq G$ (normal)

$\eta(K) \trianglelefteq G/H$ $\quad\quad \eta : G \to G/H$

then $(aH)\,\eta(K)\,(aH)^{-1} = \eta(a)\,\eta(K)\,\eta(a^{-1})$

$= \eta(aKa^{-1})$

$= \eta(K)$

Similarly if $\bar{K} \trianglelefteq G/H$, then $\eta^{-1}(\bar{K}) \trianglelefteq G$.

Third isomorphism Thm:

G group $\quad H$ a normal subgroup of G

$$H \trianglelefteq G$$

and K is a normal subgroup of G that contains H.

$$H \trianglelefteq K \trianglelefteq G$$

Then $\left({G/H} \Big/ {K/H} \right) \cong G/K$

Pf: $\varphi : G \xrightarrow[\text{map}]{\text{quotient}} G/H \xrightarrow[\text{map}]{\text{quotient}} (G/H)\big/(K/H)$

φ is a group hom, surjective

$$\ker \varphi = \{ g \in G \mid \varphi(g) = 1 \}$$
$$= K$$

$$\Rightarrow \boxed{G/K \cong (G/H)/(K/H)}$$

Example: $G = \mathbb{Z}$

$H = 8\mathbb{Z}$

$K = 2\mathbb{Z}$

$$8\mathbb{Z} \trianglelefteq 2\mathbb{Z} \trianglelefteq \mathbb{Z}$$

$G/H \cong \mathbb{Z}/8\mathbb{Z}$ $\quad$ $K/H \cong 2\mathbb{Z}/8\mathbb{Z}$

$$(G/H)/(K/H) \cong \mathbb{Z}/2\mathbb{Z}$$