MATH 370 ALGEBRA, SPRING 2024, HOMEWORK 8

Problem 1 Find generators for the kernels of the following maps.

- $\mathbb{R}[x, y] \to \mathbb{R}$ defined by $f(x, y) \mapsto f(0, 0)$.
- $\mathbb{R}[x] \to \mathbb{C}$ defined by $f(x) \mapsto f(2 + \iota)$.
- $\mathbb{Z}[x] \to \mathbb{R}$ defined by $f(x) \mapsto f(1 + \sqrt{2})$.
- $\mathbb{Z}[x] \to \mathbb{C}$ defined by $f(x) \mapsto f(\sqrt{2} + \sqrt{3})$.

Problem 2

- An element x of a ring R is called nilpotent if some power of x is 0. Prove that if x is nilpotent, then $1 + x$ is a unit.
- Suppose that R has prime characteristic p not equal to 0. Prove that if a is nilpotent, then $1 + a$ is unipotent i.e., some power of $1 + a$ is 1.

Problem 3 Let a be an element of a ring R and let R' be the quotient ring $R[x]/(ax-1)$ obtained by adjoining an inverse of a to R. Let $[x]$ denote the class of x in R'.

- Show that every element β of R' can be written in the form $\beta = [x]^k b$ with b in R.
- Prove that the kernel of the map $R \to R'$ is the set of elements $b \in R$ such that $a^n b = 0$ for some n bigger than 0.
- Prove that R' is the zero ring if and only if a is nilpotent.

Problem 4 Factor the following polynomials into irreducible factors in $\mathbb{Z}/p\mathbb{Z}[x]$.

- $x^3 + x^2 + x + 1, p = 2$
- $x^2 - 3x - 3, p = 5$
- $x^2 + 1, p = 7$

Date: Friday 5$^{\text{th}}$ April, 2024.