# MATH 370

## §. MATRICES

A $m \times n$ matrix is a rectangular array of real numbers arranged in $m$ rows and $n$ columns.

$$A = \begin{bmatrix} & & \vdots & & \\ - & - & (a_{ij}) & - & - \\ & & \vdots & & \end{bmatrix}$$

entry in $i$th row and $j$th column of $A$

**Examples:** Let $A$ be a $m \times n$ matrix in the following cases:

1) If $m=1$, then $A$ is called a row matrix.

2) If $n=1$, then $A$ is called a Column matrix.

3) If $m=n$, then $A$ is called a Square matrix.

4) Suppose $A$ is a square matrix, if $a_{ij}=0$ whenever $i \neq j$ then $A$ is called a diagonal matrix.

5) $A$ is a Scalar matrix if $A$ is a diagonal matrix with all the diagonal entries same.

Operations on Matrices:

1) You can add two matrices $A$ and $B$ if they have same number of rows and columns.

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

2) Let $A$ be a $m \times n$ matrix and $B$ be a $r \times p$ matrix. You can multiply $A$ and $B$ only when $n = r$.

3) You can multiply a matrix with a scalar.

Properties of these operations:

1) Matrix addition and multiplication is associative i.e, $(AB)C = A(BC)$ and $(A+B)+C = A+(B+C)$

2) Matrix addition is commutative, i.e,

$$(A+B) = (B+A)$$

3) For every $m \times n$ matrix $A$,

$$A+0 = 0+A = A$$

4) Given a $m \times n$ matrix $A$, there exists a $m \times n$ matrix $B$ s.t

$$A + B = B + A = 0.$$

(Just take $B = -A$)

5) For every $m \times n$ matrix $A$,

$$A I_{n \times n} = A = I_{m \times m} A$$

6) For matrices $A$, $B$ and $C$

$$A(B+C) = AB + AC$$

and $(A+B)C = AC + BC$

7) For matrices $A$, $B$ and a scalar $c$, $c(AB) = (cA)B = A(cB)$

## § Row Echelon form of a matrix

Let $A$ be a $m \times n$ matrix.

$$\begin{bmatrix} 1 & * & 0 & 0 & \text{---} & & & \\ 0 & 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & - & - & - & - & 0 \end{bmatrix}$$

We say that $A$ is a <u>row echelon matrix</u> if it has the following properties:

i) If $i$-th row of $A$ is zero, then $j$-th row of $A$ is zero for every $j > i$.

ii) The first non-zero entry in a non-zero row is 1. It is called a pivot. All the entries below and above a pivot (column-wise) should be zero.

iii) The pivot in $(i+1)$-th row should be to the right of pivot in ith-row.

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To reduce a $m \times n$ matrix $A$ into its row echelon form perform the following operations called row operations:

i) Interchange row $i$ with

row $j$.

ii) Multiply a row with a number.

iii) Multiply row $i$ with a number and add it to row $j$.

Example: Let $A = \begin{bmatrix} 0 & 2 & 5 \\ 3 & 2 & 10 \\ 0 & 0 & 5 \end{bmatrix}$

① $R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} 3 & 2 & 10 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

② $R_2 \to R_2 - R_3$

$$\begin{bmatrix} 3 & 2 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

③ $R_2 \to \left(\frac{1}{2}\right) R_2$ ④ $R_3 \to \left(\frac{1}{5}\right) R_3$

$$\begin{bmatrix} 3 & 2 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⑤ $R_1 \to R_1 - 2R_2$ ⑥ $R_1 \to R_1 - 10R_3$

$$\begin{bmatrix} 3 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⑦ $R_1 \to \left(\frac{1}{3}\right) R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Solving system of equations using row echelon form

Consider the system (*)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We can write this system into its matrix form as follows:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then the system of equations * is just $Ax = b$

Write the marix $[A|b]$ in its row echelon form;

i) There is a pivot in the last column or there is a row of the form $[0 \cdots 0 | 1]$ if and only if there is no solution to $Ax = b$.

ii) Suppose (i) is not true. If $m<n$, then there are infinitely many solutions of $Ax=b$. If $m=n$ and there is a pivot in last row and last column of $A$ then there is a unique solution to $Ax=b$ else there are infinitely many solutions. If $m>n$, then if the number of non-zero rows is $n$, the system has unique solution else it has infinitely many solutions.

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§ Invertible Matrices

Let $A$ be a square matrix of size $n \times n$.

Defn: We say that $A$ is invertible if there is a matrix $B$ such that $AB = BA = I_n$.

We say that $B$ is the inverse of $A$.

Example: Assume that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Calculate its inverse using row operations.

i) Suppose $a \neq 0$

$$R_1 \to (1/a) R_1$$

$$\begin{pmatrix} 1/a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix}$$

ii) $R_2 \to R_2 - cR_1$

$$\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/a \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

iii) $R_2 \to \left(\frac{a}{ad-bc}\right) R_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & \frac{ad-bc}{a} \end{pmatrix} = \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$$

iv) $R_1 \to R_1 - (b/a) R_2$

$$\begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\dfrac{b}{ad-bc} \\ 0 & \dfrac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ -c/a & 1 \end{pmatrix}$$

$$\|$$

$$\begin{pmatrix} \dfrac{1}{a} + \dfrac{bc}{a(ad-bc)} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

$$\frac{1}{a} + \frac{bc}{a(ad-bc)} = \frac{ad-bc+bc}{a(ad-bc)} = \frac{d}{ad-bc}$$

$$\boxed{\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}}$$

Lemma 1: Suppose that $A$ is a square matrix of size $n \times n$. Let $A'$ be its row echelon form. Then $A' = I$ or bottom row of $A'$ is $0$.

Proof: Suppose the bottom row of $A'$ is not $0$. We will show that $A' = I$.

Since bottom row is not zero, there are $n$ pivots each in 1 row. Since there are $n$ columns, each column has exactly 1 pivot.

So, $A = I$.

**Lemma 2:** Suppose $A$ is a square matrix and $B$ is a matrix satisfying $AB = I$. Then $BA = I$.

**Proof:** Let $P$ be a matrix that converts $A$ to its row echelon form $A'$;

i.e. $PA = A'$.

Exc: Check that $P$ is invertible.

Now, $P = PI = P(AB)$
$= (PA)B$
$= A'B$.

Since $P$ is invertible, its bottom row is not zero. Therefore, bottom row of $A'$ is non-zero. By lemma 1, $A' = I$.

Since $PA = A'$, we get $PA = I$.

We will show that $P = B$.

$$P = PI = P(AB)$$
$$= (PA)B$$
$$= IB$$
$$= B.$$