Cayley's Theorem

Thm: If G is a finite group, G is isomorphic to a subgroup of S_n for some value of n that depends on G.

Pf: Let us define a map

$$\psi : G \longrightarrow S_n \qquad n = |G|$$

$$g \longmapsto \sigma_g$$

what is σ_g here?

Let's order $G = \{g_1, g_2, \ldots, g_n\}$

& think of elements in S_n given by indices of g_i

i.e $S_n = \{1, 2, \ldots, n\}$

$1 \downarrow$ Think g_1 $\qquad$ $n \downarrow$ Think g_n

$$\sigma_g : S_n \longrightarrow S_n$$

$$i \longmapsto \text{Index of } gg_i$$

Claim: ψ is injective group homomorphism.

Pf of Claim: $\psi(g_1 g_2) = \sigma_{g_1 g_2}$

we want to show $= \sigma_{g_1} \circ \sigma_{g_2}$

$$\sigma_{g_1 g_2}(g_i) = g_1 g_2 (g_i)$$
$$= g_1 (g_2 g_i)$$
$$= \sigma_{g_1}(\sigma_{g_2}(g_i))$$

So, $\sigma_{g_1 g_2} = \sigma_{g_1} \circ \sigma_{g_2}$

$$\ker \psi = \{ g \in G \mid \sigma_g = \text{identity permutation} \}$$
$$= \{ g \in G \mid \sigma_g(g_i) = g_i \ \forall i \}$$
$$= \{ g \in G \mid g g_i = g_i \ \forall i \}$$
$$= \{ e \}$$

Examples:

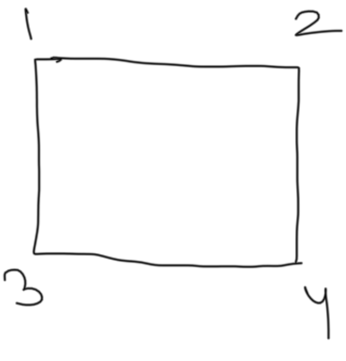
$$G = (\mathbb{Z}/4\mathbb{Z}, +)$$
$$= \{0, 1, 2, 3\}$$

$$G \hookrightarrow S_4$$

Let's look at subgroup of rotations $\{e, (1234), (13)(24), (1432)\}$

$$\cong$$
$$(\mathbb{Z}/4\mathbb{Z}, +)$$

Example: D_8 = Group of symmetries of a square



4 rotations

$e, (1342), (14)(23), (1243)$

4 reflections

$(23), (14), (12)(34), (13)(24)$

Cayley's theorem gives us an embedding

$D_8 \hookrightarrow S_8$. Can we embed it into

a smaller S_n?

Example: Quaternions $= \{\pm 1, \pm i, \pm j, \pm k\}$

$$\begin{tikzcd} & i \arrow[dr, bend left] & \\ k \arrow[ur, bend left] & & j \arrow[ll, bend left] \end{tikzcd} \qquad i^2 = j^2 = k^2 = -1$$

$$Q_8 \hookrightarrow S_8$$

Can we make this 8 smaller?

Answer: No! Let's see a proof.

Suppose $\psi : Q_8 \hookrightarrow S_n \qquad n < 8$

We get a group action

$$\varphi : Q_8 \times \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$$

$$(g, i) \longmapsto \psi(g)(i)$$

Fix any $i \in \{1, 2, \dots, n\}$.

$$\frac{|G|}{|\mathrm{Stab}_i|} = \mathrm{Orbit}(i) \leq n < 8$$

$$\Rightarrow \quad |\mathrm{Stab}_i| > 1$$

Any subgroup of Q_8 bigger than $\{1\} \supseteq \{\pm 1\}$

$$\varphi : Q_8 \hookrightarrow S_n$$

$$\operatorname{Ker} \varphi = \bigcap_{i=1}^{n} \operatorname{Stab} i \supseteq \{\pm 1\}$$

$\Rightarrow$ φ is not injective.

So, smallest possible n is 8.

Conjugation

(Example of a group action)

$$\varphi : G \times G \longrightarrow G$$

$$\varphi(g, h) \longmapsto ghg^{-1}$$

1) $\varphi(e, h) = ehe^{-1} = h$

2) $\varphi(g_1, \varphi(g_2, h)) = g_1 (g_2 h g_2^{-1}) g_1^{-1}$

$$= g_1 g_2 \, h \, (g_1 g_2)^{-1}$$

$$= \varphi(g_1 g_2, h)$$

$$\mathrm{Stab}_h = \{ g \in G \mid ghg^{-1} = h \}$$

$$= \{ g \in G \mid gh = hg \}$$

(also known as centralizer)

$$\mathrm{Orbit}(h) = \{ ghg^{-1} \mid g \in G \}$$

(also known as conjugacy class)

Observation: Conjugacy classes are disjoint.

If $h_1 \neq g h_2 g^{-1}$ for any $g \in G$, then we want to show that $\text{Orbit}(h_1) \cap \text{Orbit}(h_2) = \phi$

Say $x \in \text{Orbit}(h_1)$
$x \in \text{Orbit}(h_2)$

$$x = g_1 h_1 g_1^{-1} = g_2 h_2 g_2^{-1}$$

$$\Rightarrow g_2^{-1} g_1 h_1 g_1^{-1} g_2 = h_2$$

$$\Rightarrow g_2^{-1} g_1 h_1 (g_2^{-1} g_1)^{-1} = h_2$$

$$\Rightarrow h_2 \in \text{Orbit}(h_1)$$

Contradiction!

$$G = \bigsqcup \text{Conjugacy classes}$$

$$= C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$$

$$|G| = |C_1| + |C_2| + \cdots + |C_k|$$

↳ Class equation of G.

Qn: If G is abelian, what is its class equation?

Qn: How many 1's appear in a class equation?

Centre of a group

$$Z(G) = \{x \in G \mid gxg^{-1} = x \text{ for all } g \in G\}$$

$$|G| = \underbrace{1 + 1 + \cdots + 1}_{|Z(G)| \text{ times}} + \cdots$$

<u>Permutation</u> <u>Groups</u>

$$|S_2| = 1 + 1$$

$$S_3 = \{e\}, \ \{(123), (132)\}, \ \{(12), (23), (13)\}$$

$$|S_3| = 1 + 2 + 3$$

Lemma: $\sigma, \tau \in S_n$ are in same conjugacy class $\iff$ they have same cycle type.

Pf: Example: $(12345)\ (67)$

$$\sigma\ (12345)\ (67)\ \sigma^{-1}$$

$$\underbrace{\sigma(12345)\sigma^{-1}}_{\text{cycle of length } 5}\ \underbrace{\sigma(67)\sigma^{-1}}_{\text{cycle of length } 2}$$

So, $\sigma\tau\sigma^{-1}$ has same cycle type for $\sigma, \tau \in S_n$.

$\Leftarrow$: Suppose σ_1, σ_2 have same cycle type.

Let's construct an element $\tau \in S_n$ such that $\tau\sigma_1\tau^{-1} = \sigma_2$

$\sigma_1 = (1\,2\,3\,4\,5)$

$\sigma_2 = (1\,4\,5\,3\,2)$

$\tau(1) = 1$ $\quad \tau(2) = 4$ $\quad \tau(3) = 5$

$\tau(4) = 3$ $\quad \tau(5) = 2$

$\tau = (2\,4\,3\,5)$

$\tau(4)$

$$\sigma_1 = (1\ 2\ 3\ 4\ 5)$$

$\tau(4)$ $\qquad$ $\tau\sigma_1(4)$

$$\sigma_2 = (1\ 4\ 5\ 3\ 2)$$

$\sigma_2\tau(4)$

$$\sigma_2 \circ \tau(i) = \tau\sigma_1(i) \quad \forall\, i \in \{1, 2, \dots, n\}$$

True in general, as well.

<u>Exc:</u> let's work out a class equation for S_4.