MATH 370 ALGEBRA, SPRING 2024, HOMEWORK 4

Problem 1 Let G be a group. Let H and K be subgroups of G. We have defined HK in class. Show that HK is a subgroup of G if and only if $HK = KH$ if and only if KH is a subgroup of G.

Problem 2 Let G be a group. Let N and M be normal subgroups of G. Show that if $N \cap M = \{1\}$, then NM is isomorphic to $N \times M$.

Problem 3 Let a and b be natural numbers. Let $m := lcm(a, b)/a$ and $n := b/gcd(a, b)$. Show that $a\mathbb{Z}/lcm(a, b)\mathbb{Z}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and $gcd(a, b)\mathbb{Z}/b\mathbb{Z}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. From this deduce the formula $lcm(a, b)gcd(a, b) = ab$.

Problem 4 Show that the quotient group $\mathbb{Q}/\mathbb{Z}$ contains exactly one cyclic subgroup of order n, for each $n \geq 1$.

Problem 5 Find all homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. Find all homomorphisms $\mathbb{Q}/\mathbb{Z} \to \mathbb{Z}$.

Problem 6 Let $n \geq 2$. What are the groups G such that there exists a surjective homomorphism $\mathbb{Z}/n\mathbb{Z} \to G$?

Problem 7 If G is an abelian group of order pq, where p and q are distinct primes, show that G is cyclic.

Date: Friday 23$^{\text{rd}}$ February, 2024.