MATH 370 ALGEBRA, SPRING 2024, MIDTERM 1

Problem 1 [5+5 points]

- Let G be a cyclic group. Show that every subgroup H of G is cyclic.
- Give an example of a group G whose every proper subgroup is cyclic but G is not cyclic.

Problem 2 [10 points] Let G be a finite group of cardinality 4. Show that G is either isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Problem 3 [8+2 points]

- Let G be a finite group of order $2n$. Suppose that exactly a half of G consists of elements of order 2 and the rest forms a subgroup. i.e., $G = S \sqcup H$, where S is the set of all elements of order 2 in G and H is a subgroup of G. The cardinalities of S and H are both n. Show that H is an abelian normal subgroup of odd order.
- Give an example of a group G and a subgroup H of G such that index of H is 3 and H is not normal in G.

Problem 4 [2+8 points] Let $n \geq 1$ be an odd integer.

- Show that 2 has a multiplicative inverse, denoted r in $\mathbb{Z}/n\mathbb{Z}$.
- Suppose $r^2a^2 - b$ is a square in $\mathbb{Z}/n\mathbb{Z}$, i.e., there exists $k \in \{0, \ldots, n-1\}$ such that $k^2 = r^2a^2 - b$ in $\mathbb{Z}/n\mathbb{Z}$. Show that $x^2 + ax + b = 0$ has a solution in $\mathbb{Z}/n\mathbb{Z}$.

Problem 5 [5+5 points] Let $\simeq$ be the equivalence relation on the set $\mathbb{N} \times \mathbb{N}$ defined by the rule $(a, b) \simeq (c, d)$ if and only if $a + d = b + c$. You can take for granted that this is indeed an equivalence relation. Let S be the set of equivalence classes of $\mathbb{N} \times \mathbb{N}$. Let $\gamma\colon \mathbb{N} \times \mathbb{N} \to S$ be the natural map $(a, b) \mapsto [(a, b)]$.

- Let $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be defined by $f((a, b)) = (b, a)$. Show that there exists a function $g\colon S \to S$ such that $g \circ \gamma = \gamma \circ f$.
- Let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be defined by $h((a, b)) = (2a, b)$. Show that there does not exist a function $g\colon S \to S$ such that $g \circ \gamma = \gamma \circ h$.

Date: Tuesday 4^{th} June, 2024.