MATH 314 (Lecture 2)

Topics to be discussed today

Subspaces, intersection & sum

Textbook sections (2.2, 6.6)

Recall that a subspace of a vector space is a subset that is vector space in its own right.

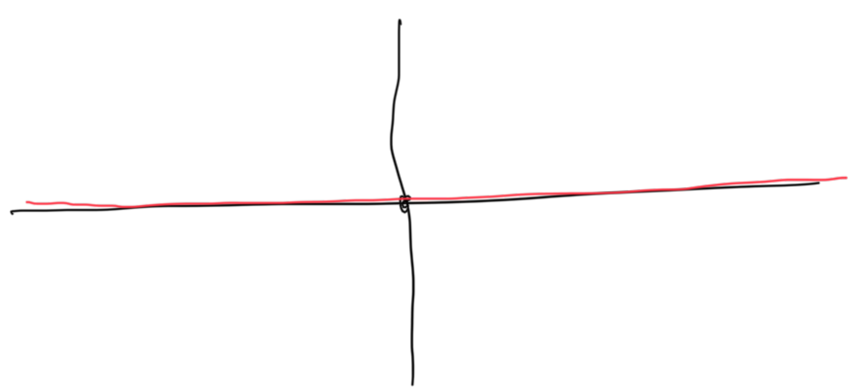
We proved last time that

A non-empty subset W of V is a subspace $\iff$ for every $x, y \in W$ & $\alpha \in F$
$\alpha x + y \in W$

Examples:

1) $\{0\} \subseteq V$

2) $V = \mathbb{R}^2$
$W = \{(x, 0) \mid x \in \mathbb{R}\}$



Discuss: Can we describe all subspaces of $\mathbb{R}^2$? What about $\mathbb{R}^n$?

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3) Consider the set of all $n \times n$ matrices with entries in a field F, call it

$M_{n\times n}(F)$

$W = \{\text{Set of all symmetric matrices}\}$

W is a subspace of $M_{n \times n}(F)$.

Thm: Intersection of any collection of subspaces is a subspace.

Pf: Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be collection of subspaces. Call $\cap V_\alpha = W$

We want to show that W is a subspace.

(1) $W \neq \phi$ as $0 \in W$.

It suffices to check that $cx+y \in W$ $\forall x, y \in W$ & $\forall c \in F$.

Let $x, y \in W$ & $c \in F$.

Then $cx+y \in V_\alpha$ $\forall \alpha$ $(W \subseteq V_\alpha)$

So, $cx+y \in W$.

Span

V vector space over F

$v_1, v_2, \ldots, v_n$ a finite set of vectors in V

<u>Defn:</u> The <u>Subspace spanned by</u> $\{v_1, v_2, \ldots, v_n\}$ is the smallest subspace of V that contains $\{v_1, \ldots, v_n\}$.

Examples:

1) $\text{Span}(\{0\}) = \{0\}$

2) $\text{Span}(V) = V$

3) Take $V = \mathbb{R}^2$

$v_1 = (1,0)$

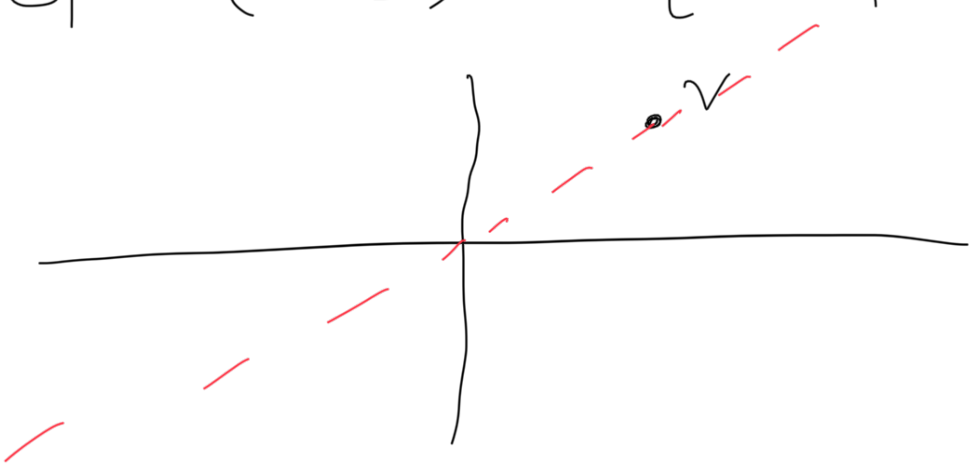
$\text{Span}(\{v_1\}) = \{(x,0) \mid x \in \mathbb{R}\}$

$v_2 = (0,1)$

$\text{Span}(\{v_2\}) = \{(0,y) \mid y \in \mathbb{R}\}$

Let v be a non-zero vector in V.

$\text{Span}(\{v\}) = \{cv \mid c \in \mathbb{R}\}$



Thm: If $S = \{v_1, v_2, \dots, v_n\}$

Then $\text{Span}(S) = \left\{ c_1v_1 + c_2v_2 + \dots + c_nv_n \,\middle|\, c_i \in F \right\}$

Pf: We will show that

(1) $\text{Span}(S) \subseteq \{ c_1v_1 + \dots + c_nv_n \mid c_i \in F\}$

and (2) $\{ c_1v_1 + \dots + c_nv_n \mid c_i \in F\} \subseteq \text{Span}(S)$

Let us try to show (1).

$\{c_1v_1 + \dots + c_nv_n \mid c_i \in F\}$ contains $\{v_1, \dots, v_n\}$ because

$$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n$$

(similarly $v_2, \dots, v_n$ are there).

Further $\{c_1v_1 + \dots + c_nv_n \mid c_i \in F\}$ is a subspace, so $\text{Span}(S) \subseteq \{c_1v_1 + \dots + c_nv_n \mid c_i \in F\}$

(2) Span (S) is a subspace that contains $\{v_1, \ldots, v_n\}$.

So, $c_i v_i \in \text{Span}(S) \quad \forall\, c_i \in F$
$1 \leq i \leq n$

(Closed under scalar multiplication)

$$\sum c_i v_i \in \text{Span}(S)$$

(closed under addition)

Combining (1) & (2) we get

$$\text{Span}(S) = \{c_1 v_1 + \ldots + c_n v_n \mid c_i \in F\}$$

Let's see some examples.

$V = \mathbb{R}^2$

$v_1 = (1, 2)$

$v_2 = (2, 3)$

Is $(3, 5)$ in span of $\{v_1, v_2\}$?

We want to check whether we can write

$$(3,5) = c_1 (1,2) + c_2 (2,3)$$

for some $c_1, c_2 \in \mathbb{R}$.

$$3 = c_1 + 2c_2 \quad — (1)$$

$$5 = 2c_1 + 3c_2 \quad — (2)$$

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Inverse of $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is $\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$v_1 = (1,1) \quad v_2 = (2,2)$

Is $(2,3)$ in $\operatorname{Span}(\{v_1, v_2\})$?

$$A = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{not invertible}$$

$$\left.\begin{aligned} 2 &= c_1 + 2c_2 \\ 3 &= c_1 + 2c_2 \end{aligned}\right\} \quad \text{No solution}$$

Sum of subspaces

$W_1 + W_2 + \cdots + W_n$ is set

$$\{ w_1 + w_2 + \cdots + w_n \mid w_i \in W_i \}$$

$V = \mathbb{R}^2$

$W_1 = \{ (x,0) \mid x \in \mathbb{R} \}$

$W_2 = \{ (0,y) \mid y \in \mathbb{R} \}$

$W_1 + W_2 = \mathbb{R}^2$ because

$$(a,b) = \underset{\substack{\uparrow \\ W_1}}{(a,0)} + \underset{\substack{\uparrow \\ W_2}}{(0,b)}$$

Another example

$W_1 = \{ (x,0) \mid x \in \mathbb{R} \}$

$W_2 = \{ (c,c) \mid c \in \mathbb{R} \}$

What is $W_1 + W_2$?

$V = \mathbb{R}^2$

$W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$

$W_2 = \{(0, y) \mid y \in \mathbb{R}\}$

$W_1 \cap W_2 = \{(0,0)\}$

In this case, $W_1 + W_2$ is called <u>direct sum</u> of W_1 & W_2.

<u>More examples:</u>

$V = M_{n\times n}(F)$

$A \in M_{n\times n}(F)$

$$A = \underbrace{\left(\frac{A + A^T}{2}\right)}_{\text{Symmetric}} + \underbrace{\left(\frac{A - A^T}{2}\right)}_{\text{Skew-Symmetric}}$$

$W_1 = \{$ set of all symmmetric matrices $\}$

$W_2 = \{$ set of all skew-symmmetric matrices $\}$

1. Are W_1 & W_2 subspaces?

2. Is $V = W_1 + W_2$?

3. Is V a direct sum of W_1 & W_2?

$V = \mathbb{R}^3$

$W_1 = \{ (x, 0, 0) \mid x \in \mathbb{R} \}$

$W_2 = \{ (0, y, 0) \mid y \in \mathbb{R} \}$

$W_3 = \{ (0, 0, z) \mid z \in \mathbb{R} \}$

$V = W_1 + W_2 + W_3$