MATH 314 (Lecture 14)

Topics to be discussed today

Multilinear forms, Permutations

Recap: V $\quad V^*$ $\quad V^{**}$

(V vector space over F) $\quad$ (V dual)

$$V^* = \{\varphi : V \to F \mid \varphi \text{ linear}\}$$

V has basis $\{b_1, \dots, b_n\}$

Dual basis is a basis of V^* obtained from $\{b_1, \dots, b_n\}$ as

$$\varphi_i(b_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

V & V^{**} are canonically isomorphic

$$V \longrightarrow V^{**}$$
$$\alpha \longmapsto L_\alpha$$

$$L_\alpha : V^* \longrightarrow F$$
$$L_\alpha(\varphi) \longmapsto \varphi(\alpha)$$

Bilinear forms

<u>Defn</u>: A map $\varphi: V \times V \to F$ is called <u>bilinear</u> if :

i) $\varphi(v_1+v_1', v_2) = \varphi(v_1, v_2) + \varphi(v_1', v_2)$ $\forall\, v_1, v_1', v_2 \in V$

$\varphi(cv_1, v_2) = c\,\varphi(v_1, v_2)$ $\forall\, v_1, v_2 \in V$ $\forall\, c \in F$

ii) $\varphi(v_1, v_2+v_2') = \varphi(v_1, v_2) + \varphi(v_1, v_2')$ $\forall\, v_1, v_2, v_2' \in V$

$\varphi(v_1, cv_2) = c\,\varphi(v_1, v_2)$ $\forall\, v_1, v_2 \in V$ $\forall\, c \in F$

Let $\varphi : V \times V \to F$ &

$\psi : V \times V \to F$ be two bilinear forms.

We can add them, defined as follows:

$$(\varphi + \psi)(v_1, v_2) = \varphi(v_1, v_2) + \psi(v_1, v_2)$$

We can also define scalar multiplication as follows:

$$(c\varphi)(v_1, v_2) = c\,\varphi(v_1, v_2)$$

With these operations, the set of all bilinear maps from $V \times V \to F$ is a vector space over F, denoted as $(V^*)^{\otimes 2}$.

Let $\varphi : V \times V \to F$ be a bilinear form.

<u>Defn</u>: ① We say φ is <u>symmetric</u> if

$$\varphi(v_1, v_2) = \varphi(v_2, v_1) \text{ for all } v_1, v_2 \in V.$$

② We say that φ is <u>skew-symmetric</u> if

$$\varphi(v_1, v_2) = -\varphi(v_2, v_1) \quad \forall\, v_1, v_2 \in V.$$

$$\mathrm{Sym}^2 V^* = \{ \varphi \in (V^*)^{\otimes 2} \mid \varphi \text{ is symmetric} \}$$

$$\Lambda^2 V^* = \{ \varphi \in (V^*)^{\otimes 2} \mid \varphi \text{ is skew symmetric} \}$$

<u>Lemma:</u> $\mathrm{Sym}^2 V^*$ & $\Lambda^2 V^*$ are subspaces of $(V^*)^{\otimes 2}$.

<u>Pf:</u> Suppose $\varphi, \psi \in \mathrm{Sym}^2 V^*$.

Then $(c\varphi+\psi)(v,w) = (c\varphi)(v,w) + \psi(v,w)$

$$= c(\varphi(v,w)) + \psi(v,w)$$

$$= c(\varphi(w,v)) + \psi(w,v)$$

$$= (c\varphi+\psi)(w,v)$$

So, $c\varphi+\psi$ is symmetric as well.

Hence $\mathrm{Sym}^2 V^*$ is a subspace.

Similarly, if $\varphi, \psi \in \Lambda^2 V^*$, then

$$(c\varphi+\psi)(v,w) = c(\varphi(v,w)) + \psi(v,w)$$

$$= -c\varphi(w,v) - \psi(w,v)$$

$$= -(c\varphi+\psi)(w,v).$$

Thm: $(V^*)^{\otimes 2} = \mathrm{Sym}^2 V^* \oplus \Lambda^2 V^*$

Pf: If $\varphi \in \mathrm{Sym}^2 V^* \cap \Lambda^2 V^*$

then
$$\varphi(v,w) = \varphi(w,v) \quad \forall v,w \in V$$
$$\varphi(v,w) = -\varphi(w,v)$$

$$\Rightarrow 2\varphi(v,w) = 0 \Rightarrow \varphi(v,w) = 0$$
$$\forall v,w \in V$$

So, if we show that

$$(V^*)^{\otimes 2} = \mathrm{Sym}^2 V^* + \Lambda^2 V^*$$

then we are done.

Since $\mathrm{Sym}^2 V^*$ & $\Lambda^2 V^*$ are subspaces

$$\mathrm{Sym}^2 V^* + \Lambda^2 V^* \subseteq (V^*)^{\otimes 2}$$

We will now show that

$$(V^*)^{\otimes 2} \subseteq \mathrm{Sym}^2 V^* + \Lambda^2 V^*$$

Let $\varphi \in (V^*)^{\otimes 2}$

Define $\hat{\varphi} : V \times V \to F$ as follows

$$\hat{\varphi}(v, w) = \varphi(w, v)$$

$\hat{\varphi}$ is a bilinear form.

observe that $\frac{\varphi + \hat{\varphi}}{2}$ is Symmetric

because

$$\left(\frac{\varphi + \hat{\varphi}}{2}\right)(v, w) = \frac{\varphi(v, w) + \varphi(w, v)}{2}$$

$$\left(\frac{\varphi + \hat{\varphi}}{2}\right)(w, v) = \frac{\varphi(w, v) + \varphi(v, w)}{2}$$

Observe that $\left(\frac{\varphi - \hat{\varphi}}{2}\right)$ is skew-symmetric

because

$$\left(\frac{\varphi - \hat{\varphi}}{2}\right)(v, w) = \frac{\varphi(v, w) - \varphi(w, v)}{2}$$

$$\left(\frac{\varphi - \hat{\varphi}}{2}\right)(w, v) = \frac{\varphi(w, v) - \varphi(v, w)}{2}$$

$$\varphi = \underbrace{\left(\frac{\varphi + \hat{\varphi}}{2}\right)}_{\text{Symmetric}} + \underbrace{\left(\frac{\varphi - \hat{\varphi}}{2}\right)}_{\text{Skew-Symmetric}}$$

Hence, $(V^*)^{\otimes 2} \subseteq \mathrm{Sym}^2 V^* + \Lambda^2 V^*$.

Multilinear forms

Let V be a vector space over a field F.

A **multilinear k-form** on V is a map

$$H : \underbrace{V \times \cdots \times V}_{k\text{-times}} \longrightarrow F$$

Such that it is linear in each argument.

We say H is symmetric if for each i, and all $v_1, \ldots, v_k$

$$H(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \ldots, v_k)$$

$$= H(v_1, \ldots, v_{i-1}, v_{i+1}, v_i, \ldots, v_k)$$

We say H is <u>alternating</u> or <u>skew symmetric</u> if for each i, & $v_1, \dots, v_k$

$$H(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_k)$$
$$= -H(v_1, \dots, v_{i-1}, v_{i+1}, v_i, \dots, v_k)$$

<u>Notation:</u> Vector space of all k-forms $(V^*)^{\otimes k}$

Subspace of all symmetric k-forms $\mathrm{Sym}^k V^*$

and

Subspace of all skew-symmetric k-forms $\Lambda^k V^*$.

Permutations

A permutation is a bijection from $\{1, \dots, k\}$ to itself.

A permutation that interchanges exactly two elements is called a transposition.

Every permutation is composition of transpositions.

For example

$$\sigma : \{1,2,3\} \longrightarrow \{1,2,3\}$$

$$1 \mapsto 2$$
$$2 \mapsto 3$$
$$3 \mapsto 1$$

$$\sigma = \sigma_1 \sigma_2$$

where $\sigma_1 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$$1 \mapsto 1$$
$$2 \mapsto 3$$
$$3 \mapsto 2$$

$\sigma_2 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$$1 \mapsto 3$$
$$3 \mapsto 1$$
$$2 \mapsto 2$$

<u>Lemma:</u> If H is symmetric & σ is a permutation, then

$$H(v_1, \ldots, v_k) = H(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

If σ is composition of r transpositions then

$$\text{signature}(\sigma) = (-1)^r$$

Lemma: If H is skew-symmetric

$$H(v_1, \dots, v_k) = \text{signature}(\sigma) \cdot H(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$