MATH 314 (Lecture 21)

Topics to be discussed today

Jordan normal form for nilpotent endomorphisms (section 7.3)

Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C})$,

$\downarrow$

Jordan Normal form

but A is not diagonalizable over $\mathbb{C}$ because

$$\mathrm{char}_A(x) = (x-1)^2 = \mathrm{min}_A(x)$$

From last time

$$[T]_B = \begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix}$$

So, it suffices to analyze T on $\mathcal{E}_\lambda$ (Generalized eigen space)

$$\mathcal{E}_\lambda = \mathrm{Ker}\,(T-\lambda)^d$$

Nilpotent Matrices

Defn: A linear map $T: V \to V$ (or a matrix A) is called nilpotent if $T^k = 0$ for some $k \geq 0$. The smallest such k is called the exponent of T.

Lemma: Let $T: V \to V$ be a nilpotent map. Let $v \in V$ be of exponent k. Then the vectors
$v, Tv, T^2 v, \ldots, T^{k-1} v$ are linearly independent.

Proof: Suppose

$$c_1 v + c_2 T v + \cdots + c_k T^{k-1} v = 0 \quad (1)$$

If all c_i are 0, then we are done. Otherwise, let c_i be the first non-zero

Coefficient in 1.

Apply T^{k-i} to 1 to get

$$c_i T^{k-1} v + c_{i+1} T^{k} v + c_{i+2} T^{k+1} v + \cdots = 0 \quad \text{— } 2$$

but $T^k v$, $T^{k+1} v \ldots$ are 0 so

2 becomes

$$c_i T^{k-1} v = 0 \implies c_i = 0 \text{ because}$$

$T^{k-1} v \neq 0$, this is a contradiction.

Defn: Cyclic subspace generated by v of exponent k is

$\text{Span}\{v, Tv, \ldots, T^{k-1} v\}$, we will denote it as $Cyc(v)$.

Lemma: 1) $Cyc(v)$ is T-invariant.

2) $T|_{Cyc(v)} : Cyc(v) \to Cyc(v)$ is nilpotent of exponent k.

Pf: 1) It follows because
$$T(T^i v) = T^{i+1} v \in Cyc(v)$$

2) $$T^k(u) = 0 \quad \forall\, u \in Cyc(v)$$
and $T^{k-1} \neq 0$ because $T^{k-1} v \neq 0$.

The matrix of T with respect to the basis $\{T^{k-1}v, \ldots, v\}$ is

Jordan block

$$\begin{aligned} T^{k-1}v &\longmapsto 0 \\ T^{k-2}v &\longmapsto T^{k-1}v \\ &\vdots \\ v &\longmapsto Tv \end{aligned} \qquad \begin{pmatrix} 0 & 1 & 0 & & 0 \\ \vdots & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

Thm: Let V be a vector space of dim n. Let $T: V \longrightarrow V$ be nilpotent of exponent m. Then

$$V = Cyc(v_1) \oplus \cdots \oplus Cyc(v_r)$$

$r = \dim \ker T$

Pf: We will use induction on $\dim V$.

If $\dim V = 1$, then since T is nilpotent

$r = 1$.

$$V = \text{Span}\{1\} = Cyc(1)$$

Suppose the theorem holds for $n-1$.

Inductive step: Since T is nilpotent, $r > 0$, so $\dim(\text{Image } T) < n$.

Construct a $n-1$ dimensional subspace of V that contains Image (T), say it is V_{n-1}.

$\text{Image}(T) \subseteq V_{n-1}$

and $T(V_{n-1}) \subseteq \text{Image}(T)$

So, V_{n-1} is T-invariant.

$$V_{n-1} = Cyc(x_1) \oplus \cdots \oplus Cyc(x_k)$$

Choose a vec tor $y \notin V_{n-1}$.

$$Ty = v_1 + \cdots + v_k \qquad v_i \in Cyc(x_i)$$

For each i we can assume that either $v_i = 0$ or $v_i \notin T(Cyc(x_i))$.

If $v_i \notin T(Cyc(x_i))$, then we are done otherwise tweak y to $y - u_i$, $T(u_i) = v_i$

$y - v_i \notin V_{n-1}$

& $T(y - u_i) = v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_k$

$\downarrow$

ith component 0

If $v_i = 0 \quad \forall i$, then

$$V = \mathrm{Span}\{y\} \oplus V_{m-1}$$
$$= \mathrm{Span}\{y\} \oplus \mathrm{Cyc}(x_1) \oplus \cdots \oplus \mathrm{Cyc}(x_k)$$

If some $v_i \neq 0$, then

$$V = \underbrace{\mathrm{Span}(\{y, Ty, \ldots, T^m y\})}_{} \oplus \mathrm{Cyc}(x_2) \oplus \cdots \oplus \mathrm{Cyc}(x_k)$$

m is exponent of T_y

because exponent of T_y is maximum of exponents of v_i,

Assume v_1 has largest exponent.

It is enough to show that

$$\mathrm{Span}(\{y, \ldots, T^m y\}) \cap \left(\mathrm{Cyc}(x_2) \oplus \cdots \oplus \mathrm{Cyc}(x_k)\right) = \{0\}$$

Suppose not

$$c_1 y + c_2 Ty + \cdots + c_{m+1} T^m y$$
$$\in Cyc(x_2) \oplus \cdots \oplus Cyc(x_k)$$
$$\subset V_{n-1}$$

y was chosen such that $y \notin V_{n-1}$
and V_{n-1} contains Image T

$\Rightarrow \quad c_1 y \in V_{n-1} \quad \Rightarrow \quad c_1 = 0$

Since $c_2 Ty + \cdots + c_{m+1} T^m y \in Cyc(x_2) \oplus \cdots$

$Cyc(x_1)$ component is zero which is

$$c_2 v_1 + c_3 T v_1 + \cdots + c_{m+1} T^{m-1} v_1 = 0$$

but $v_1, Tv_1, \ldots, T^{m-1} v_1$ form a basis
for $Cyc(x_1)$, so $c_i = 0 \quad i \geq 2$.

Suppose $T: V \longrightarrow V$ is any linear map

(not necessarily nilpotent)

$\mathcal{E}_\lambda$ — Generalized Eigen space of T corresponding to λ

$(T-\lambda)\big|_{\mathcal{E}_\lambda}$ is nilpotent

Suppose $u \in \mathcal{E}_\lambda$ is a vector of exponent k.

Let U be the cyclic subspace generated by

$\{u, (T-\lambda)u, \ldots, (T-\lambda)^{k-1}u\}$

Matrix of $T|_U$ with respect to the basis $\{(T-\lambda)^{k-1}u, \ldots, u\}$ is

$$\begin{aligned} T(T-\lambda)^{k-1}u &= (T-\lambda)^{k-1}Tu \\ &= (T-\lambda)^{k-1}((T-\lambda)u+\lambda u) \\ &= (T-\lambda)^{k}u + \lambda(T-\lambda)^{k-1}u \\ &= \lambda(T-\lambda)^{k-1}u \end{aligned}$$

$$
\begin{aligned}
T(T-\lambda)^{k-2}u &= (T-\lambda)^{k-2}Tu \\
&= (T-\lambda)^{k-2}\big((T-\lambda)u + \lambda u\big) \\
&= (T-\lambda)^{k-1}u + \lambda(T-\lambda)^{k-2}u
\end{aligned}
$$

$$\vdots$$

$$
\begin{pmatrix}
\lambda & 1 & 0 & 0 & & 0 \\
0 & \lambda & 1 & 0 & & \vdots \\
\vdots & 0 & \lambda & 1 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \lambda & & \vdots \\
0 & 0 & 0 & 0 & & \lambda
\end{pmatrix}
$$

$\downarrow$

Jordan block of type λ

<u>Quick Summary:</u>

Last time we saw that

$$
[T]_B = \begin{bmatrix}
\boxed{J_1} & & & \\
& \boxed{J_2} & & \\
& & \ddots & \\
& & & \boxed{J_k}
\end{bmatrix}
$$

J_i - Jordan blocks on $\mathcal{E}_{\lambda_i}$

E_{λ_1} can be decomposed as direct sum of cyclic subspaces and on each cyclic subspace the matrix looks like Jordan block of type λ.

Defn: A $n \times n$ matrix is said to be of Jordan form if A is block diagonal

$$A = \begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \ddots & & \\ & & & & J_k \end{pmatrix}$$

each J_i is Jordan block of type λ.

Thm: Let $T: V \to V$ be a linear map. Then there exists a basis in which T has Jordan form.

Examples: $A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$

Find Jordan form of A.

Step 1 : Find characteristic polynomial of A and factorize it, over $\mathbb{C}$ to get all factors.

$$\mathrm{Char}_A(x) = \det \begin{pmatrix} -2-x & 2 & 1 \\ -7 & 4-x & 2 \\ 5 & 0 & -x \end{pmatrix}$$

$$= (-2-x)\Big((4-x)(-x)\Big) - 2\Big(7x-10\Big) + \Big((x-4)5\Big)$$

$$= (-2-x)\left(-4x+x^2\right) - 14x + 20 + 5x - 20$$

$$= 8x - 2x^2 + 4x^2 - x^3 - 14x + 5x$$

$$= -x^3 + 2x^2 - x = -x\left(x^2-2x+1\right)$$

$$= -x\,(x-1)^2$$

<u>Step 2:</u> Find all eigenvalues of $char_A(x)$.

0 and 1

<u>Step 3:</u> Find dimension of eigenspaces

$\dim(\ker(A-\lambda))$ for λ eigenvalue

$$\dim(\ker A) = 1$$

$$\dim(\ker(A-I)) = 1 < 2$$

We compute $\dim(\ker(A-I)^2) = 2$

$$\text{Jordan form} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Inner Products

Example: Dot Product on $\mathbb{R}^n$

$$\alpha \cdot \beta \in \mathbb{R}$$

Defn: An inner product on V is a $(V/\mathbb{C})$ function which assigns to each ordered pair of vectors (v_1, v_2) a scalar, such that the following hold:

i) $(v_1 + v_2, v_3) = (v_1, v_3) + (v_2, v_3)$

ii) $(cv_1, v_2) = c(v_1, v_2)$

iii) $(v_2, v_1) = \overline{(v_1, v_2)}$

iv) $(v, v) \geq 0$

Example: 1) $\alpha = (x_1, x_2) \in \mathbb{R}^2$

$\beta = (y_1, y_2) \in \mathbb{R}^2$

$$\langle \alpha, \beta \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$$

2) $V = M_{n \times n}(\mathbb{C})$

$$\langle A, B \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} A_{jk} \overline{B_{jk}}$$

$$\parallel$$

$$\operatorname{Trace}(AB^*)$$

$$B^*_{kj} = \overline{B_{jk}}$$