MATH 314 (Lecture 12)

Topics to be discussed today

Linear forms, duality

MIDTERM SOLUTIONS

1. $T : V \to V$

$T \circ T = T$

$$\varphi : V \longrightarrow \ker T \oplus \operatorname{Im} T$$
$$v \longmapsto (v - Tv \;,\; Tv)$$

1) φ is linear

$$v_1 + v_2 \longmapsto (v_1 + v_2 - Tv_1 - Tv_2 \;,\; Tv_1 + Tv_2)$$
$$= (v_1 - Tv_1 \;,\; Tv_1)$$
$$+ (v_2 - Tv_2 ,\; Tv_2)$$

2) φ is injective

$$(v - Tv ,\; Tv) = (0, 0)$$
$$\Rightarrow \quad Tv = 0 \;\&\; v = Tv = 0$$

So, $\ker \varphi = \{0\}$

3) φ is surjective

$$\varphi : V \longrightarrow \operatorname{Ker} T \oplus \operatorname{Im} T$$
$$v \longmapsto (v - Tv,\ Tv)$$

Suppose $(w_1, w_2) \in \operatorname{Ker} T \oplus \operatorname{Image} T$

Given $\begin{cases} w_2 = T(v) & \text{for some } v \\ T(w_1) = 0 \end{cases}$

$$T(w_1 + w_2) = Tw_1 + Tw_2 = Tw_2$$
$$= T(Tv)$$
$$= Tv$$

$$w_1 + w_2 \longmapsto \left(w_1 + w_2 - T(w_1 + w_2),\ T(w_1 + w_2)\right)$$
$$= (w_1, w_2)$$

Surjectivity holds.

2) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

a) $\mathbb{R}^n \supseteq \text{Image}(T) \supseteq \text{Image}(T^2) \dots$

$$\text{Image}(T^i) \supseteq \text{Image}(T^{i+1})$$

$v \in \text{Image}(T^{i+1})$

$$v = T^{i+1}(w) = T^i(Tw)$$

$$\Downarrow$$

$$v \in \text{Image}(T^i)$$

b) Dimensions either stay constant or strictly derease at each step, it cannot decrease forever so it must be equal at some step.

$$\text{Image}(T^m) = \text{Image}(T^{m+1})$$

If $k > m$

$v \in \text{Image}(T^k)$

$$
\begin{aligned}
v &= T^{k}(w) \\
&= T^{k-m} \circ T^{m}(w) \\
&= T^{k-m} \circ T^{m+1}(w') \\
&= T^{k+1}(w'), \quad v \in \text{Image}(T^{k+1})
\end{aligned}
$$

$v \in \text{Image}(T^{k+1})$

$$
\begin{aligned}
v &= T^{k+1}(w) \\
&= T^{k-m} \circ T^{m+1}(w) \\
&= T^{k-m} \circ T^{m}(w') \\
&= T^{k}(w')
\end{aligned}
$$

$v \in \text{Image}(T^{k})$

So, $\text{Image}(T^{k}) = \text{Image}(T^{k+1})$

c) $W = \text{Image}(T^{m})$

If $w \in W$, $w = T^{m}(v)$

$$= T(T^{m} v)$$

$= T(w') \quad w' \in W$

So, $T: W \to W$ is surjective

Problem 3)

$$M_k = \begin{pmatrix} 1 & k-5 \\ 0 & 10-k \\ 1 & 5-k \\ -k-3 & 0 \end{pmatrix}$$

a) $$B_k = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{10-k} & 0 & 0 \end{pmatrix} \quad k \neq 10$$

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{10} & 0 & \frac{-1}{10} & 0 \end{pmatrix} \quad k = 10$$

b) $M_k B_k = I$ not possible

$\text{Rank}(M_k B_k) \leq 2$

$\text{Rank}(I) = 4$

Problem 4) $AB = BA$ for all invertible matrices B

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ & & \ddots & 0 \\ & 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -a_{11} & a_{12} & \cdots & a_{1n} \\ -a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ -a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

$a_{12} = a_{13} = \cdots = a_{1n} = 0$, $a_{21} = 0 = \cdots = a_{n1}$

Proceeding in a similar fashion, show that A is a diagonal matrix.

$$A = \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix}$$

$$\begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & - & - & 0 \\ 0 & 0 & 1 & - & 0 \\ & & \vdots & & \\ 0 & 0 & - & - & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & c_1 & 0 & \cdots & 0 \\ c_2 & 0 & - & - & 0 \\ 0 & 0 & c_3 & - & 0 \\ \vdots & 0 & & \ddots & 0 \\ \vdots & - & - & - & c_n \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdot & - & 0 \\ 0 & 0 & 1 & 0 & 0 \\ & & \ddots & & 0 \\ & 0 & & \ddots & 1 \end{pmatrix} \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & c_2 & - & - & \cdot & 0 \\ c_1 & 0 & - & - & - & 0 \\ 0 & 0 & c_3 & - & - & 0 \\ \vdots & 0 & & \ddots & 0 & \vdots \\ \vdots & - & - & - & - & c_n \end{pmatrix}$$

$c_1 = c_2$

Similarly, we can show that

$c_1 = c_2 \cdots = c_n$.

Problem 5: $AB - BA = I$

$$\text{Trace}(AB - BA) = \text{Trace}(AB) - \text{Trace}(BA)$$

$$\text{Trace}(AB) = \text{Trace}(BA)$$

$$\text{Trace}(AB - BA) = 0$$

$$\text{Trace}(I) = n \quad \text{not possible}$$

$$\text{Trace}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ji} A_{ij}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} B_{ji} A_{ij} = \sum_{j=1}^{n} (BA)_{jj}$$

$$= \mathrm{Trace}(BA)$$

Quotient Spaces

V vector space

W subspace

$$V/W = \{ v + W \mid v \in V \}$$

Examples: 1) $W = \{0\}$

$$V/\{0\} = V$$

2) $V = \mathbb{R}^2 \qquad W = \{(0,y) \mid y \in \mathbb{R}\}$

$$(a,b) \sim (c,d) \iff (a-c,\ b-d) \in W$$
$$\iff a = c$$

So, coset is determined by its x-co-ordinate.

$$V/W = \{(x,0) \mid x \in \mathbb{R}\}$$

3) $V = \mathbb{C}$ $\quad W = \mathbb{R}$

V/W

$$a+bi \sim c+di \iff a-c+(b-d)i \in \mathbb{R}$$
$$\iff b=d$$

$$V/W = \{ bi \mid b \in \mathbb{R} \}$$

Thm: Let $T: V_1 \longrightarrow V_2$ be a linear map.

Then $V_1/\ker T \cong \text{Image}(T)$

$\cong$ isomorphic

Proof: Define $\varphi: V_1/\ker T \longrightarrow \text{Image}(T)$

$$v + \ker T \longmapsto T(v)$$

1) Need to check φ is well defined

if $v_1 + \ker T = v_2 + \ker T$

then $v_1 - v_2 \in \ker T$

$T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$

So $\varphi(v_1 + \ker T) = \varphi(v_2 + \ker T)$

2) We show that φ is linear

$$\varphi((v_1 + \ker T) + (v_2 + \ker T))$$
$$= \varphi((v_1 + v_2) + \ker T) = T(v_1 + v_2)$$
$$= Tv_1 + Tv_2 = \varphi(v_1 + \ker T) + \varphi(v_2 + \ker T)$$

3) We will show that φ is injective

If $\varphi(v + \operatorname{Ker} T) = T(v) = 0$

$\Rightarrow v \in \operatorname{Ker} T \Rightarrow v + \operatorname{Ker} T$ is the zero coset, $0 + \operatorname{Ker} T$

4) φ is surjective

Suppose $w \in \operatorname{Image}(T)$, i.e $w = Tv$

Then $w = \varphi(v + \operatorname{Ker} T)$

Corollary: Rank - Nullity Thm :

$$\dim\left(V / \operatorname{Ker} T \right) = \dim(\operatorname{Image} T)$$

$$\dim V - \dim(\operatorname{Ker} T) = \dim(\operatorname{Image} T)$$

$$\dim V = \dim(\operatorname{Ker} T) + \dim(\operatorname{Image} T)$$

Linear Functionals

Let V be a vector space over a field F.

A *linear functional* is a linear transformation $\varphi : V \longrightarrow F$.

Examples: 1) Suppose $V = F^n$

Fix $a_1, a_2, \ldots, a_n \in F$.

$$\varphi : F^n \longrightarrow F$$

$$(x_1, \ldots, x_n) \longmapsto a_1 x_1 + \cdots + a_n x_n$$

In fact, every linear functional is of this form. If we fix a basis $\{b_1, \ldots, b_n\}$ for F^n and say

$$\varphi(b_i) = a_i$$

then $\varphi(v) = \varphi\left(\sum c_i b_i\right) = \sum c_i a_i$

2) Trace: $M_{n\times n}(F) \to F$

$$A \mapsto \text{Trace}(A)$$

3) Evaluation_a: Vector space of all polynomials with coefficients in F $\longrightarrow F$

Fix some $a \in F$

$$f(t) \longmapsto f(a)$$

4) V = Vector space of all continuous fns on $[a,b]$

$$\varphi: V \longrightarrow \mathbb{R}$$

$$f \longmapsto \int_a^b f(x)\,dx$$

$$V \longrightarrow V^*$$

Vector Space $\qquad\qquad$ Dual Space

$$V^* = \left\{ \varphi : V \to F \;\middle|\; \varphi \text{ linear functional} \right\}$$

$$= L(V, F)$$

We already know that V^* is a vector space and $\dim(V^*) = (\dim V)(\dim F)$
$= \dim V.$

Let's see an explicit basis of V^*.

Fix a basis $\{b_1, \ldots, b_n\}$ of V.

Define $f_i : V \to F$ as follows

$$f_i(b_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\{f_1, \ldots, f_n\}$ is linearly independent because if

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

$$\Rightarrow (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)(b_i) = 0 \quad \text{for } i \in \{1, \ldots, n\}$$

$$\Rightarrow \quad c_i = 0 \quad \text{for } i \in \{1, \ldots, n\}$$

If $\varphi : V \to F$ is a linear map, then

Say $\varphi(b_i) = a_i$

<u>Claim:</u> $\varphi = \sum_{i=1}^{n} a_i f_i$

<u>Pf:</u> $\varphi(b_j) = \sum a_i f_i(b_j)$

$$= a_j$$

Since $\varphi = \sum_{i=1}^{n} a_i f_i$ on basis, they are equal everywhere.