MATH 314 (Lecture 1)

Topics to be discussed today

Fields, Vector Spaces

Textbook sections 1.1, 2.1, 2.2 up to Thm 1

Example of rational numbers $(\mathbb{Q})$

$$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; \begin{array}{l} m \in \mathbb{Z} \\ n \in \mathbb{Z},\ n \neq 0 \end{array} \right\}$$

Addition $(+)$	Multiplication $(\cdot)$
1) $\forall\, x, y \in \mathbb{Q},\ x+y \in \mathbb{Q}$ Closure Property	$\forall\, x, y \in \mathbb{Q},\ x \cdot y \in \mathbb{Q}$
2) $x+(y+z) = (x+y)+z$ Associative	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
3) Existence of additive identity $x+0 = 0+x = x$	$x \cdot 1 = 1 \cdot x = x$
4) Existence of additive inverse $x+(-x) = (-x)+x = 0$	If $x \neq 0$ $x \cdot \left(\frac{1}{x}\right) = 1 = \left(\frac{1}{x}\right) \cdot x$
5) Commutative $x+y = y+x$	$x \cdot y = y \cdot x$

These two operations interact with each other via <u>distributivity</u>.

$$x\cdot(y+z) = x\cdot y + x\cdot z \quad \forall\, x, y, z \in \mathbb{Q}$$

The set of rational numbers is an example of <u>field</u>.

<u>Defn:</u> A set $(F, +, \cdot)$ is called a field if it satisfies all of the above properties.

<u>Excercise:</u> Consider the following sets. Which of them is a field?

i) $(\mathbb{Z}, +, \cdot)$ ii) $(\mathbb{R}, +, \cdot)$ iii) $(\mathbb{C}, +, \cdot)$

iv) Set of 2x2 matrices over integers with usual addition and multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

<u>Lemma</u> 1) In a field F, additive identity & multiplicative identity are unique.

<u>Pf:</u> Suppose there are two additive identities 0_1 & 0_2. Then

$$0_1 + 0_2 = 0_1 = 0_2$$

Similarly, if there are two multiplicative identities 1_1 & 1_2, then

$$1_1 \cdot 1_2 = 1_1 = 1_2$$

Hence, shown.

Lemma 2 : Let F be a field. For a fixed $x \in F$, there are no two additive inverses. If $x \neq 0$, there are no two multiplicative inverses.

Pf: Suppose there are, call them x_1 & x_2. We want to show that $x_1 = x_2$.

$$x + x_1 = 0 = x + x_2$$

Add x_1 throughout to get

$$x_1 + x + x_1 = x_1 + 0 = x_1 + x + x_2$$

$$0 + x_1 = x_1 = 0 + x_2$$

$$\Rightarrow \boxed{x_1 = x_2}$$

Similarly we can show that if there are two multiplicative inverses, then they are same.

$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

$\mathbb{Q}$ is a subfield of $\mathbb{R}$ & $\mathbb{C}$.

$\mathbb{R}$ ———————— $\mathbb{C}$.

<u>Defn:</u> A subfield of a field F is a subset of F that is a field in its own right.

<u>Discussion qun:</u>

What is the smallest field that you can think of?

Answer depends on 1 (the multiplicative identity of F)

This leads to notion of characteristic of a field.

1

1+1

1+1+1

⋮

Do you ever get 0?

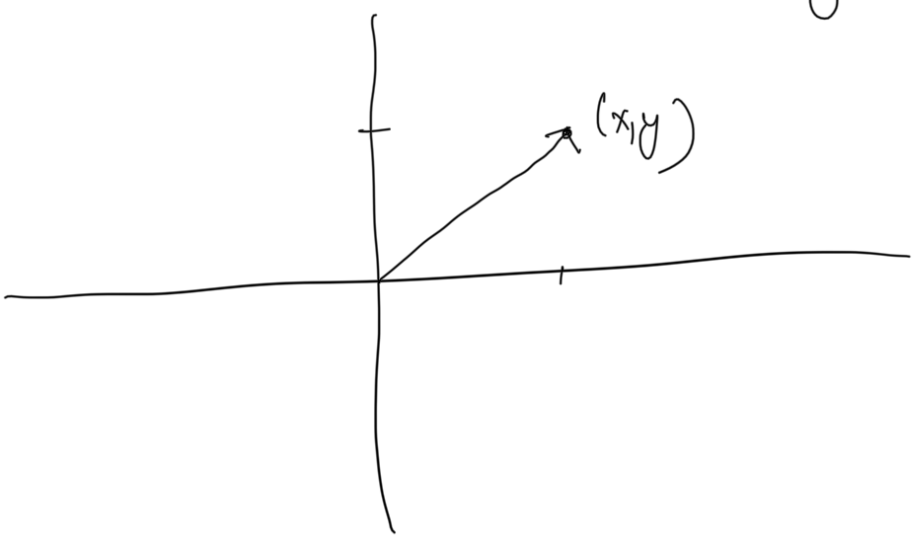
Answer depends on the field, in $\mathbb{Q}$ no.

Such fields are called Characteristic 0 field.

We now come to one of the central objects of study in this class, Vector Spaces.

Example $\mathbb{R}^2$

Each point can be thought of as a vector



Like we had addition & multiplication in $\mathbb{R}$ we have <u>vector addition</u> & <u>scalar multiplication</u> in $\mathbb{R}^2$.

Vector addition $(x,y) + (a,b) = (x+a, y+b)$

Associative $((x,y)+(a,b))+(c,d) = (x,y) + ((a,b)+(c,d))$

Commutative

$(x,y) + (a,b) = (a,b) + (x,y)$

Existence of zero vector

$$(x,y) + (0,0) = (x,y)$$

Existence of additive inverse

$$(x,y) + (-x,-y) = (0,0)$$

Scalar Multiplication

for $\alpha \in \mathbb{R}$, $\alpha \cdot (x,y) = (\alpha x, \alpha y)$

$$1 \cdot (x,y) = (x,y)$$

$$(\alpha\beta)(x,y) = \alpha(\beta(x,y))$$

$$\alpha((x,y)+(a,b)) = \alpha(x,y) + \alpha(a,b)$$

$$(\alpha+\beta)(x,y) = \alpha(x,y) + \beta(x,y)$$

A vector space V over a field F is a set that satisfies all of these properties.

<u>Example</u> $\mathbb{P}_n = \{$ Set of all polynomials of degree up to n with Coefficients in $\mathbb{R}$ $\}$

$$\begin{array}{r} a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\ + \; b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \\ \hline (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n \end{array}$$

$$\alpha \left(a_0 + a_1 x + \cdots + a_n x^n \right) = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n$$

<u>Example:</u> $\mathbb{C}$ is a vector space over $\mathbb{R}$.

$$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$$

Think of $\mathbb{C}$ as $\mathbb{R}^2$

We will end with concept of subspaces that will be used a lot.

A subspace of a vector space is a subset that is vector space in its own right.

Examples: $\mathbb{R} \subseteq \mathbb{C}$

$$\mathbb{P}_{n-1} \subseteq \mathbb{P}_n$$

Thm: A non-empty subset W of V is a subspace of V $\Longleftrightarrow$ for every $x, y \in W$ and each scalar $\alpha \in F$ we have $\alpha x + y \in W$

Pf: $\Rightarrow$ $\alpha x \in W$ (Scalar multiplication)
$\alpha x + y \in W$ (addition)

$\Leftarrow$: Take $\alpha = 1$, $x + y \in W$ (closure)
Associativity & Commutative property follows by W being a subset.

Take $\alpha = -1$

$(-1)x$ is additive inverse of x

$$x + (-1)x = (1 + (-1))x$$
$$= 0 \cdot x$$

$$0 \cdot x = (0 + 0) \cdot x$$
$$= 0 \cdot x + 0 \cdot x$$

Add additive inverse of $0 \cdot x$ to both sides to get

$$0 \cdot x = 0$$

Finally 0 is in W because W is non-empty, choose $x \in W$

then $-x \in W$

Hence $\underbrace{x + (-x)}_{0} \in W$

$\alpha x \in W \quad \forall x \in W, \alpha \in F$

Closed under scalar multiplication.