MATH 314 (Lecture 16)

Topics to be discussed today

Determinant of an endomorphism

Recall: Determinant is an alternating function

$$Det : \underbrace{F^n \times \cdots \times F^n}_{n \text{ copies}} \longrightarrow F$$

Such that $Det(e_1, e_2, .., e_n) = 1$.

Using this definition we can compute

$$Det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\shortparallel$$

$$Det\big((a,c),(b,d)\big)$$

More generally,

$$Det(A_1, A_2, \ldots, A_n) =$$

$$\sum_{\substack{\sigma \\ \text{Permutation on} \\ n \text{ letters}}} \text{Signature}(\sigma)\, A_{\sigma(1),1} A_{\sigma(2),2} \cdots A_{\sigma(n),n}$$

Some Properties of Determinants :

1) $\text{Det}(A^t) = \text{Det}(A)$

$$\text{Det}(A^t) = \sum_{\sigma} \text{Signature}(\sigma) A^t_{\sigma(1),1} \cdots A^t_{\sigma(n),n}$$

$$= \sum_{\sigma} \text{Signature}(\sigma) A_{1,\sigma(1)} \cdots A_{n,\sigma(n)} \quad \text{—— (1)}$$

Say $\sigma(i) = j \iff i = \sigma^{-1}(j)$

$$(i, \sigma(i)) = (\sigma^{-1}(j), j)$$

$$A_{1,\sigma(1)} \cdots A_{n,\sigma(n)} = A_{\sigma^{-1}(1),1} \cdots A_{\sigma^{-1}(n),n}$$

Since $\text{Signature}(\sigma\sigma^{-1}) = \text{Signature}(id) = 1$

$$\text{Signature}(\sigma)\,\text{Signature}(\sigma^{-1}) = 1$$

$$\Rightarrow \text{Signature}(\sigma^{-1}) = \text{Signature}(\sigma)$$

As σ varies over all permutations, So does σ^{-1}

Rewrite ① as

$$\text{Det}(A^t) = \sum_{\sigma^{-1}} \text{Signature}(\sigma^{-1}) A_{\sigma^{-1}(1),1} \cdots A_{\sigma^{-1}(n),n}$$

$$= \text{Det}(A)$$

2) $\text{Det}(AB) = \text{Det}(A)\,\text{Det}(B)$

Fix a matrix B.

Define $H : M_{n\times n}(F) \to F$

$$H(A) = \text{Det}(AB)$$

If Rows of A are $R_1, R_2, \ldots, R_n$, then Rows of AB are $R_1B, \ldots, R_nB$.

$$H(R_1, \ldots, R_n) = \text{Det}(R_1B, \ldots, R_nB)$$

Claim: ① H is multilinear function.

For any $i \in \{1, 2, \ldots, n\}$

$$H(R_1, \ldots, cR_i^1 + R_i^2, \ldots, R_n)$$

$$= \mathrm{Det}\left(R_1B, \dots, (cR_i^1 + R_i^2)B, \dots, R_n\right)$$

$$= \mathrm{Det}\left(R_1B, \dots, c(R_i^1 B) + R_i^2 B, \dots, R_n\right)$$

$$\shortparallel$$

$$c\,\mathrm{Det}\left(R_1B, \dots, R_i^1 B, \dots, R_n\right)$$

$$+$$

$$\mathrm{Det}\left(R_1B, \dots, R_i^2 B, \dots, R_n\right)$$

(2) H is alternating, because Det is alternating.

(1) & (2) together imply that

$$H(A) = \mathrm{Det}(A)\, H(I)$$

but $H(I) = \mathrm{Det}(B)$

$$\Rightarrow \quad \mathrm{Det}(AB) = \mathrm{Det}(A)\,\mathrm{Det}(B)$$

Computing Determinants

① Using Row Reduction

Row Operations	Effect on Det
Switching two rows	Multiplying by -1
Multiplying a row with a scalar	Multiplying by that scalar
Adding two rows	No change

↓

this is because

$$\text{Det}(R_1 + cR_2, R_2, R_3) = \text{Det}(R_1, R_2, R_3) + \text{Det}(cR_2, R_2, R_3)$$

$$= \text{Det}(R_1, R_2, R_3)$$

Example:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

① $R_1 \longleftrightarrow R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = P_0 \qquad \text{Det}(P_0) = -\text{Det } A$$

② $R_2 \to R_2 + R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = P_1 \qquad \text{Det}(P_1) = \text{Det}(P_0) = -\text{Det } A$$

③ $R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = P_2 \qquad \text{Det}(P_2) = \text{Det}(P_1) = -\text{Det}(A)$$

(4) $R_2 \to R_2 - R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P_3 \qquad \operatorname{Det}(P_3) = \operatorname{Det}(P_2)$$

$$= -\operatorname{Det}(A)$$

$P_3 = I$, so $\operatorname{Det}(P_3) = 1$

$$\Rightarrow \boxed{\operatorname{Det}(A) = -1}$$

Inverse of a matrix

We can compute inverses using adjoints.

Thm: If A is invertible, its inverse is

$$A^{-1} = (\operatorname{adj} A)(\det A)^{-1}$$

Example:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Remove i-th row & j-th column and take determinant of remaining 2×2 matrix, multiply that by $(-1)^{i+j}$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A^{-1} = \Big(0$$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A' = \left(\begin{matrix} 0 & 0 & \\ & & \\ & & \end{matrix} \right.$$

... Continue like this

$$A' = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\text{adj } A = (A')^T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = -\text{adj } A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det A(j|i)$$

$$((\operatorname{adj} A) A)_{ij} = \sum_{k=1}^{n} (\operatorname{adj} A)_{ik} A_{kj}$$

$$= \underbrace{\sum_{k=1}^{n} (-1)^{i+k} \det A(k|i) A_{kj}}_{*}$$

If $i \neq j$ $*$ is 0

If $i = j$ $*$ is $\det A$

$$(\operatorname{adj} A) A = (\det A) I$$

If A is invertible, $\det A \neq 0$

$(\operatorname{adj} A)(\det A)^{-1}$ is the inverse of A.

Another method of computing inverse is to solve a system of n^2 equations.

Suppose one wants to solve

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

We want to solve $AX = b$

If $\quad Ax = b$

then $\quad (\operatorname{adj} A) A X = (\operatorname{adj} A) b$

$$(\det A) X = (\operatorname{adj} A) y$$

↳ this method is useless if $\det A = 0$

Revisiting Calculus : Alternating forms

A differential form on an open Subset of $\mathbb{R}^n$ is an expression of the form $f_1(x_1, \ldots, x_n)\,dx_1 + f_2(x_1, \ldots, x_n)\,dx_2 + \cdots + f_n(x_1, \ldots, x_n)\,dx_n$

Wedge products give us 2-forms

Example: $(x\,dx + y\,dy + z\,dz) \wedge (e^x\,dx + x\,dy + 3\,dz)$

$\downarrow$

Alternating form

$$xe^x\,dx\wedge dx + x^2\,dx\wedge dy + 3x\,dx\wedge dz$$
$$+ e^x y\,dy\wedge dx + yx\,dy\wedge dy + 3y\,dy\wedge dz$$
$$+ e^x z\,dz\wedge dx + zx\,dz\wedge dy + 3z\,dz\wedge dz$$

$$= (x^2 - e^{x}y)\, dx \wedge dy + (3x - e^{x}z)\, dx \wedge dz + (3y - 2x)\, dy \wedge dz$$

Two-forms on $\mathbb{R}^3$ are combinations of $dx \wedge dy$, $dx \wedge dz$, $dy \wedge dz$.

Similarly three-forms on $\mathbb{R}^3$ are combinations of $dx \wedge dy \wedge dz$.

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k-forms on $\mathbb{R}^3$ for $k > 3$ vanish.