MATH 314 LINEAR ALGEBRA, SPRING 2024, MIDTERM 1

Problem 1 [5 points] Let V be a finite dimensional vector space. Let $T\colon V \to V$ be a linear map that satisfies $T \circ T = T$. Construct a linear map between vector spaces V and $ker(T) \oplus image(T)$ that is both injective and surjective.

Problem 2 [2+5+2 points] Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

- Show that there is the following containment of subspaces
$$\mathbb{R}^n \supseteq Image(T) \supseteq Image(T^2) \supseteq Image(T^3) \supseteq \ldots.$$
- Show that for some positive integer $m \geq 1$, there is equality
$$Image(T^k) = Image(T^{k+1})$$
for all $k \geq m$.
- Let $W = Image(T^m)$ for the m in the previous part. Show that T when restricted to W is surjective.

Problem 3 [3+3 points] For any real number k, consider the 4×2 matrix M_k
$$\begin{pmatrix} 1 & k-5 \\ 0 & 10-k \\ 1 & 5-k \\ -k-3 & 0 \end{pmatrix}$$

- For every value of k find a 2×4 matrix B_k such that $B_k M_k$ is the identity matrix I_2.
- Show that for every value of k there exists no 2×4 matrix A_k such that $M_k A_k$ is the identity I_4.

Problem 4[5 points] Let A be an $n \times n$ matrix. If $AB = BA$ for all invertible matrices B, show that $A = cI$ for some scalar c.

Problem 5[5 points] There are no square matrices A, B with the property that $AB - BA = I$. Either prove this statement or provide a counterexample.

Date: Tuesday 4^{th} June, 2024.