MATH 314 (Lecture 10)

Topics to be discussed today

Ranks, System of equations, congruences

Rank

Recall: $A_{m\times n}$ matrix with entries in F

$Row(A) = \{\text{Span of rows of } A\}$
$\subseteq F^n$

$\dim(Row(A)) = rank(A)$
$\quad\;\; \|$
$\dim(Col(A))$

└ Column Space
$\quad\;\; \|$
$\{\text{Span of Columns of } A\}$

→ How do we actually compute $Row(A)$?

Ans: Row reduce A to its echelon form and observe that

$Row(\text{Echelon form of } A) = Row(A)$

Example: Finding inverses

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$a \neq 0$

$R_1 \to \frac{1}{a} R_1$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 - cR_1$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - \frac{cb}{a} & -c/a & 1 \end{array}\right)$$

$a = 0$

$R_1 \longleftrightarrow R_2$

$$\left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right)$$

If $b = 0$, not invertible

else

$R_2 \to \frac{1}{b} R_2$

$$\left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & 1 & 1/b & 0 \end{array}\right)$$

If $c = 0$, not invertible

If $ad-bc=0$,
matrix is not invertible

If $ad-bc \neq 0$

$R_2 \to \frac{a}{ad-bc} R_2$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$R_1 \to R_1 - b/a\, R_2$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{b}{a}\left(\frac{-c}{ad-bc}\right) & \frac{-b}{a}\left(\frac{a}{ad-bc}\right) \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$\longrightarrow$ Inverse

else $R_1 \to \frac{1}{c} R_1$

$$\left(\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right)$$

$R_1 \to R_1 - \frac{d}{c} R_2$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{-d}{bc} & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right)$$

$\downarrow$

Inverse

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, Row space is $\mathbb{R}^2$.

else Row Space is $\{0\}$ or 1-dimensional.

Thm: Suppose W is a subspace of F^n and $\dim W \leq m$. Then, there is exactly one $m \times n$ row reduced echelon matrix over F which has W as its row space.

Proof: Example: $W = \text{Span}\{(a,b)\}$

$a=0,\ b\neq 0$ $\quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$a\neq 0,\ b=0$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$a\neq 0,\ b\neq 0$ $\quad \begin{bmatrix} 1 & b/a \\ 0 & 0 \end{bmatrix}$

$a=0,\ b=0$ $\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

In general, if β is a non-zero vector in W, then

$$\beta = (0, \ldots, 0, b_t, \ldots, b_n), \quad b_t \neq 0$$

Collect all indices i s.t there exist a non-zero vector in W, with its first non-zero coordinate in i-th position.

Arrange them in increasing order

$$i_1 < i_2 < \cdots < i_r$$

Let ρ_1 be the vector $(0, 0, \ldots, \underset{i_1}{1}, 0, \ldots, 0)$

ρ_2 be the vector $(0, 0, \ldots, 0, \underset{i_2}{1}, \ldots, 0)$

$\vdots$

$$R_{m\times n} = \begin{bmatrix} - \rho_1 - \\ - \rho_2 - \\ \vdots \\ - \rho_r - \\ - 0 - \\ \vdots \\ - 0 - \end{bmatrix}$$

Corollary: Row reduced echelon form is unique.

Corollary: A & B are row equivalent
$\Longleftrightarrow$ A & B have same row space.

Towards Quotient of vector spaces

Recall: When we discussed proof of "Every vector space has a bases" we talked about Partially Ordered Sets (POSETS).

(RELATIONS) Let X be a set. A relation R on X is a subset of $X \times X$.

For example: if $X = \{1, 2, 3\}$

No. of relations on X = No. of Subsets of $X \times X$

$$= 2^9$$

We say that $x \sim y \iff (x, y) \in R$.

if $R = \{(1,1), (1,2), (1,3)\}$, then

$1 \sim 1$, $1 \sim 2$, $1 \sim 3$ but $3 \not\sim 1$.

EQUIVALENCE RELATION

<u>Defn:</u> We say a relation R is an equivalence relation if the following hold:

1) R is reflexive. ($x \sim x \ \forall x \in R$)
2) R is symmetric (If $x \sim y$, then $y \sim x$)
3) R is transitive (If $x \sim y$, $y \sim z$, then $x \sim z$)

<u>Examples or non-examples:</u>

1) $X = \mathbb{R}$, $x \sim y \iff x < y$
2) $X = \mathbb{R}$, $x \sim y \iff x = y$
3) $X = \mathbb{R}^2$, $x \sim y \iff x = Ay$ for some $A \in GL_2(\mathbb{R})$

4) $X = \mathbb{Z}$

Fix a natural number N.

$x \sim y \iff N \mid x - y$.

EQUIVALENCE CLASSES

Let X be a set. Suppose R is an equivalence relation on X.

Equivalence class of $a = \{ y \in X \mid a \sim y \}$

We will denote equivalence class of a as $\overline{a}$.

Some observations.

1) $a \in \overline{a}$

2) If $a \not\sim b$, then
$\overline{a} \cap \overline{b} = \phi$

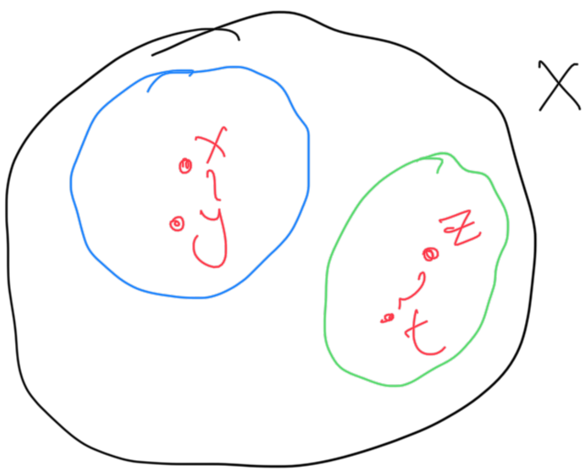
because if $z \in \bar{a} \Rightarrow a \sim z$

& $z \in \bar{b} \Rightarrow b \sim z$

$b \sim z \Rightarrow z \sim b$ (Reflexive)

$a \sim z$ and $z \sim b \Rightarrow a \sim b$ (Transitive)

Contradiction!



So, we can write X as a disjoint union of equivalence classes.

$$X = \bigsqcup_{\alpha \in I} \bar{a}_\alpha$$

I — indexing set of X

Let us compute equivalence classes for examples above.

1) $X = \mathbb{R}$, $\quad x \sim y \iff x = y$

$$\bar{x} = \{x\}$$

2) $X = \mathbb{R}^2$, $\quad x \sim y \iff x = Ay$ for some $A \in GL_2(\mathbb{R})$

If $x = (0,0)$, then

$\bar{x} = x$

If $x = (a,b)$ $\quad ab \neq 0$

then $x \sim (1,0)$ Take $A = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ if $a \neq 0$

$A = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ if $b \neq 0$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

So, $\bar{x} = \mathbb{R}^2 - \{(0,0)\}$

$$\mathbb{R}^2 = \overline{(0,0)} \sqcup \overline{(1,0)}$$

3) $X = \mathbb{Z}$, $\quad x \sim y \iff N \mid x-y$

What are its equivalence classes?

If $a, b \in \mathbb{Z}$ $\quad a < b$, $a \neq 0$ then

$b = aq + r \qquad r \in \{0, 1, 2, \dots, a-1\}$

Suppose $N = 2$

$$\bar{1} = \{\text{Odd integers}\}$$

$$\bar{0} = \{\text{Even integers}\}$$

$$\mathbb{Z} = \bar{0} \sqcup \bar{1}$$

$N = 3$

$$\bar{0} = \{\text{all multiples of } 3\}$$

$$\bar{1} = \{3m+1 \mid m \in \mathbb{Z}\}$$

$$\bar{2} = \{3m+2 \mid m \in \mathbb{Z}\}$$

$$\mathbb{Z} = \bar{0} \sqcup \bar{1} \sqcup \bar{2}$$

In general, we have

$$\mathbb{Z} = \bar{0} \sqcup \bar{1} \sqcup \bar{2} \cdots \sqcup \overline{N-1}$$

In the context of vector spaces

Let V be a vector space over field F. Let W be a subspace of V.

Define R on V as follows

We say $v_1 \sim v_2 \iff v_1 - v_2 \in W$

(We say that $v_1 \equiv v_2 \mod W$ if $v_1 \sim v_2$)

Qn: Is it an equivalence relation?

Equivalence classes are known as <u>COSETS</u> of W.

$$\alpha \in V, \quad \bar{\alpha} = \{\beta \in V \mid \beta - \alpha \in W\}$$
$$= \{\beta \in V \mid \beta = \alpha + w \text{ for some } w \in W\}$$

$$\bar{\alpha} = \underset{\substack{\downarrow \\ \text{vector}}}{\alpha} + \underset{\substack{\downarrow \\ \text{set}}}{W}$$

$$= \{\alpha + w \mid w \in W\}$$

<u>Notation:</u> We will denote the set of COSETS of W by V/W.

$$\bar{0} = W$$

Take two cosets $\alpha_1 + W$ and $\alpha_2 + W$

Define addition
$$(\alpha_1 + W) + (\alpha_2 + W) = (\alpha_1 + \alpha_2) + W$$

Scalar multiplication $\quad c(\alpha_1 + W) = c\alpha_1 + W$

Qn: Are these operations well-defined?

Qn: Is V/W a vector space with respect to these operations?