MATH 314 (Lecture 4)

Topics to be discussed today

Basis, exchange lemma

Let V be a vector space over a field F.

Let $S \subseteq V$ be a subset of V.

$$\text{Span}(S) = \left\{ \sum_{i=1}^{n} c_i v_i \;\middle|\; \begin{array}{l} v_i \in S \\ c_i \in F \end{array} \right\}$$

We will say that S is linearly independent if every finite subset of S is linearly independent.

(Recall from last time that a set $\{v_1, \ldots, v_n\}$ is linearly independent if the following statement is true

$$\sum c_i v_i = 0 \iff c_i = 0$$
)

Basis is a subset B of V such that

1) $\text{Span}(B) = V$

2) B is linearly independent.

Examples: 1) Consider $\mathbb{C}$ over $\mathbb{C}$

$B = \{1\}$ is a basis b/c

$\text{Span}(B) = \{c \cdot 1 \mid c \in \mathbb{C}\} = \mathbb{C}$

B is linearly independent.

Qn: Is B the only basis for $\mathbb{C}$ over $\mathbb{C}$?

If not, what are some other bases?

2) Consider $\mathbb{C}$ over $\mathbb{R}$.

$B = \{1, i\}$ is a basis.

3) Consider $\mathbb{R}^n$ over $\mathbb{R}$.

$$B = \left\{ \begin{array}{l} (1,0,\dots,0), \\ (0,1,\dots,0), \\ (0,0,1,\dots,0), \\ \quad\vdots \\ (0,0,\dots,1) \end{array} \right\} \text{ is a basis.}$$

4) Consider $M_{n\times n}(\mathbb{R})$ over $\mathbb{R}$

Define A_{ij} as matrix whose ij-th entry is 1 & every other entry is 0.

Then $\{A_{ij}\}_{i,j=1}^{n}$ form a basis.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

5) Consider the set of all polynomials with coefficients in $\mathbb{R}$. It is a vector space over $\mathbb{R}$. Then $B = \{1, x, x^2, \ldots\}$ forms a basis, because if f is a polynomial then

$$f = c_0 + c_1 x + \cdots + c_n x^n \qquad c_i \in \mathbb{R}$$

So, B spans.

If $$a_1 x^{i_1} + a_2 x^{i_2} + \cdots + a_n x^{i_n} = 0$$

$$i_1 < i_2 < \cdots < i_n$$

$\downarrow$

(Think of this as a polynomial of degree i_n)

then $a_1 = a_2 = \cdots = a_n = 0$

<u>Lemma</u> If $S \subseteq V$ is linearly independent &

$T \subseteq S$, then T is also linearly independent.

<u>Pf:</u> If $c_1 t_1 + c_2 t_2 \cdots + c_n t_n = 0$ for $t_i \in T$

& $c_i \in F$

then $c_i = 0$ for $i \in \{1, 2, \ldots, n\}$ because

$t_i \in S$ for $i \in \{1, 2, \ldots, n\}$ and S is

linearly independent.

<u>Lemma:</u> If $S \subseteq V$, is such that $\text{Span}(S) = V$

and $S \subseteq T$, then $\text{Span}(T) = V$

<u>Pf:</u>

$$S \subseteq T$$

$$\text{Span}(S) \subseteq \text{Span}(T) \subseteq V$$

$$\text{Span}(S) = V$$

$$\Rightarrow \boxed{\text{Span}(T) = V}$$

Thm: Let V be a vector space over F. Suppose $S = \{v_1, \ldots, v_n\}$ spans V, ie, $\text{Span}(S) = V$. Then any independent set of vectors is finite and contains no more than n elements.

Pf: We will show that a set of more than n elements is linearly dependent. Let $S = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be one such. $(m > n)$

$$\beta_1 = c_{11} v_1 + c_{21} v_2 + \cdots + c_{n1} v_n$$

$$\beta_2 = c_{12} v_1 + c_{22} v_2 + \cdots + c_{n2} v_n$$

$$\vdots$$

$$\beta_m = c_{1m} v_1 + c_{2m} v_2 + \cdots + c_{nm} v_n$$

Define $A = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nm} \end{pmatrix}$

Since $m > n$ $Ax = 0$ has a <u>non-trivial</u> solution ie $\exists (x_1, \ldots, x_m) \neq (0, 0, \ldots 0)$

s.t. $A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$x_1 \beta_1 + x_2 \beta_2 + \cdots + x_m \beta_m$$

$$= x_1 (c_{11} v_1 + c_{21} v_2 + \cdots + c_{n1} v_n)$$
$$+ x_2 (c_{12} v_1 + c_{22} v_2 + \cdots + c_{n2} v_n)$$
$$\vdots$$
$$+ x_m (c_{1m} v_1 + c_{2m} v_2 + \cdots + c_{nm} v_n)$$

Rearrange terms to get

$$= v_1 \left(c_{11} x_1 + c_{12} x_2 + \cdots + c_{1m} x_m \right)$$
$$+ v_2 \left(c_{21} x_1 + c_{22} x_2 + \cdots + c_{2m} x_m \right)$$
$$\vdots$$
$$+ v_n \left(c_{n1} x_1 + c_{n2} x_2 + \cdots + c_{nm} x_m \right)$$

$$= 0 + 0 + \cdots + 0 = 0$$

So, S is linearly dependent.

<u>Corollary 1:</u> If V is finite dimensional, i.e it has a basis of finite size, then any two bases have same cardinality.

<u>Pf:</u> Let B_1 & B_2 be two bases.

Say $|B_1| = m$ $|B_2| = n$

by previous thm $n \leq m$ &

$m \leq n$

So $m = n$.

Corollary 2: Suppose V has a finite basis of size n. Then

i) Any subset of V that contains more than n elements is linearly dependent.

ii) If S is a set such that $\text{Span}(S) = V$ then $|S| \geq n$.

Qn: Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

Is there a finite basis for $\mathbb{R}$?

Soln: No! Consider the following infinite set

$$S = \{ \log(2), \log(3), \log(5), \ldots \log(p), \ldots \}$$

We will show that S is linearly independent.

If $c_1 \log(p_1) + c_2 \log(p_2) + \cdots + c_n \log(p_n) = 0$

$$= \log(p_1^{c_1}) + \log(p_2^{c_2}) + \cdots + \log(p_n^{c_n}) = 0$$

$$= \log(p_1^{c_1} p_2^{c_2} \cdots p_n^{c_n}) = 0$$

$$\Rightarrow p_1^{c_1} p_2^{c_2} \cdots p_n^{c_n} = 1$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

So, it cannot have a finite basis.

Qn: What is the basis for $V = \{0\}$?

Lemma: Suppose S is a linearly independent set in a vector space V. If β is a vector in V s.t $\beta \notin \text{Span}(S)$, then $S \cup \{\beta\}$ is linearly independent.

Pf: Suppose $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n + d\beta = 0$

If $d = 0$, then $c_1 = c_2 = \cdots = c_n = 0$ because S is linearly independent.

If $d \neq 0$, then rewrite above equation as

$$-d\beta = c_1 v_1 + \cdots + c_n v_n$$

$$\beta = \frac{-1}{d}\left(c_1 v_1 + \cdots + c_n v_n\right)$$

$\Rightarrow \beta \in \text{Span}(S)$, which is contradiction

Hence $d = 0$ & proof is complete.

Thm: Any linearly independent set can be extended to a basis.
(Here we are assuming that V has a basis of finite size.)

Pf: Suppose V has a basis of size n & S is a linearly independent set.

So, $|S| \leq n$. If $|S| = n$, then S is already a basis because if not, that means there exists a vector $w \in V$ s.t $w \notin \text{Span}(S)$. So, $S \cup \{w\}$ must be linearly independent, which is a contradiction.

If $|S| < n$, then S is not a basis so there exists a vector $w \in V$

Such that $w \notin \text{Span}(S)$. So, $S \cup \{w\}$ is linearly independent. Similarly, we can keep extending till we get a set of size n.

<u>Example:</u> 1) $\mathbb{R}^2$

$$S = \{(1,0)\}$$

$$(1,1) \notin \text{Span}(S)$$

$S \cup \{(1,1)\} = \{(1,0),(1,1)\}$ forms a basis.

2) $\mathbb{R}^3$ $\quad S = \{(1,0,0)\}$

$$(1,1,0) \notin \text{Span}(S)$$

$$S' = S \cup \{(1,1,0)\} = \{(1,0,0),(1,1,0)\}$$

$$(1,1,1) \notin \text{Span}(S')$$

$\{(1,0,0), (1,1,0), (1,1,1)\}$ forms a basis for $\mathbb{R}^3$.

Some terminology (V has one basis of finite size)

$\dim(V)$ = No. of elements in any basis of V
(Dimension of V)

Can also write $\dim_F(V)$ to denote the field over which we are considering V as a vector space

Examples: $\dim_{\mathbb{C}}(\mathbb{C}) = 1$ $\quad \dim_F(F) = 1$

$\dim_{\mathbb{R}}(\mathbb{C}) = 2$

$\dim_{\mathbb{R}}(\mathbb{R}^n) = n$