# MATH 314 (Lecture 8)

Topics to be discussed today

Isomorphism theorem, Coordinates

Last time (1) defined linear maps

(2) Kernel and image

$$T : V \to W$$

$$\text{Ker}(T) = \{ v \in V \mid T(v) = 0 \}$$

$$\text{Image}(T) = \{ T(v) \mid v \in V \}$$

(3) $\mathcal{L}(V, W) = \{$ set of all linear maps from $V$ to $W$ $\}$

↳ is a vecher space of dimension $\dim(V) \dim(W)$

(4) If $V$ is a vector space over $F$ of dimension $n$, then $V$ is isomorphic to $F^n$.

# Rank - Nullity Theorem

Let $T: V \to W$ be a linear map.

Define $\text{Nullity}(T) = \dim(\text{Ker } T)$

$\text{Rank}(T) = \dim(\text{Image } T)$

Then $\text{Rank}(T) + \text{Nullity}(T) = \dim V$

Pf: We know that $\text{Ker}(T)$ is a subspace of $V$.

Suppose $\{v_1, \ldots, v_n\}$ is a basis for $\text{Ker}(T)$.

Extend $\{v_1, \ldots, v_n\}$ to a basis of $V$, say

$\{v_1, \ldots, v_n, \alpha_1, \ldots, \alpha_m\}$.

Observe that by construction $\alpha_i \notin \text{Ker}(T)$.

Claim: $\{T(\alpha_1), \ldots, T(\alpha_m)\}$ form a basis for $\text{Image}(T)$.

Take $w \in \text{Image}(T)$

$\Rightarrow w = T(v)$ for some $v \in V$

$\{v_1, \ldots, v_n, \alpha_1, \ldots, \alpha_m\}$ is a basis for $V$.

So, $v = a_1 v_1 + \cdots + a_n v_n + b_1 \alpha_1 + \cdots + b_m \alpha_m$

$T(v) = a_1 T(v_1) + \cdots + a_n T(v_n) + b_1 T(\alpha_1) + \cdots + b_m T(\alpha_m)$

$w = 0 + \cdots + 0 + b_1 T(\alpha_1) + \cdots + b_m T(\alpha_m)$

So $\{T(\alpha_1), \ldots, T(\alpha_m)\}$ is a spanning set for $\text{Image}(T)$.

Suppose $c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_m T(\alpha_m) = 0$

$\Rightarrow T(c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_m \alpha_m) = 0$

$\Rightarrow c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_m \alpha_m \in \text{Ker}(T)$

$\Rightarrow c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_m \alpha_m = d_1 v_1 + \cdots + d_n v_n$

$\Rightarrow c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_m \alpha_m - d_1 v_1 - \cdots - d_n v_n = 0$

$\Rightarrow c_1 = c_2 = \cdots = c_m = d_1 = \cdots = d_n = 0$

So, $\{T(\alpha_1), \ldots, T(\alpha_m)\}$ is linearly independent.

$$\dim V = n + m$$
$$= \text{Nullity} + \text{Rank}$$

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<u>Application</u> of <u>Rank-Nullity Theorem</u>:

(1) Polynomial Interpolation

$$V = \{p \in \mathbb{C}[x] \mid \deg p \leq n-1\}$$

$$W = \mathbb{C}^n$$

Pick distinct $s_1, \ldots, s_n \in \mathbb{C}$ and define

$$T : V \to W$$
$$p \mapsto (p(s_1), \ldots, p(s_n))$$

$T$ is a linear map.

$$\dim(W) = n$$
$$\dim(V) = n$$

$\ker T = \{ P \mid P(S_1) = P(S_2) = \cdots = P(S_n) = 0 \}$

P is a polynomial of degree at most $n-1$

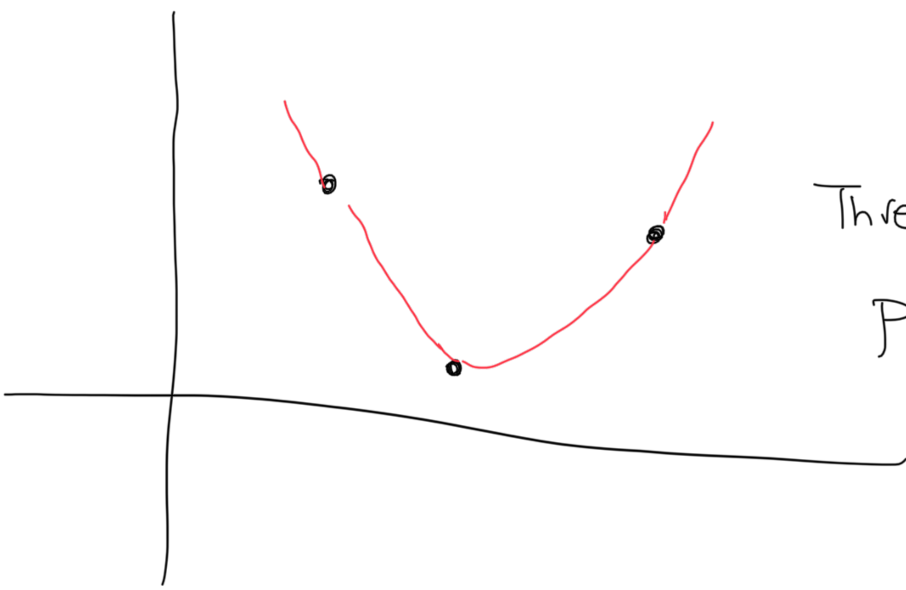
If it has $n$ distinct roots $\Rightarrow P = 0$

So, $\ker T = \{0\}$

$\Rightarrow$ T is surjective

$(\text{Nullity} + \text{Rank} = n \Rightarrow \text{Rank} = n)$

$n$ points determine a degree $\leq n-1$ polynomial



Three points determine a parabola

(2) Average value of a polynomial

$$V = \{ p \in \mathbb{R}[x] \mid \deg p \leq n-1 \}$$
$$W = \mathbb{R}^n$$

Let $I_1, I_2, \ldots, I_n \subseteq \mathbb{R}$ be pairwise disjoint intervals.

Say $I_1 = (a,b)$ or $[a,b]$

$$|I_1| = b-a$$

<u>Average value</u> of a polynomial $p$ over $I_j$ is

$$\overline{P_j} = \frac{1}{|I_j|} \int_{I_j} p(t)\, dt$$

Define the linear function

$$T : V \longrightarrow W$$
$$p \longmapsto (\overline{P_1}, \ldots, \overline{P_n})$$

$\dim V = \dim W = n$

$\ker T = \{ p \in V \mid \overline{P_1} = \overline{P_2} = \cdots = \overline{P_n} = 0 \}$

$\Rightarrow$ $p$ changes sign in $I_j$ for every $j \in \{1, 2, \dots, n\}$

$\Rightarrow$ $p$ has a root in $I_j$

$\Rightarrow$ $p$ has $n$ distinct roots

$\Rightarrow$ $p = 0$.

So, $T$ is an isomorphism.

There is exactly one polynomial with specified average values.

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Aside: Checking injectivity of $T: V \to W$

Propn: $T$ is injective $\Longleftrightarrow$ $T$ sends every linearly indepent subset to a linearly independent subset.

Pf: $\Rightarrow$: (Covered last time)

$\Leftarrow$ Want to show $\operatorname{Ker} T = \{0\}$

Say $v \in \ker T$, $v \neq \{0\}$

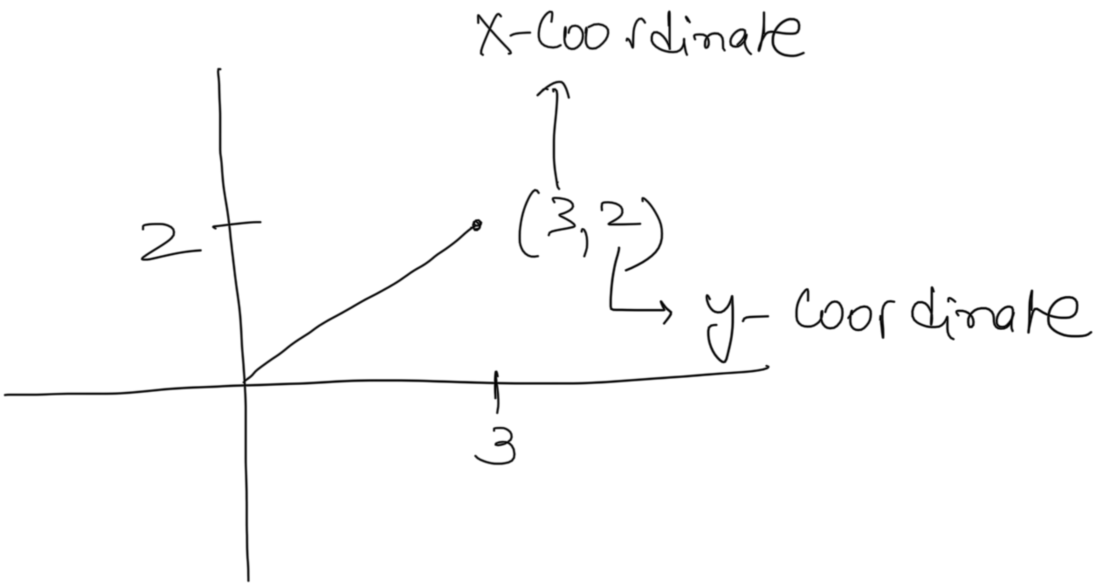
then $\{v\}$ is linearly independent

$\Rightarrow$ $\{Tv\}$ is also ______

In particular $Tv \neq 0$, Contradiction.

So, $\ker T = \{0\}$.

# Coordinates




This can be generalized.

When we think of coordinates in "usual" sense we are thinking of this linear combination

$$(3,2) = 3\underbrace{(1,0) + 2(0,1)}_{\text{Standard basis vectors}}$$

This can be generalized!

Say $V$ is a vector space of dimension $n$.

Fix a basis $\underset{\mathcal{B}}{\{v_1, \dots, v_n\}}$ of $V$.

Any vector $v \in V$ can be written as a linear combination

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

The scalars are unique in this combination.

Suppose
$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = b_1 v_1 + b_2 v_2 + \dots + b_n v_n$$

$$\Rightarrow (a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \dots + (a_n - b_n) v_n = 0$$

Linear independence gives us

$$a_1 = b_1$$
$$a_2 = b_2$$
$$\vdots$$
$$a_n = b_n$$

$(a_1, a_2, \dots, a_n)$ are coordinates of $v$ with respect to $B$.

We will denote the coordinate vector of $v$ with respect to $B$ as $[v]_B$.

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Suppose $B'$ is another basis for $V$.

<u>Qn:</u> How are two vectors $[v]_B$ and $[v]_{B'}$ related to each other?

Say $B = \{v_1, \dots, v_n\}$

$B' = \{v_1', \dots, v_n'\}$

$$v_1' = a_{11} v_1 + a_{12} v_2 + \dots + a_{1n} v_n$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$v_n' = a_{n1} v_1 + a_{n2} v_2 + \dots + a_{nn} v_n$$

Let's see an example.

$V = \mathbb{R}^2$

$B = \{(1,0), (0,1)\}$

$B' = \{(1,1), (1,0)\}$

$(1,1) = 1(1,0) + 1(0,1)$

$(1,0) = 1(1,0) + 0(0,1)$

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If $v = c_1 v_1' + \cdots + c_n v_n'$

$$v = c_1(a_{11} v_1 + a_{12} v_2 + \cdots + a_{1n} v_n)$$
$$+ c_2 ( \qquad )$$
$$\vdots$$
$$+ c_n(a_{n1} v_1 + a_{n2} v_2 + \cdots + a_{nn} v_n)$$

$$= (c_1 a_{11} + c_2 a_{21} + - + c_n a_{n1}) v_1$$

$$\vdots$$

$$+ (c_1 a_{1n} + c_2 a_{2n} + \cdots + c_n a_{nn}) v_n$$

The coordinates of v with respect to B is given by the matrix product

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & - & - & - a_{nn} \end{pmatrix}^T \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

So $[v]_B = \underbrace{C_{B \to B'}}_{\text{is an invertible matrix}} [v]_{B'}$

$$[v]_{B'} = (C_{B \to B'})^{-1} [v]_B$$

Let's see an example.

$$V = \mathbb{R}^3$$

$$B = \{(1,0,0), (1,1,0), (1,1,1)\}$$

$$B' = \{(0,1,0), (1,0,0), (0,0,1)\}$$

$\hookrightarrow$ Ordering matters.

$$(0,1,0) = -1(1,0,0) + 1(1,1,0) + 0(1,1,1)$$

$$(1,0,0) = 1(1,0,0) + 0(1,1,0) + 0(1,1,1)$$

$$(0,0,1) = 0(1,0,0) - 1(1,1,0) + 1(1,1,1)$$

Let's pick a vector in $\mathbb{R}^3$, say $(1,2,3) = v$

$$[v]_{B'} = (2,1,3)$$

$$[v]_B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$[v]_B = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

(Quick verification $-1(1,0,0) - 1(1,1,0) + 3(1,1,1) = (1,2,3)$)

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**Thm:** Coordinate mapping is an isomorphism.
i.e if $V$ is a vector space, fix a basis $B$ of $V$.

$n = \dim V$

Define $T: V \to F^n$ as follows

$$v \longmapsto [v]_B$$

Then $T$ is an isomorphism.

Pf: (1) T is linear because

$$T(v+w) = [v+w]_B$$

$$= [v]_B + [w]_B$$

$$T(cv) = [cv]_B$$

$$= c[v]_B$$

(2) $$\text{Ker}(T) = \{v \in V \mid [v]_B = 0\}$$

$$= \{0\}$$

(3) $$0 + \text{Rank}(T) = n$$

$\Rightarrow$ T is surjective.