MATH 314 (Lecture 19)

Topics to be discussed today

Minimal polynomial, Primary decomposition

Let F be a field.

$F[x]$ = Polynomials with coefficient in F

Defn: A subset $I \subseteq F[x]$ is called an ideal if

i) I is closed under addition

ii) for $\alpha \in I$, $f \in F[x]$, $f\alpha \in I$.

Examples: $I = \{ f(x)\, g(x) \mid g(x) \in F[x] \}$

i) If $\alpha_1 = f(x)\, g_1(x)$ &

$\alpha_2 = f(x)\, g_2(x) \quad \in I$

then $\alpha_1 + \alpha_2 = f(x)\left(g_1(x) + g_2(x)\right) \in I$

ii) If $\alpha = f(x)\, g(x) \in I$, then for $h(x) \in F[x]$

$$h(x)\alpha = h(x)\, f(x)\, g(x)$$

$$= f(x)\, h(x)\, g(x) \in I$$

Theorem: If $I \subseteq F[x]$ is an ideal, then

$I = \{ f(x)\, g(x) \mid g(x) \in F[x] \}$ for some

monic $f(x) \in F[x]$.

Pf: If $I = \{0\}$, then $f(x) = 0$ works.

Assume that $I \neq \{0\}$, choose a monic polynomial of smallest degree in I

Say that is $g(x)$.

For any other $f(x) \in I$, when we divide f by g we get a quotient and a remainder

$$\underset{\substack{\uparrow \\ I}}{f(x)} = g(x)\, \underset{\substack{\uparrow \\ I}}{q}(x) + r(x)$$

$\Rightarrow r(x) \in I$

So, $r = 0$ otherwise $\deg r < \deg g$

$\rightarrow\leftarrow$

So $I = \{ g(x)\, h(x) \mid h(x) \in F[x] \}$.

Minimal Polynomial

Let V be a vector space over F.

Let $T: V \to V$ be a linear map.

Choose any basis B of V, let A_B be the matrix representation of T with respect to B.

We know that $\mathrm{char}_{A_B}(x)$ does not depend on choice of bases B.

Defn: $\mathrm{char}_T(x) = \mathrm{char}_{A_B}(x)$

$\downarrow$

Characteristic polynomial of T

We know that $\text{char}_T(T) = 0$.

$\text{degree}(\text{char}_T(x)) = n^2$ (n is $\dim_F V$)

Can we find a polynomial of smaller degree that has T as its root?

Ans: Sometimes yes / Sometimes no!

$$I_T = \{ f \in F[x] \mid f(T) = 0 \}$$

Lemma: I_T is an ideal of $F[x]$.

Pf: 1) If $f, g \in I_T$ then

$$(f+g)(T) = f(T) + g(T)$$
$$= 0$$

So, $f+g \in I_T$

2) If $f \in I_T$, $g \in F[x]$

$$(gf)(T) = g(T)\, f(T) = 0$$

So, $gf \in I_T$.

We have already seen that there is a polynomial of smallest degree (monic) denoted by $\min_T(x)$ such that

$$I_T = \{\min_T(x) f(x) \mid f(x) \in F[x]\}.$$

We will call $\min_T(x)$ as the <u>minimal polynomial</u> of T.

<u>Lemma</u>: $\min_T(x)$ divides $\mathrm{Char}_T(x)$.

<u>Pf</u>: Because $\mathrm{char}_T(x) \in I_T$ the lemma follows.

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Thm: $\text{Char}_T(x)$ and $\min_T(x)$ have the Same roots, except for multiplicities.

Pf: Suppose $c \in F$. We want to show that

$$\text{Char}_T(c) = 0 \iff \min_T(c) = 0$$

If $\min_T(c) = 0 \Rightarrow \text{Char}_T(c) = 0$ because $\text{char}_T(x)$ is a multiple of $\min_T(x)$.

Assume $\text{char}_T(c) = 0$.

$\Rightarrow$ c is eigenvalue of T

$\Rightarrow$ $\exists$ a non-zero vector $\alpha \in V$ such that

$$T\alpha = c\alpha$$

If $T\alpha = c\alpha \Rightarrow \min_T(T)\alpha = \min_T(c)\alpha$

but $\min_T(T) = 0$

$\Rightarrow \min_T(c)\alpha = 0$

Since $\alpha \neq 0$ $\min_T(c) = 0$.

Examples: 1) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{R})$

$$\mathrm{char}_A(x) = \det\left(xI - A\right)$$

$$= \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$$

$$= x^2 + 1$$

This is also the minimal polynomial because x^2+1 is irreducible.

Remark: We can think of A in $M_2(\mathbb{Q})$ or $M_2(\mathbb{C})$ but that will not change minimal polynomial.

2) $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$$\text{char}_A(x) = \det(xI - A)$$
$$= x^2(x^2 - 4)$$
$$= x^2(x-2)(x+2)$$

There are only 2 possibilities for $\min_A(x)$, one is $\text{char}_A(x)$ and other is $(x-2)(x+2)x$.

Let's verify if A satisfies second polynomial

$(A-2I)(A+2I)A$

$(A-2I) = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$

$$A+2I = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

$$(A-2I)(A+2I) = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix}$$

$$(A-2I)(A+2I)(A) = \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Characterization of diagonal and triangular linear operators in terms of minimal polynomial

Invariant Subspaces

$T : V \to V$ — V fin dim vector space over F

Defn: We say that W is invariant subspace of V under T if

$T|_W : W \to W$, ie.

$T(W) \subseteq W$.

Please note that $T|_W(\alpha) = T(\alpha)$ but $T|_W$ and T are different operators because their domains &

Co-domains are different.

Example: Suppose $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\text{char}_A(x) = x^2+1$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$

Invariant subspaces of $\mathbb{R}^2$ under T are $\mathbb{R}^2$, $\{0\}$.

There is no 1-dimensional invariant subspace because assume there is, say W.

Then $W = \text{Span}(\{\alpha\})$ $\alpha \neq 0 \in \mathbb{R}^2$

$T(\alpha) = c\alpha \Rightarrow c$ is eigenvalue

but A has no eigenvalues over $\mathbb{R}$.

Matrix Representation of T_W:

$$T: V \to V$$

W is invariant subspace of V under T

$$T|_W : W \to W$$

Fix a basis $B_W = \{b_1, \ldots, b_r\}$ of W.

Extend this to a basis

$$B_V = \{b_1, \ldots, b_r, b_{r+1}, \ldots, b_n\} \text{ of } V.$$

$$T(b_1) = a_{11} b_1 + a_{21} b_2 + \cdots + a_{r1} b_r + 0 \cdot b_{r+1} \cdots + 0 \cdot b_n$$

$$\vdots$$

$$T(b_r) = a_{1r} b_1 + a_{2r} b_2 + \cdots + a_{rr} b_r + 0 \cdot b_{r+1} \cdots + 0 \cdot b_n$$

$$[T]_{B_V} = \left[\begin{array}{c|c} (A_W)_{r\times r} & C_{r\times n-r} \\ \hline O_{n-r\times r} & D_{n-r\times n-r} \end{array}\right]$$

Matrix representation of T_W with respect to B_W.

<u>Lemma:</u> $T: V \to V$

$T|_W : W \to W$ — W is invariant subspace of V under T

Then $\text{char}_{T_W}(x)$ divides $\text{char}_T(x)$

$\min_{T_W}(x)$ divides $\min_T(x)$

Pf:
$$[T]_B = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$$

$$\underbrace{\det(xI - [T]_B)}_{\mathrm{Char}_T(x)} = \det(xI - A)\underbrace{\det(xI - D)}_{\mathrm{Char}_{T_W}(x)}$$

So, $\mathrm{Char}_{T_W}(x)$ divides $\mathrm{Char}_T(x)$.

$$[T]_B^2 = \left[\begin{array}{c|c} A & C \\ \hline 0 & D \end{array}\right]\left[\begin{array}{c|c} A & C \\ \hline 0 & D \end{array}\right]$$

$$= \left[\begin{array}{c|c} A^2 & C_2 \\ \hline 0 & D^2 \end{array}\right]$$

Similarly, $$[T]_B^k = \left[\begin{array}{c|c} A^k & C_k \\ \hline 0 & D^k \end{array}\right]$$

Suppose $\min_T(x) = c_0 + c_1 x + \cdots + x^m$

$$c_0 + c_1 T + \cdots + T^m = 0$$

$$\Downarrow$$

$$c_0 + c_1 A + \cdots + A^m = 0 \quad \text{and}$$

$$c_0 + c_1 D + \cdots + D^m = 0$$

$\min_{T_W}(x)$ divides $\min_T(x)$.

Thm: $T: V \to V$

T is triangulable $\iff$ $\min_T(x)$ is of the form

$$(x-c_1)^{r_1} \cdots (x-c_k)^{r_k}$$

$$c_i \in F$$

Thm: $T: V \to V$

T is diagonalizable $\iff$ $\min_T(x)$ is of the form

$$(x-c_1) \cdots (x-c_k)$$

$$c_i \in F$$

Proofs via invariant subspaces in next class!