MATH 314 (Lecture 23)

Topics to be discussed today

Inner Product Spaces

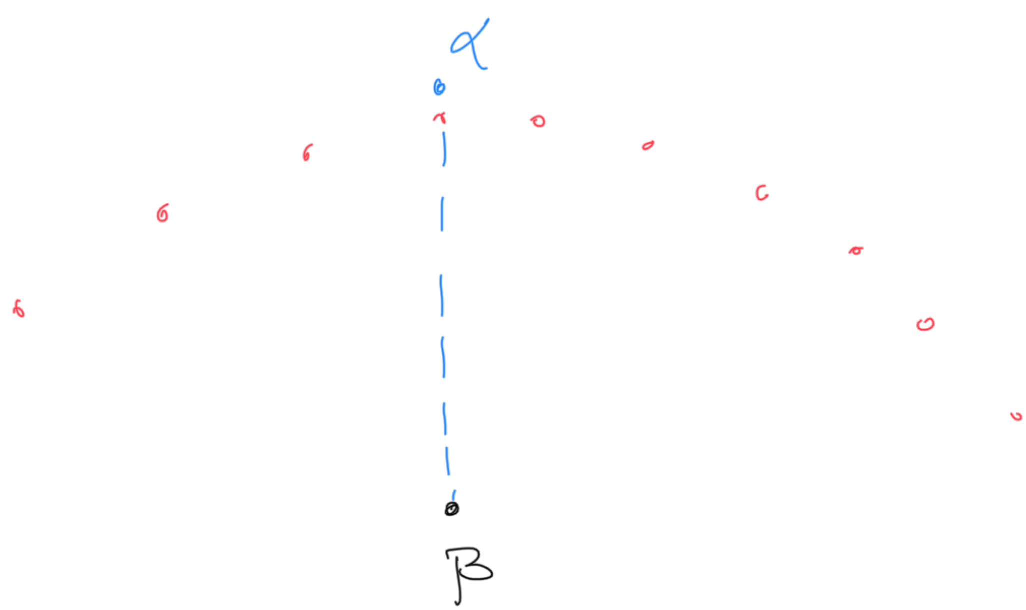
Approximation

Thm: Let W be a subspace of an inner product space V and let $\beta \in V$.

i) $\alpha \in W$ "best approximates β", i.e.

$$\|\alpha - \beta\| \leq \|\gamma - \beta\| \quad \text{for every } \gamma \in W$$

$\Longleftrightarrow$ $\alpha - \beta$ is orthogonal to every vector in W.



ii) If W is finite-dimensional and $\{\alpha_1, \ldots, \alpha_n\}$ is orthonormal basis for W, then

$$\alpha = \sum_k \langle \beta, \alpha_k \rangle \alpha_k$$

best approximates β, and is unique.

<u>Proof</u>: i) Assume α best approximates β.

Let $\gamma \in W$. We want to show $\langle \alpha - \beta, \gamma \rangle = 0$

$$\beta - \gamma = \beta - \alpha + \alpha - \gamma$$

$$\|\beta - \gamma\|^2 = \|\beta - \alpha\|^2 + \|\alpha - \gamma\|^2 + 2\,\mathrm{Re}\,\langle \beta - \alpha, \alpha - \gamma \rangle$$

$$\|\alpha - \beta\| \leq \|\beta - \gamma\|$$

$$\Rightarrow 2\,\mathrm{Re}\,\langle \beta - \alpha, \alpha - \gamma \rangle + \|\alpha - \gamma\|^2 \geq 0 \quad \forall \gamma \in W$$

$$\Updownarrow$$

$$* \quad 2\,\mathrm{Re}\,\langle \beta - \alpha, x \rangle + \|x\|^2 \geq 0 \quad \forall x \in W$$

Substitute $x = -\dfrac{\langle \beta - \alpha, \alpha - \gamma \rangle}{\|\alpha - \gamma\|^2} (\alpha - \gamma)$

* becomes

$$2\,\mathrm{Re}\left\langle \beta-\gamma,\ -\frac{\langle \beta-\alpha,\ \alpha-\gamma\rangle}{\|\alpha-\gamma\|^2}(\alpha-\gamma)\right\rangle$$

$$+\ \frac{|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2}{\|\alpha-\gamma\|^2} \geq 0$$

$$\Rightarrow\ -\frac{2}{\|\alpha-\gamma\|^2}|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2 + \frac{|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2}{\|\alpha-\gamma\|^2} \geq 0$$

$$\Rightarrow\ -\frac{|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2}{\|\alpha-\gamma\|^2} \geq 0$$

$$\Rightarrow\ \langle \beta-\alpha,\ \alpha-\gamma\rangle = 0$$

$\Rightarrow$ $\langle \beta-\alpha\rangle$ is orthogonal to every vector in W.

$\Leftarrow$ Assume $\alpha-\beta$ is orthogonal to every vector in W.

$$\beta-\gamma = \beta-\alpha+\underbrace{\alpha-\gamma}_{\in W} \quad \text{for every } \gamma\in W$$

$$\|\beta-\gamma\|^2 = \|\beta-\alpha\|^2 + \|\gamma-\alpha\|^2$$

$$\Rightarrow \|\beta-\gamma\|^2 \geq \|\beta-\alpha\|^2$$

$\Rightarrow$ α best approximates β.

ii) We want to show that

$$\beta-\sum_{k=1}^{n}\langle\beta,\alpha_k\rangle\alpha_k \text{ is orthogonal}$$

to every vector in W

Since W has orthonormal basis $\{\alpha_1,\dots,\alpha_n\}$ it suffices to show $\beta-\sum_{k=1}^{n}\langle\beta,\alpha_k\rangle\alpha_k$ is orthogonal to α_i

$$\left\langle \beta - \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k \, , \, \alpha_i \right\rangle$$

$$= \langle \beta, \alpha_i \rangle - \left\langle \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k, \alpha_i \right\rangle$$

$$= \langle \beta, \alpha_i \rangle - \langle \beta, \alpha_i \rangle = 0$$

So, done.

Uniqueness : Suppose $w \in W$ is another vector that best approximates β. We want to show that

$$w = \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k$$

This follows because $\beta - w$ is perpendicular to α_i, hence

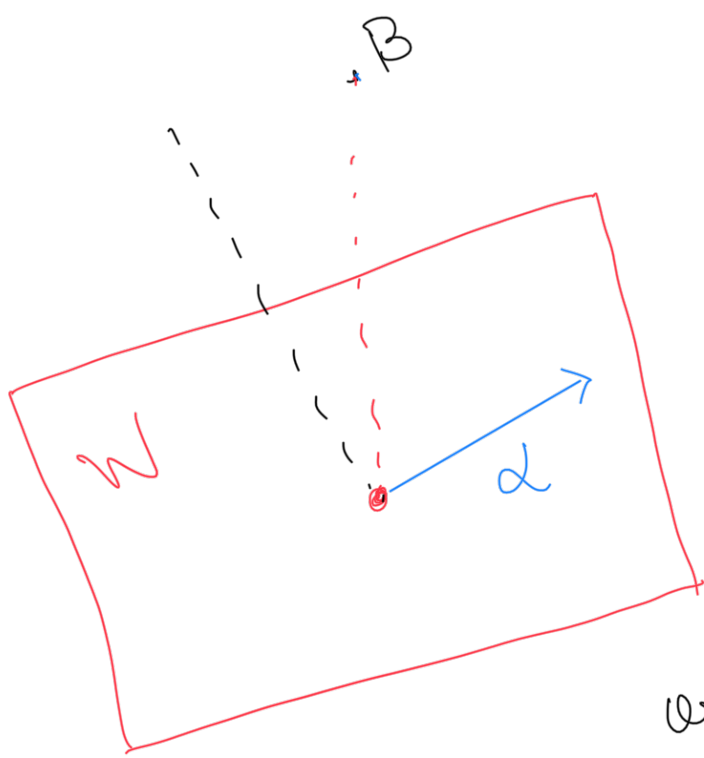
$$\langle \beta - w, \alpha_i \rangle = 0 \Rightarrow \langle \beta, \alpha_i \rangle = \langle w, \alpha_i \rangle$$

and we know that

$$w = \sum_{i=1}^{n} \langle w, \alpha_i \rangle \alpha_i$$

$$\| \quad \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle \alpha_i .$$

Hence, done.



α is "best approximation" of β

also known as orthogonal projection of β on W.

Defn: V inner product space

$S \subseteq V$

Orthogonal complement of $S = \{ v \in V \mid \langle v, s \rangle = 0 \ \forall s \in S \}$
$(S^\perp)$

Lemma: $S^\perp$ is a subspace of V.

Pf: $S^\perp \neq \phi$ because $0 \in S^\perp$.

$$\left(\begin{array}{l} \langle 0, s \rangle = \langle 0+0, s \rangle = \langle 0, s \rangle + \langle 0, s \rangle \\ \Rightarrow \langle 0, s \rangle = 0 \quad \forall s \in V \end{array} \right)$$

→ $S^\perp$ is closed under addition

If $x, y \in S^\perp$, then $\langle x+y, s \rangle$
$= \langle x, s \rangle + \langle y, s \rangle$
$= 0 + 0 = 0 \quad \forall s \in S$

→ $S^\perp$ is closed under scalar multiplication because if $x \in S^\perp$, then

$$\langle cx, s\rangle = c\langle x, s\rangle = c \cdot 0 = 0$$

$$\forall s \in S.$$

Corollary: V inner product space

W subspace of V

$E: V \longrightarrow W$

$\beta \longmapsto \hat{\beta}$ (orthogonal projection of V on W)

Then $\beta - \hat{\beta}$ is orthogonal projection of V on $W^\perp$.

Pf: If $w \in W$, then

$$\langle \beta - \hat{\beta}, w\rangle = 0 \implies \beta - \hat{\beta} \in W^\perp$$

and $\beta - (\beta - \hat{\beta}) = \hat{\beta}$ is orthogonal to $W^\perp$.

Thm: V inner product space

$W \subseteq V$ finite-dimension

E orthogonal projection of V on W.

Then E is an idempotent linear transformation of V onto W, $W^{\perp}$ is $\ker(E)$

and $V = W \oplus W^{\perp}$.

Pf: 1 To show: $E^2 = E$

$$E(E\beta) = E(\beta)$$

$E\beta \in W$

because projection of any vector already in W is that vector itself.

2 To show $\ker(E) = W^{\perp}$

$$\ker(E) = \{ v \in V \mid E(v) = 0 \}$$

$$= \{ v \in V \mid v \in W^{\perp} \}$$

$$= W^{\perp}$$

(3) $V = W \oplus W^\perp$

If $w \in W$ and $w \in W^\perp$

$\Rightarrow \langle w, w \rangle = 0 \Rightarrow w = 0$

Further $\dim(W \oplus W^\perp) = \dim W + \dim W^\perp$

$= \dim(\text{range}(E)) + \dim(\ker E)$

$= \dim V$

So, $V = W \oplus W^\perp$.

(n need not be $\dim V$)

Corollary: Let $\{\alpha_1, \ldots, \alpha_n\}$ be an orthogonal set of non-zero vectors in V. If $\beta \in V$, then

$$\sum_{k=1}^{n} \frac{|\langle \beta, \alpha_k \rangle|^2}{\|\alpha_k\|^2} \leq \|\beta\|^2$$

and equality holds $\Longleftrightarrow$

$$\beta = \sum_{k=1}^{n} \frac{\langle \beta, \alpha_k \rangle}{\|\alpha_k\|^2} \alpha_k$$

Pf: Let W be the subspace of V spanned by $\{\alpha_1, \dots, \alpha_n\}$.

$$\beta = \underset{W}{\underset{\cap}{\hat{\beta}}} + \underset{W^\perp}{\underset{\cap}{\beta^\perp}}$$

$$\hat{\beta} = \sum_{k=1}^{n} \frac{\langle \beta, \alpha_k \rangle \alpha_k}{\|\alpha_k\|^2}$$

$$\|\beta\|^2 = \|\hat{\beta}\|^2 + \|\beta^\perp\|^2$$

So, $\|\beta\|^2 \geq \|\hat{\beta}\|^2$

and $\|\hat{\beta}\|^2 = \displaystyle\sum_{k=1}^{n} \frac{|\langle \beta, \alpha_k \rangle|^2}{\|\alpha_k\|^2}$

Equality holds $\iff \beta^\perp = 0$

$$\left\{ \begin{array}{c} \text{Linear} \\ \text{functionals} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Inner product} \\ \text{with a fixed} \\ \text{vector} \end{array} \right\}$$

V inner product space

Fix $\beta \in V$.

Define $f: V \longrightarrow \mathbb{C}$

$$\alpha \longmapsto \langle \alpha, \beta \rangle$$

f is a linear functional.

Let us discuss the converse of this, i.e. does every linear functional come from an inner product?

Thm: Let V be a finite dimensional inner product space and f a linear functional on V. Then there exists a unique vector β_f in V such that

$$f(\alpha) = \langle \alpha, \beta_f \rangle \quad \forall \alpha \in V.$$

Pf: Choose an orthonormal basis of V, say $\{v_1, \ldots, v_n\}$.

Define $\beta_f = \sum_{i=1}^{n} \overline{f(v_i)}\, v_i$

$$\langle \alpha, \beta_f \rangle = \left\langle \alpha, \sum_{i=1}^{n} \overline{f(v_i)}\, v_i \right\rangle$$

$$= \overline{\left\langle \sum_{i=1}^{n} \overline{f(v_i)}\, v_i, \alpha \right\rangle}$$

$$= \overline{\sum_{i=1}^{n} \langle \overline{f(v_i)}\, v_i, \alpha \rangle}$$

$$= \sum_{i=1}^{n} f(v_i) \overline{\langle v_i, \alpha \rangle}$$

$$\alpha = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

$$\langle v_i, \alpha \rangle = \langle v_i, \alpha_1 v_1 + \cdots + \alpha_n v_n \rangle$$

$$= \overline{\langle \alpha_1 v_1 + \cdots + \alpha_n v_n, v_i \rangle}$$

$$= \overline{\alpha_i}$$

$$\langle \alpha, \beta_f \rangle = \sum_{i=1}^{n} f(v_i) \alpha_i$$

$$= \sum_{i=1}^{n} f(\alpha_i v_i)$$

$$= f\left(\sum_{i=1}^{n} \alpha_i v_i \right)$$

$$= f(\alpha)$$

This is not necessarily true for infinite dimensional vector spaces.

$V = \{$ Polynomials with coefficients in $\mathbb{C}$ $\}$

$$\langle f, g \rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt$$

Let $L : V \longrightarrow \mathbb{C}$ $\quad$ z fixed Complex number

$p \longmapsto p(z)$

Is there a polynomial g such that

$$L(p) = p(z) = \langle p, g \rangle \quad \forall p \in V$$

No!

Suppose there is such a polynomial

$$p(z) = \int_0^1 p(t)\,\overline{g(t)}\,dt \quad \forall p \in V$$

Take $h(x) = x - z$

$(hf)(z) = 0 \qquad \forall f \in V$

$$0 = \int_0^1 h(t) f(t) \overline{g(t)} \, dt \quad \forall f \in V$$

Take $f = \bar{h} g$

$$0 = \int_0^1 h(t) \overline{h(t)} \, g(t) \overline{g(t)} \, dt$$

$$0 = \int_0^1 |h(t)|^2 |g(t)|^2 \, dt$$

$\Rightarrow \quad hg = 0 \quad$ but $\quad h \neq 0 \quad$ so

$g = 0$

$\Rightarrow \quad L = 0 \quad$, contradiction.

Adjoint operators

Thm: $T: V \to V$ V finite dimensional inner product space

There exists a $T^*: V \to V$ such that

$$\langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle$$

(T^* is called adjoint of T)

Pf: For a fixed vector $\beta \in V$

$\alpha \mapsto \langle T\alpha, \beta \rangle$ is a linear functional

So, $\exists$ β' such that

$$\langle T\alpha, \beta \rangle = \langle \alpha, \beta' \rangle \quad \forall \alpha \in V$$

Let $T^*: V \to V$

$\beta \mapsto \beta'$

We want to show that T^* is linear.

$$\langle \alpha, T^*(c\beta+\gamma)\rangle = \langle T\alpha, c\beta+\gamma\rangle$$

$$= \overline{\langle c\beta+\gamma, T\alpha\rangle}$$

$$= \overline{\langle c\beta, T\alpha\rangle} + \overline{\langle \gamma, T\alpha\rangle}$$

$$= \bar{c}\langle T\alpha, \beta\rangle + \langle T\alpha, \gamma\rangle$$

$$= \bar{c}\langle \alpha, T^*\beta\rangle + \langle \alpha, T^*\gamma\rangle$$

$$= \langle \alpha, cT^*\beta\rangle + \langle \alpha, T^*\gamma\rangle$$

$$= \langle \alpha, cT^*\beta + T^*\gamma\rangle$$

$$\Rightarrow T^*(c\beta+\gamma) = cT^*\beta + T^*\gamma$$

<u>Corollary</u>: The matrix of T^* is the Conjugate transpose of matrix of T.

Pf: $B = \{v_1, \ldots, v_n\}$ orthonormal basis

$A = [T]_B$

$A' = [T^*]_B$

$$Tv_j = \sum_{k=1}^{n} A_{kj} v_k$$

$$Tv_j = \sum_{k=1}^{n} \langle Tv_j, v_k \rangle v_k$$

$$\Rightarrow A_{kj} = \langle Tv_j, v_k \rangle$$

So, $A_{kj} = \langle Tv_j, v_k \rangle$

$$A'_{kj} = \langle T^* v_j, v_k \rangle$$

$$= \langle v_j, Tv_k \rangle = \overline{\langle Tv_k, v_j \rangle}$$

$$= \overline{A_{jk}}$$

Adjoints need not exist in infinite dimensional vector spaces.

Some properties of adjoints:

1) $(T+U)^* = T^* + U^*$

2) $(cT)^* = \bar{c}\, T^*$

3) $(TU)^* = U^* T^*$

4) $(T^*)^* = T$

<u>Defn:</u> If $T = T^*$, T is called self-adjoint or Hermitian.