# MATH 314 (Lecture 13)

Topics to be discussed today

Linear forms, duality

$V \longrightarrow V^*$

Vector Space — Dual Space

$$V^* = \{ \varphi : V \to F \mid \varphi \text{ linear functional} \}$$
$$= L(V, F)$$

We already know that $V^*$ is a vector space and $\dim(V^*) = (\dim V)(\dim F)$
$= \dim V.$

Let's see an explicit basis of $V^*$.

Fix a basis $\{b_1, \ldots, b_n\}$ of $V$.

Define $f_i : V \to F$ as follows

$$f_i(b_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Dual basis

$\{f_1, \ldots, f_n\}$ is linearly independent because if

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

$$\Rightarrow (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)(b_i) = 0 \quad \text{for } i \in \{1, \ldots, n\}$$

$$\Rightarrow \quad c_i = 0 \quad \text{for } i \in \{1, \ldots, n\}$$

If $\varphi : V \to F$ is a linear map, then

Say $\varphi(b_i) = a_i$

<u>Claim:</u> $\varphi = \sum_{i=1}^{n} a_i f_i$

<u>Pf:</u> $\varphi(b_j) = \sum a_i f_i(b_j)$

$$= a_j.$$

Since $\varphi = \sum_{i=1}^{n} a_i f_i$ on basis, they are equal everywhere.

# Pairing and duality

Given $f \in V^* = L(V, F)$

and $v \in V$

we will denote $f(v)$ as $\langle f, v \rangle$.

$$\langle \, , \rangle : V^* \times V \longrightarrow F$$
$$(f, v) \longmapsto f(v)$$

The map $\langle , \rangle$ is a pairing, i.e. it satisfies the following properties.

1) $\langle f_1 + f_2 , v \rangle = \langle f_1, v \rangle + \langle f_2, v \rangle$

Pf: $\langle f_1 + f_2, v \rangle = (f_1 + f_2)(v)$
$$= f_1(v) + f_2(v)$$
$$= \langle f_1, v \rangle + \langle f_2, v \rangle$$

2) $\langle f, v_1 + v_2 \rangle = \langle f, v_1 \rangle + \langle f, v_2 \rangle$

$\langle f, v_1 + v_2 \rangle = f(v_1 + v_2) = fv_1 + fv_2$

$$= \langle f, v_1 \rangle + \langle f, v_2 \rangle$$

3) $\langle \lambda f, v \rangle = \lambda \langle f, v \rangle$

$$\langle \lambda f, v \rangle = (\lambda f) v = \lambda (f(v))$$

$$= \lambda \langle f, v \rangle$$

4) $\langle f, \lambda v \rangle = \lambda \langle f, v \rangle$

$$f(\lambda v) = \lambda f(v) = \lambda \langle f, v \rangle$$

# Orthogonal Complement

$$\begin{array}{ccc} & V & & V^* & \\ & \cup\!| & & \cup\!| & \\ \text{Subspace} & W & & W^0 & \text{(Orthogonal complement)} \\ & & & \| & \\ & & & \{f \in V^* \mid \langle f, v \rangle = 0 \ \forall v \in W\} & \end{array}$$

__Lemma:__ $W^0$ is a subspace of $V^*$.

__Pf:__ If $f_1, f_2 \in W^0$, then

$$(cf_1 + f_2)(v) = c(f_1(v)) + f_2(v)$$
$$= 0$$

$\Rightarrow cf_1 + f_2 \in W^0$.

$W^0$ is called annihilator of $W$.

Thm: $\dim W + \dim W^0 = \dim V$

Pf: Suppose $n = \dim V$

$k = \dim W$

Choose a basis $\{v_1, \dots, v_k\}$ for $W$.
We complete it to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for $V$.

For $j \in \{k+1, \dots, n\}$ define

$f_j : V \to F$ as follows

$$f_j(v_i) = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}$$

$f_j \in W^0$.

We already saw that $\{f_j\}_{j=k+1}^{n}$ is linearly independent.

If $f \in W^0$

then say $f(v_i) = a_i \quad a_i \in F$

$a_i = 0$ for $i \in \{1, \dots, k\}$

Then $f = a_{k+1} f_{k+1} + \dots + a_n f_n$

So, $\{f_{k+1}, \dots, f_n\}$ spans $W^0$.

Hence, it is a basis.

So, $\dim W + \dim W^0 = k + n - k$

$= n$

$= \dim V$

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<u>Corollary of proof:</u>

$W$ is the intersection of $n-k$ hyperspaces in $V$.

(null space of a single linear functional $f$)

Corollary: $W_1 = W_2 \iff W_1^0 = W_2^0$

Pf: $W_1 = W_2 \Rightarrow W_1^0 = W_2^0$

$\Leftarrow$ Assume $W_1 \neq W_2$, then there exists $\alpha$ in $W_1$, $\alpha \notin W_2$ or $\exists\ \alpha \in W_2$, $\alpha \notin W_1$.

Take a fn $f$ s.t $f \in W_2^0$ and $f(\alpha) \neq 0$ then $f \notin W_1^0$ or take $f$ s.t $f \in W_1^0$ and $f(\alpha) \neq 0$ then $f \notin W_2^0$.

In either case $W_1^0 \neq W_2^0$.

$$V \rightsquigarrow V^*$$

Basis $\{b_1, \dots, b_n\}$ — Dual basis $\{f_1, \dots, f_n\}$

$$f_i(b_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Qn: Is every basis of $V^*$ dual of some basis of $V$?

Double Dual

$$V \xrightarrow[\text{Dual}]{} V^* \longrightarrow V^{**}$$

Double Dual

$\alpha$ — $L_\alpha$

$$L_\alpha : V^* \longrightarrow F$$

$$L_\alpha(f) \longmapsto f(\alpha)$$

$$\begin{aligned} L_\alpha(cf+g) &= (cf+g)(\alpha) \\ &= cf(\alpha) + g(\alpha) \\ &= c L_\alpha(f) + L_\alpha(g) \end{aligned}$$

Thm: The map $V \longrightarrow V^{**}$

$$\alpha \longmapsto L_\alpha$$

is an isomorphism.

Pf: $c\alpha+\beta \longmapsto L_{c\alpha+\beta}$

We want to show that $L_{c\alpha+\beta} = cL_\alpha + L_\beta$

$$L_{c\alpha+\beta}(f) = f(c\alpha+\beta) = c f(\alpha) + f(\beta)$$
$$= c L_\alpha(f) + L_\beta(f).$$

So, this map is linear.

Suppose $L_\alpha = 0$, i.e. $L_\alpha(f) = 0 \quad \forall f \in V^*$
$$\overset{\shortparallel}{f(\alpha)} = 0 \quad \forall f \in V^*$$

Consider $W = \operatorname{Span}(\{\alpha\})$

$W^0 = V^*$

$\dim W + \dim W^0 = \dim V^*$

$\Rightarrow \dim W = 0 \Rightarrow \alpha = 0.$

$\dim V = \dim V^* = \dim V^{**}$

$\Rightarrow$ The map is an isomorphism.

Corollary: Each basis for $V^*$ is the dual of some basis for $V$.

pf: Suppose $\{f_1, \ldots, f_n\}$ is a basis for $V^*$.

Construct $\{L_1, \ldots, L_n\} \subseteq V^{**}$ such that

$$L_i(f_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

It is a basis for $V^{**}$.

But from previous thm, we know that

$L_i = L_{\alpha_i}$, Consider

$B = \{\alpha_1, \ldots, \alpha_n\} \subseteq V$

$B$ is linearly independent.

$$c_1\alpha_1 + \cdots + c_n\alpha_n = 0$$

$$\Rightarrow \quad L_{c_1\alpha_1 + \cdots + c_n\alpha_n} = 0$$

$$\shortparallel$$

$$c_1 L_{\alpha_1} + \cdots + c_n L_{\alpha_n} = 0$$

$$\Rightarrow \quad c_i = 0 \,.$$

We have a linearly independent set of size $n$, so it is a basis.

$$f_i(\alpha_j) = L_{\alpha_j}(f_i) = L_j(f_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

So, $\{f_1, \ldots, f_n\} \subseteq V^*$ is dual of

$\{\alpha_1, \ldots, \alpha_n\} \subseteq V.$

# Transpose as dual

**Thm:** Let $V$ and $W$ be vector spaces over $F$. For each $T: V \to W$, there is a unique linear transformation

$$T^t : W^* \to V^* \quad \text{such that}$$

$$(T^t g)\,\alpha = (g \circ T)(\alpha)$$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[dr, "g \circ T"'] & W \arrow[d, "g"] \\
 & F
\end{tikzcd}$$

**Pf:** Suppose there are two $T_1^* : W^* \to V^*$, $T_2^* : W^* \to V^*$

s.t. $T_1^*(g) = g \circ T = T_2^*(g)$

$\Rightarrow T_1^* = T_2^*$

$$T^t(cg_1+g_2)(\alpha) = ((cg_1+g_2)\circ T)(\alpha)$$
$$= (cg_1+g_2)(T\alpha)$$
$$= cg_1(T\alpha) + g_2(T\alpha)$$
$$= c(T^t g_1)(\alpha) + (T^t g_2)(\alpha)$$

So, $T^t$ is linear.

$T^t$ is called <u>transpose</u> or <u>adjoint</u> of $T$.

<u>Thm:</u> $V$ $W$

$B$ ordered basis $B'$ ordered basis

$B^*$ dual basis $(B')^*$ dual basis

$$T: V \longrightarrow W$$

& $A$ is the matrix relative to $B, B'$

Then $A^t$ is the matrix of

$$T^t: W^* \longrightarrow V^*$$

with respect to $(B')^*$, $B^*$

Pf: $B = \{v_1, \dots, v_m\}$ $B' = \{w_1, \dots, w_m\}$

$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$$

$$(T^t g_j)(v_i) = g_j T(v_i)$$

$$= g_j \left( \sum_{k=1}^{m} A_{ki} w_k \right)$$

$$= \sum_{k=1}^{m} A_{ki} g_j(w_k)$$

$$= A_{ji}$$

$$\Rightarrow (T^t g_j) = \sum_{k=1}^{n} (A^t)_{kj} f_k$$

because

$$(T^t g_j)(v_i) = \sum_{i=1}^{n} A_{jk} f_k(v_i)$$

$$= A_{ji}$$