MATH 314 (Lecture 15)

Topics to be discussed today

Determinant of an endomorphism

Review of Permutations

Defn: A permutation of a finite set $\{1, 2, \ldots, n\}$ is a bijective map from $\{1, 2, \ldots, n\}$ to itself.

Examples: 1)

$$\begin{tikzcd} 1 \arrow[dr] & 2 \arrow[dr] & 3 & \cdots & n \arrow[dllll] \\ 1 & 2 & 3 & \cdots & n \end{tikzcd}$$

2)

$$\begin{tikzcd} 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \cdots & n \arrow[d] \\ 1 & 2 & 3 & \cdots & n \end{tikzcd}$$

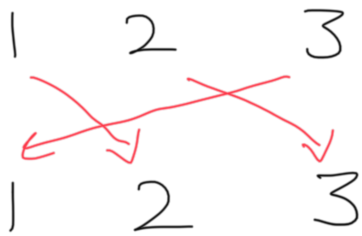
3)

$$\begin{tikzcd} 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \cdots & i \arrow[dr] & i+1 \arrow[dl] & \cdots & n \arrow[d] \\ 1 & 2 & 3 & \cdots & i & i+1 & \cdots & n \end{tikzcd}$$

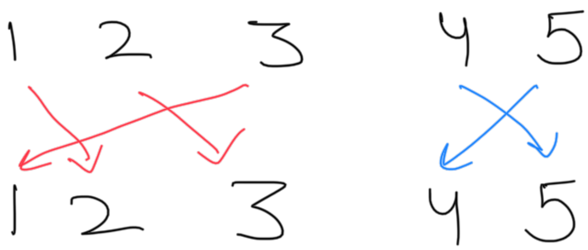
Transposition

Signature of a Permutation

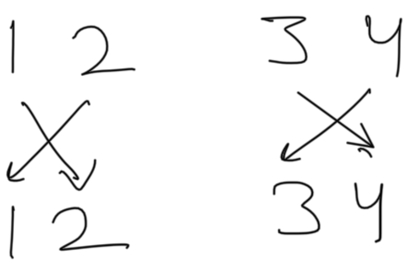
Cycle notation of a permutation



$(1\,2\,3)$



$(1\,2\,3)\,(4\,5)$



$(1\,2)\,(3\,4)$

Observation: Every permutation is a product of cycles.

Thm: Every permutation is product of transpositions.

Pf: Since a permutation is product of cycles, it suffices to prove this theorem for cycles. Let σ by a cycle of length n, we will use induction on n.

For $n=2$, σ is already a transposition. Assume the result holds for all cycles of length $n-1$.

Inductive Step: Suppose σ has length n.

$$\sigma = (a_1 \, a_2 \cdots a_n)$$

$$\sigma = \underbrace{(a_1 \, a_2 \cdots a_{n-1})}_{\text{Use induction}} (a_{n-1} \, a_n)$$

Suppose σ is product of r transpositions

$$\text{Signature}(\sigma) = (-1)^r$$

Defn: We say σ is odd if Signature(σ) $= -1$ & σ is even if Signature (σ) $= 1$.

Propn: Signature ($\sigma\tau$) = Signature (σ) Signature (τ)

Pf: Suppose σ is product of r transpositions

and τ ———— s ————

then $\sigma\tau$ ———— $r+s$ ————

Example: 1) $\sigma = (123) = (12)(23)$

$$\tau = (456) = (45)(56)$$

$$\sigma\tau = (123)(456) = (12)(23)(45)(56)$$

2) $\sigma = (123) = (12)(23)$

$\tau = (132) = (13)(23)$

$\sigma\tau = (123)(132) = (12)(23)(13)(23)$

So, signature $(\sigma\tau) = (-1)^{r+s}$

$= (-1)^r (-1)^s$

$=$ signature (σ) signature (τ)

Alternating multilinear form

V finite dimensional vector space over a field F.

Defn: We say that a function

$$H : \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}} \to F$$ is a

multilinear k-form if H is linear in each variable.

Defn: Let $H : \underbrace{V \times V \times \cdots \times V}_{k\text{-copies}} \to F$ be a

multilinear k-form. We say H is alternating if

$$H(v_1, \ldots, v_k) = -H(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

$\forall\, v_i \in V$ & all transpositions σ.

Lemma: Let H be an alternating k-form.

The following are true:

1) $H(v_1, v_2, \dots, v, v, \dots, v_k) = 0 \quad \forall v_i, v \in V$

2) Let σ be any permutation.

$$H(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{Signature}(\sigma)\, H(v_1, \dots, v_k)$$
$$\forall v_i \in V$$

Proof: 1) Since H is alternating

$$H(v_1, v_2, \dots, v, v, \dots, v_k) = -H(v_1, \dots, \underbrace{v, v}_{\text{switched}}, \dots, v_k)$$

$$\Rightarrow 2H(v_1, \dots, v, v, \dots, v_k) = 0$$

$$\Rightarrow H(v_1, \dots, v, v, \dots, v_k) = 0$$

2) If σ is product of r transpositions,

say $\sigma = \sigma_1 \cdots \sigma_r$

$$H(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = -H(v_{\sigma_1 \cdots \sigma_{r-1}(1)}, \dots, v_{\sigma_1 \cdots \sigma_{r-1}(k)})$$
$$\dots = (-1)^r H(v_1, \dots, v_k)$$

Determinants

Defn: Let F be a field. Let Det be a function from the set of all $n \times n$ matrices over F to F.

$$\text{Det} : M_{n \times n}(F) \longrightarrow F$$

We say that Det is a <u>determinant</u> function if

1) Det is alternating.

2) $\text{Det}(I) = 1$.

(Think of columns of matrix as n variables)

i.e. if $A = \begin{pmatrix} | & & | \\ A_1 & - - & A_n \\ | & & | \end{pmatrix}$

$$\text{Det}(A) = \text{Det}\underbrace{(A_1, \ldots, A_n)}_{n\text{-linear map}}$$

Examples for small values of n:

If $n=1$

$$A = (a)$$

We know that $\mathrm{Det}(I) = 1$, and Det is linear so $\mathrm{Det}(A) = a$.

If $n=2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad A_2 = \begin{pmatrix} b \\ d \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathrm{Det}(A_1, A_2) = \mathrm{Det}(ae_1 + ce_2,\ be_1 + de_2)$$

$$= \mathrm{Det}(ae_1 + ce_2,\ be_1) + \mathrm{Det}(ae_1 + ce_2,\ de_2)$$

$$= \mathrm{Det}\left(ae_1, be_1 \right) + \mathrm{Det}\left(ce_2, be_1 \right)$$

$$+ \mathrm{Det}\left(ae_1, de_2 \right) + \mathrm{Det}\left(ce_2, de_2 \right)$$

$$\mathrm{Det}\left(ae_1, be_1 \right) = a\,\mathrm{Det}\left(e_1, be_1 \right)$$
$$= a\,b\,\mathrm{Det}\left(e_1, e_1 \right)$$
$$= 0 \quad (\text{because Det is alternating})$$

Similarly, $\mathrm{Det}\left(ce_2, de_2 \right) = 0$

$$\mathrm{Det}\left(ae_1, de_2 \right) = \mathrm{Det}\,ad \quad (e_1, e_2) = ad$$
$$(\text{because } \mathrm{Det}(I) = 1)$$

$$\mathrm{Det}\left(ce_2, be_1 \right) = cb\,\mathrm{Det}\left(e_2, e_1 \right)$$
$$= -cb\,\mathrm{Det}(e_1, e_2) = -cb$$

So $\mathrm{Det}(A) = ad - bc \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Thm: There is exactly one determinant function

$$\text{Det} : M_{n\times n}(F) \longrightarrow F.$$

If D is any alternating n-form on $M_{n\times n}(F)$, then

$$D(A) = \text{Det}(A)\, D(I).$$

Pf: Let D be a alternating n-form on $M_{n\times n}(F)$.

Say $A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix}$

$$A_1 = \sum_{i=1}^{n} A_{i1} e_i$$

$$\vdots$$

$$A_j = \sum_{i=1}^{n} A_{ij} e_i$$

$$\vdots$$

$$D(A_1, \ldots, A_j, \ldots, A_n)$$

$$= D\left(\sum_{i=1}^{n} A_{i1} e_i, \ldots, \sum_{i=1}^{n} A_{ij} e_i, \ldots\right)$$

$$= \sum_{i=1}^{n} A_{i1} D\left(e_1, \ldots, \sum_{i=1}^{n} A_{ij} e_i, \ldots\right)$$

Similarly, using linearity in each variable we ultimately get

$$D(A) = \sum_{k_1, k_2, \ldots, k_n} A_{k_1 1} A_{k_2 2} \cdots A_{k_n n} \, D(e_{k_1}, \ldots, e_{k_n})$$

If two indices k_i, k_j are equal then $D(e_{k_1}, \ldots, e_{k_i}, e_{k_j}, \ldots, e_{k_n}) = 0$

So, in $D(A)$ the only indices that will survive are permutations of k_i.

So,

$$D(A) = \left[\sum_{\sigma} (\text{signature } \sigma) A_{\sigma(1),1} \cdots A_{\sigma(n),n} \right]$$

σ permutation of $\{1, 2, \ldots, n\}$

$$\times D(I).$$

If we further impose that $D(I) = 1$

then

$$D(A) = \det(A)$$

$$= \sum_{\sigma} (\text{signature } \sigma) A_{\sigma(1),1} \cdots A_{\sigma(n),n}$$

Example of 3×3 matrix:

$$A = \begin{pmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{pmatrix}$$

$$\mathrm{Det}(A) = \mathrm{Det}(A_1, A_2, A_3)$$

$$= \mathrm{Det}\begin{pmatrix} a_{11}e_1 + a_{21}e_2 + a_{31}e_3, \\ a_{12}e_1 + a_{22}e_2 + a_{32}e_3, \\ a_{13}e_1 + a_{23}e_2 + a_{33}e_3 \end{pmatrix}$$

$$= \mathrm{Det}(a_{11}e_1, A_2, A_3) + \mathrm{Det}(a_{21}e_2, A_2, A_3) + \mathrm{Det}(a_{31}e_3, A_2, A_3)$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32})$$
$$+ a_{21}(-a_{12}a_{33} + a_{32}a_{13})$$
$$+ a_{31}(a_{12}a_{23} - a_{22}a_{13})$$