MATH 314 (Lecture 18)

Topics to be discussed today

Eigenvalues, characteristic polynomial

$\mathbb{Z}$	$F[x]$
Absolute value	Degree
Division algorithm	Division algorithm
primes	irreducibles
Unique factorization into primes	Unique factorization into irreducibles.

Division algorithm to find GCD

Example: $GCD(21, 36)$

$$36 = 21(1) + 15$$

(*)

$$21 = 15(1) + 6$$

$$15 = 6(2) + 3 \leftarrow \text{3 is the GCD}$$

$$6 = 3(2) + 0$$

Thm: Applying (*) algorithm produces GCD of (a,b).

Pf: Let r be the GCD of a and b, i.e. r is the last non-zero remainder in (*) algorithm.

① Let us show that r is a common divisor of a and b.

$$a = b q_0 + r_0$$

$$b = r_0 q_1 + r_1$$

$$r_0 = r_1 q_2 + r_2$$

$$\vdots$$

$$r_{i-1} = r_i q_{i+1} + r \quad \longleftarrow \text{GCD}$$

$$r_i = r \; q_{i+2} + 0$$

n-th step

From last step we know that $r \mid r_i$,

from $(n-1)$-th step ———————— $r \mid r_{i-1}$

Similarly if we trace back, we can deduce that $r \mid b$ and $r \mid a$.

Hence r is a common divisor of a and b.

(2) We will show that r is greatest among all common divisors of a & b,

i.e. if $c \mid a$ and $c \mid b$ then $c \mid r$.

From (*) we can deduce that we can write

$$r = ax + by$$ where x & y are some integers.

Let's see an example.

$$36 = 21(1) + 15$$

$$21 = 15(1) + 6$$

$$15 = 6(2) + 3 \leftarrow$$

$$6 = 3(2) + 0$$

$$\begin{aligned} 3 &= 15 - 6(2) \\ &= 15 - (21-15)(2) \\ &= 15 - 21(2) + 15(2) \\ &= 15(3) - 21(2) \\ &= (36-21)(3) - 21(2) \\ &= 36(3) - 21(5) \end{aligned}$$

$r = ax + by$

So, if $c \mid a$ and $c \mid b$ $\Rightarrow$ $c \mid r$

Eigen values

Let F be a field. Let V be a vector space over a field F.

Let $T: V \longrightarrow V$ be a linear map.

Motivating qun: Does there exist a basis B for V such that matrix of T with respect to B is diagonal?

Examples: 1) Consider $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$e_1 \longmapsto -e_2$$
$$e_2 \longmapsto e_1$$

So, matrix of T with respect to standard basis is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longleftarrow \text{not diagonal}$$

Suppose there exists some basis B with respect to which T is diagonal.

That new matrix is

$$A' = C_{B \to \text{Standard basis}} \, A \left(C_{B \to \text{Standard bases}} \right)^{-1}$$

$\Rightarrow$ $\det A' = \det A$

Trace A' = Trace A

$$A' = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

$c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$c_1 c_2 = 1 \Rightarrow c_1(-c_1) = 1$

$\downarrow$

No real number satisfies this!

2) Consider

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$e_1 \longmapsto e_2$$

$$e_2 \longmapsto e_1$$

$B' = \{e_1 + e_2, \; e_1 - e_2\}$

B' is a basis.

$$e_1+e_2 \mapsto e_1+e_2$$
$$e_1-e_2 \mapsto -(e_1-e_2)$$

So, matrix of T with respect to B' is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Defn: 1) An eigenvalue of T is a scalar $c \in F$ such that there exists a non-zero $\alpha \in V$ satisfying $T\alpha = c\alpha$.

2) If c is an eigenvalue of T, then any α such that $T\alpha = c\alpha$ is called eigenvector associated to c.

3) Eigenspace associated with c is

$$\{\alpha \in V \mid \alpha \text{ is eigenvector of } T \text{ with eigenvalue } c\}$$

Similarly we can define eigenvalues, eigenvectors, eigenspaces of a matrix A.

Defn: Characteristic polynomial, denoted as $\mathrm{char}_A(x)$ is defined as $\det(xI - A)$.

Example: $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

$$xI - A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} x-1 & 0 \\ -1 & x-2 \end{pmatrix}$$

$$\det(xI - A) = (x-1)(x-2)$$
$$= x^2 - 3x + 2$$

Some observations

1) $\mathrm{Char}_A(A) = \det(AI - A)$

$= \det(0) = 0$

2) For $c \in F$, $\mathrm{Char}_A(c) = 0$

$\iff c$ is an eigenvalue of A.

<u>Proof:</u> $\mathrm{Char}_A(c) = 0 \iff \det(cI - A) = 0$

$\iff T : V \to V$ is not

$v \mapsto (cI - A)v$

invertible $\iff \ker T \neq \{0\}$

$\iff \exists$ non-zero $\alpha \in V$ s.t

$T(\alpha) = 0$

$\iff (cI - A)\alpha = 0 \iff c\alpha = A\alpha$

$\iff c$ is an eigenvalue of A.

Lemma: Similar matrices have same characteristic polynomial, hence same eigenvalues.

Pf: Suppose $B = PAP^{-1}$

$$\det(xI - B) = \det(xI - PAP^{-1})$$
$$= \det(xPP^{-1} - PAP^{-1})$$
$$= \det(P(xP^{-1} - AP^{-1}))$$
$$= \det(P(xI - A)P^{-1})$$
$$= \det(P)\det(xI - A)\det(P^{-1})$$
$$= \det(xI - A)$$
$$= \text{Char}_A(x).$$

They need not have same eigen vectors!

If $\quad B\alpha = c\alpha \leftarrow \alpha$ eigen vector

$PAP^{-1}(\alpha) = c\alpha$

$A(P^{-1}(\alpha)) = c(P^{-1}\alpha) \leftarrow P^{-1}\alpha$ eigen vector

Point to note: Eigenvalues depend on the field. For example, take

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathrm{char}_A(x) = \det(xI - A) = \det\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$$

$$= x^2+1$$

x^2+1 is irreducible in $\mathbb{R}[x]$ but is reducible over $\mathbb{C}[x]$.

Over $\mathbb{C}$, $x^2+1 = (x+i)(x-i)$

So, A has no eigenvalues over $\mathbb{R}$ but has 2 eigenvalues over $\mathbb{C}$, $i, -i$.

Lemma: Let $T: V \to V$ be a linear map.

Let $c_1, \dots, c_m$ be distinct eigenvalues of T. Let $W_1, \dots, W_m$ be the corresponding eigenspaces. Then

$$\dim(W_1 + \dots + W_m) = \dim(W_1) + \dots + \dim(W_m)$$

Pf: We will show that if

$$w_1 + w_2 + \dots + w_m = 0 \qquad w_i \in W_i$$

then $\quad w_i = 0 \quad$ for $\; i \in \{1, 2, \dots, m\}$.

Choose polynomials $f_1, \dots, f_m$ such that

$$f_i(c_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

For $i \in \{1, 2, \dots, m\}$

$$0 = f_j(T)\, 0 = f_j(T)(w_1 + \dots + w_m)$$

$$= f_i(T)(w_1) + \cdots + f_i(T)(w_m)$$

$$= f_i(c_1)(w_1) + \cdots + f_i(c_m)(w_m)$$

$$= w_i$$

So, done.

Thm: Let $T: V \to V$ be a linear map.

Let $c_1, \ldots, c_m$ be distinct eigenvalues of T and let $W_1, \ldots, W_m$ be corresponding eigenspaces.

The following are equivalent.

i) There is a basis of V with respect to which T is diagonal.

ii) The characteristic polynomial of T is $(x - c_1)^{d_1} \cdots (x - c_m)^{d_m}$

iii) $\dim V = \dim W_1 + \cdots + \dim W_m$

Pf: (i) $\Rightarrow$ (ii)

Let A be the matrix repn of T with respect to which T is diagonal.

$$A = \begin{bmatrix} c_1 & & & & & & & \\ & \ddots & & & & & 0 & \\ & & c_1 & & & & & \\ & & & c_2 & & & & \\ & & & & \ddots & & & \\ & & & & & c_2 & & \\ & 0 & & & & & c_m & \\ & & & & & & & \ddots \\ & & & & & & & & c_m \end{bmatrix}$$

$$\det(XI - A) = (X - c_1)^{d_1} \cdots (X - c_m)^{d_m}$$

(ii) $\Rightarrow$ (iii)

Let us show that $\dim W_i \geq d_i$

We will show it for $i = 1$, for remaining i's it is similar.

c_1 appears in first d_1 columns.

So, $Ae_1 = c_1 e_1, \ldots, Ae_{d_1} = c_1 e_{d_1}$

$\{e_1, \ldots, e_{d_1}\}$ are eigenvectors.

So, $\dim W_1 \geq d_1$.

$$d_1 + d_2 + \cdot + d_m = n = \dim V$$

$$\dim W_1 + - - + \dim W_m \leq n$$

but each $\dim W_i \geq d_i$

So

$$\underset{\substack{\parallel \\ n}}{d_1 + \cdot + d_m} \leq \dim W_1 + \cdot + \dim W_m \leq n$$

So, $d_i = \dim W_i$ & (iii) is proved.

(iii) $\Rightarrow$ (i)

We know that $V = W_1 + - - + W_m$

Let B_i be ordered bases for W_i

then $\{B_1, \cdots, B_m\}$ is ordered bases for V. (follows from direct sum).

with respect to $\{B_1, \cdots, B_m\}$, T is diagonal.