MATH 314 (Lecture 5)

Topics to be discussed today

Dimension formula, Exchange theorem

Recall: V vector space over a field F

A basis B is a subset of V such that

i) $\text{Span}(B) = V$

ii) B is linearly independent

Examples: 1) $V = \mathbb{R}^n$

$$B = \left\{ \begin{array}{c} (1,0,0,\dots,0) \\ (0,1,0,\dots,0) \\ \vdots \\ (0,0,\dots,1) \end{array} \right\}$$

2) $V = \mathbb{C}$ over $\mathbb{R}$

$$B = \{1, i\}$$

B need not be unique and B need not be finite!

Last time we showed that

1) If V has a basis of finite size, say m then every basis has size m.

2) If V has a basis of finite size, say m then any linearly independent set can be extended to a basis.

(Proof strategy: Start with a linearly independent set, say S if $|S| = m$ then S is already a basis otherwise take $S \cup \{v\}$ where $v \notin \text{Span}(S)$, repeat this till we get a set of size m.)

Today we will show that every vector space has a basis.

Thm: Every vector space has a basis.

To prove this, we need more background.

Partially Ordered Set (POSET)

Defn: Let S be a set. A relation R is a subset of $S \times S$. We say that $a \leq b$ (or a is related to b) if $(a,b) \in R$.

(Partial Order) We say that a relation R on S is a partial order if

1) $a \leq a$ for all $a \in S$ (Reflexivity)

2) If $a \leq b$ & $b \leq a$ then $a = b$
(Anti-Symmetry)

3) If $a \leq b$ & $b \leq c$ then $a \leq c$.

Defn: We say that $(S, \leq)$ is a POSET if $\leq$ is a partial order on S.

Examples: 1) $S = \mathbb{N}$

$\leq$ = division i.e, we say that $a \leq b \iff a \mid b$

i) $a \mid a$ for all $a \in \mathbb{N}$

ii) If $a \mid b$ & $b \mid a$ then $a = b$

iii) If $a \mid b$ & $b \mid c$ then $a \mid c$.

2) Let V be a vector space over F.

S = Set of all subsets of V

$\leq$ = inclusion, i.e; we say that $a \leq b \iff a \subseteq b$

i) $a \subseteq a$ for all $a \in S$

ii) If $a \leq b$ & $b \leq a$ then $a = b$

iii) If $a \leq b$ & $b \leq c$ then $a \leq c$

<u>Defn (CHAIN)</u> A <u>chain</u> is a totally ordered subset of a POSET, i.e. any two elements in a chain are related to each other.

<u>Defn (UPPER BOUND)</u> We say that a chain C has an <u>upper bound</u> x if $y \leq x$ for all $y \in C$.

<u>(ZORN'S LEMMA)</u> Let S be a POSET such that every chain in S has an upper bound in S. Then S has a maximal element, i.e., there exists an $m \in S$ such that $x \leq m$ for all $x \in S$.

Thm: Every vector space has a basis.

Proof: Let V be a vector space over a field F.

Case I: $V = \{0\}$, then ϕ is a basis for V.

Case II: $V \neq \{0\}$, $P = \{S \subseteq V \mid S \text{ is linearly independent}\}$

Define a relation R on P as follows

$$S_1 \leq S_2 \iff S_1 \subseteq S_2$$

R is a partial order.

We will now show that every chain has an upper bound.

Let $C = \{S_\alpha\}_{\alpha \in I}$ be a chain.

Consider $\bigcup_{\alpha \in I} S_\alpha$, this is an upper bound for C because $S_\alpha \leq \bigcup_{\alpha \in I} S_\alpha$ for all $\alpha \in I$.

To apply Zorn's lemma, we need to show that $\bigcup_{\alpha \in I} S_\alpha$ is linearly independent.

Suppose $c_1 x_1 + c_2 x_2 + \cdots + c_r x_r = 0$

for some x_i's in $\bigcup_{\alpha \in I} S_\alpha$

and c_i's in F.

Pick an S_α that contains $x_1, x_2, \ldots, x_r$

Since S_α is linearly independent

$$c_1 = c_2 = \cdots = c_r = 0.$$

So by Zorn's lemma there exists a maximal element B in P.

Claim: B is basis for V.

pf: Since B lies in P, it is linearly independent. We need to show that

B spans V.

Suppose it doesn't. Then there exists a $v \in V$ such that $v \notin \text{Span}(B)$. Then

$B \cup \{v\}$ is linearly independent.

(We showed this in last class.)

But B is maximal so, $B = B \cup \{v\}$

This is a contradiction. So, B spans V.

Hence, B is a basis for V.

This is an existential proof. It does not tell us how to construct a basis.

Dimension

We say that V is <u>finite dimensional</u> over F if there exists a basis of finite size. Otherwise, we say that V is <u>infinite dimensional</u>.

(Recall that if V is finite dimensional over F, then every basis has same size and we denote this number by $\dim V$ or $\dim_F V$)

<u>Lemma:</u> If W is a <u>proper subspace</u> of a finite dimensional vector space V, then $\dim W < \dim V$.

<u>Pf:</u> If $B_W = \{v_1, \ldots, v_r\}$ is a basis for W, then

(1) B_W does not span V because otherwise $\text{Span}(B_W) = W = V$ but W is a proper subspace of V.

(2) B_W is a linearly independent subset of V.

In last class, we showed that any linearly independent subset can be extended to a basis.

So, B_W can be extended to a basis of V.

Hence $\dim W < \dim V$.

<u>Thm:</u> If W_1 & W_2 are finite-dimensional subspaces of V, then $W_1 + W_2$ is finite-dimensional and

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Pf: $\dim(W_1 \cap W_2) \leq \dim W_1$ and

$\dim(W_1 \cap W_2) \leq \dim W_2$

Suppose $\{\alpha_1, \ldots, \alpha_r\}$ is a basis for $W_1 \cap W_2$.

We extend $\{\alpha_1, \ldots, \alpha_r\}$ to a basis for W_1

Say $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}$, and also

to a basis for W_2 say

$$\{\alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_t\}$$

$W_1 + W_2$ is spanned by $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t\}$

because if $w_1 + w_2 \in W_1 + W_2$ then

$$w_1 = c_1\alpha_1 + \cdots + c_r\alpha_r + d_1\beta_1 + \cdots + d_s\beta_s$$

$$w_2 = e_1\alpha_1 + \cdots + e_r\alpha_r + f_1\gamma_1 + \cdots + f_t\gamma_t$$

$$\begin{aligned} w_1 + w_2 = {} & (c_1 + e_1)\alpha_1 + \cdots + (c_r + e_r)\alpha_r \\ & + d_1\beta_1 + \cdots + d_s\beta_s \\ & + f_1\gamma_1 + \cdots + f_t\gamma_t \end{aligned}$$

We will show that this set is also linearly independent.

Suppose

$$\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_k \gamma_k = 0 \quad (1)$$

Then

$$\underbrace{-\sum z_k \gamma_k}_{\in W_2} = \underbrace{\sum x_i \alpha_i + \sum y_j \beta_j}_{\in W_1}$$

$$\sum z_k \gamma_k \in W_1 \cap W_2$$

So, $\sum z_k \gamma_k = \sum c_i \alpha_i$ for some $c_i \in F$

$$\sum z_k \gamma_k - \sum c_i \alpha_i = 0$$

Since $\{\alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_t\}$ is a linearly independent set

$$z_k = 0 \quad \forall k$$

Equation 1 reduces to

$$\sum x_i \alpha_i + \sum y_j \beta_j = 0$$

Since $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}$ is linearly independent $x_i = 0 \quad \forall i$

$y_j = 0 \quad \forall j$

$\dim(W_1 + W_2) = r + s + t$

$\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$= r + s + r + t - r$

$= r + s + t$

Hence, proved.