MATH 314 (Lecture 6)

Topics to be discussed today

Matrices, System of equations, row-reduced form, elementary row operations

An $m \times n$ matrix looks like

$$\begin{pmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ \\ a_{m1} & a_{m2} & - & - & - & a_{mn} \end{pmatrix}$$

Rows are vectors in $\mathbb{R}^n$

& Columns are vectors in $\mathbb{R}^m$

Product of Matrices

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{pmatrix}$$

(First column of A) b_{11} + (Second column of A) b_{21}

(First column of A) b_{12} + (Second column of A) b_{22}

$$\begin{pmatrix} (\text{First Row of } B)\, a_{11} + (\text{Second Row of } B)\, a_{12} \\ (\text{First Row of } B)\, a_{21} + (\text{Second row of } B)\, a_{22} \end{pmatrix}$$

In general, a column of AB is a linear combination of columns of A & a row of AB is a linear combination of rows of B.

<u>Defn</u>: 1) <u>Column Space</u> denoted as $Col(A)$ is the span of columns of A.

If A is $m\times n$ matrix

$$\underbrace{Col(A) \subseteq \mathbb{R}^m}_{\text{Subspace}}$$

2) <u>Row Space</u> denoted as $Row(A)$ is the span of rows of A.

If A is $m\times n$ matrix

$$Row(A) \subseteq \mathbb{R}^n$$

Lemma: $\text{Row}(A^T) = \text{Col}(A)$

A^T: Transpose of A

and $\text{Col}(A^T) = \text{Row}(A)$

Solving a system of equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Put this into augmented matrix

$$A|b = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Apply row operations to put it into row reduced echelon form and then solve it using backward substitution.

Row operations

1. Interchange two rows
2. Add a multiple of one row to another
3. Multiply a row by a non-zero scalar

Row reduced echelon form

1) Leading non-zero entry of a non-zero row is 1, we will call it pivot.

2) If a column contains 1, all other entries of that column must be 0.

3) Zero rows should be below non-zero rows.

4) Pivots are arranged in an increasing fashion, i.e

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & \\ 0 & \cdots & 1 & \vdots & \\ \vdots & & \vdots & 1 & \\ 0 & & 0 & 0 & \ddots \end{pmatrix}$$

Example:

$$x_1 + x_2 + x_3 = 5$$
$$2x_1 + x_2 - x_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 1 & -1 & 0 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -1 & -3 & -10 \end{array}\right)$$

$R_1 \to R_1 + R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & -1 & -3 & -10 \end{array}\right)$$

$R_2 \to -R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 10 \end{array}\right)$$

$$x - 2z = -5$$

$$y + 3z = 10$$

$$x = -5 + 2z, \quad y = 10 - 3z$$

$$\{(-5+2z,\ 10-3z,\ z) \mid z \in \mathbb{R}\}$$

are all the solutions.

Thm: If A is $m \times n$ matrix and $m < n$, then the system $Ax = 0$ has at least one non-zero solution.

Pf: Ax is linear combination of columns of A.

Now $Col(A) \subseteq \mathbb{R}^m$

There are n columns of A & $n > m$

So, columns of A are linearly dependent. So, a non-trivial combination of columns of A is 0.

Thm: If A is $n \times n$ matrix, then

A can be row-reduced to identity matrix $\Longleftrightarrow$ $Ax = 0$ has only the trivial solution

Pf: $\Rightarrow$ $(A \mid 0) \sim (I \mid 0)$ only trivial solution

$\Leftarrow$ If $Ax = 0$ has only the trivial Solution, then Columns of A are linearly independent

There are n columns, linearly independent so they form a basis for $\mathbb{R}^n$.

$\Rightarrow$ In row reduced echelon form each Column must Contain a Pivot

$\Rightarrow$ Each row has a pivot.

So, $A \sim I$.

<u>Qn:</u> Find a basis for Column space of

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -5 & 3 & -7 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

$R_3 \to R_3 - R_1$

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -5 & 3 & -7 \\ 0 & -2 & 1 & -1 \end{pmatrix}$$

$R_2 \to \left(\frac{-1}{5}\right) R_2$

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & 1 & -3/5 & 7/5 \\ 0 & -2 & 1 & -1 \end{pmatrix}$$

$R_1 \to R_1 - 3R_2$

$R_3 \to R_3 + 2R_2$

$$A = \begin{pmatrix} 1 & 0 & 4/5 & -1/5 \\ 0 & 1 & -3/5 & 7/5 \\ 0 & 0 & -1/5 & 9/5 \end{pmatrix}$$

$R_3 \to -5 R_3$

$R_2 \to R_2 + 3/5 R_3$

$R_1 \to R_1 - 4/5 R_3$

$$A = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -9 \end{pmatrix}$$

Pivot columns

Basis for $\mathrm{Col}(A) = \{$ Column 1, Column 2, Column 3 $\}$

A $m \times n$ matrix

Thm: Set of all solutions of $Ax = 0$ is a subspace of $\mathbb{R}^n$.

Pf: If x_1 & x_2 satisfy $Ax_1 = 0$ & $Ax_2 = 0$

then $A(cx_1 + x_2) = A(cx_1) + Ax_2$

$= cAx_1 + Ax_2 = 0$

Defn Null space of A denoted as

$$\text{Null}(A) = \text{Set of all solutions of } Ax = 0$$

Defn: $\text{Rank}(A) = \dim(\text{Col } A)$

$\text{Nullity}(A) = \dim(\text{Null } A)$

Rank- Nullity Thm If A is $m \times n$ matrix, then

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

Example: $A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{pmatrix}$

Row reduced form $A = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -9 \end{pmatrix}$

The Columns that contain Pivot form a basis for Col (A).

Null (A)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & -9 & 0 \end{array}\right)$$

$$x_1 + 7x_4 = 0$$
$$x_2 - 4x_4 = 0$$
$$x_3 - 9x_4 = 0$$

$$(-7x_4, 4x_4, 9x_4, x_4)$$

$\{(-7, 4, 9, 1)\}$ is a basis for Null (A).

Columns that don't contain pivot give rise to Null (A).

In general, if there are n Columns, each column is either part of basis for Col (A) or part of basis for Null (A).

So, rank (A) + Nullity (A)

= no. of columns = n

Thm: $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$
$= \text{rank}(A)$

(A is $m\times n$ matrix)

Pf: Suppose $\{C_1, C_2, \ldots, C_r\}$ form a basis for Col (A). Let C_i be any column, then

$$C_i = a_{1i}C_1 + a_{2i}C_2 + \cdots + a_{ri}C_r$$

Define $B = \begin{pmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_r \\ | & | & & | \end{pmatrix}_{m\times r}$

$$X = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rn} \end{pmatrix}_{r\times n}$$

$$A = BX$$

Rows of A are linear combinations of rows of X.

So, $\text{Row}(A) \subseteq \text{Row}(X)$

$$\dim(\text{Row}(A)) \leq \dim(\text{Row}(X))$$
$$\leq r$$
$$\parallel$$
$$\dim(\text{Col}(A))$$

Apply the above argument to A^T, to get

$$\dim(\text{Row}(A^T)) \leq \dim(\text{Col}\, A^T)$$
$$\parallel \qquad\qquad \parallel$$
$$\dim(\text{Col}\, A) \leq \dim(\text{Row}\, A)$$

So, $\boxed{\dim(\text{Col}\, A) = \dim(\text{Row}\, A) = \text{rank}}$