MATH 314 (Lecture 3)

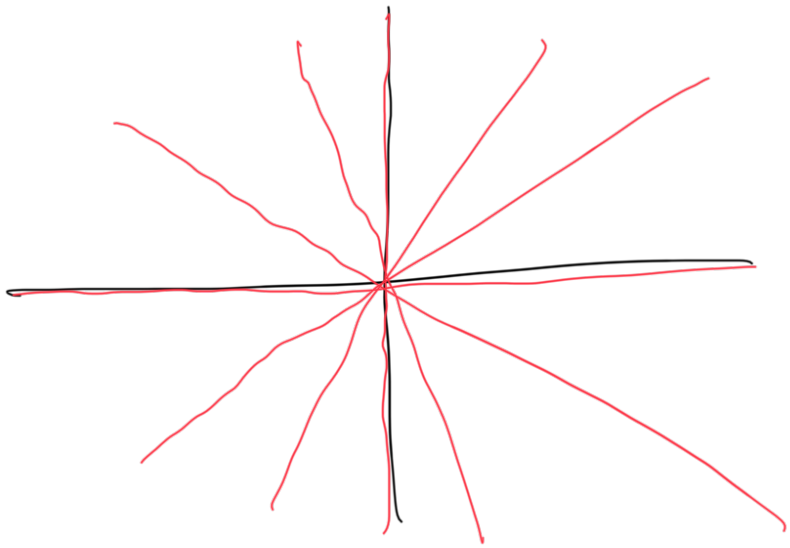
Topics to be discussed today

Direct sum, Complements & linear independence / dependence

Recall from last time

We discussed Subspaces

Example:



Subspaces of $\mathbb{R}^2$ are $\{0\}$, $\mathbb{R}^2$ & lines passing through origin

→ We saw that intersection of any arbitrary collection of subspaces is again a subspace.

→ Sum of Subspaces

→ Direct sum of two subspaces

→ Span of a set of vectors

Basis of a vector space

V vector space over field F

(Recall) Span of $\{v_1, \dots, v_n\}$ $v_i \in V$

There are two ways to think of Span of $\{v_1, \dots, v_n\}$

1. It is the intersection of all subspaces of V that contain $\{v_1, \dots, v_n\}$.

2. It is equal to $\{c_1 v_1 + \dots + c_n v_n \mid c_i \in F\}$

These two statements are equivalent.

Defn: (Linear independence) We say that a set $\{v_1, \dots, v_n\} \subseteq V$ is linearly independent if $\sum_{i=1}^{n} c_i v_i = 0$ holds only when $c_i = 0$ for $i \in \{1, 2, \dots, n\}$.

Examples: (1) $v_1 = (1,1)$
$v_2 = (1,2)$ $\in \mathbb{R}^2$

Is $\{v_1, v_2\}$ linearly independent?

Suppose $c_1 v_1 + c_2 v_2 = 0$

$$c_1 (1,1) + c_2 (1,2) = 0$$

$$c_1 + c_2 = 0 \quad \text{---} \quad (1)$$

$$c_1 + 2c_2 = 0 \quad \text{---} \quad (2)$$

Solving (1) & (2) for c_1 & c_2 we get

$c_1 = -c_2$
& $c_1 = -2c_2$ $\Rightarrow$ $-c_2 = -2c_2$

$\Rightarrow c_2 = 0$
$\Rightarrow c_1 = 0$

Yes! They are linearly independent.

Defn: (linear dependence) We say that a set $\{v_1, \ldots, v_n\} \subseteq V$ is linearly dependent if it is not linearly independent.

Qn: Let us take set of two vectors in $\mathbb{R}^n$, $S = \{v_1, v_2\}$. When is S linearly independent / dependent?

Example: $S = \{(1,1), (1,2), (1,3)\} \subseteq \mathbb{R}^2$

Is S dependent or independent?

Soln: $c_1(1,1) + c_2(1,2) + c_3(1,3) = (0,0)$

$$c_1 + c_2 + c_3 = 0$$
$$c_1 + 2c_2 + 3c_3 = 0$$

One solution is $c_1 = 1$, $c_2 = -2$, $c_3 = 1$

Defn (Basis) Let V be a vector space over a field F. We say that a set S forms a basis for V if

① S is linearly independent.

② $\text{Span}(S) = V$

Observation: Given a vector space, there can be more than one basis for it.

Example: $V = \mathbb{R}^2$

$$B_1 = \{(1,0), (0,1)\}$$

$$B_2 = \{(1,1), (1,2)\}$$

$\text{Span}(B_i) = V$

B_i is linearly independent.

Direct Sums

Recall: We discussed direct sum of two subspaces. The sum $W_1 + W_2$ is direct sum

$\iff W_1 \cap W_2 = \{0\}$.

Lemma: Let W_1, W_2 be two subspaces of V such that $W_1 \cap W_2 = \{0\}$. Then for any $w_1 \in W_1$ & $w_2 \in W_2$

$$w_1 + w_2 = 0 \iff w_1 = w_2 = 0.$$

Proof: $\Leftarrow$: Assume that $w_1 = w_2 = 0$

Then $w_1 + w_2 = 0$.

$\Rightarrow$ Assume that $w_1 + w_2 = 0$

So, $w_1 = -w_2 \Rightarrow w_2 \in W_1$

$\Rightarrow w_2 \in W_1 \cap W_2$

$\Rightarrow w_2 = 0$

$\Rightarrow w_1 = 0$

This lemma helps us to generalize direct sums for more than two subspaces.

<u>Lemma:</u> Let $W_1, \dots, W_n$ be subspaces of V. Let $W = W_1 + \dots + W_n$. The following statements are equivalent.

1) If $w_1 + w_2 + \dots + w_n = 0$ for $w_i \in W_i$ then $w_i = 0 \ \forall \ i \in \{1, 2, \dots, n\}$.

2) For each j, $j \in \{2, \dots, n\}$ we have

$$W_j \cap (W_1 + \dots + W_{j-1}) = \{0\}$$

<u>Pf:</u> (1) $\Rightarrow$ (2)

Fix any $j \in \{2, \ldots, n\}$

Say $\alpha \in W_j$ &

$\alpha \in W_1 + \cdots + W_{j-1}$

We want to show that $\alpha = 0$.

Since $\alpha \in W_j$, $-\alpha \in W_j$

$$\underset{\substack{\cap \\ W_1+\cdots+W_{j-1}}}{\alpha} + \underset{\substack{\cap \\ W_j}}{(-\alpha)} = 0$$

Using ① we know that $\alpha = 0$.

(2) $\Rightarrow$ (1)

Assume that

$$\alpha_1 + \cdots + \alpha_n = 0 \qquad \alpha_i \in W_i$$

$$\underset{\substack{\cap \\ W_n}}{\alpha_n} = \underset{\substack{\cap \\ W_1+\cdots+W_{n-1}}}{-(\alpha_1 + \cdots + \alpha_{n-1})}$$

$$\alpha_n \in W_n \cap (W_1 + \cdots + W_{n-1})$$

For $j = n$, using (2) we know that

$$\alpha_n = 0$$

Now we are left with $\alpha_1 + \cdots + \alpha_{n-1} = 0$

Repeat the above argument to show that each $\alpha_i = 0$.

Defn: (Direct Sum of more than two subspaces)

We say that $W = W_1 + \cdots + W_n$ is a direct sum of $W_1, \ldots, W_n$ if any of the above conditions is satisfied.

Lemma: Let $W = W_1 + \cdots + W_n$ be sum of $W_1, \ldots, W_n$. Then W is a direct sum of $W_1, \ldots, W_n \iff$ each $\alpha \in W$ can be expressed uniquely as $\alpha = \alpha_1 + \cdots + \alpha_n$ for $\alpha_i \in W_i$.

Pf: $\Rightarrow$ Suppose $\alpha = \alpha_1 + \cdots + \alpha_n$
$$= \beta_1 + \cdots + \beta_n$$

So, $0 = (\alpha_1 - \beta_1) + \cdots + (\alpha_n - \beta_n)$

$\Rightarrow \alpha_i - \beta_i = 0$ for $i \in \{1, 2, \ldots, n\}$

$\Rightarrow \alpha_i = \beta_i$

$\Leftarrow$ We want to show that if $\alpha_1 + \cdots + \alpha_n = 0$ then $\alpha_i = 0$

but, $0 + \cdots + 0 = 0$

So, $\alpha_i = 0$ for $i \in \{1, 2, \ldots, n\}$

Notations: $W = W_1 + \cdots + W_m$ (Sum)

$W = W_1 \oplus \cdots \oplus W_m$ (Direct Sum)

Examples:

① $W_1 = \{(x, 0, 0) \mid x \in \mathbb{R}\}$

$W_2 = \{(0, y, 0) \mid y \in \mathbb{R}\}$

$W_3 = \{(0, 0, z) \mid z \in \mathbb{R}\}$

$$\mathbb{R}^3 = W_1 \oplus W_2 \oplus W_3$$

② In general if $W_i = \{(0, \ldots, x_i, \ldots 0) \mid x_i \in \mathbb{R}\}$ then

$$\mathbb{R}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

Are these decompositions unique?

No! Consider $\mathbb{R}^2$

$$W_1 = \{(x,x) \mid x \in \mathbb{R}\}$$

$$W_2 = \{(0,y) \mid y \in \mathbb{R}\}$$

$\mathbb{R}^2 = W_1 + W_2$ because any $(a,b) \in \mathbb{R}^2$

$$(a,b) = (a,a) + (0, b-a)$$

Moreover, if $\alpha \in W_1 \cap W_2$

$$\alpha = (x,x) \ \&$$

$$\alpha = (0,y)$$

for some $x, y \in \mathbb{R}$

$$\Rightarrow (x,x) = (0,y)$$

$$\Rightarrow x = 0$$

$$\Rightarrow \alpha = (0,0).$$

So, W_1 & W_2 form a direct sum.

Example: ($n \times n$ matrices)

$$W_{i,j} = \left\{ \begin{pmatrix} 0 & \cdots & \vdots & \cdots & 0 \\ \vdots & & X_{ij} & & \vdots \\ 0 & \cdots & \vdots & \cdots & 0 \end{pmatrix} \;\middle|\; X_{ij} \in \mathbb{R} \right\}$$

$$\begin{aligned} M_{n\times n}(\mathbb{R}) = \; & W_{1,1} \oplus W_{1,2} \oplus \cdots \oplus W_{1,n} \\ & \oplus W_{2,1} \oplus W_{2,2} \cdots \oplus W_{2,n} \\ & \vdots \qquad \vdots \qquad\qquad \vdots \\ & \oplus W_{n,1} \oplus W_{n,2} \cdots \oplus W_{n,n} \end{aligned}$$