MATH 314 (Lecture 10)

Topics to be discussed today

Ranks, System of equations, congruences

EQUIVALENCE RELATION

Defn: We say a relation R is an equivalence relation if the following hold:

1) R is reflexive. ($x \sim x \ \forall x \in R$)
2) R is Symmetric (If $x \sim y$, then $y \sim x$)
3) R is transitive (If $x \sim y$, $y \sim z$, then $x \sim z$)

Examples or non-examples:

1) $X = \mathbb{R}$, $x \sim y \iff x < y$
2) $X = \mathbb{R}$, $x \sim y \iff x = y$
3) $X = \mathbb{R}^2$, $x \sim y \iff x = Ay$ for some $A \in GL_2(\mathbb{R})$

4) $X = \mathbb{Z}$

Fix a natural number N.

$x \sim y \iff N \mid x - y$.

EQUIVALENCE CLASSES

Let X be a set. Suppose R is an equivalence relation on X.

Equivalence class of $a = \{ y \in X \mid a \sim y \}$

We will denote equivalence class of a as $\overline{a}$.

Some observations.

1) $a \in \overline{a}$

2) If $a \not\sim b$, then
$$\overline{a} \cap \overline{b} = \phi$$

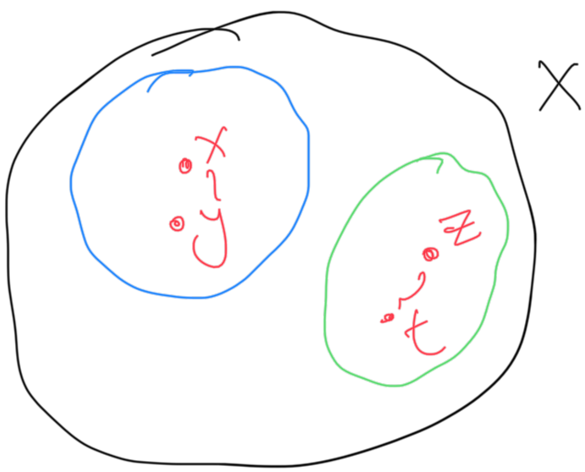
because if $z \in \overline{a} \Rightarrow a \sim z$

& $z \in \overline{b} \Rightarrow b \sim z$

$b \sim z \Rightarrow z \sim b$ (Reflexive)

$a \sim z$ and $z \sim b \Rightarrow a \sim b$ (Transitive)

Contradiction!



So, we can write X as a disjoint union of equivalence classes.

$$X = \bigsqcup_{\alpha \in I} \overline{a_\alpha}$$

indexing set of X

Let us compute equivalence classes for examples above.

1) $X = \mathbb{R}$, $x \sim y \iff x = y$

$$\bar{x} = \{x\}$$

2) $X = \mathbb{R}^2$, $x \sim y \iff x = Ay$ for some $A \in GL_2(\mathbb{R})$

If $x = (0,0)$, then
$\bar{x} = x$

If $x = (a,b)$ $ab \neq 0$

then $x \sim (1,0)$ Take $A = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ if $a \neq 0$

$A = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ if $b \neq 0$

$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

So, $\bar{x} = \mathbb{R}^2 - \{(0,0)\}$

$\mathbb{R}^2 = \overline{(0,0)} \sqcup \overline{(1,0)}$

3) $X = \mathbb{Z}$, $x \sim y \iff N \mid x - y$

What are its equivalence classes?

If $a, b \in \mathbb{Z}$ $a < b$, $a \neq 0$ then

$b = aq + r$ $r \in \{0, 1, 2, \ldots, a-1\}$

Suppose $N = 2$

$$\bar{1} = \{\text{Odd integers}\}$$

$$\bar{0} = \{\text{Even integers}\}$$

$$\mathbb{Z} = \bar{0} \sqcup \bar{1}$$

$N = 3$

$$\bar{0} = \{\text{all multiples of } 3\}$$

$$\bar{1} = \{3m+1 \mid m \in \mathbb{Z}\}$$

$$\bar{2} = \{3m+2 \mid m \in \mathbb{Z}\}$$

$$\mathbb{Z} = \bar{0} \sqcup \bar{1} \sqcup \bar{2}$$

In general, we have

$$\mathbb{Z} = \bar{0} \sqcup \bar{1} \sqcup \bar{2} \cdots \sqcup \overline{N-1}$$

In the context of vector spaces

Let V be a vector space over field F. Let W be a subspace of V.

Define R on V as follows

we say $v_1 \sim v_2 \iff v_1 - v_2 \in W$

(We say that $v_1 \equiv v_2 \mod W$ if $v_1 \sim v_2$)

Qn: Is it an equivalence relation?

Equivalence classes are known as <u>COSETS</u> of W.

$$\alpha \in V, \quad \bar{\alpha} = \{ \beta \in V \mid \beta - \alpha \in W \}$$
$$= \{ \beta \in V \mid \beta = \alpha + w \text{ for some } w \in W \}$$

$$\bar{\alpha} = \underset{\substack{\downarrow \\ \text{vector}}}{\alpha} + \underset{\substack{\downarrow \\ \text{set}}}{W}$$

$$= \{ \alpha + w \mid w \in W \}$$

<u>Notation:</u> We will denote the set of COSETS of W by V/W.

$$\bar{0} = W$$

Take two cosets $\alpha_1 + W$ and $\alpha_2 + W$

Define addition
$$(\alpha_1 + W) + (\alpha_2 + W) = (\alpha_1 + \alpha_2) + W$$

Scalar multiplication $\quad c(\alpha_1 + W) = c\alpha_1 + W$

__Qn:__ Are these operations well-defined?

__Qn:__ Is V/W a vector space with respect to these operations?

Answer: Let us show that addition is well-defined, i.e. we want to show that if $\alpha + W = \beta + W$ and $\gamma + W = \delta + W$ then

$$\alpha + \gamma + W = \beta + \delta + W$$

$$\begin{aligned}
\alpha + \gamma + W &= \{\alpha + \gamma + w \mid w \in W\} \\
&= \{\beta + w_0 + \delta + w_1 + w \mid w \in W\} \\
&= \{\beta + \delta + w_0 + w_1 + w \mid w \in W\} \\
&= \{\beta + \delta + w' \mid w' \in W\} \\
&= \beta + \delta + W
\end{aligned}$$

because $\alpha + W = \beta + W$

Similarly, we can show that scalar multiplication is well-defined.

$$(\alpha_1 + W) + (\alpha_2 + W) = (\alpha_1 + \alpha_2) + W$$

$$c(\alpha + W) = c\alpha + W$$

V/W is a vector space with addition & scalar multiplication defined above.

V/W is also called <u>quotient</u> <u>space</u>.

Some properties of V/W

① There exists a natural linear map

$$\eta : V \longrightarrow V/W$$
$$\alpha \longmapsto \alpha + W$$

$$\eta(\alpha+\beta) = (\alpha+\beta) + W$$
$$= (\alpha+W) + (\beta+W)$$
$$= \eta(\alpha) + \eta(\beta)$$

$$\eta(c\alpha) = c\alpha + W$$
$$= c(\alpha+W) = c\eta(\alpha)$$

which is surjective because any $\alpha + W \in V/W$
$= \eta(\alpha)$

$$\ker \eta = \{\alpha \in V \mid \alpha + W = 0 + W\}$$
$$= \{\alpha \in V \mid \alpha \in W\} = W$$

$\Rightarrow \dim V = \dim W + \dim \left(V/W \right)$

(2) $V = W \oplus W' \iff \eta|_{W'} : W' \to V/W$

is an isomorphism.

$\Rightarrow$ Suppose $V = W \oplus W'$.

Then $\dim V = \dim W + \dim W'$

$\Rightarrow \dim \left(V/W \right) = \dim W'$

$\ker \left(\eta|_{W'} \right) = W' \cap W = \{0\}$

$\Rightarrow \eta|_{W'}$ is surjective, so it is an isomorphism.

$\Leftarrow$: $\eta|_{W'} : W' \to V/W$ is an isomorphism

$W' \cap W = \{0\}$

Let $v \in V$, then the coset $v + W = \eta|_{W'}(a)$

for some $\alpha \in W'$.

$\Rightarrow \quad v + W = \alpha + W \qquad \alpha \in W'$

$\Rightarrow \quad v = \alpha + w \quad$ for some $w \in W$

$\Rightarrow \quad V = W + W'$.

Hence $\quad V = W \oplus W'$.