MATH 314 (Lecture 20)

Topics to be discussed today

Generalized Eigenspaces, Jordan normal form for nilpotent endomorphisms (section 7.3)

Thm 1: $T: V \to V$

T is triangulable $\iff$ $\min_T(x)$ is of the form

$$(x-c_1)^{r_1} \cdots (x-c_k)^{r_k}$$

$$c_i \in F$$

Thm 2: $T: V \to V$

T is diagonalizable $\iff$ $\min_T(x)$ is of the form

$$(x-c_1) \cdots (x-c_k)$$

$$c_i \in F$$

Defn: Let W be an invariant subspace of V with respect to $T: V \to V$. Let $\alpha \in V$. The T-conductor of α into V is

$$\{ g(x) \in F[x] \mid g(T)\alpha \in W \}.$$

Lemma 1. The T-conductor of α into V is an ideal of $F[X]$.

Pf: 1) Suppose f_1, f_2 lie in T-conductor.

Then $(f_1+f_2)(T)\alpha = (f_1(T) + f_2(T))\alpha$

$$= \underset{W}{\underset{\cap}{f_1(T)\alpha}} + \underset{W}{\underset{\cap}{f_2(T)\alpha}}$$

So, $(f_1+f_2)(T)\alpha \in W$.

2) If $g \in F[X]$ and $f \in$ T-conductor

then $(gf)(T)\alpha = g(T)(f(T)\alpha)$

$= g(T)\,w$

$\in W$ because W is T-invariant.

The unique monic generator of T-conductor of α into W will be called T-conductor.

Lemma 2: Suppose minimal polynomial of T factors as

$$\min_T(x) = (x-c_1)^{r_1} \cdots (x-c_k)^{r_k} \quad c_i \in F$$

Let W be a proper invariant subspace of V. Then

i) $\exists\ \alpha \in V$ such that $\alpha \notin W$

ii) $(T-cI)\alpha \in W$ for some eigenvalue c of T.

Proof: Let β be any vector that lies in V and not in W.

Now $\min_T(T) = 0$ so $\min_T(T)\beta \in W$

$\Rightarrow$ $\min_T(x)$ lies in T-conductor of β into W.

$\Rightarrow$ The T-conductor say g divides $\min_T(x)$.

Also g is not constant, otherwise since $g(T)\beta \in W \Rightarrow \beta \in W \longrightarrow\!\leftarrow$.

So g is of the form

$$(X-c_1)^{e_1} \cdots (X-c_k)^{e_k}$$ where $e_i \leq f_i$ and some $e_i > 0$.

$$g = (X-c_i)\, h$$

but $h(T)\beta \notin W$

Define $\alpha := h(T)\beta$

$$(T-c_iI)\alpha = (T-c_iI)\,h(T)\beta$$
$$= g(T)\beta \in W$$

<u>Pf</u> of <u>Theorem 1</u>: If T is triangulable

$$[T]_B = \begin{bmatrix} c_1 & & & & & & & \\ & \ddots & & & & & & \\ & & c_1 & & & & & \\ & & & c_2 & & & & \\ & & & & \ddots & & & \\ & & & & & c_2 & & \\ & & & & & & \ddots & \\ & & & & & & & c_r \\ & & & & & & & & \ddots \\ & & & & & & & & & c_r \end{bmatrix}$$

$\Rightarrow \min_T(x) = (X-c_1)^{e_1} \cdots (X-c_r)^{e_r}$

$\Leftarrow$: Suppose $\min_T(x) = (X-c_1)^{e_1} \cdots (X-c_r)^{e_r}$

Apply Lemma 2 to $W = \{0\}$.

There exists a vector $\alpha_1 \neq 0$ such that

$$(T - cI)\alpha_1 \in W$$

$$(T - cI)\alpha_1 = 0$$

$$\Rightarrow T\alpha_1 = c\alpha_1$$

Build a basis by repeated application of lemma 2 $\quad B = \{\alpha_1, \ldots$

Take $W = \text{Span}\{\alpha_1\}$

Apply lemma 2 again $\exists\ \alpha_2 \notin W$ s.t

$$(T - c'I)\alpha_2 = d\alpha_1$$

$$T\alpha_2 = d\alpha_1 + c'\alpha_2$$

$$B = \{\alpha_1, \alpha_2, \ldots$$

and continue like this till we get n vectors in B.

$[T]_B$ is upper triangular.

Pf of Thm 2: Assume T is diagonalizable.

Then V has a basis of eigenvectors say $\{\alpha_1, \dots, \alpha_n\}$.

Consider $p(x) = (x - c_1) \cdots (x - c_k)$

c_i are distinct eigenvalues of T.

$$p(T)(\alpha_1) = \left((T - c_1) \cdots (T - c_k)\right)(\alpha_1)$$

$$= (T - c_1)\alpha_1 \cdots (T - c_k)\alpha_1$$

$$= 0$$

Similarly $p(T)\alpha_i = 0$ for $i \in \{1, 2, \dots, n\}$

So $p(T) = 0$

Hence p is the minimal polynomial for T.

$\Leftarrow$: Suppose $\min_T(x) = (x - c_1) \cdots (x - c_k)$

c_i are distinct eigenvalues.

Assume that T is not diagonalizable.

Let W be subspace spanned by eigenvectors

So, $W \neq V$. W is T-invariant, so

apply lemma 2. There exists $\alpha \notin W$

s.t $(T - c_j I)\alpha \in W$.

$$\min_T(x) = (x - c_j)\, q(x)$$

We will show that $q(c_j) = 0$ which is a contradiction.

$$\beta := (T - c_j I)\alpha = \beta_1 + \cdots + \beta_k \qquad T(\beta_i) = c_i \beta_i$$

For any polynomial $g(x) \in F[x]$

$$\begin{aligned} g(T)\beta &= g(T)\beta_1 + \cdots + g(T)\beta_k \\ &= g(c_1)\beta_1 + \cdots + g(c_k)\beta_k \\ &\in W \end{aligned}$$

Consider the polynomial $q(x) - q(c_j)$

$$q(x) - q(c_j) = (x - c_j)\, h(x)$$

$$q(T)\alpha - q(c_j)\alpha = (T - c_j)\, h(T)\alpha = h(T)(T - c_j)\alpha$$

$$q(T)\alpha - q(c_j)\alpha \in W \quad \text{—} \quad (1)$$

$$0 = \min_T(T)\alpha = (T - c_j)\, q(T)\alpha$$

$$\Rightarrow \quad q(T)\alpha \in W \quad \text{—} \quad (2)$$

From (1) & (2) $\quad q(c_j)\alpha \in W$

If $q(c_j) \neq 0 \Rightarrow \alpha = \dfrac{w}{q(c_j)} \qquad w \in W$

$$\Rightarrow \alpha \in W \rightarrow\leftarrow$$

So, $q(c_j) = 0 \longrightarrow \leftarrow$.

So, T is diagonalizable.

Generalized Eigenvectors

$T : V \to V$ $\quad V$ dim n

Let $c \in F$.

Defn: A generalized eigenvector is $\alpha \in V$ satisfying $(T - cI)^k \alpha = 0$ for some $k \geq 0$.
The smallest such k is called the exponent of α.

Example: $V = \left\{ \begin{array}{l} \text{Set of all differentiable fns} \\ \text{on } \mathbb{R} \end{array} \right\}$

$$T : V \longrightarrow V$$
$$f \longmapsto \frac{df}{dx}$$

$\lambda \in \mathbb{R}$

$f(x) = e^{\lambda x}$ is an eigenvector

$f(x) = \underbrace{P(x)}_{\text{Polynomial}} e^{\lambda x}$ generalized eigenvector

$$(T-\lambda) f(x) = T(f(x)) - \lambda f(x)$$
$$= p'(x) e^{\lambda x} + \lambda e^{\lambda x} p(x) - \lambda p(x) e^{\lambda x}$$
$$= p'(x) e^{\lambda x}$$

So, $p(x) e^{\lambda x}$ is a generalized eigenvecher of exponent $\deg p + 1$.

Lemma: Define $\mathcal{E}_\lambda = \{$ Generalized eigenvechers corresponding to $\lambda \}$

Then

i) $\mathcal{E}_\lambda$ is a T- invariant subspace.

ii) $\mathcal{E}_\lambda \neq 0 \iff \lambda$ is an eigenvalue of T

Pf: if v_1 & $v_2 \in \mathcal{E}_\lambda$ say

$$(T-\lambda)^{k_1} v_1 = 0 \quad \& \quad (T-\lambda)^{k_2} v_2 = 0$$

Say $k_1 \geq k_2$, then

$$(T-\lambda)^{k_1} v_1 = 0 \quad \& \quad (T-\lambda)^{k_1} v_2 = 0$$

$$(T-\lambda)^{k_1}(v_1+v_2) = 0$$

So, $v_1+v_2 \in \mathcal{E}_\lambda$.

Similarly it is closed under scalar multiplication.

$T(\mathcal{E}_\lambda) \subseteq \mathcal{E}_\lambda$ because if

$$(T-\lambda)^k v = 0, \text{ then}$$

$$(T-\lambda)^k Tv = T(T-\lambda)^k v = 0$$

ii) $\Leftarrow$ if λ is an eigenvalue of T, let v be its eigenvector. Then, $v \in \mathcal{E}_\lambda$.

$\Rightarrow$: Suppose v is a generalized eigenvector of exponent d, i.e.

$$(T-\lambda)^d v = 0 \ \& \ (T-\lambda)^{d-1} v \neq 0$$

$$\beta := (T-\lambda)^{d-1} v$$

$$(T-\lambda)\beta = 0 \Rightarrow \lambda \text{ is eigenvalue.}$$

Properties of Generalized eigenspaces:

i) $\mathcal{E}_\lambda$ is the union of subspaces

$$\ker(T-\lambda) \subset \ker(T-\lambda)^2 \subset \cdots$$

Since V is finite dimensional, this sequence stops at some d &

$$\mathcal{E}_\lambda = \ker(T-\lambda)^d \quad \text{for some } d \geq 0$$

This d is called depth of λ.

ii) Choose a basis of $\ker(T-\lambda)$, complete it to a basis of $\ker(T-\lambda)^2$ . . . and so on till we get a basis of V.

if e_1 is the first basis vector, then

$$Te_1 = \lambda e_1$$

Say e_i is the next basis vector that does not come from $\ker(T-\lambda)$

$$\Rightarrow \quad (T-\lambda)^2 e_i = 0$$

$$\Rightarrow \quad (T-\lambda)((T-\lambda)e_i) = 0$$

$$\Rightarrow \quad (T-\lambda) e_i = c_1 e_1$$

$$T e_i = c_1 e_1 + \lambda e_i$$

and similarly ..

So matrix of T with respect to $\{e_1, \dots, e_i, \dots\}$ is upper triangular with diagonals λ.

iii) If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues then $\mathcal{E}_{\lambda_1}, \dots, \mathcal{E}_{\lambda_k}$ form a direct sum.

Pf: If $v_1 + \dots + v_k = 0 \quad v_i \in \mathcal{E}_{\lambda_i}$

then we will show that $v_i = 0$.

We will use induction on k, for $k=1$ nothing to show.

Suppose it holds for $k-1$.

Apply $(T-\lambda_k)^d (v_1 + \dots + v_k) = 0$

(d is exponent of v_k)

$$\underset{\in \mathcal{E}_{\lambda_1}}{(T-\lambda_k)^d v_1} + \cdots + \underset{\in \mathcal{E}_{\lambda_{k-1}}}{(T-\lambda_k)^d v_{k-1}} = 0$$

By induction $(T-\lambda_k)^d v_i = 0 \quad i \in \{1, \ldots, k-1\}$

If $v_i \neq 0$ for some i, $\Rightarrow$ we will get an eigenvector in $\mathcal{E}_{\lambda_i}$ with eigenvalue $\lambda_k \rightarrow \leftarrow$.

So, $v_i = 0$ for $i \in \{1, \ldots, k-1\}$. Done.

Corollary: If $\text{char}_T(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$

then $V = \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_k}$

Pf: $\dim \mathcal{E}_{\lambda_i} = r_i$

Choose a basis S_i for $\mathcal{E}_{\lambda_i}$

$B = \{S_1, S_2, \ldots, S_k\}$ forms a basis for V.

$$[T]_B = \begin{pmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_k} \end{pmatrix}$$

Block matrices

matrix of $T|_{\mathcal{E}_{\lambda_k}}$

Therefore, to analyze T it suffices to analyze T on generalized eigenspaces.

Next time: Nilpotent matrices and Jordan normal form.