MATH 314 (Lecture 6)

Topics to be discussed today

Linear Maps

HW Problem

Qn: Suppose $V = W_1 + W_2 + \cdots + W_n$

and $\dim V = \dim(W_1) + \cdots + \dim(W_n)$

then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$

Soln: let's see the solution for $n=3$.

$$\dim V = \dim(W_1 + W_2 + W_3)$$

$$= \dim W_1 + \dim W_2 + \dim W_3$$
$$- \dim(W_1 \cap W_2) - \dim(W_2 \cap W_3)$$
$$- \dim(W_1 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3)$$

$$\dim(W_1 \cap W_2 \cap W_3) = \dim(W_1 \cap W_2) + \dim(W_2 \cap W_3)$$
$$+ \dim(W_1 \cap W_3)$$

$$W_1 \cap W_2 \cap W_3 \subseteq W_1 \cap W_2$$

$$\dim(W_1 \cap W_2 \cap W_3) \leq \dim(W_1 \cap W_2)$$

If it is a strict inequality we have a contradiction

$$\dim(W_1 \cap W_2 \cap W_3) = \dim(W_1 \cap W_2)$$

$$\Rightarrow \dim(W_2 \cap W_3) = 0$$

$$\& \dim(W_1 \cap W_3) = 0$$

Similarly $\dim(W_1 \cap W_2) = 0$

& $\dim(W_1 \cap W_2 \cap W_3) = 0$

Choose distinct bases $\{\alpha_1, \dots, \alpha_r\}$ for W_1

$\{\beta_1, \dots, \beta_s\}$ for W_2

$\{\gamma_1, \dots, \gamma_t\}$ for W_3

$\{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t\}$ is a spanning set in V.

Pf:

Suppose $v \in V$

$$v = w_1 + w_2 + w_3 \qquad w_i \in W_i$$

$$= \sum c_i \alpha_i + \sum d_j \beta_j + \sum e_k \gamma_k$$

Therefore, $\left\{\begin{matrix} \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \\ \gamma_1, \ldots, \gamma_t \end{matrix}\right\}$ is

also linearly independent.

Suppose $v = c_1\alpha_1 + \cdots + c_r\alpha_r$
$+ d_1\beta_1 + \cdots + d_s\beta_s$
$+ e_1\gamma_1 + \cdots + e_t\gamma_t$

& $v = c_1'\alpha_1 + \cdots + c_r'\alpha_r$
$+ d_1'\beta_1 + \cdots + d_s'\beta_s$
$+ e_1'\gamma_1 + \cdots + e_t'\gamma_t$

$(c_1 - c_1')\alpha_1 + \cdots + (e_t - e_t')\gamma_t = 0$

$\Rightarrow \quad c_1 = c_1'$
$\vdots$
$e_t = e_t'$

So, it forms a direct sum.

Linear Maps

Suppose V_1, V_2 are two vector spaces.

Defn: We say that $T: V_1 \to V_2$ is a linear transformation if

$$T(cv_1 + v_2) = cT(v_1) + T(v_2)$$

for all $c \in F$, $v_1 \in V_1$, $v_2 \in V_2$

Examples: (1) V = Vector space of all polynomials with coefficients in $\mathbb{R}$

$$T: V \to V$$
$$p(x) \mapsto \frac{d\,p(x)}{dx}$$

(2)

$$T: V \to V$$
$$p(x) \mapsto \int p(x)\, dx$$

(3) $V = \mathbb{R}^n$

Fix a matrix $A \in M_{n\times n}(\mathbb{R})$

$$T : V \longrightarrow V$$
$$x \longmapsto Ax$$

(Think of x as a column vector)

(4) $V = M_{m\times n}(\mathbb{R})$ Fix P $m\times m$ matrix

Q $n\times n$ matrix

$$T : V \longrightarrow V$$
$$A \longmapsto PAQ$$

Qn: What are all the linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$? $\mathbb{R}^3$ to $\mathbb{R}^3$? $\mathbb{R}^n$ to $\mathbb{R}^n$?

Properties of linear maps :

1) $T(0) = 0$

$$\left(\begin{aligned} & T(0+0) = T(0) + T(0) \\ \Rightarrow \ & T(0) = T(0) + T(0) \\ & \text{Add } -T(0) \text{ to both sides} \\ & 0 = T(0) \end{aligned} \right)$$

2) If $T : V_1 \longrightarrow V_2$

Suppose $\{\beta_1, \ldots, \beta_n\}$ is a basis for V_1

then it is sufficient to define T on β_i's.

Say $T(\beta_i) = w_i$

then if $v = c_1 \beta_1 + \cdots + c_n \beta_n$

$$T(v) = T(c_1 \beta_1 + \cdots + c_n \beta_n)$$
$$= c_1 w_1 + \cdots + c_n w_n$$

Kernel and Image

Suppose $T: V \to W$

Defn: $Ker(T) = \{v \in V \mid T(v) = 0\}$

$Image(T) = \{T(v) \mid v \in V\}$

1. $Ker(T)$ is a subspace of V.

If $v_1, v_2 \in V$ then

$$T(cv_1 + v_2) = cT(v_1) + T(v_2)$$
$$= c(0) + 0 = 0$$

2. $Image(T)$ is a subspace of W.

Say $w_1, w_2 \in W$

$$cw_1 + w_2 = cT(v_1) + T(v_2)$$
$$= T(cv_1 + v_2)$$

So, $cw_1 + w_2 \in W$.

$T: V \longrightarrow W$

① T is one-to-one $\Longleftrightarrow$ $Ker(T) = \{0\}$
(or injective)

Suppose $T(v_1) = T(v_2)$

$\Rightarrow T(v_1) - T(v_2) = 0$

$\Rightarrow T(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 = 0$

$\Rightarrow v_1 = v_2$

② T is onto $\Longleftrightarrow$ $Image(T) = W$
(or surjective)

<u>Defn:</u> ① We say that $T: V \to W$ is an isomorphism $\Longleftrightarrow$ T is one-to-one & T is onto.

② We say that V & W are isomorphic

as vector spaces if there exists an isomorphism between them.

(Remark: This notion of isomorphism is different from what you see in topology.)

Lemma: If T is one-to-one, then T maps linearly independent sets to linearly independent sets.

Pf: $T: V \to W$

Say $\{v_1, \dots, v_n\} \subseteq V$ is linearly independent in V.

Want to show $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is linearly independent in W.

If $c_1 T v_1 + \cdots + c_n T v_n = 0$

then $T(c_1 v_1 + \cdots + c_n v_n) = 0$

T is one-to-one

$\Rightarrow$ $c_1 v_1 + \cdots + c_n v_n = 0$

$\Rightarrow$ $c_1 = 0, c_2 = 0, \ldots, c_n = 0.$

Qn: What about linearly dependent Subspaces?

Operations on linear transformations

Suppose $T_1 : V \to W$, $T_2 : V \to W$ are two linear maps

Define $T_1 + T_2 : V \to W$ as follows

$$(T_1 + T_2)(v) = T_1(v) + T_2(v)$$

For a scalar $c \in F$, define $cT : V \to W$ as

$$cT(v) = c(T(v))$$

Exc: These operations make the set of all linear maps from V to W, denoted as $\mathcal{L}(V,W)$ into a vector space over F.

Thm: $\dim(\mathcal{L}(V,W)) = \dim V \dim W$

Pf: Suppose V has basis $\{v_1, \dots, v_n\}$

W has basis $\{w_1, \dots, w_m\}$.

Define $T_{i,j}(v_k) = \begin{cases} 0 & \text{if } j \neq k \\ w_i & \text{if } j = k \end{cases}$

for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$

We can extend $T_{i,j}$ to linear map on V.

for each (i,j).

It suffices to show that the set of all $T_{i,j}$ is linearly independent and spans $\mathcal{L}(V,W)$.

Suppose $T: V \to W$ is a linear map

$$T(v_1) = a_{11} w_1 + a_{12} w_2 + \cdots + a_{1m} w_m$$

$$T(v_2) = a_{21} w_1 + a_{22} w_2 + \cdots + a_{2m} w_m$$

$$\vdots$$

$$T(v_n) = a_{n1} w_1 + a_{n2} w_2 + \cdots + a_{nm} w_m$$

$$T(v_1) = a_{11} T_{1,1}(v_1) + a_{12} T_{2,1}(v_1) + \cdots + a_{1m} T_{m,1}(v_1)$$

Similarly,

$$T(v_i) = a_{i1} T_{1,i}(v_i) + a_{i2} T_{2,i}(v_i) + \cdots + a_{im} T_{m,i}(v_i)$$

So, $T = a_{11} T_{1,1} + a_{12} T_{2,1} + \cdots + a_{1m} T_{m,1}$

$$\vdots$$

$$+ a_{i1} T_{1,i} + \cdots + a_{im} T_{m,i}$$

$$\vdots$$

Linearly independent

Suppose

$$a_{11} T_{1,1} + a_{12} T_{2,1} + \cdots + a_{1,m} T_{m,1}$$

$$\vdots \qquad = 0$$

$$+ a_{i1} T_{1,i} + \cdots + a_{im} T_{m,i}$$

$$\vdots$$

So, the above map sends every vector to 0.

$$a_{11} T_{1,1}(v_1) + a_{12} T_{2,1}(v_1) + \cdots + a_{1m} T_{m,1}(v_1) = 0$$

$$\Rightarrow a_{11} w_1 + a_{12} w_2 + \cdots + a_{1m} w_m = 0$$

but $\{w_1, \ldots, w_m\}$ is linearly independent

$$\Rightarrow a_{11} = a_{12} = \cdots = a_{1m} = 0$$

Similarly, we can show each row of scalars is 0.

Thm: Let F be a field. Suppose V is a vector space over F of dimension n. Then V is isomorphic to F^n.

Pf: Say V has basis $\{v_1, \dots, v_n\}$

Define $T: V \to F^n$

$$v_1 \mapsto e_1 = (1, 0, \dots, 0)$$

$$\vdots$$

$$v_n \mapsto e_n = (0, \dots, 0, 1)$$

Extend T by linearity to V.

If $T(v) = 0$

$\Rightarrow T(c_1 v_1 + \dots + c_n v_n) = 0$

$\Rightarrow c_1 T v_1 + \dots + c_n T v_n = 0$

$\Rightarrow c_1 e_1 + \dots + c_n e_n = 0$

$\Rightarrow c_1 = c_2 = \dots = c_n = 0$

$\Rightarrow T$ is injective.

We will now show that T is onto

Suppose $(c_1, \dots, c_n) \in F^n$

$$(c_1, \dots, c_n) = c_1 e_1 + \dots + c_n e_n$$

$$c_1 e_1 + \dots + c_n e_n = c_1 T v_1 + \dots + c_n T v_n$$

$$= T(c_1 v_1 + \dots + c_n v_n)$$

So, $(c_1, \dots, c_n) \in \text{Image } T$.

Hence, done.