MATH 314 (Lecture 9)

Topics to be discussed today

Matrix representation of linear maps

Recall: A linear map

$T: V \to W$ is a function

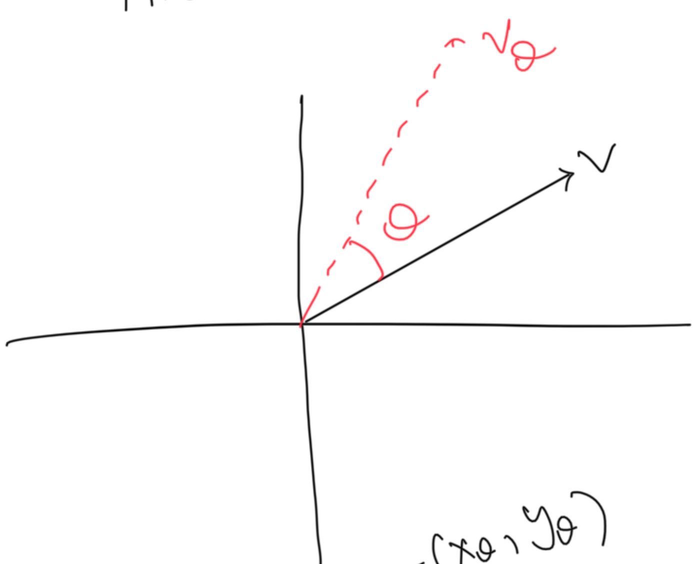
that satisfies

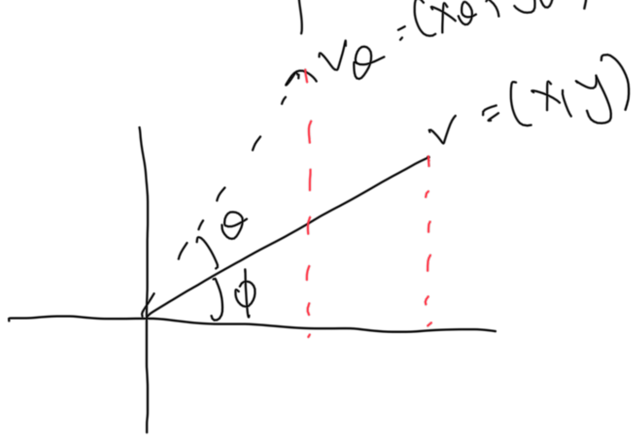
1) $T(v_1 + v_2) = T(v_1) + T(v_2)$

2) $T(cv) = cT(v)$

Examples: 1) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad v \mapsto v_\theta$

Fix $\theta \in (0, 2\pi]$





$$\tan(\theta + \phi) = \frac{y_\theta}{x_\theta}$$

$$\tan \phi = y/x$$

$$\frac{\sin\phi}{\cos\phi} = y/x$$

$$\tan(\theta+\phi) = \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$

$y = R\sin\phi \quad x = R\cos\phi \quad R$ is constant

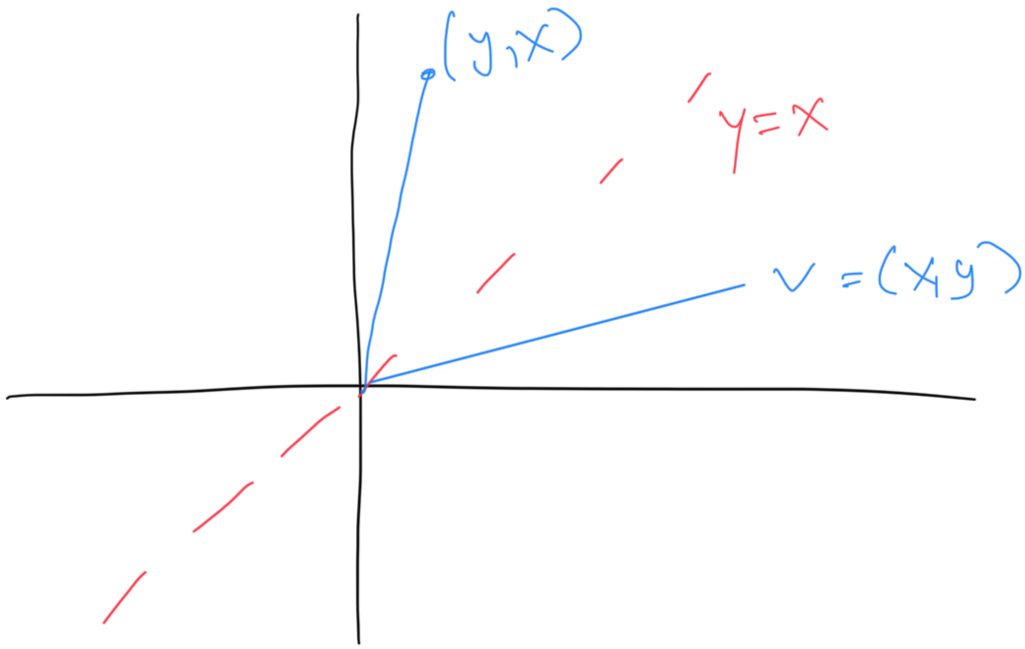
$$R = |v| = |v_\theta|$$

$$y_\theta = R(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$= \sin\theta\, x + y\cos\theta$$

$$x_\theta = R(\cos\theta\cos\phi - \sin\theta\sin\phi)$$
$$= \cos\theta\, x - y\sin\theta$$

$$(x, y) \longmapsto (\cos\theta\, x - y\sin\theta,\ \sin\theta\, x + y\cos\theta)$$
$$\shortparallel$$
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

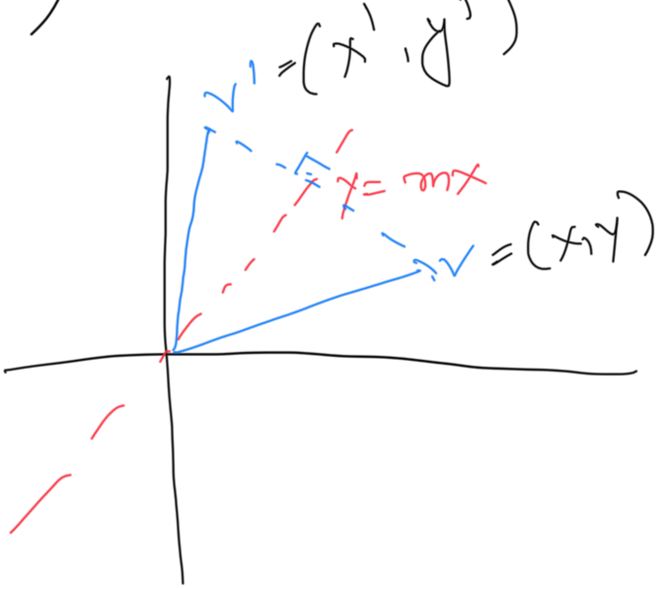
2) Reflection across the line $y = x$



$$(x, y) \longmapsto (y, x)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

3) Reflection across the line $y = mx$



Line joining v' & v must be perpendicular to $y = mx$

$$\frac{y' - y}{x' - x} = -\frac{1}{m} \quad \text{--- (1)}$$

Also Mid point of line joining v' & v must lie on $y = mx$.

$$\text{Mid point} = \left(\frac{x'+x}{2} , \frac{y'+y}{2} \right)$$

$$\left(\frac{y'+y}{2} \right) = \frac{m}{2} (x'+x)$$

$$y' = m(x'+x) - y \quad \text{———} \quad 2$$

$$y'-y = \frac{-1}{m} (x'-x)$$

$$y' = \frac{-1}{m} (x'-x) + y$$

$$mx' + mx - y = \frac{-1}{m} x' + \frac{x}{m} + y$$

$$\left(m + \frac{1}{m} \right) x' = \left(\frac{1}{m} - m \right) x + 2y$$

$$x' = \frac{m}{m^2+1} \left(\left(\frac{1}{m} - m \right) x + 2y \right)$$

Substitute this value of x' in ② to find y'

$$y' = \frac{-1}{m}\left(\frac{m}{m^2+1}\left(\left(\frac{1}{m}-m\right)x+2y\right)-x\right)+y$$

$$= -\frac{1}{m^2+1}\left[\left(\frac{1-m^2}{m}\right)x+2y\right]+\frac{x}{m}+y$$

$$= -x\left(\frac{1-m^2}{m(m^2+1)}-\frac{1}{m}\right)+y\left(\frac{-2}{m^2+1}+1\right)$$

$$= -x\left(\frac{1-m^2-m^2-1}{m(m^2+1)}\right)+y\left(\frac{-2+m^2+1}{m^2+1}\right)$$

$$= x\left(\frac{2m^2}{m(m^2+1)}\right)+y\left(\frac{m^2-1}{m^2+1}\right)$$

So, reflection across the line $y = mx$ is given by the matrix multiplication

$$(x,y) \longmapsto \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{m^2+1} \\ \frac{2m}{(m^2+1)} & \frac{m^2-1}{m^2+1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Let V, W be vector spaces over F.

Recall that $\mathcal{L}(V,W)$ is the set of all linear maps from V to W. We proved that it is a vector space over F of dimension $\dim(V)\dim(W)$. Say $\dim(V) = n$, $\dim(W) = m$.

Consider $M_{m\times n}(F) = \left\{ \begin{array}{l} \text{Set of all } m\times n \\ \text{matrices with} \\ \text{entries in } F \end{array} \right\}$

$\dim(M_{m\times n}(F)) = mn$.

Let's construct an explicit isomorphism between $\mathcal{L}(V, W)$ and $M_{m\times n}(F)$.

Fix an ordered basis $B = \{v_1, \ldots, v_n\} \subseteq V$

$B' = \{w_1, \ldots, w_m\} \subseteq W$.

Goal: Given a $T: V \to W$, want to construct a matrix A such that $[Tv]_{B'} = A[v]_B$

$\downarrow$ coordinate vector

Recipe: $T(v_1) = a_{11}w_1 + a_{12}w_2 + \cdots + a_{1m}w_m$

$\vdots$

$T(v_n) = a_{n1}w_1 + a_{n2}w_2 + \cdots + a_{nm}w_m$

Claim $A = \begin{pmatrix} a_{11} & & & a_{n1} \\ a_{12} & - & - & \vdots \\ \vdots & & & \vdots \\ a_{1m} & & & a_{nm} \end{pmatrix}$

Pf of the claim: We want to show that

$$[Tv]_{B'} = A[v]_B \quad \forall\, v \in V.$$

Let's show it for basis vectors first.

$$[Tv_1]_{B'} = \begin{pmatrix} a_{11} \\ \vdots \\ \vdots \\ a_{1m} \end{pmatrix} \qquad A[v_1]_B = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ \vdots \\ a_{1m} \end{pmatrix}$$

Similarly $[Tv_i]_{B'} = A[v_i]_B$ for $i \in \{2, \dots, n\}$.

Let's show this for any $v \in V$.

Suppose $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$$Tv = c_1 Tv_1 + c_2 Tv_2 + \cdots + c_n Tv_n$$

$$
\begin{aligned}
c_1 T v_1 &= c_1 \left(a_{11} w_1 + a_{12} w_2 + \cdots + a_{1m} w_m \right) \\
+ c_2 T v_2 &\quad + c_2 \left(a_{21} w_1 + a_{22} w_2 + \cdots + a_{2m} w_m \right) \\
\vdots &\qquad \vdots \\
+ c_n T v_n &\quad + c_n \left(a_{n1} w_1 + a_{n2} w_2 + \cdots + a_{nm} w_m \right)
\end{aligned}
$$

$$
\begin{aligned}
= &\left(c_1 a_{11} + c_2 a_{21} + \cdots + c_n a_{n1} \right) w_1 \\
&+ \\
&\vdots \\
&+ \\
&\left(c_1 a_{1m} + c_2 a_{2m} + \cdots + c_n a_{nm} \right) w_m
\end{aligned}
$$

$$
[Tv]_{B'} = A \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = A [v]_B
$$

Thm: $\mathcal{L}(V,W)$ & $M_{m\times n}(F)$ are isomorphic.

Pf: Fix a basis B of V & B' of W

$$\varphi : \mathcal{L}(V,W) \longrightarrow M_{m\times n}(F)$$
$$T \longmapsto A \quad \text{where}$$

A satisfies $[Tv]_{B'} = A[v]_B \ \forall v \in V$

We want to show that φ is linear, injective & surjective.

$$\varphi(T_1 + T_2) = A_1 + A_2$$

because

$$\begin{aligned}
[(T_1+T_2)(v)]_{B'} &= [T_1v + T_2v]_{B'} \\
&= [T_1v]_{B'} + [T_2v]_{B'} \\
&= A_1[v]_B + A_2[v]_B
\end{aligned}$$

$$= (A_1 + A_2)[v]_B$$

and $A_1 + A_2 = \varphi(T_1) + \varphi(T_2)$

Similarly $\varphi(cT) = c\varphi(T) \quad \forall c \in F$

Let us show that φ is injective.

Suppose $\varphi(T_1) = 0$

ie $[T_1 v]_{B'} = 0 \quad \forall v \in V$

$\Rightarrow T_1 v = 0 \quad \forall v \in V$

$\Rightarrow T_1 = 0$

Let us show that φ is surjective.

Start with an $A \in M_{m\times n}(F)$.

Define $[Tv_i]_{B'}$ using i-th column of A . . . .

So, far we saw if $T: V \longrightarrow W$ and if we fix ordered bases B & B' for V & W respectively, then $A \in M_{m \times n}(F)$ s.t

$$[Tv]_{B'} = A[v]_B \quad \forall v \in V.$$

This matrix A depends on choice of basis.

How do things change if we change bases?

For simplicity let's fix B and change B' to B''.

(Notation from lecture 8 notes)

$$[Tv]_{B''} = C_{B'' \to B'} [Tv]_{B'}$$

$$= C_{B'' \to B'} A [v]_B$$

Now suppose B'' is fixed and we change B to B_1.

$$[T\vec{v}]_{B''} = C_{B''\to B'} A [\vec{v}]_B$$

$$[\vec{v}]_{B_1} = C_{B_1\to B} [\vec{v}]_B$$

$$\Rightarrow [\vec{v}]_B = (C_{B_1\to B})^{-1} [\vec{v}]_{B_1}$$

$$\boxed{[T\vec{v}]_{B''} = C_{B''\to B'} A (C_{B_1\to B})^{-1} [\vec{v}]_{B_1}}$$

Pictorial Summary

$$T: \underset{(B)}{V} \longrightarrow \underset{(B')}{W} \qquad A$$

$$T: \underset{(B_1)}{V} \longrightarrow \underset{(B'')}{W} \qquad C_{B''\to B'} A (C_{B_1\to B})^{-1}$$

Motivated by this, here is a definition

Defn: We say two matrices A and B are similar $\iff$ there exists P s.t $PAP^{-1} = B$.