# MATH 314 (Lecture 22)

Topics to be discussed today

Inner Product Spaces

# Inner Products

**Example:** Dot Product on $\mathbb{R}^n$

$$\alpha \cdot \beta \in \mathbb{R}$$

$(V/\mathbb{C})$

**Defn:** An inner product on $V$ is a function which assigns to each ordered pair of vectors $(v_1, v_2)$ a scalar, such that the following hold:

i) $(v_1 + v_2, v_3) = (v_1, v_3) + (v_2, v_3)$

ii) $(c v_1, v_2) = c(v_1, v_2)$

iii) $(v_2, v_1) = \overline{(v_1, v_2)}$

iv) $(v, v) \geq 0$

Example: 1) $\alpha = (x_1, x_2) \in \mathbb{R}^2$

$\beta = (y_1, y_2) \in \mathbb{R}^2$

$$\langle \alpha, \beta \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$$

2) $V = M_{n \times n}(\mathbb{C})$

$$\langle A, B \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} A_{jk} \overline{B_{jk}}$$

$$\| $$

$$\text{Trace}(AB^*)$$

$$B^*_{kj} = \overline{B_{jk}}$$

(Please look section 8.1 for more examples.)

Defn: An inner product space is a real or complex vector space, together with a specified inner product on that space.

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We use symbol $\|\alpha\|$ to denote norm of a vector $\alpha \in V$, $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$.

Thm: Let $V$ be an inner product space, then the norm function satisfies following properties:

(i) $\|c\alpha\| = |c| \, \|\alpha\|$ for scalar $c$ and vector $\alpha$

(ii) $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \, \|\beta\|$

(iii) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

Pf: i) $\|c\alpha\|^2 = \langle c\alpha, c\alpha\rangle$

$$= c^2\langle\alpha,\alpha\rangle$$

$$= c^2\|\alpha\|^2$$

$$\Rightarrow \|c\alpha\| = |c|\,\|\alpha\|$$

ii) Define $\gamma = \beta - \frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2}\alpha$

$$\langle\alpha,\gamma\rangle = \left\langle\alpha, \beta - \frac{\langle\alpha,\beta\rangle\alpha}{\|\alpha\|^2}\right\rangle$$

$$= \langle\alpha,\beta\rangle - \frac{\langle\alpha,\beta\rangle\langle\alpha,\alpha\rangle}{\|\alpha\|^2}$$

$$= 0$$

$$0 \le \|\gamma\|^2 = \left\langle \beta - \frac{\langle\alpha,\beta\rangle\alpha}{\|\alpha\|^2}, \beta - \frac{\langle\alpha,\beta\rangle\alpha}{\|\alpha\|^2}\right\rangle$$

$$= \langle\beta,\beta\rangle - \frac{\langle\alpha,\beta\rangle\langle\alpha,\beta\rangle}{\|\alpha\|^2}$$

$$- \frac{\langle \alpha, \beta \rangle \langle \alpha, \beta \rangle}{\|\alpha\|^2} + \frac{\langle \alpha, \beta \rangle^2}{\|\alpha\|^2}$$

$$= \langle \beta, \beta \rangle - \frac{\langle \alpha, \beta \rangle^2}{\|\alpha\|^2}$$

$$\geq 0$$

So, $\|\beta\|^2 \geq \dfrac{\langle \alpha, \beta \rangle^2}{\|\alpha\|^2}$

$$\|\beta\|^2 \|\alpha\|^2 \geq \langle \alpha, \beta \rangle^2$$

$$\Rightarrow \; |\langle \alpha, \beta \rangle| \leq \|\alpha\| \|\beta\|$$

iii) $\|\alpha + \beta\|^2 = \langle \alpha + \beta, \alpha + \beta \rangle$

$$= \|\alpha\|^2 + 2\langle \alpha, \beta \rangle + \|\beta\|^2$$

$$\leq \|\alpha\|^2 + 2\|\alpha\| \|\beta\| + \|\beta\|^2$$

$$= (\|\alpha\| + \|\beta\|)^2$$

So, $\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$

$\hookrightarrow$ Cauchy - Schwarz inequality

— x — x — x —

Orthogonality

$V$ inner product space

Defn: 1) We say that $\alpha$ is orthogonal to $\beta$ if $\langle \alpha, \beta \rangle = 0$.

2) We say that $\alpha$ is an orthonormal vector if $\|\alpha\| = 1$.

3) A set $S$ is called an orthogonal set if all pairs of distinct vectors in $S$ are orthogonal.

4) A set $S$ is called orthonormal if

it is <u>orthogonal</u> and satisfies $\|\alpha\| = 1$ for all $\alpha \in S$.

<u>Example:</u> 1) $S = \{(1,0), (0,1)\} \subseteq \mathbb{R}^2$ is orthonormal, (think dot product on $\mathbb{R}^2$).

2) $V = M_{n \times n}(\mathbb{C})$

$$(E^{pq})_{ij} = \begin{cases} 1 & \text{if } (i,j) = (p,q) \\ 0 & \text{otherwise} \end{cases}$$

$$\langle E^{pq}, E^{rs} \rangle = \operatorname{trace}(E^{pq} E^{sr})$$

$$= \delta_{qs} \operatorname{trace}(E^{pr})$$

$$= \delta_{qs} \delta_{pr}$$

So $\left\{ E^{pq} \;\middle|\; \begin{array}{l} p \in \{1, \dots, n\} \\ q \in \{1, \dots, n\} \end{array} \right\}$ is orthonormal.

3) $V = \{ f : [0,1] \to \mathbb{C} \mid f \text{ is continuous} \}$

$$\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} \, dx$$

$$f_n(x) = \sqrt{2} \cos 2\pi n x$$

$$g_n(x) = \sqrt{2} \sin 2\pi n x$$

$\{1, f_1, g_1, f_2, g_2, \ldots\}$ is an orthonormal set.

(Pf: Exc!)

<u>Lemma:</u> An orthogonal set of non-zero vectors is linearly independent.

<u>Pf:</u> Let $v_1, \ldots, v_n$ be orthogonal.

Suppose $c_1 v_1 + \cdots + c_n v_n = 0$ — (1)

Take inner product of LHS in (1) with $v_1$

$$\langle c_1 v_1 + \cdots + c_n v_n, v_1 \rangle = 0$$

$$\parallel$$

$$\langle c_1 v_1, v_1 \rangle + \langle c_2 v_2, v_1 \rangle + \cdots + \langle c_n v_n, v_1 \rangle$$

$$= 0$$

$$\langle c_k v_k, v_1 \rangle = c_k \langle v_k, v_1 \rangle \quad (k \neq 1)$$

$$= 0$$

$$c_1 \langle v_1, v_1 \rangle = 0$$

$\Rightarrow$ $c_1 = 0$ because $v_1 \neq 0$

Similarly all $c_i = 0$.

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Gram – Schmidt Orthogonalization

Linearly independent set $\rightsquigarrow$ Orthogonal set

$v_1, \ldots, v_n$ linearly independent vectors

$$\alpha_1 := v_1$$

$$\alpha_2 := v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$\langle \alpha_2, \alpha_1 \rangle = \left\langle v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1 \,,\, v_1 \right\rangle$$

$$= 0$$

$$\alpha_3 := v_3 - \frac{\langle v_3, \alpha_2 \rangle}{\|\alpha_2\|^2} \alpha_2 - \frac{\langle v_3, \alpha_1 \rangle}{\|\alpha_1\|^2} \alpha_1$$

$$\vdots$$

Proceed Similarly.

After we get an orthogonal set we can normalize vectors to make their length 1.

A very useful feature of working with orthonormal bases is that coordinates are easy to figure out.

$V$ vector space

$v_1, \dots, v_n$ orthonormal basis

Suppose $v = c_1 v_1 + \cdots + c_n v_n$

$\Rightarrow \langle v, v_1 \rangle = c_1 \langle v_1, v_1 \rangle$

$\Rightarrow c_1 = \langle v, v_1 \rangle$

$\vdots$

$c_i = \langle v, v_i \rangle$

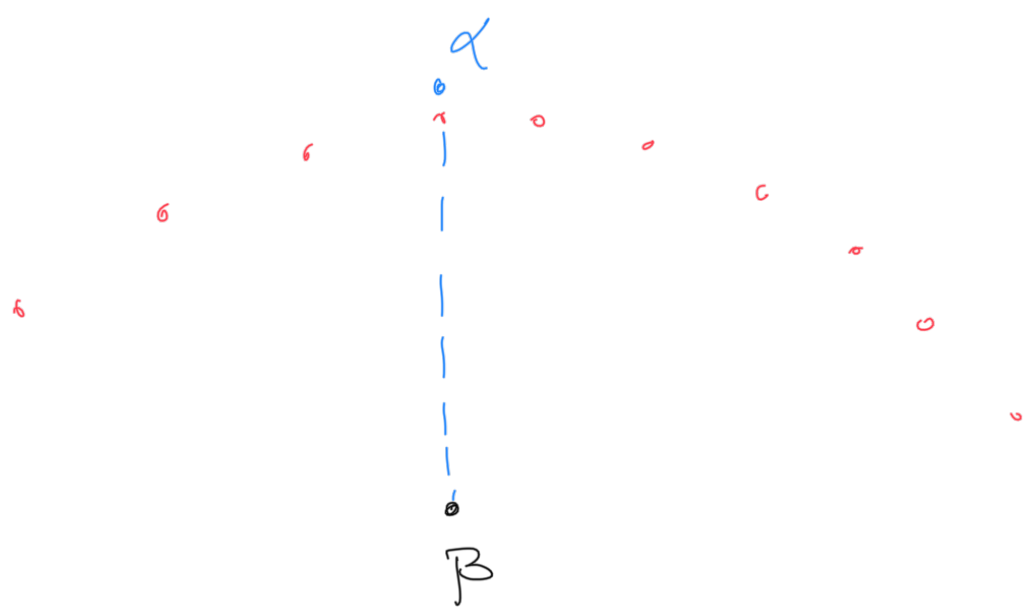
# Approximation

**Thm:** Let $W$ be a subspace of an inner product space $V$ and let $\beta \in V$.

i) $\alpha \in W$ "best approximates $\beta$", i.e.

$$\|\alpha - \beta\| \leq \|\gamma - \beta\| \quad \text{for every } \gamma \in W$$

$\Longleftrightarrow$ $\alpha - \beta$ is orthogonal to every vector in $W$.




ii) If $W$ is finite-dimensional and $\{\alpha_1, \ldots, \alpha_n\}$ is orthonormal basis for $W$, then

$$\alpha = \sum_k \langle \beta, \alpha_k \rangle \alpha_k$$

best approximates $\beta$, and is unique.

<u>Proof</u>: i) Assume $\alpha$ best approximates $\beta$.

Let $\gamma \in W$. We want to show $\langle \alpha - \beta, \gamma \rangle = 0$

$$\beta - \gamma = \beta - \alpha + \alpha - \gamma$$

$$\|\beta - \gamma\|^2 = \|\beta - \alpha\|^2 + \|\alpha - \gamma\|^2 + 2\,\mathrm{Re}\,\langle \beta - \alpha, \alpha - \gamma \rangle$$

$$\|\alpha - \beta\| \leq \|\beta - \gamma\|$$

$$\Rightarrow 2\,\mathrm{Re}\,\langle \beta - \alpha, \alpha - \gamma \rangle + \|\alpha - \gamma\|^2 \geq 0 \quad \forall \gamma \in W$$

$$\Updownarrow$$

$$* \quad 2\,\mathrm{Re}\,\langle \beta - \alpha, x \rangle + \|x\|^2 \geq 0 \quad \forall x \in W$$

Substitute $x = -\dfrac{\langle \beta - \alpha, \alpha - \gamma \rangle}{\|\alpha - \gamma\|^2} (\alpha - \gamma)$

* becomes

$$2\,\mathrm{Re}\left\langle \beta-\gamma,\ -\frac{\langle \beta-\alpha,\ \alpha-\gamma\rangle}{\|\alpha-\gamma\|^2}(\alpha-\gamma)\right\rangle$$

$$+\ \frac{|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2}{\|\alpha-\gamma\|^2} \geqslant 0$$

$$\Rightarrow\ -\frac{2\,|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2}{\|\alpha-\gamma\|^2} + \frac{|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2}{\|\alpha-\gamma\|^2} \geqslant 0$$

$$\Rightarrow\ -\frac{|\langle \beta-\alpha,\ \alpha-\gamma\rangle|^2}{\|\alpha-\gamma\|^2} \geqslant 0$$

$$\Rightarrow\ \langle \beta-\alpha,\ \alpha-\gamma\rangle = 0$$

$\Rightarrow$ $\langle \beta-\alpha\rangle$ is orthogonal to every vector in W.

$\Leftarrow$ Assume $\alpha - \beta$ is orthogonal to every vector in $W$.

$$\beta - \gamma = \beta - \alpha + \underbrace{\alpha - \gamma}_{W} \quad \text{for every } \gamma \in W$$

$$\|\beta - \gamma\|^2 = \|\beta - \alpha\|^2 + \|\gamma - \alpha\|^2$$

$$\Rightarrow \|\beta - \gamma\|^2 \geq \|\beta - \alpha\|^2$$

$\Rightarrow$ $\alpha$ best approximates $\beta$.

ii) We want to show that

$$\beta - \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k \text{ is orthogonal}$$

to every vector in $W$

Since $W$ has orthonormal basis $\{\alpha_1, \ldots, \alpha_n\}$ it suffices to show $\beta - \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k$ is orthogonal to $\alpha_i$

$$\left\langle \beta - \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k \,,\, \alpha_i \right\rangle$$

$$= \langle \beta, \alpha_i \rangle - \left\langle \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k, \alpha_i \right\rangle$$

$$= \langle \beta, \alpha_i \rangle - \langle \beta, \alpha_i \rangle = 0$$

So, done.

Uniqueness : Suppose $w \in W$ is another vector that best approximates $\beta$. We want to show that

$$w = \sum_{k=1}^{n} \langle \beta, \alpha_k \rangle \alpha_k$$

This follows because $\beta - w$ is perpendicular to $\alpha_i$, hence

$$\langle \beta - w, \alpha_i \rangle = 0 \Rightarrow \langle \beta, \alpha_i \rangle = \langle w, \alpha_i \rangle$$

and we know that

$$w = \sum_{i=1}^{n} \langle w, \alpha_i \rangle \alpha_i$$

$$\| \quad \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle \alpha_i .$$

Hence, done.