MATH 3500 Spring 2023 (Lecture 2)

WRITING PROOFS

<u>Problem</u>: Show that sum of two consecutive integers is an odd integer.

Understand the problem first

Consecutive integers

Defn: We say that a & b, $a<b$ are consecutive integers if and only if $b = a+1$.

Defn: We say that b is an odd integer iff $b = 2k+1$ for some integer k.

Direct Proof:

Let $a < b$ be two consecutive inregers.

So, $b = a+1$. Then $a+b = 2a+1$

which is odd by definition.

So, done.

PROOF BY CONTRADICTION

If p, then q. Assume p is true and q is false and arrive at a $\underbrace{\text{contradiction}}$

↓

a false statement

Show that sum of two consecutive integers is odd.

Assume a, b $a<b$ are consecutive.
Assume $a+b$ is even.

Since a & b are consecutive $b = a+1$
We have assumed $a+b = 2m$ for some integer m.
Also $a+b = 2a+1 = 2m$
So, $1 = 2(m-a)$ $\rightarrow\leftarrow$

PROOF BY CONTRAPOSITIVE

Want to show $P \Rightarrow q$

In this method we show $\neg q \Rightarrow \neg P$

not of q

a & b are consecutive $\Rightarrow$ a+b is odd

Assume

a+b is even

$\Rightarrow$ a & b are not consecutive

$a+b$ is even

$$\left.\begin{array}{ll} a \text{ even} & b \text{ even} \\ a \text{ odd} & b \text{ odd} \end{array}\right\} \text{Cannot be Consecutive}$$

. - . - Try to write down Complete proof using well formed sentences.

<u>Proof</u> by <u>induction</u> is used to prove results about natural numbers, see example below:

Show that $1+2+\dots+n = \frac{n(n+1)}{2}$

(1) This holds for $n=1$.

(2) Assume it holds for $n=k$.

(3) Now show that the statement holds for $n=k+1$

$$1+2+\dots+k+k+1 = \frac{k(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2}.$$

LINEAR DIOPHANTINE EQUATION

Degree

1

involves variables and coefficients

Coefficients are integers

Example:

$$7x^2 + 11xy + 21z^2 = 0$$

This is diophantine equation but not linear!

Example: (Linear Diophantine equation)

$$2x + 3y = 5$$

Solving linear diophantine equations

It is easy to show that there are no solutions than to find them all

THU JAN 12, 2023

Are there any integer solutions to

$\rightarrow \quad 12x + 15y = 5$?

No, because 3 divides LHS but not RHS

$\rightarrow \quad 2x + 3y = 1$

Yes $x = 2$, $y = -1$ and many more.

GCD of two integers

Defn: Let a, b be two integers. We say that d is the GCD of a & b iff

(i) $d > 0$

(ii) $d \mid a$ & $d \mid b$

(iii) If c is an integer such that $c \mid a$ & $c \mid b$ then $c \mid d$.

Find $GCD(2,3)$

Common divisors are $1, -1$.

$\downarrow$

GCD.

Find $GCD(12, 15)$.

Divisors of 12 $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Divisors of 15 $\pm 1, \pm 3, \pm 5, \pm 15$

$GCD = 3$

Find GCD(720, 134).

$\downarrow$

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Most effective & useful way to Compute GCD is by using Euclidean algorithm.

Next time!

Euclidean Algorithm

Can be used to find gcd of two integers

$GCD(72, 40)$

$72 = 40(1) + 32$

$40 = 32(1) + 8$ is the GCD

$32 = 8(4) + 0$

Stop when you get 0 as a remainder

Find GCD of 45 and 144

201 and 335

$144 = 45(3) + 9$

$45 = 9(5) + 0$

GCD

$335 = 201(1) + 134$

$201 = 134(1) + 67$

$134 = 67(2) + 0$

GCD

(1) Why does Euclidean algorithm terminate in finitely many steps?

(because divisor is decreasing at every step)

(2) Why do we get GCD in the end?

Let d be the positive number that we get in the end. Observe that $d = ax + by$ for some $x, y \in \mathbb{Z}$.

Tracing back the steps $d | a$ & $d | b$.

If $c | a$ & $c | b$ then $c | ax + by \Rightarrow c | d$.

Fix two integers a and b, look at all the possible combinations

$$ax + by = c \quad (x, y \text{ are integers})$$

which is the smallest positive c in this combination? GCD

Is it possible that c is not a multiple of $\mathrm{GCD}(a,b)$? No!

a and b are co-prime $\Longleftrightarrow$ $ax+by=1$ for some $x,y \in \mathbb{Z}$.

(Why?) because 1 is the smallest positive integer that you can get as a combination.

$\rightarrow$ Find GCD of $(n, n+1)$. 1

$\rightarrow$ Show that $GCD(a,b)=1 \Longleftrightarrow GCD(a^2,b^2)=1$

Argue via prime factorization

→ Show that $GCD(ka, kb) = |k| \, GCD(a, b)$

→ Show that $GCD(a^2, b^2) = GCD(a, b)^2$.

Fibonacci Numbers

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

Sum of previous two terms

$$a_n = a_{n-1} + a_{n-2}$$

Perform Euclidean algorithm on two consecutive Fibonacci numbers. What do you observe?

Hint: Use the property that $GCD(a+b, b)$

$= GCD(a, b)$.

Why is $GCD(a+b, b) = GCD(a, b)$?