Last time we showed that there are infinitely many primes of the form $4k+3$.

[Trick : Consider $N = 4p_1p_2 \cdots p_n - 1$]

Suppose $a = p_1^{e_1} \cdots p_r^{e_r}$

What do divisors of a look like?

Each divisor looks like $p_1^{f_1} \cdot p_r^{f_r}$

where $0 \le f_i \le e_i$

Q1: How many divisors are there?

(denoted by $\sigma_0(a)$)

Q2: What is the sum of all divisors?

(denoted by $\sigma_1(a)$)

$\sigma_0(a)$ = No. of divisors of a

There are e_i+1 choices for each power of p_i, so we get $\prod (e_i+1)$ divisors

Q1: There are e_i+1 choices for each i, so no of divisors

$$= (e_1+1)(e_2+1)\dots(e_r+1)$$

Q2: The sum of all divisors can be expressed as

$$(1+p_1+p_1^2+\dots+p_1^{e_1}) = (p_1^{e_1+1}-1)/(p_1-1)$$

$$\times(1+p_2+p_2^2+\dots+p_2^{e_2}) = (p_2^{e_2+1}-1)/(p_2-1)$$

$$\vdots$$

$$\times(1+p_r+p_r^2+\dots+p_r^{e_r}) = (p_r^{e_r+1}-1)/(p_r-1)$$

$$\boxed{\text{Sum of all divisors} = \prod_{i=1}^{r}\frac{(p_i^{e_i+1}-1)}{(p_i-1)}}$$

GEOMETRIC SERIES

$$1+p+p^2+\dots+p^r = \frac{p^{r+1}-1}{p-1}$$

in bijection with
{Divisors of a} × {Divisors of b}

Qn: Suppose $(a,b) = 1$.
What can you say about divisors of ab?

MULTIPLICATIVE FUNCTIONS

A function $f : \mathbb{N} \to \boxed{}$
Integer, real or Complex

is called multiplicative if
$$f(ab) = f(a) f(b)$$ whenever a and b are Coprime.

Examples: σ_0, σ_1

Qn: If $(a,b)=1$, then what is

$(a^m, b^n) = ?$ (here m,n are natural numbers)

$= 1$

Can you give some more examples of multiplicative fns?

For $n = 100$, compute $100 = 2^2 5^2$

$\sigma_0(n) = 3 \times 3 = 9$

$\sigma_1(n) = (1+2+2^2)(1+5+5^2)$

$\sigma_2(n) = (1+2^2+2^4)(1+5^2+5^4)$

$\sigma_3(n) = (1+2^3+2^6)(1+5^3+5^6)$

PERFECT NUMBERS

Sum of divisors of $n = 2n$

$\Updownarrow$

Sum of proper divisors of $n = n$

Examples: 6 ; $1+2+3+6 = 12$

28 ; $1+2+4+7+14+28 = 56$

496

8128

$\vdots$

How many are there? (No one knows!)

Let us show a connection between perfect numbers & Mersenne Primes.

→ If n is a natural number and 2^n-1 is a prime #, then n is prime number.

Pf: $2^{ab}-1 = (2^a-1)(1+2^a+2^{2a}+\cdots+2^{(b-1)a})$

→ Suppose 2^n-1 is prime, then $N = 2^{n-1}(2^n-1)$ is perfect.

$\sigma_1(N) = \sigma_1(2^{n-1})\,\sigma_1(2^n-1) = (2^n-1)(2^n)$

$2N = 2^n(2^n-1)$

→ Suppose N is even + perfect.

Then there exists primes p & q such that $q = 2^p - 1$

$$N = 2^{p-1} q$$

Pf: $N = 2^{e_2} 3^{e_3} 5^{e_5} \cdots$

Define $p = e_2 + 1$

$$q = 3^{e_3} 5^{e_5} \cdots$$

By construction $N = 2^{p-1} q$

Want to show that p & q are primes.

$$\sigma_1(N) = \sigma_1(2^{p-1}) \sigma_1(q)$$

$$2N = (2^p - 1) \sigma_1(q)$$

$$2^p q = (2^p - 1)(\sigma_1(q))$$

$$\sigma_1(q) = \left(\frac{2^p}{2^p - 1}\right) q = \left(\frac{(2^p - 1) + 1}{2^p - 1}\right) q$$

$$\sigma_1(q) = q + \frac{q}{2^p - 1}$$

$$\downarrow$$

$$\frac{q}{2^p-1} = \sigma_1(q) - q$$

$$\downarrow$$

divisor of q & proper

$$\sigma_1(q) = q + \frac{q}{2^p-1}$$

What can you say about $\frac{q}{2^p-1}$?

It is 1

q is prime

$q = 2^p - 1$ is prime

$\Rightarrow p$ is prime