Math 350 Midterm II

Instructions and Information

1. Read these instructions carefully, and sign the Statement on Academic Integrity below once you have completed the exam.
2. This test contains a cover page and 4 problems.
3. Each question is worth the same number of points.
4. In addition to accuracy, you will be graded on the **correctness, completeness, clarity, and neatness of your accompanying work and explanations.** While we appreciate if you type your responses using LaTeX, you are not required to do so, and it is better to turn in good hand-written work than hastily-prepared solutions written in TeX.
5. This midterm is due on April 7.
6. You may consult your notes, the official course text, and any class recordings posted on Canvas. You may not use any other external resources, such as calculators.
7. Please print this cover sheet, sign the academic integrity statement, and upload it to Canvas along with your completed exam. If you prefer not to print the cover sheet, you may include a typed or handwritten sheet of paper on which you copy and sign the academic integrity statement below.

Statement on Academic Integrity

I acknowledge that this exam is my own work, completed with the aid of only those sources permitted above. My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

____
Name

______________________________ ____________
Signature Date

1. Observe that $2021 = (43)(47)$. Is 1163 a quadratic residue modulo 2021?

2. Cicadas are insects that live most of their lives underground, then emerge in their final summer. 17-year cicadas emerge every 17 years, and will next emerge in the mid-Atlantic in the summer of 2021. 13-year cicadas will next emerge in the mid-Atlantic in the summer of 2024. In what year will the two broods next emerge simultaneously?

3. Throughout this problem, suppose that n is a square-free integer greater than 1. In other words, n is not divisible by any perfect square other than 1.

   Prove that if $\gcd(a, n) = 1$, $a^{\phi(n)} \equiv 1 \mod n$. Then show that there is an element of cycle-length $\phi(n)$ if and only if $n = 2$, $n = p$, or $n = 2p$, where p is an odd prime number.

4. Let $q(T)$ and $r(T)$ be polynomials in $\mathbb{F}_p[T]$ with no common nonconstant factors. For any polynomial $s(T)$, let $\mathbb{F}_p[T]/s(T)$ denote the set of remainders when polynomials are divided by $s(T)$, and define addition and multiplication of remainders analogously to modular arithmetic with natural representatives.

   Find a function between $\mathbb{F}_p[T]/(q(T)r(T))$ and the set

   $$\{(a(T), b(T)) : a(T) \in \mathbb{F}_p[T]/q(T),\ b(T) \in \mathbb{F}_p[T]/r(T)\},$$

   with the following properties:

   - Your function is bijective. (Recall that a function is bijective if it is both injective/one-to-one and surjective/onto.)
   - If $f(T) \mapsto ((a(T), b(T))$ and $g(T) \mapsto (c(T), d(T))$, then $f(T) + g(T) \mapsto (a(T) + c(T), b(T) + d(T))$.
   - If $f(T) \mapsto ((a(T), b(T))$ and $g(T) \mapsto (c(T), d(T))$, then $f(T)g(T) \mapsto (a(T)c(T), b(T)d(T))$.

   Prove that your function satisfies all three properties.

5. (Not for credit) What to all four of the above problems have in common?