Midterm 2 last question

$$\mathbb{Z}/15\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$$

$$a \longrightarrow \left(a \pmod 3,\ a \bmod 5 \right)$$

For eg

$$11 \longrightarrow (2,\ 1)$$

Fix a prime p

$$\mathbb{F}_p[T] \longrightarrow \frac{\mathbb{F}_p[T]}{q(T)} \times \frac{\mathbb{F}_p[T]}{s(T)}$$

$q(T)\,s(T)$

$$g \longrightarrow \left(g \bmod q(T),\ g \bmod s(T) \right)$$

Recall: $Q(x,y) = ax^2 + bxy + cy^2$

Discriminant $(Q) = b^2 - 4ac$

$(\Delta(Q))$

If Q & Q' are equivalent, then $\Delta(Q) = \Delta(Q')$

Converse is false.

$\Delta(Q) = -100$

$Q_1(x,y) = x^2 + 25y^2$

$Q_2(x,y) = 5x^2 + 5y^2$

$\Delta(Q_1) = -4(25) = -100$

$\Delta(Q_2) = -4(25) = -100$

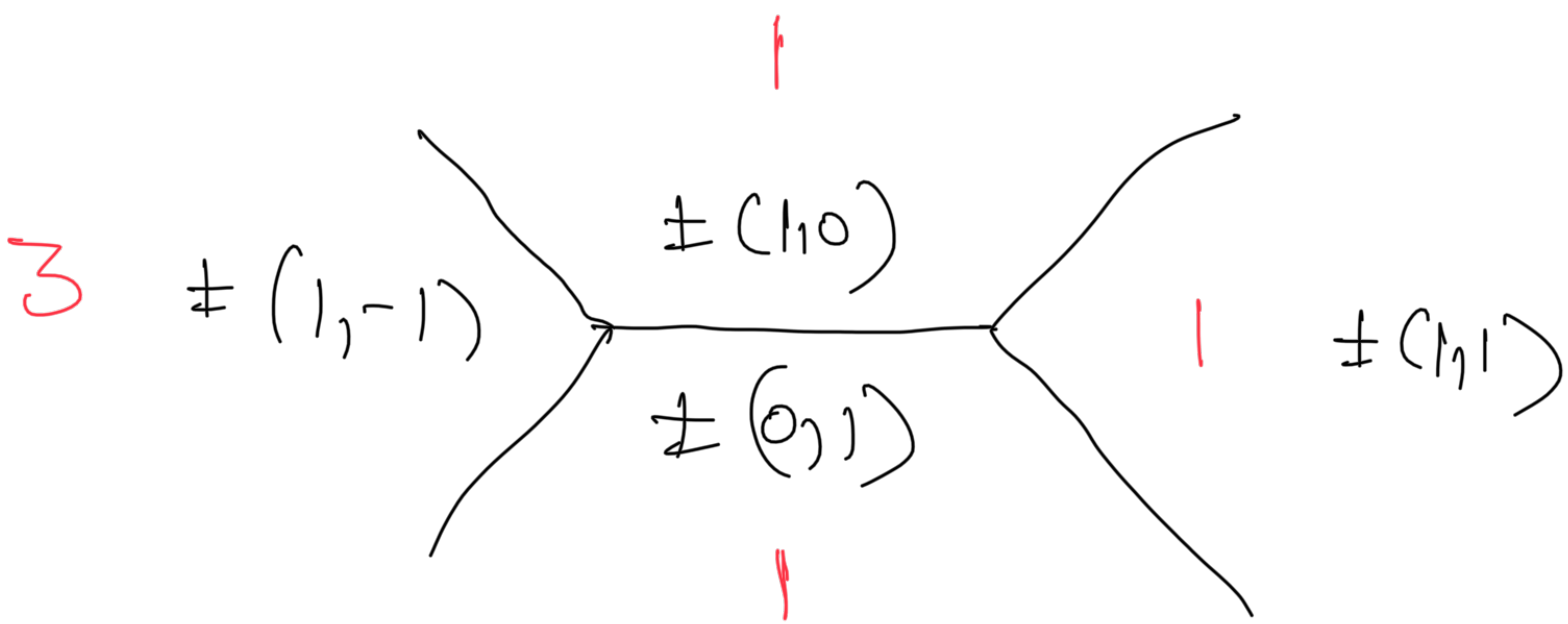
Q_1 & Q_2 are not equivalent

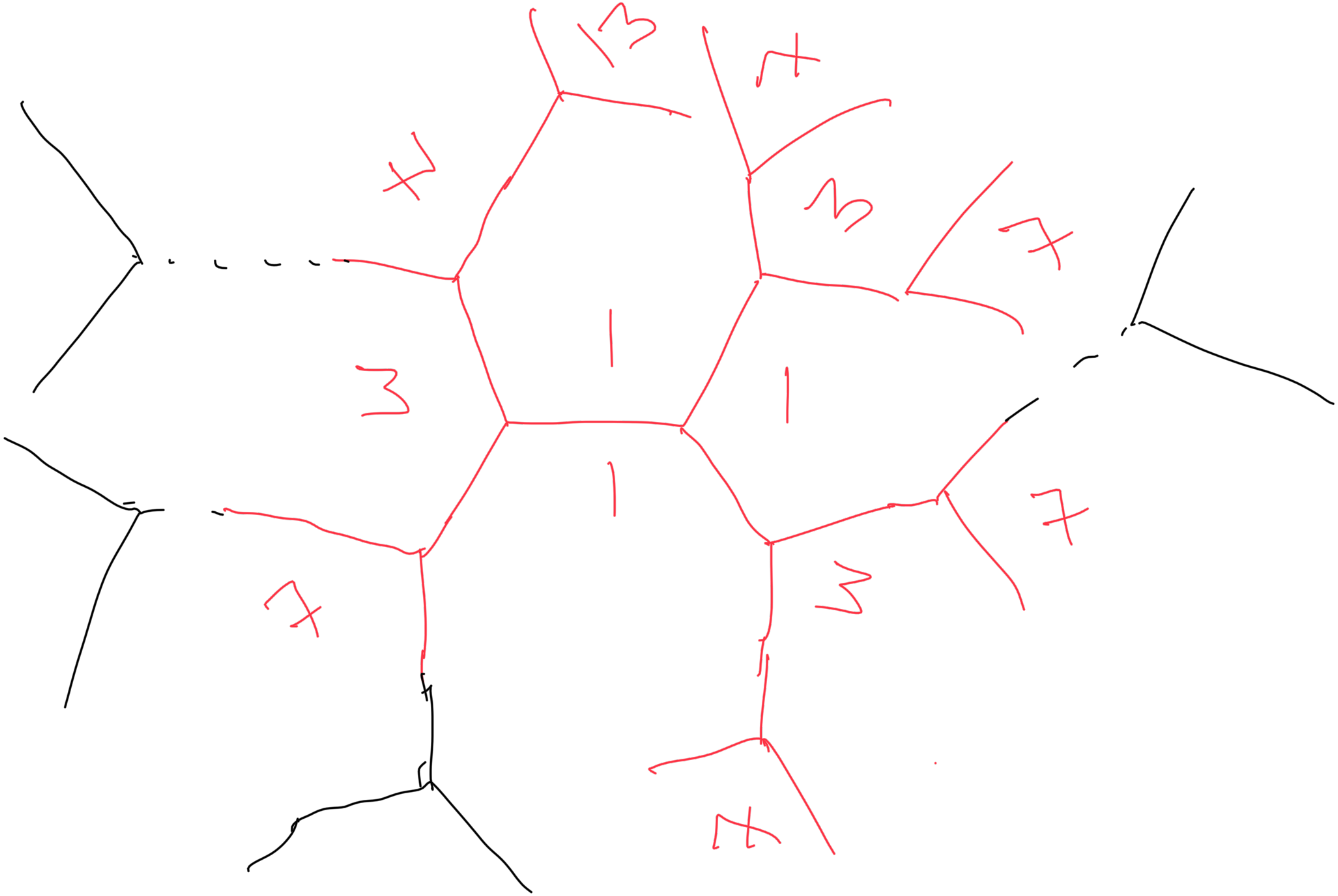
$Q_1(1,0) = 1$

$Q_2(x,y) \neq 1$ for any $(x,y) \in \mathbb{Z}^2$.

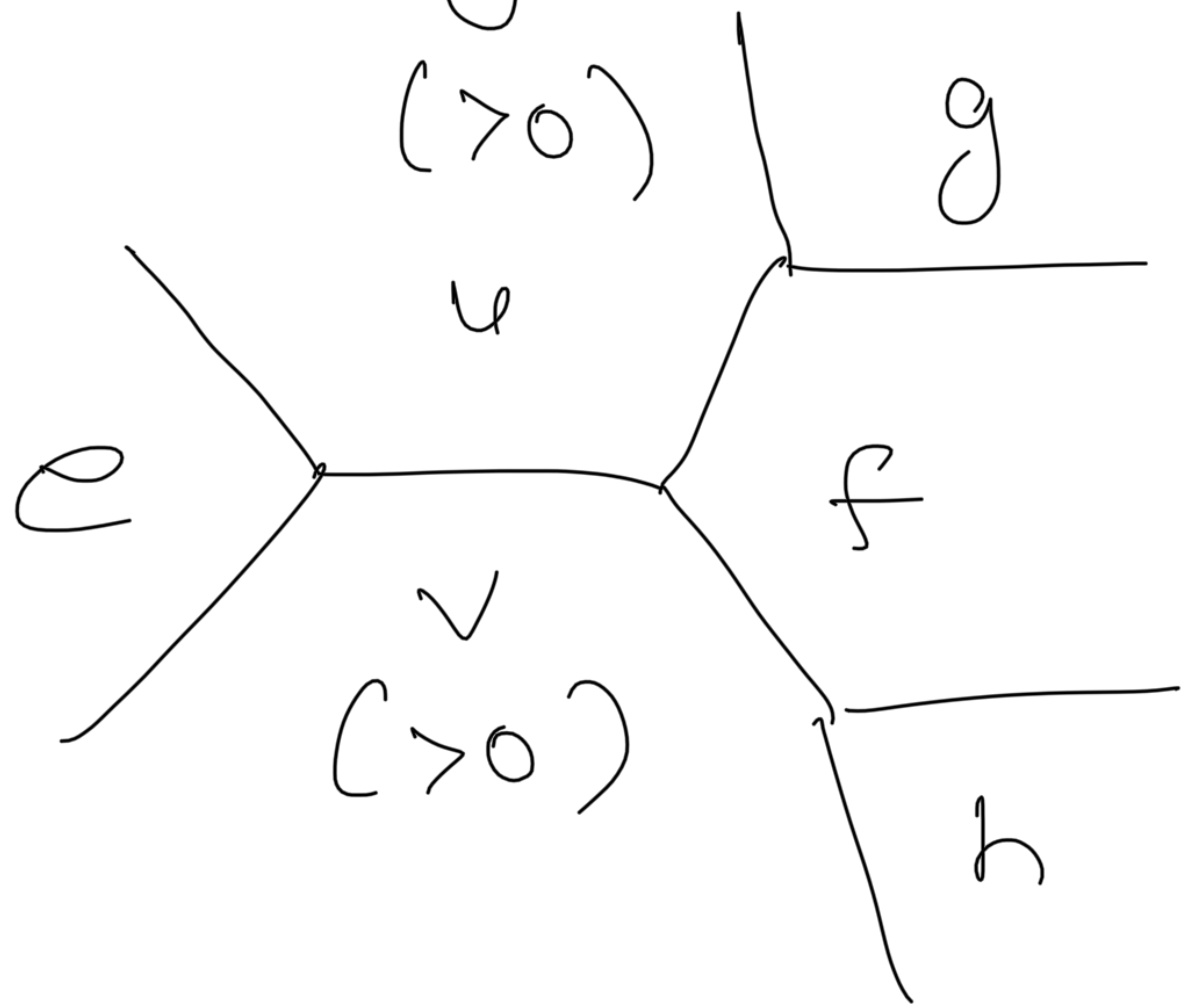
Let's analyze range topograph of

$x^2 - xy + y^2$




13
3
1
1
1
3
7
7

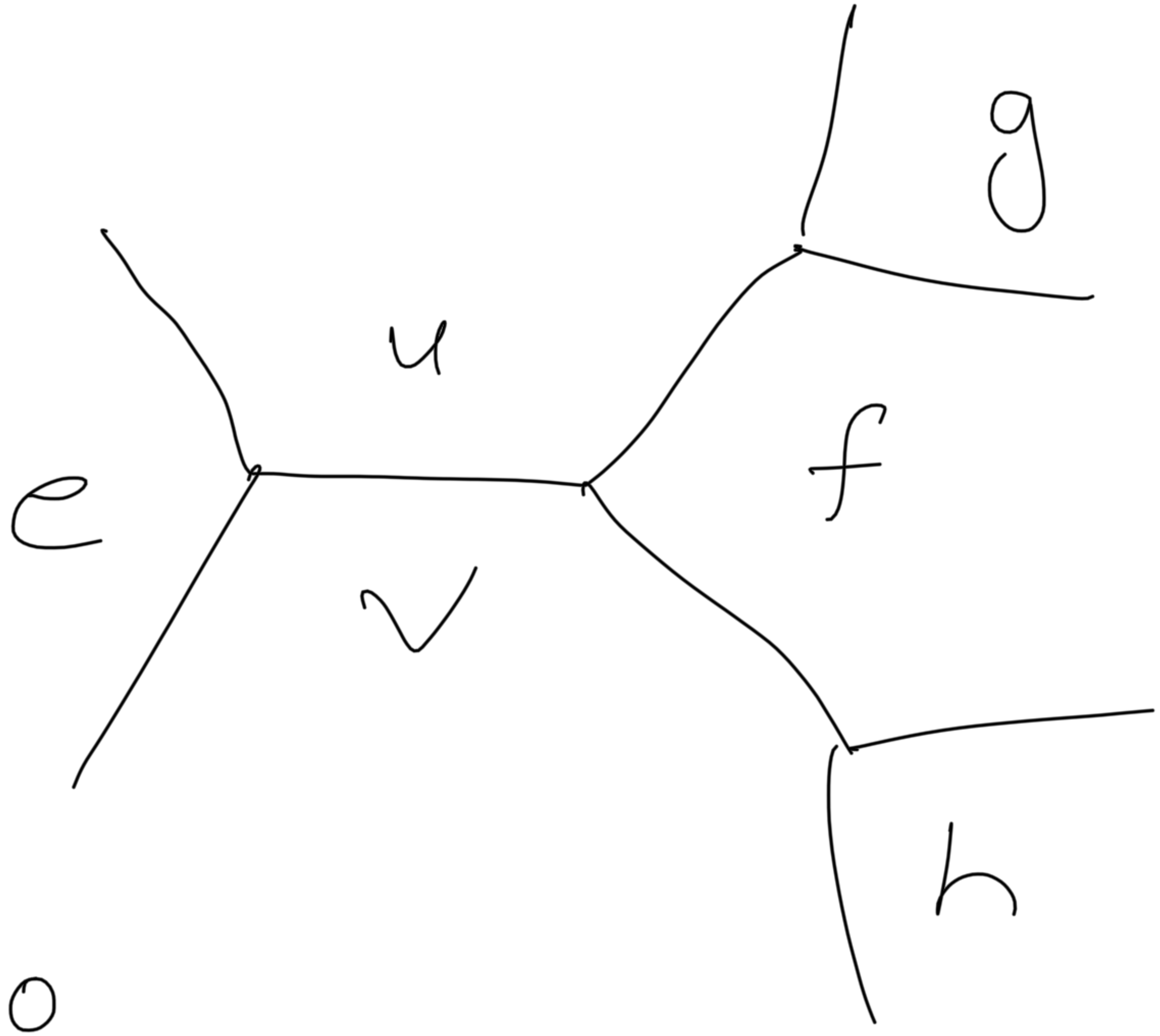
Conway's Climbing Principle



Suppose $e, \ u+v, \ f$ increases

$e < u+v < f$

then $v < u+f < g$ & $v < v+f < h$



$u > 0$

$v < u + u + v < u + f$

$v < u + f < g$

$u < u + v + v$

$< v + f$

$< h$

Find all Solutions of

$$x^2 - xy + y^2 = 7.$$

$\pm(1,-2)$

$\pm(3,2)$

$\pm(2,-1)$

$\pm(2,3)$

$\pm(3,1)$

$\pm(1,3)$

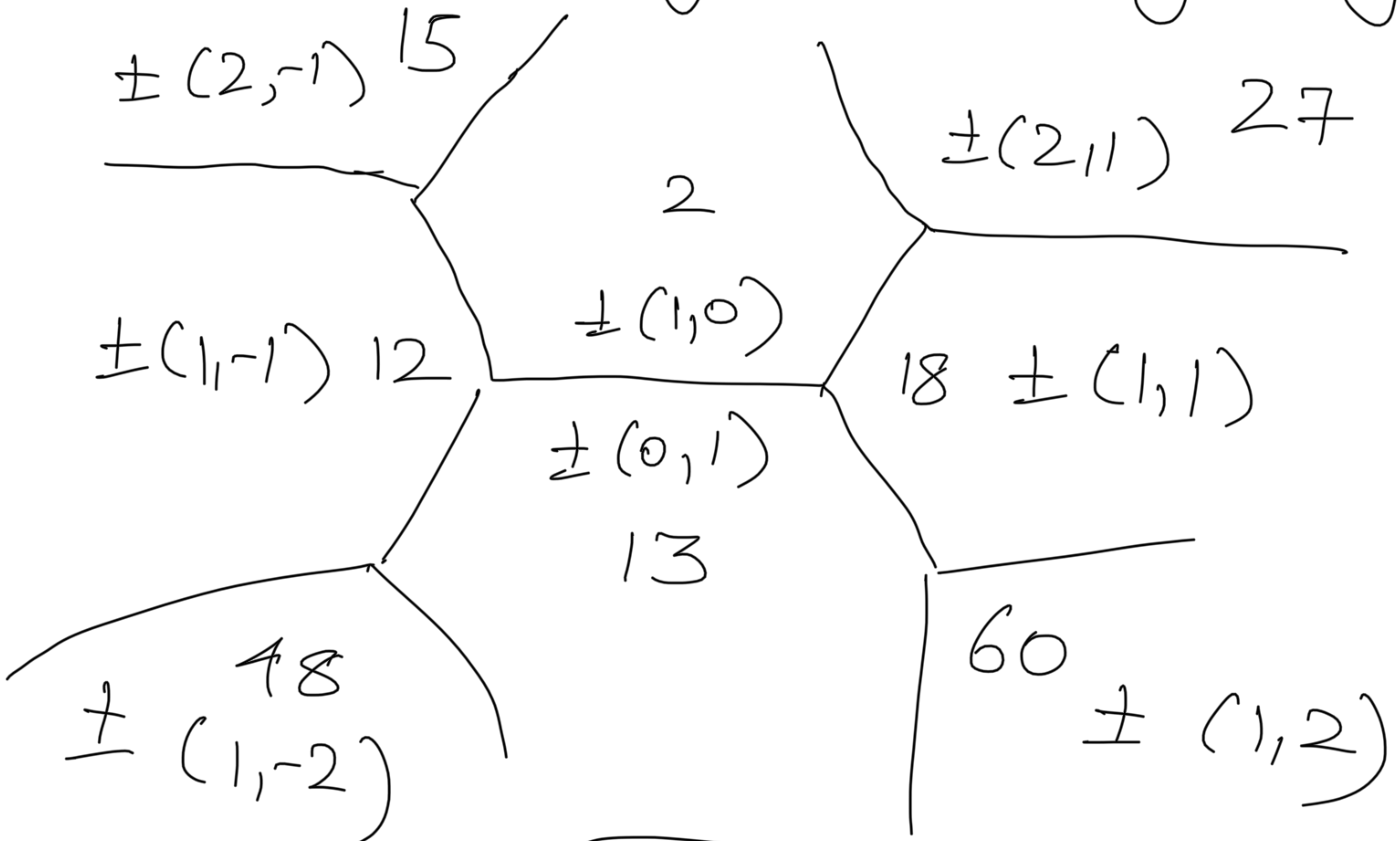
Is that all?

Find all solutions of $x^2 - xy + y^2 = 4$

$$\left.\begin{array}{l} \pm(1,0) \\ \pm(0,1) \\ \pm(1,1) \end{array}\right\} \quad 1$$

$$\left.\begin{array}{l} \pm(2,0) \\ \pm(0,2) \\ \pm(2,2) \end{array}\right\} \quad 4$$

Find all solutions of $2x^2+3xy+13y^2=18$



$\boxed{\pm(1,1)}$

$\boxed{\pm(3,0)}$

If $Q(x,y) > 0 \quad \forall \ (x,y)$ positive definite

If $Q(x,y) < 0 \quad \forall \ (x,y)$ negative definite

$\Delta < 0$ (positive or negative definite)

$\Delta > 0$ Indefinite

$\Delta = -100$

$Q_1(x,y) = x^2 + 25y^2$

not equivalent

$Q_2(x,y) = 5x^2 + 5y^2$

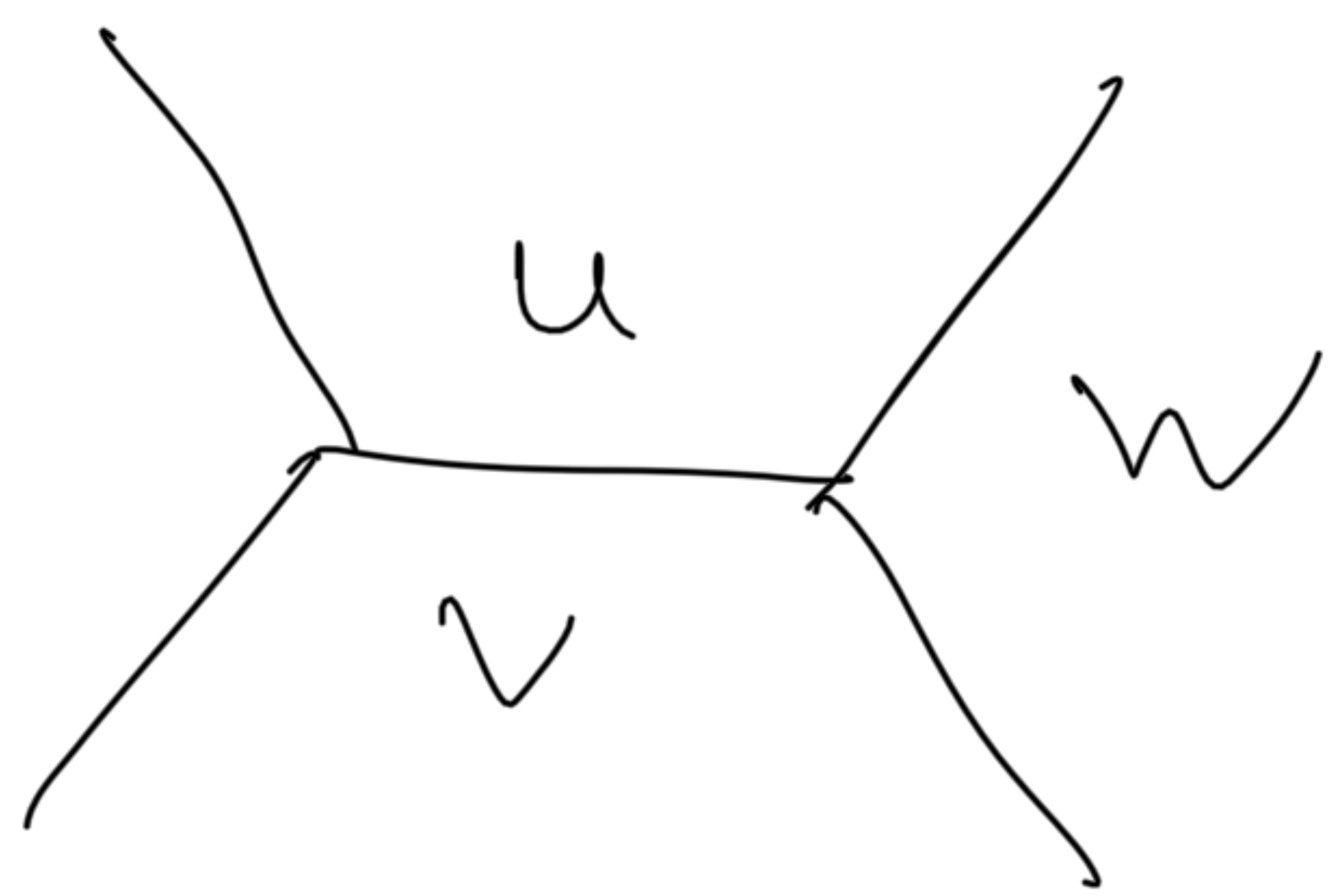
Are these all ?

How do we find all of them ?

By looking at possible range topographs !

Suppose $Q(x,y) > 0 \quad \forall\, x,y$

Let u be the minimum value among all $Q(x,y)$



$u \le v \le w$

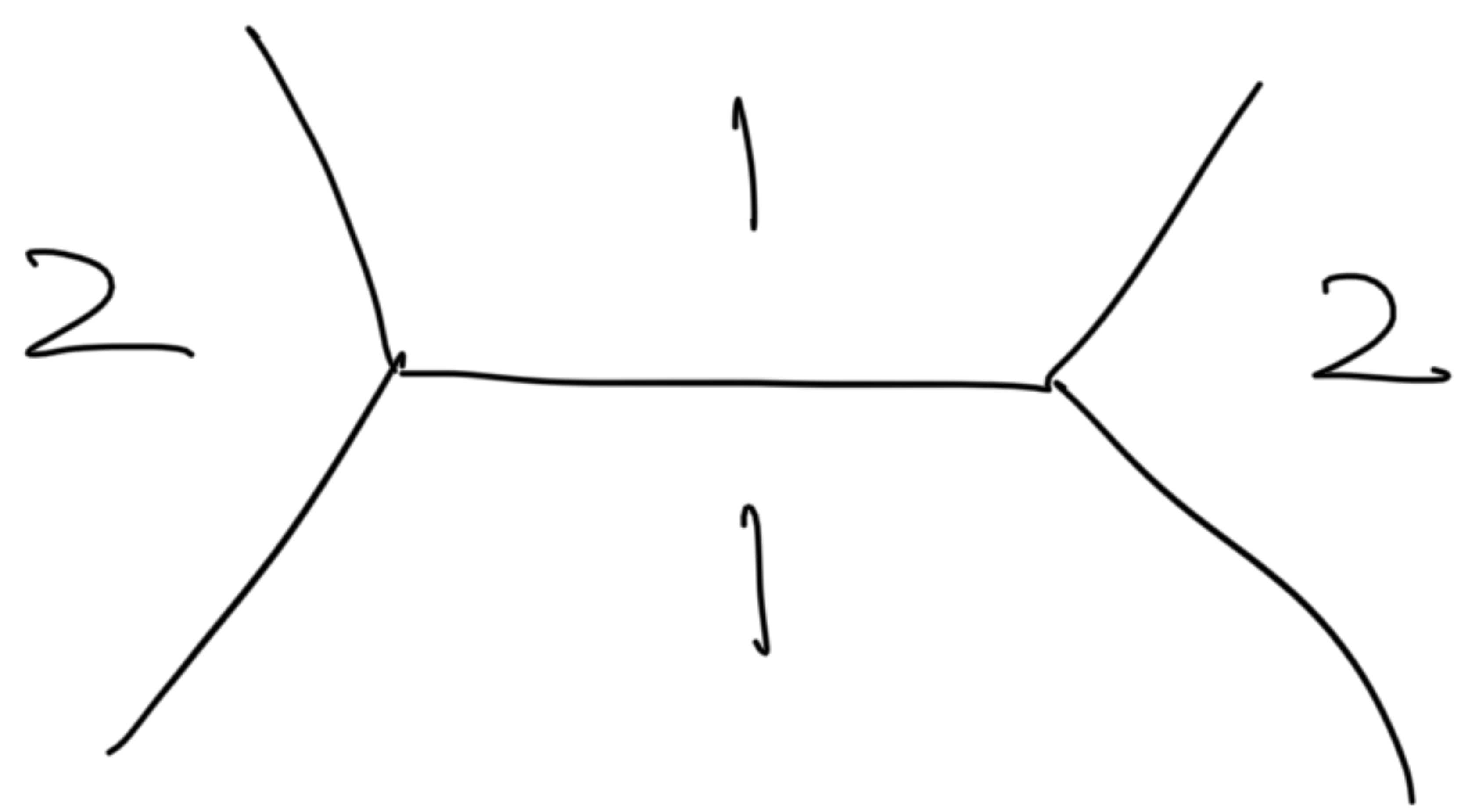
$w \le u+v$

either $u+v = w$ or $u+v > w$

Double well — Single well

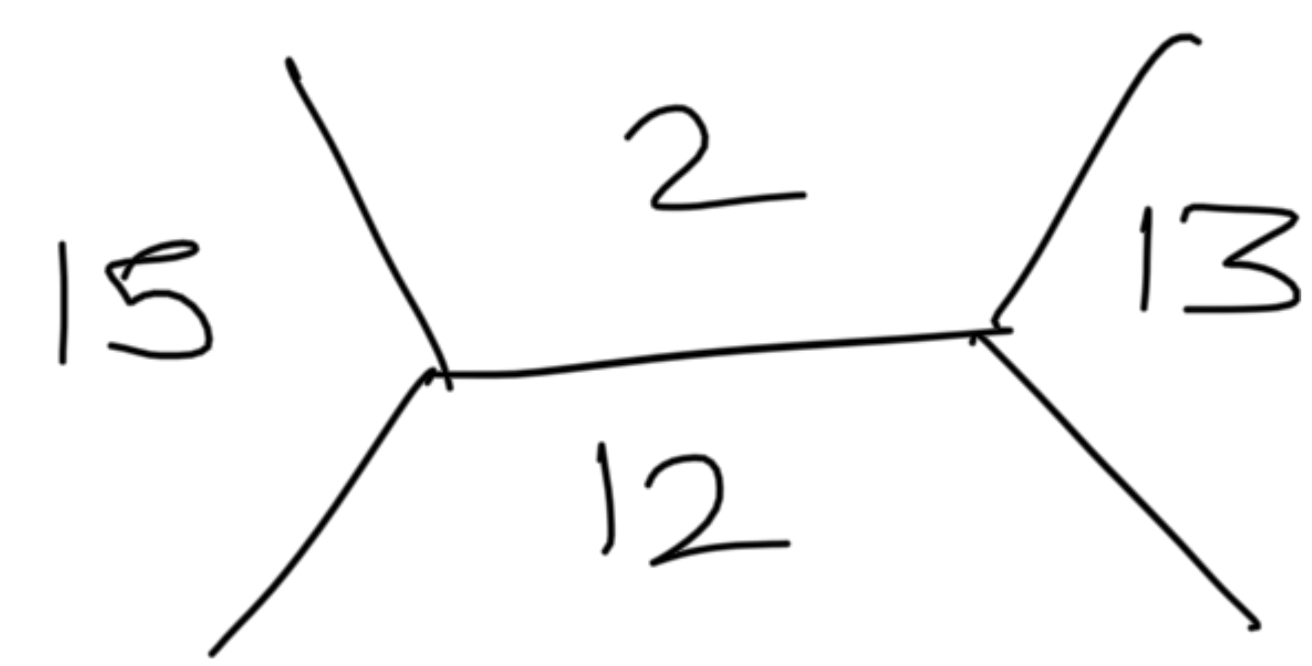
Example of Double well:

$$Q(x,y) = x^2 + y^2$$



Example of Single well:

$$Q(x,y) = 2x^2 + 3xy + 13y^2$$

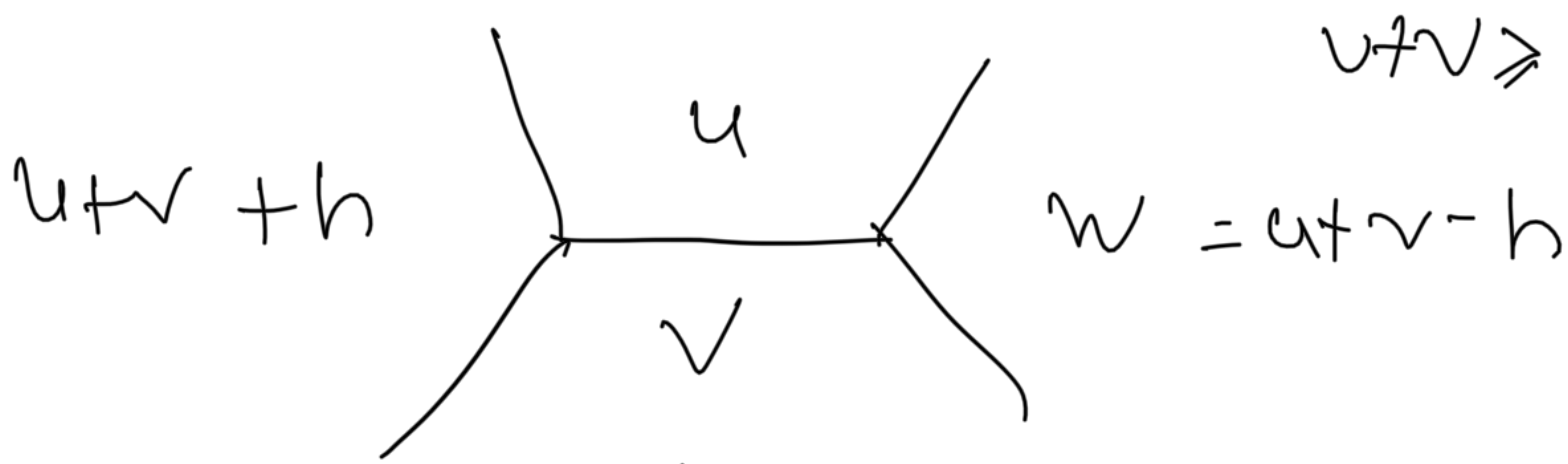


Thm: Let Q be a positive definite binary quadratic form with discriminant Δ. Let u be the smallest nonzero value that is taken by Q. Then $u \leq \sqrt{\frac{|\Delta|}{3}}$

Pf: There is exactly one well in range topograph of Q.

$u \leq v \leq w$

$u + v \geq w$



Recall discriminant =

$$= u^2 + v^2 + w^2 - 2uv - 2vw - 2uw$$

$$= h^2 - 4uv < 0$$

Since $\underset{u+v-h}{\underset{\shortparallel}{w}} \le u+v \Rightarrow h \geq 0$

$$u \le \underline{v \le u+v-h}$$

$$\Rightarrow 0 \le u-h \Rightarrow h \le u$$

So, $\boxed{0 \le h \le u \le v}$

$$|\Delta| = 4uv - h^2 \geq 4uv - u^2 \geq 4u^2 - u^2 = 3u^2$$

$$3u^2 \leq |\Delta|$$

$$\Rightarrow \boxed{u \leq \sqrt{\frac{|\Delta|}{3}}}$$

The following is true:

1) $1 \leq u \leq \sqrt{\frac{|\Delta|}{3}}$

2) $0 \leq b \leq u$

3) $\Delta = b^2 - 4uv$

4) Δ even $\Rightarrow$ b even

Δ odd $\Rightarrow$ b odd

Qn: What are all inequivalent binary quadratic forms of $\Delta = -100$?

→ Positive definite, let's start by listing all wells.

$$\sqrt{\frac{101}{3}} = \sqrt{33.33\cdots} < 6$$

$$1 \le u \le 5$$

h must be even, $0 \le h \le u$

$$\Delta = h^2 - 4uv$$

$$-100 = 4 - 4(2v)$$

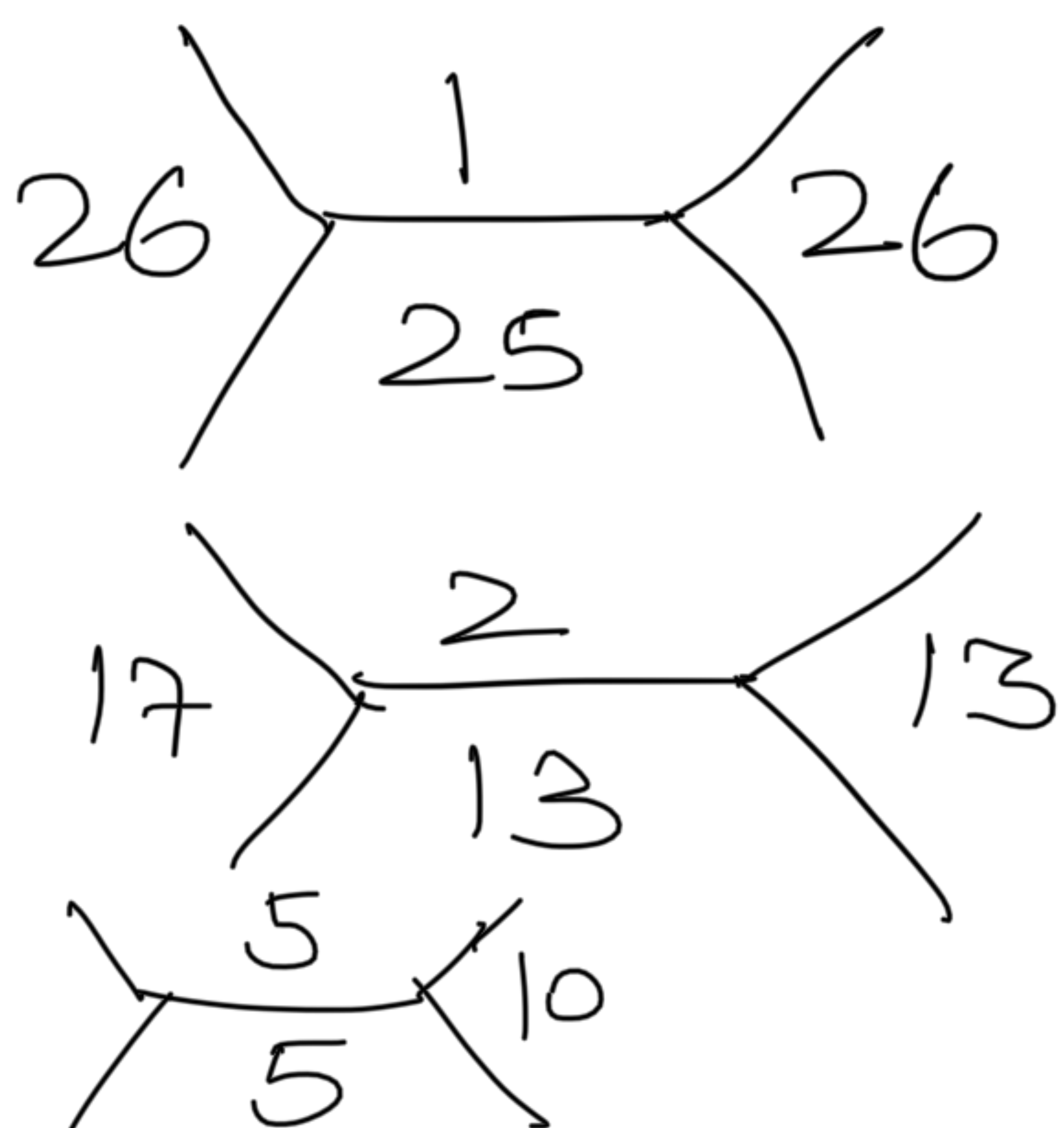
-26

u	h	v	
1	0	25	
2	0	25/2	not an integer
2	2	13	
3	0	100/12	not integer
3	2	104/12	
4	0	100/16	
4	2	104/16	not integer
4	4	116/16	

5	0	5	
5	2	104/20	not integer
5	4	116/20	

Three inequivalent forms

u	h	v
1	0	25
2	2	13
5	0	5



$x^2 + 25y^2$

$2x^2 - 2xy + 13y^2$

$5x^2 + 5y^2$