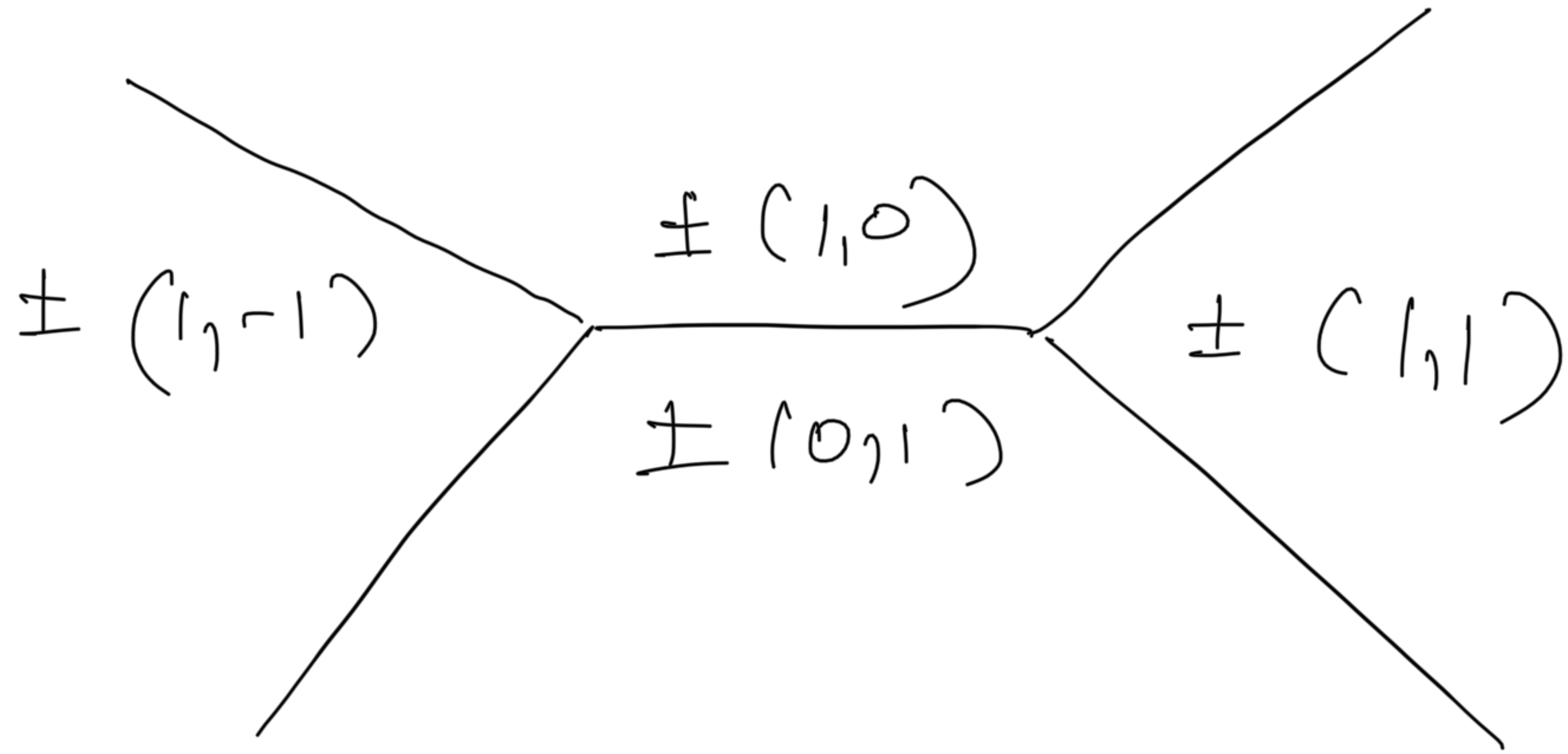
Recall from last time



This is domain topograph, can be extended infinitely in both directions.
There is a symmetry in this, so it is important to get a sense of

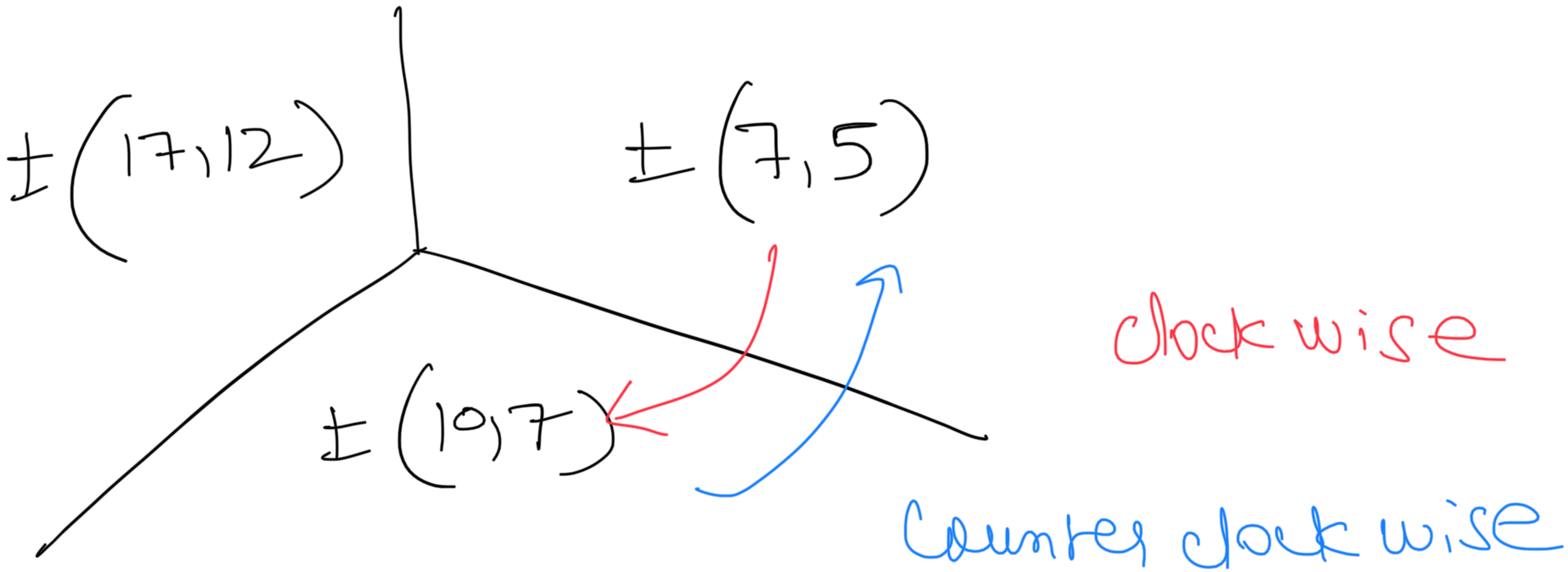
Orientation.

Determining Orientation

A <u>super basis</u> is a set $\{u, v, w\}$ of vectors such that $u+v+w=0$ & any two of them form a basis.

e.g. $\{(-17,-12), (10,7), (7,5)\}$

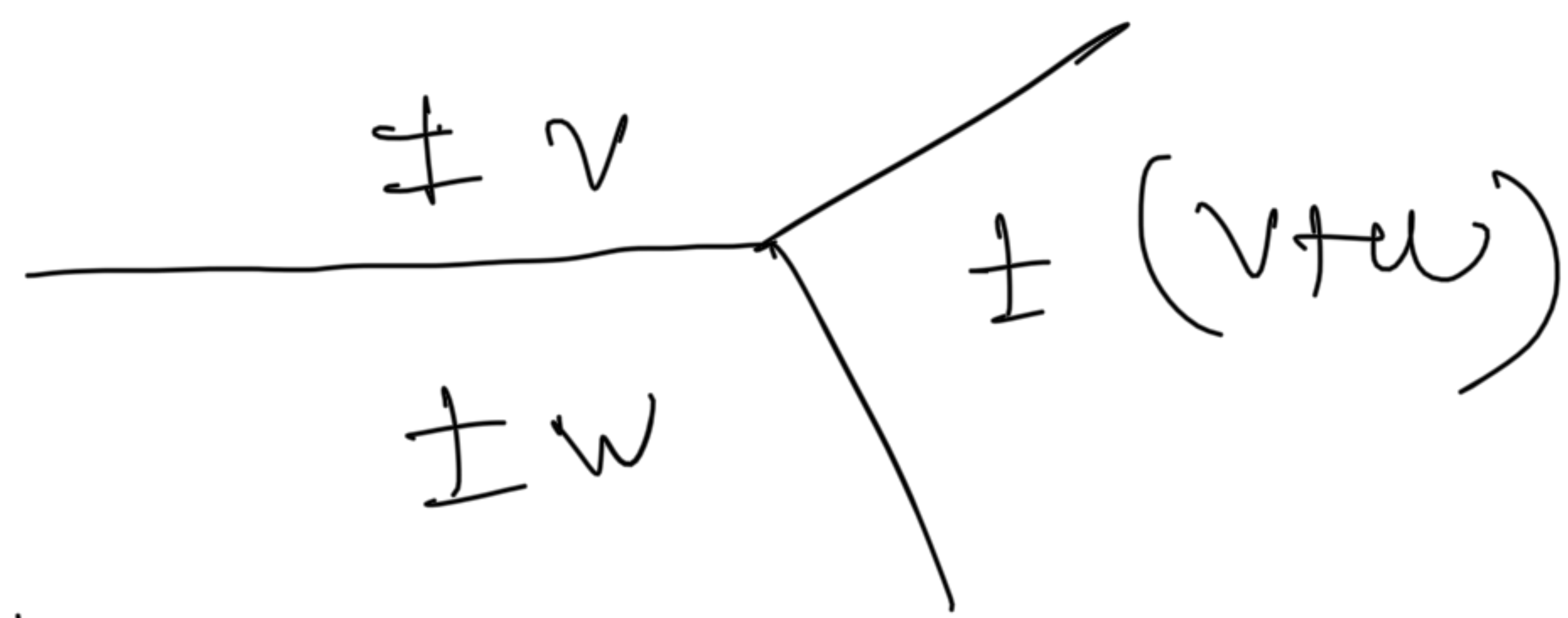
$$\begin{vmatrix} 10 & 7 \\ 7 & 5 \end{vmatrix} = 1$$



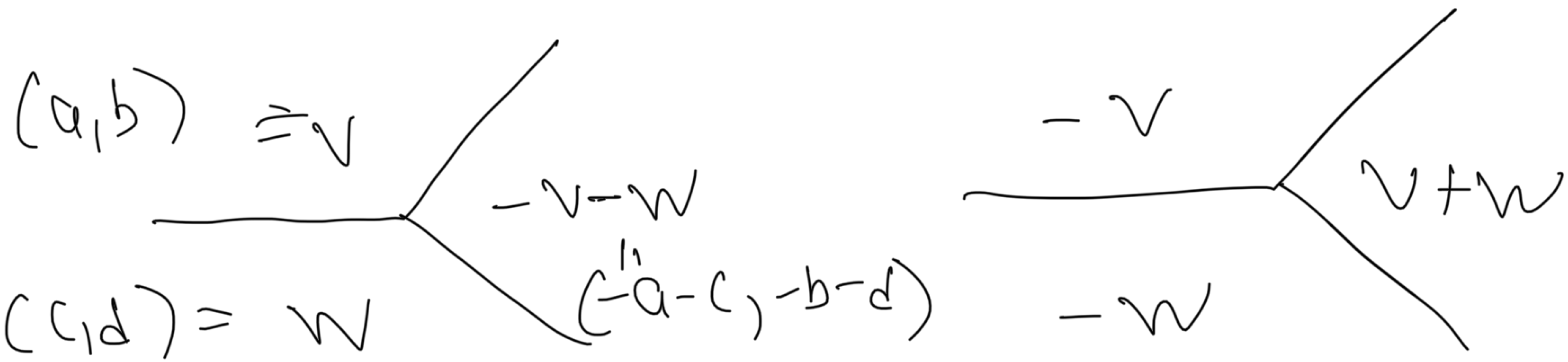
$$\begin{vmatrix} 7 & 5 \\ 10 & 7 \end{vmatrix} = -1$$

Thm: All counterclockwise determinants equal 1.

Pf: 1 Choosing a superbasis from a lax basis

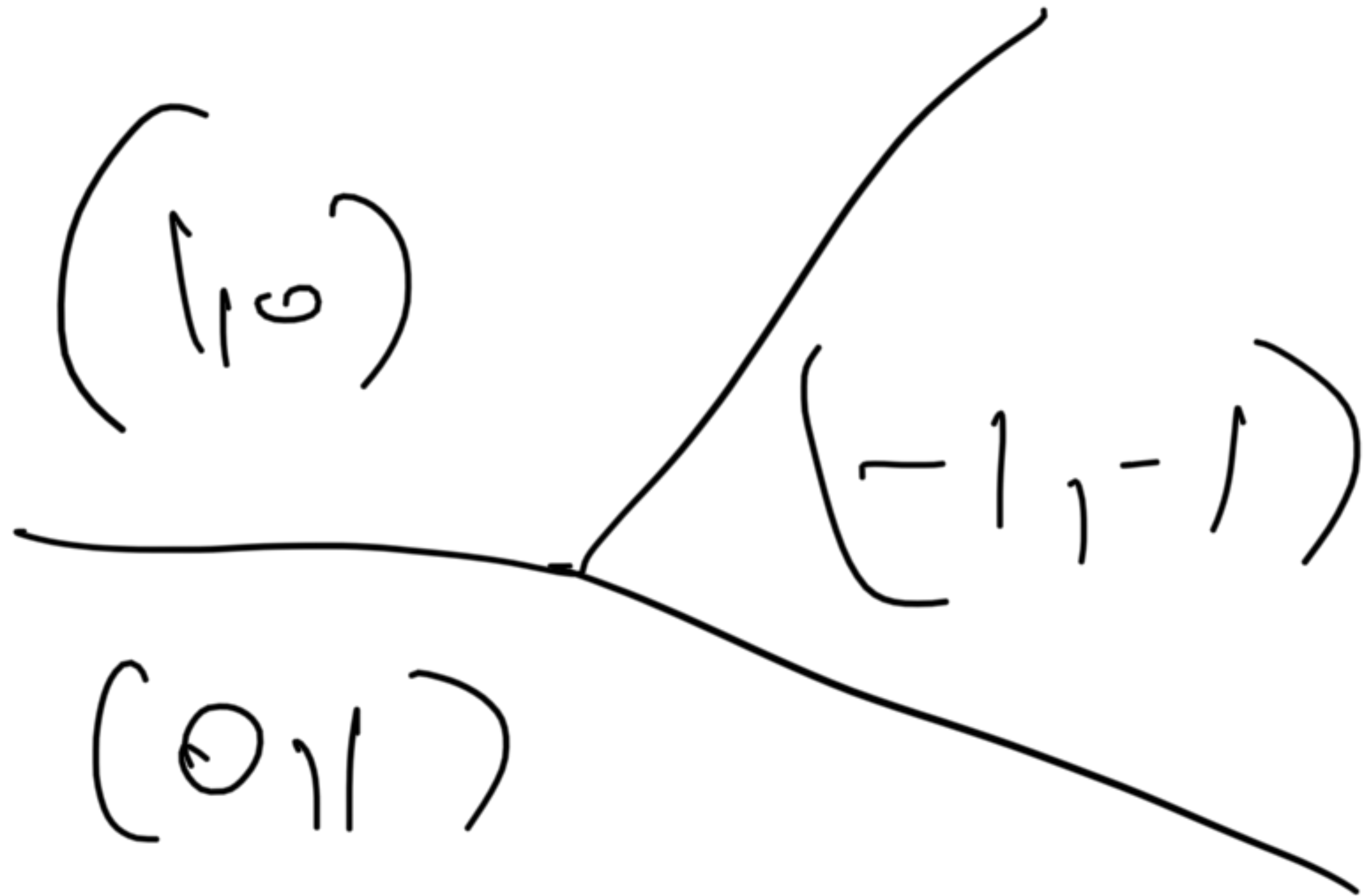


$$\left.\begin{matrix} v, w, -v-w \\ -v, -w, v+w \end{matrix}\right\}$$ Only two choices of super basis



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ -a-c & -b-d \end{vmatrix} = \begin{vmatrix} -a-c & -b-d \\ a & b \end{vmatrix}$$

$$= \begin{vmatrix} -a & -b \\ -c & -d \end{vmatrix} = \begin{vmatrix} -c & -d \\ a+c & b+d \end{vmatrix} = \begin{vmatrix} a+c & b+d \\ -a & -b \end{vmatrix}$$



Counter-clock wise

clock-wise

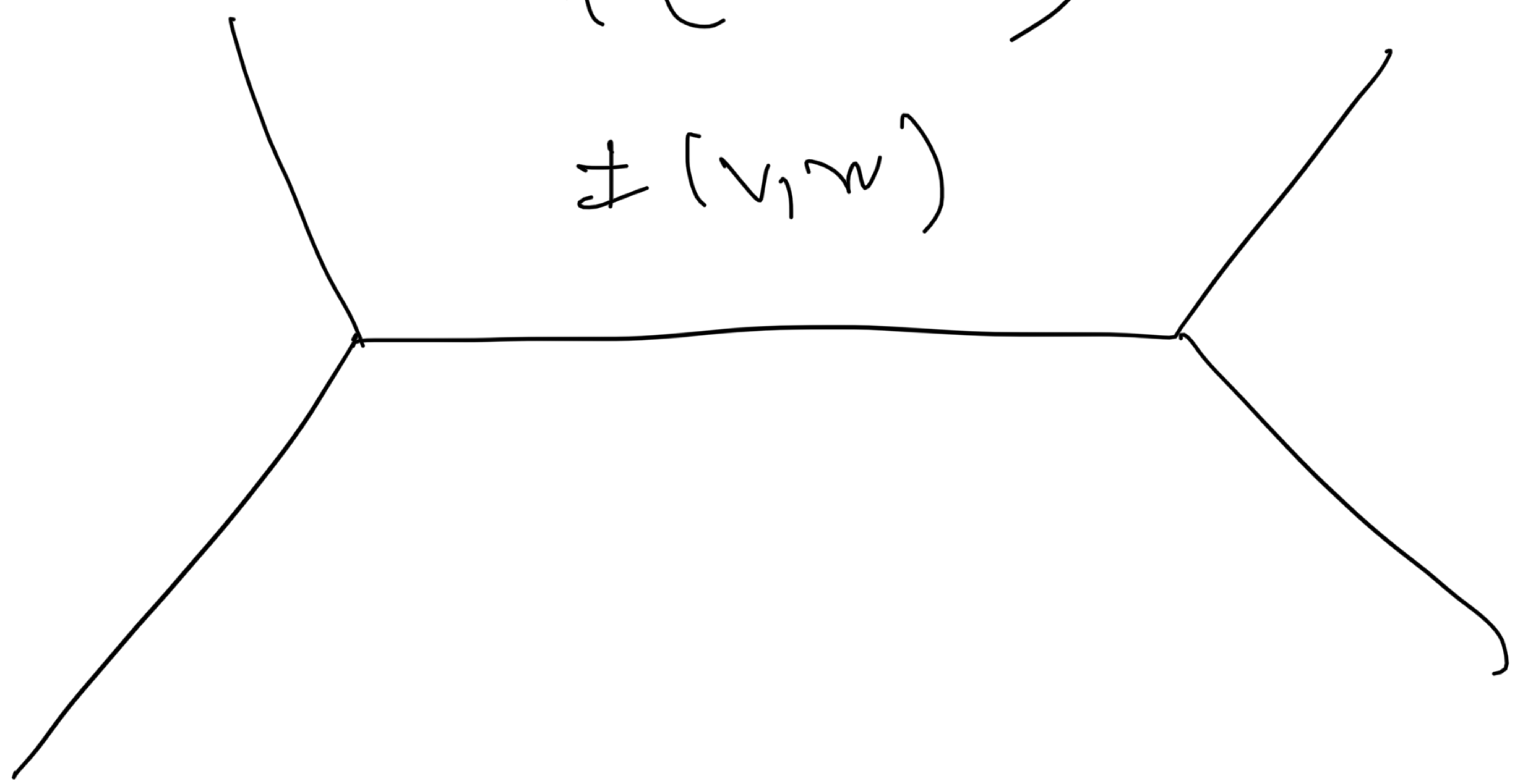
det = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

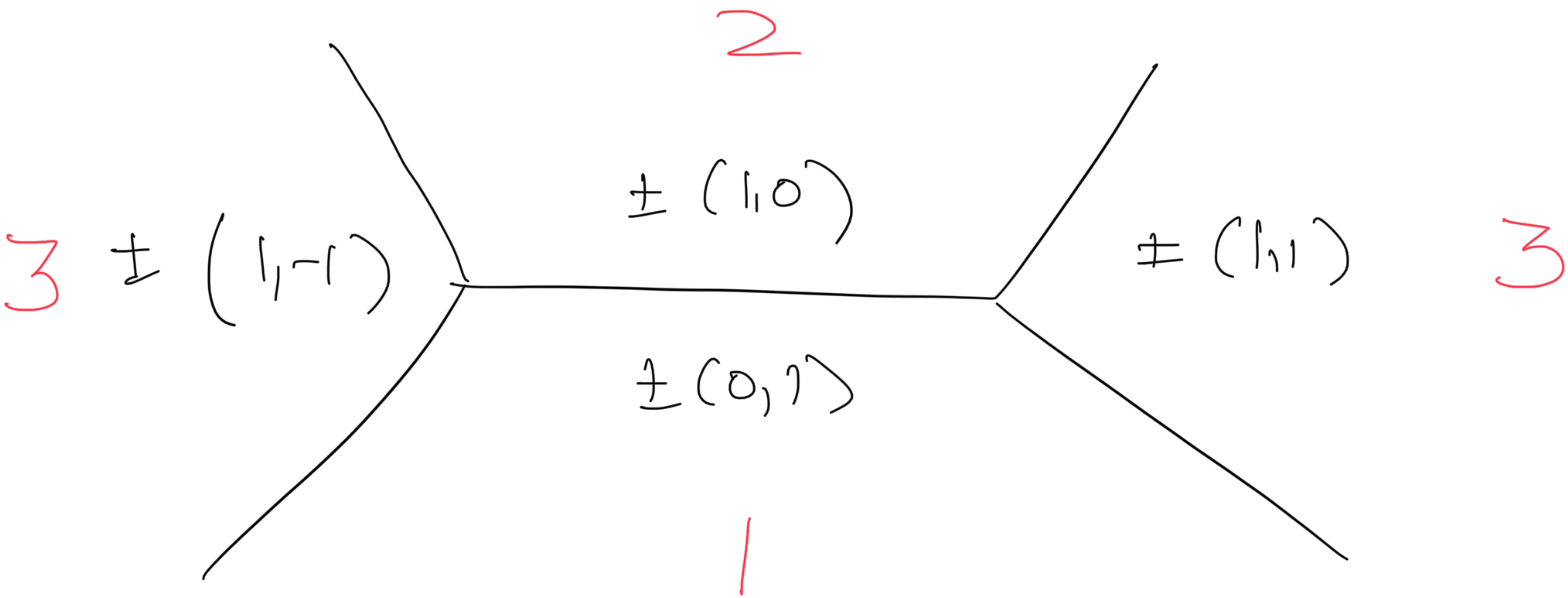
Range Topograph

$$Q(x,y) = ax^2 + bxy + cy^2$$

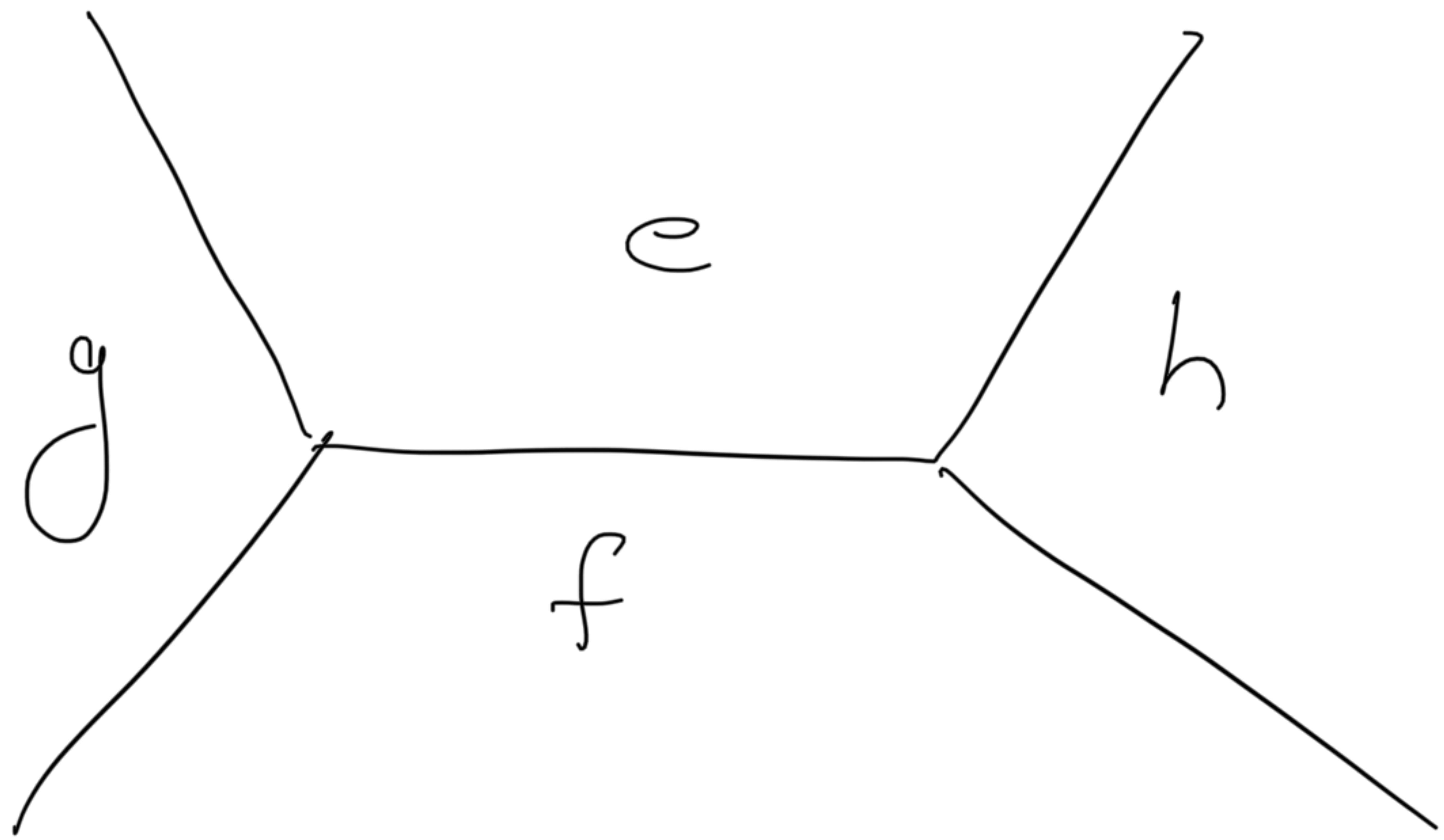
$$Q(\pm(v,w)) \in \mathbb{Z}$$



Example: $Q(x,y) = 2x^2+y^2$



Thm:



Range Topograph

Then g, $e+f$, h form arithmetic progression.

$$Q(x,y) = ax^2 + bxy + cy^2$$

Say $Q(u_1, u_2) = e = au_1^2 + bu_1u_2 + cu_2^2$

$Q(v_1, v_2) = f = av_1^2 + bv_1v_2 + cv_2^2$

$Q(u_1 + v_1, u_2 + v_2) = h$

$= au_1^2 + av_1^2 + 2au_1v_1 + bu_1u_2$

$+ bu_1v_2 + bv_1u_2 + bv_1v_2$

$+ cu_2^2 + cv_2^2 + 2cu_2v_2$

$Q(u_1 - v_1, u_2 - v_2) = g$

$= au_1^2 + av_1^2 - 2au_1v_1 + bu_1u_2 - bu_1v_2 - bv_1u_2$

$+ bv_1v_2 + cu_2^2 + cv_2^2 - 2cu_2v_2$

$$e+f-g = 2a\,u_1v_1 + b\,u_1v_2 + b\,v_1u_2 + 2c\,u_2v_2$$

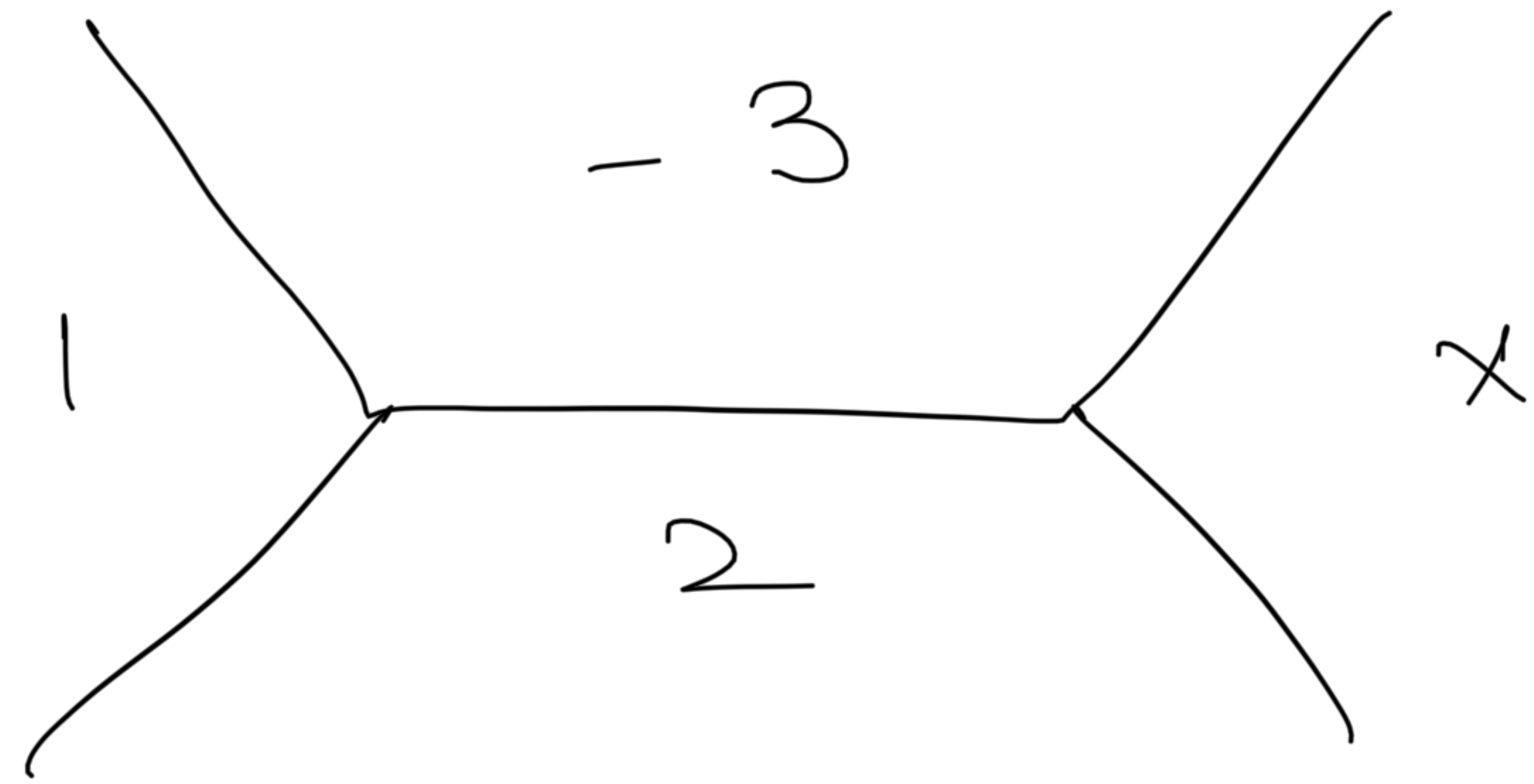
$$h-(e+f) = 2a u_1v_1 + b\,u_1v_2 + b\,v_1u_2 + 2c\,u_2v_2$$

$$(e+f)-g = h-(e+f)$$

So, g, $e+f$, h form arithmetic progression.

Using this rule, we can complete range topograph without explicity evaluating at quadratic form.

Example:



$1, -1, x$ are in arithmetic progression

$\Rightarrow x+1 = -2$

$\Rightarrow x = -3$

Qn: Can two different quadratic forms have same range topograph?

Yes!

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{Z}$$

$$\det M = \pm 1$$

$$Q(x,y) = Ax^2 + Bxy + Cy^2$$

$$Q(ax+by,\ cx+dy) = Q'(x,y)$$

Q & Q' are said to be <u>equivalent</u> to each other.

If $\det M = 1$, we say they are <u>properly</u> <u>equivalent</u> to each other.

(proper isometry)

Discriminant

Suppose $Q(x,y) = ax^2 + bxy + cy^2$

If we know $Q(1,0)$, $Q(0,1)$, $Q(1,1)$

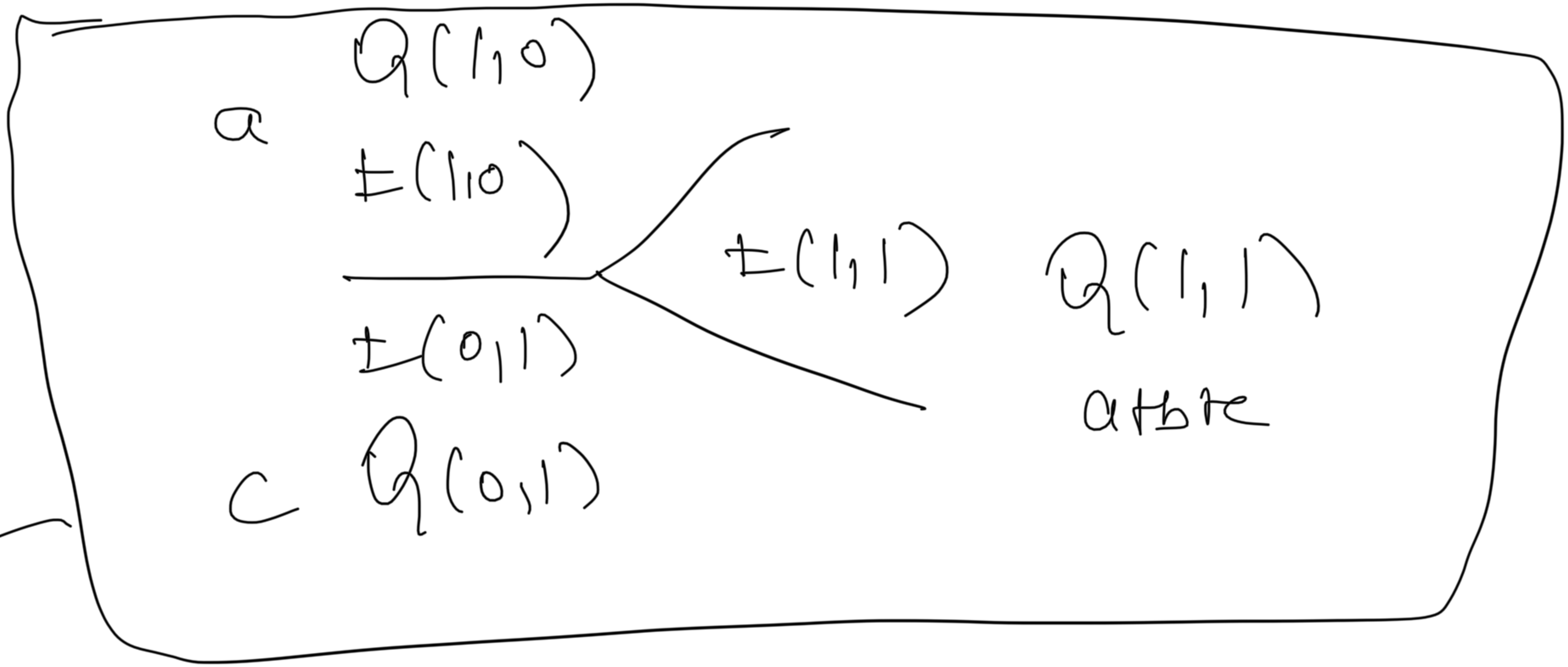
then we know a, b & c.

$$Q(1,0) = a$$

$$Q(0,1) = c$$

$$Q(1,1) = a+b+c$$

So, Q gets determined from



this part of range topograph.

We associate a number to this

$b^2-4ac = \Delta(Q)$ (Discriminant)

Thm: If Q, Q' are equivalent, then

$$\Delta(Q) = \Delta(Q').$$

Pf: Via range topographs