MATH 350, SPRING 2023

HOMEWORK 9, DUE APRIL 17

(1) Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions with positive denominator whose Ford circles are tangent. Show that the matrix
$$\begin{pmatrix} a & a+c \\ b & b+d \end{pmatrix}$$
has an inverse with integer entries.

(2) Fix a Ford circle C. Prove that there is a parabola P such that the center of all Ford circles tangent to C have centers lying on P. *Hint: Consider the focus and directrix of the parabola. You may need to look up these definitions.*

(3) Although it is not completely apparent until page 244, we are building up to analyzing binary integer quadratic forms, which are expressions of the form $ax^2 + bxy + cy^2$. We encountered a few examples earlier in the course, when we considered norms of Gaussian and Eisenstein integers. This problem will hopefully help you connect what you have been reading with where we are going. Fix an integer d. Suppose there exist x and y such that $ax^2+bxy+cy^2 = d$. Show that for any t, there exist z and w such that $az^2+bzw+cw^2 = dt^2$. Then explain why it is sufficient to understand the values of quadratic form only on pairs (x, y) such that $\gcd(x, y) = 1$.

(4) Find a lax vector $\pm v$ so that $\{\pm(20, 17), \pm v\}$ is a lax basis. Then find a walk from your lax basis to home base.

(5) Prove that there cannot be a set of *four* lax vectors, any two of which form a basis.(*Hint: Use Lemma 9.16*)

(6) Let a_n be the n^{th} Fibonacci number. Prove that $\{(a_n, a_{n+1}), (a_{n+1}, a_{n+2})\}$ forms a (strict) basis. *Hint: Proceed by induction. For the inductive step, multiply by the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Use your argument to find the length of a walk from $\{(a_n, a_{n+1}), (a_{n+1}, a_{n+2})\}$ to home base.

(7) *PAR problem #9.* Show that every invertible 2×2 matrix with integer entries can be expressed as a product of the matrices
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$
(Note that some or all of these matrices may appear more than once as factors, possibly nonconsecutively.)