MATH 350, SPRING 2023

HOMEWORK 1, DUE JANUARY 23

(1) Let a, b, c, k, and ℓ be integers. Suppose $a \mid b$ and $a \mid c$. Show that $a \mid kb + \ell c$.
(2) If $x \leq y$, $x + 5 \leq y + 5$. Is the same true with $\leq$ replaced by $|$?
(3) If $x \leq y$, $5x \leq 5y$. Is the same true with $\leq$ replaced by $|$?
(4) Prove that if $x \mid x^2 + 1$, $x = \pm 1$.
(5) A *partially ordered set*, also called a *poset*, is a set S together with a binary relation $\leq$ satisfying
   - reflexivity: For each $a \in S$, $a \leq a$;
   - antisymmetry: For all distinct a and b in S, if $a \leq b$, then $b \not\leq a$. (The symbol $\not\leq$ denotes the negation of $\leq$); and
   - transitivity: For all a, b, and c in S, if $a \leq b$ and $b \leq c$, then $a \leq c$.

   (a) Show that the positive integers, with the "divides" relation, is a partially ordered set.
   (b) What about the set of nonnegative integers, with the same relation?
   (c) What about the set of integers, with the same relation?

   We say that a partially ordered set S is a *totally ordered set* if for all a and b in S, either $a \leq b$ or $b \leq a$.

   (d) Show that the set of positive integers, with the "divides" relation, is not totally ordered.
   (e) List at least four other posets, and determine which of your examples are totally ordered. Include at least one example that is totally ordered and one that is not.
(6) If you can hop 12 units to the left or right, and skip 18 units to the left or right, is it possible to travel exactly 9 units? Why or why not? Express your answer as the existence, or nonexistence, of a solution to a Diophantine equation.
(7) What is $\gcd(0,0)$? Prove your answer. *Note, you should give a proof without using the Euclidean algorithm, which we defined only for positive integers.* Conclude that for any pair of natural numbers a and b, $\gcd(a,b)$ exists.
(8) Use the Euclidean algorithm to find
   (a) $\gcd(69, 372)$
   (b) $\gcd(792, 275)$
   (c) $\gcd(57970, 10353)$
(9) For each part of the previous problem, write the greatest common divisor you found in terms of the numbers given.
(10) Recall from calculus that there is also a notion of division with remainder for polynomials. Use the Euclidean algorithm to find the greatest common divisor of $x^5 + x^4 - x - 1$ and $x^8 - 1$. Then write the greatest common divisor as $p(x)(x^5 + x^4 - x - 1) + q(x)(x^8 - 1)$ for polynomials p and q.
(11) *This is your first PAR problem. You will write this problem up separately from the others, and turn in your rough draft online by January 25. We will talk in class about the PAR process and schedule. For now, it will be helpful to think about solving this problem, but you need not start writing up your solution.*

The Fibonacci sequence is defined recursively by the formula

$$a_n = \begin{cases} 1 & n = 1 \\ 1 & n = 2 \\ a_{n-2} + a_{n-1} & n \geq 3 \end{cases}$$

Use the Euclidean algorithm to prove that the greatest common divisor of any two consecutive Fibonacci numbers is 1.