

MATH 3500 Spring 2023 (lecture 2)

WRITING PROOFS

Problem: Show that sum of two consecutive integers is an odd integer.

Understand the problem first

Consecutive integers

Defn: We say that a & b , $a < b$ are consecutive integers if and only if $b = a + 1$.

Defn: We say that b is an odd integer iff $b = 2k+1$ for some integer k .

Direct Proof:

Let $a < b$ be two consecutive integers.

So, $b = a+1$. Then $a+b = 2a+1$

which is odd by definition.

So, done.

PROOF BY CONTRADICTION

If P , then Q .

Assume P is true

and Q is false and

arrive at a contradiction



a false
Statement

Show that Sum of two consecutive integers is odd.

Assume a, b $a < b$ are consecutive.

Assume $a+b$ is even.

Since $a \& b$ are consecutive $b = a+1$
We have assumed $a+b = 2m$ for some integer m .

$$\text{Also } a+b = 2a+1 = 2m$$

$$\text{So, } 1 = 2(m-a) \quad \rightarrow \leftarrow$$

PROOF By CONTRA POSITIVE

Want

to show $P \Rightarrow q$

Show

In this method we
show

$$\neg q \Rightarrow \neg P$$

a & b are
consecutive \Rightarrow

$a+b$ is
odd

not of 2
Assume
 $a+b$ is even

\Rightarrow a & b are
not consecutive

$a+b$ is even

a even

b even

a odd

b odd



Cannot be
consecutive

Try to write down
complete proof using well
formed sentences.

Proof by induction is used to prove results about natural numbers, see example below:

Show that $1+2+\dots+n = \frac{n(n+1)}{2}$

- ① This holds for $n=1$.
- ② Assume it holds for $n=k$.
- ③ Now show that the statement holds for $n=k+1$

$$1+2+\dots+k+k+1 = \frac{k(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2}$$

LINEAR DIOPHANTINE EQUATION

Degree

1

involves variables
and Coefficients

Coefficients are

integers

$$7x^2 + 11xy + 21z^2 = 0$$

Example:

This is diophantine equation but not linear!

Example: (Linear Diophantine equation)

$$2x + 3y = 5$$

x

x

x

Solving linear diophantine equations

It is easy to show that there are
no solutions than to find them all

Are there any integer solutions to

$$\rightarrow 12x + 15y = 5 ?$$

No, because 3 divides LHS but
not RHS

$$\rightarrow 2x + 3y = 1$$

Yes $x=2, y=-1$ and many more.

GCD of two integers

Def'n: Let a, b be two integers.

We say that d is the GCD of $a \& b$ iff

(i) $d > 0$

(ii) $d | a \& d | b$

(iii) If c is an integer such that $c | a \& c | b$ then $c | d$.

Find $\text{GCD}(2, 3)$

Common divisors are 1, -1.
↓
GCD.

Find $\text{GCD}(12, 15)$.

Divisors of 12 $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Divisors of 15 $\pm 1, \pm 3, \pm 5, \pm 15$

$\text{GCD} = 3$

find $\text{GCD}(720, 134)$.

↓
2

Most effective & useful way to
compute GCD is by using
Euclidean algorithm.

Next time !

Euclidean Algorithm

Can be used to find gcd of two integers

$$\text{GCD}(72, 40)$$

$$72 = 40(1) + 32$$

$$40 = 32(1) + 8 \quad \text{is the GCD}$$

$$32 = 8(4) + 0$$

Stop when you get
0 as a remainder

Find GCD of 45 and 144

201 and 335

$$\begin{aligned}144 &= 45(3) + 9 \\45 &= 9(5) + 0\end{aligned}$$

GCD

$$\begin{aligned}335 &= 201(1) + 134 \\201 &= 134(1) + 67 \\134 &= 67(2) + 0\end{aligned}$$

GCD

① Why does Euclidean algorithm terminate
in finitely many steps?
(because divisor is decreasing at every step)

② Why do we get GCD in the end?

Let d be the positive number that we get in
the end. Observe that $d = ax + by$ for some
Tracing back the steps $d \mid a$ & $d \mid b$.
 $x, y \in \mathbb{Z}$.

If $c \mid a$ & $c \mid b$ then $c \mid ax + by \Rightarrow c \mid d$.

Fix two integers a and b , look at all the possible combinations

$$ax + by = c$$

(x, y are integers)

which is the smallest combination? GCD

positive c in this

Is it possible that c is not a multiple of $\text{GCD}(a, b)$? No!

$$a \text{ and } b \text{ are co-prime} \iff ax+by = 1$$

(Why?) for some $x, y \in \mathbb{Z}$.

because 1 is the smallest positive integer that you can get as a combination.

→ Find GCD of (n_1, n_2) . 1

→ Show that $\text{GCD}(a, b) = 1 \iff \text{GCD}(a^2, b^2) = 1$

Argue via prime factorization

→ Show that $\text{GCD}(R_a, R_b) = |k| \text{GCD}(a, b)$

→ Show that $\text{GCD}(a^2, b^2) = \text{GCD}(a, b)^2$.

There exists integers x, y s.t.

$$ax + by = \text{GCD}(a, b)$$

$$kax + kb y = k \text{GCD}(a, b) \Rightarrow \text{GCD}(ka, kb) \mid k \text{GCD}(a, b)$$

$$|k| \text{GCD}(a, b) \mid ka \& kb \Rightarrow |k| \text{GCD}(a, b) \mid \text{GCD}(ka, kb)$$

→ Let $\text{GCD}(a, b) = d$

$$\text{GCD}\left(\frac{a}{d}, \frac{b}{d}\right) = 1$$

$$\Leftrightarrow \text{GCD}\left(\frac{a^2}{d^2}, \frac{b^2}{d^2}\right) = 1$$

$$d^2 \text{GCD}\left(\frac{a^2}{d^2}, \frac{b^2}{d^2}\right) = \text{GCD}(a^2, b^2)$$
$$= d^2$$

Fibonacci Numbers

0, 1, 1, 2, 3, 5, 8, 13, 21, ...
↓
Sum of
Previous
two terms

$$q_n = q_{n-1} + q_{n-2}$$

Perform Euclidean algorithm on two consecutive Fibonacci numbers. What do you observe ? They are coprime.

Hint: Use the Property that $\text{GCD}(a+b, b) = \text{GCD}(a, b)$.

Why is $\text{GCD}(a+b, b) = \text{GCD}(a, b)$?

Show that they divide each other

Diophantine Equations

①

When does $ax + by = c$ has a solution?
 $\Leftrightarrow \text{GCD}(a,b) | c$

②

Find one solution of

$$71x + 83y = 2$$

③

Find all the solutions to

$$71x + 83y = 2$$

$$83 = 71(1) + 12$$

$$71 = 12(5) + 11$$

$$12 = 11(1) + 1$$

$$11 = 1(11) + 0$$

$$1 = 12 - 11(1)$$

$$= 12 - 71 + 12(5) = 12(6) - 71$$

$$2 = 83(12) - 71(14) = 83(6) - 71(7)$$

All the other solutions are of the form
 $(12+71m, -14-83m)$

$$71x_0 + 83\% = 2$$

$$71x_0 + 71(83)\% - 71(83)\% + 83y_0 = 2$$

$$\underline{71(x_0 + 83\%)} + \underline{83(y_0 - 71\%)} = 2$$

Are these all ? Yes

In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949.

How many pages are torn off?

Sum of torn page numbers = $4k - n$
(n is the # pages torn)

$$\Rightarrow n \in \{3, 7, 11, \dots, 3\}$$
$$4k - n = 5050 - 4949 = 101$$

Further let a_1, a_2, \dots, a_n be page numbers of pages term. Then,

$$1 + 2 + \dots + n \leq a_1 + \dots + a_m \\ + (n+1) + \dots + (2m)$$

$$\Rightarrow \frac{2n(2n+1)}{2} \leq 101$$

$$\Rightarrow n(2n+1) \leq 101$$

$$\Rightarrow n = 3$$