MATH 350, SPRING 2023

HOMEWORK 4, DUE FRIDAY, FEBRUARY 19

(1) Let p be a prime number. Prove that $\sqrt{p}$ is irrational. Deduce if $\frac{a}{b}$ is a reduced fraction with a and b both positive, $\sqrt{\frac{a}{b}}$ is rational if and only if a and b are each perfect squares.

(2) Fermat's Last Theorem states that $a^n + b^n = c^n$ has no solutions other than 0's and ± 1's for integers a, b, and c and natural numbers $n > 2$. Explain why it suffices to prove Fermat's last theorem only when n is an odd prime or equal to 4. (Fermat himself gave a correct proof in the case $n = 4$.)

(3) Show that the product of two Gaussian integers is a Gaussian integer, and the product of two Eisenstein integers is an Eisenstein integer.

(4) Show that the norm of a product of two Gaussian integers is the product of their norms, and likewise for Eisenstein integers.

(5) Factor 10 into a product of Gaussian primes. Factor 12 into a product of Eisenstein primes. (*Hint: to determine whether an element is prime, consider its norm and the norms of its possible factors.*)

(6) Find the prime factorizations for 2, 3, 5, 7, 11, and 13, in $\mathbb{Z}[i]$. Draw the primagon for the prime factors of each prime. Complex conjugate pairs should appear on the same sketch. Indicate which Gaussian primes are of type (S), (I), and (R). Formulate a conjecture about which integer primes split, are inert, and ramify in $\mathbb{Z}[i]$. You may wish to compute additional primagons in order to gather enough data to make your conjecture.

(7) Repeat the above question with $\mathbb{Z}[\omega]$ in place of $\mathbb{Z}[i]$.