Midterm 2 Problem 1

Jacobi Symbol

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

$\gcd(a, n) = 1$

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{a_1} \cdots \left(\frac{a}{p_r}\right)^{a_r}$$

Legendre Symbol

$\left(\frac{a}{p}\right) = 1 \iff x^2 \equiv a \pmod{p}$ has a solution

Jacobi Symbol

$x^2 \equiv a \pmod{n}$ has a solution

$\implies \left(\frac{a}{n}\right) = 1$

$\not\Longleftarrow$ not true

Revisit Fermat's 2 square thm

Thm: $p \equiv 1 \pmod 4 \iff p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$

pf: Alternate proof using binary quadratic forms.

Let p be a prime, $p \equiv 1 \pmod 4$.

$\exists\, r, u$ s.t. $pu + r^2 = -1$

Consider $Q(x, y) = px^2 + 2rxy - uy^2$

$\Delta = 4r^2 + 4pu = 4(-1) = -4$

$Q(1,0) = P$

So, Q is positive definite.

What are all positive definite quadratic forms (up to equivalence) such that $\Delta = -4$?

Minimum value $u \leq \sqrt{4/3} \leq 1$

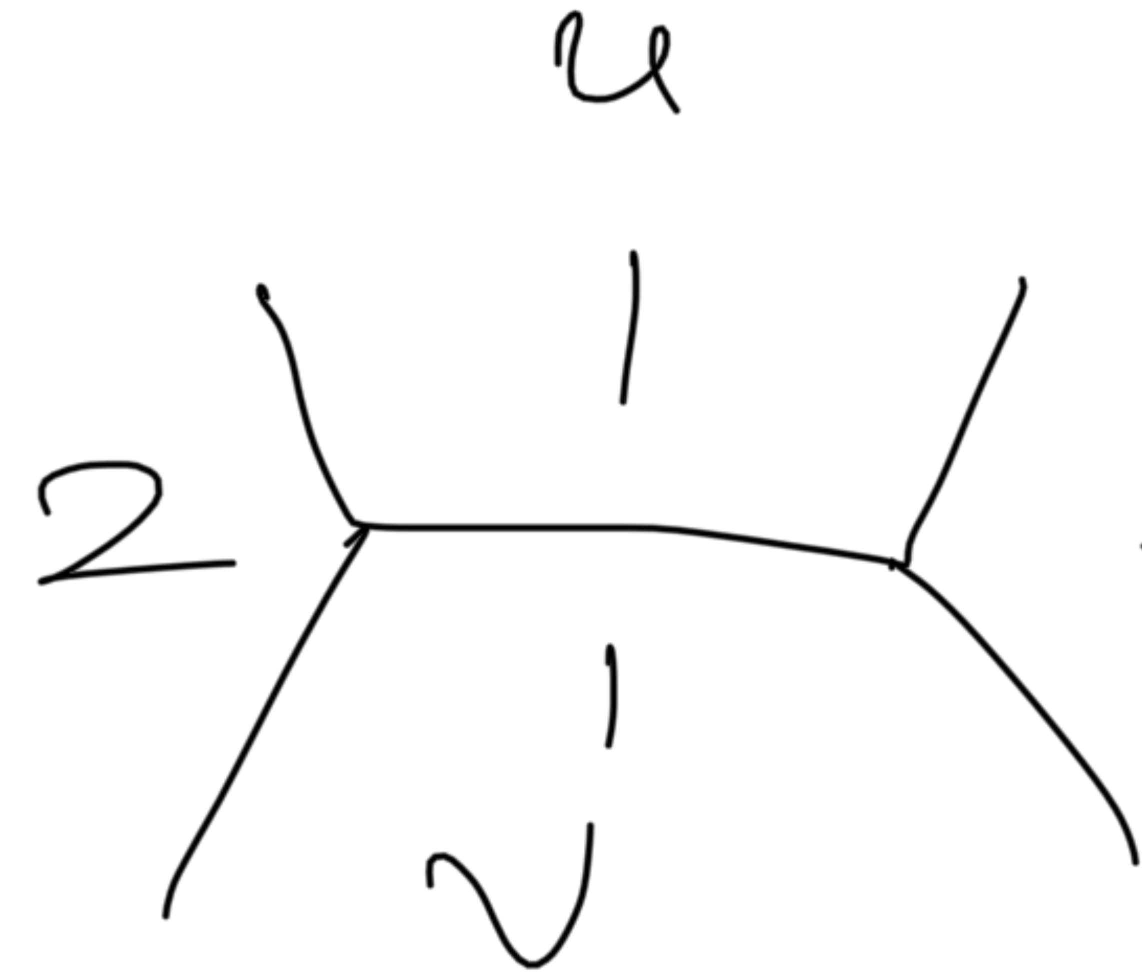
So, $u = 1$,

$0 \leq h \leq u$

$\Delta = -4 \implies h$ even, $h = 0$

$$h^2 - 4uv = -4$$

$$uv = 1 \Rightarrow v = 1$$



double well

$u+v-h$

$$\boxed{x^2+y^2}$$

So, Q is equivalent to x^2+y^2.

Q represents P

So, $\boxed{x^2+y^2 \text{ represents } P.}$

Thm: If $p \equiv 1 \pmod 3$, then

$x^2 + xy + y^2 = p$ has a solution.

Pf: $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)$

$= \left(\frac{-1}{p}\right)\left(\frac{p}{3}\right)(-1)^{\left(\frac{3-1}{2}\right)\left(\frac{p-1}{2}\right)}$

$= \left(\frac{-1}{p}\right)\left(\frac{-1}{p}\right)\left(\frac{p}{3}\right) = \left(\frac{p}{3}\right) = 1$

$\exists$ r, u s.t. $-3 \equiv r^2 \pmod p$

$-3 = r^2 + pu$

$$Q(x,y) = px^2 + 2rxy - uy^2$$

$$\Delta(Q) = 4r^2 + 4pu$$
$$= 4(r^2 + pu) = -12$$

(positive def)

$x^2 + xy + y^2$ has $\Delta = -3$

Need to find another quadratic form.

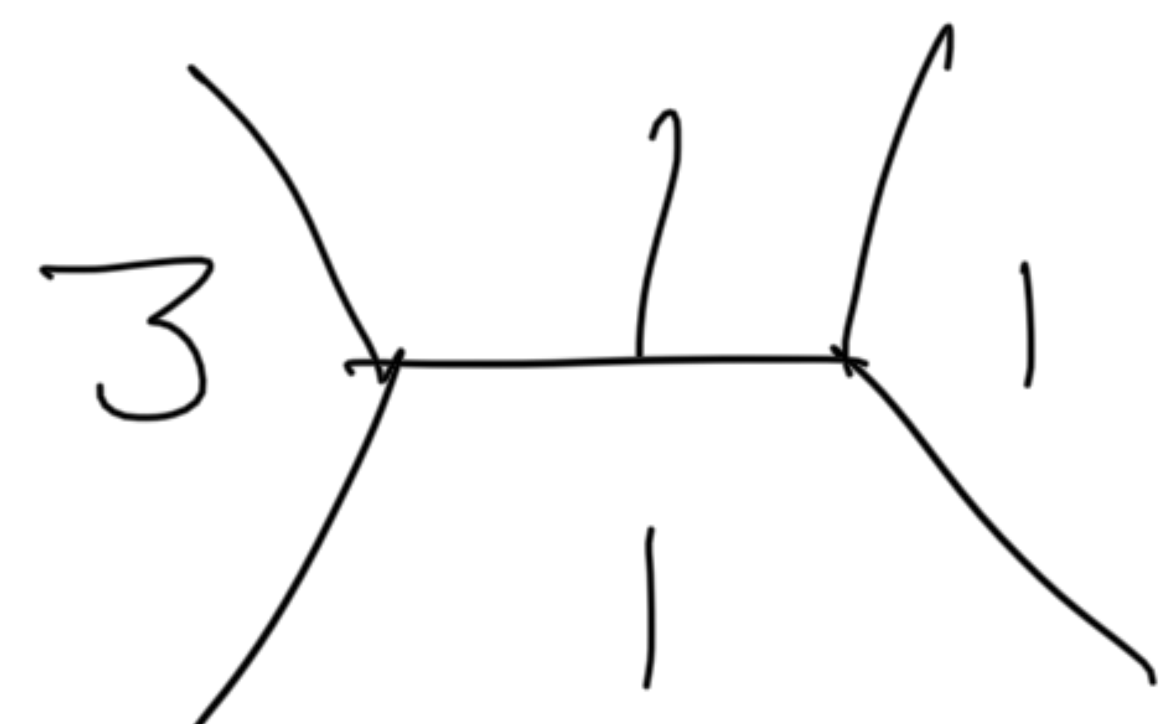
$$-3 = r^2 + pu$$

$$Q(x,y) = px^2 + rxy - (u/4)y^2$$

$$\Delta(q) = r^2 + 4(p)\left(\frac{u^2}{4}\right)$$
$$= r^2 + pu^2 = -3$$

$$u \leq \sqrt{\frac{|\Delta|}{3}} = \sqrt{1} = 1$$

$u=1$ $\quad$ $h=1$ $\quad$ $v=1$



$\sim x^2 + xy + y^2$

Practice Problems

1. Show that every positive definite binary quadratic form of $\Delta = -12$ is equivalent to

$$x^2 + 3y^2 \quad \text{or to} \quad 2x^2 + 2xy + 2y^2$$

Soln: We start with classifying wells

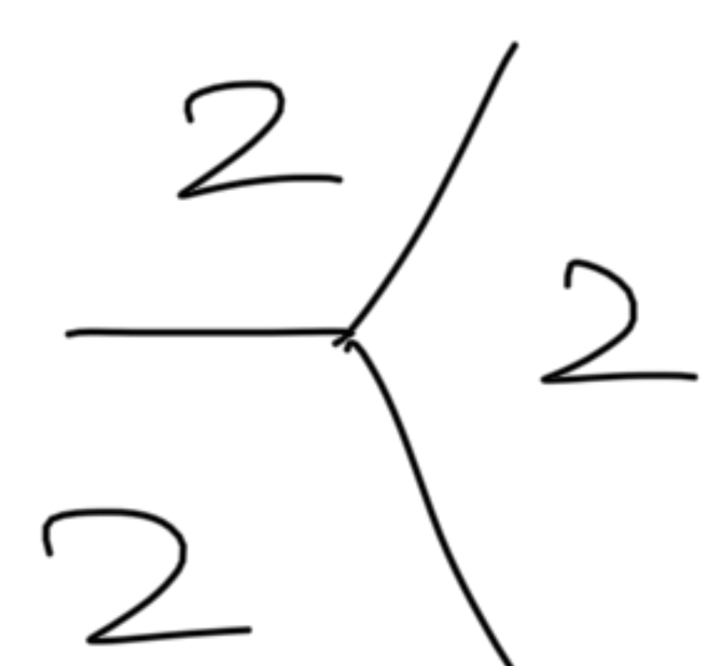
1) $0 < u \leq \sqrt{\dfrac{|\Delta|}{3}} = \sqrt{\dfrac{12}{3}} = 2$

2) $0 \leq h \leq u$

3) $\Delta = h^2 - 4uv$

u	h	v	
1	0	3	
2	0	3/2	← not an integer
2	2	2	

x^2+3y^2



$2x^2+2xy+2y^2$

2. $Q_1(x,y) = 41x^2 + 70xy + 30y^2$

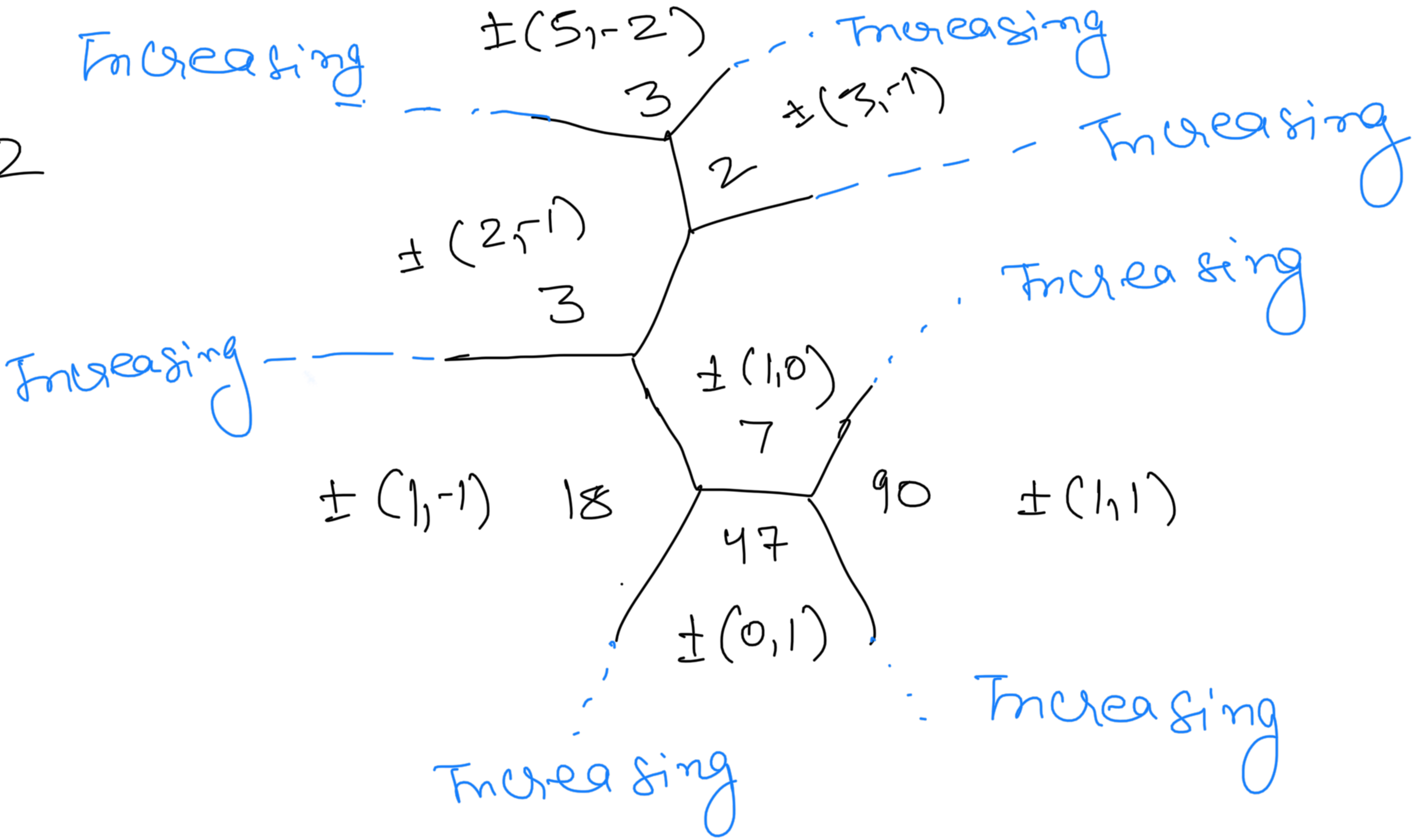
$Q_2(x,y) = 7x^2 + 36xy + 47y^2$

$Q_3(x,y) = 23x^2 + 76xy + 63y^2$

All of them have same discriminant -20. Which of them are properly equivalent?

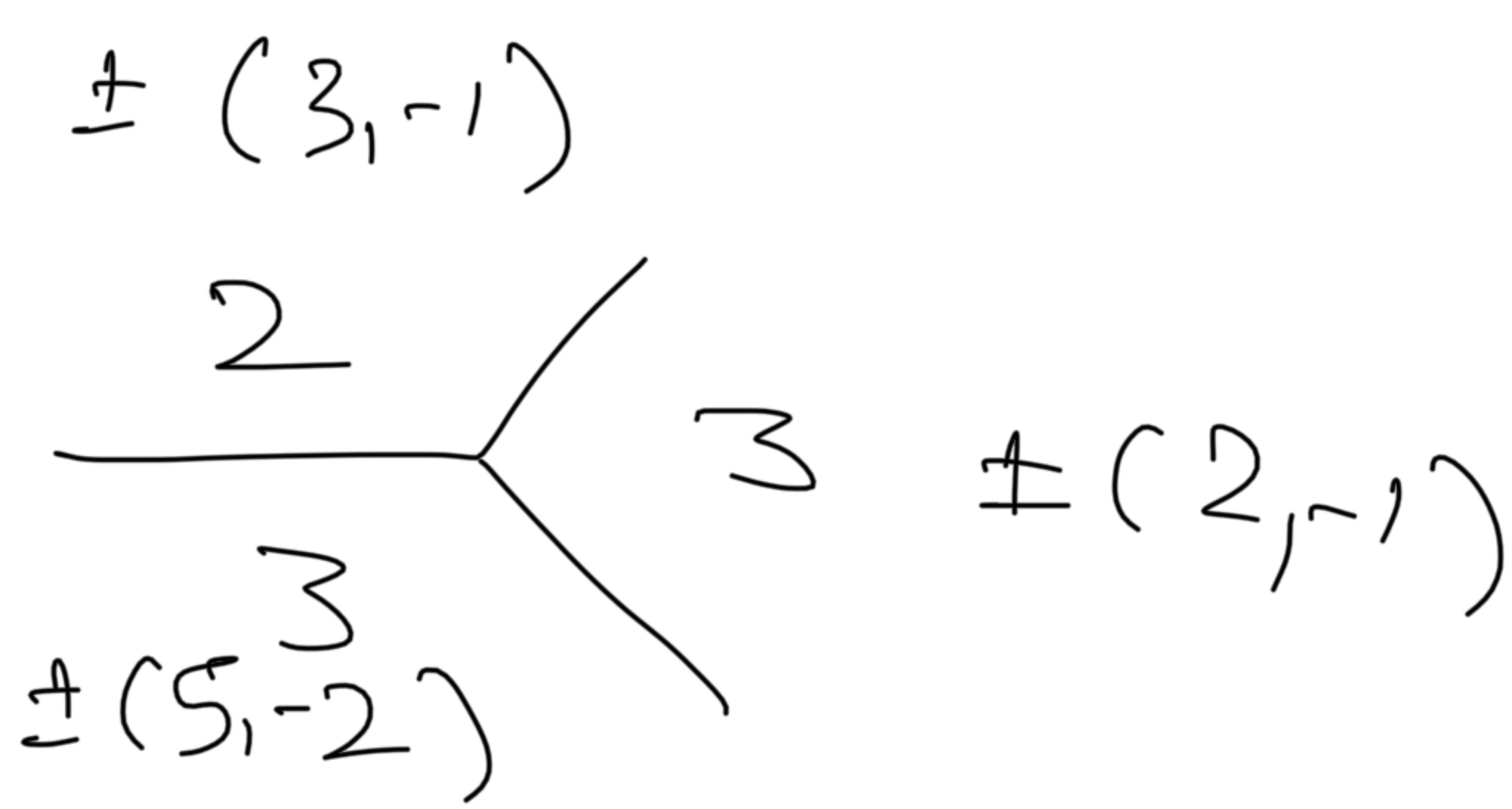
Let us start by finding wells in their range topographs

Q22



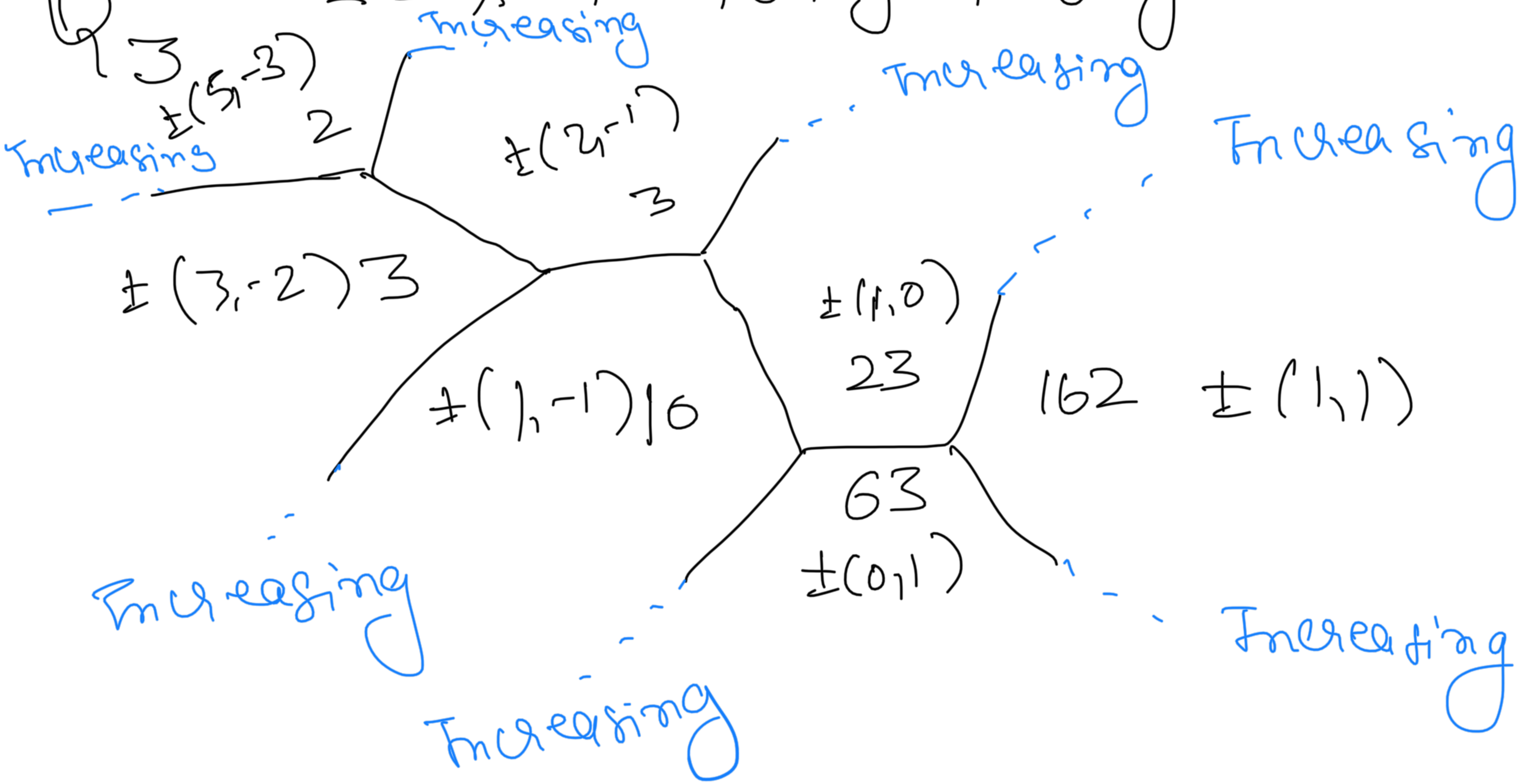
Increasing
±(5,-2)
3
Increasing
±(3,-1)
2
Increasing
±(2,-1)
3
Increasing
Increasing
±(1,0)
7
±(1,-1)
18
90
±(1,1)
47
±(0,1)
Increasing
Increasing

Well

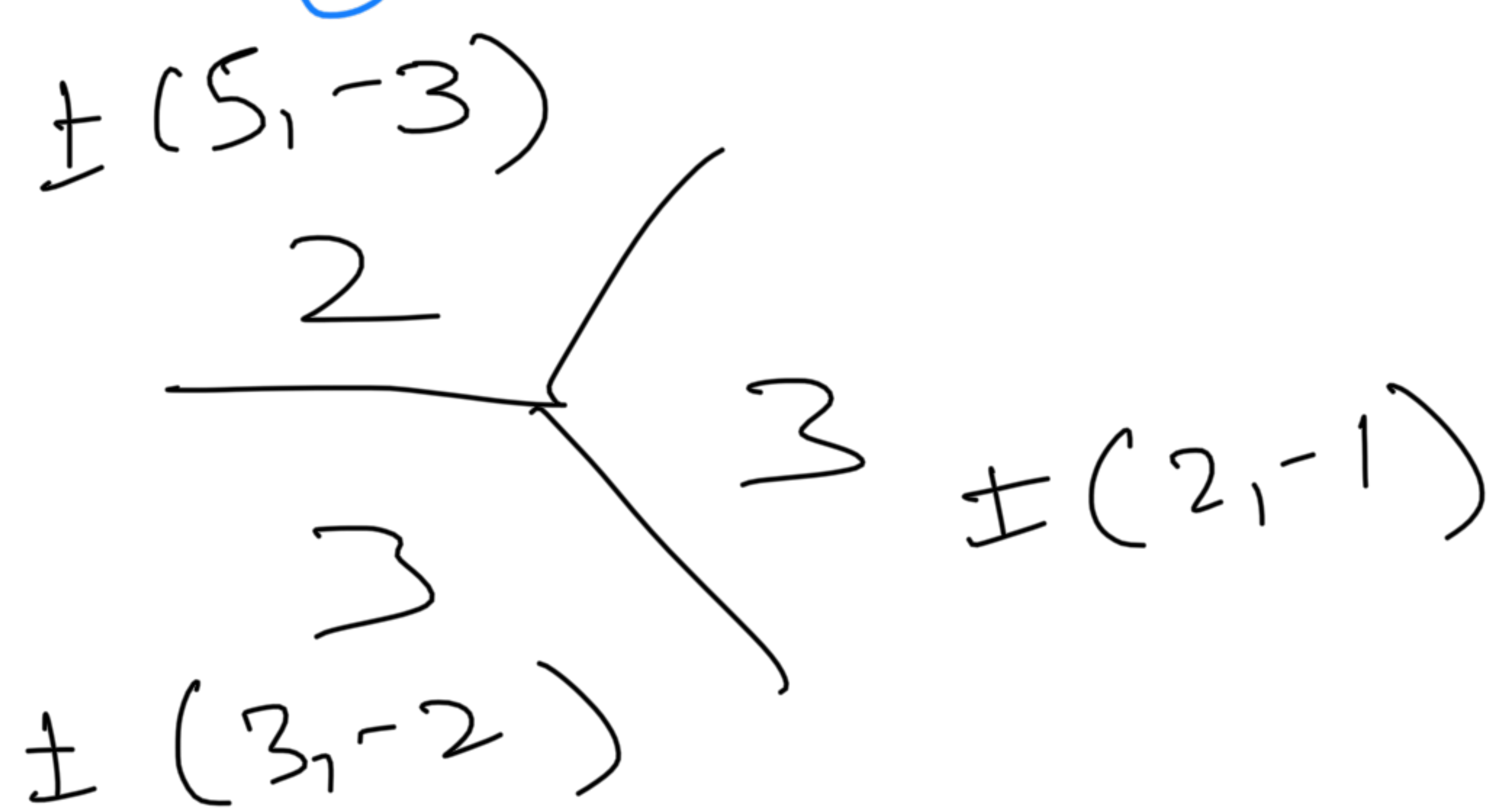


±(3,-1)
2
3
±(2,-1)
3
±(5,-2)

Q3 $23x^2 + 76xy + 63y^2$



Well



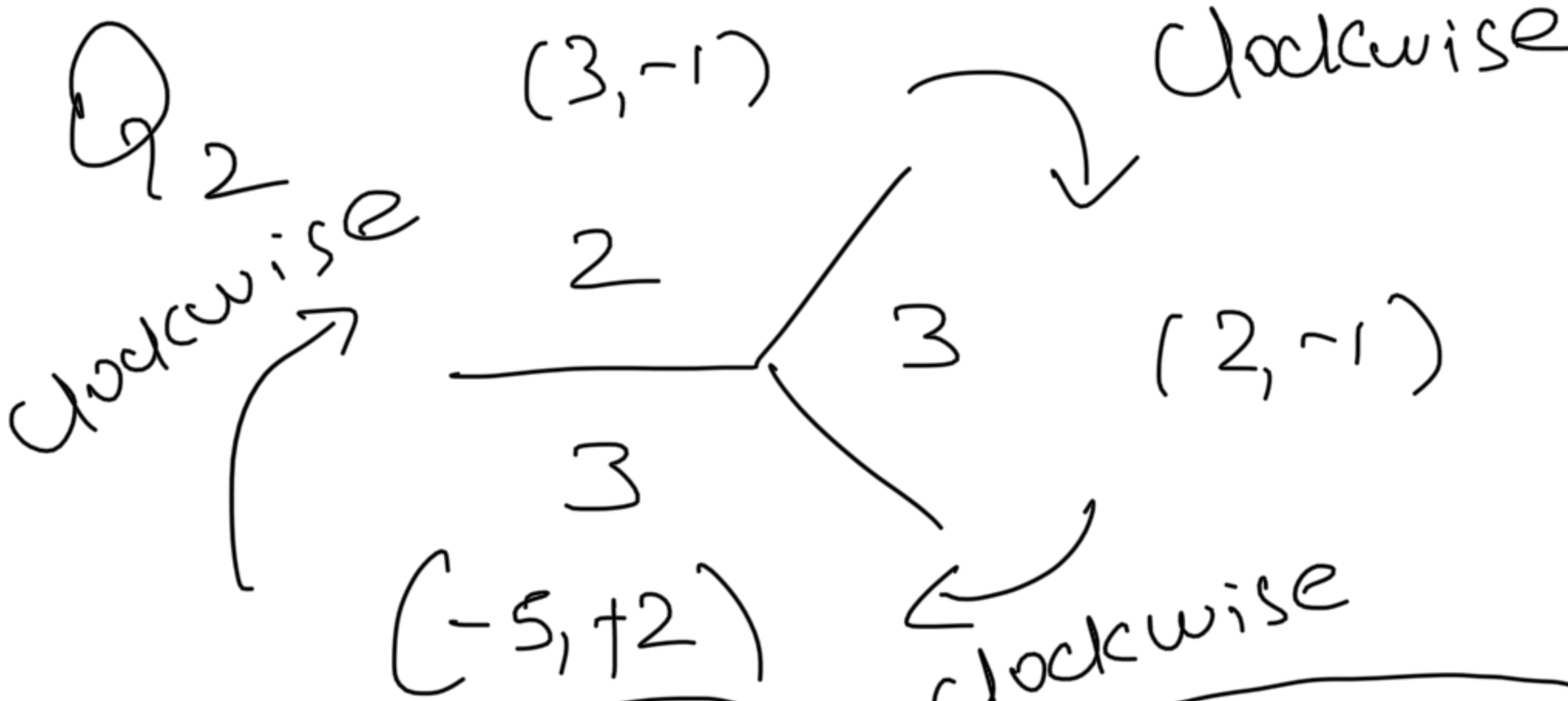
So, Q_2 & Q_3 are equivalent.

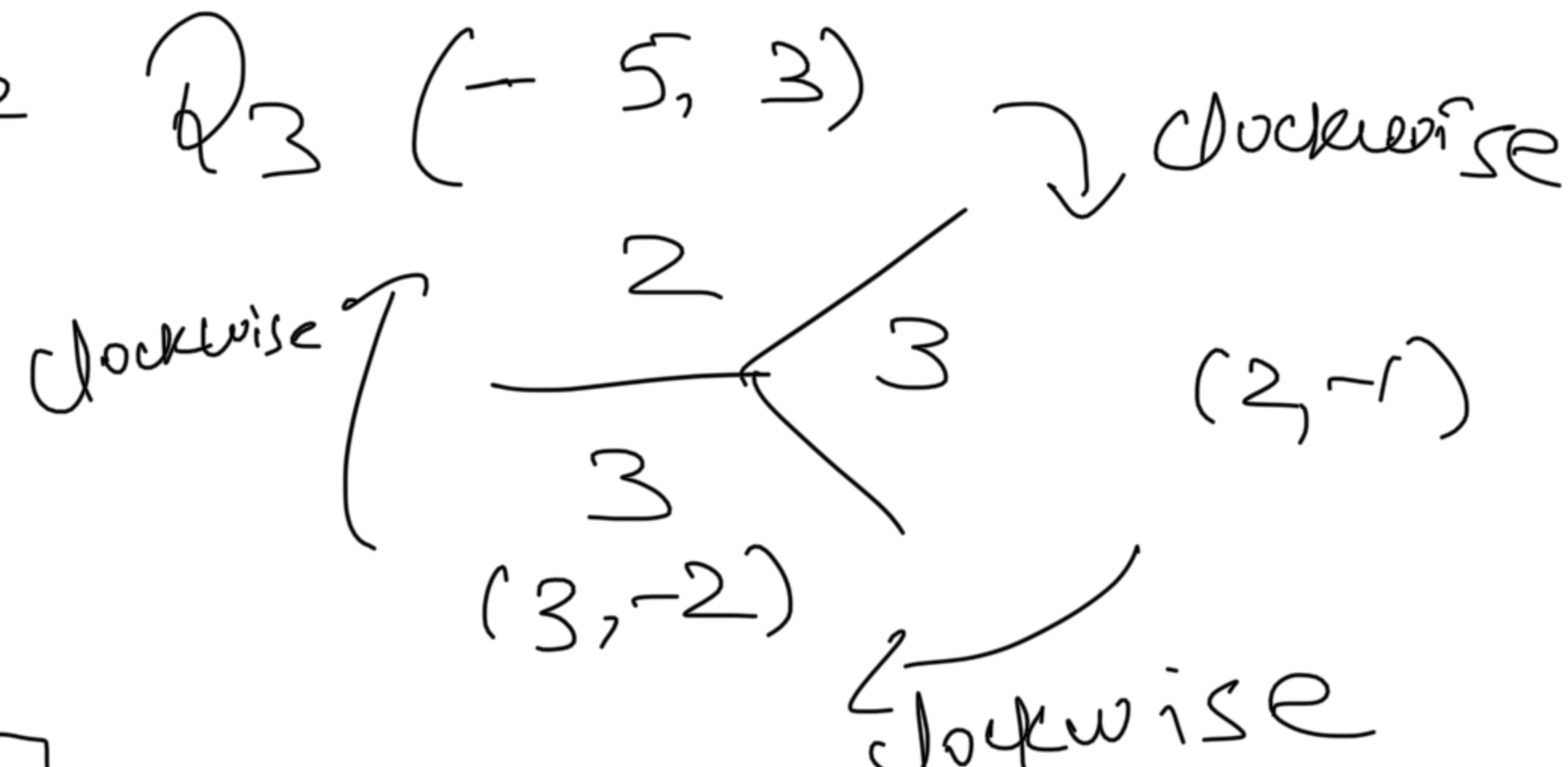
By observation $Q_1(1,-1) = 41(1)^2 + 70(-1) + 30(-1)^2$

$= 41 - 70 + 30 = 1$

So, Q_1 is not equivalent to Q_2 or Q_3.

Let's look at orientation of Q_2 & Q_3's wells.





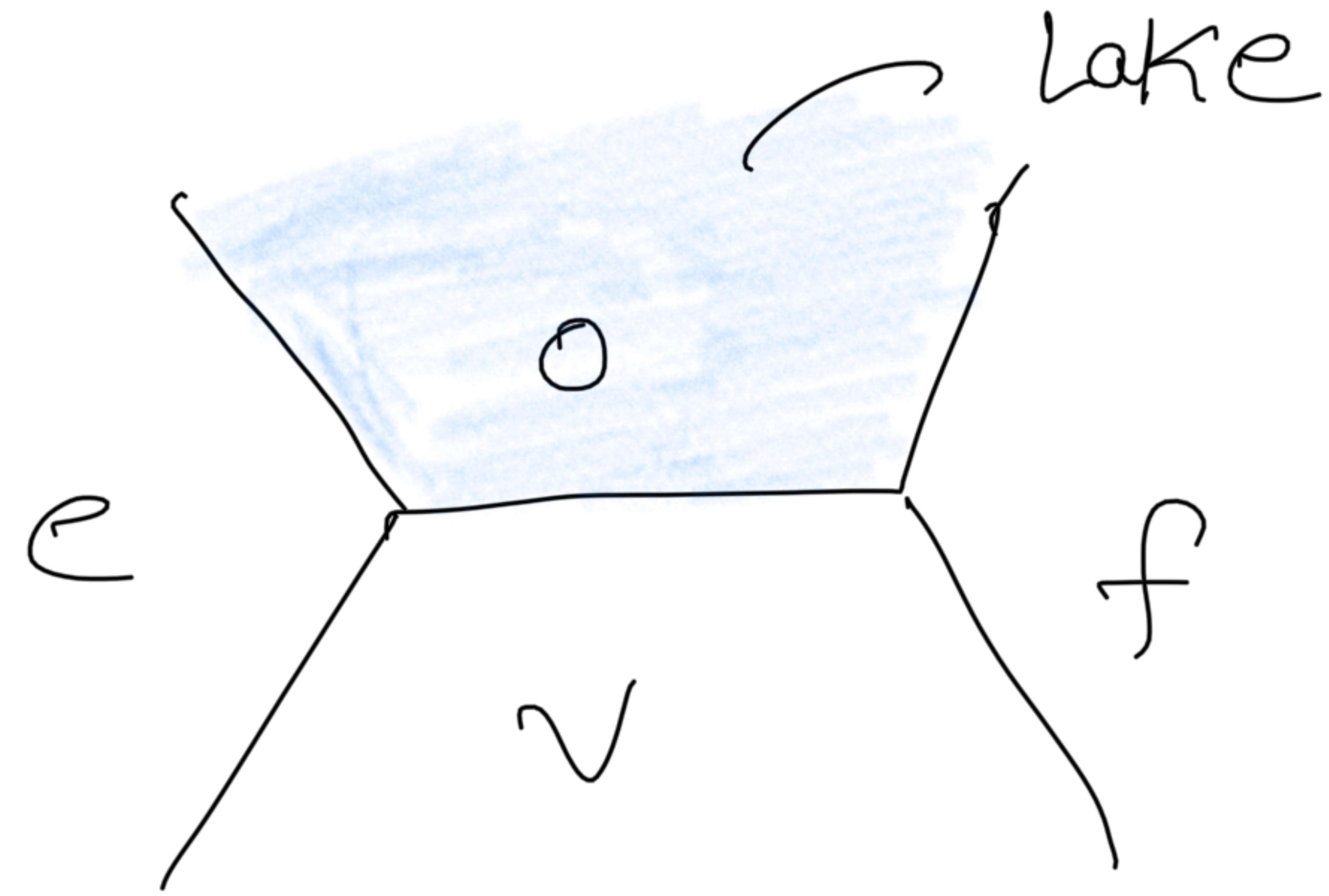
$\boxed{\text{Properly equivalent}}$

We saw that if $Q(x,y)$ is positive definite then its range topograph contains exactly one well.

What happens if $Q(x,y)$ is positive semi-definite

$$Q(x,y) \geq 0$$

Suppose Q is positive semi definite.



e v f are in arithmetic progression

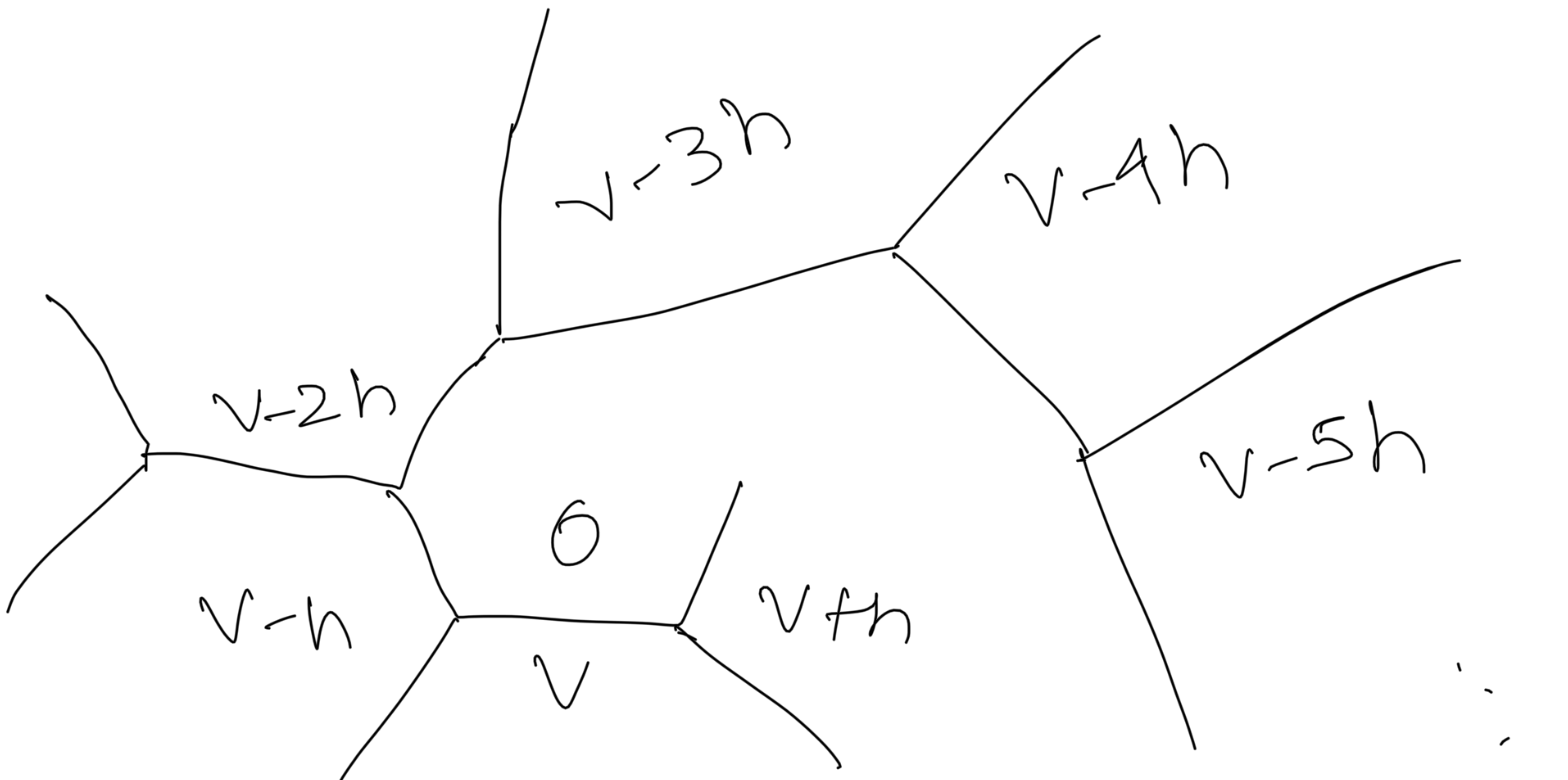
$f = v + h$

$\Delta(Q)$

$= f^2 + v^2 - 2vf$

$\Delta(Q) = (f - v)^2 = h^2$

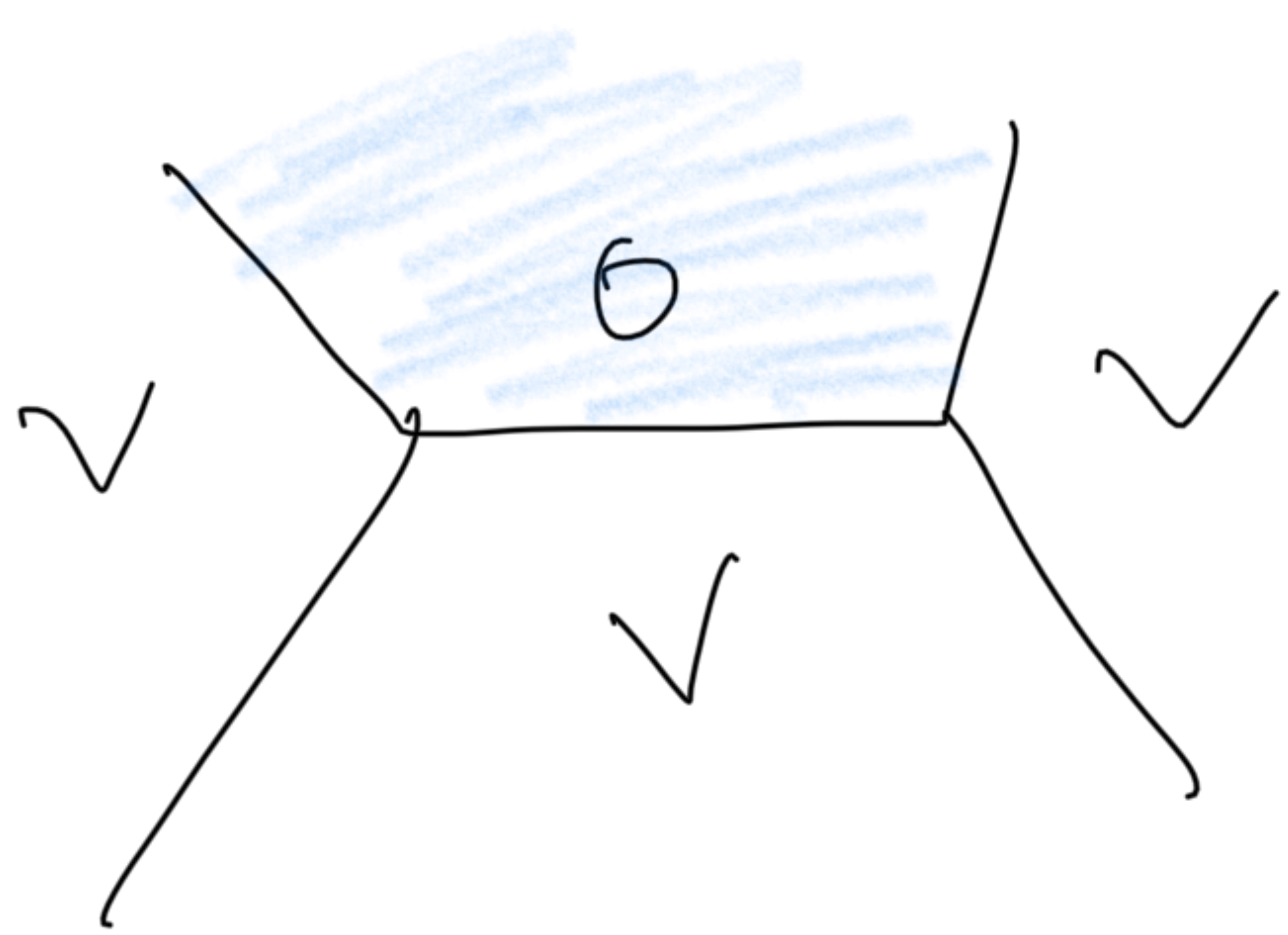
Let's explore this h.



$\Rightarrow \quad h = 0$

So, if Q is positive semidefinite

$$\Delta(Q) = 0.$$



$$Q(x,y) = vx^2$$

Q positive semidefinite $\Rightarrow$ Q is equivalent to vx^2, $v \in \mathbb{Z}$.