Last time we showed that there are infinitely many primes of the form $4k+3$.

[Trick : Consider $N = 4p_1p_2 \cdots p_n - 1$]

Suppose $a = p_1^{e_1} \cdots p_r^{e_r}$

What do divisors of a look like?

Each divisor looks like $p_1^{f_1} \cdots p_r^{f_r}$

where $0 \le f_i \le e_i$

Q1: How many divisors are there?

(denoted by $\sigma_0(a)$)

Q2: What is the sum of all divisors?

(denoted by $\sigma_1(a)$)

Q1: There are e_i+1 choices for each i, so no of divisors

$$= (e_1+1)(e_2+1)\cdots(e_r+1)$$

Q2: The sum of all divisors can be expressed as

$$(1+p_1+p_1^2+\cdots+p_1^{e_1})$$
$$\times(1+p_2+p_2^2+\cdots+p_2^{e_2})$$
$$\vdots$$
$$\times(1+p_r+p_r^2+\cdots+p_r^{e_r})$$

GEOMETRIC SERIES

$$1+p+p^2+\cdots+p^r = \frac{p^{r+1}-1}{p-1}$$

Qn: Suppose $(a,b) = 1$.

What can you say about divisors of ab?

MULTIPLICATIVE FUNCTIONS

A function $f: \mathbb{N} \to \boxed{}$

Integer, real or complex

is called multiplicative if

$$f(ab) = f(a)\, f(b)$$ whenever a and b are coprime.

Examples: σ_0, σ_1

Qn: If $(a,b)=1$, then what is

$(a^m, b^n) = ?$ (here m,n are natural numbers)

Can you give some more examples of multiplicative fns?

For $n = 100$, compute

$\sigma_0(n) =$

$\sigma_1(n) =$

$\sigma_2(n) =$

$\sigma_3(n) =$

PERFECT NUMBERS

Sum of divisors of $n = 2n$

$\Updownarrow$

Sum of proper divisors of n $= n$

Examples:

6 ; $1+2+3+6 = 12$

28 ; $1+2+4+7+14+28 = 56$

496

8128

$\vdots$

How many are there?

Let us show a connection between perfect numbers & Mersenne Primes.

→ If n is a natural number and 2^n-1 is a prime #, then n is prime number.

Pf: $2^{ab}-1 = (2^a-1)(1+2^a+2^{2a}+ \cdots \cdots + 2^{(b-1)a})$

→ Suppose 2^n-1 is prime, then

$N = 2^{n-1}(2^n-1)$ is perfect.

$\sigma_1(N) =$

$2N =$

→ Suppose N is even + perfect.

Then there exists primes p & q such that

$$q = 2^p - 1$$
$$N = 2^{p-1} q$$

Pf: $N = 2^{e_2} 3^{e_3} 5^{e_5} \cdots$

Define

$$p = e_2 + 1$$
$$q = 3^{e_3} 5^{e_5} \cdots$$

By construction $N = 2^{p-1} q$

Want to show that p & q are primes.

$$\sigma_1(N) = \sigma_1(2^{p-1}) \sigma_1(q)$$
$$2N = (2^p - 1) \sigma_1(q)$$
$$2^p q = (2^p - 1)(\sigma_1(q))$$
$$\sigma_1(q) = \left(\frac{2^p}{2^p - 1}\right) q = \left(\frac{(2^p - 1) + 1}{2^p - 1}\right) q$$

$$\sigma_1(q) = q + \frac{q}{2^p - 1}$$

$$\downarrow$$

$$\frac{q}{2^p - 1} = \sigma_1(q) - q$$

$$\downarrow$$

divisor of q & proper

$$\sigma_1(q) = q + \frac{q}{2^p - 1}$$

What can you say about $\frac{q}{2^p - 1}$?

q is prime

$q = 2^p - 1$ is prime

$\Rightarrow p$ is prime

Divisibility test for 7

$$\begin{array}{r} 1428 \end{array} \qquad \begin{array}{r} 142 \\ -\ 16 \\ \hline 126 \end{array} \qquad \begin{array}{r} 12 \\ -12 \\ \hline 0 \\ \hline \end{array}$$

Divisible by 7

$$N_0 = a_0 + 10a_1 + 100a_2 + \cdots + 10^n a_n$$

$$(1428 = 8 + 20 + 400 + 1000)$$

$$N_1 = (10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + a_1) - 2a_0$$

Look at $N_0 - 10N_1$

$= 21a_0$ divisible by 7

$$\boxed{7 \mid N_0 \iff 7 \mid 10N_1 \iff 7 \mid N_1}$$

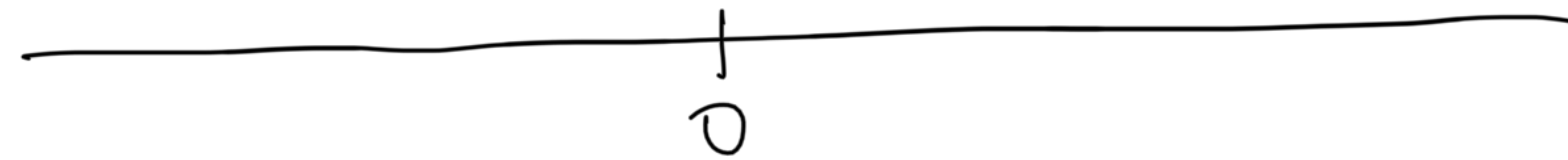
COMPLEX NUMBERS

$\mathbb{N}$
(Natural numbers) $= \{1, 2, 3, 4, 5, 6, \ldots\}$

$\mathbb{Z}$
(Integers) $= \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q}$
(Rational numbers) $= \left\{ \dfrac{a}{b} \,\middle|\, \begin{array}{l} a, b \in \mathbb{Z} \\ b \neq 0 \end{array} \right\}$

$\mathbb{R}$
(real numbers)



(Not easy to construct $\mathbb{R}$ from $\mathbb{Q}$!)

$x^2-5=0$ has no solutions over $\mathbb{Q}$!

(why?)

$x^2-5=0$ has two solutions over $\mathbb{R}$

$\sqrt{5}, \ -\sqrt{5}$

$x^2+1=0$ has no solutions over $\mathbb{R}$

This is where complex numbers come into picture!

Call $\sqrt{-1}$ as i (iota)

$\mathbb{C}$ (complex numbers) $= \{a+bi \mid a,b \in \mathbb{R}\}$

Addition

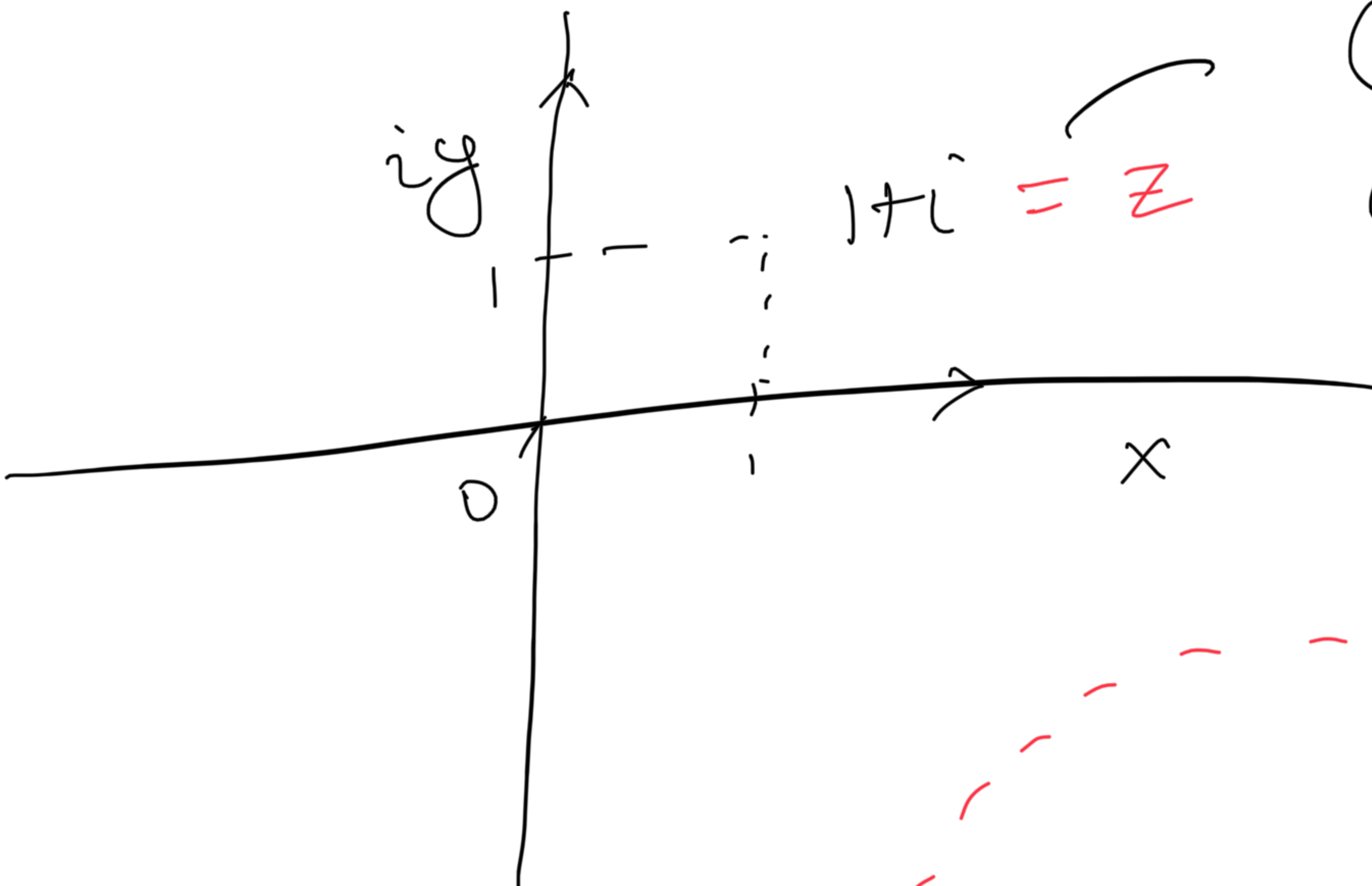
$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

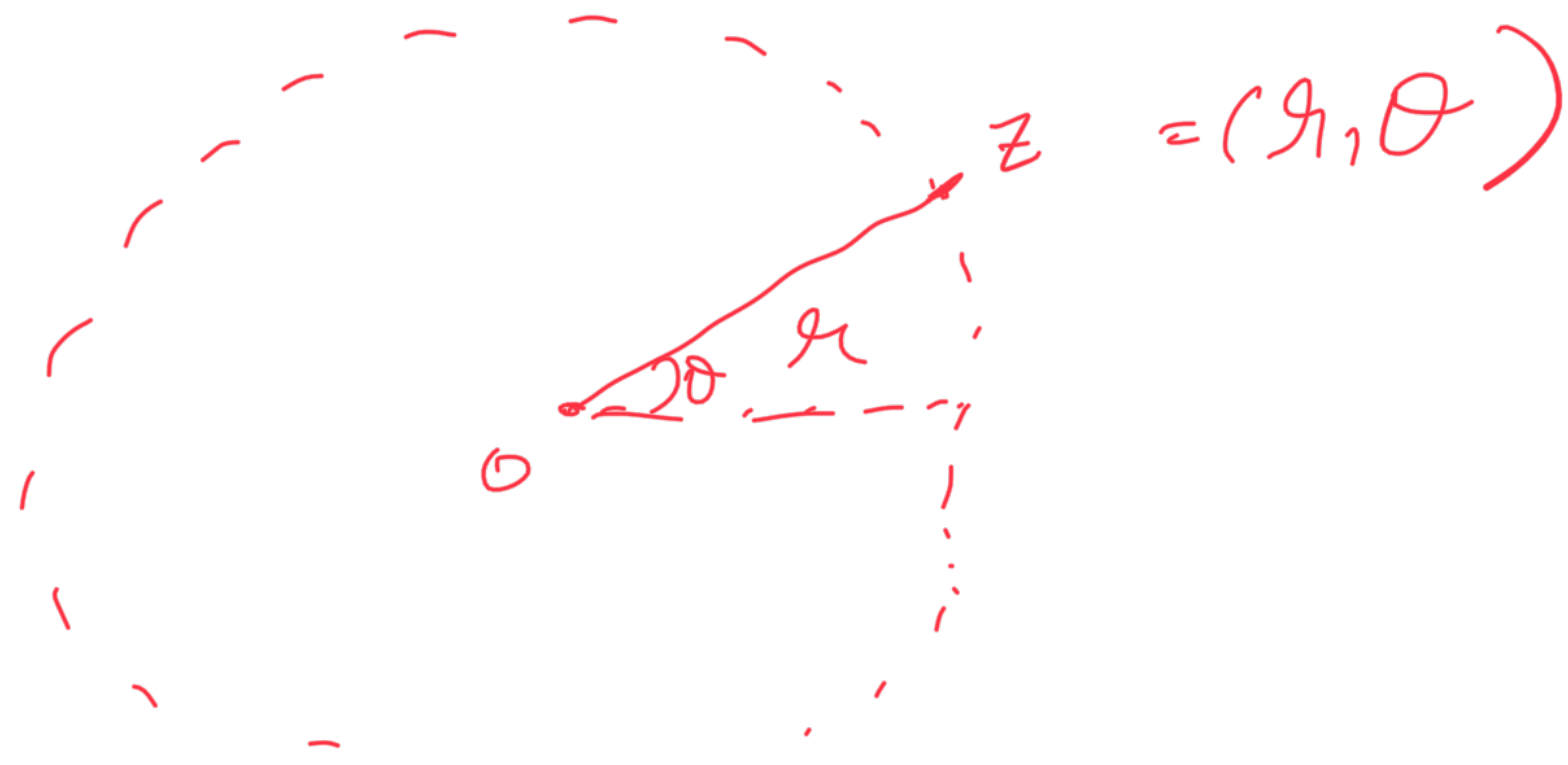
Multiplication

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$$

Cartesian Coordinates



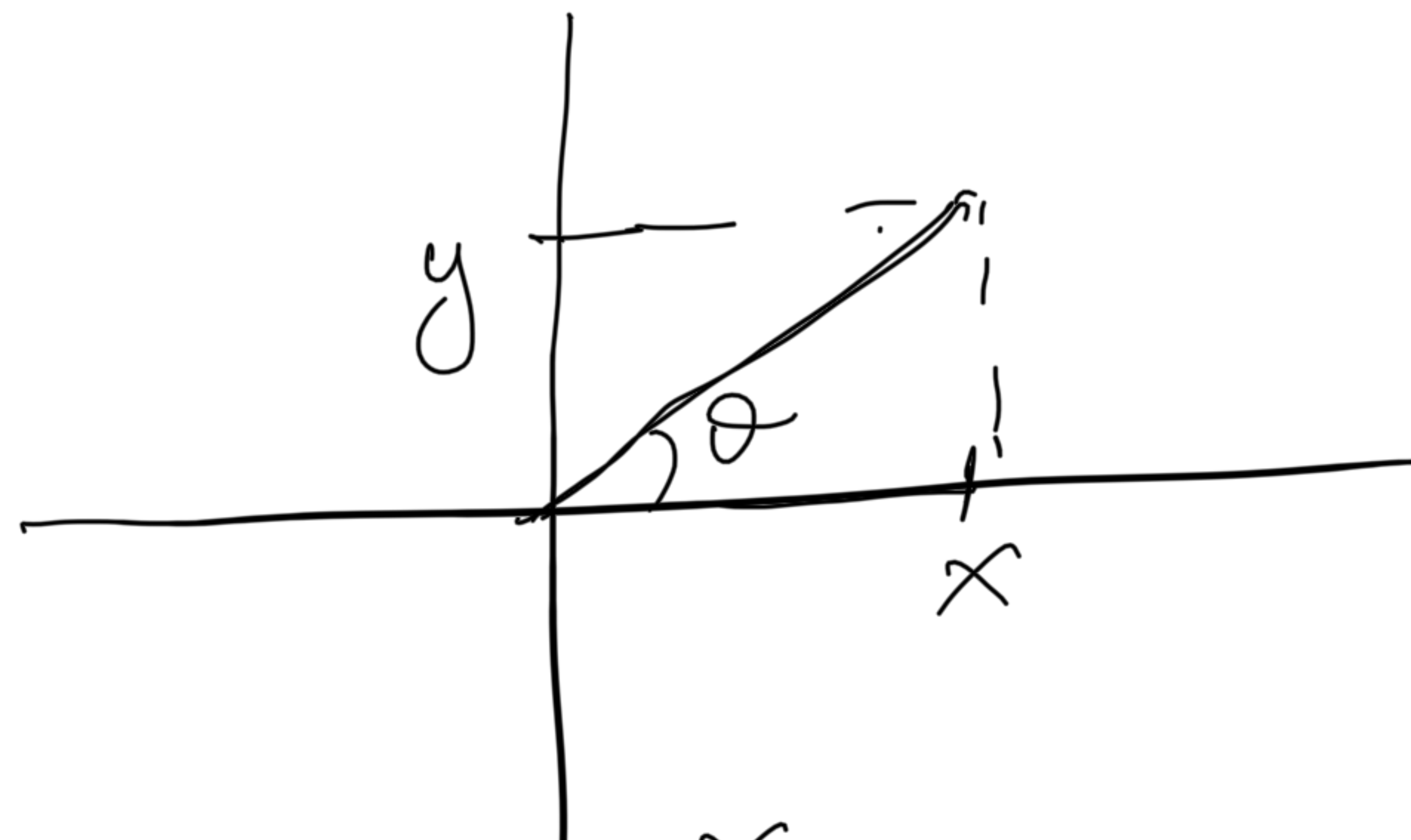


$z = x + iy$

In polar coordinates

$r = \sqrt{x^2 + y^2}$ (Pythagoras thm)

(distance from origin to z)



$\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$ $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

In polar form

$$Z = |r| (\cos\theta + i\sin\theta)$$

$$= |r| e^{i\theta}$$

Polar coordinates (r, θ)

$$1+i = \sqrt{2}\, e^{\pi/4}$$

Multiplication in polar coordinates is easy

$$(r,\theta)\,(s,\phi) = (rs,\ \theta+\phi)$$