MATH 350, SPRING 2023

HOMEWORK 3, DUE FEBRUARY 6

(1) Prove that there are infinitely many prime numbers that are 3 more than a multiple of 4. *Hint: Follow Euclid's proof that there are infinitely many primes, but multiply by* 4 *before* subtracting 1. (In fact, it turns out that "half" of all prime numbers are of this form, and the other "half" are 1 less than a multiple of 4.)

(2) Suppose $\gcd(a, b) = 1$. Show that divisors of ab are in bijective correspondence with ordered pairs (u, v) where $u \mid a$ and $v \mid b$. Then give a counterexample when $\gcd(a, b) \neq 1$.

(3) Prime Gaps

  (a) Find all pairs of primes p and $p + 1$. Prove your answer.

  The twin primes conjecture asserts that are infinitely many primes of the form p and $p+2$. Work by Zhang and Maynard shows that there are infinitely pairs of consecutive primes p and q with $|p - q| \leq 246$.

  (b) Prove that for any $n \geq 3$, $n! + 2$, $n! + 3$, $\ldots$, and $n! + n$ are all composite. Conclude that that there are arbitrarily long gaps between consecutive primes.

(4) Let $T = \{1, 4, 7, 10, 13, 16, 19, \ldots\}$ be the set of all natural numbers that have a remainder of 1 when divided by 3.

  (a) Show that if quotient of two elements of T is an integer, the quotient is in T.

  (b) Call an element of T *irreducible* if it cannot be factored into two elements of $T \backslash \{1\}$. Find all irreducible elements of T that are less than 100. *Hint: Perform a sieve. That is, write out all elements of T up to* 100*, then remove the multiples of* 4*, then the multiples of* 7*, then the multiples of* 10*, etc.*

  (c) Find an element of T that can be factored into irreducible elements in more than one way.

(5) Consider the following subset of the complex numbers: $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. This is an example of a ring: like the integers, $\mathbb{Z}[\sqrt{-5}]$ has addition, subtraction, and multiplication that obey the usual axioms like commutativity, associativity, and distributivity.

  (a) Define the norm map $N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}$ by $N(a + b\sqrt{-5}) = a^2 + 5b^2$. Show that $N(xy) = N(x)N(y)$.

  (b) Show that the element $6 = 6+0\sqrt{-5}$ can be nontrivially factored in two different ways.

  (c) Use the norm map to show that the factors you found in the previous part cannot be factored further. Conclude that $\mathbb{Z}[\sqrt{-5}]$ does not have unique factorization.

(6) Euclid's Lemma states that a prime number p satisfies, for any natural numbers a and b, that if $p \mid ab$, $p \mid a$ or $p \mid b$.

  The following definition is standard in abstract algebra courses: *An integer $p \neq \pm 1$ is* prime *if for any integers a and b, $p \mid ab$ implies $p \mid a$ or $p \mid b$.* In other words, we use Euclid's Lemma as the definition of a prime number. Classify all prime integers according to this definition. *Careful: This definition includes at least one integer that you probably don't think of as a prime number!*

(7) *PAR problem.* Prove that if a, b, and n are positive integers, then $a \mid b$ if and only if $a^n \mid b^n$.