Lecture Notes in Economics and Mathematical Systems 448

Günter Fandel Tomas Gal (Eds.)
In collaboration with Thomas Hanne

Multiple Criteria Decision Making

Proceedings of the Twelfth International Conference, Hagen (Germany)

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Preface

The organizers of the **12th International Conference on Multiple Criteria Decision Making (MCDM)** held June 19-23, 1995 in Hagen received the second time the opportunity to prepare an international conference on MCDM in Germany; the first opportunity has been the 3rd International Conference on MCDM in Königswinter, 1979. Quite a time ellapsed since then and therefore it might be interesting to compare some indicators of the development of the **International Society on MCDM**, which has been founded in Königswinter. Stanley Zionts has been elected first president and all 44 participants of that Conference became founding members. Today our Society has over 1200 members and its own Journal (**MCDM World Scan**). In Hagen, 1996, we had 152 participants from 34 countries.

It is interesting to mention that also other Groups established their organization, like the European Working Group on Multiple Criteria Decision Aid, the German Working Group on Decision Theory and Applications, the Multi-Objective Programming and Goal Programming Group, ESIGMA, and some others. It is also interesting to note that the intersection of members of all these Groups and Societies is not empty and there is quite a cooperation among them.

It has not been the first time that in such a Conference researchers from East and West are meeting and exchanging their experiences and knowledge; it is however the first time after the Big Change in the Eastern countries that quite a number of distinguished researchers from those countries could participate, present their papers and freely exchange opinions. This was one of the reasons to celebrate "Twenty Years of East-West Cooperation in MCDM" which became the motto of this Conference. We were proud and happy to have about 30% participants from the Eastern countries.

An outstanding event has been the Conference Banquet during which we had a really international entertainment program prepared and performed by the participants from various countries. This evening witnessed also the decoration for some people by awards: Yacov Y. Haimes (USA), Hirotaka Nakayama (Japan) and Bernard Roy (France) for various kinds of outstanding works into the theoretical development of MCDM; the 1995 Wiley Prize for the best application-oriented paper was awarded to Valerie Belton (Scotland), Fran Ackerman (Scotland), and Ian Shepherd (UK) for their paper "COPE-ing with VISA"; Honorable Mention went to Jifa Gu (P.R. of China) and Xijin Tang (P.R. of China) for their paper "An Application of MCDM in Water Resources Problems". Also the organizers have been awarded.

Another outstanding event of the Conference has been the first Evening of the PhD's, in which PhD students presented the ideas of their dissertations and discussed the corresponding topics. We are indebted very much to Mikael Lind and Bartel van der Walle who not only came up with the idea but also helped very actively to organize the evening.

A pleasant evening has also been the banquet to which the mayor of the city Hagen invited all the participants. The Queens hotel in Hagen turned out a very pleasant and well organized place for our Conference; not only the quick changing of the meeting rooms from big to small size and vice versa has been well organized but also the food has been enjoyed every day by the participants.

It is our pleasure to thank very much Ms. Kirsten Lauter, Holger Kruse, Dr. Johannes Wolf and Luke Stevens for their devoted help in the preparation and performation of the Conference.

We especially want to thank Thomas Hanne for his enthusiastic and devoted help in preparing and performing the Conference and editing the Proceedings. The editors / organizers cannot imagine to have managed in time all the needed work without his help.

We would like to stress that we are deeply indebted to our Sponsors, AIP GmbH, Hagen, Deutsche Forschungsgemeinschaft, Bonn, FernUniversität Hagen, Ministerium für Wissenschaft und Forschung Nordrhein-Westfalen, Düsseldorf, and Volksbank Hagen, without the aid of which this Conference could have never been been realized and also for their generous sponsoring the participants from the Eastern Countries.

We are also indebted very much to all session chairmen/-ladies, who did their job with enthusiasm. The thoroughly refereed papers had sometimes been a problem for the authors: We would like to thank to all contributors which revised their papers according to the referee's comments sometimes even two times. Also the activities of the participants themselves contributed to the scientific and social success as well as to the high scientific level of the Conference. Thanks to all!

The scientific areas of this Conference have been subdivided into three main parts: Theory, Methods and Applications. In this place we wish to thank the topic coordinators who have put in much work and time to prepare and organize special sessions in their field. In the theoretical part (Johannes Jahn) contributions concerning still open problems in the mathematical theory of MCDM have been collected, in the methodological part (Wojtek Michalowski, Mehrdad Tamiz, Philippe Vincke) papers concerning methodological concepts and algorithms have been presented. The Applications (Wolfram Stadler) have been subdivided into Engineering and Management problems. Also Behavioral Issues in MCDM (Pekka Korhonen, Jyrki Wallenius) have been a topic for panel discussions.

In the present Proceedings the papers are ordered within each of the Subdivisions according to the alphabetic order of the first authors. On p. iv there is a list of the participants and on p. vi the program of the Conference.

Hagen, September 1996

Günter Fandel and Tomas Gal

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Part 1
Theory

An Algorithm for Vectorial Control Approximation Problems

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Abstract

We consider a convex vectorial control approximation problem and derive necessary and sufficient optimality conditions for solutions of a corresponding scalarized problem using the subdifferential calculus. These optimality conditions can be solved by a proximal point algorithm introduced in [3]. This fact together with some stability results form the base of a dialogue algorithm to generate approximate solutions of the vectorial problem.

Keywords. Approximation problem, stability, dialogue algorithm

1 Introduction

Several authors have studied vectorial and scalar location and approximation problems from the theoretical as well as the computational point of view (see [5], [6], [7], [12], [13], [16], [18], [21], [22], [23]).
The aim of this paper is to extend the results of Michelot and Lefebvre [16] and Idrissi, Lefebvre and Michelot [12] to a general vectorial control approximation problem. In order to formulate this problem we suppose that

(A) a^i, $x \in X$, $\alpha_i \geq 0$, $\beta_i \geq 1$, $C \in L(X \to R^n)$, $A^i \in L(X \to Y_i)$, $(i = 1, ..., n)$, ($L(X \to Y)$ denotes the space of linear continuous operators from X to Y),

(B) $D_j \subset X \quad (j = 1, ..., m)$ are closed and convex sets, $D_0 \subseteq X$ is a closed linear subspace and $D = \bigcap_{j=0}^{m} D_j$ is non-empty and bounded.

(C) One of the following conditions $(C1)$ or $(C2)$ is fulfilled:

(C1) H is a Hilbert space and $X = Y_i = H$ for all $i = 1, ..., n$

(C2) $X = R^k$ and $Y_i = R^{m_i}$ for all $i = 1, ..., n$.

(D) $K \subset R^n$ is a convex cone with $cl\ K + (R^n_+ \setminus \{0\}) \subset int\ K$.

Now, we consider the following vectorial control approximation problem

$$(P): \qquad \text{Compute the set} \qquad Eff(f[D], K),$$

where

$$f(x) := C(x) + \begin{pmatrix} \alpha_1 \parallel A^1(x) - a^1 \parallel^{\beta_1} \\ \cdots \\ \alpha_n \parallel A^n(x) - a^n \parallel^{\beta_n} \end{pmatrix} \tag{1}$$

and

$$Eff(f[D], K) := \{f(x) \mid x \in D, \quad f[D] \cap (f(x) - (K \setminus \{0\})) = \emptyset\}.$$

Remark 1:
Problem (P) can be considered as an extension of a scalar problem studied by Idrissi, Lefebvre and Michelot [12]:

$$\sum_{i=1}^{n} \alpha_i \parallel x - a^i \parallel_i \longrightarrow \min_{x \in D}$$

with $D_j \subset R^p$ $(j = 0, 1, ..., m)$, $x, a^i \in R^p$ $(i = 1, ..., n)$.

Remark 2:
The class of problems (P) contains many practically important special cases:

1. Linear vector optimization problems $(\alpha_i = 0,\ i = 1, ..., n)$.
2. Surrogate problems for linear vector optimization problems

   $$Eff(f_1[\mathcal{D}], K),$$

   where

   $$f_1(x) := C(x), \qquad \mathcal{D} = \{x \in D \mid A^i(x) = a^i (i = 1, ..., n)\},$$

   for which the feasible set $\mathcal{D}$ is empty but D is nonempty.
3. Perturbed respectively regularized linear vector optimization problems.
4. Vector-valued optimal control problems of the form (see Benker and Kossert [2]):

   $$Eff(f_2[U], R^2_+),$$

   with

   $$f_2(u) := \begin{pmatrix} \parallel Au - a \parallel^p \\ \parallel u \parallel^p \end{pmatrix}, \qquad u \in U \subset X,$$

   where $C = 0$, $n = 2$, $\beta_i = p$ $(i = 1, 2)$, H_1 and H_2 are Hilbert spaces, $A \in L(H_1 \longrightarrow H_2)$, $a \in H_2$, $U \subset H_1$ is a nonempty closed convex set and R^2_+ denotes the usually ordering cone in R^2. Here u denotes the so called control variable, the image $z = Au$ denotes the state variable.

5. Vectorial approximation and location problems (cf. Jahn [14], Gerth and Pöhler [8], Wanka [21]).

In order to derive a primal-dual algorithm we assume in the following that any suitable constraint qualification (generalized Slater condition, stability ...) is fulfilled (cf. [9]). Moreover, we suppose that the inverse operator $(A^{iT})^{-1}$ of the adjoint operator A^{iT} to A^i exists for all $i = 1, ..., n$.
Furthermore, $N_{D_j}(x^0)$ denotes the normal cone to D_j at $x^0 \in X$ defined by

$$N_{D_j}(x^0) = \begin{cases} \{x \in X^* : \quad (x, x' - x^0) \leq 0 \quad \forall x' \in D_j\} & : \quad x^0 \in D_j \\ \emptyset & : \quad \text{otherwise.} \end{cases}$$

B^0 denotes the unit sphere associated with the dual norm of the norm $|| \,.\, ||$.
$P_{\mathcal{D}}(e)$ denotes the projection of the element $e \in X$ onto the set $\mathcal{D}$.
The paper is organized as follows:
In Chapter 2 we introduce a suitable scalarization of the vectorial control approximation (P). Furthermore, in Chapter 3 we derive optimality conditions for the scalarized problem using the subdifferential calculus. These optimality conditions are useful for applying Spingarn's proximal point algorithm (see [12], [16] and [17]) for solving problem (P) in Chapter 4. Finally, in Chapter 5 we derive a dialogue algorithm for the vectorial control approximation problem using a surrogate parametric optimization problem and taking into account stability results of this special parametric optimization problem.

2 A suitable scalarization of the vector optimization problem

Under the given assumptions it is easy to see that the vector-valued objective function $f : X \longrightarrow R^n$ in (1) is (R^n_+) - **convex**, i.e., for all $x_1,\ x_2 \in X$, $\mu \in [0, 1]$ it holds $\mu f(x_1) + (1 - \mu) f(x_2) \in f(\mu x_1 + (1 - \mu)x_2) + R^n_+$.
Then we can show (compare [14]) that for each element $f(x^0) \in Eff(f[D], K)$ there exists a parameter $\lambda \in int\ K^*$ (K^* denotes the dual cone to K) such that x^0 solves the real-valued optimization problem

$$(P(\lambda)) : \qquad f(x, \lambda) := \sum_{i=1}^{n} \lambda_i (C_i(x) + \alpha_i \ || \ A^i(x) - a^i \ ||^{\beta_i}) \longrightarrow \min_{x \in D}.$$

In the following for the case $\alpha_i > 0$ for all $i = 1, ..., n$ without loss of generality we replace $(P(\lambda))$ by

$$(P_s') : \qquad (c, x) + \sum_{i=1}^{n} || \ A^i(x) - a^i \ ||^{\beta_i} \longrightarrow \min_{x \in D},$$

where $c \in X^*$. Using the *indicator functions* χ_{D_j} of D_j defined by $\chi_{D_j}(x) = 0$ if $x \in D_j$ and $\chi_{D_j}(x) = +\infty$ otherwise the problem (P_s') is equivalent to the following unconstrained minimization problem

$$(P_s): \quad F(x) = (c,x) + \sum_{i=1}^{n} \|A^i x - a^i\|^{\beta_i} + \sum_{j=0}^{m} \chi_{D_j}(x) \to \min_{x\in X}. \qquad (2)$$

3 Optimality conditions for the scalarized problem

At first we consider the case of $((A),(B),(C1))$. Obviously, under the given assumptions the functional F in (2) is convex. Therefore, the subdifferential-condition

$$0 \in \partial F(x^0) \qquad (3)$$

is necessary and sufficient for the optimality of a feasible x^0 (compare Zeidler [24]). Applying the rule for sums of subdifferentials and the known formula for the subdifferential of a indicator function $(\partial\chi_{D_j}(\cdot) = N_{D_j}(\cdot))$ to the functional F in (2) we can write (3) equivalently in the form that there are q_i $(i = 1,...,n)$, r_j $(j = 1,...,m)$ with

$$q_i \in \partial(\|A^i x^0 - a^i\|^{\beta_i}), \qquad i = 1,2,...,n, \qquad (4)$$

$$r_j \in N_{D_j}(x^0), \qquad j = 1,2,...,m, \qquad (5)$$

$$c + \sum_{i=1}^{n} q_i + \sum_{j=1}^{m} r_j \in D_0^{\perp}, \qquad (6)$$

(compare [3]).
In order to reformulate the optimality conditions (4) - (6) in a more practical way we introduce a *Hilbert space* $E := (H)^{n+m+1}$ *with the inner product* $(u,v) := \sum_{i=1}^{n+m+1} (u_i, v_i)$ *for* $u = (u_1,...,u_{n+m+1})$ *and* $v = (v_1,...,v_{n+m+1}) \in E$,

the subspaces

$$\mathcal{A} := \{y \in E \mid y = (y_1,...,y_{n+m+1}) \; ; \; y_j = x \, , \; j = 1,...,n+m+1 \; ; \; x \in D_0\},$$

$$\mathcal{B} := \left\{ p \in E \mid p = (p_1,...,p_{n+m+1}) \, , \sum_{i=1}^{n+m+1} p_i \in D_0^{\perp} \right\}$$

and the operator T *defined on* E *by*

$$T(y) := (T^1(x), ..., T^{n+m+1}(x)), \quad y = (x, ..., x), \tag{7}$$

where

$$T^i(x) := \partial(\left\|A^i x - a^i\right\|^{\beta_i}), \qquad i = 1, ..., n,$$

$$T^{n+j}(x) := N_{D_j}(x), \qquad j = 1, ..., m,$$

$$T^{n+m+1}(x) := c.$$

With this notations the optimality conditions (4)-(6) can be rewritten as:

$$\text{Find} \qquad (y^0, p^0) \in \mathcal{A} \times \mathcal{B} \qquad \text{such that} \qquad p^0 \in T(y^0). \tag{8}$$

Remark 3: Under the assumptions $((A), (B), (C2))$ we can define the space E and the subspaces $\mathcal{A}$ and $\mathcal{B}$ analogously with R^k instead of H.

4 Application of the proximal point algorithm

In [3] we have shown that under $((A), (B), (C1))$ the assumptions for a successful application of the proximal point algorithm of Spingarn [17] for the solution of (8) are fulfilled. This algorithm has the form

$$z^{k+1} = (I + c_k T_{\mathcal{A}})^{-1} z^k, \qquad k = 1, 2, ..., \tag{9}$$

where the starting point z^1 is arbitrary and $\{c_k\}$ is a sequence of positive real numbers and $T_{\mathcal{A}}$ is the partial inverse of T with respect to $\mathcal{A}$. If $\{c_k\}$ is bounded away from zero (for simplicity we set $c_k = 1$) then either $\{z^k\}$ converges weakly to a zero of $T_{\mathcal{A}}$ or $\left\|z^k\right\| \to \infty$ and $T_{\mathcal{A}}$ has no zeros (see [17]). Taking into account the structure of the subdifferential of the norm and some relations of this subdifferential to certain normal cones and projections we have derived in [3] the following proximal point algorithm **PPACAP** for solving (P_s) in the special $\beta_i = 1$ for all $i = 1, ..., n$. For the case $\beta_i = \beta \geq 1$ compare subsection 4.1.2 in [3], where we have transformed the problem in such a way that we can use the following algorithm too.
Algorithm (PPACAP):

- Choose arbitrary initial points

$$x^1 \in D_0 \qquad \text{and} \qquad p^1 \in \mathcal{B}. \tag{10}$$

- Compute x^{k+1} and p^{k+1} from

$$x^k - x^{k+1} + p_i^{k+1} = P_M(x^k + p_i^k - \left(A^i\right)^{-1} a^i), \qquad i = 1, ..., n, \tag{11}$$

$$x^{k+1} + p_{n+j}^k - p_{n+j}^{k+1} = P_{D_j}(x^k + p_{n+j}^k), \qquad j = 1, ..., m, \tag{12}$$

$$x^k - x^{k+1} + p_{n+m+1}^{k+1} = c, \tag{13}$$

with

$$p_i^{k+1} \in \mathcal{B} \qquad \text{and} \qquad x^{k+1} \in D_0, \tag{14}$$

where P_M is the projection operator onto the set

$$M = \{ A^{iT} z_i^* \mid z_i^* \in B^0 \}$$

Remark 4: According to Remark 3 under the assumptions $(A), (B), (C2))$ it is possible to derive the algorithm in the same way. In this case the weak convergence of the sequence $\{z^k\}$ implies even the norm convergence.

5 A dialogue algorithm for the vectorial control approximation problem (P)

In this section we present a dialogue algorithm, in which we have to solve the special parametric optimization problem from Chapter 2 under the assumptions $((A), (B), (C1))$ or $((A), (B), (C2))$:

$$(P(\lambda)): \qquad f(x,\lambda) = \sum_{i=1}^{n} \lambda_i (C_i(x) + \alpha_i \parallel A^i(x) - a^i \parallel^{\beta_i}) \longrightarrow \min_{x \in D},$$

where $\lambda \in \Lambda \subset int\ K^*$.

Stability results for parametric optimization problems are important for an effective dialogue algorithm. With other words, we need various types of semicontinuity of the **optimal value function**

$$\varphi(\lambda) := inf\{f(x,\lambda) \mid x \in D\},$$

of the **optimal set mapping**

$$\psi(\lambda) := \{x \in D \mid f(x,\lambda) = \varphi(\lambda)\}$$

or of the **ϵ-optimal set mappings**

$$\psi_\epsilon(\lambda) := \{x \in D \mid f(x,\lambda) < \varphi(\lambda) + \epsilon\}$$

and

$$\bar{\psi}(\lambda, \epsilon) := \{x \in D \mid f(x,\lambda) \le \varphi(\lambda) + \epsilon\}$$

By stability of the mappings φ, ψ, ψ_ϵ and $\bar{\psi}$ we mean in particular certain continuity attributes of these mappings. We use the concept of semicontinuity according to **Berge** ([4]):

Definition 1 *A point-to-set mapping* $\Gamma : \Lambda \longrightarrow 2^X$, *where* (X, d_X) *and* (Λ, d_Λ) *are metric spaces, is called:*

1. ***upper semicontinuous*** *in the sense of Berge at a point* λ^o, *if for each open set* Ω *containing* $\Gamma(\lambda^o)$ *there exists a* δ*- neighbourhood* $V_\delta\{\lambda^o\}$ *of* λ^o *such that*
$$\Gamma(\lambda) \subset \Omega \qquad \text{for all} \qquad \lambda \in V_\delta\{\lambda^o\};$$

2. ***lower semicontinuous*** *in the sense of Berge at a point* λ^o , *if for each open set* Ω *satisfying* $\Omega \cap \Gamma(\lambda^o) \neq \emptyset$ *there exists a* δ*-neighbourhood* $V_\delta\{\lambda^o\}$ *of* λ^o *such that*
$$\Gamma(\lambda) \cap \Omega \neq \emptyset \qquad \text{for all} \qquad \lambda \in V_\delta\{\lambda^o\}.$$

The existence of continuous selection functions is closely related to the condition that ψ is lower semicontinuous in the sense of Berge and all optimal sets are non-empty and convex. If one considers ϵ- optimal solutions then the strong condition of lower semicontinuity of ψ may be avoided.

Theorem 1 *Let the assumptions* $((A), (B), (C1))$ *are fulfilled. Then for each* $\epsilon > 0$ *the* ***ϵ-optimal set mapping*** ψ_ϵ *is lower semicontinuous in the sense of Berge.*

Proof: The assumptions of Theorem 4.2.4. in [1] are fulfilled, since f is continuous on $X \times \Lambda$, D is closed and does not depend on the parameter λ and φ is continuous regarding the continuity of f on the set D. Theorem 4.2.4 in [1] yields that for each $\epsilon > 0$ the ϵ-optimal set mapping ψ_ϵ is lower semicontinuous .

In the finite dimensional case we can derive some additional results:

Theorem 2 *We consider the problem* $(P(\lambda))$ *subject to the assumptions* $((A), (B)$ ***and*** $(C2))$. *Suppose that* $\psi(\lambda^0)$ *is non-empty and bounded. Then*

- *the the optimal value function* φ *is continuous at* λ^0,
- *the optimal set mapping* ψ *is upper semicontinuous in the sense of Berge at* λ^0,
- *the* ϵ*-optimal set mapping* ψ_ϵ *is lower semicontinuous in the sense of Berge at* λ^0 *for each* $\epsilon > 0$,
- *the mapping* $\bar{\psi}$ *defined by*
$$\bar{\psi}(\lambda, \epsilon) := \{x \in M \mid f(x, \lambda) \leq \varphi(\lambda) + \epsilon\}, \lambda \in \Lambda, \epsilon \geq 0,$$
is upper semicontinuous in the sense of Berge at $(\lambda^0, 0)$.

Proof: The results follow immediately from Theorem 9 in Hogan [11].

Remark 5: The last property is of interest if the problems corresponding to the parameters λ^t, $t = 1, 2, ..., \lambda^t \longrightarrow \lambda^0$, constitute certain substitute problems with respect to the problem to be solved, $(P(\lambda^0))$, and the problems $(P(\lambda^t))$ are solved with increasing accuracy $\epsilon_t \longrightarrow 0$. Then each sequence $\{x^t\}$ of ϵ_t-optimal solutions of $(P(\lambda^t))$ possesses an accumulation point and each of its accumulation points is contained in the solution set $\psi(\lambda^0)$.

Moreover, under the assumptions of Theorem 2 a Continuous Selection Theorem of Michael [15] can be used if we additionally assume the compactness of Λ.

Theorem 3 *Assume $((A), (B), (C))$ and Λ is a compact set. Then there exists a function $g \in C(\Lambda, X)$ such that*

$$g(\lambda) \in cl\ \psi_\epsilon(\lambda) \quad \forall \lambda \in \Lambda.$$

Proof: We can conclude from Theorem 1 and 2 respectively that ψ_ϵ is lower semicontinous in the sense of Berge. Moreover, the image sets $\psi_\epsilon(\lambda)$ are nonempty and convex for all $\lambda \in \Lambda$. Then we get the desired result from the Continuous Selection Theorem of Michael [15].

Using these stability statements we can derive the following dialogue algorithm for the vectorial control approximation problem (P) with the help of them at least under $((A), (B), (C2))$ it is possible to seek for elements of a neighbourhood of the set of proper efficient points which corresponds to the individual interest of the decision maker (compare [10], [20]).

Step 1: Choose $\bar{\lambda} \in \Lambda$. Compute an approximate solution (x^0, p^0) with the primal-dual algorithm **(PPACAP)**. If (x^0, p^0) is accepted by the decision-maker, then **stop**. Otherwise go to Step 2.

Step 2: Put $k = 0$, $t_0 = 0$. Choose $\hat{\lambda} \in \Lambda$, $\hat{\lambda} \neq \bar{\lambda}$. Go to Step 3.

Step 3: Choose t_{k+1} with $t_k < t_{k+1} \leq 1$ and compute an approximate solution (x^{k+1}, p^{k+1}) of

$$P(t_{k+1}, \bar{\lambda}, \hat{\lambda}) : \quad \min_{x \in D}\{\sum_{i=1}^{n}(\bar{\lambda}_i + t_{k+1}(\hat{\lambda}_i - \bar{\lambda}_i))(C_i(x) + \alpha_i \parallel A^i(x) - a^i \parallel^{\beta_i}\}$$

with the algorithm **PPACAP** and use (x^k, p^k) as starting point. If an approximate solution of $P(t, \bar{\lambda}, \hat{\lambda})$ cannot be found for $t > t_k$, then go to Step 1. Otherwise go to Step 4.

Step 4: The point (x^{k+1}, p^{k+1}) is to be evaluated by the decision-maker. If it is accepted by the decision-maker, then **stop**. Otherwise go to Step 5.

Step 5: If $t_{k+1} \geq 1$, then go to Step 1. Otherwise set $k = k + 1$ and go to Step 3.

Remark 6: Under the assumptions $((A), (B), (C2))$ of Theorem 3 a sufficiently good approximation of a solution of $(P(t_{k+1}, \bar{\lambda}, \hat{\lambda})$ can be generated if we use an approximate solution of $(P(t_k, \bar{\lambda}, \hat{\lambda}))$ as starting point.

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Multiple Criteria Models with the Linear Pseudoboolean Functions and Disjunctive Restrictions

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1 Introduction and Statement of the Problem

The combination of Artificial Intelligence and Operation Research methods has been significant in recent years. The MCDM models with Boolean variables are most suitable to use them in the Intelligence Decision Making Systems, but these models are insufficiently studied [1].

Functions of the form $f\colon B^n \to R$ are called pseudo-Boolean. We will use the notation $PS_2(n)$ to denote the class of such functions. Functions of $PS_2(n)$ are defined on the set of vertices of the unit n-dimensional cube $B^n = \{0,1\}^n$.

A problem of the form

$$\operatorname{extr} f(\tilde{x}), \tilde{x} = (x_1,\ldots,x_n) \in B^n, f \in PS_2(n) \tag{1}$$

are referred to as pseudo-Boolean scalar conditional optimization. For any pseudo-Boolean optimization problem of the form (1), there is an equivalent representation with the *disjunctive restriction* [2]:

$$\operatorname{extr} f(\tilde{x}), \quad \bigvee_{j=1}^{m} x_{j_1}^{\sigma_{j_1}} \& \ldots \& x_{j_{r_j}}^{\sigma_{j_{r_j}}} = 1, \tag{2}$$

where literals x^σ are defined as

$$x^\sigma = \begin{cases} x, \sigma = 1, \\ \bar{x}, \sigma = 0, \end{cases}$$

$\bar{x}$ is a boolean inversion; $\vee$ and $\&$ denote boolean disjunction and conjunction.

The class $LPS_2(n) \subset PS_2$ of pseudo-Boolean functions is called *linear*: $LPS_2(n) = \{f \in PS_2(n)\colon f(\tilde{x}) = a_0 + a_1x_1 + \ldots + a_nx_n\}$, where $a_0, a_1, \ldots, a_n \in R$ with operations of addition and multiplication in the real number field.

A set $\mathfrak{R}^r = \{\tilde{x} \in B^n\colon x_{i_1}^{\sigma_{i_1}} \& \ldots \& x_{i_r}^{\sigma_{i_r}} = 1\}$ is called *interval* of rank r. Dimension of $\mathfrak{R}^r$ is (n-r). A set $\{\sigma_{i_1},\ldots,\sigma_{i_r}\}$ is called a *code of interval*, $\{i_1,\ldots,i_r\} \in \{1,2,\ldots,n\}$.

Multiple criteria pseudoboolean optimization problem with disjunctive restriction is defined as follows.

$$\begin{cases} extr\, f_1(\tilde{x}), extr\, f_2(\tilde{x}), \dots, extr\, f_s(\tilde{x}), \\ \bigvee_{j=1}^{m} x_{j_1}^{\sigma_{j_1}} \& \dots \& x_{j_{r_j}}^{\sigma_{j_{r_j}}} = 1, f_k \in \mathrm{PS}_2(\mathrm{n}), \mathrm{k} = \overline{1,\mathrm{s}}, \tilde{x} \in \mathrm{B}^{\mathrm{n}}. \end{cases} \tag{3}$$

Conventional scalar or multiple criteria conditional pseudoboolean optimization problems are defined with restrictions in the form of equations or inequalities. In these cases the reduction of the problem to an equivalent form with a disjunctive restriction can have nonpolynomial complexity [2]. It is important to notice that problems of pseudoboolean optimization are not purposely reduced to a form with the disjunctive restriction. They arise in that form when cybernetic methods of synthesizing discrete decision-making models are used [3].

The scalar pseudoboolean optimization problems with a disjunctive restriction and corresponding optimization algorithms were considered in [2].

In this paper we consider a problem with the linear functions:

$$\begin{cases} max\, f_1(\tilde{x}), max\, f_2(\tilde{x}), \dots, max\, f_s(\tilde{x}), \\ \bigvee_{j=1}^{m} x_{j_1}^{\sigma_{j_1}} \& \dots \& x_{j_{r_j}}^{\sigma_{j_{r_j}}} = 1, \\ f_k \in \mathrm{LPS}_2(\mathrm{n}), \mathrm{k} = \overline{1,\mathrm{s}}, \tilde{x} \in \mathrm{B}^{\mathrm{n}}. \end{cases} \tag{4}$$

It is required to construct a Pareto set $\mathcal{P}$ for the problem (4) in the *disjunctive normal form* (DNF). This DNF is called the *logical description of* $\mathcal{P}$. We consider an approach to choose a solution $\tilde{x}^* \in \mathcal{P}$. The proposed approach uses the necessary condition for $\tilde{x}^* \in \mathcal{P}$ and a Branch-and-Bound method.

2 The Necessary Condition for $\tilde{x} \in \mathcal{P}$ and Unconditional Optimization Problem

Let P_k be a set of variable's numbers with positive coefficients in f_k and N_k is a set of variable's numbers with negative coefficients in the linear pseudo-Boolean function f_k, $k = \overline{1,s}$. We denote $P_0 = P_1 \cap P_2 \cap \dots \cap P_S; N_0 = N_1 \cap N_2 \cap \dots \cap N_S$.

Theorem 1 Let P_0 and N_0 be the sets defined for the unconditional multi-criteria optimization problem

$$\begin{cases} \boldsymbol{max}\, f_1(\tilde{x}), \boldsymbol{max}\, f_2(\tilde{x}), \dots, \boldsymbol{max}\, f_s\,(\tilde{x}), \\ \tilde{x} \in B^n; f_1, \dots, f_s \in LPS_2(n) \end{cases} \tag{5}$$

If P_0 and N_0 are not empty and $\tilde{x}^*$ is a Pareto point then $\tilde{x}^*$ satisfies the equation

$$\left(\underset{i \in P_0}{\&} x_i\right) \& \left(\underset{i \in N_0}{\&} \overline{x_i}\right) = 1. \tag{6}$$

Proof Let $\tilde{\alpha}$ be any point which does not satisfy the equation (6). Then the number i exists so that $\alpha_i = 0$ when $i \in P_1 \cap P_2 \cap \ldots \cap P_S$ or $\alpha_i = 1$ when $i \in N_1 \cap N_2 \cap \ldots \cap Ns$. Let
$\tilde{\beta} = (\alpha_1, \ldots, \alpha_{i-1}, \overline{\alpha_i}, \alpha_{i+1}, \ldots, \alpha_n)$. Then $f_1(\tilde{\beta}) > f_1(\tilde{\alpha})$, $f_2(\tilde{\beta}) > f_2(\tilde{\alpha})$, ... , $f_s(\tilde{\beta}) > f_s(\tilde{\alpha})$ and $\tilde{\alpha}$ is not a Pareto point □

We have the following necessary conditions for $\tilde{x}$ to be Pareto point for the problem (5) (see the table 1):

Table 1. Necessary conditions for $\tilde{x}$ to be Pareto point

P_0 and N_0 sets	Necessary conditions
$P_0 \neq \varnothing$, $N_0 \neq \varnothing$	$\underset{i \in P_0}{\&} x_i = 1$ $\underset{i \in N_0}{\&} \overline{x_i} = 1$
$P_0 \neq \varnothing$, $N_0 = \varnothing$	$\underset{i \in P_0}{\&} x_i = 1$
$P_0 = \varnothing$, $N_0 \neq \varnothing$	$\underset{i \in N_0}{\&} \overline{x_i} = 1$
$P_0 = \varnothing$, $N_0 = \varnothing$	—

Condition (6) is not sufficient. To be convinced let's consider the following.

Example 1

$$\begin{cases} \boldsymbol{max}(f_1(\tilde{x}) = 25x_1 + x_2 - x_3 - x_4); \\ \boldsymbol{max}(f_2(\tilde{x}) = -x_1 + x_2 - x_3 + 25x_4); \\ \tilde{x} \in B^4 \end{cases}$$

We find $P_1 \cap P_2=\{2\}$; $N_1 \cap N_2=\{3\}$. The necessary condition (6) is $x_2\overline{x_3} = 1$. The point $\tilde{\beta}=(0,1,0,0)$ satisfes this condition, but $\tilde{\beta}$ is not a Pareto point: for $\tilde{\gamma}=(1,1,0,1)$ we have $25=f_1(\tilde{\gamma})>f_1(\tilde{\beta})=1$ and $25=f_2(\tilde{\gamma})>f_2(\tilde{\beta})=1$.

We introduce the following definitions to apply a Branch-and-Bound method for the determination of the Pareto set.

Definition 1 The vector

$$\left(\min_{x\in X} f_1(\tilde{x}), \min_{x\in X} f_2(\tilde{x}), \ldots, \min_{x\in X} f_S(\tilde{x})\right)$$

$$\left(\left(\max_{x\in X} f_1(\tilde{x}), \max_{x\in X} f_2(\tilde{x}), \ldots, \max_{x\in X} f_S(\tilde{x})\right)\right)$$

is called the lower (upper) vector estimator of the admissible set X.

Definition 2 Vector $(a_1, a_2, \ldots, a_s)$ is called majorized by vector $(b_1, b_2, \ldots, b_s)$ if $a_k \leq b_k$ for all $k = \overline{1,s}$ and such k exists, so that $a_k<b_k$.

Definition 3 Let $\tilde{\mathfrak{R}} = \{\mathfrak{R}_1, \ldots, \mathfrak{R}_q\}$ be a collection of intervals from the admissible set $X \subset B^n$ and $\tilde{\gamma} \in X$. The vector $\tilde{f}(\tilde{\gamma}) = (f_1(\tilde{\gamma}), \ldots, f_S(\tilde{\gamma}))$ is called the *record* of the collection $\tilde{\mathfrak{R}}$ if there is no such point $\tilde{\alpha} \in X$ that $\tilde{f}(\tilde{\gamma})$ is majorized by vector $\tilde{f}(\tilde{\alpha})$ and $\tilde{f}(\tilde{\gamma})$ is not majorized by lower vector estimator for any interval from $\tilde{\mathfrak{R}}$.

We will make branching steps by fixing values 0 and 1 for variables x_i, $i=\overline{1,n}$. On any branching step B^n is separated and some collection of intervals $\tilde{\mathfrak{R}}_t$ is appeared. These intervals are to be investigated.

The interval $\mathfrak{R} \in \tilde{\mathfrak{R}}_t$ must be removed from $\tilde{\mathfrak{R}}_t$ in the following cases:

a) the upper vector estimator of the interval $\mathfrak{R}$ is majorized by the record of $\tilde{\mathfrak{R}}_t$;

b) an interval $\mathfrak{R}' \in \tilde{\mathfrak{R}}_t$ exists so that the upper vector estimator of $\mathfrak{R}$ is majorized by the lower vector estimator of $\mathfrak{R}'$.

The interval must be separated if it is not removed and it's upper and lower estimators are different. The choice of the interval and variable for branching is a heuristic element of the suggested method.

Example 2 We consider the determination of the Pareto set for the unconditional optimization problem from the example 1. We use the necessary condition for the Pareto points: $x_2\overline{x}_3 = 1$. The initial tree is shown in Fig.1.

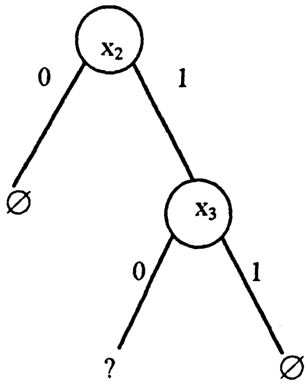

Fig. 1. Initial tree

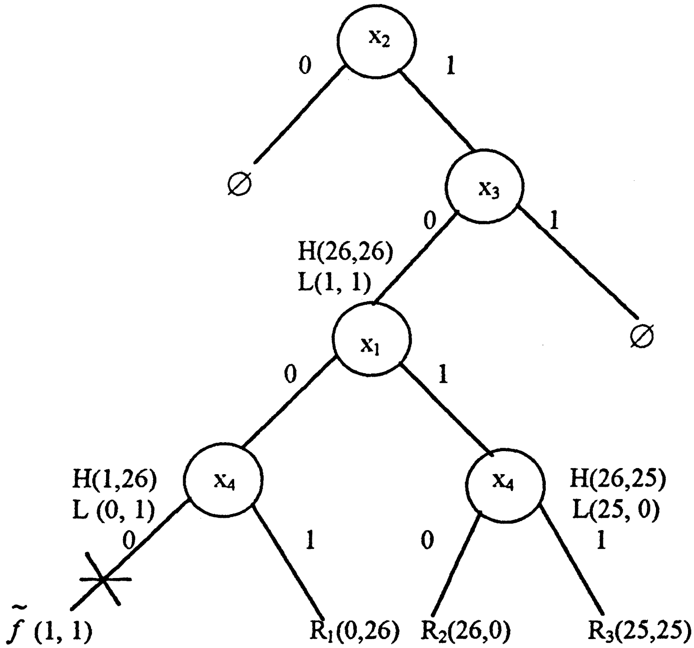

Fig. 2. The branching process from Example 2

Symbol "$\varnothing$" denotes the absence of the Pareto points in the interval corresponding to the marked branch; symbol "?" means the next branching is necessary; symbol "x" denotes that the interval must be removed.

Let $H(y_1,y_2)$ be an upper and $L(y_1,y_2)$ be a lower vector estimators for $\tilde{f}=(f_1,f_2)$ and R(a,b) is a current record. Figure 2 illustrates the branching process.

The Pareto set consists of the following three record's points: {(0,1,0,1),(1,1,0,0),(1,1,0,1)} (R_1, R_2, R_3). The logical description of $\mathcal{P}$ is of the form $\mathcal{P}=\{\tilde{x}: x_1x_2\bar{x}_3 \vee x_2\bar{x}_3x_4 = 1\}$

3 Problems With a Disjunctive Restriction and Branch-and-bound

Theorem 2 Let the problem $\boldsymbol{max}\, f_1(\tilde{x}),\dots,\boldsymbol{max}\, f_S(\tilde{x}), \tilde{x} \in B^n$, $f_1,\dots,f_S \in LPS_2(n)$ have a Pareto set $\mathcal{P} \neq \varnothing$ and the restriction $\tilde{x} \in \Omega, \Omega \neq \varnothing, \Omega \subset B^n, \Omega \neq B^n$, is added. A set $\{\mathcal{P} \cap \Omega\}$, if it is not empty, contains only Pareto points for this problem with additional restriction.

Proof Let $\tilde{x}^* \in \{\mathcal{P} \cap \Omega\}$. Then $\tilde{x}^* \in \mathcal{P}$ and $\tilde{x}^*$ is not majorized by any point from B^n, therefore so $\tilde{x}^*$ is not majorized by any point from $\Omega \subset B^n$. Whereas $\tilde{x}^* \in \Omega$, $\tilde{x}^*$ is an admissible point, hence $\tilde{x}^*$ is a Pareto point for the problem with additional restriction.

Remark Condition $\tilde{x} \in \{\mathcal{P} \cap \Omega\}$ is sufficient for $\tilde{x}$ to be a Pareto point if $\mathcal{P} \cup \Omega \neq \varnothing$. But this condition is not necessary. Indeed, for the problem

$$\begin{cases} \boldsymbol{max}(f_1(\tilde{x}) = 25x_1 + x_2 - x_3 - x_4); \\ \boldsymbol{max}(f_2(\tilde{x}) = -x_1 + x_2 - x_3 + 25x_4); \\ \tilde{x} \in \Omega = \{\tilde{x}: \bar{x}_1x_2 = 1\} \subset B^4 \end{cases}$$

the two points (0,1,0,0) and (0,1,0,1) are Pareto points, $\tilde{f}$ (0,1,0,0)=(1,1), $\tilde{f}$ (0,1,0,1)=(0,26) but $\{0,1,0,0\} \notin \{P \cap \Omega\}$.

Using theorem 2 we can find, generally speaking, a subset $\{\mathcal{P} \cap \Omega\}$ of the Pareto set $\mathcal{P}_\Omega$ for the problem with restriction $\tilde{x} \in \Omega \subset B^n$, where $\mathcal{P}$ is the Pareto set for the unconditional multi-criteria optimization problem □.

If a set $\mathcal{P}$ and the logical description of $\mathcal{P}$ were found (as in Example 2) then a set $\{\mathcal{P} \cap \Omega\}$ can be found by multiplying of two corresponding DNFs because $\tilde{x} \in \Omega$ is a disjunctive restriction.

A disjunctive restriction in the multi-criteria problem (4) can be presented in the form $\bigvee_{q=1}^{m} K_q = 1$, where $K_q=K_q(\tilde{x})$ are elementary conjunctions.

The domain of admissible solutions Ω of the problem (4) can be represented in the form $\Omega = \bigcup_{q=1}^{m} \mathfrak{R}_{K_q}$, $\mathfrak{R}_{K_q} = \{\tilde{x}: K_q(\tilde{x}) = 1\}$, where $\mathfrak{R}_{K_q}, q = \overline{1,m}$ are intervals corresponding to conjunctions K_q.

Generally speaking, each interval of $\mathfrak{R}_{K_1},\ldots,\mathfrak{R}_{K_m}$ must be investigated to find Pareto points. Branch-and-Bound in this case is like expounded above. The branching tree is built separately for each interval taking into account fixed variable's values of the interval code.

Estimation, branching, cutting and completion testing are carried taking into account all m trees and all obtained and not removed intervals for each step.

Example 3 Let consider the problem with functions f_1 and f_2 from example 1 with additional restriction: $x_1x_2 \vee x_3x_4 = 1$. Conjunction x_1x_2 does not contradict to the condition $x_2\bar{x}_3 = 1$. Therefore, the interval $\mathfrak{R}_{x_1x_2\bar{x}_3}$ contains only Pareto points. The tree corresponding for this interval has the representation shown on the Fig. 3a.

Two records R_1 (26,0) and R_2(25,25) are obtained.

The second conjunction x_3x_4 contradicts to the condition $x_2\bar{x}_3 = 1$ but it must be investigated (See Fig. 3b).

The interval $\mathfrak{R}_{x_3x_4}$ does not contain Pareto points because its upper vector estimator H(24,25) is majorized by the record R_2 (25,25). The Pareto set consists of two points $\{(1,1,0,0),(1,1,0,1)\}=\mathcal{P}_\Omega$ and has the logical description $D_{\mathcal{P}_\Omega} = x_1x_2\bar{x}_3$.

4 A Choice of the Intervals and Variables for the Branching

A time of the problem solution finding depends on the series of intervals and variables choices. The branching strategies are heuristics, for example:

a) At first we separate (by the variable number i) those intervals, which correspond to conjunction without literal of variable x_i so that the replacement of the value of this variable from 1 to 0 gives as more as possible number of decreasing scalar functions.

b) We separate the interval $\mathfrak{R}$, which has a greatest difference d(H,L) between upper and lower vector estimators $H=(h_1,\ldots,h_s)$ and $L=(l_1,\ldots,l_s)$ of $\mathfrak{R}$:

$$d(H,L) = \min_{1\le j\le s}\left(\frac{h_j - l_j}{M_j - \mu_j}\right)$$

where

$$M_j = \max_{x\in\Omega} f_j(\tilde{x}), \mu_j = \min_{x\in\Omega} f_j(\tilde{x})$$

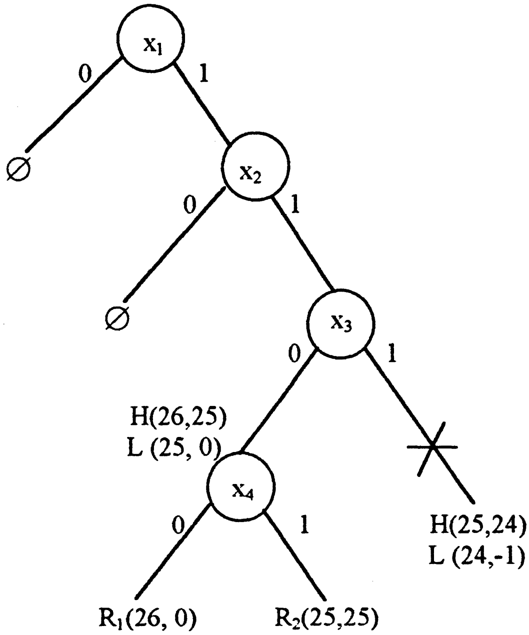

Fig. 3a. The branching process from Example 3; $\mathfrak{R}_{x_1 x_2 \bar{x}_3}$

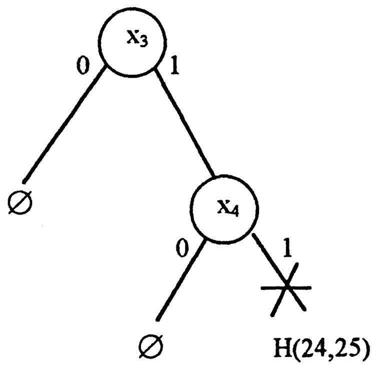

Fig. 3b. The branching process from Example 3; $\mathfrak{R}_{x_3 x_4}$

Note that $M_j - \mu_j > 0$ because in another case the scalar function f_j may be removed.

The computational complexity of M_j and μ_j finding can be estimated as O(nm) [2].

It is easy to build a "pathological" multi-criteria pseudoboolean optimization problem with a disjunctive restriction so that the complexity of its solution will be equal to the volume of a complete inspection by the considered method. In these cases we consider the following approximation.

Let $\varepsilon_1, \ldots, \varepsilon_s \in R^+$ be fixed. Two vectors $(a_1, \ldots, a_s)$ and $(b_1, \ldots, b_s)$ are said $\tilde{\varepsilon}$*-equal* if $|a_i - b_i| \leq \varepsilon_i, i = \overline{1,s}$.

If the upper and lower vector estimators of the interval $\mathfrak{R}$ are $\tilde{\varepsilon}$-equal then the interval $\mathfrak{R}$ is called $\tilde{\varepsilon}$*-interval.*

If the lower vector estimator of an $\tilde{\varepsilon}$-interval is majorized by the record or by the lower vector estimator of another interval then the $\tilde{\varepsilon}$-interval must be removed from $\tilde{\mathfrak{R}}_t$.

Collection of the record's points and nonmajorized ε-intervals is an approximation of the Pareto set.

Thus, using a structure of the branch-and-bound tree allows to synthesize the logical description $D_{\mathcal{P}}$ of the Pareto set $\mathcal{P}$.

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Optimality Conditions in Set-Valued Vector Optimization

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Abstract. In this paper we discuss set-valued vector optimization problems and present optimality conditions with the aid of so-called contingent epiderivatives.

Keywords. Set-valued analysis, vector optimization

1 Introduction to Set-Valued Vector Optimization

In set-valued vector optimization one considers vector optimization problems with a set-valued objective function which has to be minimized or maximized. In the standard vector optimization theory it is always assumed that the objective function is exactly given. In practice this function is sometimes not exactly given or its values may vary in a certain range. Problems with uncertain objectives can be found in stochastic programming and fuzzy set optimization. In set-valued optimization it is assumed that the objective function is set-valued and that this map (or the range in which the objective values can vary) is explicitly known. Hence, there are close relationships between these three types of vector optimization problems.

For our investigations we have the following standard assumption.

Assumption 1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let S be a nonempty subset of X, let Y be partially ordered by a convex cone $C \subset Y$ (then $\leq_C$ is the corresponding partial order), and let $F : S \to 2^Y$ be a set-valued map.

Under this assumption we consider the set-valued vector optimization problem

$$\min_{x \in S} F(x). \tag{1}$$

A minimizer of this problem is introduced as follows (for instance, see [8]):

Definition 1. Let Assumption 1 be satisfied, and let $F(S) := \bigcup_{x \in S} F(x)$ denote the image set of F. Then a pair $(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ is called a *minimizer* of the problem (1), if $\bar{y}$ is a minimal element of the set $F(S)$, i.e.

$$(\{\bar{y}\} - C) \cap F(S) \subset \{\bar{y}\} + C.$$

Example 1. Let Assumption 1 be satisfied.

(a) Assume that $f, g : S \to Y$ are given vector functions. Then $F : S \to 2^Y$ with

$$F(x) := \{y \in Y \mid f(x) \le_C y \le_C g(x)\}$$

is a possible set-valued map which may be used as an objective. If $f = g$ and C is pointed (i.e., $C \cap (-C) = \{0\}$), then at every $x \in S$ a corresponding image y is uniquely determined, otherwise the values of y vary in the order interval $[f(x), g(x)]$ (see Fig. 1).

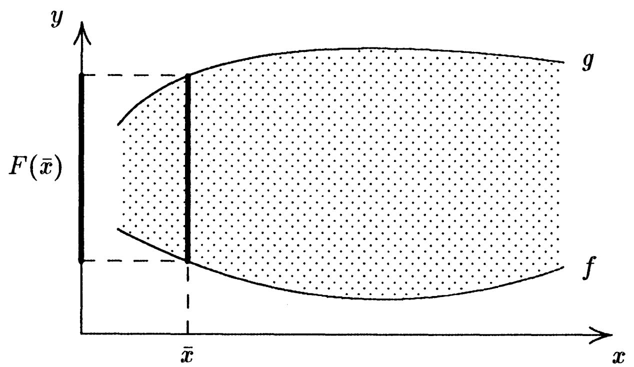

Fig. 1. Illustration of the set-valued map F in Example 1,(a)

(b) One special case of the previous example is obtained if a vector function $\varphi : S \to Y$ is known and the y-values vary around $\varphi(x)$, i.e. we have

$$F(x) := \{y \in Y \mid \varphi(x) - \alpha \le_C y \le_C \varphi(x) + \beta\}$$

where $\alpha, \beta \in C$.

(c) Another special case appears if we admit relative errors around $\varphi(x)$. Again, we assume that a vector function $\varphi : S \to Y$ is known, and for an arbitrary $\epsilon > 0$ we define

$$\begin{aligned} F(x) &:= \{y \in Y \mid \varphi(x) - \epsilon\varphi(x) \le_C y \le_C \varphi(x) + \epsilon\varphi(x)\} \\ &= \{y \in Y \mid (1-\epsilon)\varphi(x) \le_C y \le_C (1+\epsilon)\varphi(x)\}. \end{aligned}$$

It can be expected that the minimization of the set-valued map F in Example 1,(a) has something to do with the minimization of f. Therefore, under the assumptions given in Example 1,(a) we consider the single-valued vector optimization problem

$$\min_{x \in S} f(x). \tag{2}$$

Recall that a minimal solution $\bar{x}$ of this problem is a preimage of a minimal element of the image set $f(S)$.

Theorem 1. *Let Assumption 1 be satisfied, let C be pointed, let $f : S \to Y$ be a given function, and let $F : S \to 2^Y$ be defined as*

$$F(x) := \{y \in Y \mid f(x) \le_C y\} \text{ for all } x \in S.$$

(a) If $(\bar{x}, \bar{y})$ is a minimizer of the problem (1), then $\bar{y} = f(\bar{x})$ and $\bar{x}$ is a minimal solution of the problem (2).

(b) If $\bar{x}$ is a minimal solution of the problem (2), then $(\bar{x}, f(\bar{x}))$ is a minimizer of the problem (1).

Proof.

(a) Since $(\bar{x}, \bar{y})$ is a minimizer of the problem (1) and C is pointed we have

$$(\{\bar{y}\} - C) \cap F(S) = \{\bar{y}\}. \tag{3}$$

Obviously it is

$$\bar{y} \in F(\bar{x}) \subset \bigcup_{x \in S} F(x) = F(S),$$

and therefore we conclude

$$(\{\bar{y}\} - C) \cap F(\bar{x}) = \{\bar{y}\}. \tag{4}$$

If we assume that $\bar{y} \neq f(\bar{x})$, we obtain because of $f(\bar{x}) \le_C \bar{y}$ a contradiction to (4). Consequently, $\bar{y} = f(\bar{x})$, and by the equation (3) and $f(\bar{x}) \in f(S) \subset F(S)$ we get $(\{f(\bar{x})\} - C) \cap f(S) = \{f(\bar{x})\}$, i.e. $\bar{x}$ is a minimal solution of the problem (2).

(b) Assume that $(\bar{x}, f(\bar{x}))$ is not a minimizer of the problem (1). Then there is an $\tilde{x} \in S$ with

$$(\{f(\bar{x})\} - C) \cap F(\tilde{x}) \neq \{f(\bar{x})\}. \tag{5}$$

So we have for some $y \in F(\tilde{x})$

$$y \leq_C f(\bar{x}),\ y \neq f(\bar{x}) \ \text{(by (5))}$$

and

$$f(\tilde{x}) \leq_C y \text{ (by the definition of } F(\tilde{x})).$$

Hence we get

$$f(\tilde{x}) \leq_C f(\bar{x}),\ f(\tilde{x}) \neq f(\bar{x}).$$

But then $\bar{x}$ is not a minimal solution of the problem (2). □

The preceding theorem shows that in the special case discussed in Example 1,(a) the set-valued optimization problem (1) is equivalent to the vector optimization problem (2) being simpler than the problem (1). Therefore, it is not necessary to work with such a general set-valued theory in this special case. Hence, the theory of optimality conditions makes only sense for set-valued maps whose lower boundary cannot be described by a function f as it is done in Example 1,(a).

2 Contingent Epiderivatives

For the formulation of optimality conditions one needs an appropriate differentiability concept. Here we use the notion of contingent epiderivatives introduced in [2] for real-valued functions and in [7] for set-valued maps. The notion of contingent derivatives introduced by Aubin [2] plays an important role in set-valued analysis (e.g., see [4]). But it turns out that the contingent epiderivative is the better tool for the formulation of necessary and sufficient optimality conditions (see [7]).

Definition 2. Let Assumption 1 be satisfied.

(a) The set

$$\text{epi}(F) := \{(x, y) \in X \times Y \mid x \in S, y \in F(x) + C\}$$

is called the *epigraph* of F.

(b) Let a pair $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ be given. A single-valued map $DF(\bar{x}, \bar{y}) : X \to Y$ whose epigraph equals the contingent cone to the epigraph of F at $(\bar{x}, \bar{y})$, i.e.

$$\text{epi}(DF(\bar{x}, \bar{y})) = T(\text{epi}(F), (\bar{x}, \bar{y})),$$

is called *contingent epiderivative* of F at $(\bar{x}, \bar{y})$.

For the definition of contingent cones we refer to the standard special literature (e.g., see [6]). A simple existence result for contingent epiderivatives can be given in the special case $Y = \mathbf{R}$.

Theorem 2 ([7, Thm. 1]). *Let Assumption 1 be satisfied with $Y = \mathbf{R}$, and let $(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ be given. Assume that there are functions $f, g : X \to \mathbf{R}$ with $\text{epi}(f) \supset T(\text{epi}(F), (\bar{x}, \bar{y})) \supset \text{epi}(g)$. Then the contingent epiderivative $DF(\bar{x}, \bar{y})$ is given as*

$$DF(\bar{x}, \bar{y})(x) \;=\; \min\{y \in \mathbf{R} \mid (x, y) \in T(\text{epi}(F), (\bar{x}, \bar{y}))\} \;\; \forall \; x \in X.$$

The contingent epiderivative has interesting properties. We present here the sublinearity in the case of a convex set-valued map.

Definition 3.

(a) Let X be a real linear space, let Y be a real linear space partially ordered by a convex cone $C \subset Y$, and let S be a nonempty convex subset of X. A set-valued map $F : S \to 2^Y$ is called *C-convex*, if for all $x_1, x_2 \in S$ and $\lambda \in [0, 1]$

$$\lambda F(x_1) + (1 - \lambda) F(x_2) \subset F(\lambda x_1 + (1 - \lambda) x_2) + C.$$

(b) Let X be a real linear space, and let Y be a real linear space partially ordered by a convex cone $C \subset Y$. A map $f : X \to Y$ is called *sublinear* if

(i) $f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$ and all $x \in X$ (positive homogenity),

(ii) $f(x_1 + x_2) \in \{f(x_1) + f(x_2)\} - C$ for all $x_1, x_2 \in X$ (subadditivity).

Theorem 3 ([7, Thm. 4]). *Let Assumption 1 be satisfied, and, in addition, let C be pointed, let $S = X$, and let F be C-convex. If the contingent epiderivative $DF(\bar{x}, \bar{y})$ exists, then it is sublinear.*

The rather technical proof of this assertion generalizes the known result that a functional is sublinear if and only if its epigraph is a convex cone.

In order to compare this differentiability notion with known concepts in analysis, we consider the case of a real- and single-valued function.

Theorem 4 ([7, Thm. 6]). *Let $(X, \|\cdot\|_X)$ be a real normed space, and let $F : X \to \mathbf{R}$ be a single-valued function being continuous at an $\bar{x} \in X$ and convex. Then the contingent epiderivative equals the directional derivative of F at $\bar{x}$.*

For the definition of the standard notion of a directional derivative we refer, for instance, to [6].

Next, we consider again the set-valued map in Example 1,(a). The contingent epiderivative of this map can be given with the aid of the contingent epiderivative of f.

Theorem 5. *Let Assumption 1 be satisfied, let $F : S \longrightarrow 2^Y$ be given as*

$$F(x) := \{y \in Y \mid f(x) \leq_C y \leq_C g(x)\}$$

with $f, g : S \longrightarrow Y$, and let $\bar{x} \in S$ be arbitrarily given. If the contingent epiderivative $DF(\bar{x}, f(\bar{x}))$ exists, then

$$DF(\bar{x}, f(\bar{x})) = Df(\bar{x}, f(\bar{x})).$$

Proof. Because of the definition of F we have

$$\text{epi}(F) = \{(x, y) \in X \times Y \mid x \in S, f(x) \leq_C y\} = \text{epi}(f),$$

and therefore, we conclude

$$\begin{aligned}
\text{epi}(DF(\bar{x}, \bar{y})) &= T(\text{epi}(F), (\bar{x}, \bar{y})) \\
&= T(\text{epi}(f), (\bar{x}, \bar{y})) \\
&= \text{epi}(Df(\bar{x}, \bar{y})).
\end{aligned}$$

This leads to the assertion. □

Corollary 1. *Let $(X, \|\cdot\|_X)$ be a real normed space, let $F : X \longrightarrow 2^{\mathbb{R}}$ be given as*

$$F(x) := \{y \in \mathbb{R} \mid f(x) \leq y \leq g(x)\}$$

with $f, g : X \longrightarrow \mathbb{R}$, let $\bar{x} \in X$ be arbitrarily given, and let f be continuous at $\bar{x}$ and convex. Then the contingent epiderivative of F at $(\bar{x}, f(\bar{x}))$ exists and equals the directional derivative of f at $\bar{x}$.

Proof. It is obvious that $\text{epi}(F) = \text{epi}(f)$. Since f is a convex functional, its epigraph is convex and we get by a standard result on contingent cones (for instance, see [6])

$$\text{epi}(f) = \text{epi}(F) \subset T(\text{epi}(F), (\bar{x}, f(\bar{x}))) + \{(\bar{x}, f(\bar{x}))\}.$$

f is continuous at $\bar{x}$ and convex and, therefore, there is a subgradient l of f at $\bar{x}$ (see [6]) with

$$\text{epi}(h) \supset T(\text{epi}(F), (\bar{x}, f(\bar{x}))) + \{(\bar{x}, f(\bar{x}))\}$$

for

$$h(x) := l(x - \bar{x}) + f(\bar{x}) \text{ for all } x \in S.$$

Hence, the assumptions of Theorem 2 are fulfilled and we conclude that the contingent epiderivative $DF(\bar{x}, f(\bar{x}))$ exists. By Theorem 5 $DF(\bar{x}, f(\bar{x}))$ equals $Df(\bar{x}, f(\bar{x}))$ which, by Theorem 4, equals the directional derivative of f at $\bar{x}$. □

3 Optimality Conditions

Now we turn our attention to optimality conditions for the set-valued optimization problem (1). These conditions use the concept of the contingent epiderivative discussed in the previous section. We give only a short survey on results from [7].

It is well-known from vector optimization that it is simpler to formulate optimality conditions for the weak minimality notion than for the minimality notion. Therefore, we recall a similar concept for the set-valued case (see also [8]).

Definition 4. Let Assumption 1 be satisfied, and let $F(S) := \bigcup_{x \in S} F(x)$ denote the image set of F.

(a) Let C have a nonempty interior int(C). A pair $(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ is called a *weak minimizer* of the problem (1), if $\bar{y}$ is a weakly minimal element of the set $F(S)$, i.e.

$$(\{\bar{y}\} - \text{int}(C)) \cap F(S) = \emptyset.$$

(b) A pair $(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ is called a *strong minimizer* of the problem (1), if $\bar{y}$ is a strongly minimal element of the set $F(S)$, i.e.

$$F(S) \subset \{\bar{y}\} + C.$$

Optimality conditions in set-valued optimization were given by Corley [1] and Luc [8], [9] using contingent derivatives. Oettli [10] introduced a differentiability notion which generalizes the Neustadt derivative. In this paper we discuss conditions being based on the concept of contingent epiderivatives.

Theorem 6 ([7, Thm. 7]). *Let Assumption 1 be satisfied, and let C have a nonempty interior. If $(\bar{x}, \bar{y})$ is a weak minimizer of the problem (1) and the contingent epiderivative $DF(\bar{x}, \bar{y})$ exists, then*

$$DF(\bar{x}, \bar{y})(x - \bar{x}) \notin -int(C) \text{ for all } x \in S. \tag{6}$$

This necessary condition generalizes the known necessary optimality condition in single-valued vector optimization (for instance, see [5, Thm. 7.6]).

Theorem 7 ([7, Thm. 8]). *Let Assumption 1 be satisfied, let C have a nonempty interior, and, in addition, let S be a convex set and let F be C-convex. If the contingent epiderivative $DF(\bar{x}, \bar{y})$ exists at an $\bar{x} \in S$ and a $\bar{y} \in F(\bar{x})$ and the condition (6) is fulfilled, then $(\bar{x}, \bar{y})$ is a weak minimizer of the problem (1).*

Hence, under the assumptions given in Theorem 7 the condition (6) is a necessary and sufficient optimality condition for a weak minimizer. Such a complete characterization cannot be formulated with the aid of the notion of the contingent derivative.

Theorem 8 ([7, Thm. 9]). *Let Assumption 1 be satisfied, and, in addition, let C be closed, let S be a convex set and let F be C-convex. Let the contingent epiderivative $DF(\bar{x}, \bar{y})$ exist at an $\bar{x} \in S$ and a $\bar{y} \in F(\bar{x})$. The pair $(\bar{x}, \bar{y})$ is a strong minimizer of the problem (1) if and only if*

$$DF(\bar{x}, \bar{y})(x - \bar{x}) \in C \text{ for all } x \in S.$$

Aubin and Ekeland [3] published a necessary and sufficient optimality condition for strong minimizers using the contingent derivative and the very strong assumption that the graph of F is convex. The previous theorem presents a similar result in the contingent epidifferentiability case under the weaker assumption that F is C-convex.

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A Multiple Objective Approach to Nash Equilibria in Bimatrix Games

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Abstract. Nash equilibria of bimatrix games may be found by solving a nonconvex quadratic multiple objective programming problem over a linear constraint set. The advantages over traditional approaches are explored. Every efficient solution is a Nash equilibrium point, so one may easily obtain multiple equilibria, which is a capability not found in other approaches. Since it is known that Nash equilibria exist, one also obtains a proof that efficient solutions exist for these nonconvex quadratic multiple objective programming problems. Finally, there is an interesting new interpretation of Nash equilibria obtained, namely, that a multiple objective referee of the game exists, who ensures the optimal play by the two participants.

Keywords: Nash equilibrium, bimatrix games, efficient solutions, multiple objective quadratic programming, linear complementarity problems

1 Introduction

The study of Nash equilibria is nearly fifty years old now, and it has lead to some important results, which are more appreciated in recent years. John Nash and Carlton Lemke were awarded the von Neumann Theory Prize. The award of the Nobel prize in Economics to Nash, Selten and Harsanyi in 1994 again brings the study to the forefront. Not less important is the penetration of the Nash equilibrium concept into modern industrial uses [1]. Yet, there are some serious computational issues which arise when one wants to compute Nash equilibria. One of these, which many people say is a main stumbling block to the application of Nash equilibria, is the multiplicity of solutions. Within this stumbling block is the concept of selection of certain more desirable Nash equilibria and the capablity to exclude the ones which, for some reason, are not interesting. The reason is quite dependent on the application, so one may not wish to design a single generic elimination scheme for processing the multiple equilibria. Rather it seems to be an advantage to be able to examine multiple equilibria points, as necessary, to obtain a subset of the most interesting ones for the decision makers.

In most cases, the algorithms which have been proposed to date [7,8] have the capability to compute only one Nash equilibrium point. This difficiency, especially relative to Lemke & Howson algorithm was noted in several papers [12,14,15]. This difficiency is due to the fact that they approach the equilibrium point by means of a linear complementarity problem, and by complementary pivoting. Such pivoting does not allow for deviation from a unique path of vertices, which ironically, is one of its main strengths relative to being able to obtain one equilibrium solution. Lemke and Howson gave a proof of the existence of an odd number of equilibria and indicated the multiplicity of solutions as an area of research yet to be investigated. Tamir [13] noted that it is possible for a complementarity problem to have an exponential number of solutions, but his examples were not related to bimatrix game models. Le Van [9] gave another proof that (under a nondegeneracy assumption) the number of solutions to a bimatrix game is an odd number, by invoking topological degree theory. His paper sketches a way in which all the equilibria may be found using decomposition of the hypercube and exhaustive search and degree computations on the elementary sets of the decomposition. This type of searching, unguided by any type of objective function, is likely to produce an unacceptably large number of subdivisions, with each subdivision requiring a type of multi-dimensional integration to evaluate the degree of the mapping. Mangasarian [10] proposed an algorithm which can compute all the vertices of the convex polyhedral feasible set, in order to determine all equilibria. It was stated that computer running times may be excessive if $m \times n > 100$. The main idea of the current paper is to introduce another way to search for equilibria, which is more likely to lead to a successful conclusion, namely the computation of a subset of all equilibria having some desired property.

The approach which will be developed here is related to the idea proposed in Kostreva and Wiecek [4] and generalized by Ebiefung [2]. Those papers show the equivalence of finding a solution of a complementarity problem to the problem of searching for efficient points for a nonlinear multiple objective programming problem with a linear constraint set. Of course, all difficulties do not disappear. These nonlinear multiple objective programming problems are not trivial, as they have nonconvex, but polynomial objective functions. The recent research on such problems [3] demonstrates that searching for efficient points is a well founded approach. Note that there are now other techniques which can handle such problems. Among those are the integral based global optimization methods developed in [5,6]. Hence there is much evidence that a proposal to solve nonconvex multiple objective problems is based on a strong mathematical foundation.

As a benefit of viewing the game in this way, a new interpretation of the equilibrium arises. This is a multiple objective referee for which finding efficient points corresponds to finding the Nash equilibrium points. The concept of a referee in a noncooperative game has been used before, but not as a multiple objective referee. For example, Pau [11] introduces another player (a referee) in a differential game, and then forms a highly nonlinear objective function which is used by the referee to find a single Nash equilibium point. This objective function is constructed as the product of n differential terms for an n-player game, with each term formed as a dot product of a gradient and a direction vector. The optimal control problem which arises is solved by a decomposition algorithm. The multiple objective referee proposed here does not create something highly nonlinear and more easily accomodates the multiple equilibria which are of interest. It seems to be novel to view the bimatrix game from the multiple objective point of view, and not without its advantages.

2 Formulations

In this section the basic formulations will be described and their inter-relationships detailed.

We are given a bimatrix game defined by the $m \times n$ matrices $\hat{A}, \hat{B}$ and where the players r, c belong to $\mathbf{R}^m, \mathbf{R}^n$ and satisfy

$$r \geq 0, \quad c \geq 0, \quad \sum_i r_i = 1, \quad \sum_j c_j = 1; \tag{1}$$

r_i, c_i are the components of r and c.

A Nash equilibrium point is a point $z_0 = (r_0, c_0) \in \mathbf{R}^m \times \mathbf{R}^n$, such that

$$\begin{array}{ll} z_0 \text{ satisfies (1)}, & r_0^T \hat{A} c_0 \geq r^T \hat{A} c_0, \\ r_0^T \hat{B} c_0 \geq r_0^T \hat{B} c, & \forall z = (r, c) \text{ satisfying (1)}. \end{array} \tag{2}$$

Let e denote the vector of 1's (whose dimension will be understood from the context). Observe that the problem (2) is equivalent to the following:

$$\begin{aligned} z_0 &= (r_0, c_0) \text{ satisfies (1)}, \\ \hat{B}^T r_0 &\leq \left(r_0^T \hat{B} c_0\right) e, \\ \hat{A} c_0 &\leq \left(r_0^T \hat{A} c_0\right) e. \end{aligned} \tag{3}$$

Now, let

$$E = ee^T$$

be the matrix with all 1's, so that, for (r, c) satisfying (1),

$$Ec = e \quad \text{and} \quad r^T E c = 1.$$

Let k be fixed and large enough so that

$$kE - \hat{B}^T > 0 \quad \text{and} \quad kE - \hat{A} > 0.$$

Consider now solutions to

$$\begin{aligned} \left(kE - \hat{B}^T\right) r &\geq e, \quad r \geq 0, \\ \left(kE - \hat{A}\right) c &\geq e, \quad c \geq 0, \\ c^T \left[\left(kE - \hat{B}^T\right) r - e\right] &= 0, \\ r^T \left[\left(kE - \hat{A}\right) c - e\right] &= 0. \end{aligned} \tag{4}$$

Lemke and Howson [7] have shown that there exists a one-one correspondence between the solutions (r_0, c_0) of (3) and the solutions (r, c) of (4). The correspondence is given by

$$\begin{aligned} r_0 &= r/(r^T e), \quad c_0 = c/(c^T e), \\ k - 1/(r^T e) &= r_0^T \hat{B} c_0, \\ k - 1/(c^t e) &= r_0^T \hat{A} c_0. \end{aligned}$$

So we are interested in solutions (r, c)

$$\begin{aligned} B^T r - e &\geq 0, \quad r \geq 0, \\ Ac - e &\geq 0, \quad c \geq 0, \\ c^T(B^T r - e) &= 0, \\ r^T(Ac - e) &= 0, \end{aligned} \tag{5}$$

where A, B are the $m \times n$ matrices, whose elements a_{ij}, b_{ij} are strictly positive.

Notice that the last system of equations and inequalities is a special case of the linear complementarity problem:

Find x in $\mathbf{R}^n$ satisfying

$$y = Mx + q, \tag{6}$$
$$y_i \geq 0, \tag{7}$$
$$x_i \geq 0, \tag{8}$$
$$y_i \cdot x_i = 0, \tag{9}$$

for all $i, i = 1, \cdots, n$.

Multiple objective programming may be formulated as follows:

$$\text{minimize } f(x) \tag{10}$$
$$\text{subject to } x \in X \tag{11}$$

where $f(x) = [f_1(x), f_2(x), \cdots, f_p(x)]^T, p \geq 2$, is a vector-valued function defined over the feasible set X which is given by

$$X = \{x \in \mathbf{R}^n | g_i(x) \geq 0, \forall i \in I\}, \tag{12}$$

where $I = \{1, 2, \cdots, m\}$, and $\forall i \in I, g_i : \mathbf{R}^n \to \mathbf{R}$. Also, let $J = \{1, 2, \cdots, p\}$ and for any $k \in J$, let J_k denote the set $(J - \{k\})$.

We seek solutions that are efficient in the following sense:

Definition 2.1. A points x^0 is said to be an *efficient solution* of problem (10) - (12) if $x^0 \in X$ and if $f_i(x) < f_i(x^0)$ for some $x \in X$ and some $i \in J$ implies that there exists at least one $j \in J_i$ such that $f_j(x) > f_j(x^0)$. Let X_E denote the set of all efficient solutions for problem (10) - (12).

For bimatrix games, a certain version of the multiple objective problem (10) - (12) is of central interest:

$$\begin{aligned} &\text{minimize } [y_1 x_1, y_2 x_2, \cdots, y_n x_n]^T \\ &\text{subject to } x \in X \end{aligned} \tag{13}$$

where

$$X = \{x | x \geq 0, y = Mx + q \geq 0\}. \tag{14}$$

Let us note here for clarity that the above problem, defined in (13) - (14) has quadratic objective functions, linear constraints and each objective is bounded below by zero. This problem (13) - (14) is a special case of the polynomial problems studied in Kostreva, Ordoyne and Wiecek [3] and it has an unsusual structure which we can exploit. For example, observe the following efficient solutions:

Definition 2.2. If $x \in X_E$, and for $i = 1, 2, \cdots, p, f_i(x) = 0$, then we say that x is a *zero-efficient solution*.

The implication of such solutions is now noted in the following theorem: Kostreva and Wiecek [4].

Theorem 2.1. *The point $x \in X_E$ is a zero-efficient solution of (13) - (14) if and only if x is a solution of $LCP(q, M)$ (6) - (9).*

Proof. Let $x \in X_E$ be zero-efficient. Certainly $y_i \geq 0, x_i \geq 0$, and by zero-efficiency, $y_i \cdot x_i = 0$ for all $i, i = 1, \cdots, n$. By (14), $y = Mx + q$, so that x solves $LCP(q, M)$ (6) - (9).

Suppose x is a solution of $LCP(q, M)$ (6) - (9). Then $y_i \geq 0, x_i \geq 0$, and by complementarity, $y_i \cdot x_i = 0$. Thus x solves (14), and for $i = 1, 2, \cdots, n, f_i(x) = 0$. Since $y_i \geq 0, x_i \geq 0$ and $f_i(x) = y_i \cdot x_i$, it is impossible to have $z \in X$ with $f_i(z) < 0$. Hence $x \in X_E$, and x is zero-efficient. $\square$

3 Results

The implications of the above formulations are now investigated. All of the machinery of multiple objective optimization may be brought to bear on the problem of finding zero efficient solutions of the quadratic multiple objective programming problem. Since the above quadratic functions are, in general, neither convex nor concave, global approaches are appropriate and may be required. It is possible to use the specialized polynomial method of Kostreva, Ordoyne and Wiecek [3], or the integral global optimization approach as in Kostreva, Zheng and Zhuang [5,6]. Other global methods will also apply, after appropriate scalarization of the objective functions, and these methods will benefit from the simplicity of the linearly constrained feasible solution set. It is possible also to transfer results from game theory to the quadratic multiple objective problem. The two subjects are generally investigated separately, and the techniques of achieving mathematical theories are similar but not the same. Hence some unexpected results are possible.

Theorem 3.1. *There exists at least one zero efficient solution to (13)-(14) when*

$$M = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix} \quad q = \begin{bmatrix} -e \\ -e \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} r \\ c \end{bmatrix} \in \mathbf{R}^m \times \mathbf{R}^n.$$

Proof: Follows from the existence of a solution to the bimatrix game. □

It is difficult to see how to achieve a proof of this existence result with the current theory of multiple objective programming. The constraint set is linear, but not necessarily compact. The objective functions, being neither concave nor convex, are also not easily handled by current theory. Even though the objective functions are bounded below, general quadratic functions may exhibit hyperbolic behavior and may have asymptotes, thereby causing complications for proving existence.

Theorem 3.2: *If*

$$M = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix} \quad q = \begin{bmatrix} -e \\ -e \end{bmatrix} \quad \textit{and} \quad x = \begin{bmatrix} r \\ c \end{bmatrix} \in \mathbf{R}^m \times \mathbf{R}^n.$$

and the linear system is nondegenerate, then there exists a discrete efficient set containing an odd number of efficient solutions to (13)-(14).

Proof: Due to the structure of the problem, only zero efficient solutions are efficient. Each zero efficient solution is in one-to-one correspondence with the solutions of the bimatrix game, which are of odd parity. □

It seems that this is the first instance of having a proof of the existence of an odd number of efficient points for any multiple objective programming problem. The parity property seems to be quite worthwhile investigating for more general multiple objective programming problems, and it may be that this first result sheds some light on how to proceed with such an investigation.

Next we proceed with the interpretation of the complementarity condition, which is also the interpretation of the zero-efficient condition. This condition comes in two forms, corresponding to the two players:

$$\begin{aligned} (r_i) \cdot (Ac - e)_i &= 0 \text{ for } i = 1, 2, \cdots, m. \\ (c_i) \cdot (B^T r - e)_i &= 0 \text{ for } i = 1, 2, \cdots, n. \end{aligned}$$

Looking at the first form, if player one is using a certain strategy (i.e. the r variable is positive), then the linear constraint is tight. This constraint being tight has the interpretation that the payoff achieved is maximal. This follows from equation (3). So for each strategy for player one, if the strategy is positive in the equilibrium mixed strategy, the maximal payoff must be achieved. So by taking a mixture of such strategies, the same maximal property is achieved. If the strategy is not used, the payoff need not be maximal. The analysis for player two is completely symmetrical.

By introducing the multiple objective quadratic programming model for the bimatrix game, and by searching for zero-efficient points, one sees that there is a new viewpoint for the game operations. If the multiple objective programming model is the decision making model of a single individual, this individual may be thought of as a referee. Such a referee will only accept positive values for the components of vectors r and c if the result is a maximal payoff for the corresponding player. It means that the players must be able to play optimally, not in some off-hand or errroneous way. Of course, this is completely compatible with the Nash equilibrium concept, but it is yet another way to view the Nash concept.

It is an interesting consideration that the players need not know how to optimize their play. They simply visit the referee, who solves the multiple objective quadratic programming problem. The referee can also tell them what the proportion of use of each strategy should be corresponding to each zero-efficient solution, and hence to each Nash quilibrium. Thus we see another way to approach Nash equilibria, without appeal to the Lemke Algorithm, or to the fixed point approach suggested by Nash's original work. Those two approaches, although guaranteed to find one equilibrium point, generally go no further. With the multiple objective programming solution, one may attempt to compute all or part of the efficient set, by a collection of methods which has now been established in the literature. Finding any efficient point, one finds a zero- efficient point and hence a Nash equilibrium point.

4 Conclusions

Nash Equilibria of a bimatrix game provides an interesting new multiple objective programming problem. Since the objective functions thus obtained are quadratic and bounded below by zero, and the constraint set is linear, such problems may be solved by existing techniques which handle global optimization. The theory involved in transforming the bimatrix game into a multiple objective program has been recently developed, and has potential for further application. It provides an existence proof for efficient points, and information about the structure of the efficient set, together with a new interpretation of for Nash equilibria in bimatrix games.

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The Bargaining Model for Characteristic-function Game

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Abstract. This paper deals with generalized n-person cooperative market games without side payments derived from unbalanced pure exchange economies. The economic motivation for such investigation is a problem of fair sharing rules in situation of deficit on the market of commodities. Our model allows to take into account as a conflict of interests and an acceptable compromise of them.

1 The formal model

Consider a market with a set $I = \{1, 2, \ldots, n\}$ of economical agents (countries, firms, individuals and so on) and a set $K = \{1, 2, \ldots, m\}$ of commodities. Let R^m_+ be the commodities space. The agent $i \in I$ has a supply-vector $a_i = (a^1_i, \ldots, a^m_i) \in R^m_+$ and a demand-vector $b_i = (b^1_i, b^2_i \ldots, b^m_i) \in R^m_+$. For simplicity suppose $a^k_i b^k_i = 0,\ k = 1, \ldots, m$, so if $a^k_i \neq 0$ then the agent i is a seller of a commodity k, and if $b^k_i \neq 0$ then he is a buyer of this commodity. We suppose, that utility function of an agent is additive and homogeneous with respect to commodities and so we can restrict ourself by definition of the utilities for unit quantities of commodities. Let the player i has utility-vector $u_i = (u^1_i, u^2_i \ldots, u^m_i) \in R^m_+$ as a seller, and utility vector $w_i = (w^1_i, w^2_i \ldots, w^m_i) \in R^m_+$ as a buyer. We can interpret u^k_i as a minimal price for selling of a commodity k (for example , it's production cost) by the agent i, and, analogous w^k_i – as a maximal price of a commodity k for buying. As in our model every agent has only one role (seller or buyer) we put $u^k_i = 0$ if $a^k_i = 0$ and $w^k_i > \max_{j: a^j_i > 0} w^j_i$ if $b^k_i = 0$.

Denote the market

$$M = \langle \{a_i\}_{i \in I}, \{b_i\}_{i \in I}, \{u_i\}_{i \in I}, \{w_i\}_{i \in I}, \{c_i\}_{i \in I} \rangle,$$

where $c_i \in R^1_+$ is the initial capital of the agent i.

Propose that u^k_i, w^k_i and $c_i,\ i \in I,\ k \in K$ are expressed in the same monetary units.

2 The properties of the cooperative game generated by the market

Associate the cooperative game with this market.

Let $\xi_i = (\xi_i^1, \ldots, \xi_i^m)$, $\xi_i^k \leq a_i^k$, $k = 1, \ldots, m$
$\eta_i = (\eta_i^1, \ldots, \eta_i^m)$, $\eta_i^k \leq b_i^k$, $k = 1, \ldots, m$, $i \in I$
$\sum\limits_{i=1}^{n} \xi_i^k = \sum\limits_{i=1}^{n} \eta_i^k$, $k = 1, \ldots, m$,
where ξ_i^k is the quantity of a commodity k given by the player i to another ones, and η_i^k is the quantity of commodity k received by player i.

The collection $\{(\xi_i, \ldots, \xi_n); (\eta_i, \ldots, \eta_n)\}$ is named **the distribution** and is denoted (ξ, η). The set of all distributions is denoted $D(I)$.

The subsets of I are called **coalitions** in cooperative game theory. Let $S \subset I$ be the coalition and denote

$$D(S) = \{(\xi, \eta) \in D(I) | \xi_i^k = \eta_i^k = 0,\ k = 1, \ldots, m,\ i \notin S\}.$$

If the distribution $(\xi, \eta) \in D(S)$, we name it S-**distribution** and denote (ξ^S, η^S). In accordance with distribution (ξ, η) the coalition S has the total profit

$$v(S, \xi, \eta) = \sum_{k=1}^{m} \sum_{i \in S} \eta_i^k w_i^k - \sum_{k=1}^{m} \sum_{i \in S} \xi_i^k u_i^k$$

Denote $v(S) = \max\limits_{(\xi,\eta) \in D(S)} v(S, \xi, \eta)$. Put $v(\emptyset) = 0$

Then there is the cooperative game $\Gamma(M) =< I, v >$. If $a_i^k = b_i^k = 0$ for $i \in I$, $k \neq k_0$, then we name this game one-product and denote Γ^{k_0}. We will mark all notions for Γ^{k_0} by the letter k_0 $(D^{k_0}(I), D^{k_0}(S), v^{k_0}(S)$ and so on)

S_0 far as the distribution of every commodity can be choose independently from others it is easy to verify the following.

Proposition 1.

$$v(S) = \sum_{k=1}^{m} v^k(S)$$

Let us research one-product game in detail. We will omit the marker k_0 in this part of the paper.

Let $N_1 \subset N$ is a set of a sellers, and $N_2 \subset N$ is a set of a buyers of the product, and $N_1 \cup N_2 = N$.

For simplicity, we suppose that

$$N_1 = \{1, \ldots, n_0\}, N_2 = \{n_0 + 1, \ldots, n\},\ 1 < n_0 < n,$$

$$u_1 \leq u_2 \leq \ldots u_{n_0}$$

$$w_{n_0+1} \leq w_{n_0+2} \leq \cdots \leq w_n$$

The utilities u_k, $k > n_0$, and w_k, $k < n_0$ are not considered.

$$D(I) = \left\{ \begin{array}{rcl|l} (\xi,\eta) & = & & 0 \le \xi_i \le a_i \\ & = & (\xi_1,\ldots,\xi_{n_0}; & 0 \le \eta_j \le b_j \\ & & \eta_{n_0+1},\ldots,\xi_n) \in R^n & \sum_{i=1}^{n_0} \xi_i = \sum_{j=n_0+1}^{n} \eta_j \end{array} \right.$$

$$D(S) = D(I) \cap \{x \in R^n | x_i = 0 \quad i \notin S\}$$

$$v(S,\xi,\eta) = \sum_{j\in S\cap N_2} \eta_j w_j - \sum_{i\in S\cap N_1} \xi_i u_i$$

$$v(S) = \max_{(\xi,\eta)\in D(S)} v(S,\xi,\eta)$$

If $S \cap N_1 = \emptyset$ or $S \cap N_2 = \emptyset$ then $D(S) = \{(0,\ldots,0)\}$, and $v(S) = 0$. Thus $v(\{i\}) = 0$. The coalition S is named **active**, if $v(S) \ne 0$.

The game Γ is called

(a) the game with deficit, if $\sum_{i=1}^{n_0} a_i < \sum_{j=n_0+1}^{n} b_j$

(b) the balanced game, if $\sum_{i=1}^{n_0} a_i = \sum_{j=n_0+1}^{n} b_j$

(c) the game with surplus, if $\sum_{i=1}^{n_0} a_i > \sum_{j=n_0+1}^{n} b_j$

3 The trivial S-distribution

For every active coalition $S \subseteq I$ we will describe now the algorithm of calculating a S-distribution $(\bar{\xi}^S, \bar{\eta}^S)$, that we will name a **trivial S-distribution** (we use the terminology from [1]).

Step 1. Let i_1, j_1 are such that

$$u_{i_1} = \min_{i\in S\cap N_1} u_i,\ w_{j_1} = \max_{j\in S\cap N_2} w_j$$

$$x_{i_1,j_1} = \begin{cases} \min(a_{i_1} b_{j_1}) & \text{if } u_{i_1} \le w_{j_1} \\ 0 & \text{otherwise} \end{cases}$$

Step 2. If $x_{i_1j_1} = 0$ then "stop"

If $x_{i_1j_1} = a_{i_1}$ then $S \Rightarrow S \setminus \{i_1\}$, $b_{j_1} = b_{j_1} - a_{j_1}$

If $x_{i_1j_1} = b_{j_1}$ then $S \Rightarrow S \setminus \{j_1\}$, $a_{i_1} = a_{i_1} - b_{j_1}$

Step 3. If $S \cap N_1 = \emptyset$ or $S \cap N_2 = \emptyset$ then "stop"

else goto step 1 with the new S, a_{i_1}, b_{j_1}

Let l be a number of iteration on that the process stops. It will occur in one of the three cases:
(a) $x_{i_l j_l} = 0$ and $u_{i_l} > w_{i_l}$
(b) $S \cap N_1 = \emptyset$;
(c) $S \cap N_2 = \emptyset$.
We put

$i(S) = i_{l-1},\ j(S) = j_{l-1}$ in the case (a)

$i(S) = i_l,\ j(S) = j_l$ in the case (b),(c)

$$S_1 = \{i \in S \cap N_1 | i \leq i(S)\}$$

$$S_2 = \{j \in S \cap N_2 | j \geq j(S)\}$$

The pair $(i(S), j(S))$ is named S-**marginal.**
Let

$$\overline{\xi}_i^S = \sum_{j \in S_2} x_{ij}, \quad i \in S_1$$

$$\overline{\eta}_j^S = \sum_{i \in S_1} x_{ij}, \quad j \in S_2$$

$$\overline{\xi}_i^S = \overline{\eta}_j^S = 0 \text{ for other } i, j \in N$$

It is obvious that $(\overline{\xi}^S, \overline{\eta}^S)$ is S-distribution and we call it **trivial.**
Proposition 2. The trivial S-distribution has the following properties:
(a) $\overline{\xi}_i^S = a_i, \quad i \in S_1,\ i \neq i(S)$

(b) $\overline{\eta}_j^S = b_j, \quad j \in S_2,\ j \neq j(S)$

(c) $\sum\limits_{i \in S \cap N_1} \overline{\xi}_i^S = \sum\limits_{j \in S \cap N_2} \overline{\eta}_j^S = \min\left(\sum\limits_{i \in S_1} a_i, \sum\limits_{j \in S_2} b_j\right)$

(d) $\overline{\xi}_{i(S)}^S = a_{i(S)}$ or $\overline{\eta}_{j(S)}^S = b_j(S)$ but both equalities are true only if $\sum\limits_{i \in S_1} a_i = \sum\limits_{j \in S_2} b_j$

(e) $v(S, \overline{\xi}^S, \overline{\eta}^S) = \max\limits_{(\xi,\eta) \in D(S)} v(S, \xi, \eta) = v(S)$

Proof. Properties (a)-(d) obviously follows from algorithm. Let consider (e). Denote $x^S = ||x_{ij}||_{i,j \in S}$ the plan of exchanges between the members of coalition S, where x_{ij} is a quantity of product transmitted from agent i to agent j. If $i \notin S \cap N_1$, $j \notin S \cap N_2$, or $u_i > w_j$ then $x_{ij} = 0$.

It is clear, that

$$\xi_i^S = \sum_j x_{ij},\ \eta_j^S = \sum_i x_{ij}$$

$$v(S,\xi,\eta) = \sum_{j \in S \cap N_2} \eta_j^S w_j - \sum_{i \in S \cap N_1} \xi_i^S u_i = \sum_{i,j \in S} x_{ij}(w_j - u_i)$$

We have an optimization problem:

$$\max \sum_{i,j \in S} x_{ij}(w_j - u_i)$$

$$\sum_i x_{ij} \leq b_j,\ j \in S \cap N_2$$

$$\sum_j x_{ij} \leq a_i,\ i \in S \cap N_1$$

$$x_{ij} > 0, \quad i,j \in S$$

$$x_{ij} = 0, \text{ if } i \in S \setminus N_1, \text{ or } j \in S \setminus N_2, \text{ or } u_i > w_j$$

Let the plan $\overline{x}^S$ be a solution of this problem. $\overline{\xi}_i^S = \sum_j \overline{x}_{ij}; \overline{\eta}_j^S = \sum_i \overline{x}_{ij}$.

It is easy to show, that $(\overline{\xi}_i^S, \overline{\eta}_j^S)$ is the trivial S-distribution. In such form our problem is the sort of transport one, and $\overline{x}^S$ is so-called North-West plan for it. This plan is optimal in our case, although that is not so for arbitrary transport problem . Q.E.D.

Remark, that if all positive $c_{ij} = w_j - u_i$ are different then trivial S-distribution is unique solution of the corresponding optimization problem, and so we have a rather strict way of achievement of maximal total profit for coalition.

For the market with m commodities the trivial I-distribution $(\overline{\xi}, \overline{\eta})$ is a collection of similar distributions for every commodities, that is $(\overline{\xi}, \overline{\eta}) = \{(\overline{\xi}^k, \overline{\eta}^k)\}_{k=1}^m$, and it is unique optimal distribution for whole market.

4 Balanced distribution and balanced prices

Now consider the market with the fixed price-vector $p = (p^1, \ldots, p^m)$. The prices with the distribution define **allocation** $x = (x_1, \ldots, x_n)$ of the total profit, where

$$x_i = x_i(p,\xi,\eta) = \sum_{k=1}^m \xi_i^k (p^k - u_i^k) + \sum_{k=1}^m \eta_i^k (w_i^k - p^k)$$

$$H(\Gamma) = \{x(p,\xi,\eta);\ (\xi,\eta) \in D(I)\}$$

is the set of allocations.

For one-product game we have

$$x_i(p, \xi, \eta) = \xi_i(p - u_i), \ i \in N_1$$

$$x_j(p, \xi, \eta) = \eta_j(w_j - p), \ j \in N_2$$

We call the allocation vector x **balanced** if

$$\sum_{i \in S} x_i \geq v(S), \quad \text{for all } S \subset I, \ S \neq I$$

The corresponding distribution (ξ^*, η^*) and the price-vector p^* also named balanced.

Theorem 1. ([1]) The trivial I-distribution $(\bar{\xi}, \bar{\eta})$ is balanced one. The balanced price-vector p^* can be chosen arbitrary from the set

$$P^* = \left\{ (p^1, \ldots, p^m) | u^k_{i^k(I)} \leq \underline{p}^k \leq p^k \leq \bar{p}^k \leq w^k_{j^k(I)} \right\}$$

It is desirable to connect the balanced prices and the balanced distribution with competitive equilibrium in the perfect market with transferable utilities and money.

Put

$$U_i(\xi, \eta) = \sum_{k=1}^{m} \left[u^k_i (a^k_i - \xi^k_i) + w^k_i \eta^k_i + p^k \xi^k_i - p^k \eta^k_i \right]$$

$$B(p, c) = \left\{ (\xi, \eta) \in D(I) | \sum_{k=1}^{m} (p^k \eta^k_i - p^k \xi^k_i) \leq c_i \right\}$$

$B(p, c)$ is named a **budget set** .

Theorem 2. A vector C^* exists such that for $c_i \geq c^*_i$, $i = 1, \ldots, n$ the trivial distribution $(\bar{\xi}, \bar{\eta})$ and corresponding balanced price-vector p^* form a competitive equilibrium in the market M with utility functions $U_i(\xi, \eta)$, $i = 1, \ldots, n$ and budget set $B(p, c)$.

5 Computer experiments and concluding remarks

The trivial distribution is accessible only for perfect market, and the problem remains to look for another suitable distributions, for example another balanced distributions.

Let us consider the following optimization problem

$$\text{extr} f(\xi, \eta)$$

$$\sum_{i\in S}\sum_{k=1}^{m}\left(\xi_i^k(p^k-u_i^k)+\eta_i^k(w_i^k-p^k)\right)\geq v(S)\quad \text{for } S\subset I,\ S\neq I$$

$$(\xi,\eta)\in D(I);\ p^k\geq 0,\ k=1,\dots,m$$

This problem is rather difficult because we deal with the large system of nonlinear inequalities. Our purpose of the optimization would be one of the following:

(a) $\max v(I,\xi,\eta)$; (b) $\min v(I,\xi,\eta)$; (c) $\min\limits_{k=1}^{m} \alpha_k p^k$; and so on.

We know the answer in the case (a) – trivial distribution and corresponding prices. In other cases we have the results of computer experiments. The problem becomes linear if we consider vector p as a parameter-vector. But the method of linear programming is effective only for a small n. The another way is one of the methods of random search. But the calculating difficulties become hopeless already for a market with 15-20 agents.

The next idea is the introduction of a coalition structure. The existence of such structure could be explained by communicating restrictions or incomplete information for real market.

Our computer system includes the complex of procedures which allow for given market M and determined coalition structure to construct trivial distribution and the set of corresponding balanced prices, to calculate $v(S)$ for all possible coalitions, to build another balanced distributions, to solve in dialogue would the given price-vector p be balanced for any distribution and to answer another questions about given market M.

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CALCULUS OF CHOQUET BOUNDARIES USING PARETO SETS

by

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Abstract. One of the aim of this paper is to present new properties for the efficient (Pareto) point sets in separated locally convex spaces. Thus, for any non-empty and compact subset, the coincidence result between the set of corresponding Pareto points with respect to a convex cone and the Choquet boundary with respect to a convenient cone of real continuous functions established by us in a previous research work gives interesting topological properties of Pareto sets. An important result shows immediate applications of spline optimal interpolation in multiple criteria decision making problems and two numerical examples based on spline functions in H-locally convex spaces illustrate the possibility of calculus for Choquet boundaries through the agency of Pareto sets and conversely.

Keywords. Efficient point, Pareto set, Choquet boundary, H-locally convex space, spline function.

1. Introduction

One of the main direction of research concerning with the efficient point (Pareto) sets is the study of the properties for these sets. Following it, in [2] we established an important result of coincidence between the Pareto set corresponding to a non-empty, compact subset of an ordered Hausdorff locally convex space and the Choquet boundary of such a set as this with respect to the convex cone of real, continuous and increasing functions (Theorem 2.3). This is a new connection between two great fields of Mathematics : Vector Optimization and Potential Theory. At the same time, this coincidence result offers a new way for the investigations of Pareto sets using the Choquet boundaries and conversely (see Theorem 2.3 and Theorem 2.4). Besides these theorems, here we give two theorems (Theorem 2.1 and Theorem 2.2) which suggest new ways for the study of the efficient points in linear spaces ordered by convex cones and we specify the sets of solutions for some vectorial optimization problems in H-locally convex spaces through the agency of spline functions (Theorem 2.5) for which we indicate their expressions in two numerical examples.

2. Pareto Sets and Choquet Boundaries

Let (X,τ) be a Hausdorff locally convex space ordered by a convex cone K. If A is a non-empty subset of X and $a_0 \in A$, then

Definition 2.1. We say that a_0 is an efficient (Pareto minimum) point for A with respect to K, in notation, $a_0 \in MIN_K(A)$ if it satisfies one of the following equivalent conditions :

(i) $A \cap (a_0 - K) \subseteq \{a_0\} + K$; (ii) $K \cap (a_0 - A) \subseteq -K$;

(iii) $(A+K) \cap (a_0 - K) \subseteq \{a_0\} + K$; (iv) $K \cap (a_0 - A - K) \subseteq -K$

In a similar manner one defines the set of Pareto maximum points by $MAX_K(A) = MIN_{-K}(A)$. $MIN_K(A)$ will be called the Pareto (minimum) set of A with respect to K.

The results below are straightforward but suggest several possibilities for the study of the efficient points.

Theorem 2.1. The following conditions are equivalent:

(i) $a_0 \in MIN_K(A)$

(ii) $a_0 \in MIN_K(A+K)$

(iii) a_0 is a fixed point for one of the following multifunctions defined using the (conical) sections of A : $F_1, F_2, F_3, F_4 : A \to A$,

$F_1(t) = \{a \in A : A \cap (a-K) \subseteq \{t\} + K\}$, $F_2(t) = \{a \in A : A \cap (t-K) \subseteq \{a\} + K\}$,

$F_3(t) = \{a \in A : (A+K) \cap (a-K) \subseteq \{t\} + K\}$, $F_4(t) = \{a \in A : (A+K) \cap (t-K) \subseteq \{a\} + K\}$, that is, $a_0 \in F_i(a_0)$ for some $i = \overline{1,4}$

Theorem 2.2. If K is pointed, i.e., $K \cap (-K) = \{0\}$ then the next statements are equivalent:

(i) $a_0 \in MIN_K(A)$;

(ii) $A \cap (a_0 - K) = \{a_0\}$;

(iii) $K \cap (a_0 - A) = \{0\}$

(iv) $A \cap (a_0 - \varepsilon - K) = \emptyset$ whenever $\varepsilon \in K \setminus \{0\}$, that is, a_0 is an ε-efficient point of A for any ε in $K \setminus \{0\}$;

(v) $A \cap (a_0 - K \cap V - K) = \emptyset$ for some neighbourhood V of the origin;

(vi) a_0 is a critical point for the generalized dynamical system $F : A \to A, F(a) = A \cap (a-K)$, that is $F(a_0) = \{a_0\}$.

In all further considerations we suppose that K is a closed, convex, pointed cone. Under these assumptions, on the vector space X we consider the usual order relation $\leq_K$ associated with K as follows : for $x, y \in X$ one defines $x \leq_K y$ iff there exists $k \in K$ with $y = x + k$.

Clearly, this order relation on X is closed, that is, the set G_K given by $G_K = \{(x,y) \in X \times X : x \leq_K y\}$ is a closed subset of $X \times X$.

Let now A be a non-empty compact subset of X. Before we give our main result which is the basis for the examples, we recall the definition of the Choquet boundary for A with respect to a convex cone of continuous functions on A. Thus, we remember that if S is a convex cone of real continuous functions on A such that the constant functions on A belong to S, it is min-stable (i.e., for every $f_1, f_2 \in S$ it follows $inf(f_1, f_2) \in S$) and it separates the points of A, then on the set $M_+(A)$ of all positive Radon measures on A we associate the following order relation: if $\mu, \nu \in M_+(A)$, then $\mu \leq_S \nu$ iff $\mu(s) \leq \nu(s)$ for all $s \in S$.

Following [4] a measure $\mu \in M_+(A)$ is minimal with respect to the above order relation if for any continuous function $f: A \rightarrow R$ we have $\mu(Q_S f) = \mu(f)$ where by definition

$$Q_S f = inf\{s \in S: f \leq s\}.$$

Particularly, if $a \in A$, then the Dirac measure $\varepsilon_a \in M_+(A)$ defined by $\varepsilon_a(f) = f(a)$ for every real continuous function f on A is minimal iff $\varepsilon_a(Q_S f) = \varepsilon_a(f)$, that is, $Q_S f(a) = f(a)$ $(a \in A)$ for every continuous function $f: A \rightarrow R$.

The set of all points $a \in A$ such that ε_a is a minimal measure with respect to $\leq_S$ is named the Choquet boundary of A with respect to S and it is denoted by $\partial_S A$.

Hence, if $C(A)$ is the set of all real continuous functions on A, then

$$\partial_S A = \{a \in A: Q_S f(a) = f(a), \ \forall f \in C(A)\}.$$

An important connection established by us in [2] between Vector Optimization and Potential Theory is the next coincidence result of Pareto sets and Choquet boundaries in separated locally convex spaces.

Theorem 2.3.[2]. If K is a convex, closed, pointed cone and A is a non-empty, compact subset in X, then $MIN_K(A)$ coincides with the Choquet boundary of A with respect to the convex cone of all real, continuous functions on A which are increasing with respect to the order relation induced by K. Consequently, the set $MIN_K(A)$ endowed with the trace topology τ_A generated by the locally convex topology τ of X is a Baire space. Moreover, if (A,τ_A) is metrizable, then $MIN_K(A)$ is a G_δ set of (A,τ_A).

Remark 2.1. Compactness is the strongest demand on a given set which ensures the existence of the efficient points with respect to a convex cone, the largest class of convex cones ensuring the existence of the efficient points for compact (cone-compact) sets in Hausdorff topological vector spaces being introduced in [9]. So, if we want to obtain existence criteria in a less restrictive class of sets we must impose stronger conditions on the cone (for recent results in this direction see, for example, [3] and [7], respectively). Since there are a lot of cases in which to discover the efficient points sets is easier than find the Choquet boundaries indicated by Theorem 2.3, we think that it is interesting to know if the Theorem 2.3 can be considered as a starting point for a modality of extension the Choquet boundary to non-compact sets which have efficient points. For all we know, this is an open problem.

Remark 2.2. In general, under the hypotheses of Theorem 2.3, the set $MIN_K(A)$ coincides with the Choquet boundary of A only with respect to the convex cone of all real, continuous and K-increasing functions on A. Thus, for example, if A is a non-empty, compact and convex subset in X then, taking into account the Theorem 2.2 in the first paragraph of Chapter 2 [1], the Choquet boundary of A with respect to the convex cone of all real, continuous and concave functions on A coincides with the set of all extreme points for A, that is, with the set of $a \in A$ such that if $a_1, a_2 \in A$, $\lambda \in (0,1)$ and $a = \lambda a_1 + (1-\lambda) a_2$, then $a = a_1 = a_2$. But it is easy to see that, even in finite dimensional cases, an extreme point for a compact convex set is not necessary an efficient point and conversely.

In the same background specified before the Theorem 2.3, a closed subset B of a non empty compact set A in X is called <u>S-absorbent</u> if $b \in B$ and $\mu \leq_S \varepsilon_b$ implies that $\mu(A \setminus B) = 0$. If one denotes $A\_ = \{a \in A : \exists s \in S \text{ with } s(a) < 0\}$, then there exists a topology on A such that the closed sets in this topology coincide with the S-absorbent subsets of $A\_$ or with A. Such a topology as this is usually named <u>the Choquet topology</u> on $\partial_S A$. Taking into account Theorem 2.3 and theorems 2.11, 2.12 in the second Chapter of [1] , we have

Theorem 2.4. <u>If S is the convex cone of all real, continuous - increasing functions on a non-empty, compact subset A of X, then the sets $MIN_K(A)$ and $MIN_K(A) \cap \{a \in A : s(a) \leq 0\}$ are compact in the Choquet topology whenever $s \in S$</u>.

3. Applications

In [5] we introduced the concept of spline function in H-locally convex spaces which were defined by T. Precupanu [8] as Hausdorff locally convex spaces with the seminorms satisfying the parallelogram law and in [6] we presented the best simultaneous and vectorial approximation in H-locally convex spaces by the linear subspaces of spline functions. Here we consider some vectorial optimization problems in H-locally convex spaces for which our splines give the only elements of the corresponding Choquet boundaries and, in accordance with Theorem 2.3, these are the only solutions. Following this way, one offers concrete examples for applications of spline approximations and optimal interpolations to solve appropriate multiple criteria decision making programs.

Let $(X,\mathcal{P}=\{p_\alpha:\alpha\in I\})$ be a H - locally convex space with each seminorm p_α being induced by a scalar semiproduct $(.,.)_\alpha(\alpha\in I)$ and M a closed linear subspace of X for which there exist a H - locally convex space $(Y,\mathcal{Q}=\{q_\alpha:\alpha\in I\})$ with every seminorm q_α generated by a scalar semiproduct $<.,.>_\alpha(\alpha\in I)$ and a linear continuous operator U: $X\text{->}Y$ such that $M=\{x\in X:(x,y)_\alpha=<Ux,Uy>_\alpha,\forall y\in X,\alpha\in I\}$.

The space of spline functions with respect to U was defined in [5] as the U - orthogonal of M, that is, $M^\perp=\{x\in X:<Ux,U\varsigma>_\alpha=0,\forall\varsigma\in M,\alpha\in I\}$. Clearly, $M^\perp$ is the orthogonal of M in the H - locally convex sense.

If we consider R^I endowed with the topology generated by the family $\mathcal{S}=\{s_i:i\in I\}$ of seminorms defined by $s_i(x)=|x_i|$ for every $x=(x_i)\in R^I$ and we denote $R^I_+=\{(x_i)\in R^I:x_i\geq 0,\forall i\in I\}$, then, by virtue of Theorem 2.3 combined with the splines properties of best approximation and optimal interpolation given in [5], [6], we obtain the result below.

Theorem 3.1. If $X_1=M\oplus M^\perp$ is the direct sum of $M,M^\perp$ and for any $x\in X_1$ we denote its projection onto $M^\perp$ by s_x, then for every non-empty compact set A in X the corresponding Choquet boundary of each of the following sets for which we search the efficient points with respect to the convex cone of all real,continuous increasing functions on it is singleton and coincides with the set of solutions :

(i) $MIN_{R^I_+}\{(q_\alpha(U(\eta-s))):\eta\in X_1\cap A\ and\ \eta-\sigma\in M\}=\begin{cases}\{(q_\alpha(U(\sigma-s)))\}\ if\ \sigma\in A\\ \varnothing\ if\ \sigma\notin A\end{cases},\ \forall\sigma,s\in M^\perp;$

(ii) $MIN_{R^I_+}\{(q_\alpha(U(\eta-x))):\eta\in M^\perp\cap A\}=\begin{cases}\{(q_\alpha(U(s_x-x)))\}\ if\ s_x\in A\\ \varnothing\ if\ s_x\notin A\end{cases};$

(iii) $MIN_{R^I_+}\{(p_\alpha(x-y)):y\in M^\perp\cap A\}=\begin{cases}\{(p_\alpha(x-s_x))\}\ if\ s_x\in A\\ \varnothing\ if\ s_x\notin A\end{cases};$

(iv) $MIN_{R^I_+}\{(q_\alpha(Uy)):y-x\in M\cap A\}=\begin{cases}\{(q_\alpha(Us_x))\}\ if\ x-s_x\in A\ or\ s_x-x\in A\\ \varnothing\ if\ x-s_x,\ s_x-x\notin A\end{cases};$

In the following two numerical examples we specify the expressions of the splines and we show that our abstract construction of spline functions in H-locally convex spaces is useful also to find orthogonal decompositions for H-locally convex spaces which, in general, are not easy to be obtained .

Example 3.1

Let

$X = H^m(R) = \{f \in C^{m-1}(R): f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2_{loc}(R)\}$, $m \geq 1$ endowed with the H - locally convex topology generated by the scalar semiproducts

$$(x,y)_k = \sum_{h=0}^{m-1}[x^{(h)}(k)y^{(h)}(k)+x^{(h)}(-k)y^{(h)}(-k)]+\int_{-k}^{k}x^{(m)}(t)y^{(m)}(t)dt \quad , \; k=0, \; 1, \; 2, \; \ldots.$$

and $Y = L^2_{loc}(R)$ with the H - locally convex topology induced by the scalar semiproducts

$$<x,y>_k = \int_{-k}^{k}x(t)y(t)dt \quad , \; k=0, \; 1, \; 2, \ldots$$

If U: $X->Y$ is the derivation operator of order m, then $M = \{x \in H^m(R): x^{(h)}(v) = 0, \forall h = \overline{0,m-1}\}$ and

$$M^{\perp} = \{s \in H^m(R): \int_{-k}^{k}s^{(m)}(t)x^{(m)}(t)dt = 0, \; \forall x \in M, \, k = 0, 1, 2, \ldots\}.$$

We proved in [5] that

$M^{\perp} = \{s \in H^m(R): s_{(v,v+1)}$ is a polinomial function of degree 2m-1 at most$\}$ and if $y = (y_v), y' = (y'_v), y'' = (y''_v), \ldots, y^{(m-1)} = (y_v^{(m-1)})$ are m sequences of real numbers, then there exists an unique spline $S \in M^{\perp}$ satisfying the following conditions of interpolation: $S^{(h)}(v) = y_v^{(h)}$ whenever $h = \overline{0,m-1}$ and $v \in Z$. Any spline function such as this is defined by

$$S(x) = p(x) + \sum_{h=0}^{m-1}c_1^{(h)}(x-1)_+^{2m-1} + \sum_{h=0}^{m-1}c_2^{(h)}(x-2)_+^{(2m-1)} + \ldots + \sum_{h=0}^{m-1}c_0^{(h)}(-x)_+^{(2m-1)} + \ldots$$

where $u_+ = \frac{|u|+u}{2}$ for every real number u, p is a polinomial function of degree $2m-1$ at most perfectly determined by the conditions $p^{(h)}(0) = y_0^{(h)}$ and $p^{(h)}(1) = y_1^{(h)}$ for all $h = \overline{0,m-1}$ and the coefficients $c_v^{(h)}(h = \overline{0,m-1}, v \in Z)$ are successively given by the cardinal interpolation. Therefore, for every function $f \in H^m(R)$, there exists an unique function denoted by $S_f \in M^{\perp}$ such that $S_f^{(h)}(v) = f^{(h)}(v), \forall h = \overline{0,m-}$ and $v \in Z$. Hence M and $M^{\perp}$ realize an orthogonal decomposition of the space $H^m(R)$.

Example 3.2

Let

$X=F_m=\{f\in C^{m-1}(R): f^{(m-1)}$ is locally absolutely continuous and $f^{(m)}\in L^2(R)\}$ with the H-locally convex topology induced by the scalar semiproducts

$(x,y)_v = x(v)y(v)+\int_R x^{(m)}(t)y^{(m)}(t)dt,\ v\in Z.$ $Y=L^2(R)$ with the topology generated by the inner product $<x,y>_v=\int_R x(t)y(t)dt,\ v\in Z.$ and $:X\to Y$ be the derivation operator of order m. Then $M=\{x\in F_m: x(v)=0$ for all $v\in Z\}$ and

$$M^{\perp}=\{s\in F_m:\int_R s^{(m)}(t)x^{(m)}(t)dt=0 \text{ for every } x\in M\}.$$

In a similar manner as in Example 3.1 it may be prooved that $M^{\perp}$ coincides with the class of piecewise polynomial functions of order 2m (degree 2m-1) at most) having their knots at the integer points. Moreover, for every function f in F_m there exists an unique spline function $S_f\in M^{\perp}$ which interpolates f on the set Z of all integer numbers, that is, S_f satisfies the equalities $S_f(v)=f(v)$ for every $v\in Z$, being defined by

$$S_f(x)=p(x)+a_1(x-1)_+^{2m-1}+a_2(x-2)_+^{(2m-1)}+\ldots+a_0(-x)_+^{(2m-1)}+a_{-1}(-x-1)_+^{2m-1}+\ldots$$

where u_+ has the same signification as in Example 3.1, the coefficients $a_v(v\in Z)$ are successively and completely determined by the interpolation conditions $S_f(v)=f(v), v\in Z\setminus\{0,1\}$ and p is a polynomial function satisfying the conditions $p(0)=f(0)$ and $p(1)=f(1)$. The uniqueness of S_f is ensured by Theorem 2 in [5]. Thus, M and $M^{\perp}$ give an orthogonal decomposition of the space F_m and, as in the preceding example, $M^{\perp}$ is simultaneous and vectorial proximinal [6] with respect to the family of seminorms generated by the scalar semiproducts defined above.

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Generalized Mond-Weir Duality for Multiobjective Nonsmooth Programming

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Abstract. A general Mond-Weir dual for nonlinear nonsmooth multiobjective programming problem is introduced and some duality results are given.

Keywords: Nonsmooth programming, efficient and properly efficient solutions, optimality conditions, duality theorem

1. Introduction

Diewert [4] developed some very simple necessary conditions for nonsmooth single- objective programming problems and showed that these conditions are also sufficient in case when the objective function is pseudoconcave and the constraint functions are quasiconvex without assuming that one-sided directional derivatives exist. In his paper he uses classical Dini-derivatives. Recently, some optimality and duality results for multiobjective nonsmooth programming problems were obtained. In Bhatia and Aggarwal [3], on the line of Egudo [5], the results of Diewert [4] are extended for multiobjective nonsmooth programming problem where the concept of efficient solution and concept of proper efficient solution are studied. Weir and Mond [9], by considering the concept of weak minima extended different scalar duality results to multiobjective programming problems. They established a number of weak , strong and converse duality theorems under a variety of generalized convexity conditions. On this line and using the approach of Diewert [4], in Bector, Bhatia and Jain [2], by considering the concept of weak maximum, necessary and sufficient conditions, for a nonsmooth multiobjective programming problem, have been developed. Also the duality results have been obtained for Wolfe dual in terms of classical Dini-derivative. In the present paper, we consider duality results for a generalized Mond-Weir dual for a nonsmooth multiobjective programming problem.Thus we obtain generalizations for some results from Egudo, Diewert and Bhatia and Aggarwal.

2. Definitions and Preliminaries

Now we consider the following nonsmooth multiobjective programming problem.

(P) $\quad$ *Maximize* $f(x)$
subject to $x \in X$

where $X = \{x \in X_0; g(x) \leqq 0\}$, X_0 is a convex subset of R^n, $f : X_0 \longrightarrow R^k$, $g : X_0 \longrightarrow R^m$ with $f(x) = (f_1(x), ..., f_k(x))$, $g(x) = (g_1(x), ..., g_m(x))$, are nondifferentiable proper functions.

Definition 2.1 (Diewert [4]) A function $h : R^n \longrightarrow R$ is said to be a proper function if $h(x) < +\infty$ for all $x \in R^n$ and $h(x) > -\infty$ for at least one $x \in R^n$.

Definition 2.2 (Diewert [4]) The set of feasible directions at a point $x \in R^n$ for (P) is defined as the set

$F(x) = \{v/v \in R^n, v^T v = 1,$ there exists $\delta > 0$ such that $g(x + tv) \leqq 0$, $0 \leqq t \leqq \delta \}$

Definition 2.3 (Diewert [4]) Let $h : R^n \longrightarrow R$. Let $x \in R^n$, $v \in R^n$ with $v^T v = \sum_{i=1}^n v_i^2 = 1$. The Dini derivatives of h at x in the direction v are defined as

$$D_v^{+u} h(x) = \sup_{\{t_n\}} \lim_{n\to\infty} \left\{ \frac{h(x + t_n v) - h(x)}{t_n} ; 0 < t_n \leqq \frac{1}{n} \right\}$$

$$D_v^{+l} h(x) = \inf_{\{t_n\}} \lim_{n\to\infty} \left\{ \frac{h(x + t_n v) - h(x)}{t_n} ; 0 < t_n \leqq \frac{1}{n} \right\}$$

$$D_v^{-u} h(x) = \sup_{\{t_n\}} \lim_{n\to\infty} \left\{ \frac{h(x - t_n v) - h(x)}{-t_n} ; 0 < t_n \leqq \frac{1}{n} \right\}$$

$$D_v^{-l} h(x) = \inf_{\{t_n\}} \lim_{n\to\infty} \left\{ \frac{h(x - t_n v) - h(x)}{-t_n} ; 0 < t_n \leqq \frac{1}{n} \right\}$$

Infinite limits are also allowed in the above definitions. Here $D_v^{+u} h(x)$ is the upper right derivative, $D_v^{+l} h(x)$ is the lower right derivative, $D_v^{-u} h(x)$ is the upper left derivative, $D_v^{-l} h(x)$ is the lower left derivative of h evaluated at x in the direction of v.

According to Diewert [4] and Bector et al. [2] we have

$D_v^{+u} h(x) \geqq D_v^{+l} h(x)$; $D_v^{-u} h(x) \geqq D_v^{-l} h(x)$;

$D_v^{-u} h(x) = -D_{-v}^{+l} h(x)$; $D_v^{-l} h(x) = -D_{-v}^{+u} h(x)$;

$D_v^{+u}(-h(x)) = -D_v^{+l} h(x)$; $D_v^{-u}(-h(x)) = -D_v^{-l} h(x)$;

$D_v^{+u}(\alpha_1 h_1(x) + \alpha_2 h_2(x)) \leqq \alpha_1 D_v^{+u} h_1(x) + \alpha_2 D_v^{+u} h_2(x)$,

for $\alpha_1, \alpha_2 \in R$ and $h_1, h_2 : X_0 \longrightarrow R$.

Definition 2.4 (Aggarwal and Bhatia [1]) A point $x^0 \in X$ is said to be an efficient solution of (P) if there exists no $v \in R^n$ satisfying $v^T v = 1, x^0 + tv \in X$, $0 < t \leqq \delta$, such that

$f_r(x^0+tv) > f_r(x^0)$, and $f_i(x^0+tv) \geqq f_i(x^0)$ for $i = 1,2,...,k,\ i \neq r$.

Definition 2.5 (Bhatia and Aggarwal [3]) A point $x^0 \in X$ is said to be a properly efficient solution of (P) if it is efficient and there exists a scalar $M > 0$ such that for each $i \in K = \{1,2,...,k\}$ and $v \in R^n$ satisfying $v^T v = 1$, $0 < t \leqq \delta$ for some $\delta > 0$, $x^0 + tv \in X$, $f_i(x^0+tv) > f_i(x^0)$:

$$\frac{f_i(x^0+tv) - f_i(x^0)}{f_j(x^0) - f_j(x^0+tv)} \leqq M,$$

for some $j \in K$ such that $f_j(x^0+tv) < f_j(x^0)$

Definition 2.6 (Diewert [4]) The function $h : X_0 \longrightarrow R$ is said to be a pseudoconcave function on X_0 iff $x^0 \in X_0$, $v \in R^n$, satisfying $v^T v = 1$, for every $t > 0$, $x^0 + tv \in X_0$, $D_v^{+u} h(x^0) \leqq 0$ implies $h(x^0+tv) \leqq h(x^0)$, or equivalently $h(x^0+tv) > h(x^0)$ implies $D_v^{+u} h(x^0) > 0$.

Geoffrion [6] characterized properly efficient solutions of (P) in terms of solutions of the following single-objective programming problem:

$$(P_\lambda) \qquad \begin{array}{ll} \textit{Maximize} & \lambda^T f(x) \\ \textit{subject to} & x \in X \end{array}$$

where $\lambda \in R^k_+$ is a preassigned vector.

Lemma 2.1 (Geoffrion [6]) Let $\lambda \in R^k_+$ be fixed. If x^0 is an optimal solution for (P_λ), then x^0 is a properly efficient solution for (P).

Lemma 2.2 (Geoffrion [6]) Let f_i, $1 \leqq i \leqq k$, be concave and g_j, $1 \leqq j \leqq m$, be convex functions on X_0. If x^0 is an efficient solution for (P) then there exists a $\lambda \in R^k_+$ such that x^0 is an optimal solution for (P_λ).

In Bhatia and Aggarwal [3] some necessary and /or sufficient optimality conditions for (P) are given.

Theorem 2.3 (Bhatia and Aggarwal [3]) Suppose x^0 is an efficient solution for (P). Then there exists a $\lambda \in R^k_+$ such that the following necessary condition holds

$$v \in F(x^0) \Longrightarrow D_v^{+u}(\lambda^T f(x^0)) \leqq 0 \tag{2.1}$$

Corollary 2.4 (Bhatia and Aggarwal [3]) Let x^0 be a feasible solution for (P) and let $B(x^0)$ be the set of binding constraints, i.e. $j \in B(x^0)$ iff $g_j(x^0) = 0$. Then

$$D_v^{+u} g_j(x^0) < 0 \text{ for } j \in B(x^0), \text{ and } D_v^{+u} g_j(x^0) < \infty \text{ for } \\ j \notin B(x^0) \Longrightarrow v \in F(x^0)$$

Theorem 2.5 (Bhatia and Aggarwal [3]) Suppose for some $\lambda \in R^k_+$, $\lambda^T f(x)$ is pseudoconcave, $g_1, g_2, ..., g_m$ are quasiconvex and x^0 is a feasible solution for (P) which satisfies condition (2.1). Then x^0 is an efficient solution for (P).

Theorem 2.6 (Bhatia and Aggarwal [3]) Suppose that, for some $\lambda \in R^k_+$, $\lambda^T f(x)$ is proper and pseudoconcave function , $g_1, g_2, ..., g_m$ are proper and quasiconvex functions. Suppose there exists a vector of multipliers $\mu^0 = (\mu^0_1, ..., \mu^0_m) \in R^m$and $x^0 \in X_0$ such that $f(x)$ is finite and the following conditions are satisfied:

$$\text{for every direction } v, \quad D^{+u}_v[\lambda^T f(x^0) - \textstyle\sum_{j=1}^m \mu^0_j g_j(x^0)] \leqq 0 \tag{2.2}$$
$$g(x^0) \leqq 0$$
$$\mu^0 \geqq 0$$
$$\textstyle\sum_{j=1}^m \mu^0_j g_j(x^0) = 0$$

Then x^0 is an efficient solution for (P).

Corollary 2.7 (Bhatia and Aggarwal [3]) Theorem 2.6 remains valid if (2.2) is replaced by

$$v \in F(x^0) \Longrightarrow D^{+u}_v[\lambda^T f(x^0) - \sum_{j=1}^m \mu^0_j g_j(x^0)] \leqq 0$$

Corollary 2.8 (Bhatia and Aggarwal [3]) Theorem 2.6 remains valid if (2.2) is replaced by

$$v \in F(x^0) \Longrightarrow D^{+u}_v(\lambda^T f(x^0)) - D^{+u}_v(\sum_{j \in B(x^0)}^m \mu^0_j g_j(x^0)) \leqq 0$$

Let I_t ,$1 \leqq t \leqq r$, be a partition of $M = \{1, 2, ..., m\}$, i.e., $I_s \cap I_t = \Phi$ with $s \neq t$ and $\cup_{t=1}^r I_t = M$. We denote $\mu^T_{I_k} g_{I_k}(x^0) = \sum_{j \in I_k} \mu^0_j g_j(x^0)$, for $0 \leqq k \leqq r$. Now we have

Theorem 2.9 Let x^0 be an efficient solution of (P), and the constraints being assumed to satisfy the conditions of Corollary 2.4 at x^0. Then exist $v \in F(x^0)$, $\lambda \in R^k_+$, $\mu^0 = (\mu^0_1, ..., \mu^0_m) \in R^m$ such that the following conditions are satisfied:

$$D^{+u}_v(\lambda^T f - \mu^{0T}_{I_0} g_{I_0})(x^0) - \textstyle\sum_{k=1}^r D^{+u}_v(\mu^{0T}_{I_k} g_{I_k}(x^0)) \leqq 0$$
$$g(x^0) \leqq 0$$
$$\mu^0 \geqq 0$$
$$\textstyle\sum_{j=1}^m \mu^0_j g_j(x^0) = 0$$
$$\lambda^T e = 1, \quad e = (1, 1, ...1)^T \in R^k$$

Proof. Let x^0 be an efficient solution of (P) , and let the constraints of (P) satisfy the conditions of Corollary 2.4 for some $v \in R^n$. Then $v \in F(x^0)$ and using the line of Theorem 1 of [2] we get that there exist $\lambda \in R^k_+$, $\lambda^T e = 1$

and $\mu_1^0, ..., \mu_m^0$ such that

$$\sum_{i=1}^{k} \lambda_i D_v^{+u} f_i(x^0) - \sum_{j=1}^{m} \mu_j^0 D_v^{+l} g_j(x^0) = 0$$
$$\mu^0 \geqq 0$$
$$\sum_{j=1}^{m} \mu_j^0 g_j(x^0) = 0$$

Since $-D_v^{+u}(g_j(x)) = D_v^{+l}(-g_j(x))$ and I_t ,$1 \leqq t \leqq r$, is a partition of M={1,2,....,m}, we get

$$\sum_{i=1}^{k} \lambda_i D_v^{+u} f_i(x^0) + \sum_{j\in I_0} \mu_j^0 D_v^{+u}(-g_j)(x^0) + \sum_{k=1}^{r} \sum_{j\in I_k} \mu_j^0 D_v^{+u}(-g_j)(x^0) = 0$$

Using $\mu^0 \geqq 0$ and $D_v^{+u}(\alpha_1 h_1(x) + \alpha_2 h_2(x)) \leqq \alpha_1 D_v^{+u} h_1(x) + \alpha_2 D_v^{+u} h_2(x)$, for $\alpha_1, \alpha_2 \in R$ we obtain

$$D_v^{+u}(\lambda^T f - \mu_{I_0}^{0T} g_{I_0})(x^0) + \sum_{k=1}^{r} D_v^{+u}(-\mu_{I_k}^{0T} g_{I_k})(x^0) \leqq 0$$

Now, because $D_v^{+l}(-\mu_{I_k}^{0T} g_{I_k})(x^0) \leqq D_v^{+u}(-\mu_{I_k}^{0T} g_{I_k})(x^0)$, and $D_v^{+l}(-\mu_{I_k}^{0T} g_{I_k})(x^0) \leqq -D_v^{+u}(\mu_{I_k}^{0T} g_{I_k})(x^0)$ we obtain

$$D_v^{+u}(\lambda^T f - \mu_{I_0}^{0T} g_{I_0})(x^0) - \sum_{k=1}^{r} D_v^{+u}(\mu_{I_k}^{0T} g_{I_k})(x^0) \leqq 0$$

3. Duality

Relative to problem (P) we consider a general Mond-Weir dual, corresponding to every $v \in F(x)$,

$$\text{(D)} \quad \begin{array}{l} \textit{Minimize } f(w) - \mu_{I_0}^T g_{I_0}(w) e \\ \textit{subject to} \\ D_v^{+u}(\lambda^T f - \mu_{I_0}^T g_{I_0})(w) - \sum_{k=1}^{r} D_v^{+u}(\mu_{I_k}^T g_{I_k})(w) \leqq 0 \\ \mu_{I_k}^T g_{I_k}(w) \geqq 0, 1 \leqq k \leqq r \\ \mu \geqq 0 \\ \lambda > 0, \lambda^T e = 1 \end{array}$$

where $e = (1, 1, ..., 1)^T$.

Also, we consider the following single objective parametric program (D_λ) relative to dual problem (D).

$$(D_\lambda)\qquad \begin{array}{l} \textit{Minimize } \lambda^T f(w) - \mu_{I_0}^T g_{I_0}(w) \\ \textit{subject to} \\ \quad D_v^{+u}(\lambda^T f - \mu_{I_0}^T g_{I_0})(w) - \sum_{k=1}^r D_v^{+u}(\mu_{I_k}^T g_{I_k})(w) \leqq 0 \\ \qquad\qquad \mu_{I_k}^T g_{I_k}(w) \geqq 0, 1 \leqq k \leqq r \\ \qquad\qquad\qquad \mu \geqq 0 \\ \qquad\qquad \lambda > 0, \lambda^T e = 1 \end{array}$$

We first give a sufficient condition for properly efficient solution in (D) in terms of problem (D_λ).

Theorem 3.1 If for fixed $\overline{\lambda} \in R_+^k$,$(\overline{w}, \overline{\mu})$ solves the problem $(D_{\overline{\lambda}})$,then $(\overline{w}, \overline{\lambda}, \overline{\mu})$ is a properly efficient solution of program (D).

Proof. First we show that $(\overline{w}, \overline{\lambda}, \overline{\mu})$ is efficient for (D). If $(\overline{w}, \overline{\lambda}, \overline{\mu})$ is not efficient for (D), then there exists an $(s, q, r) \in R^{n+k+m}$ with $(s, q, r)^T(s, q, r) = 1$ such that

$$f_i(\overline{w} + ts) - \mu_{I_0}^T g_{I_0}(\overline{w} + ts) \leqq f_i(\overline{w}) - \mu_{I_0}^T g_{I_0}(\overline{w})$$

for any $i = 1, 2, ..., k$, with a strict inequality for some i, where $0 < t \leqq \delta$ for some $\delta > 0$ such that $(\overline{w} + ts, \overline{\lambda} + tq, \overline{\mu} + tr) \in Y(v)$, where $Y(v)$ is the set of feasible solutions for (D). Since $\overline{\lambda} > 0$, $\overline{\lambda}^T e = 1$, we get

$$\overline{\lambda}^T f(\overline{w} + ts) - \mu_{I_0}^T g_{I_0}(\overline{w} + ts) \leqq \overline{\lambda}^T f(\overline{w}) - \mu_{I_0}^T g_{I_0}(\overline{w})$$

which contradicts the optimality of $(\overline{w}, \overline{\mu})$ in $(D_{\overline{\lambda}})$. Thus $(\overline{w}, \overline{\lambda}, \overline{\mu})$ is efficient for (D). Using the lines from Geoffrion [6] and Bhatia and Aggarwal [3] and the relations $\overline{\lambda} > 0, \overline{\lambda}^T e = 1$, we obtain that $(\overline{w}, \overline{\lambda}, \overline{\mu})$ is properly efficient for (D).

Now we state some duality results between the problems (P) and (D).

Theorem 3.2 (Weak Duality) Let $w \in X$ and $(w, \lambda, \mu) \in Y(v)$. If $\lambda^T f - \mu_{I_0}^T g_{I_0}$ is pseudoconcave at w, then

$$\lambda^T f(w + tv) \leqq \lambda^T f(w) - \mu_{I_0}^T g_{I_0}(w)$$

for all $v \in R^n$ with $v^T v = 1$, $0 < t \leqq \delta$ for some $\delta > 0$ such that $w + tv \in X$.

Proof. We have that (w, λ, μ) is a feasible solution for (D). Hence, corresponding to $v \in F(w)$, i.e., $v \in R^n$ with $v^T v = 1$, $0 < t \leqq \delta$ for some $\delta > 0$ such that $w + tv \in X$, we have

$$\begin{array}{c} D_v^{+u}(\lambda^T f - \mu_{I_0}^T g_{I_0})(w) - \sum_{k=1}^r D_v^{+u}(\mu_{I_k}^T g_{I_k})(w) \leqq 0 \\ \mu_{I_k}^T g_{I_k}(w) \geqq 0, 1 \leqq k \leqq r \\ \mu \geqq 0 \\ \lambda > 0, \lambda^T e = 1 \end{array}$$

Since $w + tv \in X$ and $\mu \geqq 0$ we have

$$\mu_{I_k}^T g_{I_k}(w + tv) \leqq 0$$

for any $0 \leqq k \leqq r$. Hence

$$\mu_{I_k}^T \frac{g_{I_k}(w + tv) - g_{I_k}(w)}{t} \leqq 0$$

for any t such $0 < t \leqq \delta$ .We get

$$D_v^{+u}(\mu_{I_k}^T g_{I_k})(w) \leqq 0$$

for any $1 \leqq k \leqq r$. Now we obtain

$$D_v^{+u}(\lambda^T f - \mu_{I_0}^T g_{I_0})(w) \leqq 0$$

Using the pseudoconcavity of $\lambda^T f - \mu_{I_0}^T g_{I_0}$ at w, it follows

$$\lambda^T f(w + tv) - \mu_{I_0}^T g_{I_0}(w + tv) \leqq \lambda^T f(w) - \mu_{I_0}^T g_{I_0}(w)$$

Because $\mu_{I_0}^T g_{I_0}(w + tv) \leqq 0$, the last inequality implies

$$\lambda^T f(w + tv) \leqq \lambda^T f(w) - \mu_{I_0}^T g_{I_0}(w)$$

Thus the theorem is proved.

Theorem 3.3 (Strong Duality) Let x^0 be a properly efficient solution of (P) such that

i1) the constraints satisfy conditions of Corollary 2.4 ;

i2) for any $\lambda \in R_+^k$ such that $\lambda^T e = 1$, $\lambda^T f - \mu_{I_0}^T g_{I_0}$ is proper and pseudoconcave ;

i3) $g_1, g_2, ..., g_m$ are proper and quasiconvex functions.

Then there exist $\lambda^0 \in R_+^k$ such that $\lambda^{0T} e = 1$ and $\mu^0 \in R^m$ such that (x^0, λ^0, μ^0) is properly efficient solution for (D).

Proof. It is clear that x^0 is also an efficient solution for (P). Using Lemma 2.1 we have that there exists a $\lambda^0 \in R_+^k$ such that $\lambda^{0T} e = 1$ and x^0 is an optimal solution for (P_{λ^0}). Now by Theorem 2.9 , there exists $\mu^0 \in R^m$ such that for every $v \in F(x^0)$

$$D_v^{+u}(\lambda^{0T} f - \mu_{I_0}^{0T} g_{I_0})(x^0) - \sum_{k=1}^r D_v^{+u}(\mu_{I_k}^{0T} g_{I_k})(x^0) \leqq 0$$
$$g(x^0) \leqq 0$$
$$\mu^0 \geqq 0$$
$$\sum_{j=1}^m \mu_j^0 g_j(x^0) = 0$$

We get that (x^0, μ^0) is a feasible solution for (D_{λ^0}). Also we obtain that $\mu_{I_k}^T g_{I_k}(x^0) = 0$ for any $0 \leqq k \leqq r$. Also we have that (x^0, μ^0) is an optimal solution for (D_{λ^0}). Indeed, if (x^0, μ^0) is not an optimal solution for (D_{λ^0}) , then there exists some $(s, q) \in R^{n+m}$ with $(s, q)^T(s, q) = 1$, $0 < t \leqq \delta$ for some $\delta > 0$ such that $(x^0 + ts, \mu^0 + tq)$ is a feasible solution for (D_{λ^0}) and

$$\lambda^{0T} f(x^0 + ts) - \mu_{I_0}^{0T} g_{I_0}(x^0 + ts) \leqq \lambda^{0T} f(x^0) - \mu_{I_0}^{0T} g_{I_0}(x^0)$$

Since $\mu_{I_0}^{0T} g_{I_0}(x^0) = 0$ we get

$$\lambda^{0T} f(x^0 + ts) - \mu_{I_0}^{0T} g_{I_0}(x^0 + ts) \leqq \lambda^{0T} f(x^0)$$

which contradicts weak duality . Therefore (x^0, μ^0) is an optimal solution for (D_{λ^0}). Using $\lambda^{0T} e = 1$ and Theorem 3.1 we obtain that (x^0, λ^0, μ^0) is properly efficient solution for (D). Thus the theorem is proved.

Remark 3.1 If $I_0 = \Phi$ and $I_k = M$ for some k,$1 \leqq k \leqq r$ we obtain the dual and some results obtained by Bhatia and Aggarwal [3]. Further if $k = 1$ we obtain the results from Diewert [4].

Remark 3.2 Our results are stated on the lines from Diewert [4], Egudo [5] and Bhatia and Aggarwal [3].

Remark 3.3 Our dual (D) for problem (P) is a general Mond-Weir dual [8].

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The Nucleolus in Multiobjective n-person Cooperative Games

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Abstract. In this paper, multiobjective n-person cooperative games are defined, and the nucleolus is considered in such games. First, a multiobjective game is reduced to single-objective games by using the scalarizing methods of multiobjective programming, i.e., the weighting coefficients method and the weighted minimax method. Second, the nucleolus is defined directly in a multiobjective n-person cooperative game. Computational methods for deriving the nucleolus are shown when excess functions are defined as distance from the ideal point or a Pareto optimal point.

Keywords Multiobjective n-person cooperative games, Nucleolus, Multiobjective programming

1 Introduction

We intend to consider the nucleolus, a lexicographical solution concept in multiobjective n-person cooperative games. Research on multiobjective two-person zero-sum games has been made since mid-'50s [3], [12] and multiobjective two or n-person non-zero-sum games have also been studied since mid-'80s [8], [14], [15]. Especially, for two-person games, computational methods for obtaining minimax solutions and equilibrium solutions have been developed [4], [10], [9].

Bergstresser and Yu [2] considered some solutions concepts using domination structure in a conventional n-person cooperative game and in an n-person cooperative game with a characteristic function which associates a subset of players with its real vector value.

Recently, a framework of multiobjective n-person cooperative games has been presented by Tanino, Muranaka and Tanaka [13]. They extend a characteristic function to a characteristic mapping which associates a coalition with its set of real vector value.

Multiobjective n-person cooperative games are based on a concept essentially different from that of n-person cooperative games with nontransferable utility (NTU-games), while NTU-games have been studied as games also defined by a pair of a set of players and sets in a payoff space. We clearly describe different points between multiobjective n-person cooperative games and NTU-games. In n-person cooperative games with transferable utility (TU-games), a payoff for each player is represented by a real value and it is assumed that the payoff can freely be transferred among players. There are many cases where the payoff is supposed to be a transferable utility. Therefore, on condition that there exists a common exchangeable unit among players, interpersonal utility comparison and cardinal utility are assumed. Games without such conditions are NTU-games, which are developed by Aumann and Peleg [1]. In NTU-games, a characteristic function is extended from a viewpoint that a payoff cannot be transferred among players. Thus, a coalition is corresponded to a set of payoffs dimensions of which is a number of players belonging to the coalition in NTU-games, while it is corresponded to a scalar real value in TU-games.

In contrast, in multiobjective n-person cooperative games, it is not assumed that a payoff is a value of utility, but multiple attribute values handled in a real world are directly dealt with as a vector of payoffs in multiobjective games. Consequently, a payoff is represented as a real vector value dimension of which is a number of objectives or attributes considered in a decision making problem under conflict.

On the other hand, the nucleolus, which is defined by Schmeidler [11] in TU-games, is a lexicographical solution concept related to the bargaining set and the kernel. Since the nucleolus always exists and consists of a single point, it is suitable for an application to managerial and public decision making problems.

In this paper, we define multiobjective n-person cooperative games in a way different from Tanino, Muranaka and Tanaka [13], and extend a concept of the nucleolus. In Section 2, multiobjective n-person cooperative games are defined and reduced to a single-objective n-person cooperative games by using two scalarizing methods; the weighting coefficients method and the weighted minimax method. In Section 3, basic concept such as the individual rationality, the group rationality and an excess function are considered, and the nucleolus is directly defined on the basis of the concept of excess functions in multiobjective n-person cooperative games. Finally, using illustrative examples of excess functions, computational methods for deriving the nucleolus are developed.

2 Basic definitions and reduction to a single-objective game

2.1 Basic definitions

While a coalition value $v(S)$ is represented as a scalar value in a single-objective n-person cooperative game, a set of vector value is used for a corresponding mathematical representation in a multiobjective game.

Let n be a fixed positive integer, $N = \{1, 2, \ldots, n\}$, and Λ denote a family of nonempty subsets of N. An element $i \in N$ are called a player and an element $S \in \Lambda$, a coalition. Let ℓ be a fixed positive integer, $K = \{1, 2, \ldots, \ell\}$, and an element $k \in K$ denote an objective.

For a coalition $S \in \Lambda$, consider a set V_S satisfying the following conditions:

(i) V_S is a nonempty closed subset of R^ℓ.

(ii) if $u \leqq v$ holds for $v \in V_S$ and $u \in R^\ell$, then $u \in V_S$ holds, where $u \leqq v$ means $u^k \leqq v^k, k = 1, 2, \ldots, \ell$.

Second condition (ii) of V_S means comprehensiveness.

For a family of sets $V = \{V_S \mid S \in \Lambda\}$, suppose that, for each objective, a vector of multiple payoffs $v = (v^1, v^2, \ldots, v^\ell) \in V_S$ can be shared by members of a coalition S, and that, for a payoff variable $x = (x^1, x^2, \ldots, x^\ell) \in R^{n\times\ell}$, $x^k = (x_1^k, x_2^k, \ldots, x_n^k) \in R^n$, $v^k \geqq \sum_{i\in S} x_i^k$, $k = 1, 2, \ldots, \ell$. Then, an n-person cooperative game with ℓ-objectives can be represented by (N, V, K).

2.2 Reduction to a single-objective game

Before we consider solution concepts in multiobjective n-person cooperative games, we examine how to reduce a multiobjective game to single-objective games from the advantage that several solution concepts in conventional (single-objective) games can be used without any modification.

We consider the scalarizing methods in order to reduce a multiobjective n-person cooperative game (N, V, K) into ℓ single-objective games (N, v^k), $k = 1, 2, \ldots, \ell$. For any coalition $S \in \Lambda$, if V_S is a set having only one element, such a game (N, V, K) can be directly reduced to ℓ single-objective games (N, v^k), $k = 1, 2, \ldots, \ell$, i.e., if $V_S = \{v_S\} \in R^\ell$, the multiobjective game (N, V, K) can be reduced to single-objective games (N, v^k), $k = 1, 2, \ldots, \ell$ such that $v^k(S) = v_S^k$, $S \in \Lambda$, $k = 1, 2, \ldots, \ell$. However, since V_S is a set satisfying the conditions (i) and (ii), selecting one point from V_S is necessary for reduction to single-objective games. In this paper, using basic techniques in multiobjective decision making, a multiobjective game is reduced to single-objective games. By using weighting coefficients for objectives, we select one point satisfying Pareto optimality from V_S. In the study of von Neumann and Morgenstern [7], a coalition value $v(S)$ in single-objective games is interpreted as a maximin value of a two-person game played between S and $N\backslash S$, assuming that these two coalitions form. Thus $v(S)$ is the amount of payoff that the members of S can obtain from the game, whatever the remaining players $N\backslash S$ may do.

From this point of view, it is required that the one point selected from V_S satisfies Pareto optimality. Pareto maximal points for a coalition S is

$$EV_S = \{v \in V_S \mid (V_S - v) \cap R_{++}^\ell = \emptyset\}, \tag{1}$$

where $R_{++}^\ell = \{u = (u^1, u^2, \ldots, u^\ell) \in R^\ell \mid u \geq \mathbf{0}\}$, $\emptyset$ is an empty set, and $u \geq \mathbf{0}$ means $u^k \geqq 0$, $k = 1, 2, \ldots, \ell$ and there is at least one k such that $u^k > 0$. EV_S has an infinite number of elements, but in many cases, we can select one point from EV_S by using weighting coefficients for objectives.

We employ the weighting coefficients method and the weighted minimax method. If V_S is convex, every Pareto optimal point associate with at least one positive weighting coefficients.

The weighting coefficients method is well-known technique for solving multiobjective optimization problems. In this case, a value function, which represents preference of a coalition, is formed by summing the ℓ weighted objectives. We can obtain a point in the set V_S such as minimizes the value function. This problem can also be regarded as a maximization problem of ℓ_1-norm. Let weighting coefficients

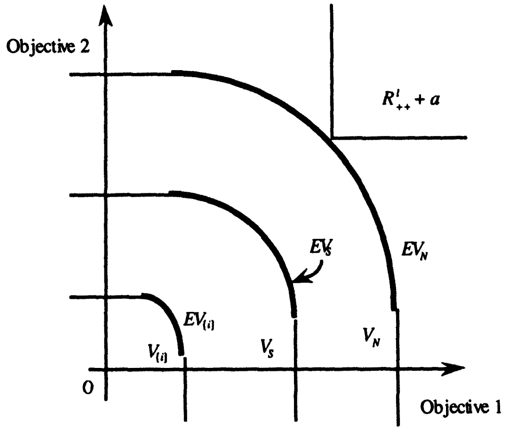

Fig. 1. A set V_S and Pareto maximal points.

be $w_S \in W \triangleq \{w \in R^\ell \mid w^k > 0,\ k = 1, 2, \ldots, \ell,\ \sum_{k=1}^{\ell} w^k = 1\}$. Then the problem is represented as:

$$\left.\begin{array}{ll} \text{maximize} & w_S v \\ \text{subject to} & v \in V_S. \end{array}\right\} \tag{2}$$

Let $\hat{v}_S$ denote an optimal solution to the problem (2).

For the weighted minimax method, a value function is formed by a minimal objective function among the ℓ weighted objective functions, and we can obtain a point minimizing the value function in the set V_S. This problem can also be regarded as a maximization problem of ℓ_∞-norm, and then it is represented as:

$$\left.\begin{array}{ll} \text{maximize} & \varepsilon \\ \text{subject to} & v^k / w_S^k \geqq \varepsilon,\ k = 1, 2, \ldots, \ell, \\ & v \in V_S. \end{array}\right\} \tag{3}$$

Let $\tilde{v}_S$ denote an optimal solution to the problem (3).

Multiobjective n-person cooperative games with weighting coefficients $\bar{W} = \{w_S, \forall S \in \Lambda\}$ can be reduced to ℓ single-objective cooperative games (N, v^k), $k = 1, 2, \ldots, \ell$ by setting $v^k(S) = \hat{v}_S^k$, $k = 1, 2, \ldots, \ell$ in the case of the maximization of the linear weighted sum, and by setting $v^k(S) = \tilde{v}_S^k$, $k = 1, 2, \ldots, \ell$ in the case of the weighted minimax method. After the above reduction, we can apply solution concepts in single-objective games such as the core and the nucleolus to the reduced ℓ single-objective games (N, v^k), $k = 1, 2, \ldots, \ell$.

3 The nucleolus in multiobjective n-person cooperative games

For the reduction to single-objective games, there is the advantage that several solution concepts in single-objective games can be used without any modification, but there is a possibility that some information in the multiobjective game will be lost.

In this section, we extend basic concepts in single-objective games to those in multiobjective games, define the nucleolus based on the extended concepts, and develop computational methods for deriving the nucleolus.

3.1 Extension of basic concepts

The individual rationality in multiobjective n-person cooperative games is represented as:

$$IR(N, V, K) = \left\{ x \in R^{n\times\ell} \;\middle|\; x \in \bigcap_{i\in N} \{V_N \backslash \mathrm{int} V_{\{i\}}\} \right\}, \tag{4}$$

where $V_N \backslash \mathrm{int} V_{\{i\}} = \{z \in V_N \mid z \notin \mathrm{int} V_{\{i\}}\}$. The group rationality also is represented by:

$$GR(N, V, K) = \left\{ x \in R^{n\times\ell} \;\middle|\; v^k = \sum_{i\in N} x_i^k,\, k = 1, 2, \ldots, \ell,\, (V_N - v) \cap R_{++}^{\ell} = \emptyset \right\}. \tag{5}$$

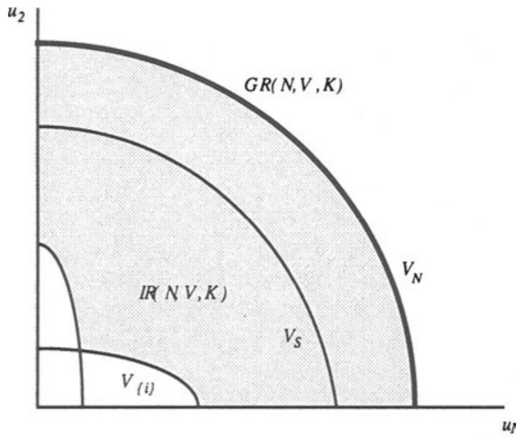

Fig. 2. Individual rational points and group rational points.

In single-objective games, the excess function plays a vital role for defining the nucleolus. We consider excess functions in multiobjective n-person cooperative games. It is proper for the excess functions $E(S, \cdot)$ to satisfy the following conditions:

[Condition 1] If $x, y \in R^{n\times\ell}$ satisfy $\sum_{i\in S} x_i^k = \sum_{i\in S} y_i^k$ for every $k \in K$, then

$$E(S, x) = E(S, y). \tag{6}$$

[Condition 2] If $x, y \in R^{n\times\ell}$ satisfy $\sum_{i\in S} x_i^k < \sum_{i\in S} y_i^k$ for every $k \in K$, then

$$E(S,x) > E(S,y). \tag{7}$$

[Condition 3] $E(S,x)$ is jointly continuous with respect to x and V.

Let us consider examples of the excess function.

Example 1. By using the maximum in V_S for the objective k and weighting coefficients $w_S \in W$, an excess function of a coalition S for a payoff vector x is characterized. The excess function is defined as the weighted distance from the ideal point to a payoff vector x. For example, using ℓ_1-norm,

$$E(S,x) = \sum_{k\in K} w_S^k e^k(S,x), \tag{8}$$

where

$$e^k(S,x) = \max_{v\in V_S} v^k - \sum_{i\in S} x_i^k. \tag{9}$$

An excess function using ℓ_∞-norm can also be represented as:

$$E(S,x) = \max_{k\in K} w_S^k e^k(S,x). \tag{10}$$

Example 2. By using a certain Pareto optimal point in V_S and weighting coefficients $w_S \in W$, an excess function of a coalition S for a payoff vector x is characterized. The excess function is defined as the weighted distance from the Pareto optimal point to a payoff vector x. A set of Pareto optimal points in V_S can be represented by:

$$EV_S = \{v \in R^\ell \mid (V_S - v) \cap R^\ell_{++} = \emptyset\}. \tag{11}$$

If V_S is convex, we can select one point from EV_S by maximizing ℓ_1 or ℓ_∞-norm in a way similar to Example 1. In the case of ℓ_1-norm,

$$\{\hat{v}_S\} = \arg\max_{v\in V_S}\{w_S v\}, \tag{12}$$

and an excess function is represented as:

$$E(S,x) = \sum_{k\in K} w_S^k e^k(S,x), \tag{13}$$

where

$$e^k(S,x) = \hat{v}_S^k - \sum_{i\in S} x_i^k. \tag{14}$$

In the case of ℓ_∞-norm,

$$\{\tilde{v}_S\} = \arg\max_{v\in V_S}\min_{k\in K}\left\{v^k/w_S^k\right\}, \tag{15}$$

and an excess function is represented as follows:

$$E(S,x) = \min_{k\in K} w_S^k e^k(S,x), \tag{16}$$

where

$$e^k(S,x) = \tilde{v}_S^k - \sum_{i \in S} x_i^k. \tag{17}$$

Next, we consider the nucleolus as a solution concept in multiobjective n-person cooperative games. In multiobjective games, we can also define the nucleolus, which is a payoff vector minimizing excesses in lexicographical order. Let $H_{2^n-2} : R^{2^n-2} \to R^{2^n-2}$ denote a mapping which arranges entries of a $2^n - 2$-dimensional vector in order of decreasing magnitude. Then, the nucleolus is defined as

$$\begin{aligned} \mathrm{N}(N,V,K) &= \{x \in X \mid H_{2^n-2}(E(S_1,x), E(S_2,x), \ldots, E(S_{2^n-2},x)) \\ &\qquad \leqq_L H_{2^n-2}(E(S_1,y), E(S_2,y), \ldots, E(S_{2^n-2},y)), \forall y \in X\}, \end{aligned} \tag{18}$$

where X is a set of all the imputations of the game (N,V,K), i.e., $X = IR(N,V,K) \cap GR(N,V,K)$, and, in the case of the pre-nucleolus, $X = GR(N,V,K)$, and the lexicographical order $\leqq_L$ is defined in the following. Let $r(x)$ be a vector arranged in order of decreasing magnitude, i.e., if $i < j$, $r_i(x) \geqq r_j(x)$. Then, for any pair of payoff vectors x and y, if $x = y$ or, for the first entry h in which they differ, $r_h(x) < r_h(y)$, x is smaller than or equal to y in lexicographical order.

If $E(S,x)$ is continuous jointly in x and V, and X is compact, then $\mathrm{N}(N,V,K)$ is not empty. It can be proven by induction in a procedure similar to the proof of the existence of the nucleolus in NTU-games by Kalai [5] because the following set is not empty:

$$\left\{ x \in X \;\middle|\; \max_{S \in \Lambda} E(S,x) \leqq \max_{S \in \Lambda} E(S,y), \forall y \in X \right\}. \tag{19}$$

3.2 Computational methods for the nucleolus

We consider methods for computing the nucleoli which are defined by using the excess functions in Examples 1 and 2.

First, the case of Example 1 is examined. When ℓ_1-norm is employed, the excess function can be represented as:

$$E(S,x;w) = \sum_{k \in K} w_S^k \left(\tilde{v}_S^k - \sum_{i \in S} x_i^k \right), \tag{20}$$

where

$$\{\bar{v}_S^k = (\bar{v}_S^1, \bar{v}_S^2, \ldots, \bar{v}_S^\ell)\} = \arg\max_{v \in V_S} v^k. \tag{21}$$

Then, a mathematical programming problem which yields a solution minimizing the maximal excess, which is called the least core in TU-games, is formulated as:

$$\left.\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & \displaystyle\sum_{k=1}^{\ell} w_S^k \left(\bar{v}_S^k - \sum_{i \in S} x_i^k \right) \leq \varepsilon, \ \forall S \neq \emptyset, N, \\ & x \in X. \end{array}\right\} \tag{22}$$

If an optimal solution (x^*, ε^*) to (22) is unique, then the solution x^* is the nucleolus; otherwise the active inequality constraints are replaced with the equality constraints $\sum_{k=1}^{\ell} w_S^k \left(\bar{v}_S^k - \sum_{i\in S} x_i^k\right) = \varepsilon^*$ and the updated mathematical programming problem is solved. By repeating the procedure, an optimal solution x^* can be obtained and it is the nucleolus in the game (N, V, K).

When ℓ_∞-norm is employed, the excess function can be represented by:

$$E(S, x; w) = \max_{k\in K} w_S^k \left(\bar{v}_S^k - \sum_{i\in S} x_i^k\right). \tag{23}$$

Then, a mathematical programming problem which yields a solution minimizing the maximal excess is formulated as:

$$\left.\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & \bar{v}_S^1 - \sum_{i\in S} x_i^1 \leq \varepsilon,\ \forall S \neq \emptyset, N, \\ & \bar{v}_S^2 - \sum_{i\in S} x_i^2 \leq \varepsilon,\ \forall S \neq \emptyset, N, \\ & \vdots \\ & \bar{v}_S^\ell - \sum_{i\in S} x_i^\ell \leq \varepsilon,\ \forall S \neq \emptyset, N, \\ & x \in X. \end{array}\right\} \tag{24}$$

The nucleolus can also be obtained in a similar procedure.

Second, in the case of Example 2, when ℓ_1-norm is employed, the excess function can be represented as:

$$E(S, x; w) = \sum_{k\in K} w_S^k \left(\hat{v}_S^k - \sum_{i\in S} x_i^k\right). \tag{25}$$

When ℓ_∞-norm is employed, the excess function can be represented as:

$$E(S, x; w) = \max_{k\in K} w_S^k \left(\tilde{v}_S^k - \sum_{i\in S} x_i^k\right). \tag{26}$$

A similar formulation can also be done in the case of Example 2.

Assuming the linearity of the constraint of payoff vectors X, we can develop a computational method using the technique of the linear programming. The constraint of payoff vectors X is represented as $X = IR(N, V, K) \cap GR(N, V, K)$. Even if both of $IR(N, V, K)$ and $GR(N, V, K)$ can be represented by linear constraints, because X is not always convex, by the techniques of the linear programming, it is difficult to solve a mathematical programming problem such as (22) and (24) which yields a solution minimizing the maximal excess. However, unless the weighting coefficients are not ill-balanced, the optimal value to the mathematical programming problem such as (22) and (24) does not vary by substituting $\overline{IR}(N, V, K)$ such that

$$\begin{aligned} IR(N, V, K) \supset \overline{IR}(N, V, K) = \{x \in R^{n\times\ell} \mid \bar{v}_{\{i\}}^k &\leqq x_i^k, \\ i = 1, 2, \ldots, n,\ k &= 1, 2, \ldots, \ell\} \end{aligned} \tag{27}$$

for $IR(N, V, K)$. Moreover, since $GR(N, V, K)$ is the set of Pareto maximal points of V_N, assuming that V_N is a convex set represented as linear constraints:

$$g_j(x) = a_j^1(x_1^1 + \cdots + x_n^1) + \cdots + a_j^\ell(x_1^\ell + \cdots + x_n^\ell) + a_j^{\ell+1} \leqq 0, \quad j = 1, 2, \ldots, m, \tag{28}$$

for a certain $j = \hat{j}$, a payoff vector being the nucleolus must satisfy a equality constraint, i.e.,

$$a_{\hat{j}}^1(x_1^1 + \cdots + x_n^1) + \cdots + a_{\hat{j}}^\ell(x_1^\ell + \cdots + x_n^\ell) + a_{\hat{j}}^{\ell+1} = 0. \tag{29}$$

Consider the case of ℓ_1-norm in Example 1 as an excess function. Then, we formulate a linear programming problem:

$$\left.\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & \displaystyle\sum_{k=1}^{\ell} w_S^k(\bar{v}_S^k - \sum_{i \in S} x_i^k) \leq \varepsilon, \ \forall S \neq \emptyset, N \\ & \bar{v}_{\{i\}}^k \leqq x_i^k, \ i = 1, 2, \ldots, n, \ k = 1, 2, \ldots, \ell \\ & a_j^1(x_1^1 + \cdots + x_n^1) + \cdots + a_j^\ell(x_1^\ell + \cdots + x_n^\ell) + a_j^{\ell+1} \leqq 0, \\ & \hfill j = 1, 2, \ldots, m. \end{array}\right\} \tag{30}$$

Solve the problem (30) and let an optimal solution to the problem be denoted by x^*. If, for a certain $j = \hat{j}$, x^* satisfies the equality constraint (29), it minimizes the maximal excess. If not, solve m linear programming problems:

$$\left.\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & \displaystyle\sum_{k=1}^{\ell} w_S^k(\bar{v}_S^k - \sum_{i \in S} x_i^k) \leq \varepsilon, \ \forall S \neq \emptyset, N \\ & \bar{v}_{\{i\}}^k \leqq x_i^k, \ i = 1, 2, \ldots, n, \ k = 1, 2, \ldots, \ell \\ & a_j^1(x_1^1 + \cdots + x_n^1) + \cdots + a_j^\ell(x_1^\ell + \cdots + x_n^\ell) + a_j^{\ell+1} \leqq 0, \\ & \hfill j = 1, 2, \ldots, m, \ j \neq \hat{j} \\ & a_{\hat{j}}^1(x_1^1 + \cdots + x_n^1) + \cdots + a_{\hat{j}}^\ell(x_1^\ell + \cdots + x_n^\ell) + a_{\hat{j}}^{\ell+1} = 0 \end{array}\right\}, \quad \hat{j} = 1, 2, \ldots, m. \tag{31}$$

An optimal solution x^{**} which has the minimal optimal value among m problems is a payoff vector minimizing the maximal excess. If x^* or x^{**} is a unique optimal solution to (30) or (31), x^* or x^{**} is the nucleolus. If not, the active inequality constraints are replaced with the equality constraints and the updated mathematical programming problem is solved. The procedure is repeated at most $2^n - 1$ times. Because the dimension of payoff vectors is $n \times \ell$ and the number of all the coalitions is $2^n - 1$, the nucleolus is uniquely obtained if $2^n - 1 \geqq n \times \ell$, otherwise it is not always unique. The above consideration is that of the case of ℓ_1-norm, and for the case of ℓ_∞-norm, the nucleolus can also obtained by formulating similar linear programming problems.

4 Conclusions

In this paper, we have defined multiobjective n-person cooperative games in a way different from Tanino, Muranaka and Tanaka, and extend a concept of the nucleolus. First, multiobjective n-person cooperative games have been defined and reduced to a single-objective n-person cooperative games by using the scalarizing methods of the multiobjective optimization. Second, basic concepts such as the individual rationality, the group rationality and the excess function have been extended, and on the basis of the concepts, the nucleolus has been directly defined in multiobjective n-person cooperative games. Finally, assuming illustrative excess functions, computational methods for deriving the nucleolus have been developed.

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Sufficient Conditions in the Vector-Valued Maximin Problems

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Abstract. Constructive sufficient conditions for existence of Slater maximin in strictly convex (by uncertainty) multicriteria problems without restrictions are introduced.

Key words. Multicriterion, convex, Slater, maximin.

1 Problem Statement

Consider the multicriterion problem with uncertainty

$$\langle \mathbf{X},\ \mathbf{Y},\ f(x,y) \rangle, \tag{1}$$

where the set $\mathbf{X}\subseteq\mathbf{R}^n$ of solutions $x=(x_1,\dots,x_n)$ is given, for uncertainties y (noises, perturbations, errors of measurements, etc.) only the boundaries of changes are known. The realization of any uncertainty $y=(y_1,\dots,y_m)\in\mathbf{Y}\subseteq\mathbf{R}^m$ is equally probable. The components $f_i(x,y)$, $i\in\mathbf{N}=\{1,2,\dots,N\}$, of a vector criterion $f(x,y)=(f_1(x,y),\dots,f_N(x,y))$ are supposed to be <u>twice continuously differentiable</u> on $\mathbf{X}\times\mathbf{Y}$.

Let's put in correspondence to each solution $x\in\mathbf{X}$ the set $\mathbf{Y}_S(x)$ of Slater minimal solutions $y(x)$ of the multicriterion problem

$$\langle \{x\},\ \mathbf{Y},\ f(x,y) \rangle, \tag{2}$$

i.e.

$$\mathbf{Y}_S(x)=\{\ y(x)\in\mathbf{Y}\ |\ f(x,y(x))\not> f(x,y),\ \forall y\in\mathbf{Y}\ \}. \tag{3}$$

Notice that problem (2) is obtained from problem (1) fixing the solution $x\in\mathbf{X}$.

<u>Definition</u>. The solution $x^*\in\mathbf{X}$ is called Slater maximin for problem (1) if there exists an uncertainty $\hat{y}(x^*)\in\mathbf{Y}(x)$, such that

$$f(x^*,\hat{y}(x^*)) \not< f(x,y(x)),\ \forall x\in\mathbf{X},\ y(x)\in\mathbf{Y}_S(x). \tag{4}$$

In this case vector $f(x^*,\hat{y}(x^*))$ is called Slater maximin for problem (1).

The investigations of Slater maximins and other vector maximins (by Pareto, Geoffrion, etc.) have been intensively carried out in Russia [7], Georgia [8], Italy [2], Japan [6] (the review of the results is given in [7, pp. 4-5]). The existence of Slater maximin (in the case $\mathbf{X}\in\mathrm{comp}\mathbf{R}^n$ and $\mathbf{Y}\in\mathrm{comp}\mathbf{R}^m$ and continuous on $\mathbf{X}\times\mathbf{Y}$ scalar functions $f_i(x,y)$, $i\in\mathbf{N}$) has been determined; "geometric" method for constructing Slater maximins in the case of "separated" criteria $f_i(x,y)=\varphi_i(x)+\psi_i(y)$, $i\in\mathbf{N}$, has been suggested. The monogragh [9] has been dedicated to a dynamic variant of problem (1) with positional strategies-solutions. But, up to the present, there have not yet been obtained constructive sufficient conditions of vector maximin existence (which should have "worked" also in case of "unseparated" criteria). The object of this article is to liquidate this gap in case of strictly convex (in y) problem (1).

Proceeding from the given definition, the construction of a Slater maximin solution x^* is reduced to sequential solution of two problems:

a) Problem of internal vector minimum: to construct the whole set $\mathbf{Y}_S(x)$ of Slater minimal uncertainties $y(x)$ of problem (2) for each solution $x\in\mathbf{X}$;

b) Problem of external vector maximum: to construct a Slater maximal pair $(x^*,\hat{y}(x^*))$, satisfying condition (4).

Further, suppose that the following condition is fulfilled:

Condition 1. The sets $\mathbf{X}=\mathbf{R}^n$, $\mathbf{Y}=\mathbf{R}^m$, the functions $f_i(x,y)$, $i\in\mathbf{N}$, are twice continuously differentiable and the hessians

$$\frac{\partial^2 f_i(x,y)}{\partial y^2}>0,\ \forall(x,y)\in\mathbf{R}^{n+m},\ i\in\mathbf{N}. \tag{5}$$

So, in the present paper, we are dealing only with problem (1) without any limitations and, as it is shown below, strictly convex in y.

2 Problem of Internal Vector Minimum

Proceeding from definition [1, p.99], problem (1) is called strictly convex in y, if for every $x\in\mathbf{X}$ and $i\in\mathbf{N}$ all criteria $f_i(x,y)$ are strictly convex in y, i.e. the set $\mathbf{Y}$ is convex and for any $x\in\mathbf{X}$ the functions $f_i(x,y)$ are strictly convex in y:

$$f_i(x,\lambda y^{(1)}+(1-\lambda)y^{(2)}) < \lambda f_i(x,y^{(1)})+(1-\lambda)f_i(x,y^{(2)}) \tag{6}$$

with every $y^{(1)}\neq y^{(2)}$, $y^{(k)}\in\mathbf{Y}$, $(k=1,2)$ and $\lambda\in(0,1)$.

"Geometrically" this fact means that for every $x\in\mathbf{X}$ the segment, which connects two arbitrary uncoinciding points on the surface $z=f(x,y)$, is above this surface with every

$$y=\lambda y^{(1)}+(1-\lambda)y^{(2)},\ 0<\lambda<1.$$

If $\mathbf{Y}$ is convex and for every $x\in\mathbf{X}$ and $i\in\mathbf{N}$

$$f_i(x,\lambda y^{(1)}+(1-\lambda)y^{(2)}) \leqslant \lambda f_i(x,y^{(1)})+(1-\lambda)f_i(x,y^{(2)}) \tag{7}$$

with every $y^{(k)}\in\mathbf{Y}$, $(k=1,2)$ and $\lambda\in[0,1]$, then problem (1) is called **convex** in y. It is evident that from strict convexity in y of problem (1) follows its convexity. Apparently, a contrary statement is not true.

Lemma 1. If condition 1 is fulfilled, problem (1) is strictly convex in y.

Proof is being carried out by the scheme given in [1, pp. 69-70]. That is, as R^m together with arbitrary uncoinciding points $y^{(1)}$ and $y^{(2)}$ contains the whole segment $[y^{(1)},y^{(2)}]$ as well, then by Tailor's theorem

$$f_i(x,y^{(2)})=f_i(x,y^{(1)})+(y^{(2)}-y^{(1)})'\frac{\partial f_i(x,y^{(1)})}{\partial y}+$$

$$+\frac{1}{2}(y^{(2)}-y^{(1)})'\frac{\partial^2 f_i(x,y^{(1)}+\theta(x)(y^{(2)}-y^{(1)}))}{\partial y^2}\cdot(y^{(2)}-y^{(1)})$$

for some $\theta(x)\in(0,1)$. From (5) and $y^{(1)}\neq y^{(2)}$ then follows

$$f_i(x,y^{(2)})-f_i(x,y^{(1)}) > (y^{(2)}-y^{(1)})'\frac{\partial f_i(x,y^{(1)})}{\partial y}.$$

From this inequality for the point $y^{(3)}=\lambda y^{(1)}+(1-\lambda)y^{(2)}$ and arbitrary $\lambda=\text{const}\in(0,1)$, we get (since $y^{(3)}\neq y^{(1)}$ and $y^{(3)}\neq y^{(2)}$)

$$f_i(x,y^{(1)})-f_i(x,y^{(3)}) > (y^{(1)}-y^{(3)})'\frac{\partial f_i(x,y^{(3)})}{\partial y},$$

$$f_i(x,y^{(2)})-f_i(x,y^{(3)}) > (y^{(2)}-y^{(3)})'\frac{\partial f_i(x,y^{(3)})}{\partial y}.$$

Finally, multiplying the first of these inequalities by λ, the second - by $(1-\lambda)$ and summing them up, we are coming to the chain of correlations

$$\lambda f_i(x,y^{(1)})+(1-\lambda)f_i(x,y^{(2)})-f_i(x,y^{(3)}) >$$

$$> [\lambda y^{(1)}+(1-\lambda)y^{(2)}-y^{(3)}]\cdot\frac{\partial f_i(x,y^{(3)})}{\partial y} = 0$$

owing to the choice $y^{(3)}=\lambda y^{(1)}+(1-\lambda)y^{(2)}$. Then

$$\lambda f_i(x,y^{(1)})+(1-\lambda)f_i(x,y^{(2)}) > f_i(x,\lambda y^{(1)}+(1-\lambda)y^{(2)}),$$

i.e. inequality (6) takes place.

Lemma 2. If problem (1) is strictly convex in y, then the function

$$\varphi(x,y,\alpha)=\sum_{i\in N}\alpha_i f_i(x,y) \qquad (8)$$

is strictly convex in y for any $\alpha=(\alpha_1,\dots,\alpha_N)\in A=\{\alpha\in R^N \mid \sum_{i\in N}\alpha_i=1,$ $\alpha_i\geq 0,\ i\in N\}$ and $x\in X$.

Actually, with every $y^{(1)} \neq y^{(2)}$, $y^{(k)} \in \mathbb{Y}$ $(k=1,2)$, $\alpha \in A$, $\lambda \in (0,1)$ the following chain is valid:

$$\varphi(x, \lambda y^{(1)} + (1-\lambda) y^{(2)}, \alpha) = \sum_{i \in \mathbb{N}} \alpha_i f_i(x, \lambda y^{(1)} + (1-\lambda) y^{(2)}) <$$

$$< \sum_{i \in \mathbb{N}} \alpha_i [\lambda f_i(x, y^{(1)}) + (1-\lambda) f_i(x, y^{(2)})] = \lambda \sum_{i \in \mathbb{N}} \alpha_i f_i(x, y^{(1)}) +$$

$$+ (1-\lambda) \sum_{i \in \mathbb{N}} \alpha_i f_i(x, y^{(2)}) = \lambda \varphi(x, y^{(1)}, \alpha) + (1-\lambda) \varphi(x, y^{(2)}, \alpha).$$

Remark 1. From strict convexity of problem (1) in y it follows its convexity in y, i.e. performance of condition (7).

Gurvitz Theorem [4]. Let the set $\mathbb{Y}$ be convex and problem (1) be convex in y. Then, in order that the uncertainty $y(x,\alpha) \in \mathbb{Y}_S(x)$ (as in (3)), it is necessary and sufficient that there should exist a vector $\alpha \in A$ with which an equality

$$\sum_{i \in \mathbb{N}} \alpha_i f_i(x, y(x,\alpha)) = \min_{y \in \mathbb{Y}} \sum_{i \in \mathbb{N}} \alpha_i f_i(x,y) \tag{9}$$

is valid or in notations (8)

$$\varphi(x, y(x,\alpha), \alpha) = \min_{y \in \mathbb{Y}} \varphi(x,y,\alpha). \tag{10}$$

The proof, which differs from the one given in [4] and based on the construction of an effectively convex set, is given in [5, pp. 104-105].

Lemma 3. If the function $\varphi(x,y,\alpha)$ is strictly convex in y for every $\alpha \in A$ and $x \in \mathbb{X}$, then with every fixed $\alpha \in A$ the function $y(x,\alpha)$, satisfying (10), is single-valued.

Actually, if with at least one pair $(x,\alpha) \in \mathbb{X} \times A$ minimum in (9) could be achieved in two points $y^{(1)}(x,\alpha)$ and $y^{(2)}(x,\alpha)$, then in the middle of the segment, connecting these points, the functiom $\varphi(x,y,\alpha)$ could get less values than

$$\min_{y \in \mathbb{Y}} \varphi(x,y,\alpha)$$

because of strict convexity of a scalar function $\varphi(x,y,\alpha)$ in y with every fixed pair $(x,\alpha) \in \mathbb{X} \times A$. Lemma is proved.

Using the given statements let's go over to the solution of the problem of the internal vector minimum when the condition 1 is fulfilled.

Statement 1. If the condition 1 is fulfilled, then the set $\mathbb{Y}_S(x)$ from (3) can be presented in the following form:

$$\mathbb{Y}_S(x) = \bigcup_{\alpha \in A} y(x,\alpha), \tag{11}$$

where $y(x,\alpha)$ is defined by (9).

Proof. From the condition 1, lemmas 1 and 2 there follows the convexity in y of the problem (1). Then, according to Gurvitz theorem condition (11) takes place. For this, with fixed $x\in\mathbb{X}$ we get the whole set $\mathbb{Y}_S(x)$ of Slater minimal solutions of problem (2) by solving minimization problem (9) with every $\alpha\in A$. It should be taken into account (lemma 3) that to every $\alpha\in A$ there responds a single solution $y(x,\alpha)$ of problem (9).

Remark 2. So, from the statement 1 and its proof we get the following procedure for the solution of the problem of internal vector minimum:

- for every N-vector $\alpha=(\alpha_1,\dots,\alpha_N)$ from the compact $A=\{\alpha\in\mathbb{R}^N|\sum_{i\in\mathbb{N}}\alpha_i=1,\alpha_i\geqslant 0,i\in\mathbb{N}\}$ to find (in the analytical form) the solution $y(x,\alpha)$ of the problem

$$\sum_{i\in\mathbb{N}}\alpha_i f_i(x,y(x,\alpha)) = \min_{y\in\mathbb{R}^m}\sum_{i\in\mathbb{N}}\alpha_i f_i(x,y). \qquad (12)$$

Owing to the condition 1, the strict convexity of a scalar function $\varphi(x,y,\alpha)=\sum_{i\in\mathbb{N}}\alpha_i f_i(x,y)$ in y and lemmas 2 and 3 the solution of problem (12) exists and is unique for every $\alpha\in A$.

According to the strict convexity of $\varphi(x,y,\alpha)$ in y, a sufficient condition of $y(x,\alpha)$ existence is implementation of the system of m equalities

$$\sum_{i\in\mathbb{N}}\alpha_i\frac{\partial f_i(x,y(x,\alpha))}{\partial y} = 0_m. \qquad (13)$$

If (13) does not allow to find $y(x,\alpha)$ immediately and clearly, then for the construction of $y(x,\alpha)$ the theorem about derivative of an unexplicit function [3, p. 462] can be used:

$$\frac{\partial y_k}{\partial x_j} = -\frac{\det[\sum_{i\in\mathbb{N}}\alpha_i\frac{\partial^2 f_i(x,y)}{\partial y\partial(y_{M\setminus\{k\}},x_j)}]}{\det[\sum_{i\in\mathbb{N}}\alpha_i\frac{\partial^2 f_i(x,y)}{\partial y^2}]} = F_{kj}(x,y) \qquad (14)$$

$(k=1,\dots,m,\ j=1,\dots,n).$

Owing to definite positiveness (5) of the quadratic form $z'\frac{\partial^2 f_i(x,y)}{\partial y^2}z$, $i\in\mathbb{N}$, for $z\in\mathbb{R}^m$, and definition of the set A ($\sum_{i\in\mathbb{N}}\alpha_i>0$, $\alpha_i\geqslant 0$, $i\in\mathbb{N}$) the quadratic form $z'[\sum_{i\in\mathbb{N}}\alpha_i\frac{\partial^2 f_i(x,y)}{\partial y^2}]z$

will be also definite positive with every $(x,y)\in\mathbf{R}^{n+m}$. Then

$$\det\left[\sum_{i\in\mathbf{N}}\alpha_i\frac{\partial^2 f_i(x,y)}{\partial y^2}\right]\neq 0,\ \forall(x,y)\in\mathbf{R}^{n+m}.$$

Taking into account this inequality and twice continuous differentiability of $f_i(x,y)$, $i\in\mathbf{N}$, (condition 1) system (14) has a continuous solution $y(x,\alpha)$ in "sufficiently small" neighbourhood of an any point $x\in\mathbf{R}^n$ (with each fixed $\alpha\in A$). Its uniqueness is provided by lemma 3. Finally, in case of scalar $x=x_1\in\mathbf{R}^1$, for continuity of solution (14) on the interval $(-\infty,+\infty)$ it is sufficient to require, for example, the fulfillment of Filippov's condition:

$$\|\mathbf{F}(x,y)\|\leqslant\gamma(1+\|y\|),\ \gamma=\mathrm{const}>0,$$

where $\mathbf{F}(x,y)=(F_{11}(x,y),\dots,F_{m1}(x,y))$.

Let's introduce $m\times n$-matrix (it will be used further):

$$\frac{\partial y}{\partial x}=\begin{bmatrix}\dfrac{\partial y_1}{\partial x_1},\dots,\dfrac{\partial y_1}{\partial x_n}\\ \dots\dots\dots\dots\\ \dfrac{\partial y_m}{\partial x_1},\dots,\dfrac{\partial y_m}{\partial x_n}\end{bmatrix},\tag{15}$$

where $\dfrac{\partial y_k}{\partial x_j}$ have been defined in (14).

3 Problem of External Vector Maximum

Statement 2. If the condition 1 is fulfilled and there exists such a vector $\beta=(\beta_1,\dots,\beta_N)\in A$, that

$$\max_{x\in\mathbf{X}}\max_{\alpha\in A}\sum_{i\in\mathbf{N}}\beta_i f_i(x,y(x,\alpha))=\sum_{i\in\mathbf{N}}\beta_i f_i(x^*,y(x^*,\alpha^*)),\tag{16}$$

where $y(x,\alpha)$ is the solution of problem (9), then x^* is a Slater maximin solution of problem (1), and $y(x^*,\alpha^*)=\hat{y}(x^*)$ is a corresponding uncertainty from $\mathbf{Y}_S(x^*)$.

Proof. Note that we may rearrange the order of getting maximums in (16) because of independence of the sets $\mathbf{X}$ and A. The proof of the statement 2 is based on the statement 1. Namely, according to (11) and Gurvitz theorem, with every fixed $x\in\mathbf{X}$, to each element $y(x)\in\mathbf{Y}_S(x)$ there responds "its" vector α, such that $y(x)=y(x,\alpha)$ is the solution of (9). And vice versa, to each $\alpha\in A$ corresponds "its" single-valued function $y(x,\alpha)$ - the solution of (9). Therefore, correlation (4) (from the definition of a Slater maximin solution of problem (1)) may be represented as follows:

$$f(x^*,y(x^*,\alpha^*)) \nless f(x,y(x,\alpha)),\ \forall x\in\mathbb{X},\ \alpha\in A, \qquad (17)$$

i.e. the pair (x^*,α^*) is Slater maximin in a multicriterion problem

$$< \mathbb{X}\times A,\ f(x,y(x,\alpha)) >, \qquad (18)$$

where $y(x,\alpha)$ is the solution of problem (9). There is a whole series of sufficient conditions of a Slater maximality [5, pp. 66-70]. We'll make use of one of them [5, p. 68], where we'll use increasing by > on $\mathbb{R}^N$ function $\sum_{i\in\mathbb{N}} \beta_i f_i$, $\beta=(\beta_1,\ldots,\beta_N)$ $\in A$. According to [5, p.68], the pair (x^*,α^*), having been found by (16), will be Slater maximal for problem (18), i.e. it satisfies the correlation (17) and, consequently, x^* will be a maximin solution of problem (1), and $y(x^*,\alpha^*)$ is a corresponding to it uncertainty.

Corollary 1. In the case $\beta_i=1$, $\beta_j=0$ $(j\in\mathbb{N}\setminus\{i\})$, the condition (16) is reduced to

$$\max_{x\in\mathbb{X}} \max_{\alpha\in A} f_i(x,y(x,\alpha)) = f_i(x^*,y(x^*,\alpha^*)), \qquad (19)$$

i.e. when the condition 1 is fulfilled a Slater maximal solution x^* (and corresponding uncertainty $\hat{y}(x^*)=y(x^*,\alpha^*)$) might be found from (19), where $y(x,\alpha)$ is the solution of problem (9).

Let's consider the solution of the problem

$$\max_{x\in\mathbb{R}^n} \sum_{i\in\mathbb{N}} \beta_i f_i(x,y(x,\alpha)) = \sum_{i\in\mathbb{N}} \beta_i f_i(x^*,y(x^*(\alpha),\alpha)) \qquad (20)$$

with fixed $\alpha\in A$ and $\beta\in A$ separately.

Statement 3. Let the condition (1) be fulfilled, some $\alpha\in A$ is fixed and $y(x,\alpha)$ is the solution of problem (9). Then, if there exist N-vector $\beta\in A$ and $x^*\in\mathbb{R}^n$, such that

$$\sum_{i\in\mathbb{N}} \beta_i \frac{\partial f_i(x^*,y(x^*,\alpha))}{\partial x} = 0_n,$$

$$\sum_{i\in\mathbb{N}} \beta_i \left(\frac{\partial^2 f_i(x,y(x,\alpha))}{\partial x^2} + \frac{\partial^2 f_i(x,y(x,\alpha))}{\partial x \partial y} \cdot \frac{\partial y(x,y(x,\alpha))}{\partial x}\right)<0,$$
$$\forall x\in\mathbb{X},\ \alpha\in A, \qquad (21)$$

where $\frac{\partial y}{\partial x}$ has been defined in(15), then $x^*(\alpha)$ is the solution of problem (20).

Actually, fulfillment of the requirements (21) is [3, pp. 422-425] a sufficient condition of existence of the solution x^*, satisfying equality (20). The second condition in (21) is composed by taking into account the rules of complex functions differentiation.

Remark 3. From the statements 1 and 3 we get the following algorithm for the construction of a Slater maximin solution x^* of problem (1) in the case when the condition 1 is fulfilled.

1. Find the functions $y(x,\alpha)$ from (9).

2. Construct the solution (x^*,α^*) of problem (16) with at least one $\beta\in A$.

Then x^*, which has been obtained, is a Slater maximin solution of problem (1).

Remark 4. Union of (16) and (20) leads to the following problem:

$$\max_{\alpha\in A}\sum_{i\in\mathbb{N}}\beta_i f_i(x(\alpha),y(x(\alpha),\alpha))=\sum_{i\in\mathbb{N}}\beta_i f_i(x^*(\alpha^*),y(x^*(\alpha^*),\alpha^*)). \quad (22)$$

Getting α^* from (22) is needed for the construction of a Slater maximin $f(x^*(\alpha^*),\ y(x^*(\alpha^*),\alpha^*))$ and a Slater maximin solution $x^*(\alpha^*)$ of problem (1).

4 Problem of Weakly Separated Quadratic Programming

Consider a multicriterion problem of quadratic programming with uncertainty

$$< \mathbb{R}^n,\ \mathbb{R}^m,\ \{f_i(x,y)\}_{i\in\mathbb{N}} >. \quad (23)$$

Here $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, criteria

$$f_i(x,y) = x'A_ix+2\varepsilon x'B_iy+y'C_iy,\ i\in\mathbb{N}, \quad (24)$$

where A_i, B_i, C_i are constant matrices of corresponding dimensions, A_i and C_i being symmetric, prime above means the transposition operation, $\varepsilon>0$ is a small parameter. The criteria (24) are not separated by x and y, as in [7, pp. 68-70]. Namely, the items $\varepsilon x'B_iy$ "prevent" to separate the criteria $f_i(x,y)$ into the items, containing only x and only y. These items are small owing to a small parameter $\varepsilon>0$ (hence we have the title "weakly separated").

Statement 4. Let

$$C_i>0,\ i\in\mathbb{N}, \quad (25)$$

and there exists, at least, one number $j\in\mathbb{N}$, such that

$$A_j<0. \quad (26)$$

Then with sufficiently small values of the parameter $\varepsilon>0$ the Slater maximin solution of problem (23)-(24) is $x^*=0_n$, the corresponding uncertainty $\hat{y}(x^*)=0_m$ and the Slater maximin $f(x^*,\hat{y}(x^*))=0_N$.

Remark 5. In conclusion it would certainly present a great interest if the approach proposed here for the cases of multicriteria problems with constraints and undifferentiable criteria should be expanded and generalized.

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Error Estimates for the Crude Approximation of the Trade-off Curve

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Abstract. The crude global search that is used in Parameter Space Investigation provides an approximation of the trade-off curve. An estimate of the approximation error is obtained and numerical expcriments confirm the estimate.

Keywords. Multiple criteria decision making, parameter space investigation, Pareto optimality, trade-off curve, optimization, LP_τ -sequences, quasi-random search, global search

1.Introduction. Consider a nonlinear problem with multiple objectives of the form

$$f_1(x) \to \min, \dots, f_k(x) \to \min; \quad x \in I^n, \tag{1}$$

where $x = (x_1, \dots, x_n)$ is a point of the n-dimensional unit cube I^n, so that $0 \le x_j \le 1,\ 1 \le j \le n$.

The Parameter Space Investigation (PSI) method was developed [3,4] as an interactive method for constructing the set of admissible solutions (that is solutions with acceptable objective function values for all objectives simultaneously). However PSI can be used also for obtaining a crude approximation of the set E of efficient (or nondominated, or Pareto-optimal) points. Its algorithm is easy to use and very reliable though the convergence is in general slow.

In this paper for the first time error estimates for the approximate trade-off curve are derived. If N trial points are used in PSI, the worst error estimate is of the order $N^{-1/n}$ but it may be much better in particular cases.

2.The computation algorithm. Trial points $x^{(1)}, x^{(2)}, \dots$ are selected that fill uniformly the cube. At each of these points, say $x^{(i)}$, the criterion vector $(f_1(x^{(i)}, \dots, f_k(x^{(i)}))$ is computed. If this vector is dominated by any currently retained criterion vector, the point $x^{(i)}$ is discarded. If not, the point $x^{(i)}$ is retained while all currently retained points with criteria vectors dominated by the new one must be discarded.

Thus for $i = N$, we obtain $N_0 \le N$ trial points that are called approximately efficient. The finite set of these N_0 points is denoted by E_N and can be regarded as an approximation to E.

3. Objective functions satisfying a general Lipschitz condition. Assume that all the functions $f_p(x)$, $1 \le p \le k$, satisfy a common general Lipschitz condition: for arbitrary x and x' in I^n

$$|f_p(x') - f_p(x)| \le \rho(x, x'), \tag{2}$$

where

$$\rho(x, x') = \sum_{j=1}^{n} L_j |x'_j - x_j| \tag{3}$$

and all $L_j \ge 0$. The word "general" is used to stress that the Lipschitz constants L_j may be different. Some L_j may even be zero.

In [1] the ρ-dispersion d_N of the points $x^{(1)}, \dots, x^{(N)}$ was introduced:

$$d_N = \sup_{x \in I^n} \min_{1 \le s \le N} \rho(x, x^{(s)}). \tag{4}$$

Then for an arbitrary point $x \in I^n$ a trial point $x^{(s)}$ exists, so near to x that

$$|f_p(x^{(s)}) - f_p(x)| \le d_N \tag{5}$$

for all $1 \le p \le k$ simultaneously. Of course, the same is true when x is an efficient point, $x \in E$.

Now, the trial point $x^{(s)}$ in (5) corresponding to an efficient point x is not necessarily approximately efficient. If we retain trial points from E_N only, a weaker assertion can be made: *For an arbitrary $x \in E$ a trial point $x^{(i)} \in E_N$ can be found so that*

$$\min_{1 \le p \le k} |f_p(x^{(i)}) - f_p(x)| \le d_N. \tag{6}$$

Indeed, if the trial point $x^{(s)}$ satisfying (5) is in E_N then clearly (5) implies (6) with $x^{(i)} = x^{(s)}$. If $x^{(s)}$ is not in E_N then it is dominated by another trial point $x^{(i)} \in E_N$. Since x is an efficient point, inequalities

$$f_p(x^{(i)}) \le f_p(x)$$

cannot hold for all $1 \le p \le k$, and for at least one index $p = q$

$$f_q(x) \le f_q(x^{(i)}) \le f_q(x^{(s)}).$$

It follows from (5) that

$$|f_q(x^{(i)}) - f_q(x)| \le d_N$$

and this implies (6).

4.Quasirandom trial points. A new estimate of the lower bound for d_N was introduced in [1], namely

$$c_N = \frac{1}{2}\max(s!L_{j_1}\dots L_{j_s}/N)^{1/s}; \tag{7}$$

the maximum in (7) is extended over all sets $1 \le j_1 < \dots < j_s \le n$ and $s = 1, 2, \dots, n$.

It was proved in [1] that

1)For arbitrary points $x^{(1)}, \dots, x^{(N)} \in I^n$

$$d_N \ge c_N.$$

2)For an arbitrary P_τ-net in I^n

$$d_N \le Ac_N,$$

where $A = A(n,\tau)$ depends neither on N nor on $L_1, \dots, L_n$.

Clearly, c_N defines the best possible order of convergence of d_N as $N \to \infty$.

Various optimization theories consider only Lipschitz classes with equal Lipschitz constants $L_j = L$ for $1 \le j \le n$. In this case (7) implies that

$$c_N = \frac{1}{2}(n!)^{1/n} L\, N^{-1/n}. \tag{8}$$

At large n, the order of convergence (8) is poor. However, if there are only t positive constants among the L_j, $t < n$, then $c_N \sim N^{-1/t}$, which can be much better than (8). One may expect that if the L_j are of different orders of magnitude, (7) will be a much more realistic estimate than (8).

In practical problems, the total number n of decision variables may be large, but individual objectives often depend heavily on a few of these variables and are not very sensitive to the others; therefore most of the L_j are very small indeed.

As a rule in PSI, points of a quasirandom $L\,P_\tau$-sequence are used as trial points [2,3,4]. Initial sections $x^{(1)}, \dots, x^{(N)}$ of such sequences containing $N = 2^\nu$ points are P_τ-nets at all sufficiently large integers ν. Therefore it is advisable to monitor the convergence comparing results obtained at successive ν.

Points of $L\,P_\tau$-sequences can be easily computed using computer codes published in [5] for $n \le 51$, $N < 2^{30}$. The languages are "C" or FORTRAN-77.

5.Bi-criterial problems. Consider the problem

$$f(x) \to \min, \quad g(x) \to \min; \quad x \in I^n. \tag{9}$$

Assume that for arbitrary x and x' in I^n

$$|f(x') - f(x)| \le \sum_{j=1}^{n} M_j |x'_j - x_j|, \tag{10}$$

$$|g(x') - g(x)| \le \sum_{j=1}^{n} R_j |x'_j - x_j|. \tag{11}$$

Then both objective functions, f and g, satisfy (2)–(3) with $L_j = \max(M_j; R_j)$.

The functions $f(x)$ and $g(x)$ define a mapping of I^n into the criterial plane (f, g) and the image of E is called the trade-off curve. In practice, the trade-off curve is often regarded as a final solution of problem (9).

The image of a point $x^{(i)} \in E_N$ is the point $Q_i = (f_i, g_i)$ with coordinates $f_i = f(x^{(i)})$, $g_i = g(x^{(i)})$. The points Q_i can be easily ordered since increasing f_i correspond to decreasing g_i. A monotone polygonal line connecting all the points Q_i is called an approximate trade-off curve. Its convergence to the exact trade-off curve was established in [4].

Theorem. *Let $x^{(i)} \in E_N$ and the straight lines $f = f_i$ and $g = g_i$ intersect the trade-off curve in points P_1 and P_2 (Fig.1). Assume that the arc P_1P_2 of the trade-off curve is continuous and consider the stripe between this arc and its translation obtained by adding the vector $(\vec{i} + \vec{j})d_N$. Then the point $Q_i = (f_i, g_i)$ falls inside the stripe.*

Proof. Assume the contrary: Q_i is outside the stripe (Fig.1). Clearly, there are no trial points inside the curvilinear triangle $Q_iP_1P_2$. Therefore the point P in Fig.1 has a preimage $x \in E$ whose properties contradict (6).

Corollary. *Denote the distances $|Q_iP_1| = \Delta g_i$, $|Q_iP_2| = \Delta f_i$. Then*

$$\min(\Delta f_i; \Delta g_i) \le d_N. \tag{12}$$

The theorem suggested the following measure of proximity of the approximate trade-off curve to the exact one:

$$\Delta_N = \max_i \min(\Delta f_i; \Delta g_i), \tag{13}$$

where the maximum is extended over all i corresponding to trial points $x^{(i)} \in E_N$.

If the trial points $x^{(1)}, \ldots, x^{(N)}$ are points of a P_τ-net, one can expect that

$$\Delta_N \leq A\, c_N .$$

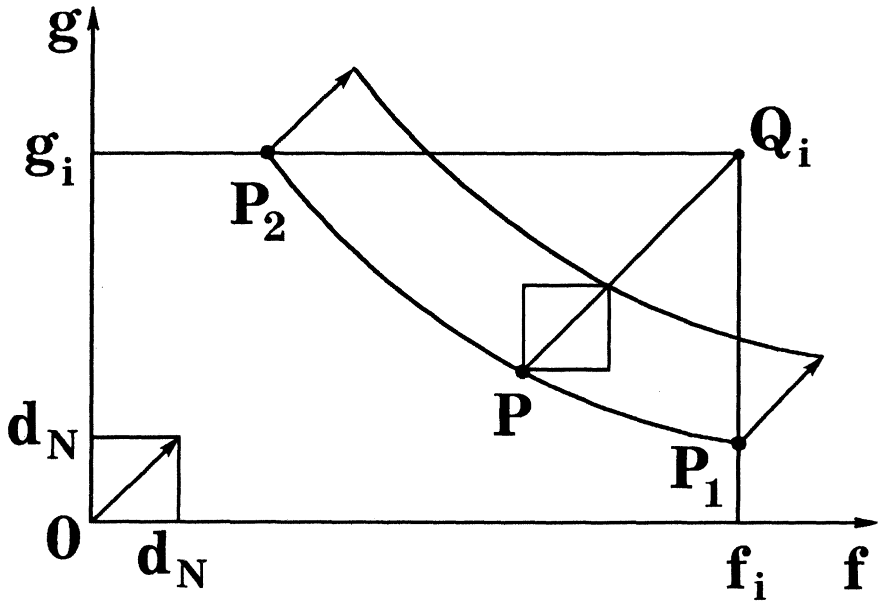

Fig. 1. Proof of the theorem.

6.Numerical experiment. The main goal of our experiment was to verify whether Δ_N really behaves as c_N. First, will Δ_N be of order $N^{-1/n}$ if in one of the conditions (10) or (11) all the Lipschitz constants are equal? And second, will the convergence of Δ_N improve if the number of large values $L_j = \max(M_j; R_j)$ is smaller than n?

Let $n = 8$ and consider problem (9) with

$$f = \sum_{j=1}^{n} A_j x_j^2, \quad g = \sum_{j=1}^{n} B_j (1 - x_j)^2.$$

In this case the equation grad$f = -\lambda\,$grad$g$ yields a parametric equation of E:

$$x_j = \lambda B_j/(A_j + \lambda B_j), \quad 1 \leq j \leq n, \quad 0 \leq \lambda \leq \infty.$$

The trade-off curve in parametric form:

$$f = \sum_{j=1}^{n} A_j \left(\frac{\lambda B_j}{A_j + \lambda B_j} \right)^2, \qquad g = \sum_{j=1}^{n} B_j \left(\frac{A_j}{A_j + \lambda B_j} \right)^2. \tag{14}$$

The conditions (2) - (3) will be satisfied if we set

$$L_j = 2\ \max(A_j; B_j).$$

We have solved numerically eight problems that were divided into two series. The corresponding values of A_j and B_j are listed in Table 1 together with the number m, that occurs in the analytical expression of the trade-off curve obtained by elimination of λ from equations (14):

$$\sqrt{f} + \sqrt{g} = \sqrt{m}.$$

Table 1

Ser	Prob	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	m
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	8
	2	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	4
	3	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	2
	4	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1
2	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0
	2	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	4
	3	1	1	1	1	0	0	0	0	1	1	0	0	0	0	0	0	2
	4	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	2

For all the problems in Ser 1 as well as for problem 1 in Ser 2 the common Lipschitz constants are $L_1 = \ldots = L_8 = 2$ and the common $c_N \sim N^{-1/8}$. In Ser 2 for problems 2 and 3 we have $L_1 = \ldots = L_4 = 2, L_5 = \ldots = L_8 = 0$, so that $c_N \sim N^{-1/4}$. And for problem 4 in Ser 2 only $L_1 = L_2 = 2$ are positive, so that $c_N \sim N^{-1/2}$.

7.Numerical results. Fig.2 shows the Δ_N values for Ser 1. Here $x = \log_2 N$, $y = \lg \Delta_N$, $+Prob1, \times Prob2, \diamondsuit Prob3, \square Prob4$. The straight line without markers shows the values $y = \lg(c_N/5)$ that are common to all these problems. The average slopes of all the Δ_N graphs are near to the slope of the straight line.

Fig.3 contains the Δ_N values for Ser2. The notations and markers are the same as in Fig.2. The three straight lines without markers show the values $y = \lg(c_N/6)$ for Prob 1, for Probs 2 and 3, and for Prob 4. Again, the slopes of these lines agree with the average slopes of the corresponding Δ_N graphs.

In problem 1 Ser 2 the trade-off curve degenerates into one point $f = 0$, $g = 0$. Therefore the computation of Δ_N was carried out for a perturbed problem described below.

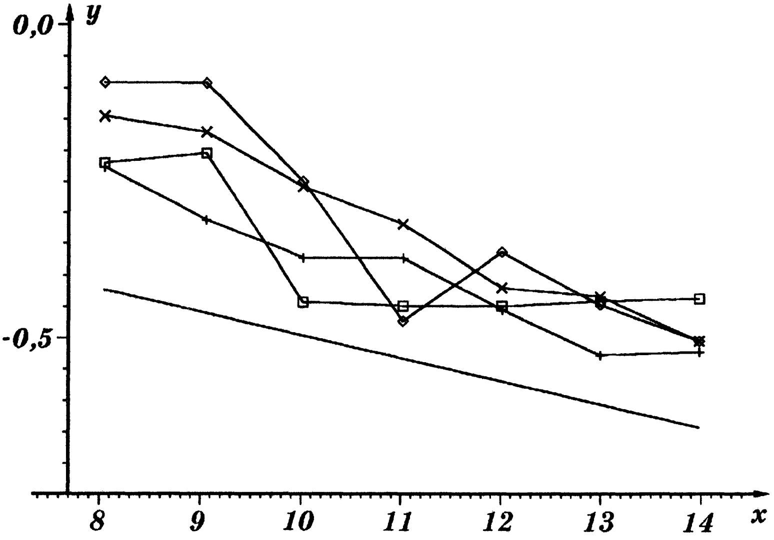

Fig. 2. Δ_N values for Ser1 ($x = \log_2 N$, $y = \lg \Delta_N$).

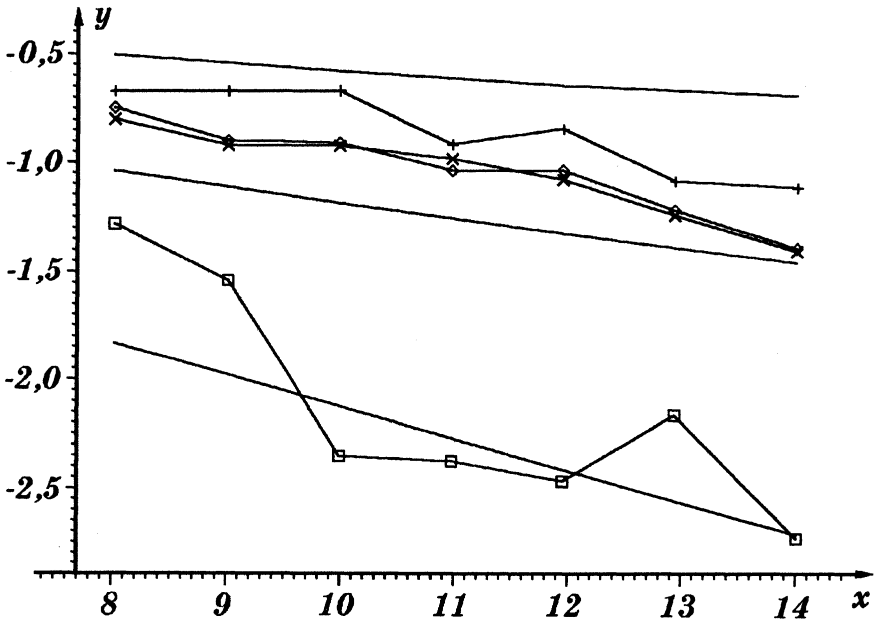

Fig. 3. Δ_N values for Ser2 ($x = \log_2 N$, $y = \lg \Delta_N$).

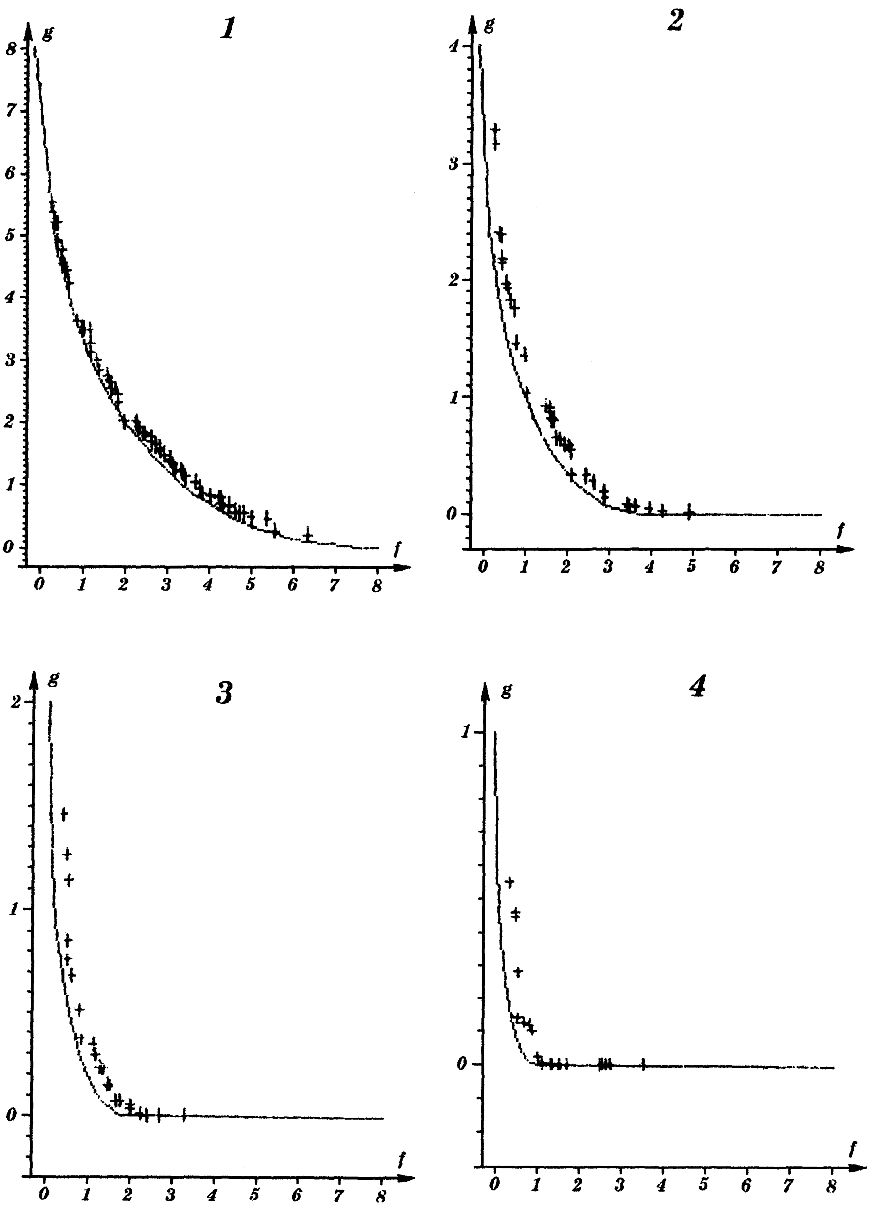

Fig. 4. Trade-off curves for Ser1 and their approximations at $N = 8192$.

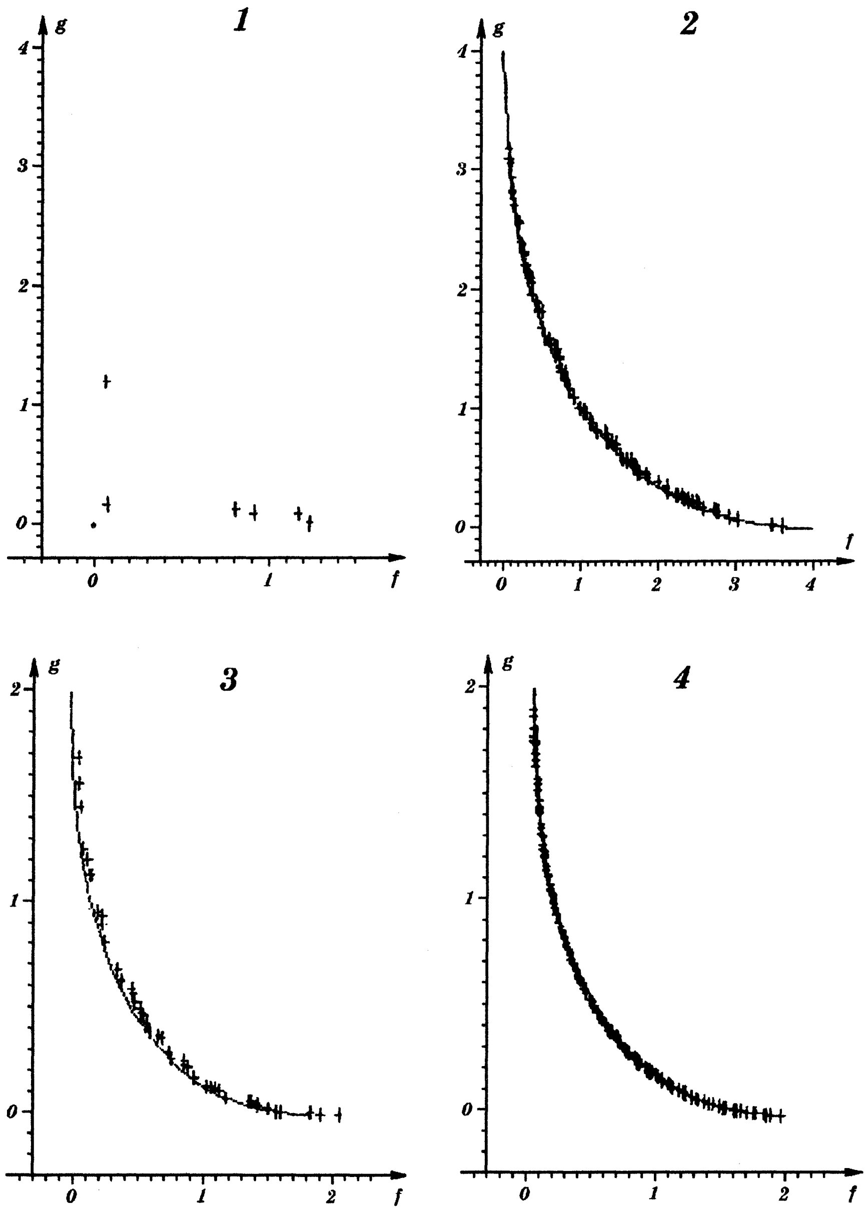

Fig. 5. Trade-off curves for Ser2 and their approximations at $N = 2048$.

Figures 4 and 5 show exact trade-off curves and computed points (f_i, g_i) at $N = 8192$ for Ser 1, and at $N = 2048$ for Ser 2.

In our experiment the number N_0 of approximately efficient points was roughly proportional to N^{β}, where β varied for different problems from 0 to 0.60.

For checking the stability of solutions, eight perturbed problems (9) were computed: in Table 1 all the zeros were replaced by $\varepsilon = 0.001$. For these problems, exact trade-off curves were constructed numerically from formulas (14) using 20 values of λ and linear interpolation. The results were practically the same.

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Stability and Sensitivity Analysis in Noncooperative Games

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Abstract: In this paper we discuss stability and sensitivity analysis of Nash equilibria which are fundamental solutions in noncooperative games. The results concerning stability are also closely related to the refinement of Nash equilibria studied actively in these days. In finite two-person noncooperative games (bimatrix games), particularly, we can obtain some interesting results about stability and sensitivity of Nash equilibria by applying the implicit function theorem to the corresponding linear complementarity problems.

Keywords: bimatrix games, Nash equilibria, stability and sensitivity analysis, linear complementarity problems

1 Introduction

Game theory, particularly noncooperative game theory has been developed as a mathematical tool for dealing with decision making situations in which there exist multiple decision makers whose objectives (payoffs) often conflict each other. It can be applied to many problems from economics, political science, evolutionary biology, engineering, computer science and so on ([3], [6]).

In this paper we concentrate on two-person nonzero-sum games with finite pure strategy sets (bimatrix games), which form a very important class of noncoperative games with many practical applications. The concept of Nash equilibria is the most fundamental solution concept in those games. It is quite important to analyze the behavior of a Nash equilibrium when the values of payoffs change because of several reasons. Qualitative analysis leads to investigating stability of a Nash equilibrium and it is closely related to the refinement of the equilibrium concept. Quantitative anlysis leads to sensitivity analysis of an equilibrium and it provides an interesting information on robustness of the equilibrium.

The paper is organized as follows. In section 2, we formulate a bimatrix game considered in this paer and define a Nash equilibrium of the bimatrix game. Section 3 is devoted to various refinements of the Nash concept closely related to stability of the equilibrium. In section 4, we describe a one-to-one correspondence between Nash equilibria of a bimatrix game and the solution of a certain linear complementarity problem. In section 5, it is shown that stability and sensitivity analysis of a Nash equilibrium is possible by applying the implicit function theorem to the corresponding linear complementarity problem.

2 Bimatrix Games and Nash Equilibria

In this paper we consider a bimatrix game (two-person nonzero-sum game) $\Gamma(A,B)$ defined as follows. There are two players 1 and 2, and the set of pure strategies of player 1 (2) is denoted by $\mathrm{N}_m = \{1,2,\dots,m\}$ ($\mathrm{N}_n = \{1,2,\dots,n\}$). The set of all mixed strategies of player 1 (2) is given by

$$\begin{array}{rcl} S^m & = & \{p \in R^m | p_i \geq 0 \ \forall i \in \mathrm{N}_m, \ \sum_{i=1}^m p_i = 1\} \\ (S^n & = & \{q \in R^n | q_j \geq 0 \ \forall j \in \mathrm{N}_n, \ \sum_{j=1}^n q_j = 1\}). \end{array}$$

The payoff of player 1 is given by a_{ij} when player 1 takes the pure strategy i and player 2 takes the pure strategy j. The $m \times n$ matrix $A = (a_{ij})$ is the payoff matrix of player 1 and the expected payoff player 1 when he takes the mixed strategy $p \in S^m$ and player 2 takes the mixed strategy $q \in S^n$ is given by $p^T Aq$. Similarly the payoff matrix of player 2 is given by $B(m \times n)$.

The most fundamental concept of solutions in this game $\Gamma(A,B)$ is the following Nash equilibria.

Definition 1 *A pair $(\bar{p},\bar{q}) \in S^m \times S^n$ is called an Nash equilibrium (point) of the bimatrix game $\Gamma(A,B)$ if*

$$\bar{p}^T A\bar{q} = \max_{p\in S^m} p^T A\bar{q} \quad \textit{and} \quad \bar{p}^T B\bar{q} = \max_{q\in S^n} \bar{p}^T Bq$$

The set of all Nash equilibrium points of (A,B) is denoted by $E(A,B)$.

The following notations are also used in this paper.

$$\begin{array}{rcl} C(p) & = & \{i \in \mathrm{N}_m | p_i > 0\} \\ C(q) & = & \{j \in \mathrm{N}_n | q_j > 0\} \end{array}$$

$$\begin{array}{rcl} D(A;q) & = & \{i \in \mathrm{N}_m | e_i^T Aq = \max_{k \in \mathrm{N}_m} e_k^T Aq\} \\ D(p;B) & = & \{j \in \mathrm{N}_n | p^T Be_j = \max_{l \in \mathrm{N}_n} p^T Be_l\} \end{array}$$

where e_i denotes the ith unit vector in R^m or R^n.

Proposition 1 *A pair $(p,q) \in S^m \times S^n$ is a Nash equilibrium of the game $\Gamma(A,B)$ if and only if $C(p) \subset D(A;q)$ and $C(q) \subset D(p;B)$.*

Definition 2 *A subset $S \subset E(A,B)$ is called a Nash subset for the game $\Gamma(A,B)$ if for all $(p,q),(p',q') \in S$ also $(p,q'),(p',q) \in S$. A Nash subset S is called a maximal Nash subset for the game $\Gamma(A,B)$ if there exists no Nash susbet properly containing S.*

Definition 3 *A Nash equilibrium (p,q) of a bimatrix game $\Gamma(A,B)$ is called isolated if there exists a neighborhood V of (p,q) such that $V \cap E(A,B) = \{(p,q)\}$.*

Proposition 2 *A Nash equilibrium (p,q) of a bimatrix game $\Gamma(A,B)$ is isolated if and only if $\{(p,q)\}$ is a maximal Nash subset for $\Gamma(A,B)$.*

Definition 4 *A Nash equilibrium (p,q) of a bimatrix game $\Gamma(A,B)$ is called quasi-strict if $C(p) = D(A;q)$ and $C(q) = D(p;B)$.*

3 Refinement and Stability of Nash Equilibria

In this section we review several refinements of Nash equilibria in the bimatrix game, which are closely related to the stability of those solutions. First the following perfect equilibrium was introduced by Selten [5]. The perfectness concept is closely related to the stability with respect to mistakes with a small probability made by each player.

Definition 5 *Let $\Gamma(A,B)$ be a bimatrix game and let $\xi, \eta, P(\xi)$ and $Q(\eta)$ be defined by*

$$\begin{array}{llll} \xi \in R^m & \textit{with } \xi_i > 0 & \textit{for all } i \in \mathrm{N}_m & \textit{and } \sum_{i=1}^m \xi_i < 1 \\ \eta \in R^n & \textit{with } \eta_j > 0 & \textit{for all } j \in \mathrm{N}_m & \textit{and } \sum_{i=1}^n \eta_j < 1, \end{array}$$

$$\begin{array}{rcl} P(\xi) & = & \{p \in S^m | p_i \geq \xi_i \ \textit{ for all } \ i \in \mathrm{N}_m\} \\ Q(\eta) & = & \{q \in S^n | q_j \geq \eta_j \ \textit{ for all } \ j \in \mathrm{N}_n\}. \end{array}$$

The perturbed game $\Gamma(\xi,\eta)$ is infinite two-person normal form game with the restricted strategy sets $P(\xi)$ and $Q(\eta)$ instead of S^m and S^n. The set of Nash equilibria of $\Gamma(\xi,\eta)$ is denoted by $E(A,B,\xi,\eta)$.

Definition 6 *An (Nash) equilibrium (p,q) of a bimatrix game $\Gamma(A,B)$ is called a perfect equilibrium of Γ if there exist sequences $\{(p^k,q^k)\}$ and $\{(\xi^k,\eta^k)\}$ with $(p^k,q^k) \in E(A,B,\xi,\eta)$ for all k, and such that $(p^k,q^k) \to (p,q)$, $\xi^k \to 0$, $\eta^k \to 0$ as $k \to \infty$.*

Okada [4] slightly strengthened this concept.

Definition 7 *For $(\hat{\xi},\hat{\eta}) \in R^m \times R^n, (\hat{\xi},\hat{\eta}) > 0$ let $\hat{U} = \{(\xi,\eta) \in R^m \times R^n | 0 < (\xi,\eta) < (\hat{\xi},\hat{\eta})\}$. An equlibrium (p,q) of a bimatrix game $\Gamma(A,B)$ is called a strictly perfect equilibrium if there exists some $(\hat{\xi},\hat{\eta}) \in R^m \times R^n$ and for each $(\xi,\eta) \in \hat{U}$ some $(p(\xi,\eta),q(\xi,\eta)) \in E(A,B,\xi,\eta)$ such that $\lim_{(\xi,\eta)\to 0}(p(\xi,\eta),q(\xi,\eta)) = (p,q)$.*

The following essential equilibrium concept, introduced by Wu Wen-Tsuün and Jiang Jia-He [7], and the concept of strongly stable equilibrium, introduced by Kojima et al. [2], are based on the idea that a reasonable equilibrium should be stable against slight perturbations in the payoff matrices.

Definition 8 *An equlibrium (p,q) of a bimatrix game $\Gamma(A,B)$ is called an essential equilibrium if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for every game $\Gamma(A',B')$ with $\|A - A'\| < \delta, \|B - B'\| < \delta$ there exists some $(p',q') \in E(A',B')$ with $\|(p,q) - (p',q')\| < \epsilon$.*

Definition 9 *An equlibrium (p,q) of a game $\Gamma(A,B)$ is called a strongly stable equilibrium if for some $\epsilon^* > 0$ and each $\epsilon \in (0,\epsilon^*]$ there exists a $\delta > 0$ such that whenever $\|a - a'\| \leq \delta, \|B - B'\| \leq \delta$,*
1) there is a point $(p',q') \in E(A',B')$ such that $\|(p,q) - (p',q')\| < \epsilon$,
2) (p',q') is a unique Nash equilibrium of $\Gamma(A',B')$ satisfying $\|(p,q) - (p',q')\| < \epsilon^*$.

The following relationships hold among the above refinements of the Nash equilibrium (van Damme [6], Jansen [1]).

Theorem 1 *For a bimatrix game (A,B), the following assertions are equivalent:*

1) (p,q) is a strongly stable equilibrium.
2) (p,q) is an isolated essential equilibrium.
3) (p,q) is an isolated and strictly perfect equilibrium.
4) (p,q) is an isolated and quasi-strict equilibrium.
5) (p,q) is an essential quasi-strict equilibrium.

4. The Linear Complimentarity Problem Corresponding to a Bimatrix Game

In this section we describe a well-known correspondence between the Nash equilibria of a bimatrix game and the solutions of a certain linear complementarity problem. Let $\Gamma(A, B)$ be a bimatrix game. Because of the strategic equivalence of games, we may assume without loss of generality that every element of the payoff matrices A and B is negative, *i.e.* $A < 0, B < 0$. Let $r = \begin{pmatrix} -1_m \\ -1_n \end{pmatrix} \in R^{m+n}$ where 1_m denotes the m-dimensional vector whose components are all 1, and

$$M = \begin{pmatrix} 0 & -A \\ -B^T & 0 \end{pmatrix} \ : \ (m+n) \times (m+n) \text{ matrix}$$

We call the linear complementarity problem

$$\begin{aligned} &z \geq 0_{m+n} \\ &w = Mz + r \geq 0_{m+n} \\ &w^T z = 0 \end{aligned}$$

the linear complimentarity problem corresponding to the bimatrix game $\Gamma(A, B)$ and denote it by LCP(A, B). If we put

$$\begin{aligned} z &= (x, y), \quad x \in R^m, \quad y \in R^n, \\ w &= (u, v), \quad u \in R^m, \quad v \in R^n, \end{aligned}$$

the above linear complemntarity problem can be rewritten as follows:

$$u_i = -\sum_{j=1}^{n} a_{ij} y_j - 1, \quad i = 1, \ldots, m \tag{1}$$

$$v_j = -\sum_{i=1}^{m} b_{ij} x_i - 1, \quad j = 1, \ldots, n \tag{2}$$

$$u_i x_i = 0, \ i = 1, \ldots, m \tag{3}$$

$$v_j y_j = 0, \ j = 1, \ldots, n \tag{4}$$

$$x_i \geq 0, \quad u_i \geq 0, \quad i = 1, \ldots, m \tag{5}$$

$$y_j \geq 0, \quad v_j \geq 0, \quad j = 1, \ldots, n \tag{6}$$

Lemma 1 *Let $\Gamma(A,B)$ be a bimatrix game with $A < 0, B < 0$. If $(p,q) \in E(A,B)$, then $(-\frac{p}{p^TBq}, -\frac{q}{p^TAq})$ is a solution of LCP (A,B). Conversely, if $(x,y) \in R^m \times R^n$ is a solution of LCP (A,B), then $x \neq 0, y \neq 0$ and $(\frac{x}{\sum_{i=1}^m x_i}, \frac{y}{\sum_{j=1}^n y_j}) \in E(A,B)$.*

Thus, in order to obtain the Nash equilibria of the bimatrix game $\Gamma(A,B)$, we may solve the linear complementarity problem LCP(A,B).

5. Sensitivity Analysis of Nash Equilibria

In this section we consider sensitivity analysis of Nash equilibria of the bimatrix game via its corresponding linear complementarity problem. Let $(\hat{p},\hat{q})$ be a Nash equilibrium of a bimatrix game $\Gamma(A,B)$ and $(\hat{u},\hat{v},\hat{x},\hat{y})$ be the corresponding solution of LCP(A,B). Suppose that the partition $I,\bar{I}$ of N_m (*i.e.*, $I \cup \bar{I} = \mathrm{N}_m, I \cap \bar{I} = \emptyset$) and the partition $J,\bar{J}$ of N_n satisfy

$$\hat{u}_i = 0,\ \forall i \in I; \qquad \hat{x}_i = 0,\ \forall i \in \bar{I} \tag{7}$$

$$\hat{v}_j = 0,\ \forall j \in J; \qquad \hat{y}_j = 0,\ \forall j \in \bar{J} \tag{8}$$

Of course $C(\hat{p}) \subset I \subset D(A;\hat{q})$ and $C(\hat{q}) \subset I \subset D(\hat{p};B)$. Rearrange the elements of $u,x(v,y)$ according to the partition $I,\bar{I}(J,\bar{J})$ as

$$u = \begin{pmatrix} u_I \\ u_{\bar{I}} \end{pmatrix},\ x = \begin{pmatrix} x_I \\ x_{\bar{I}} \end{pmatrix};\ (v = \begin{pmatrix} v_J \\ v_{\bar{J}} \end{pmatrix},\ y = \begin{pmatrix} y_J \\ y_{\bar{J}} \end{pmatrix})$$

The submatrix of A, whose rows (resp. collumns) correspond to those in I (resp. J), is denoted by A_{IJ}. $A_{\bar{I}J}, B_{IJ}$ and so on denote the analogous submatrices.

In the bimatrix game, the perturbation of the payoff matrices can be also represented by a pair of matrices (Φ,Ψ). Let us consider the following system of equations.

$$\begin{aligned} (A_{IJ} + \Phi_{IJ})y_J + 1 &= 0 & (9)\\ u_{\bar{I}} + (A_{\bar{I}J} + \Phi_{\bar{I}J})y_J + 1 &= 0 & (10)\\ (B_{IJ}^T + \Psi_{IJ}^T)x_I + 1 &= 0 & (11)\\ v_{\bar{J}} + (B_{I\bar{J}}^T + \Psi_{I\bar{J}}^T)x_I + 1 &= 0 & (12) \end{aligned}$$

$$u_I = 0,\ v_J = 0,\ x_{\bar{I}} = 0,\ y_{\bar{J}} = 0 \tag{13}$$

where the vector of appropriate dimension whose elementws are all 1(0) is also denoted by 1(0) for simplicity.

The solution of the above system of equations satisfies equations (1)-(4) for the linear complementarity problem corresponding to the perturbed game $\Gamma(A+\Phi, B+\Psi)$. Therefore, when we deal with the stability of the Nash equilibrium $(\hat{p}, \hat{q})$ of $\Gamma(A, B)$ with respect to the perturbation (Φ, Ψ), we should consider solutions of the above system of equations for possible partitions I, J based on the corresponding solution $(\hat{u}, \hat{v}, \hat{x}, \hat{y})$ of LCP (A, B). We should also note that the above system of equations does not include the inequalities (5) and (6).

Now we shall try to apply the implicit function theorem to the system of equations (9)-(13) with the independent variables Φ, Ψ and the dependent variables u, v, x, y. Since $u_I = 0,\ v_J = 0,\ x_{\bar{I}} = 0,\ y_{\bar{J}} = 0$ from equation (13), we deal with equations (9)-(12) and the variables $u_{\bar{I}},\ v_{\bar{J}},\ x_I,\ y_J$. The Jacobian matrix obtained by differentiating the left hand side of (9)-(12) with respect to $u_{\bar{I}},\ v_{\bar{J}},\ x_I,\ y_J$ is given by

$$G = \begin{pmatrix} 0 & 0 & 0 & A_{IJ} \\ E & 0 & 0 & A_{\bar{I}J} \\ 0 & 0 & B_{IJ}^T & 0 \\ 0 & E & B_{I\bar{J}}^T & 0 \end{pmatrix} \tag{14}$$

where E denotes the unit matrix of appropriate dimension. If G is nonsingular, we can apply the implicit function theorem.

Theorem 2 *The matrix G of equation (14) is nonsingular if and only if $|I| = |J|$ and both A_{IJ} and B_{IJ} are nonsingular, where $|I|$ denotes the number of elements in the set I.*

Now, if the conditions in the above theorem are satisfied, there exist the unique functions $u(\Phi,\Psi)$, $v(\Phi,\Psi)$, $x(\Phi,\Psi)$, $y(\Phi,\Psi)$ of class C^1 defined on a ceratin neighborhood of $\Phi = \Psi = 0$ satisfying equations (9)-(13) and

$$u(0,0) = \hat{u},\ v(0,0) = \hat{v},\ x(0,0) = \hat{x},\ y(0,0) = \hat{y} \tag{15}$$

Moreover, since $G^{-1} = \begin{pmatrix} -A_{\bar{I}J}A_{IJ}^{-1} & E & 0 & 0 \\ 0 & 0 & -B_{I\bar{J}}^T B_{IJ}^{-T} & E \\ 0 & 0 & B_{IJ}^{-T} & 0 \\ A_{IJ}^{-1} & 0 & 0 & 0 \end{pmatrix}$, the derivatives of these functions at $\Phi = \Psi = 0$ are obtained as follows:

$$\frac{\partial x_I}{\partial \varphi_{ij}} = 0, \quad \forall i, j \tag{16}$$

$$\frac{\partial x_I}{\partial \psi_{ij}} = -B_{IJ}^{-T} \frac{\partial \Psi_{IJ}^T}{\partial \psi_{ij}} \hat{x}_I = \begin{cases} -B_{IJ}^{-T} \begin{pmatrix} 0 \\ \hat{x}_i \\ 0 \end{pmatrix}, & i \in I,\ j \in J \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

$$\frac{\partial y_J}{\partial \varphi_{ij}} = -A_{IJ}^{-1} \frac{\partial \Phi_{IJ}}{\partial \varphi_{ij}} \hat{y}_J = \begin{cases} -A_{IJ}^{-1} \begin{pmatrix} 0 \\ \hat{y}_j \\ 0 \end{pmatrix}, & i \in I,\ j \in J \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

$$\frac{\partial y_J}{\partial \psi_{ij}} = 0, \ \ \forall i, j \tag{19}$$

$$\frac{\partial u_{\bar{I}}}{\partial \varphi_{ij}} = [-\frac{\partial \Phi_{\bar{I}J}}{\partial \varphi_{ij}} + A_{\bar{I}J} A_{IJ}^{-1} \frac{\partial \Phi_{IJ}}{\partial \varphi_{ij}}] \hat{y}_J = \begin{cases} 0, & j \in \bar{J} \\ -\begin{pmatrix} 0 \\ \hat{y}_j \\ 0 \end{pmatrix}, & i \in \bar{I}, j \in J \\ A_{\bar{I}J} A_{IJ}^{-1} \begin{pmatrix} 0 \\ \hat{y}_j \\ 0 \end{pmatrix}, & i \in I, j \in J \end{cases} \tag{20}$$

$$\frac{\partial u_{\bar{I}}}{\partial \psi_{ij}} = 0, \ \ \forall i, j \tag{21}$$

$$\frac{\partial v_{\bar{J}}}{\partial \varphi_{ij}} = 0, \ \ \forall i, j \tag{22}$$

$$\frac{\partial v_{\bar{J}}}{\partial \psi_{ij}} = [-\frac{\partial \Psi_{I\bar{J}}^T}{\partial \psi_{ij}} + B_{I\bar{J}}^T B_{IJ}^{-T} \frac{\partial \Psi_{IJ}^T}{\partial \psi_{ij}}] \hat{x}_I = \begin{cases} 0, & i \in \bar{I} \\ -\begin{pmatrix} 0 \\ \hat{x}_i \\ 0 \end{pmatrix}, & i \in I, j \in \bar{J} \\ B_{I\bar{J}}^T B_{IJ}^{-T} \begin{pmatrix} 0 \\ \hat{x}_i \\ 0 \end{pmatrix}, & i \in I, j \in J \end{cases} \tag{23}$$

Thus we have guaranteed the existence of the solution of equations (9)-(13) under appropriate conditions. Now we must check whether they satisfy the inequalities

$$x_i(\Phi, \Psi) \geq 0, \ \ u_i(\Phi, \Psi) \geq 0 \ \ i = 1, \ldots, m \tag{24}$$

$$y_j(\Phi, \Psi) \geq 0, \ \ v_j(\Phi, \Psi) \geq 0 \ \ j = 1, \ldots, n \tag{25}$$

of LCP $(A+\Phi, B+\Psi)$. If $i \in \bar{I}$, then $x_i(\Phi,\Psi) = 0$, and if $\hat{x}_i > 0$, then $x_i(\Phi,\Psi) \geq 0$, as long as Φ and Ψ are sufficiently close to 0. Thus it suffices to consider only the case $i \in I$ and $\hat{x}_i = 0$.

In this case, if the directional derivative of $x_i(\Phi,\Psi)$ at $(0,0)$ in the direction of perturbation is negative, then $x_i(\Phi,\Psi) < 0$ and therefore inequlity (24) is not satified. This implies that there exists no solution of LCP$(A+\Phi, B+\Psi)$ corresponding to this partition if the perturbation is given in the above direction. On the contrary, if the perturbation is given in the direction in which the directional derivative of x_i is positive, then $x_i(\Phi,\Psi) > 0$ and therefore inequlity (24) is satified. Similar discussions should be made about the other variables u, v, y. Moreover, if there are several possibilities of the partition satisfying (7)(8) and the conditions in Theorem 2, we must repeat the above consideration in all those cases.

Finally we should note that the derivatives of the Nash equilibrium with respect to the perturbation Φ, Ψ can be calculated as in the following by using Lemma 1.

$$\frac{\partial p_k}{\partial \varphi_{ij}} = 0; \ \frac{\partial p_k}{\partial \psi_{ij}} = \frac{\frac{\partial x_k}{\partial \psi_{ij}} \sum_i x_i - x_k \sum_i \frac{\partial x_i}{\partial \psi_{ij}}}{(\sum_i x_i)^2} \tag{26}$$

$$\frac{\partial q_l}{\partial \varphi_{ij}} = \frac{\frac{\partial y_l}{\partial \varphi_{ij}} \sum_j y_j - y_l \sum_j \frac{\partial y_j}{\partial \varphi_{ij}}}{(\sum_j y_j)^2}; \ \frac{\partial q_l}{\partial \psi_{ij}} = 0 \tag{27}$$

6. An Example

In this section we consider the following bimatrix game $\Gamma(A, B)$ along with the perturbation matrices (Φ, Ψ) as an illustrative example.

$$A = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}, \ B = \begin{pmatrix} -2 & -2 \\ -2 & -3 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \Psi = \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}$$

We can easily find

$$E(A, B) = \{((1,0), (\alpha, 1-\alpha)) \mid \frac{1}{2} \leq \alpha \leq 1\}$$

$$E(A+\Phi, B+\Psi) = \begin{cases} \{((1,0),(1,0))\} & \text{if } \theta < 0 \\ \{((\frac{1}{1+\theta}, \frac{\theta}{1+\theta}), (\frac{1}{2}, \frac{1}{2}))\} & \text{if } \theta > 0 \end{cases}$$

Let us consider the solution

$$\hat{x} = (\frac{1}{2}, 0),\ \hat{y} = (\frac{1}{5}, \frac{1}{5}),\ \hat{u} = (0,0),\ \hat{v} = (0,0)$$

of LCP (A, B) corresponding to $((1,0), (\frac{1}{2}, \frac{1}{2})) \in E(A, B)$. In this case the partition satisfying (7)(8) and the conditions in Theorem 2 is uniquely determined as follows:

$$I = \{1,2\},\ \bar{I} = \{\},\ J = \{1,2\},\ \bar{J} = \{\}$$

We should pay attention to $\hat{x}_2 = 0$. Since $\frac{\partial x_2}{\partial \theta} = \frac{1}{2}$ at $\theta = 0$, there exists a Nash equilibrium depending on θ smoothly when θ moves in the positive direction. On the contrary, however, if $\theta < 0$, then there does not exist such a solution. In this case the directional derivative in the direction $\theta > 0$ is given by

$$\frac{\partial p}{\partial \theta} = (-1, 1),\ \frac{\partial q}{\partial \theta} = (0, 0)$$

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Limiting Solution Set Structure for Converging Multiple Objective Dynamic Problems Sequence

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Abstract. For a given dynamic multiple objective problem a special family of dynamic multiple objective problems is constructed. For this family there is constructed the infinite intersection of the sets of all right ends of the quasimotions, generated from some initial position if all the Slater-maximal strategies for each problem are examined. It is shown that this intersection (or the limiting set) coincides with the set of all right ends of the quasimotions generated from the fixed initial position if all the Slater-maximal strategies of the special problem are examined.

Key words. Quasimotion, dynamic problem, multicriteria optimization, Slater-maximal strategy, matrix, set.

1 Introduction

The creation of new technologies and progressive methods in economy demanded a research of new problems in the controlled systems theory. Namely, there arose a necessity of simultaneous keeping in mind the following two factors: the multicriterialness and dynamic of the processes. In 1963 Zadeh [11] presented the need to design control systems in a multiple objective optimization framework. It is impossible to describe the controlled system's functioning quality only by one criterion. For example, at the economy system one has to estimate natural parameters (reliability, prime cost payoff, quality of the article) as well as ecological damages, the optimal connections between suppliers, etc. Also the controlled systems should be considered in their movement, i.e. their variation in time has to be got into account.

In last two decades dynamic vector optimization theory has been widely developed. The pioneer ideas of this theory were given in the monograph [6], where Salukvadze proposed how to solve dynamic problems with multiple criteria. Later an improved computational approach to Salukvadze's problem over a finite time interval has been proposed in [5], and different methods, algorithms and procedures for dynamic vector optimization were suggested (see, for example, [1], [2], [10]). Some generalizations and new approaches are given in the monograph [14]. They are similar to the results, which were earlier obtained for static optimization problems [12], [13].

The object of this article is to examine the structure of the solution set for the "limiting" problem of a family of multiple objective dynamic problems. The obtained results are similar to some static ones given in [9].

2 Definitions

Consider a multicriterial problem

$$\Gamma_0 = \ll \Sigma,\ \mathbf{V},\ \mathbf{F}(x[\theta])\gg. \tag{1}$$

The state vector x[t] is assumed to vary with the system Σ given by the following equation

$$\dot{x} = f(t,x,v), \tag{2}$$

$x\in\mathbb{R}^n$; the initial value of the state vector x_0, the times when the process starts $t_0\geq 0$ and ends $\theta>t_0$ are fixed; and the control action $v\in\mathbb{R}^q$.

Strategies V are associated with the functions $v(t,x)\subseteq Q\subset\mathbb{R}^q$ (for every possible position $(t,x)\in[t_0,\theta)\times\mathbb{R}^n$). The set of strategies $V\div v(t,x)$ will be denoted by $\mathbf{V}$ and is assumed throughout the paper to be a compactum in $\mathbb{R}^q$.

We'll assume throughout the paper that the following conditions hold:

1.The function f(t,x,v) is continuous and "locally" Lipschitz with respect to x, $\|f(t,x,v)\|\leq\gamma(1+\|x\|)$, and γ=const>0, and the set Q is a compactum in $\mathbb{R}^q$.

2. The vector-function $\mathbf{F}(x)$ has continuous components $F_i(x)$, $i\in\mathbf{N}=\{1,\dots,N\}$, with respect to x.

Now we'll define quasimotions (piecewise-continuous motions) $x[\cdot,t_0,x_0,V]=\{x[t,t_0,x_0,V],\ t_0\leq t\leq\theta\}$ of system (2) generated by the strategy $V\in\mathbf{V}$ from initial position $(t_0,x_0)\in[0,\theta)\times\mathbb{R}^n$ [14, section 2 of Chapter 1].

Given the initial position (t_0,x_0), the number $\alpha\in[0,1]$ and the strategy $V\div v(t,x)$, $V\in\mathbf{V}$, we cover the closed interval $[t_0,\theta)$ by a system of Δ semiintervals $\tau_j\leq t<\tau_{j+1}$ ($j=0,1,\dots,m(\Delta)-1$), $\tau_0=t_0$, $\tau_{m(\Delta)}=\theta$.

By a <u>stepwise quasimotion</u> of the system (2) generated at the initial position $(t_0,x_0)\in[0,\theta)\times\mathbb{R}^n$

a) by the strategy $V\div v(t,x)\subseteq Q$,

b) the decomposition Δ: $t_0=\tau_0<\tau_1<\dots<\tau_{m(\Delta)}=\theta$, and

c) the number $\alpha\in[0,1]$

we will understand any function

$$x(\cdot,V,\Delta,\alpha)=\{x(t,\tau_0,x_0,\hat{x}_0,V,\Delta,\alpha),\ \tau_0\leq t\leq\theta\}$$

which, at $\tau_j\leq t<\tau_{j+1}$, satisfies the quasistepwise equation

$$x(t,V,\Delta,\alpha)=\hat{x}_j+\int_{\tau_j}^{t} f(\tau,x(\tau,V,\Delta,\alpha),v(\tau_j,\hat{x}_j))d\tau \tag{3}$$

under the condition

$$\sum_{j=0}^{m(\Delta)-1} \|\hat{x}_j - x_0(\tau_j, V, \Delta, \alpha)\| \leq \alpha \tag{4}$$

where $x_0(\tau_j, V, \Delta, \alpha)$ satisfies the equality

$$x_0(\tau_j, V, \Delta, \alpha) = \hat{x}_{j-1} + \int_{\tau_{j-1}}^{\tau_j} f(\tau, x(\tau, V, \Delta, \alpha), v(\tau_{j-1}, \hat{x}_{j-1}))d\tau, \tag{5}$$

where $x(\tau_0, V, \Delta, \alpha) = \hat{x}_0$, $x_0(\tau_0, V, \Delta, \alpha) = x_0$.

A <u>quasimotion</u> $x[\cdot] = \{x(t, t_0, x_0, V),\ \tau_0 \leq t \leq \theta\}$ of system (2) generated from the initial position (t_0, x_0) by the strategy $V \in \mathfrak{V}$ is any function $x[\cdot] = \{\alpha(t),\ t_0 \leq t \leq \theta\}$ continuous over the closed interval $[t_0, \theta]$, for which there exists a sequence of stepwise quasimotions that converges to it (in the metric of the space $M_n[t_0, \theta]$):

$$x(\cdot, V, \Delta^{(r)}, \alpha^{(m)}) = \{x(t, \tau_0^{(r)}, x_0, \hat{x}_0^{(r)}, V, \Delta^{(r)}, \alpha^{(m)}),\ t_0 \leq t \leq \theta\},$$

$$\text{when diam } \Delta^{(r)} \to 0 \text{ and}$$

$$\begin{gathered} |\tau_0^{(r)} - t_0| + \|\hat{x}_0^{(r)} - x_0\| \to 0 \text{ as } r \to \infty, \\ \alpha^{(m)} \to 0 \text{ as } m \to \infty. \end{gathered} \tag{6}$$

Here $\operatorname{diam} \Delta^{(r)} = \max_j [\tau_{j+1}^{(r)} - \tau_j^{(r)}]$ and $0 \leq \alpha^{(m)} \leq 1$ $(r, m = 1, 2, \ldots)$.

Consequently, for this sequence $\{x(\cdot, V, \Delta^{(r)}, \alpha^{(m)})\}$, it is true that

$$\sup_{t_0 \leq t \leq \theta} \|x[t] - x(t, V, \Delta^{(r)}, \alpha^{(m)})\| \to 0$$

provided that (6) holds.

A bunch of quasimotions of system (2) will be denoted $\mathfrak{X}[t_0, x_0, V]$. Distinct quasimotions of a bunch are obtained if different sequences $\Delta^{(r)}$, $\alpha^{(m)}$ are used.

The performance of the system $\sum$ is estimated by an N-order $(N \geq 2)$ vector-valued goal functional

$$\mathfrak{F}(x[\theta]) = (F_1(x[\theta]), \ldots, F_N(x[\theta])). \tag{7}$$

The components $F_i(x)$ are defined for every $x \in X[\theta, t_0, x_0, V+Q]$ (where $X[\theta, t_0, x_0, V+Q]$ is the set of all right (at $t=\theta$) ends of the quasimotions $x[t, t_0, x_0, V+Q]$, $t_0 \leq t \leq \theta$, of system (2) generated by the strategy $V+Q$ from the initial position (t_0, x_0)). The coordinates of the vector $\mathfrak{F}(x[\theta])$ will be referred to as the goal functionals.

Generally speaking it is required to choose a strategy $V \in \mathfrak{V}$ which, starting from a fixed initial position (t_0, x_0), causes all the components of the vector $\mathfrak{F}(x[\theta])$ to assume the largest possible values simultaneously.

For given two vectors $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ of dimension N we shall introduce the following notations:

$$\mathbf{F}^{(1)} > \mathbf{F}^{(2)} \quad \Leftrightarrow \quad F_i^{(1)} > F_i^{(2)},\ i \in \mathbf{N};$$

$$\mathbf{F}^{(1)} \not> \mathbf{F}^{(2)} \quad \Leftrightarrow \quad \text{negation of condition } \mathbf{F}^{(1)} > \mathbf{F}^{(2)}.$$

Definition 1. A strategy $V^S \in \mathbf{V}$ is called Slater-maximal for problem (2) with the initial position $(t_0,x_0) \in [0,\theta) \times \mathbf{R}^n$ iff for every strategy $V \in \mathbf{V}$ the following system of inequalities

$$F_i(x[\theta,t_0,x_0,V]) \not> F_i(x[\theta,t_0,x_0,V^S]),\ i \in \mathbf{N}, \tag{8}$$

holds for any quasimotions $x[\cdot,t_0,x_0,V]$ and $x[\cdot,t_0,x_0,V^S]$.

3 Infinite Intersection

Let $\mathbf{A}$ be a constant $(N \times N)$-dimensional "expert" matrix with the elements a_{ij}, $i,j \in \mathbf{N}$ (see [8]).

Consider a family of multicriteria dynamic problems

$$\Gamma_l = \ll \textstyle\sum,\ \mathbf{V},\ \mathbf{F}^{(l)}(x[\theta]) \gg \tag{9}$$

corresponding to problem (1), where $\mathbf{F}^{(l)}$ is a vector-valued function with the components $F_i^{(l)}$, $i \in \mathbf{N}$, defined in the following way:

$$\mathbf{F}^{(l)} = \mathbf{A}^l \mathbf{F} \quad (l=0,1,2,3,\ldots).$$

Denote the set of Slater-maximal strategies for problem Γ_l (9) by $\mathbf{V}_l^S$, and by $X_l^S = \{x^S = x[\theta,t_0,x_0,V_l^S] \mid V_l^S \in \mathbf{V}_l^S\} = X[\theta,t_0,x_0,\mathbf{V}_l^S]$ or X_l^S is the set of all right (at $t=\theta$) ends of the quasimotions $x[\cdot,t_0,x_0,V_l^S]$ generated from the initial position (t_0,x_0) if all the Slater-maximal strategies V_l^S of $\mathbf{V}_l^S$ are examined. This set $X_l^S \subset X[\theta,t_0,x_0,V \div Q]$ (for each $l=0,1,2,\ldots$), which is the domain of reachability at time $t=\theta$ of system (2) with $x[t_0]=x_0$.

According to proposition 3.5 (see [14, p.164]) we can conclude that the following infinite chain of set inclusions

$$\mathbf{V}_0^S = \mathbf{V}^S \supset \mathbf{V}_1^S \supset \mathbf{V}_2^S \supset \ldots \supset \mathbf{V}_l^S \supset \ldots$$

is valid, and moreover the chain of set inclusions

$$\mathbf{X}_0^S \supset \mathbf{X}_1^S \supset \mathbf{X}_2^S \supset \ldots \supset \mathbf{X}_l^S \supset \ldots$$

is valid too.

According to proposition 3.6 ([14, p.166]) each set $\mathbf{X}_l^S$ is a nonempty compactum in $\mathbf{R}^n$.

Then it immediately follows that the infinite intersection of nested sets $\{X_l^S\}_{l=0}^{\infty}$ is a nonempty compact set, thus there exists a compact set X^* such that

$$X^* = \bigcap_{l=0}^{\infty} X_l^S. \tag{10}$$

According to the properties of nested sets [3] from equality (10) we conclude that

$$X^* = \operatorname*{Lim}_{l\to\infty} X_l^S. \tag{11}$$

Later we shall find a concrete structure of the limiting set X^*.

4 Some Additional Algebraic Facts

Definitions given in this part are taken from Lankaster [4].

Definition 2. A square N-order matrix A is called a stochastic matrix iff all its elements are nonnegative and their sum in each row is equal to 1.

The maximal eigenvalue of a stochastic matrix is equal to 1.

Definition 3. A square N-order matrix A is called prime iff it is similar to a diagonal square matrix.

Definition 4. A vector v, which satisfies the N-order vector equation $Av=\lambda v$, is called a right eigenvector, corresponding to the eigenvalue λ of a given square N-order matrix A; a vector u, which satisfies the N-order vector equation $A^T u=\lambda u$ is called a left eigenvector corresponding to the eigenvalue λ of a given square N-order matrix A.

Definition 5. Some matrix is called accompanying matrix of the eigenvalue λ of a given square matrix A iff it is the product of the right and the transposed left eigenvectors corresponding to λ.

In the paper [7] it was proved that for a given N-order prime matrix A with real elements and eigenvalues $\lambda_1,\dots,\lambda_k$, such that $|\lambda_1|>|\lambda_j|$, $j=2,\dots,N$, the following vector equation

$$\lim_{l\to\infty} (\lambda_1^{-1}A)^l = G_1$$

is valid, where G_1 is accompanying matrix corresponding to the maximal eigenvalue λ_1.

5 Structure of the Limiting Set

The following result is valid.

Theorem. The limiting set X^* coincides with the set X_{F^*} of all right (at $t=\theta$) ends of the quasimotions $x[\cdot,t_0,x_0,V_*^S]$

generated from the initial position (t_0,x_0) if all the Slater-maximal strategies V_*^S of $\mathbb{V}_*^S$ of the following multiple objective dynamic problem

$$\Gamma_{\mathbb{A}^*} = \ll \sum, \mathbb{V}, \mathbb{F}_*(x[\theta]) \gg$$

are examined. Here $\mathbb{F}_*(x[\cdot])=\mathbb{A}^*\mathbb{F}(x[\cdot])$ is the vector-valued function, and the matrix $\mathbb{A}^*$ is the accompanying matrix, corresponding to the maximal eigenvalue $\lambda_1=1$ of the N-order stochastic, prime matrix $\mathbb{A}$ with all positive elements.

Proof. To prove the theorem we must show that the equality

$$\mathbb{X}^*=\mathbb{X}_{F^*} \tag{12}$$

is valid. So we must obtain that the following inverse inclusions

$$\mathbb{X}^*\subset\mathbb{X}_{F^*}, \tag{13}$$

$$\mathbb{X}_{F^*}\subset\mathbb{X}^* \tag{14}$$

are fulfilled.

First, we shall prove the validity of inclusion (13). Suppose $x^*\in\mathbb{X}^*$. According to (11) we have that the equality

$$\mathbb{X}^*= \operatorname*{Lim}_{l\to\infty}\mathbb{X}_l^S$$

is valid, where each set $\mathbb{X}_l^S$, $l=0,1,2,\dots$, is a nonempty compact set. Then there exists a sequence of quasimotions $x^{(l)}[\cdot,t_0,x_0,V_l^S]$ $(x^{(l)}[\theta,t_0,x_0,V_l^S]\in\mathbb{X}_l^S)$, $l=0,1,2\dots$, each corresponding to its own problem Γ, such that

$$\lim_{l\to\infty} \|x^{(l)}[\theta,t_0,x_0,V_l^S]-x^*\|=0.$$

As V_l^S is a Slater-maximal strategy for the problem Γ_l then from inequality (8) we obtain that for every strategy $V\in\mathbb{V}$, any quasimotions $x[\cdot,t_0,x_0,V]$ and $x[\cdot,t_0,x_0,V_l^S]$ the system of strict inequalities

$$\mathbb{F}_i^{(l)}(x[\theta,t_0,x_0,V])>\mathbb{F}_i^{(l)}(x[\theta,t_0,x_0,V_l^S]),\ i\in\mathbb{N}, \tag{15}$$

is inconsistent.

Thus from (15) we get that for every strategy $V\in\mathbb{V}$ and some associated pair of quasimotions $(x[\cdot,t_0,x_0,V]$, $x[\cdot,t_0,x_0,V_l^S])$ there exists a subscript $i(V)=i_0\in\mathbb{N}$ such that

$$\mathbb{F}_{i_0}^{(l)}(x[\theta,t_0,x_0,V])\leqslant\mathbb{F}_{i_0}^{(l)}(x[\theta,t_0,x_0,V_l^S]). \tag{16}$$

We fix an arbitrary strategy $V\in\mathbb{V}$ and a quasimotion $x[t,t_0,x_0,V]$.

For every $i\in\mathbb{N}$ define the set

$$\mathbb{C}_i=\{l\in\{0,1,2,3,\ldots\}: \mathbb{F}_i^{(l)}(x[\theta,t_0,x_0,V])\leq$$
$$\leq\mathbb{F}_i^{(l)}(x^{(l)}[\theta,t_0,x_0,V_l^S])\},\qquad(17)$$

where the sequence $x^{(l)}[\theta,t_0,x_0,V_l^S]$ $(\in X_l^S)$, $l=0,1,2,3,\ldots$, is converging to x^*.

By (16) we have that

$$\bigcup_{i\in\mathbb{N}}\mathbb{C}_i=\{0,1,2,3,\ldots\}.$$

Then there exists at least one infinite subset $\mathbb{C}_{i_0}=\{l_t\}$ of the set $\{0,1,2,3,\ldots\}$. As $\mathbb{A}$ is a prime, stochastic matrix, the sequence $\{\mathbb{A}^l\}_{l=0}^{\infty}$ converges to $\mathbb{A}^*$, where $\mathbb{A}^*$ is the accompanying matrix, corresponding to the maximal eigenvalue $\lambda_1=1$ of $\mathbb{A}$. Thus,

$$\lim_{t\to\infty}\mathbb{A}^{l_t}=\mathbb{A}^*.$$

As $\mathbb{F}^{(l)}=\mathbb{A}^l\mathbb{F}$, $\mathbb{F}(\cdot)$ is a vector-valued function with continuous comp nents, $\lim_{l\to\infty}\|x^{(l)}[\theta,t_0,x_0,V_l^S]-x^*\|=0$, then we get

$$\lim_{t\to\infty}(\mathbb{F}_{i_0}^{(l_t)}(x[\theta,t_0,x_0,V])-\mathbb{F}_{i_0}^{(l_t)}(x^{(l_t)}[\theta,t_0,x_0,V_{l_t}^S])=$$
$$=[\mathbb{A}^*\mathbb{F}(x[\theta,t_0,x_0,V])-\mathbb{A}^*\mathbb{F}(x^*)]_{i_0}.\qquad(18)$$

Thus from equality (18) and the structure of the set $\mathbb{C}_i$, given by (17), we obtain the validity of the inequality

$$[\mathbb{A}^*\mathbb{F}(x[\theta,t_0,x_0,V])]_{i_0}\leq[\mathbb{A}^*\mathbb{F}(x^*)]_{i_0}.\qquad(19)$$

It is obvious as the problem $\Gamma_{\mathbb{A}^*}$ is the limiting problem (in the sense of criteria functions) for the family of problems $\{\Gamma_l\}_{l=0}^{\infty}$ that there exists a quasimotion $\hat{x}[\cdot,t_0,x_0,\hat{V}]$ of system (2) generated from the initial position (t_0,x_0) by some strategy $\hat{V}\in\mathbb{V}$ such that $\hat{x}[\theta,t_0,x_0,\hat{V}]=x^*$.

Then from inequality (19), taking into account that the strategy V and the quasimotion $x[\cdot,t_0,x_0,V]$ are arbitrary ones we can conclude that $x^*\in\mathbb{X}_{F^*}$. So inclusion (13) is valid.

Now we shall prove the validity of the inverse inclusion (14).

Suppose that $x^*\in\mathbb{X}_{F^*}$, but $x^*\notin\mathbb{X}^*$. Then there can be found an ordinal number l_0, such that for every fixed number $l\geq l_0$ $x^*\notin\mathbb{X}_l^S$.

According to proposition 3.1 ([14, p.71]) the equality

$$\mathbf{X}_l^S=\mathbf{X}_l^s \tag{20}$$

holds, where $\mathbf{X}_l^s$ is the set of all Slater-maxima weakly efficient solutions of the static multicriterial problem

$$<< \mathbf{X}[\theta,t_o,x_o,V\div Q],\ \mathbf{F}^{(l)}(x) >>. \tag{21}$$

As $x^*\notin\mathbf{X}_l^S$, $l\geq l_o$, then by (20) $x^*\notin\mathbf{X}_l^s$, $l\geq l_o$. In particular, it is true for $l=l_o$. So $x^*\notin\mathbf{X}_{l_o}^s$. Then there is a point $x_s^{(l_o)}\in$ $\mathbf{X}[\theta,t_o,x_o,V\div Q]$ such that the system of strict inequalities

$$\mathbf{A}^{l_o}\mathbf{F}(x_s^{(l_o)})>\mathbf{A}^{l_o}\mathbf{F}(x^*) \tag{22}$$

is valid.

As $\mathbf{A}$ is a prime matrix, then from the proof of theorem 4 from [7] one can conclude that

$$\mathbf{A}^{l_o} = \sum_{j=1}^{N} \lambda_j^{l_o}G_j, \tag{23}$$

where G_j is the accompanying matrix of the eigenvalue λ_j of the matrix $\mathbf{A}$, $j\in\mathbf{N}$.

By equality (23) we get from (22) that the vector inequality

$$\sum_{j=1}^{N} \lambda_j^{l_o}G_j(\mathbf{F}(x_s^{(l_o)})-\mathbf{F}(x^*)) >0 \tag{24}$$

is valid.

As the matrices G_j, $j\in\mathbf{N}$, are orthogonal and idempotent (see [4]), and $\lambda_1=1$, $G_1>0$, then multiplying by G_1 from the left both parts of inequality (24) we obtain the inequality

$$\mathbf{A}^*\mathbf{F}(x_s^{(l_o)}) > \mathbf{A}^*\mathbf{F}(x^*). \tag{25}$$

Thus from (25) it follows, that $x^*\notin\mathbf{X}_{F^*}^S$, where $\mathbf{X}_{F^*}^S$ is the set of all Slater-maxima weakly efficient solutions of the static multicriterial problem

$$<< \mathbf{X}[\theta,t_o,x_o,V\div Q],\ \mathbf{F}_*(x) >>.$$

According to proposition 3.1 ([14, p. 71]) the equality $\mathbf{X}_{F^*}^S=\mathbf{X}_{F^*}$ is valid. Then $x^*\notin\mathbf{X}_{F^*}$, and this contradicts to the supposed above condition that $x^*\in\mathbf{X}_{F^*}$. So, the inclusion (14) is valid, and thus equality (12) holds. The theorem is proved.

Remark. As the rows of the accompanying matrix $\mathbf{A}^*$ coincide (see [8]), then the problem Γ_{A^*} is a single objective dynamic problem and there can be used standard methods of optimal strategies construction from the dynamic optimization.

6 Conclusions

The proved above theorem allows one in the limit for the infinite family of multicriteria dynamic problems Γ_l, $l=0,1,2,3,\ldots$, to define the structure of the infinite intersection of the sets of all right ends of the quasimotions, generated from some initial position if all the Slater-maxima strategies for each problem are examined.

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Part 2

Methods

2.1 Methodology

Choosing and Ranking on the Basis of Fuzzy Preference Relations with the "Min in Favor"

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Abstract. In some MCDM techniques – most notably in Outranking Methods – the result of the comparison of a finite set of alternatives according to several criteria is summarized using a fuzzy preference relation. This fuzzy relation does not, in general, possess "nice properties" such as transitivity or completeness and elaborating a recommendation on the basis of such information is not an obvious task. The purpose of this paper is to study techniques exploiting fuzzy preference relations in order to choose or rank. We present a number of results concerning techniques based on the "min in Favor" score, *i.e.* the minimum level with which an alternative is "at least as good as" all other alternatives.

Keywords. MCDM, Fuzzy Preferences, Choice Rules, Ranking Rules, Outranking Methods.

1 Introduction

Consider a finite set of alternatives that are evaluated along several criteria. In order to select a subset of alternatives or to rank order them, methods related to the "Outranking Approach" (see [16] or [17]) usually proceed in two steps. The construction step consists in a pairwise comparison of the alternatives taking all criteria into account. In most methods this is done using a "concordance-discordance" principle which leads to declaring that "alternative a is at least as good as alternative b" when:

- a "sufficient" majority of criteria supports this proposition and
- the opposition of the minority is not "too strong".

In many methods, *e.g.*, in ELECTRE III (see [15]) or in PROMETHEE (see [8]), the result of these pairwise comparisons is summarized using a fuzzy relation. This means that a number between 0 and 1 is associated to each ordered pair (a, b) of alternatives indicating the credibility of the proposition "a is at least as good as b", *e.g.* the sum of the weights of the criteria favoring a in the comparison (eventually corrected to take into account the opposition of the other criteria). Such a way of modelling comparisons made along several dimensions is quite reminiscent of classical electoral techniques which summarize a ballot by an "electoral matrix" giving for each ordered pair (a, b) of candidates the number of voters having declared that "a is at least as good as b". It is well-known in Social Choice Theory that it is not easy to tell which candidate(s) should be elected on the basis of an "electoral matrix" as soon as the opinion of the voters is "sufficiently" conflictual. A similar problem occurs with fuzzy preference relations built in Outranking Methods. When the different criteria taken into account are conflictual, these relations do not, in general, possess "nice properties" such as transitivity or completeness. With the construction technique of ELECTRE III, it is shown in [6] that a stronger conclusion holds: any reflexive fuzzy preference relation may be obtained as soon as there is a "sufficient" number of "sufficiently conflicting" criteria. Therefore, it is far from being an easy task to select a subset of alternatives or to rank order them on the basis of such information. This calls for the application of specific techniques which constitute the "exploitation step" of Outranking Methods. Many such techniques have been proposed in the literature (see, *e.g.*, [16] or [17]) most often on a purely *ad hoc* basis. This has often been seen as a major weakness of Outranking Methods. The aim of this paper is to contribute to their analysis. We present a number of results concerning choice and ranking techniques that are based on the "min in Favor" (mF) score, *i.e.* the minimum level with which an alternative is "at least as good as" all other alternatives, consolidating and extending previous results appeared in [3], [5], [12], [13] and [14]. The axiomatic characterizations presented here will hopefully allow to emphasize the specific features of the techniques studied and, hence, to compare them more easily with other ones.

This paper is organized as follows. We introduce our definitions and notations in section 2. In section 3 we analyze a choice technique based on the mF score. Two ranking techniques based on the mF score are then studied in section 4. A final section, discussing the results and mentioning open problems, concludes the paper.

2 Definitions and Notations

Throughout this paper X will denote a nonempty finite set of "alternatives". A *fuzzy* (binary) *relation* T on X is a function from XxX to [0, 1]. With ELECTRE III in mind, we shall interpret fuzzy relations as "large" preference relations, the valuation of (a, b) indicating the credibility of the proposition "a is at least as good as b". Thus, all fuzzy relations in this paper will be supposed to be reflexive (a fuzzy relation T on X is reflexive if T(a, a) = 1, for all a $\in$ X). If Y $\subseteq$ X and T is a fuzzy relation on X, we denote by T/Y the restriction of T to Y, *i.e.* the fuzzy relation on Y such that for all a, b $\in$ Y, T/Y(a, b) = T(a, b). A fuzzy relation T on X such that T(a, b) $\in$ {0, 1}, for all a, b $\in$ X, is said to be *crisp*. We often write a T b instead of T(a, b) = 1 and Not(a T b) instead of T(a, b) = 0 when T is a crisp relation. We denote by $\mathcal{F}_X$ (resp. $\mathcal{U}_X$) the set of all fuzzy (resp. crisp) reflexive relations on X.

Let T be a crisp relation on X. It is said to be *complete* if [a T b or b T a] and *transitive* if [a T b and b T c $\Rightarrow$ a T c], for all a, b, c $\in$ X. A *weak order* is a crisp, complete and transitive binary relation. Let T be a weak order on X.We denote by $U_k(X, T)$ the kth equivalence class of T, *i.e.* for k = 1, 2, 3, ..., $U_k(X, T) = \{a \in X[T, k] : a\ T\ b$, for all $b \in X[T, k]\}$, where X[T, 1] = X and for k = 2, 3, ..., $X[T, k] = X[T, k\text{-}1] \backslash U_{k-1}(X, T)$. Observe that $U_1(X, T)$ is always nonempty and that all nonempty equivalence classes of a weak order are disjoint.

A *choice rule* C is a function associating with each finite set X and each fuzzy relation $R \in \mathcal{F}_X$ a choice set C(X, R) such that $C(X, R) \subseteq X$ and $C(X, R) \neq \emptyset$. A choice rule therefore allows to select a nonempty choice set on the basis of any reflexive fuzzy relation defined on a finite set. Similarly, a *ranking rule* $\gtrsim$ is a function associating with each finite set X and each fuzzy relation $R \in \mathcal{F}_X$ a weak order $\gtrsim(X, R)$ on X. Such definitions are adapted to methods such as ELECTRE III which can lead to any reflexive fuzzy relation on a finite set.

As shown in [2], a simple way to define choice and ranking rules is to make use of a scoring function (for alternative ways of building such rules, we refer to [9]). A scoring function S is a function associating a real number S(a, R, X) with each finite set X, each $R \in \mathcal{F}_X$ and each $a \in X$. We shall interpret the number S(a, R, X) as a measure of the "attractiveness" of alternative a within the set X endowed with the fuzzy relation R. Given a scoring function, selecting the alternatives with the highest score (resp. rank ordering the alternatives according to their scores) defines a choice rule (resp. a ranking rule). Formally we define the choice rule C_S and the ranking rule $\gtrsim_S$ associated to the scoring function S, letting, for all finite set X, all a, b $\in$ X and all $R \in \mathcal{F}_X$:

$C_S(X, R) = \{c \in X : S(c, R, X) \geq S(d, R, X)$ for all $d \in X\}$ and

$a \gtrsim_S(X, R)\ b \Leftrightarrow S(a, R, X) \geq S(b, R, X)$.

As argued in [1], an alternative way of defining a ranking rule also deserves interest. It consists in the (downward) iteration of a choice rule which leads to a weak order in the following way. The alternatives selected by the choice rule form the first equivalence class of the weak order. These alternatives are then removed from consideration. The alternatives selected in the reduced set form the second equivalence class and so on. Formally, the iteration of a choice rule C leads to a ranking rule $\gtrsim$ such that, for all finite set X and for all $R \in \mathcal{F}_X$, we have (T standing for $\gtrsim(X, R)$): $U_k(X, T) = C(X[T, k], R/X[T, k])$, for all integer k such that X[T, k] is nonempty. Though the ranking rule directly based on scores is much simpler than the one defined by iterated choice, the latter deserves attention since it corresponds to a very intuitive behaviour for ranking objects: the objects ranked in first place are the "best" objects (according to a choice rule), the objects ranked in second place are the best objects between those remaining and so on. Associated with a scoring function S we have thus defined the choice rule C_S, the ranking rule $\gtrsim_S$ and the ranking rule $\gtrsim_{IS}$ corresponding to the iteration of C_S. It should be observed that $\gtrsim_S$ and $\gtrsim_{IS}$ are not identical in general.

In this paper, we shall be concerned with the "min in Favor" scoring function, *i.e.* the scoring function such that, for all finite set X, all $R \in \mathcal{F}_X$ and all $a \in X$,

$$mF(a, X, R) = \min_{b \in X \setminus \{a\}} R(a, b).$$

It indicates the credibility with which an alternative is at least as good as other alternatives. This scoring function defines the min in Favor choice rule C_{mF}, the min in Favor ranking rule $\gtrsim_{mF}$ and the Iterated min in Favor ranking rule $\gtrsim_{ImF}$. The following numerical example illustrates these three rules. Let X = {a, b, c, d} and let $R \in \mathcal{F}_X$ be defined by the following table (to be read from row to column):

R	a	b	c	d
a	1	0.9	0.4	1
b	0.5	1	0.3	0.3
c	0.7	0.6	1	0.4
d	0.2	0.8	0.5	1

The min in Favor Choice rule obviously gives $C_{mF}(X, R) = \{a, c\}$. Using $\gtrsim_{mF}$, we obtain the following weak order (using obvious abbreviated notations): (ac) > b > d. Using $\gtrsim_{ImF}$, we obtain: (ac) > d > b. This shows that $\gtrsim_{mF}$ and $\gtrsim_{ImF}$ are distinct rules in spite of the fact that $U_1(X, \gtrsim_{mF}(X, R)) = U_1(X, \gtrsim_{ImF}(X, R)) = C_{mF}(X, R)$.

Though many other scoring functions have been proposed in the literature (see [2]), a remarkable feature of the min in Favor scoring function is that it leads to choice and ranking rules that do not make use of the cardinal properties of the valuations R(a, b). Though this might be seen as too radical an interpretation of fuzziness, it is not clear from the construction technique of the numbers R(a, b) in many outranking methods and especially in ELECTRE III, whether or not they convey any information beyond the fact that $R(a, b) \geq R(c, d)$ means that the proposition "a is at least as good as b" is no less credible than the proposition "c is at least as good as d". Using only the ordinal information conveyed by the fuzzy relation may thus be seen as a principle of prudence.

3 The Min in Favor Choice Rule

The aim of this section is to provide an axiomatic characterization of the min in Favor choice rule C_{mF}. Our first axiom is designed to capture the already-mentioned ordinal character of C_{mF}. We say that a choice rule C is *ordinal* if, for all finite set X, all $R \in \mathcal{F}_X$ and all strictly increasing and one-to-one transformation ϕ on [0, 1], $C(X, R) = C(X, \phi[R])$, where $\phi[R]$ is the element of $\mathcal{F}_X$ such that $\phi[R](a, b) = \phi(R(a, b))$ for all $a, b \in X$. It is not difficult to see that C_{mF} is indeed ordinal. There are many ordinal choice rules; the "Max in Favor" choice rule C_{MF} and the "Max Against" choice rule C_{MA} that respectively use the scores

$$MF(a, X, R) = \max_{b \in X \setminus \{a\}} R(a, b),$$

$$MA(a, X, R) = -\max_{b \in X \setminus \{a\}} R(b, a),$$

are both ordinal.

Consider a crisp relation $R \in \mathcal{U}_X$. If the set $G(X, R) = \{a \in X\text{: } a\, R\, b \text{ for all } b \in X\}$, *i.e.*, the set of the greatest elements in X given R, is nonempty, there exist alternatives in X that are unambiguously "at least as good as" all other alternatives. Thus, it seems that there is little point in selecting alternatives outside of G(X, R). This motivates the following axiom. We say that a choice rule C is *greatest faithful* if, for all finite set X and all $R \in \mathcal{F}_X$, $[R \in \mathcal{U}_X \text{ and } G(X, R) \neq \emptyset] \Rightarrow C(X, R) \subseteq G(X, R)$.

It is not difficult to see that C_{mF} is greatest faithful contrary to C_{MF} and C_{MA}. The conjunction of ordinality and greatest faithfulness does not characterize C_{mF} however. This is because greatest faithfulness imposes a constraint on the result of a choice rule when applied to crisp relations whereas ordinality imposes a constraint on the result of the choice rule for "ordinaly equivalent" fuzzy relations. Since no truly fuzzy relation (*i.e.* belonging to $\mathcal{F}_X \backslash \mathcal{U}_X$) can be ordinaly equivalent to a crisp relation, the conjunction of these two axioms imposes very few constraints on the behavior of C when applied to truly fuzzy relations. Furthermore it should be observed that C_{mF} cannot be seen as the largest or the smallest (w.r.t inclusion) choice rule in the set of all ordinal and faithful choice rules: a choice rule discriminating among the elements of C_{mF} according to their MF score is ordinal and greatest faithful but smaller than C_{mF}; a choice rule coinciding with C_{mF} for crisp relations and selecting all alternatives otherwise is ordinal and greatest faithful but larger than C_{mF}. This calls for axioms that would relate the result of a choice rule when applied to crisp relations and truly fuzzy ones. Hence, we introduce the following continuity requirement which is obviously fulfilled by C_{mF}.

Consider a sequence of fuzzy relations $(R_i \in \mathcal{F}_X, i = 1, 2, ...)$. We say that this sequence converges to $R \in \mathcal{F}_X$ if, for all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there is an integer k such that, for all $j \geq k$, $|R_j(a, b) - R(a, b)| < \varepsilon$, for all $a, b \in X$. A choice rule C is said to be *continuous* if, for all finite set X, all $R \in \mathcal{F}_X$ and all sequences $(R_i \in \mathcal{F}_X, i = 1, 2, ...)$ converging to R, $[a \in C(X, R_i)$ for all R_i in the sequence$] \Rightarrow [a \in C(X, R)]$.

Considering a sequence of strictly increasing transformations converging (pointwise) to a step function, it is not difficult to see that ordinality and continuity imply that if $a \in C(X, R)$ then for any $\lambda \in (0, 1]$, $a \in C(X, R_\lambda)$ where R_λ denotes the λ-cut of R, *i.e.* the crisp relation such that, for all $c, d \in X$, $c\, R_\lambda\, d \Leftrightarrow R(c, d) \geq \lambda$. The following result (first obtained in [5]) is based on this simple observation coupled with the fact that, since λ-cuts are crisp relations, the result of a choice rule with such relations may be constrained by greatest faithfulness.

Proposition 1. The min in Favor choice rule C_{mF} is the only ordinal, continuous and greatest faithful choice rule.

Proof. We already observed that C_{mF} is ordinal, continuous and greatest faithful. It remains to be shown that if a choice rule C is ordinal, continuous and greatest faithful then, for all finite set X, all $R \in \mathcal{F}_X$ and all $a, b \in X$:

$mF(a, X, R) > mF(b, X, R) \Rightarrow b \notin C(X, R)$ and (i)
$mF(a, X, R) = mF(b, X, R)$ and $b \in C(X, R) \Rightarrow a \in C(X, R)$. (ii)

In contradiction with (i), suppose that $mF(a, X, R) > mF(b, X, R)$ and $b \in C(X, R)$ for some ordinal, continuous and greatest faithful choice rule C. Let $\lambda \in (mF(b, X, R), mF(a, X, R))$. Consider any sequence of strictly increasing and one-to-one transformations (ϕ_i, i = 1, 2, ...) on [0, 1] converging pointwise to the step function ϕ on [0, 1] such that $\phi(x) = 1$ iff $x \geq \lambda$ and $\phi(x) = 0$ otherwise. By construction, the sequence ($\phi_i[R]$, i = 1, 2, ...) converges to the λ-cut R_λ of R. Ordinality implies that $b \in C(X, \phi_i[R])$ for all ϕ_i in the sequence and using continuity we obtain $b \in C(X, \phi[R]) = C(X, R_\lambda)$. The set $G(X, R_\lambda)$ is nonempty (since $a \in G(X, R_\lambda)$) and $b \notin G(X, R_\lambda)$. Using greatest faithfulness we obtain a contradiction. This proves (i).

In order to prove (ii), suppose that $mF(a, X, R) = mF(b, X, R) = \lambda$ and $b \in C(X, R)$, for some ordinal, continuous and greatest faithful choice rule C. Since $b \in C(X, R)$, we know, using (i), that $mF(b, X, R) \geq mF(c, X, R)$ for all $c \in X$. Consider a sequence ($R_i \in \mathcal{F}_X$, i = 1, 2 ...) of fuzzy relations identical to R except that $R_i(b, c) = \text{Max}(0 ; R(b, c) - 1/i)$ for all $c \in X\backslash\{b\}$ such that $R(b, c) = \lambda$ and $R_i(a, d) = \text{Min}(1 ; R(a, d) + 1/i)$ for all $d \in X\backslash\{a\}$ such that $R(a, d) = \lambda$. This sequence converges to R. For all R_i in the sequence, we have, by construction, $mF(a, X, R_i) > mF(c, X, R_i)$ for all $c \in X\backslash\{a\}$. From (i) we know that $C(X, R_i) = \{a\}$ for all R_i in the sequence. Continuity implies $a \in C(X, R)$ which proves (ii) and completes the proof. □

We conclude this section with some remarks.

a) As shown by the following examples, ordinality, continuity and greatest faithfulness are independent properties.

i- The Sum in Favor choice rule C_{SF} based on the following score:

$$SF(a, X, R) = \sum_{b \in X\backslash\{a\}} R(a, b)$$

is greatest faithful and continuous but not ordinal.

ii- C_{MF} is continuous and ordinal but not greatest faithful.

iii- Define C_L as:

$C_L(X, R) = \{a \in C_{mF}(X, R): MF(a, R) \geq MF(b, R) \text{ for all } b \in C_{mF}(X, R)\}$,

i.e. as the choice rule discriminating among the elements selected with C_{mF} according to their MF score. It is easy to see that this choice rule is ordinal and greatest faithful but not continuous.

b) The proof of proposition 1 shows that, in presence of ordinality and greatest faithfulness, the full power of continuity is not needed to prove that $C(X, R) \subseteq C_{mF}(X, R)$ for all finite set X and all $R \in \mathcal{F}_X$. Requiring continuity only for sequences of fuzzy relations converging to a crisp relation suffices. This leads to an alternative characterization of C_{mF} as the largest choice rule among the ones that are ordinal, greatest faithful and weakly continuous in the above sense.

c) Retaining ordinality and continuity, an obvious modification of greatest faithfulness, requiring for crisp relations that the choice set should always be included in the set of "unbeaten" alternatives when this set is nonempty, allows to characterize C_{MA}.

d) Our definition of continuity uses a distance between fuzzy relations that makes use of the cardinal properties of the numbers R(a, b). Though ordinality and continuity are not contradictory, as shown by proposition 1, coupling these two axioms is somewhat awkward since ordinality implies that the cardinal properties of the numbers R(a, b) should not be used. The following proposition, adapted from [14], shows that the conjunction of ordinality and continuity is equivalent to a "strong ordinality" requirement involving non-decreasing transformations (at the cost of a more complex proof, it is possible to consider only non-decreasing and continuous transformations). The use of strong ordinality avoids making explicit reference to a distance on the set of fuzzy relations.

Proposition 2. A choice rule C is ordinal and continuous if and only if it is strongly ordinal, *i.e.*, $C(X, R) \subseteq C(X, \phi[R])$, for all finite set X, all $R \in \mathcal{F}_X$ and all non-decreasing transformation ϕ on [0, 1] such that $\phi(0) = 0$ and $\phi(1) = 1$.

Proof. [Ordinality and continuity $\Rightarrow$ strong ordinality]. Let X be a finite set and $R \in \mathcal{F}_X$. Consider any non-decreasing transformation ϕ on [0, 1] such that $\phi(0) = 0$ and $\phi(1) = 1$. We can find a sequence of strictly increasing and one-to-one transformations (ϕ_i, i = 1, 2, ...) on [0, 1] that converges pointwise to ϕ and it can be supposed w.l.o.g. that ϕ_1 is the identity function on [0, 1]. By construction, the sequence ($\phi_i[R]$, i = 1, 2, ...) converges to $\phi[R]$. Let $a \in C(X, R)$. Since $\phi_1[R] = R$, ordinality implies that $a \in C(X, \phi_i[R])$ for all ϕ_i in the sequence. Using continuity, we obtain $a \in C(X, \phi[R])$, which shows that C is strongly ordinal.

[Strong ordinality $\Rightarrow$ ordinality and continuity]. Let ϕ be strictly increasing and one-to-one on [0, 1]. Thus ϕ^{-1} is also strictly increasing and one-to-one. Strong ordinality leads to $C(X, R) \subseteq C(X, \phi[R]) \subseteq C(X, \phi^{-1}[\phi[R]]) = C(X, R)$, which shows that C is ordinal. Let $R \in \mathcal{F}_X$ and consider a sequence of fuzzy relations ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converging to $R \in \mathcal{F}_X$ and such that $e \in C(X, R_i)$ for all R_i in the sequence. Let $|X| = n$. Observe that R can take at most n(n-1) distinct values on the set of the n(n-1) ordered pairs of distinct elements in X. Suppose that R takes exactly ℓ distinct values $k_1, k_2, \ldots, k_\ell$. Suppose w.l.o.g that $0 \leq k_1 < k_2 < k_3 < \ldots < k_\ell \leq 1$ and define ε as:

$$\varepsilon = \min_{j=2,\ldots,\ell} k_j - k_{j-1}.$$

Since ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converges to R, we can find an integer j such that $|R_j(a, b) - R(a, b)| < \varepsilon/2$, for all $a, b \in X$. Define a function ϕ from [0, 1] to [0, 1] such that, for all j =1, 2, , ℓ, $[|x-k_j| < \varepsilon/2] \Rightarrow \phi(x) = k_j]$ and $\phi(x) = x$ otherwise. It is clear that ϕ is non-decreasing and such that $\phi(1) = 1$ and $\phi(0) = 0$. By construction, we have $R = \phi[R_j]$. Since $e \in C(X, R_j)$, we obtain $e \in C(X, R)$ using strong ordinality, which completes the proof. □

4 The Min Ranking Rule and the Iterated Min Ranking Rule

Adapting the axioms introduced for choice rules to the case of ranking rules is straightforward. We say that a ranking rule $\gtrsim$ is:

– *ordinal* if, for all strictly increasing and one-to-one transformation ϕ on [0, 1], $\gtrsim(X, R) = \gtrsim(X, \phi[R])$,

– *continuous* if, for all sequences ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converging to R and all a, b $\in$ X, [a $\gtrsim(X, R_i)$ b for all R_i in the sequence] $\Rightarrow$ [a $\gtrsim(X, R)$ b].

– *greatest faithful* if [$R \in \mathcal{U}_X$ and $G(X, R) \neq \emptyset$] $\Rightarrow U_1[X, \gtrsim(X, R)] \subseteq G(X, R)$, for all finite set X and all $R \in \mathcal{F}_X$.

It is not difficult to see that $\gtrsim_{mF}$ is ordinal, continuous and greatest faithful. Rephrasing the proof of proposition 1 immediately leads to the following result (adapted from [3]).

Proposition 3. The min in Favor ranking rule $\gtrsim_{mF}$ is the only ranking rule that is ordinal, continuous and greatest faithful.

Some remarks on this proposition are in order.

a) Contrary to the case of choice rules, greatest faithfulness is not a particularly intuitive requirement for ranking rules (some alternative axioms may be found in [3]). A much more intuitive axiom would consist in imposing that $\gtrsim(X, T) = T$ for all weak orders T on X. It is easy to see that $\gtrsim_{mF}$ is not faithful in this sense since, when T is crisp, $\gtrsim_{mF}(X, T)$ consists of at most two equivalence classes. When T is a weak order, it is however true that $U_1(X, \gtrsim(X, T)) = U_1(X, T)$. In presence of ordinality and continuity, this last condition is not sufficient to characterize C_{mF} as shown by the following example. Define the scoring function:

$$S(a, X, R) = -\max_{\{b \in X : Not(aEb)\}} R(b, a),$$

where E is the equivalence relation on X defined by:

a E b $\Leftrightarrow$ [R(a, b) = R(b, a) and R(a, c) = R(b, c), R(c, a) = R(c, b) for all c $\in$ X\{a, b}].

The ranking rule based on this score is obviously ordinal. It is not difficult to show that it is continuous and that, when T is a weak order, $U_1(X, \gtrsim(X, T)) = U_1(X, T)$.

b) Examples similar to the ones used in section 3 show that the axioms used in proposition 3 are independent.

c) As was the case with choice rules, the conjunction of ordinality and continuity in proposition 3 is not entirely satisfactory. Similarly to what has been done in proposition 2, it is possible to replace ordinality and continuity by the following strong ordinality requirement:

a $\gtrsim(X, R)$ b $\Rightarrow$ a $\gtrsim(X, \phi[R])$ b,

for all finite set X, all $R \in \mathcal{F}_X$, all a, b $\in$ X and all non-decreasing transformation ϕ on [0, 1] such that $\phi(0) = 0$ and $\phi(1) = 1$,

which is equivalent to the conjunction of ordinality and continuity.
Alternative ways out of the ordinality-continuity puzzle are described in the next remark

d) The alternative characterization of $\gtrsim_{mF}$ presented in [13] – and anticipated in [12] – uses neither ordinality nor continuity. We briefly recall here the essential elements of this result. Consider two fuzzy relations R and R′ on a finite set X and let a, b be distinct elements of X. We say that R and R′ are related by translation on {a, b} if R′ is identical to R except that, for some $\varepsilon \in [-1, 1]$, $R(a, c) = R'(a, c) + \varepsilon$, for all $c \in X\setminus\{a\}$ and $R(b, d) = R'(b, d) + \varepsilon$, for all $d \in X\setminus\{b\}$.

A ranking rule $\gtrsim$ is said to be *translation invariant* if for all finite set X, all distinct $a, b \in X$ and all $R, R' \in \mathcal{F}_X$:
[R and R′ are related by translation on {a, b}] $\Rightarrow$ [a $\gtrsim(X, R)$ b $\Leftrightarrow$ a $\gtrsim(X, R')$ b].

A ranking rule $\gtrsim$ is said to be:
– *weakly reversible* if a $\gtrsim(X, R)$ b $\Rightarrow$ [for all $c \in X\setminus\{a\}$, there is a relation $R_c \in \mathcal{F}_X$ identical to R except that $R_c(a, c) \leq R(a, c)$ and such that b $\gtrsim(X, R_c)$ a],
– *strictly reversible* if a $\gtrsim(X, R)$ b $\Rightarrow$ [for all $c \in X\setminus\{a\}$ such that $R(b, c) \neq 0$, there is a relation $R_c \in \mathcal{F}_X$ identical to R except that $R_c(a, c) \leq R(a, c)$ and such that b $\succ(X, R_c)$ a],
for all finite set X, all $R \in \mathcal{F}_X$ and all $a, b \in X$.

Translation invariance means that adding a constant to all valuations leaving a and b does not alter their respective comparison. Weak reversibility implies that the comparison between any two alternatives may be reversed by sufficiently decreasing any of the valuations of the arcs leaving the best ranked alternative. Strict reversibility asserts this reversal may be strict as soon as there is no boundary problem. It is not difficult to show that these three conditions are independent and are satisfied by $\gtrsim_{mF}$. The proof that it is the only ranking rule satisfying these three conditions is easy once it has been observed that:
– if $\gtrsim$ is weakly reversible then $[S_{mF}(a, X, R) = 0] \Rightarrow$ [b $\gtrsim(X, R)$ a, for all $b \in X$] and
– if $\gtrsim$ is strictly reversible then $[S_{mF}(b, X, R) > S_{mF}(a, X, R) = 0] \Rightarrow$ [b $\succ(X, R)$ a], where $\succ(X, R)$ denotes the asymmetric part of $\gtrsim(X, R)$.
A thorough comparison between this characterization and the one presented in proposition 3 can be found in [13].

Using the above-mentioned consequences of weak and strict reversibility, another – new – characterization of $\gtrsim_{mF}$ can easily be derived combining strong ordinality and the two reversibility conditions. We have:

Proposition 4. The min in Favor ranking rule $\gtrsim_{mF}$ is the only ranking rule that is strongly ordinal, weakly reversible and strictly reversible.

Proof. We have already noted that $\gtrsim_{mF}$ is strongly ordinal, weakly reversible and strictly reversible. The proof will be complete showing that if a ranking rule $\gtrsim$ satisfies these conditions then, denoting by $\simeq(X, R)$ the symmetric part of $\gtrsim(X, R)$:

[mF(a, X, R) > mF(b, X, R) $\Rightarrow$ a $\succ$(X, R) b] and
[mF(a, X, R) = mF(b, X, R) $\Rightarrow$ a $\equiv$(X, R) b],
for all finite set X, all R $\in \mathcal{F}_X$ and all a, b $\in$ X.

Suppose that mF(a, X, R) > mF(b, X, R) and consider a non-decreasing transformation ϕ on [0, 1] such that $\phi(x) = 0$ if x < mF(a, X, R) and $\phi(x) = x$ otherwise. Since mF(a, X, ϕ[R]) > mF(b, X, ϕ[R]) = 0, strict reversibility implies that a $\succ$(X, ϕ[R]) b and b $\succsim$(X, R) a would contradict strong ordinality. The proof that mF(a, X, R) = mF(b, X, R) $\Rightarrow$ a $\equiv$(X, R) b is similar using weak reversibility and strong ordinality. □

Let us now turn to the study of the Iterated min in Favor ranking rule $\succsim_{ImF}$ based on the (downward) iteration of C_{mF}. This ranking rule is clearly ordinal. The iteration process may however create discontinuities as shown by the following example. Let X = {a, b, c, d} and consider the family of relations $R_\varepsilon \in \mathcal{F}_X$ defined by the following table:

R_ε	a	b	c	d
a	1	0.7	0.5	0
b	0.6	1	0.5	0
c	0.5	0.5	1	0.4 - ε
d	0.4	0.4	0.4	1

For all $\varepsilon \in$ (0, 0.4), $\succsim_{ImF}$ leads to the weak order d > (abc). When ε reaches 0, we obtain (cd) > a > b, which violates continuity. Since $U_1(\succsim_{ImF}(X, R)) = C_{mF}(X, R)$, continuity cannot be violated for the top-ranked elements however; hence the following requirement which is obviously satisfied by $\succsim_{ImF}$. A ranking rule $\succsim$ is *top continuous* if, for all finite set X, all R $\in \mathcal{F}_X$ and all sequences ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converging to R, [a $\in U_1$(X, $\succsim$(X, R_i)) for all R_i in the sequence] $\Rightarrow$ [a $\in U_1$(X, $\succsim$(X, R))].

Because $U_1(\succsim_{ImF}(X, R)) = U_1(\succsim_{mF}(X, R))$, $\succsim_{ImF}$ is greatest faithful. Moreover, it should be noticed that, contrary to the situation with $\succsim_{mF}$, it is also true for $\succsim_{ImF}$ that, when T is a weak order, $\succsim_{ImF}$(X, T) = T.

Since continuity implies top continuity and $\succsim_{ImF}$ is distinct from $\succsim_{mF}$, the conjunction of ordinality, top continuity and greatest faithfulness does not characterize $\succsim_{ImF}$. Replacing greatest faithfulness by the requirement of being faithful on weak orders does not solve the problem as shown by the example in remark a) above. This calls for an axiom that would capture the iterative character of $\succsim_{ImF}$. In order to do so, we use a condition proposed in [1], lacking any less transparent way to do so. A ranking rule $\succsim$ is said to be *top decomposable* if, for all finite set X and all R $\in \mathcal{F}_X$, $\succsim$(X, R)/Y = $\succsim$(Y, R/Y), where Y stands for X\U_1(X, $\succsim$(X, R)).

Though this condition may look complex it admits a simple interpretation. The alternatives that are ranked first in $\succsim$(X, R) are those of U_1(X, $\succsim$(X, R)). Top decomposability says the ranking of the remaining alternatives is unaltered if the alternatives U_1(X, $\succsim$(X, R)) are taken out of X and the same ranking rule is applied to X\$U_1$(X, $\succsim$(X, R)), R being restricted to this set. Top decomposability may be an important property in practice; if, for some reason, the best alternatives in a certain set become unavailable, it is not necessary to apply the ranking rule to the remaining alternatives: they will be ranked exactly as in the previous ranking. The technical interest of top decomposability is obvious. Let $\gtrsim$ be a ranking rule. To this ranking rule we associate a choice rule $C_{\gtrsim}$ such that, for all finite set X and all $R \in \mathcal{F}_X$, $C_{\gtrsim}(X, R) = U_1(X, \gtrsim_X(R))$. By definition, $\gtrsim$ is top decomposable if and only if $\gtrsim$ is identical to the ranking rule defined by the iteration of $C_{\gtrsim}$. It is clear that $\gtrsim$ is greatest faithful if and only if $C_{\gtrsim}$ is greatest faithful. If $\gtrsim$ is ordinal (resp. top continuous) then $C_{\gtrsim}$ is ordinal (resp. continuous). Given top decomposability and proposition 1, this proves:

Proposition 5. The Iterated min in Favor ranking rule $\gtrsim_{ImF}$ is the only ranking rule that is ordinal, top continuous, greatest faithful and top decomposable.

Since ordinality, continuity and greatest faithfulness are independent properties for choice rules and $\gtrsim_{mF}$ is ordinal, top continuous and greatest faithful but not top decomposable, it is easily shown that the axioms used in proposition 5 are independent. We are not presently aware of any axioms that would allow to dispense with the, strong, top decomposability requirement.

5 Discussion and Open Problems

The various results presented in section 3 and 4 raise many questions and leave open many problems. We summarize here those that appear to be the most important ones.

This paper has been concerned with the study of choice and ranking rules operating on any reflexive fuzzy relation defined on a finite set. Though we mentioned that this is an appropriate setting for ELECTRE III, this is not true for other methods such as PROMETHEE or the ones proposed in [1]. As shown in [6], though the fuzzy relations built with these methods do not possess remarkable properties, they may not lead to any fuzzy relation. Among aggregation methods leading to fuzzy relations, ELECTRE III which may produce any of them, may be considered as an exception. Since our axioms make explicit use of the richness of the set $\mathcal{F}_X$, our results cannot be used to characterize exploitation techniques to be coupled with methods that always produce fuzzy relations belonging to a proper subset of $\mathcal{F}_X$.

In order to overcome this problem, one may try to characterize the subset of $\mathcal{F}_X$ that may be obtained with a given aggregation technique and then analyze choice and ranking rules using axioms adapted to this subset (an example of such an analysis is found in [6]). Alternatively, and perhaps more fruitfully, one may try to characterize both steps of an Outranking Method, *i.e.*, a technique starting with alternatives evaluated on several criteria and leading to a choice or a ranking, therefore ignoring the intermediate step of the construction of a fuzzy relation.

Numerous examples of such characterizations can be found in the literature in Social Choice Theory (see, *e.g.*, [18]). Much work remains to be done in this direction in the area of MCDM.

We mentioned that we interpreted our fuzzy relations as "large" preference relations in accordance with their construction in ELECTRE III. Though this allows to motivate our greatest faithfulness axioms, this also raises the problem of defining the symmetric and asymmetric part of the relation, *i.e.* the indifference and the strict preference relation associated with the large preference relation. This problem is known to be difficult for fuzzy relations (on this point see [10] or [11]). It was not dealt with explicitly here. We were thus unable, for instance, to consider choice rules that would discriminate among alternatives that are "at least as good" as all other alternatives on the basis of the way they compare in terms of "indifference" or "strict preference" to these alternatives.

Let us finally mention that many scores apart from the mF score would deserve attention. If ranking and choice rules based on scores involving sums have already been well studied (see [4], [7]), the min Difference score:

$$mD(a, X, R) = \operatorname*{Min}_{b \in X \setminus \{a\}} (R(a, b) - R(b, a)$$

which seems particularly interesting (see [2]) remains to be fully explored. Similarly, a general characterization of rules using scores based on ranks is yet to be obtained.

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Models of Cooperative Decision Making

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Abstract. Many decision problems involve multiple decision makers with multiple goals. Goals can be divided into two types, goals that are mutual for all the decision makers and goals that are different and require cooperation of multiple decision makers to achieve a consensus. A cooperative decision making requires free communication among decision makers. The paper presents a problem solving approach to achieve a consensus in cooperative decision making.

Keywords. Group decision making, cooperation, models, problem solving

1 Introduction

There are many situations in which a group of people make some joint decisions. The people have some mutual interests but some interests can be different. Then the situation can lead to the conflict of interests. Many social conflicts can occur in the society. Our research is devoted to the modelling of social conflicts[1]. The main idea of the paper is that cooperation gives synergical effects in a conflict resolution. It is more efficient to pass from a conflict resolution to a joint problem solving.

The general formulation of group decision making involves a group of decision makers with multiple goals. Goals can be divided into two types, goals that are mutual for all the decision makers and goals that are different and require cooperation of multiple decision makers to achieve a consensus or an agreement. The aim of group decision making is to choose appropriate decisions for all the members of the group. A cooperative decision making requires free communication among decision makers.

A problem is a difference between a current state and a desired future state. The problem statement is: How can we close the gap between the current state and the desired state? The basic trend in the cooperative decision making is to transform a possible conflict to a joint problem. The problem solving statement

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enables us to use some formal problem solving procedures that help to find consensus for decision makers.

The motto of the conference "Twenty Years of East-West Cooperation in MCDM" gives a good example of group decision making as a joint cooperative problem solving. The group is The International Group of MCDM and decision makers are members of this group. The members have some goals that are general, individual, theoretical, practical etc. Decision making is about ideas and trends in MCDM. Communication media are conferences, working groups, journals, information, bulletins, visits, joint teams etc. The problem statement is: How can we close the gap between the current state of knowledge of MCDM and the desired state of knowledge of MCDM.

2 Cooperative decision making

Decision making is making a choice among the alternatives available to us. When two people are involved in making a decision they both have agreed before the decision is made. When more than two people are involved in making decision, the situation changes. The individual choice of each group member must be considered as a part of the group decision. One way the group decisions are made is by majority vote, another way is by consensus. Decision making by consensus is highly desirable so long as everyone has an opportunity to express his judgements.

There are two very important aspects of group decision making: assertiveness and cooperativeness. Assertiveness is satisfaction of one's own concerns and cooperativeness is a tendency to satisfy others. According to the position on the scales of assertiveness and cooperativeness the type of decision situations (Lose-Lose, Lose-Win, Win-Lose, Win-Win) and five conflict management approaches (Avoiding, Accommodating, Forcing, Compromising, Joint problem solving) can be determined (see Fig. 1).

One of the most important processes in preventing the dangerous effects of win-lose contexts is the development and the use of cooperative context. That means that we must change the character of the win-lose decision problem. That is, those interactions that are structured for competition must be restructured to be cooperative. We need to restructure our relations hips so that there are not winners or losers. The way to do this is to develop a more cooperative interaction.

A cooperative process leads to the defining of conflicting interests as a mutual problem to be solved by a collaborative effort. It facilitates the recognition of the legitimacy of each other's interests and of the necessity of searching for a solution that is responsive to the needs of all. It tends to limit rather than expand the scope of conflicting interests.

It is extremely difficult for a win-lose situation to be transformed into another kind of situation once it has become active. Thus when we are in a win-lose context or structure there cannot be a win-win result. The basic win-lose structure of the situation itself must be changed before the mutual sharing of the rewards can take place. We should carefully monitor our relationships with others so that the joint problem solving and the decision making can maximize the distribution of the rewards.

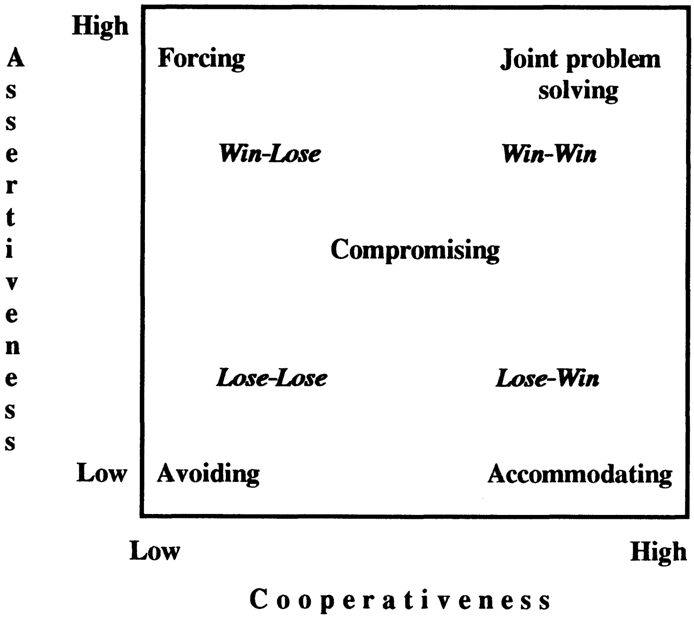

Fig. 1.

A very useful preventive system is to develop skills in the group problem solving. It includes the working out of mutual goals, exploring obstacles together, establishing the criteria that will meet the needs of both parties, exploring alternative solutions to the problem, evaluating those solutions in the terms of the criteria, and selecting the solution that meets the criteria. One of the most important of the preventive processes is the development and maintenance of an open communication between the participants. The presence of barriers and interferences to communication between people clearly encourages escalation of differences.

Negotiation process is the most used tool to resolve conflicts between different interests. The following factors appear to be relevant to develop some models of negotiation and to aid the negotiators:

a) separate the people from the problem,
b) provide communication between negotiators,
c) help negotiators identify their real interests,
d) generate options for mutual gain,
e) use objective criteria.

Negotiation is a process of conflict management aimed at fulfilling needs and interests when they are in opposition to each other. Negotiation involve mutual communication, joint decision making, coordination of interests, and the influencing of decisions.

Negotiation is a process by which a joint decision is made by two or more parties. It is a communication designed to reach an agreement when the parties have some interests that are shared and others that are opposed.

Both the communication and cooperation are required in the problem solving process of negotiation, and there are some important relationships between the communication and the cooperative behaviour. The very act of communication in negotiating involves some cooperation between the negotiators. The more cooperative they are the greater is the possibility of reaching an agreement. When the problem solving process and the negotiation are not handicapped by win-lose contexts but are marked with cooperative efforts to solve joint problems, the resolution can be more effective.

It is also suggested to develop the "Best Alternative To a Negotiated Agreement" (called BATNA) as the standard against which any proposal can be measured. BATNA represents the best you can get without any agreement.

For resolving conflicts have already been proposed some general models of negotiation process. Most of the models are based on rationality axioms. But the experiments show that neither the people exactly follows the rationality axioms settling the theories nor the information is sufficient to really build the realistic model.

It is very important to understand how human minds solve problems and make decisions. People solve problems by heuristic search thorough large problem spaces using means-ends analysis as a principal technique for guiding the search. Artificial intelligence combine human thinking with computers to achieve more efficient solutions. Problem solving was initially studied by psychologists, and now more recently by researches in artificial intelligence.

3 Problem solving

The problem solving is the attempt to find solutions to problems that confront people. The problem solving process is an extensive organization of information and attitudes surrounding a problem situation. The process involves understanding the goals, setting criteria for evaluating possible solutions, the exploration of alternative solutions, selection of the solution that best fits the criteria.

Some basic ideas of formal approaches of the problem solving can be introduced to cooperative decision making.

Problem solving is an important topic in the artificial intelligence. There are two aspects of the problem solving - representation and searching. The state space representation introduces the concepts of states and operators. An operator transforms one state into another state. A solution could be obtained by a search process that first applies operators to the initial state to produce new states and so on, until the goal state is produced. The state space is a structure (S,F), where $S = \{s_i\}$ is a set of states and $F = \{f_j\}$ are partial functions

$$f_j: S \rightarrow S.$$

The problem is defined by the start state s_0 and goal states G. The problem solving is then a sequence of operators $f_1, f_2, \ldots, f_n$, that the condition holds

$$(f_n \ldots(f_2(f_1(s_0))) \ldots) = s_n \in G.$$

The state space can be expressed as a graph, where the nodes correspond with states and the arcs correspond with operators. The problem of finding a sequence of operators is equivalent to the problem of finding a path in a graph. The graph representation is useful for discussing the various search methods.

There are several search methods. The breadth-first method expands nodes in the order in which they are generated. The depth-first search method expands the most recently generated nodes first. The depth-first search method can be completed by a backtracking procedure. The decision maker indicates the nodes as imperative or tentative. The tentative pointers indicate a path back to the start node. From the tentative nodes it is possible to search another path. The search can be blind or heuristic. It is possible to help reduce the search by heuristic information about the problem, for example the distance between an arbitrary node and the goal set of nodes.

We propose a two phases' interactive approach for solving cooperative decision making problems:

1. Finding the ideal alternative and the BATNA for individual decision makers.
2. Finding a consensus for all the decision makers.

In the first phase every decision maker search the ideal alternative by the assertivity principle.

The general formulation of an individual multicriteria decision problem is expressed as follows

$$z(x) = (z_1(x), z_2(x), \ldots, z_k(x)) \rightarrow \text{"max"} \qquad (1)$$

$$x \in X,$$

where X is a decision space, x is a decision alternative and $z_1, z_2, \ldots, z_k$ are the criteria. The decision space is defined by objective restrictions and by mutual goals of all the decision makers in the aspiration level formulation. The decision alternative x is transformed by the criteria to criteria values $z \in Z$, where Z is a criteria space. Every decision maker has his own criteria.

People appear to satisfy rather than attempting to optimize. That means substituting goals of reaching specified aspiration levels for goals of maximizing.

We denote $y^{(s)}$ aspiration levels of the criteria and $\Delta y^{(s)}$ changes of aspiration levels in the step s. We search alternatives for which it holds

$$z(x) \geq y^{(s)} \qquad (2)$$
$$x \in X.$$

According to heuristic information from results of the condition (2) the decision maker changes the aspiration levels of criteria for step s+1 :

$$y^{(s+1)} = y^{(s)} + \Delta y^{(s)}. \qquad (3)$$

We can formulate the multicriteria decision problem as a state space representation. The state space corresponds with the criteria space Z, where the states are the aspiration levels of the criteria $y^{(s)}$ and the operators are changes of the aspiration levels $\Delta y^{(s)}$. The start state is a vector of the initial aspiration levels and the goal state is a vector of the criteria levels for the best alternative.

For finding the ideal alternative we use the depth-first search method with backtracking procedure. The heuristic information is distance between an arbitrary state and the goal state.

We propose an interactive procedure ALOP (Aspiration Levels Oriented Procedure) for multiobjective linear programming problems, where the decision space X is determined by linear constraints

$$X = \{x \in R^n ; Ax \leq b, x \geq 0\} \qquad (4)$$

and $z_i = c_i x$, i=1,2, ...,k, are linear objective functions. Then z(x) = Cx, where C is a coefficient matrix of objectives.

The decision alternative $x = (x_1, x_2, \ldots, x_n)$ is a vector of n variables. The decision maker states aspiration levels $y^{(s)}$ for the criteria values. There are three possibilities for aspiration levels $y^{(s)}$. The problem (2) can be feasible, infeasible or the problem has a unique nondominated solution. We verify the three possibilities by solving the problem

$$v = \sum_{i=1}^{k} \frac{1}{\overline{z}_i} d_i^+ \rightarrow \min \tag{5}$$

$$Cx - d^+ = y^{(s)}$$

$$x \in X, \, d^+ \geq 0 \, .$$

The value of the objective function in the problem (5) can be interpreted as an increase of utility.

If it holds :

- $v>0$, then the problem is feasible and d_i^+ are proposed changes $\Delta y^{(s)}$ of aspiration levels which achieve a nondominated solution in the next step,
- $v=0$, then we obtained a nondominated solution,
- the problem is infeasible, then we search the nearest solution to the aspiration levels by solving the goal programming problem

$$v = \sum_{i=1}^{k} \frac{1}{\overline{z}_i} (d_i^+ + d_i^-) \rightarrow \min \tag{6}$$

$$Cx - d_i^+ + d_i^- = y^{(s)}$$

$$x \in X, \, d^+ \geq 0 \, , \, d^- \geq 0 \, .$$

The solution of the problem (6) is feasible with changes of the aspiration levels $\Delta y^{(s)} = d_i^+ - d_i^-$. For small changes of nondominated solutions the duality theory is applied. Dual variables to objective constraints in the problem (6) are denoted u_i, $i=1,2,\ldots,k$.

If it holds

$$\sum_{i=1}^{k} u_i \Delta y_i^{(s)} = 0 \, , \tag{7}$$

then for some changes $\Delta y^{(s)}$ the value $v=0$ is not changed and we obtained another nondominated solution. The decision maker can state k-1 small changes of the aspiration levels $\Delta y_i^{(s)}$, $i=1,2, ...,k$, $i \neq r$, then the change of the aspiration level for criterion r is calculated from (7).

The decision maker chooses a forward direction or backtracking. Results of the procedure ALOP are the path of tentative aspiration levels and the ideal solution.

The BATNA can be search by an analogical approach as the ideal alternative but on another decision space that is free separately for the decision maker. The ideal alternative represents the best criteria values and the BATNA represents the worst ones for the next cooperative problem solving.

In the second phase a consensus could be obtained by the search process and the principle of cooperativeness is applied. The heuristic information for the decision maker is the distance between his proposal and the opponent's proposal. We assume that all the decision makers found their ideal alternatives and BATNAs . We propose an interactive procedure GROUP-ALOP for searching a consensus.
For simplicity we assume the model with two decision makers

$$\begin{aligned} & z^1(x) \rightarrow \text{"max"} \\ & z^2(x) \rightarrow \text{"max"} \\ & x \in X . \end{aligned} \tag{8}$$

The decision makers search a consensus on a common decision space X. The decision makers change aspiration levels of the criteria y^1, y^2. The sets of feasible alternatives for the aspiration levels y^1 and y^2 are X^1 and X^2.

$$\begin{aligned} & z^1(x) \geq y^1 \qquad\qquad & z^2(x) \geq y^2 \\ & x \in X & x \in X . \end{aligned} \tag{9}$$

The consensus set S of the negotiations is the intersection of sets X^1 and X^2

$$S = X^1 \cap X^2 . \tag{10}$$

By changes of the aspiration levels the consensus set S is changed too. The decision makers search one element consensus set S by alternating of the consensus proposals.

The image of partner's proposal can be taken as aspiration levels in one's own criteria space. In searching for a consensus the distance between the proposals is a heuristic information. The paths of the tentative aspiration levels can be used for the backtracking procedure. The forward directions can be directed by proposed new aspiration levels in step s+1:

$$\begin{aligned} y^1(s+1) &= (1-\alpha)y^1(s) + \alpha z^1(x^2) , \\ y^2(s+1) &= (1-\beta)y^2(s) + \beta z^2(x^1) , \end{aligned}$$

where $\alpha, \beta \in <0,1>$ are the coefficients of cooperativeness.

Each decision maker applies cooperative strategy as long as his partner does the same. If the partner exploits the decision maker on a particular step, the decision maker then applies the exploitative strategy on the next step and continues to do so until the partner switches back to the cooperative strategy. Under these conditions, the problem stabilizes with the decision makers pursuing the mutually cooperative strategy and receiving the consensus.

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The Predictive Power of the Self Explicated Approach and the Analytic Hierarchy Process: A Comparison

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Abstract. The Analytic Hierarchy Process uses pairwise comparisons to determine the weights of criteria and the desirability of the levels. In the Self Explicated Approach the decision maker rates them explicitly. In this study we have compared the predictive power of these two approaches. The predictive power of both methods is tested with respect to the choice, the ranking and the preference scores of two sets of alternatives. The results of the laboratory study with 180 participants indicate that the Self Explicated approach, even with less input data, can show better results than the Analytic Hierarchy Process.

Keywords. Analytic Hierarchy Process, Self Explicated Approach, laboratory study, predictive ability

1 Introduction

Many decisions involve numerous criteria and goals, sometimes conflicting and usually partly quantitative and partly qualitative. This kind of decision-making is called Multiple Criteria Decision-Making [11],[24]. Important elements in many MCDM methods are the weights of the criteria and the scores of alternatives for these criteria. Various methods exist to derive these measures. This paper will discuss and compare two of these methods, the Analytic Hierarchy Process (AHP) and the Self Explicated Approach (SE).

The AHP requires pairwise comparisons of the criteria and the alternatives in order to determine the weights of the criteria and the scores of the alternatives. Empirical AHP research is dominated by case studies in which a specific problem is solved (e.g. [15]). Some research has been done to test the quality of the method in an experimental setting. Schoemaker and Waid [20] compare AHP to four other methods. The methods yielded significantly different weight estimates, but with respect to the predictions, the various methods performed roughly equally on average. Tscheulin [22] compares AHP to conjoint analysis and finds a similar predictive ability. Some other studies compare methods without confronting the predictions with a validation task [2],[16]. Theoretical comparisons of AHP with other methods are provided by [13] and [2]. The studies described are inconclusive with respect to the predictive ability of AHP. One reason for this inconclusiveness is the limited number of respondents (30 and 60). In this study we will compare

AHP with the Self Explicated Approach (SE). SE is an easy to understand Multiple Attribute Utility (Value) approach in which the weights of the criteria and the desirability of the criteria levels are explicitly judged by the decision maker. In this paper we will provide a brief theoretical comparison as well as an elaborate empirical comparison of both methods. The empirical comparison is based on a laboratory study with 180 participants which tested the predictive power of AHP and SE.

Section 2 will describe the theoretical comparison of both methods. The remaining sections will focus on our laboratory study. In section 3 the hypotheses underlying our experiment will be described. Section 4 will discuss the design of the research. The results of the laboratory study will be presented in section 5. The paper ends with a discussion of the resulting implications on the applicability of both methods.

2 A theoretical comparison

The Self Explicated Approach computes the overall utility for a multiple criteria alternative as a weighted sum of the alternative's criteria levels and associated value ratings, as separately and explicitly judged by the decision maker [1]. In the AHP [17],[18] a simple MCDM problem is formulated as a three level hierarchy [13]. It requires the explicit specification of the overall objective at the highest level, the criteria on the second level and the alternatives on the third level. Subsequently, the decision maker makes pairwise comparisons of the alternatives (for each criterion) to determine the scores of the alternatives. The criteria are compared pairwise in order to establish the weights of the criteria. The overall priority of an alternative is calculated as a weighted sum of the weight of a criterion and the score of an alternative to that criterion.

The Self Explicated Approach and the Analytic Hierarchy Process are both compositional methods. In modelling preferences, there are two groups of methods, compositional and decompositional methods [8]. The decompositional approach (such as conjoint analysis) begins with overall evaluations of alternatives and these evaluations are then used to explicate the values attached to criteria and their levels (criteria and attributes are used interchangeably) [1]. In contrast to decompositional methods, compositional methods measure the preferences for each criterion and criterion level separately and compute the overall utility based on a weighted sum of these values. An advantage of the compositional methods is the reduction of the information overload problem, because the respondent is questioned separately about each criterion.

Both SE and AHP are based on an additive value function to compute the overall preference of an alternative. Consequently, the methods can only be applied to problems in which the conditions for the existence of additive value function are satisfied (e.g. [13]). Also, both methods require the explicit specification of the criteria and the levels of these criteria before the analyses are conducted. The resulting hierarchies of criteria are comparable. They only differ on points of presentation [2].

Despite these similarities there are some substantial differences. The most important methodological distinction between SE and AHP is the way they determine the preference for attribute levels and the importance of attributes. SE requires decision makers to explicitly express their preference for each criterion and each level. AHP calculates these values based on pairwise comparisons. The elements in a single level (criteria or levels of a criterion) are compared with respect to a purpose from the adjacent higher level. In other words: Analytic Hierarchy Process derives preferences, Self Explicated explicitly requests them.

In a traditional AHP application, alternatives are compared based on a predefined set of criteria. Saaty [19] has proposed the absolute measurement approach in which the decision maker compares attribute levels described as relative intensities (e.g. high, medium and low). In this study we use the absolute measurement approach with absolute intensities [10]. Both the AHP and the Self Explicated Approach apply the same general model of the problem situation:

$$p_j = \sum_{i=1}^{c} w_i * x_{ij} \qquad \text{(eq. 1)} \qquad \text{with } \sum_{i=1}^{c} w_i = 1$$

p_j : the preference score of alternative j
w_i : weight of criterion i
x_{ij} : desirability score of alternative j on criterion i
c : number of criteria

The priority of alternative j is determined by means of a linear additive function in which the desirability of alternative j for criterion i (x_{ij}) is multiplied by the weight of that criterion (w_i), summed over all criteria.

An important difference between both methods when applying both methods in practice is the number of judgements (comparisons or ratings). In the absolute measurement approach of AHP, the decision maker compares each pair of criteria and each pair of levels for each criterion. In SE, the decision maker rates each criterion and each level separately. Equation 2 represents the number of pairwise comparisons in AHP; equation 3 represents the number of ratings in SE.

$$n_{AHP} = \frac{c(c-1)}{2} + \sum_{i=1}^{c} \frac{l_i(l_i-1)}{2} \qquad \text{(eq. 2)}$$

$$n_{SE} = c + \sum_{i=1}^{c} l_i \qquad \text{(eq. 3)}$$

n_{AHP} : number of pairwise comparisons in AHP
n_{SE} : number of ratings in SE
c : number of criteria
l_i : number of levels of criterion i

When the number of criteria and/or the number of levels of a criterion increases AHP requires many more judgements than SE. The difference between the number of judgements is presented in figure 1. The figure shows that the number of pairwise comparisons (AHP) when compared to the number of ratings (SE) increases if the (average) number of levels of the criteria increases. For example, in the case of 5 criteria each with 3 levels, AHP requires 10 + 5 * 3 = 25 pairwise comparisons, SE requires 5 + 5 * 3 = 20 ratings. In small problems (with only a few criteria and levels) the number of judgements is almost the same. For larger problems AHP requires substantially more judgements. This does not disqualify the method. AHP is not intended to be used for the majority of decisions we make every day, but rather for crucial decisions [5].

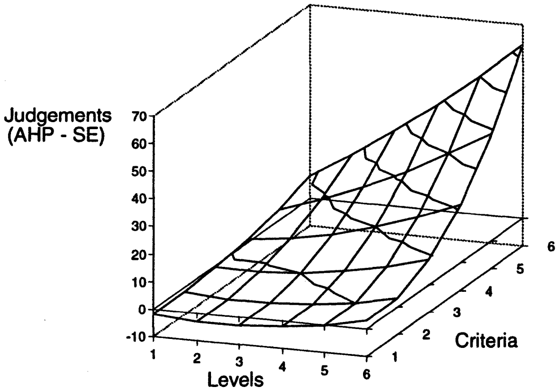

Figure 1: Difference between the number of judgements of AHP and SE

3 Hypothesis

Both AHP and SE result in an explicit model of the preference structure of the decision maker. These models should be able to predict the preferences of a decision maker in a validation task [9],[4]. The validation tasks comprise the evaluation of alternatives in order to test the predictive ability of both methods. Based on the theoretical discussion in the previous section we have no reason to expect one method to outperform the other. Consequently, we test the hypothesis that the predictive powers of AHP and SE are equal. The predictive powers are tested with respect to three aspects: the choice, the ranking of the alternatives and the preference scores of the alternatives.

The choice refers to the alternative a decision maker will select from a number of alternatives. Both AHP and SE can be applied to find out which alternative is assigned the highest score by a decision maker. We assume that the decision maker will choose the alternative with the highest preference score as computed by a method.

Hypothesis 1: AHP and SE are equally powerful when it comes to predicting which alternative a decision maker will choose from a set of alternatives.

The priority structure of the decision maker [24] results in a ranking of the alternatives considered. If the elicited preference model is a good reflection of the priority structure of the decision maker, the model should be able to predict the ranking within a set of alternatives [20],[21],[22].

Hypothesis 2: AHP and SE are equally able to predict the ranking of a set of alternatives.

Hypothesis 2 tests the ranking of the alternatives and refers to the ordinal characteristics of the preference data only. A more stringent test consists of the comparison of the actual and predicted preference scores [3],[20],[1]. The preference scores predicted by AHP and SE will be compared with the preference scores assigned to alternatives by the decision maker.

Hypothesis 3: AHP and SE are equally able to predict the preference scores within a set of alternatives.

4 Research design

To test these hypotheses we have conducted a laboratory study with 180 participants. The participants were graduate students of the University of Groningen in the Netherlands. The decision task they had to complete was the selection of a room to rent. This decision is frequently studied in related research [7],[12],[4]. We incorporated five criteria: rent, area, location, type of house, and facilities. The first two criteria are quantitative and have four levels. The other three criteria are qualitative and have three levels (Table 1). In order to test the hypotheses, we assume that the decision maker's judgements in the validation task reflect the 'right' preferences. This assumption is only valid if the decision maker can cope with the problem without decision aids. Therefore we selected a problem that is complex, to assure that MCDM methods can be applied, but not to complex, to assure that the decision maker is able to make direct evaluations. Furthermore, the problem was designed as realistic as possible by conducting a survey among students to determine the relevant criteria and the levels of criteria.

Table 1: The selected criteria and the levels of these criteria.

Criterion	Level 1	Level 2	Level 3	Level 4
Rent	Dfl 450,-	Dfl 375,-	Dfl 300,-	Dfl 225,-
Area	10 m2	15 m2	20 m2	25 m2
Location	centre	in between	near the university	
Type	small student house (3 students)	large student house (6 students)	student flats	
Facilities	garden	balcony	none	

The participants completed four tasks in this study: (1) making AHP pairwise comparisons, (2) making SE judgements, (3) comparing a small set of alternatives, and (4) rating a larger set of alternatives. The latter two tasks are validation tasks and were used to test the predictive power of AHP and SE.

1. *AHP task.* To complete the AHP task the participants compared the levels of the criteria pairwise in order to establish the desirability of the levels. Subsequently, the criteria were compared pairwise to establish the weights of the criteria. Half of the respondents made these comparisons using the verbal mode of AHP (they indicated their preference by statements like 'a room with a garden is extremely more preferred than a room with a balcony'), the others used the numerical mode (they indicated their preference by selecting a number between 1 and 9). The verbal or numerical mode was randomly assigned to the respondents. Analyses have shown there are no significant differences for the test variables considered. So the two groups were treated as one.
2. *SE task.* In the SE task the respondents divided 100 points among the five criteria. The most preferred attribute level was assigned a score of 10. The remaining attribute levels were rated on a 0 - 10 scale in comparison to the most preferred level [8].
3. *Comparing task.* The participants divided 100 points among four alternatives. This task is similar to decision tasks in practice and the traditional AHP applications. In these tasks decision makers simultaneously compare a limited number of alternatives.
4. *Rating task.* The participants evaluated 25 rooms on a 0 - 100 scale. This task is relevant because SE, AHP and other Multiple Criteria Decision Making methods assume that decision makers have an implicit decision model. Based on this model the decision maker rates a number of alternatives independent of each other.

5 Results

In the next sections we will discuss the results of the laboratory study with respect to the hypotheses.

5.1 Choice

Table 2 shows the number (and percentage) of cases in which AHP and SE predict the same choice as the decision maker made in the validation task. The number of correct predictions is not an integer value because of ties. If two alternatives were preferred most in a validation task and AHP (or SE) computed the highest preference for one of them, the value 0.5 was assigned, instead of 1 [14]. Consequently table 2 shows a conservative estimate of the predictive ability of the methods. For both the comparing and the rating task AHP predicted 50.2% of the choices correctly. SE predicted 54.3% and 54.5% of the choices correctly. For both validation tasks SE outperforms AHP, however, these differences are not significant at a $p = 0.05$ level.

Table 2: Correct choices of AHP vs. SE with respect to both validation tasks (hypothesis 1; n = 180)

Choice	Correct predictions		Differences	
	N	%	%	Z-score
Rating task				
- AHP	90.34	50.19		
- SE	97.82	54.34	4.15	0.789
Comparing task				
- AHP	90.33	50.18		
- SE	98.08	54.49	4.31	0.818

5.2 Ranking of alternatives

Kendall's tau is used to measure the extent to which the ranking of AHP and SE resembles the ranking of the comparing and rating task [6],[21],[22]. The results are displayed in Table 3. The Kendall's tau's of AHP and both validation tasks are 0.45 and 0.47. In the case of SE these values are 0.60 (twice).

Table 3: Rank order correlation coefficient between both methods and validation tasks (paired t-test; n = 180).

Rank order correlation		Difference			
Correlation between	Kendall's Tau	Mean	Standard Deviation	Standard Error	t-value
Rating task and					
AHP	.47				
SE	.60	.1359	.123	.009	14.86*
Comparing task and					
AHP	.45				
SE	.60	.1460	.354	.026	5.53*

(* $p < 0.001$)

These positive values indicate a positive correlation between the rankings of AHP and SE on the one hand, and the comparing and rating task on the other hand. A paired t-test was performed to test whether SE or AHP shows the higher average Kendall's tau's. For both validation tasks, the Kendall's tau for SE is higher than for AHP; the t-values are highly significant ($p < 0.001$). This means that SE is better able to predict the ranking order of the validation alternatives than AHP.

5.3 Preferences for alternatives

The Pearson correlation coefficient is used to measure the extent to which the calculated SE and AHP preferences resemble the assigned preferences of the comparing and rating task [1],[23]. The results are displayed in Table 4. The means of the correlation coefficients for AHP and both validation tasks are almost equal (0.58 and 0.62). In the case of SE, the correlation coefficients are 0.76 and 0.69. A t-test was applied to test the hypothesis of non-zero differences between AHP and SE. For both validation tasks, the correlation coefficients of SE are significantly higher than the coefficients of AHP ($p < 0.001$). The SE task performed better than the AHP in predicting the preferences of the validation alternatives.

Table 4: Pearson correlation coefficient between both methods and validation tasks (paired t-test; $n = 180$).

Correlation between:	Pearson correlation coefficient	Differences			
		Mean	Standard Deviation	Standard Error	t-value
Rating task and					
AHP	.62				
SE	.76	.1432	.133	.010	14.49*
Comparing task and					
AHP	.58				
SE	.69	.1079	.362	.027	4.00*

(* $p < 0.001$)

5.4 Summary of the results

In two of the three analyses, the Self Explicated Approach is better able to predict the results of the validation tasks than the Analytic Hierarchy Process. We conducted some additional analyses in order to determine the cause of these differences. These analyses have shown that the normalised weights of the criteria are similar for AHP and SE. Five pairwise t-tests revealed no significant ($p= 0.05$) differences between the weights of the criteria according to both methods (table 5).

A subsequent analysis of the normalised desirability of the levels of a criterion showed significant differences for 14 of the 17 levels (the remaining 3 levels belonged to 3 of the least important criteria). For all criteria, the range of the desirability of levels (lowest vs. highest) was larger for AHP. The difference in predictive power between AHP and SE is largely explained by the difference in the desirability of the levels of the criteria.

Table 5: Weight of criteria according to AHP and SE (paired t-test; n = 180).

Criterion	Average AHP weight	Average SE weight	Pairwise differences		
			Mean	se of mean	t-test
Rent	26.50	28.09	-1.59	.889	-1.79
Area	23.03	24.04	-1.01	.697	-1.44
Location	20.61	19.37	1.24	.721	1.73
Type	18.01	16.90	1.11	.797	1.40
Facilities	11.85	11.61	0.24	.596	0.40

6 Conclusions

In this study we have compared two compositional methods for modelling preferences, the Analytic Hierarchy Process and the Self Explicated Approach. Pairwise comparisons (AHP) and direct ratings (SE) were used to explicate a decision model. Both models were applied to two validation tasks to compare the predictive power of AHP and SE. In this study the predictive power of the Self Explicated Approach outperforms AHP. The Self Explicated Approach shows a better ability to predict the ranking of the alternatives and the preference scores of the alternatives. Based on these results, we conclude that SE, even with less input data, outperforms AHP in the elicitation of a general decision model.

The extent to which this conclusion can be generalised depends on several assumptions. For example, the decision used in this study had to be complex (involving multiple conflicting criteria) but at the same time the decision maker had to be able to rate alternatives directly in the validation tasks. One may argue that the number of criteria we used (five) is to high, or to low. Furthermore, we applied a variant of the AHP (absolute measurement with absolute intensities). Other variants of the AHP may show different results. This study therefore provides several interesting clues for further research.

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Group Decision Making and Hierarchical Modelling[1]

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Abstract. The paper presents an original procedure for selection the global compromise scenario of the group decision making problem. The procedure can be divided into two steps. The first step is a typical hierarchical modelling based on the AHP. The result of this step is a matrix of the individual priorities of the scenarios for all the decision makers. For the second step the decision makers must specify their concordance and discordance thresholds that make it possible to derive an index of concordance of the i-th decision maker with the j-th scenario, and the global threshold that expresses a necessary majority to accept the given scenario as a candidate for the global compromise scenario. It is a procedure that derives a set of candidates for the global compromise scenario. By interactive changing of the thresholds it is possible to influence the number of the elements of the set of candidates and in this way to find the global compromise scenario.

Keywords. Group decision making, conflict resolution, AHP

1 Introduction

In the process of analyzing general decision problems it is necessary to take into account all the elements of the decision problem (decision makers, criteria, alternatives, etc.) and relations among them. That is why it is a natural way to express the decision problem by means of the hierarchical structuring.

Hierarchical modelling can be used to the analyzing of a very wide scale of the complex decision problems. Often it helps to understand better multicriterial decision problems, group decision making and conflict resolution problems. In this paper we will discuss some possibilities of hierarchical modelling in group decision making and modelling of negotiation process.

The way of the structuration of the hierarchy into particular levels depends on the type of the decision problem. A typical hierarchy for group decision making problems and the modelling of the negotiation process can be expressed as follows (see Fig 1.1):

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- level 1 - main goal of the decision problem - e.g. reaching a consensus among the decision makers (the parties to the conflict),
- level 2 - the parties to the conflict taking part in the negotiation process,
- level 3 - the evaluation criteria (each of the decision makers can have its own set of criteria, obviously) - the set of criteria can be broken down into subcriteria provided that it is useful and can have them placed into the following levels,
- level 4 - the set of analyzed scenarios (alternatives, actions).

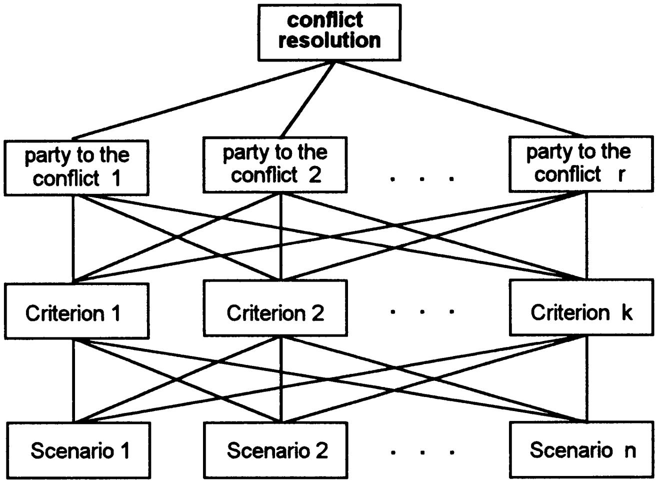

Fig 1.1. An example of a hierarchy of the conflict resolution problem

The relations in the above displayed hierarchy are apparent from Fig. 1.1. The priorities in this hierarchy, expressing the relations among its appropriate elements, can be converted into a numerical form and interpreted as follows:

- level 2 - evaluation of the importance of the parties to the conflict with respect to the given decision problem - weights of the parties to the conflict,
- level 3 - evaluation of the importance of the criteria with respect to the individual parties to the conflict - weights of the criteria,
- level 4 - evaluation of the scenarios - local priorities (with respect to the given criterion and the given decision maker) and global priorities (synthesized from the local preferences) are the direct basis for the final decision (finding the consensus, ordering of the scenarios, etc.).

The priorities of the scenarios lead apparently in the typical case to the different results when the parties to the conflict are taking into account individually. That is the basic aim of the conflict resolution is to find such approach that will make it possible, e.g. based on an interactive procedure, getting near the local priorities of a specified majority of the parties to the conflict and in this way to reach a consensus in a "best" scenario.

2 Group decision making and the AHP

In accordance with the above presented hierarchical formulation it is possible to use for solving of the conflict resolution problem the analytic hierarchy approach (AHP) developed by Saaty [5]. It is the well-known method based on structuring a decision problem into a hierarchy and afterwards in deriving local priorities of the elements in the particular levels of the hierarchy. The final step of the AHP consists in the synthesizing the local priorities into global priorities of the elements on the lowest level of the hierarchy. The basic features of the AHP are briefly mentioned in this part of the paper.

Let us suppose that in some level of a hierarchy the elements $A_1, A_2, \dots, A_k$ are defined. The decision maker must derive the local priorities of these elements with respect to the element of the preceding level. Let us denote the vector of these priorities by $v = (v_1, v_2, \dots , v_k)$. The decision maker will construct a pairwise comparison matrix $A = (a_{ij}, i,j = 1,2, \dots ,k)$ with the quantified information about the relation between the pairs of elements of the given level of the hierarchy. The element a_{ij} of the matrix A can be interpreted as an importance ratio between the elements A_i and A_j. This definition indicates immediately the basic properties of the matrix A:

- diagonal elements $a_{ii} = 1$, $i = 1,2, \dots ,k$,
- the matrix A is reciprocal - $a_{ij} = 1/a_{ji}$, $i,j = 1,2,\dots,k$, $i \neq j$.

The local priorities can be derived by using information contained in matrix A by several methods. The most often used of them are two procedures. The local priorities are developed either by solving the eigenvector problem proposed by Saaty [5] or by the logarithmic least square method, that produces a very good estimation of the right eigenvector, especially for matrices with consistency ratio less than 0.1.

Let us consider the hierarchy displayed in Fig 1.1 and denote the priorities on the particular levels of the hierarchy as follows:

- t_h, $h = 1,2, \dots , r$ - the importance of the parties to the conflict (the priorities of the decision makers) - in case this information is not available, we will suppose all the importance coefficients are set up to be equal,
- v_{hj}, $h = 1,2, \dots , r$, $j = 1,2, \dots , k$ - the weight of the j-th criterion expressed by the h-th party to the conflict,
- w_{hij} - $h = 1,2, \dots , r$, $j = 1,2, \dots , k$, $i = 1,2, \dots , n$ - the priority of the i-th scenario with respect to j-th criterion and the h-th party to the conflict.

Among these priorities the following relations hold apparently:

$$\sum_{h=1}^{r} t_h = 100 , \sum_{j=1}^{k} v_{hj} = t_h , \sum_{i=1}^{n} w_{hij} = v_{hj} , \; h = 1,2,\dots,r , j = 1,2,\dots,k .$$

A successful result of the conflict resolution problem is finding a compromise scenario acceptable for a before specified majority of the parties to the conflict. The analysis of the conflict resolution problem can be divided into several steps:

1. The expression of the individual preferences of all the parties to the conflict, deriving the individual priorities of the scenarios and the individual ordering of the scenarios.
2. The synthesizing of the individual priorities into the global priorities and finding the global ordering of the scenarios.
3. The choice of a compromise scenario.

The expressions of the individual scenarios have to be a basic condition for further participation in the conflict solution for all the parties to the conflict. The deriving the individual priorities is not still always easy for them, because they are not always able to express their preferences in the quantified form. From this point of view the AHP is a very suitable method because it makes possible a verbal expressing of the preferences. These verbal preferences are then automatically converted into the numerical scale. The main result of this process is deriving the individual priorities $u_i{}^h$ that express the weight of the i-th scenario for the h-th party to the conflict:

$$u_i^h = \sum_{j=1}^{k} w_{hij}\, ,\ h = 1,2, \ldots ,r,\ i = 1,2, \ldots ,n\, .$$

By the weighting of the individual priorities the global priorities of the scenarios u_i can be synthesized

$$u_i = \sum_{h=1}^{r} t_h u_i^h\ ,\ i = 1,2,\ldots,n. \qquad (2.1).$$

The global priorities of the scenarios define then the final rank order of the scenarios and can help in the selection of a compromise scenario.

3 Selection of a compromise scenario

The selection of a compromise scenario is the main goal in the process of solving the group decision making or conflict resolution problems. In this process all the parties to the conflict must adopt common rules that express their readiness to accept a compromise solution. In our paper we will suppose that all the parties are able to express their rate of concordance with a candidate for the compromise scenario. Providing that the rate of concordance for a scenario is greater than a before defined threshold the scenario can be taken as an acceptable alternative for the given party to the conflict. The rate of the concordance of the h-th party to the conflict with the i-th scenario will be measured by the *index of concordance* and denoted by p_{ih}, i = 1,2, ... n, h = 1,2, ... ,r. The value of the index p_{ih} depends obviously on the distance of the i-th scenario from the scenario that is individually evaluated as the best scenario for the h-th party to the conflict

- this distance will be denoted by d_{ih}. All the scenarios are individually evaluated in accordance with their individual priorities $u_i{}^h$. That is the distance d_{ih} can be expressed as

$$d_{ih} = u_0^h - u_i^h ,$$

where $u_0{}^h$ is the individual priority of the best scenario for the h-th party to the conflict. The index of concordance p_{ih} will be apparently a not increasing function of the distance d_{ih}. Values of this index can be defined over the interval <0, 1> and interpreted as follows:

$p_{ih} = 0$ - expresses an absolute discordance of the h-th party to the conflict with the i-th scenario,
$p_{ih} \approx 0$ - indicates a very strong discordance of the h-th party to the conflict with the i-th scenario,
$p_{ih} \approx 1$ - indicates a very strong concordance of the h-th party to the conflict with the i-th scenario,
$p_{ih} = 1$ - denotes an absolute concordance of the h-th party to the conflict with the i-th scenario.

For deriving the concordance indices several basic forms of the preference functions $f(d_{ih})$ are available (similar to preference functions used in the PROMETHEE class methods [1]). Fig.3.1 illustrates a shape of one of the most often used preference function.

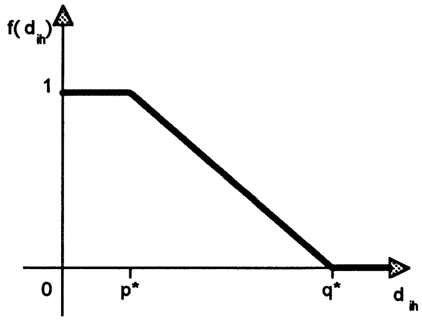

Fig 3.1. - One of the suitable preference functions

In this case the party to the conflict agrees absolutely ($p_{ih} = 1$) with the i-th scenario if the distance d_{ih} is less than the *concordance threshold* p*. On the contrary, the party to the conflict disagrees absolutely ($p_{ih} = 0$) with the i-th scenario if the distance d_{ih} is greater than the *discordance threshold* q*. In other cases the p_{ih} index can be computed as follows:

$$p_{ih} = \frac{d_{ih} - q^*}{p^* - q^*} .$$

The indices p_{ih} can be simply replaced by bivalent values s_{ih}, that can indicate acceptance (s_{ih}=1) or rejection (s_{ih}=0) of the i-th scenario by the h-th party to the conflict. The transformation of the values p_{ih} into s_{ih} is as follows:

$$s_{ih} = 1 \text{ , if } p_{ih} \geq \delta \text{ ,} \qquad (3.1)$$
$$s_{ih} = 0 \text{ , if } p_{ih} < \delta \text{ ,}$$

where δ is the *global concordance threshold* - $\delta \in$ <0.5, 1>. By using the s_{ih} values it is possible to define a subset X_S of the set of feasible scenarios as follows:

$$X_S = \{X_i \in X; \sum_h t_h s_{ih} \leq \psi\} \qquad (3.2)$$

where t_h, h=1,2,...,r are the weights of the parties to the conflict and ψ is a coefficient that specifies a necessary majority for the final acceptance of the compromise scenario. The set X_S is the set of the "candidates" for the compromise scenario. After definition of the set X_S in accordance with (3.2) the following three cases can come into consideration:

1. The set X_S contains just one element - in this case it is a compromise scenario of the decision problem.
2. The set X_S contains more than one element. In this case it is possible to reach a necessary reduction of the number of elements of this set by a proper increasing of the global concordance threshold δ. Other possibility consists in selecting the final compromise scenario by the highest value of the global priorities (2.1).
3. The set X_S does not contain any element. This situation can come relatively often. The first step in this case is a consecutive reducing of the value δ up to value 0.5. Providing that for $\delta > 0.5$ a non-empty set X_S exists it is possible to consider its elements as candidates for a compromise scenario. On the contrary it is necessary to return to the start of this procedure and try to modify individual preferences of the parties to the conflict by means of a negotiation process in order to find a non-empty set X_S. The first and the easiest step in this process that leads often to satisfactory results can consist only in modifying of the concordance and/or discordance thresholds (increasing p* and/or decreasing q*) of the particular parties of the conflict.

Table 4.1. Individual weights of the scenarios

individual weights of scenarios	party to the conflict 1	party to the conflict 2	party to the conflict 3	party to the conflict 4	party to the conflict 5
scenario X_1	2.453	1.859	1.948	1.339	1.416
scenario X_2	3.422	2.961	1.907	5.651	2.705
scenario X_3	3.778	3.055	1.988	3.268	1.540
scenario X_4	2.104	4.098	4.298	4.567	3.212
scenario X_5	6.398	5.072	3.638	2.181	3.062
scenario X_6	0.404	1.729	3.942	2.896	3.575
scenario X_7	1.441	1.227	2.280	0.098	4.491

4 A numerical illustration

Let us consider a decision problem where are 7 scenarios defined, that are evaluated by 5 criteria. Five experts participate in the resolution of this problem - all of them have the same importance, that is their weights are equal to 0.2. A scenario is accepted as a candidate for a compromise scenario provided that it is supported at least by a 60% majority of the parties to the conflict - $\psi = 0.6$. Table 4.1 contains the individual priorities of the scenarios for our hypothetical example u_i^h, $i = 1,2,...,7$, $h = 1,2,...,5$. Table 4.2 includes the concordance indices p_{ih} calculated for concordance and discordance thresholds $p^* = 0.15$ a $q^* = 0.70$.

Let us set up the global concordance threshold $\delta = 0.75$. Then it is apparent from (3.1) and (3.2), that the set of candidates for a compromise scenario X_S has two elements $X_S = \{X_4, X_5\}$. The changing of the value δ to 0.8 leads to the reducing of the set X_S. This set contains then only one scenario which is the scenario X_4. This scenario is the compromise scenario of our simple conflict resolution example.

Table 4.2. Concordance indices of the scenarios

concordance index p_{ih}	party to the conflict 1	party to the conflict 2	party to the conflict 3	party to the conflict 4	party to the conflict 5
scenario X_1	0.076	0.0	0.0	0.0	0.0
scenario X_2	0.370	0.274	0.0	1.0	0.217
scenario X_3	0.478	0.319	0.0	0.493	0.0
scenario X_4	0.0	0.812	1.0	0.918	0.516
scenario X_5	1.0	1.0	0.771	0.137	0.428
scenario X_6	0.0	0.0	1.0	0.371	0.731
scenario X_7	0.0	0.0	0.0	0.0	1.0

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PROPAGATION OF ERRORS IN MULTICRITERIA LOCATION ANALYSIS: A CASE STUDY

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Abstract. The multicriteria location problem involves a set of potential (or feasible) locational alternatives and a set of criteria (attributes) on the basis of which the alternatives are evaluated. The problem requires that a choice be made among alternatives described by their attributes and with respect to the decision maker(s) preferences. Here, it is assumed that the preferences are expressed in terms of weights assigned to the evaluation criteria and the location problem can be represented in the form of a simple additive weighting model.

Multicriteria location methods are typically applied with little consideration given to the errors in input data and propagation of those errors into the problem solution. It can be argued however that the errors may significantly influence the values of alternatives. Some experiments have shown that the actual levels of error, even for simple multicriteria analysis, are far higher than one might expect given the precision with which the multicriteria techniques are used, and in many cases are unacceptable. This paper addresses some of the error issues. It is assumed that errors arise from many sources and collectively they produce a level of uncertainty (error) in the criterion outcomes. In the context of multicriteria location problem the uncertainty can be related to inaccuracy in the spatial database and to imprecision in attribute rating and weighting. The paper focuses on these two types of errors in multicriteria location analysis. Specifically, the Taylor's series error propagation analysis is used to evaluate the effects of the errors in the measurement of the attributes and preferences (weights) on the criterion outcomes.

The error propagation method is applied to a real-world public facility location problem. The problem is to evaluate various fire station locational patterns and to select the best locational plan for the City of Sarnia-Clearwater,Ontario. The alternative locational patterns are evaluated on the basis of three criteria: the total population weighted response time, the population beyond the response time of 5 minutes, and the maximum response time. The values of the criterion function are subject to errors associated with the measurement of attributes (demand for fire services and response time) as well as with the assessment of weights assigned to the three evaluation criteria.

The results of the error propagation analysis suggest that the errors in the attributes and weights may have significant impact on the value of the alternatives and consequently on the choice of the most preferred alternative. The percentage of error in the outcome values ranges from 8.64% to 14.47%. More importantly however, there is a considerable number of alternatives that overlap one another. Two alternatives, *i* and *j*, overlap if $(z_i - s_i) \leq (z_j + s_j)$, where z is the outcome value of an alternative and s is the error associated with that outcome value. As many as 59 out of 117 feasible alternatives overlap with the the "best" alternative. The value of the "best" alternative minus the error is 0.7338 and 59 alternatives have the $z_j + s_j$ value greater than or equal to 0.7338.

Reference Distribution — An Interactive Approach to Multiple Homogeneous and Anonymous Criteria

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Abstract. There are several decision problems with multiple homogeneous and anonymous criteria where the preference model needs to satisfy the principle of anonymity (symmetry with respect to permutations of criteria). The standard reference point method cannot be directly applied to such problems. In this paper we develop, as an analogue of the reference point method, the reference distribution method taking into account both the efficiency principle and the principle of anonymity. All the solutions generated during the interactive process belong to the symmetrically efficient set which is a subset of the standard efficient set. It means, the achievement vector of the generated solution is neither dominated by another achievement vector nor by any permutation of some achievement vector.

Key Words. Symmetric Efficiency, Interactive Methods, Reference Point

1 Introduction

Consider a decision problem defined as an optimization problem with m homogeneous objective functions. For simplification we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows

$$\min \{\mathbf{F}(\mathbf{x}) \ : \quad \mathbf{x} \in Q\} \tag{1}$$

where

$\mathbf{F} = (f_1, \ldots, f_m)$	is a vector–function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$,
$Q \subset X$	denotes the feasible set,
$\mathbf{x} \in X$	denotes the vector of decision variables.

The elements of the criterion space we refer to as achievement vectors. An achievement vector $\mathbf{y} \in Y$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in Q$ ($\mathbf{y} = \mathbf{F}(\mathbf{x})$). The set of all the attainable achievement vectors is denoted by Y_a, i.e. $Y_a = \{\mathbf{y} \in Y \ : \ \mathbf{y} = \mathbf{F}(\mathbf{x}), \ \mathbf{x} \in Q\}$.

Model (1) only specifies that we are interested in minimization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to minimize multiple objective functions. The solution concepts are defined by properties of the corresponding preference model. We assume that solution concepts depend only on evaluation of the achievement vectors do not taking into account other solution properties not represented within achievement vectors. Thus, we can limit our considerations to the preference model in the criterion space Y.

The preference model is completely characterized by the relation of weak preference (c.f., [5]), denoted hereafter with $\preceq$. Namely, we say that achievement vector $\mathbf{y}' \in Y$ is (strictly) preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \prec \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \not\preceq \mathbf{y}'$. Similarly, we say that achievement vector $\mathbf{y}' \in Y$ is indifferent or equally preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \cong \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \preceq \mathbf{y}'$.

The standard preference model related to the Pareto–optimal solution concept assumes that the preference relation $\preceq$ is reflexive

$$\mathbf{y} \preceq \mathbf{y} \tag{2}$$

transitive

$$(\mathbf{y}' \preceq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \preceq \mathbf{y}''') \quad \Rightarrow \quad \mathbf{y}' \preceq \mathbf{y}''' \tag{3}$$

and strictly monotonic

$$\mathbf{y} - \varepsilon \mathbf{e}_i \prec \mathbf{y} \quad \text{for} \quad \varepsilon > 0 \tag{4}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. The last assumption expresses that for each individual objective function less means better (minimization).

We focus on multiple criteria problems with homogeneous and anonymous objective functions. Therefore, we assume also that the preference relation $\preceq$ is anonymous (or impartial), i.e.

$$(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) \cong (y_1, y_2, \ldots, y_m) \tag{5}$$

for any permutation τ of I.

There are several decision problems with multiple homogeneous and anonymous criteria. As an example one may consider location problems. The generic location problem may be stated as follows. There is given a set of m clients (spatial units). There is also given a set of n potential locations for the facilities. It may be in particular a subset (or the entire set) of points representing the clients. Further, the number p of facilities to be located is given ($p \leq n$). The main decisions to be made in the location problem can be described with the binary variables x_j ($j = 1, 2, \ldots, n$) equal to 1 if location j is to be used and equal to 0 otherwise. To meet the problem requirements the decision variables x_j have to satisfy the constraint $\sum_{j=1}^{n} x_j = p$. Further, let us assume that for each client $i = 1, 2, \ldots, m$ there is defined a function

$f_i(\mathbf{x})$ of the location pattern $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The function measures quality of the location pattern with respect to the satisfaction of client i. In typical formulations of location problems this function is usually related to the distances and thereby its less value means higher service quality and client satisfaction. Therefore, each function f_i needs to be minimized. Thus the generic location problem can be viewed as the following multiple criteria minimization problem

$$\min \{\mathbf{F}(\mathbf{x}) \ : \ \sum_{j=1}^{n} x_j = p, \quad x_j \in \{0,1\} \quad \text{for} \quad j = 1,2,\ldots,n\} \tag{6}$$

The individual objective functions f_i are usually conflicting when minimized. Therefore, (6) can be considered a multiple criteria decision problem with homogeneous objective functions. Moreover, while locating public facilities, the distribution of distances among the clients is the crucial issue and the preference model should satisfy the property of anonymity.

2 Symmetric efficiency

It is clear, or rather commonly accepted, that an achievement vector is better than another if all its individual achievements are better or at least one individual achievement is better whereas no other one is worse. In fact, it is the most general assumption about the preference model underlying the multiple criteria optimization. This assumption is equivalent to properties (2)–(4) of the preference model. It is mathematically formalized with the domination relation defined on the criterion space Y.

Definition 1 *We say that achievement vector* $\mathbf{y}' \in Y$ *dominates* $\mathbf{y}'' \in Y$*, or* $\mathbf{y}''$ *is dominated by* $\mathbf{y}'$*, if* $y_i' \le y_i''$ *for all* $i \in I$ *and for at least one index* i_0 *strict inequality holds* ($y_{i_0}' < y_{i_0}''$).

Unfortunately, there usually does not exist an attainable achievement vector that dominates all the others with respect to all the criteria. Thus, in terms of the domination relation, we cannot distinguish the best attainable achievement vector. We can only distinguish the attainable achievement vectors which are not dominated by the others.

Definition 2 *We say that achievement vector* $\mathbf{y} \in Y_a$ *is nondominated, if does not exist* $\mathbf{y}' \in Y_a$ *such that* $\mathbf{y}'$ *dominates* $\mathbf{y}$.

Definition 3 *We say that feasible solution* $\mathbf{x} \in Q$ *is an efficient (Pareto-optimal) solution of the multiple criteria problem (1), if* $\mathbf{y} = \mathbf{F}(\mathbf{x})$ *is a nondominated achievement vector.*

In our problem all the functions are equally important and the preference model satisfies the property of anonymity (5). That means we are interested in comparison rather sets of outcomes than achievement vectors. Therefore, for the problems with homogeneous and equally important objective functions we should introduce an efficiency concept based rather on the set of outcomes than on the achievement vectors. It means, we need to consider the symmetric domination relation which is not affected by any permutation of the achievement vector coefficients.

Definition 4 *We say that achievement vector* $\mathbf{y}' \in Y$ *symmetrically dominates* $\mathbf{y}'' \in Y$*, or* $\mathbf{y}''$ *is symmetrically dominated by* $\mathbf{y}'$*, if there exist permutations* τ' *and* τ'' *such that* $y'_{\tau'(i)} \leq y''_{\tau''(i)}$ *for all* $i \in I$ *and for at least one index* i_0 *strict inequality holds* ($y'_{\tau'(i_0)} < y''_{\tau''(i_0)}$).

Definition 5 *We say that feasible solution* $\mathbf{x} \in Q$ *is a symmetrically efficient solution of the multiple criteria problem (1), if* $\mathbf{y} = \mathbf{F}(\mathbf{x})$ *is symmetrically nondominated.*

The symmetric efficiency is stronger than the standard efficiency and the symmetrically efficient set is a subset of the standard efficient set. The relation of symmetric domination can be expressed as domination of the achievement vectors with coefficients ordered in the weakly decreasing order. This can be mathematically formalized with the ordering map $\Theta : R^m \to R^m$ such that $\Theta(y_1, y_2, \ldots, y_m) = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$, where $\bar{y}_1 \geq \bar{y}_2 \geq \ldots \geq \bar{y}_m$ and $\bar{y}_i = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$ for some permutation τ of I.

Definition 6 *We say that achievement vector* $\mathbf{y}' \in Y$ *dominates* $\mathbf{y}'' \in Y$ *in the ordered sense, or* $\mathbf{y}''$ *is dominated by* $\mathbf{y}'$ *in the ordered sense, if* $\bar{\mathbf{y}}' = \Theta(\mathbf{y}')$ *dominates* $\bar{\mathbf{y}}'' = \Theta(\mathbf{y}'')$*, i.e.* $\bar{y}'_i \leq \bar{y}''_i$ *for all* $i \in I$ *and for at least one index* i_0 *strict inequality holds* ($\bar{y}'_{i_0} < \bar{y}''_{i_0}$).

Proposition 1 *Achievement vector* $\mathbf{y}' \in Y$ *symmetrically dominates* $\mathbf{y}'' \in Y$ *if and only if* $\mathbf{y}'$ *dominates* $\mathbf{y}''$ *in the ordered sense.*

Proof. If $\mathbf{y}'$ dominates $\mathbf{y}''$ in the ordered sense then, obviously, it does also symmetrically dominate. Thus we only need to prove that the symmetric dominance implies the dominance in the ordered sense.

Suppose that achievement vector $\mathbf{y}' \in Y$ symmetrically dominates $\mathbf{y}'' \in Y$. It means, $\mathbf{y}' \neq \mathbf{y}''$ and there exist permutations τ' and τ'' such that the dominance inequalities are valid

$$y'_{\tau'(i)} \leq y''_{\tau''(i)} \quad \text{for} \quad i = 1, 2, \ldots, m$$

We will show that the dominance inequalities remain satisfied if one replaces permutations τ' and τ'' with permutations $\bar{\tau}'$ and $\bar{\tau}''$ sorting the corresponding achievement vectors in the weakly decreasing order. Note that any vector can be sorted by a finite number of comparisons and swappings (if necessary) made on pairs of its coefficients. Suppose that for some pair of indices $i < j$ the corresponding coefficients of one of the achievement vectors, let say $\mathbf{y}'$, violate the weakly decreasing order, i.e., $y'_{\tau'(i)} < y'_{\tau'(j)}$. If, simultaneously, $y''_{\tau''(i)} < y''_{\tau''(j)}$, then one can simply swap the corresponding coefficients of both the achievements vectors preserving the dominance inequalities. If $y''_{\tau''(i)} \geq y''_{\tau''(j)}$, then $y'_{\tau'(i)} < y'_{\tau'(j)} \leq y''_{\tau''(j)} \leq y''_{\tau''(i)}$ and therefore, one can swap the coefficients of vector $\mathbf{y}'$ preserving the dominance inequalities and the weakly decreasing order of the corresponding coefficients in the second achievement vector.

By applying this approach to the both achievement vectors, one finally gets (in a finite number of swappings) the ordered achievement vectors satisfying the dominance inequalities. □

Corollary 1 *Feasible solution $\mathbf{x} \in Q$ is a symmetrically efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the ordered multiple criteria problem*

$$\min \{\Theta(\mathbf{F}(\mathbf{x})) \ : \quad \mathbf{x} \in Q\} \tag{7}$$

There exist usually many symmetrically nondominated achievement vectors and they are incomparable each other on the basis of the specified set of objective functions. Therefore, there exist usually many symmetrically efficient solutions and they are different not only in the decision space but also in the criterion space. So, there arises a need for further analysis, or rather decision support, to help the decision maker (DM) in selection of one solution for implementation. Of course, the original objective functions do not allow one to select any symmetrically efficient solution as better than any other one. Therefore, this analysis depends usually on additional information about the DM's preferences. The DM, working interactively with a decision support system (DSS), specifies his/her preferences in terms of some control parameters and the DSS provides the DM with a symmetrically efficient solution which is the best according to the specified control parameters. For such an analysis, there is no need to identify the entire symmetrically efficient set prior to the analysis, as contemporary optimization software is powerful enough to be used on–line for direct computations at each interactive step. Thus the DSS can generate at each interactive step only one symmetrically efficient solution that meets the current preferences. Such a DSS can be used for analysis of decision problems with finite as well as infinite efficient sets. There is important, however, that the control parameters provide the completeness of the control (c.f., [7]), i.e., that varying the control parameters the DM can identify every symmetrically nondominated achievement vector.

For an interactive DSS dealing with multiple and homogeneous criteria we need parametric solution concepts generating symmetrically efficient solutions. In the case of the standard efficiency one may consider weighting of objective functions. In the case of anonymous criteria we cannot assign various weights to individual objective functions. Due to Corollary 1, the weights should be assigned rather to the specific coefficients of the ordered achievement vectors. Such an ordered weighting approach was proposed by Yager [8] in the so-called Ordered Weighted Averaging (OWA) aggregation. Applying the OWA aggregation operator to the multiple criteria problem (1) we get the following single objective problem

$$\min \{\sum_{i=1}^{m} w_i \bar{y}_i \ : \quad \bar{\mathbf{y}} = \Theta(\mathbf{F}(\mathbf{x})), \quad \mathbf{x} \in Q\} \tag{8}$$

Due to Corollary 1, the following proposition is valid.

Proposition 2 *For any positive weights w_i, any optimal solution to problem (8) is a symmetrically efficient solution of the multiple criteria problem (1).*

Unfortunately, the ordered weighting does not provide us with a complete parameterization of the entire symmetrically efficient set. It is due to the specificity of the linear weighting approach to multiple criteria. In the case when the multiple criteria problem is a discrete one (like the location problem (6)), there exist symmetrically efficient solutions that cannot be generated as optimal solutions to problem (8) with any set of positive weights. We illustrate this with a small example.

Example 1 Let us consider a simple single facility location problem with two clients (C1 and C2) and three potential locations (P1, P2 and P3). The distances between several clients and potential locations are given as follows: $d_{11} = 15$, $d_{12} = 14$, $d_{13} = 12$, $d_{21} = 10$, $d_{22} = 11$, $d_{23} = 12$.

Note that all three feasible solutions are efficient in the standard and symmetric sense. One can easily verify that while dealing with ordered weighting approach, location P2 cannot be selected for any set of positive weights. If $3w_1 < 2w_2$, then location P1 is a unique optimal solution to the problem (8). If $3w_1 > 2w_2$, then location P3 is a unique optimal solution to the problem (8). Finally, if $3w_1 = 2w_2$, then both locations P1 and P3 are optimal. Location P2 is never an optimal solution to the corresponding problem (8). □

In the case of discrete (or more general nonconvex) feasible sets, the entire efficient set can be parameterized with augmented weighted Tchebychev distance function (c.f., [4]). It is used as the basis of the reference point method [6]. Due to Corollary 1, we can apply the reference point method to the ordered problem (7) to parameterize the entire symmetrically efficient set of the original multiple criteria problem (1). In the next section we describe this approach in details.

3 Reference distribution approach

The reference point method [6] is an interactive technique for an open search for a satisficing efficient solution. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of aspiration levels for individual objective functions. Depending on the specified aspiration levels a special scalarizing achievement function is built which when minimized generates an efficient solution to the problem. The computed efficient solution is presented to the DM as the current solution allowing comparison with previous solutions and modifications of the aspiration levels if necessary.

The scalarizing achievement function not only guarantees efficiency of the solution but also reflects the DM's expectation as specified via the aspiration levels. In building the function the following assumption regarding the DM's expectations is made: the DM prefers outcomes that satisfy all the aspiration levels to any outcome that does not reach one or more of the aspiration levels. One of the simplest scalarizing functions takes the following form (c.f., [4]):

$$s(\mathbf{y}) = \max_{i=1,\dots,m} \{\lambda_i(y_i - a_i)\} + \varepsilon \sum_{i=1}^{m} \lambda_i(y_i - a_i) \tag{9}$$

where

- $\mathbf{a}$ denotes the vector of aspiration levels,
- λ is a scaling vector, $\lambda_i > 0$,
- ε is an arbitrarily small positive number.

Minimization of the scalarizing achievement function (9) over the feasible set generates an efficient solution. The selection of the solution within the efficient set depends on two vector parameters: an aspiration vector $\mathbf{a}$ and a scaling vector λ. In practical implementations the former is usually designated as a control tool for direct use by the DM during the interactive analysis. The latter is automatically calculated on the basis of some predecision analysis or adjusted during the interactive process depending on values of the reservation levels used as additional control parameters. The small scalar ε is introduced only to guarantee efficiency in the case of a nonunique optimal solution. It can be replaced by two level lexicographic minimization of the corresponding terms [3]. The reference point approach was successfully implemented in many DSS (c.f., [1]) with real–life applications including multiple criteria location decision problems (see, for example, [2]).

In order to parameterize the entire symmetrically efficient set, one may use the scalarizing achievement function

$$\bar{s}(\mathbf{y}) = \max_{i=1,\dots,m} \{\lambda_i(\bar{y}_i - \bar{a}_i)\} + \varepsilon \sum_{i=1}^{m} \lambda_i(\bar{y}_i - \bar{a}_i), \quad \bar{\mathbf{y}} = \Theta(\mathbf{y}),\ \bar{\mathbf{a}} = \Theta(\mathbf{a}) \tag{10}$$

where ε is an arbitrarily small positive parameter. Applying function (10) to the multiple criteria problem (1) we get the following parameterized single objective problem generating symmetrically efficient solutions

$$\min \{\bar{s}(\mathbf{y}) \ : \quad \mathbf{y} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in Q\} \tag{11}$$

Parametric problem (11) provides us with a complete parameterization for the symmetrically efficient set of the multiple criteria problem (1). That means, any optimal solution to problem (11) is a symmetrically efficient solution of (1) and any symmetrically efficient solution of the multiple criteria problem (1) can be generated as an optimal solution to problem (11) for some aspiration vector $\mathbf{a}$.

Ordering operator Θ used in the definition of scalarizing achievement function (10), in general, makes the scalarized problem (11) very difficult to implement. Note that even unweighted scalarizing achievement function (10) with all $\lambda_i = 1$ provides us with a complete parameterization of the entire symmetrically efficient set. If we decide to use such unweighted scalarizing achievement function we can form the corresponding scalarized problem (11) without the ordering operator in the following form

$$
\begin{aligned}
\text{minimize} \quad & \max_{i=1,\ldots,m} z_i + \varepsilon \sum_{i=1}^{m} z_i \\
\text{subject to} \quad & \mathbf{x} \in Q \\
& z_i = f_i(\mathbf{x}) - \sum_{l=1}^{m} a_l u_{il}, \quad \sum_{l=1}^{m} u_{il} = 1 \quad \text{for} \quad i = 1,2,\ldots,m \\
& \sum_{i=1}^{m} u_{il} = 1 \quad \text{for} \quad l = 1,2,\ldots,m \\
& u_{il} \in \{0,1\} \quad \text{for} \quad i = 1,2,\ldots,m; l = 1,2,\ldots,m
\end{aligned}
$$

Note that aspiration vector $\mathbf{a}$ is used in scalarizing achievement function (10) only in its ordered form $\bar{\mathbf{a}}$. Thus it is rather an aspiration set of outcomes than a vector. For problems with large number of objectives, like large location problem (6), we can consider it as an aspiration distribution of outcomes. In fact, for discrete problems with multiple homogeneous criteria we can directly deal with the distribution of outcomes. Let $V = \{v_1, v_2, \ldots, v_r\}$ ($v_1 > \cdots > v_r$) denote the set of all possible values of objective functions f_i for $\mathbf{x} \in Q$. We can introduce then integer functions $h_k(\mathbf{x})$ ($k = 1,2,\ldots,r$) expressing the number of values v_k taken in the achievement vector $\mathbf{F}(\mathbf{x})$. Analytically, functions h_k can be introduced into the model by auxiliary assignment (binary) variables u_{ik} with the following formulas

$$h_k(\mathbf{x}) = \sum_{i=1}^{m} u_{ik} \qquad \text{for} \quad k = 1,2,\ldots,r \tag{12}$$

$$f_i(\mathbf{x}) = \sum_{k=1}^{r} v_k u_{ik}, \quad \sum_{k=1}^{r} u_{ik} = 1 \qquad \text{for} \quad i = 1,2,\ldots,m \tag{13}$$

$$u_{ik} \in \{0,1\} \quad \text{for} \quad i = 1,2,\ldots,m; k = 1,2,\ldots,r \tag{14}$$

Note that in many discrete problems functions h_k can be introduced directly to the model without auxiliary variables u_{ik}. It is possible, in particular, for the location problem with explicit allocation variables.

Having defined functions h_k we can introduce cumulative distribution functions

$$\bar{h}_k(\mathbf{x}) = \sum_{l=1}^{k} h_l(\mathbf{x}) \quad \text{for} \quad k = 1, 2, \ldots, r \tag{15}$$

and consider the corresponding multiple criteria problem

$$\min \{(\bar{h}_1(\mathbf{x}), \bar{h}_2(\mathbf{x}), \ldots, \bar{h}_r(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \tag{16}$$

Proposition 3 *Feasible solution* $\mathbf{x} \in Q$ *is a symmetrically efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the multiple criteria problem (16).*

Proof. Let $\mathbf{x} \in Q$ be a symmetrically efficient solution of problem (1). Suppose that $\mathbf{x}$ is not efficient solution of the distribution problem (16). It means, there exists $\mathbf{x}^0 \in Q$ such that $\bar{h}_k(\mathbf{x}^0) \leq \bar{h}_k(\mathbf{x})$ for $k = 1, 2, \ldots, r$ where for at least one index k_0 strict inequality holds ($\bar{h}_{k_0}(\mathbf{x}^0) < \bar{h}_{k_0}(\mathbf{x})$). Then, obviously, $\Theta(\mathbf{F}(\mathbf{x}^0))$ dominates $\Theta(\mathbf{F}(\mathbf{x}))$ which contradicts symmetric efficiency of $\mathbf{x}$ for problem (1). Thus, symmetric efficiency of vector $\mathbf{x} \in Q$ for problem (1) implies its efficiency for problem (16).

Now, let $\mathbf{x} \in Q$ be an efficient solution of problem (16). Suppose that $\mathbf{x}$ is not symmetrically efficient solution of problem (1). It means, there exists $\mathbf{x}^0 \in Q$ such that $\bar{\mathbf{y}}^0 = \Theta(\mathbf{F}(\mathbf{x}^0))$ dominates $\bar{\mathbf{y}} = \Theta(\mathbf{F}(\mathbf{x}))$. Note that $\bar{h}_k(\mathbf{x}^0) = \bar{h}_k(\mathbf{x}) = 0$ if $v_k > \bar{y}_1^0$ and $v_k > \bar{y}_1$ as well as $\bar{h}_k(\mathbf{x}^0) = \bar{h}_k(\mathbf{x}) = m$ if $v_k < \bar{y}_m^0$ and $v_k < \bar{y}_m$. Moreover, for any $i \in I$ $\bar{y}_i^0 = v_{k'} \leq \bar{y}_i = v_{k''}$ implies $\bar{h}_k(\mathbf{x}^0) \leq \bar{h}_k(\mathbf{x})$ for $k' \leq k \leq k''$. So, achievement vector $\bar{H}(\mathbf{x}^0)$ dominates (in the standard sense) achievement vector $\bar{H}(\mathbf{x})$ which contradicts efficiency of $\mathbf{x}$ for problem (16). Thus, efficiency of vector $\mathbf{x} \in Q$ for problem (16) implies its symmetric efficiency for problem (1). □

Due to Proposition 3 we can apply the standard reference point method to the distribution multiple criteria problem (16) for an interactive analysis of the symmetrically efficient set of problem (1). The corresponding scalarizing achievement function takes then the following form

$$s(\mathbf{x}) = \max_{k=1,\ldots,r} \{\lambda_k(\bar{h}_k(\mathbf{x}) - \bar{q}_k)\} + \varepsilon \sum_{k=1}^{r} \lambda_k(\bar{h}_k(\mathbf{x}) - \bar{q}_k) \tag{17}$$

where

- $\bar{\mathbf{q}}$ denotes the vector of aspiration levels for the cumulative distribution of outcomes,
- λ is a scaling vector, $\lambda_k > 0$,
- ε is an arbitrarily small positive number.

Aspiration distribution vector $\bar{\mathbf{q}}$ is the main control tool for direct use by the DM during an interactive analysis. Scaling factors λ_k can be used as auxiliary control parameters and modified by the DM during the interactive process. Note that, in the case of large r, the DM does not need to deal with all the aspiration coefficients $\bar{q}_k$. As $\bar{\mathbf{q}}$ represents the reference cumulative distribution, it can be specified with only a few coefficients $\bar{q}_k$ and automatic interpolation of values for the remaining coefficients.

4 Concluding remarks

The reference point method is a very convenient technique for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing aspiration levels for the individual objective functions.

There are several decision problems with multiple homogeneous and anonymous criteria where the preference model needs to satisfy the principle of anonymity (symmetry with respect to permutations of criteria). The standard reference point method cannot be directly applied to such problems. In this paper we have developed, as an analogue of the reference point method, the reference distribution method taking into account both the efficiency principle and the principle of anonymity. All the solutions generated during the interactive process belong to the symmetrically efficient set. The interactive analysis of the symmetrically efficient set is controlled with the aspiration cumulative distribution of outcomes.

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Rank-Ordering of Alternatives in Multiattribute Decision with Incomplete Information

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Abstract. Incomplete information is a set of linear inequalities, which is the decision maker's information on both utilities and attribute weights. The decision maker under time pressure and lack of knowledge is only willing or able to provide incomplete information. We present a method for establishing pairwise dominance of alternatives under consideration, and propose an algorithm of constructing a dominance graph using the data of pairwise dominance relations. Proposed algorithm uses a graph theoretical technique, which is based on the transitivity of preferences. Dominance graph can be used to aid in selecting an optimal alternative or satisfying alternatives.

Key Words. Decision analysis, Multiple attributes, Incomplete information, Dominance

1. Introduction

As a prescriptive or normative decision technique, multiattribute utility theory [3] has widely been used as a well-known tool in the field of decision analysis. For representing a multiattribute decisionmaking (MADM) problem, we use the following notations:

- $A = \{A_i\}_{i=1,M}$: a set of M possible alternatives or courses of action.
- $J = \{j\}_{j=1,N}$: a set of indices of N additively independent attributes.
- $w = \{w_j\}_{j=1,N}$: the set of weights on the attributes.
- $u(A_i) = \{u_j(A_i)\}_{j=1,N}$: the set of utilities on attributes given alternative A_i.

The decision maker (DM)'s objective is to choose the $m(<M)$ most preferred alternatives or the most preferred alternative A^* from the set A, $A^* \in A$. In a MADM model, the expected utility $\varphi(A_i)$ of alternative A_i is given by

$$\sum_{j \in J} w_j u_j(A_i) \quad \text{or equivanently} \quad u(A_i)w^{\mathrm{T}}, \tag{1}$$

where the weights w_j satisfy non-negativity and sum-to-unity. An alternative A_h is dominated if and only if $\varphi(A_g) \geq \varphi(A_h) \; \exists \; g \neq h, A_g$ and $A_h \in A$, or

$$\sum_{j \in J} w_j d_j(A_g, A_h) \geq 0$$
$$\text{with} \quad d_j(A_g, A_h) = u_j(A_g) - u_j(A_h). \tag{2}$$

Since the likely candidates for the most preferred alternative are the non-dominated alternatives, the dominated alternatives can be removed from serious consideration. If the DM desires to choose a single alternative, he or she will select an alternative A^* such that $\varphi(A^*) \geq \varphi(A_i) \; \forall \; i$.

When the parameter value information, i.e., w and $u(A_i)$ $\forall i$, is precisely or numerically assessed by the DM, then A^* or the m most preferred alternatives can be easily determined. However, it is not simple matter actually to measure the DM's precise value information. Due to time pressure, lack of knowledge or data, and so on, the DM is only willing or able to provide incomplete or partial information. It can take the form of linear inequalities such as rankings, interval descriptions, etc.

There have been a number of studies for MADM with incomplete information [1, 2, 4-10, 12]. An excellent review can be found in [11]. These studies were concerned with establishing pairwise dominance relations for obtaining non-dominated alternatives or the most preferred alternative.

Once the pairwise dominance relations are established, a set of more than two alternatives may be rank-ordered based on the transitivity of preferences. We propose a new method for rank-ordering of alternatives, which is a dominance graph. The dominance graph can be used to select an optimal alternative or satisfying alternatives.

2. Incomplete Information

For MADM based on simple weighted-additive rule, the parameter value information from the DM can be taken in form of linear inequalities. For example, information on attribute weights is constructed by one of the following forms:

1. a weak ranking: $w_1 \geq \cdots \geq w_N$,
2. a strict ranking: $w_j - w_{j+1} \geq \alpha_j, \quad j = 1, \cdots, N-1$,
3. a ranking with multiples: $w_j \geq \alpha_j w_{j+1}, \quad j = 1, \cdots, N-1$,
4. an interval form: $\alpha_j \leq w_j \leq \alpha_j + \varepsilon_j, \quad j = 1, \cdots, N$,

5. a ranking of differences of adjacent weights based on Form 1 (F1):
$w_j - w_{j+1} \geq w_k - w_{k+1} \geq \cdots \geq w_l - w_{l+1},\ j,k,l = 1,\cdots,N$, with $w_{N+1} = 0$,
6. a mixed form using some of Forms 1 through 5,

where α_j and ε_j are non-negative constants. Information on utilities can be similarly dealt with the attribute weights, such as $u_j(A_1) \geq \cdots \geq u_j(A_M)$, $\alpha_j \leq u_j(A_i) \leq \alpha_j + \varepsilon_j$, etc. Particularly, Form 5 is a weak ranking of differences of adjacent parameters obtained by rankings among the parameters, which can be subsequently constructed based on Form 1.

Such linear partial information is denoted by incomplete information in this paper. The reasons that the DM is not able to provide precise information are 1) a decision should be made under time pressure and lack of knowledge or data, 2) many of attributes are intangible or non-monetary because they reflect social and environmental impacts, and 3) the DM has limited attention and information processing capabilities, especially on the judgment of numerical values under complex and uncertain environment.

With the incompletely identified information, the next section contains methods of establishing pairwise dominance relations.

3. Establishing Dominance

Some additional notations are used as follows.

- $W = \{\Phi_w, \Sigma w_j{=}1, w_j{\geq}0\}$: the set of constraints or all possible values on the attribute weights, $w \in W$, where Φ_w is a set derived from the DM's incomplete information regarding the relative importance of attributes.
- U : the set of constraints on the utilities obtained by the DM's information, for all attributes and A_i, $\{u_i(A_g), u_i(A_h)\} \in U$, where A_g and A_i $\in A$.
- Ω: the set of collecting dominance relations between the alternatives, $\Omega \subseteq A \times A$; for example, $(A_g, A_h) \in \Omega$ means that A_g is at least as preferred as A_h.

Let us assume throughout this paper that 1) the sets Φ_w and U are described by linear inequalities (or equalities), respectively, and these sets are not null sets respectively, 2) W is alternative invariant, and 3) Ω is transitive. This section aims at constructing the set Ω. For the purpose, the pairwise comparison with incomplete information can be generally formulated as

$$\begin{aligned} &\min z_{gh} = \varphi(A_g) - \varphi(A_h) \\ &\text{subject to} \quad W,\ U. \end{aligned} \tag{3}$$

The objective function of (3) can be replaced by max $z_{hg}=\varphi(A_h)-\varphi(A_g)$. Then, $(A_g, A_h)\in\Omega$ if and only if min $z_{gh}\geq 0$ or max $z_{hg}\leq 0$. If A^* is such that $(A^*, A_i)\in\Omega$ and $(A_i, A^*)\notin\Omega\ \forall A_i\neq A^*$, then A^* is the most preferred alternative. Additionally, if $(A_g, A_h)\in\Omega$ and $(A_h, A_g)\in\Omega$, i.e., min $z_{gh}=$ min z_{hg}, then A_g and A_h are mutually indifferent.

If the value of either attribute weights or utilities is known precisely, then the model (3) becomes a linear program (LP), thus the problem (3) can be easily solved. Assume both attribute weights and utilities are known incompletely, i.e., $w\in W$ and $\{u_j(A_g), u_j(A_h)\}\in U$. Then, (3) becomes a non-convex non-LP as

$$\min_{W,U} z_{gh} = \sum_{j\in J} w_j\left[u_j(A_g) - u_j(A_h)\right], \tag{4}$$

because the objective function of (4) is nether positive semi-definite nor negative semi-definite. Consequently, since the problem (4) is not easily solved, an appropriate solution method is needed. We present solution methods for this problem below.

Suppose that the incomplete information of utility values for each attribute are functionally independent, which is formally denoted by a notation, $U_j \perp U_k\ \forall j\neq k$, $j,k\in J$ and $U_j, U_k\subseteq U$. Then the problem (4) is separable for $j\in J$, thus yielding the following LPs:

$$\min_{W} z_{gh} = \sum_{j\in J} w_j\lambda_j(A_g, A_h) \tag{5.a}$$

$$\text{with}\quad \lambda_j(A_g, A_h) = \min_{U_j}\{u_j(A_g) - u_j(A_h)\}, j = 1,\cdots,N. \tag{5.b}$$

Then, $(A_g, A_h)\in\Omega$ if and only if min $z_{gh}\geq 0$. For constructing Ω, $M(M$-1)(1+N) LPs have to be solved, i.e., $M(M$-1)N LPs for calculating $\lambda_j(A_g, A_h)$ and $M(M$-1) LPs for z_{gh}.

If the information of utilities for each attribute are not functionally independent, $U_j \neg\perp U_k\ \exists j\neq k$ where $\neg$ is "not" operation, then (4) can be solved by the following LPs:

$$\min_{W} z_{gh} = \sum_{j\in J} w_j[\underline{u}_j(A_g) - \overline{u}_j(A_h)] \tag{6.a}$$

$$\text{with}\quad \underline{u}_j(A_g) = \min_{U} u_j(A_g),\ \text{and}$$

$$\overline{u}_j(A_h) = \max_{U} u_j(A_h),\quad j = 1,\cdots,N. \tag{6.b}$$

Then, $(A_g, A_h)\in\Omega$ if min $z_{gh}\geq 0$. Note here that min $z_{gh}\geq 0$ is not necessary but only sufficient condition for $(A_g, A_h)\in\Omega$. $2MN$ LPs have to be solved for (6.b) and $M(M$-1) LPs for (6.a).

4. Dominance Graph

After a dominance graph is defined, we examine an algorithm of constructing the dominance graph. In what follows, an example and a comparison with an earlier method are presented.

4.1. The Definition

A dominance graph is a directed graph $G(A, E)$ consisting of a set A of nodes, i.e., alternatives, and a set E of directed edges or arcs, $E \subseteq A \times A$. The set E is the set Ω, and G can be described by corresponding adjacent matrix $M_a = \|m_{ij}\|_{M \times M}$: $m_{ij}=1$ if $(A_i, A_j) \in \Omega$ or $A_i \to A_j$, and $m_{ij}=0$ otherwise.

Now, we define a hierarchical dominance graph (HDG) as $G_H(H(A), E)$ with $H(A) = [H_1, ..., H_k, ..., H_L]$, where $H_k \subseteq A$ is a set of alternatives in the kth level, L $(1 \leq L \leq M)$ the number of levels of G_H, and $H_k \neq \emptyset$ $\forall k$. The definition of the level is specifically given by: the element(s) of H_1 is (are) the most preferable alternative(s), the element(s) of H_2 is (are) less preferable alternative(s), ..., the element(s) of H_L is (are) the least preferable alternative(s). Additionally, a condition is satisfied as $H_k \cap H_i = \emptyset$ $\forall k \neq j$.

If $L=M$ (the number of alternatives), then the number of elements of H_k $\forall k$ is 1, thus the ranking of all alternatives is fully or completely established. At this time, the most preferred alternative A^* can be easily determined as the element of H_1. On the other hand, if $L=1$, the dominance among all alternatives is not established, thus choice-making of A^* is not easy. In Fig. 1 examples of HDGs are shown. Before a method of constructing G_H is presented, we briefly describe a need of G_H.

The HDG G_H is a useful tool for aiding the selection of one or more preferred alternatives. It plays a role in compactly displaying the preference relations of all alternatives under consideration in a given decision problem. When the cardinality of A is large and then the number of elements of Ω is also large, an algorithm of automatically constructing G_H is required in building a decision aiding system.

4.2. The Algorithm

Suppose Ω is already identified, and consider the following terms for conveniently describing the algorithm:

- $R= \|r_{ij}\|_{M \times M}$ is the reachability matrix of the adjacent matrix M_a ; $R = I + \sum_{i=1}^{M} M_a^{\,i}$, where I is a $M \times M$ identity matrix, and the computational operations are Boolean.
- The set of direct successors, $S(A_i)$ of alternative A_i in the R, is defined as $S(A_i) = \{A_j \in A \mid (A_i, A_j) \in E \text{ or } r_{ij}=1\}$.
- The set of direct predecessors, $P(A_i)$ of alternative A_i in the R, is defined as $P(A_i) = \{A_j \in A \mid (A_j, A_i) \in E \text{ or } r_{ji}=1\}$.

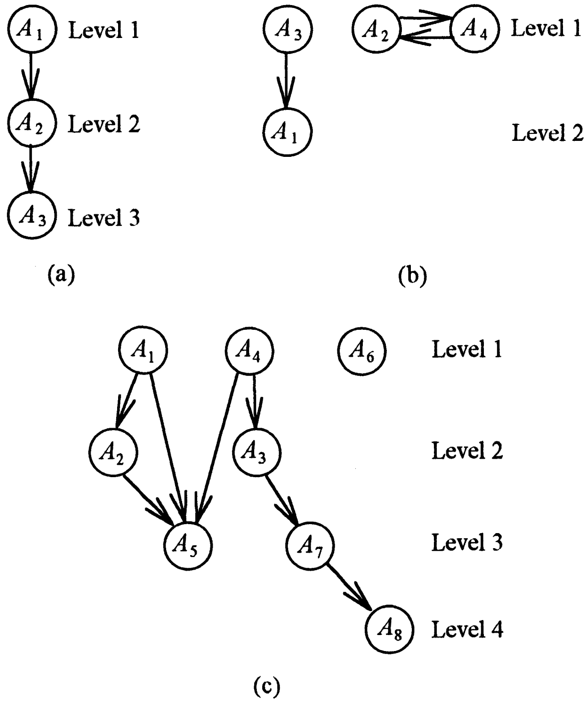

Fig. 1. Examples of hierarchical dominance graphs: (a) fully rank-ordered case, (b) an indifferent case, and (c) a complicated case.

Then, the algorithm has four steps as follows:

1. Construct the adjacent matrix M_a by using the information of Ω.
2. Compute the reachability matrix R of M_a.
3. Perform the following iterative procedure with $H_0 = \varnothing$ and k=1:
  a. Construct $B_k = A - \bigcup_{i=0}^{k-1} H_i$.
  If $B_k = \varnothing$, then set $L = k$-1 and go to Step 4.
  b. Find $H_k = \{A_i \in B_k | P_k(A_i) = P_k(A_i) \cap S_k(A_i)\}$, where $P_k(A_i)$ and $S_k(A_i)$ respectively are the sets of predecessors and successors denoted by the subgraph consisting of the elements in B_k.
  c. Set $k = k$ +1 and go to Step 3.a.
4. Display $G_H(H(A), E)$ with $H(A) = [H_1, ..., H_L]$, and stop.

The core of the above algorithm is in Step 3. The iterative procedure is stopped within M iterations because $k \leq M$. Especially the use of Step 3.b is proved by: The necessary and sufficient condition so that $A_i \in B_k$ is an element placed in the kth level, is $P_k(A_i) \subseteq S_k(A_i)$. Thus, the condition, $P_k(A_i) = P_k(A_i) \cap S_k(A_i)$, must be satisfied in order that A_i is an element of H_k.

Table 1. Summary of the results in Step 3.

$A_i \in B_k$	$S_k(A_i)$	$P_k(A_i)$	$P_k(A_i) \cap H_k$	$S_k(A_i)$
k=1				
A_1	$\{A_1, A_2, A_5\}$	$\{A_1\}$	$\{A_1\}$	
A_2	$\{A_2, A_5\}$	$\{A_1, A_2\}$	$\{A_2\}$	
A_3	$\{A_3, A_7, A_8\}$	$\{A_3, A_4\}$	$\{A_3\}$	
A_4	$\{A_3, A_4, A_5, A_7, A_8\}$	$\{A_4\}$	$\{A_4\}$	$\{A_1, A_4, A_6\}$
A_5	$\{A_5\}$	$\{A_1, A_2, A_4, A_5\}$	$\{A_5\}$	
A_6	$\{A_6\}$	$\{A_6\}$	$\{A_6\}$	
A_7	$\{A_7, A_8\}$	$\{A_3, A_4, A_7\}$	$\{A_7\}$	
A_8	$\{A_8\}$	$\{A_3, A_4, A_7, A_8\}$	$\{A_8\}$	
k=2				
A_2	$\{A_2, A_5\}$	$\{A_2\}$	$\{A_2\}$	
A_3	$\{A_3, A_7, A_8\}$	$\{A_3\}$	$\{A_3\}$	
A_5	$\{A_5\}$	$\{A_2, A_5\}$	$\{A_5\}$	$\{A_2, A_3\}$
A_7	$\{A_7, A_8\}$	$\{A_3, A_7\}$	$\{A_7\}$	
A_8	$\{A_8\}$	$\{A_3, A_7, A_8\}$	$\{A_8\}$	
k=3				
A_5	$\{A_5\}$	$\{A_5\}$	$\{A_5\}$	
A_7	$\{A_7, A_8\}$	$\{A_7\}$	$\{A_7\}$	$\{A_5, A_7\}$
A_8	$\{A_8\}$	$\{A_7, A_8\}$	$\{A_8\}$	
k=4				
A_8	$\{A_8\}$	$\{A_8\}$	$\{A_8\}$	$\{A_8\}$
k=5	L=4 and stop.			

4.3. An Example

We show an illustrative example for the above algorithm. Suppose that M=8 and $\Omega = \{(A_7, A_8), (A_2, A_5), (A_1, A_5), (A_4, A_3), (A_3, A_7), (A_4, A_5), (A_1, A_2)\}$. After constructing M_a and computing R of M_a, performing the iterative algorithm of Step 3 yields Table 1. Thus, the G_H has four levels as shown in Fig. 1(c).

5. Conclusions

We have presented methods for establishing pairwise dominance with the decision maker's information about weights and utilities that are incomplete. Note that the term "dominance" used throughout this paper is "strict dominance". With incompletely identified information, however, a final choice-making through strict dominance relations may be not made generally. When the decision maker is not willing or able to provide an additional information on parameter values, "weak dominance" concept is useful for a final choice-making (for more detailed description on strict and weak dominance, refer to Park and Kim [8]).

The hierarchical dominance graph is a useful tool for aiding the selection of preferable alternatives, because it compactly displays the preference levels of all alternatives. This paper has also proposed an algorithm for constructing the hierarchical dominance graph based upon the information of pairwise rankings. The proposed algorithm can be used for building multiattribute decision support systems.

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Structuring Techniques in Multiset Spaces

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Abstract. This paper presents methods to investigate a structure of objects described with qualitative attributes (multicriterial alternatives, textual documents and so on). Proposed algorithms of clustering and ordering are based on the theoretical model of multisets. Techniques take into account qualitative nature of object properties and decision maker preferences.

Keywords. Verbal decision analysis, multisets, clustering, ordering.

1 Introduction

The quality of decisions depends on the quality and completeness of problem investigation. A preliminary "exploratory" analysis and structurization of an initial set of alternatives is one of the important stages of decision preparing and making. When there are a lot of real objects to be compared such an analysis allows decision makers and/or experts to evaluate the reality of the decision model used, to check up an adequacy of decision maker's preferences to the limitations imposed, to examine a possibility of realization of the decision made, and (if necessary) to correct or change the decision rules so that the decision to be feasible.

The problem of a multicriterial choice is connected often with ordering and/or grouping (clustering) objects in multicriterial spaces. Difficulties of object structurization increase in a verbal decision analysis when objects are described with many qualitative attributes. These difficulties have formal and substantial bases. First, the amount of information for object specification grows largely. Second, objects' estimates are to be taken into consideration simultaneously and without any "averaging' (due to qualitative nature of attributes). Some of the difficulties mentioned above can be overcome by using new theoretical models for representing qualitative alternatives as objects in multiset spaces. In this paper effective methods for structuring multisets are considered.

2 Representation of qualitative objects

A structurization of objects deals with the identification of relations and links between them. A similarity and difference between objects, their domination or ordering are closely connected with concepts of a distance between and a value of objects. It is assumed in structuring methods that any distance or value is given a priory or introduced axiomatically. The type of space and choice of an index of value, nearness, similarity or difference between objects depend on the properties of the objects considered.

Review two cases of qualitative objects.

Let $X = \{x_1,\dots,x_k\}$ be a collection of k objects estimated by n experts using criteria $Q_1,Q_2,\dots,Q_m$ with qualitative (ordered and/or nominative) scales of estimates $\{q_r^{e_r}\}$, $r=1,\dots,m$; $e_r=1,\dots,h_r$. For instance, X is a collection of competitive R&D projects to be included in a research program. Introduce a set of criteria estimates $G = \{g_1,\dots,g_h\}$ whose elements to be defined in the following way:

$$g_1=q_1^1,\dots,g_{h_1}=q_1^{h_1},\ g_{h_1+1}=q_2^1,\dots,g_h=q_m^{h_m},\quad h=h_1+\dots+h_m. \tag{1}$$

The set G with elements g_j (1) determines the properties of projects. Each project $x_i \in X$ can be presented as

$$x_i = \{n_i(g_1)\cdot g_1,\dots,n_i(g_h)\cdot g_h\} \tag{2}$$

where $n_i(g_j)$ is a number of attribute g_j which is equal to a number of experts who gave the estimate $g_j = q_j^{e_j}$ to the project x_i.

A file of textual documents $X = \{x_i\}$ is another example of a collection of objects described with qualitative attributes. For instance, such documents can be projects, patents, scientific books, articles or their summaries related to any problem fields and so on. In this case qualitative attributes are lexical units (descriptors, keywords, terms, etc) that express a semantic substance of document. Like above present each document $x_i \in X$ in the form $x_i = \{n_i(g_j) \cdot g_j\}$ where $n_i(g_j)$ is equal to a number of lexical units g_j in the document x_i. The set of attributes $G = \{g_1, \ldots, g_h\}$ can be considered here as a problem-oriented terminological dictionary or thesaurus.

The object x_i in the form (2) is a multiset over a generic set (domain) G. The collection of objects $X=\{x_i\}$ is the multiset also. The multiset [3,7] is a group of elements where each element may occur more than once. The number of occurences of the element g_j in the multiset x_i is defined by the number (or membership) function $n_i(g_j)$. The matrix $C = ||n_i(g_j)||$ describes the relation between the multiset of objects $X=\{x_i\}$ and the set of their attributes $G=\{g_j\}$.

In case of multisets traditional set-theoretical operations can be added with new types of operations such as a summation, a multiplication on scalar, a linear combination of multisets and others [7,8]. The measure of multiset can be defined as

$$m(x_i) = \sum_j w_j \, n_i(g_j), \quad w_j > 0. \tag{3}$$

Various approaches to object grouping are possible. For instance, a collection of objects X_t can be resulted as an addition of objects $X_t = \sum_{i \in I_t} x_i$. Then the multiset $X_t \subseteq X$ is to be described by the matrix $C' = ||n'_{tj}||$ with elements

$$n'_{tj} = n'_t(g_j) = \sum_{i \in I_t} n_i(g_j). \tag{4}$$

A group of objects X_t can be also formed as a linear combination of multisets $X_t = \sum_{i \in I_t} b_i x_i$, $b_i > 0$. Elements of matrix C' are determined as

$$n'_{tj} = n'_t(g_j) = \sum_{i \in I_t} b_i n_i(g_j). \tag{5}$$

If X_t is formed as $X_t = \cup_{i \in I_t} x_i$ or $X_t = \cap_{i \in I_t} x_i$, then the matrix C' has elements

$$n'_{tj} = n'_t(g_j) = \max_{i \in I_t} n_i(g_j). \tag{6}$$

or

$$n'_{tj} = n'_t(g_j) = \max_{i \in I_t} n_i(g_j). \tag{7}$$

The expressions (4),(5) characterize the properties of all members of group X_t aggregated in different ways. The expressions (6),(7) reflect a domination of the properties of several members of group X_t.

The object x_i expressed by formula (2) is displayed graphically as a histogram where values of attributes g_j are plotted consequently along the abscissa axis, and numbers of attributes $n_i(g_j)$ are plotted along the ordinate axis. The group of objects X_t and the collection X in total are represented by histograms to be obtained as the sum, weighted sum, union or intersection of histograms corresponding to objects x_i with ordinate estimates n'_{tj} (4), (5) (6) or (7).

The theoretical model of multisets is more appropriated for representing a collection of objects described with qualitative attributes such as textual documents or multicriterial alternatives evaluated by several experts using many qualitative criteria with ordered and/or nominative scales of estimates. Multiset spaces are the non-Euclidean, in general. The axiomatic approach to metrization of multiset spaces was considered in [7,8] where properties of difference and similarity indexes were investigated. Such functions are widely used in various methods of cluster analysis.

3 Clustering multisets

In the cluster analysis an initial set of objects $X_i = \{x_i\}$ is divided into several groups $X = \{X_1,\dots,X_K\}$ basing on a difference or similarity of object properties. Principal points of cluster analysis are the following: a choice of an index determining the distance between objects; a choice of a classification algorithm; a rational interpretation of the classes formed.

Assume that indexes of difference/similarity between objects $x_a, x_b \in X$ and between groups of objects (clusters) $X_p, X_q \in X$ are the same type. According to [7,8] the following expressions can be written for distances and measures of similarity between multisets:

$$d_0(X_p,X_q)=D_{pq};\quad d_1(X_p,X_q)=D_{pq}/L;\quad d_2(X_p,X_q)=D_{pq}/M_{pq}; \tag{8}$$

$$s_1(X_p,X_q)=1-(D_{pq}/L);\quad s_2(X_p,X_q)=I_{pq}/M_{pq};\quad s_3(X_p,X_q)=I_{pq}/L. \tag{9}$$

Here

$$I_{pq} = \sum_{j=1}^{h} w_j \min(n'_{pj},\ n'_{qj});\qquad D_{pq} = \sum_{j=1}^{h} w_j |n'_{pj} - n'_{qj}|;$$

$$M_{pq} = \sum_{j=1}^{h} w_j \max(n'_{pj},\ n'_{qj});\qquad L = \sum_{j=1}^{h} w_j \sup_t n'_{tj}\ ,\quad w_j>0.$$

Elements n'_{pj}, n'_{qj} are determined by formulae (4), (5), (6) or (7). Remark that $I_{pq} + D_{pq} = M_{pq}$.

The distance d_0 is a Hamming-type distance between objects of various nature which is traditional for many applications. The functions s_1, s_2, s_3 generalize for multisets the known nonmetric measures of object similarity (the simple matching coefficient, Jaccard coefficient or Tanimoto measure, Russel and Rao measure of similarity).

The notion of a cluster center is often used in algorithms of clustering [1]. The center z_t of a cluster $X_t \subseteq X$ $(t=1,\dots,K)$ can be found as a solution of minimization problem, for instance,

$$J(z_t,X_t) = \min_z \sum_{x_i \in X_t} d(z,x_i) \tag{10}$$

where $d(z,x_i)$ is a distance in a space (X,d) of type (8).

Remark that in our case a cluster center z_t may coincide with one of the real members of the collection X or be a "phantom" which does not belong to X but possesses properties of the real existing members $x_i \in X$. For example, the center of cluster z_t can be constructed from attributes q_j and presented as the multiset in the form (2).

Review briefly major stages of hierarchical clustering for multisets when a number of generated clusters is unknown beforehand.

1. Set K=k. K is a number of clusters, k is a number of objects x_i. Then each cluster $X_i=\{x_i\}$ for all i=1,...,K.

2. Calculate distances between pairs of clusters $d(X_p,X_q)$ for all $1 \le p,q \le K$, $p \ne q$, using one of multiset space models (8).

3. Find a pair of clusters X_u, X_v such that

$$d(X_u,X_v)= \min_{p,q} d(X_p, X_q) \tag{11}$$

and form a new cluster $X_r = X_u \cup X_v$, or $X_r = X_u + X_v$, or $X_r = b_u X_u + b_v X_v$.

4. Reduce the number of clusters by unit: K=k-1. If K=1, then output the result as a dendrogram and stop. If K>1, go to the next step.

5. Recalculate new distances $d(X_p,X_r)$ for all $1 \le p \le K$, $p \ne r$. Go to step 3.

The hierarchical clustering is ended when all objects are merged in a single class. The process can be also terminated when the difference index (11) overcomes a certain threshold level. There are methods of dendrogram optimization based on searching an appropriate level $d_{opt}(X_u,X_v)$ that allow to give a comprehensible interpretation of the groups formed [5].

In methods of nonhierarchical cluster analysis a number of clusters K is considered as fixed and determined beforehand. A general framework of nonhierarchical clustering for multisets is the following.

1. Select any initial partition of objects into K clusters $X = \{X_1,...,X_K\}$.

2. Distribute all objects $x_i \in X$ into clusters X_t

$(t=1,\dots,K)$ according to a certain rule. For instance, calculate distances $d(x_i,X_t)$ between an object x_i and clusters X_t by one of the formulae (8) and allocate the object x_i into the nearest cluster X_n for which $d(x_i,X_n) = \min_{1\le t\le K} d(x_i,X_t)$. Or calculate a center z_t for each cluster X_t by solving the equality (10), and allocate each object x_i into the cluster with the nearest center, i.e. with $d(z_r,x_i) = \min_{1\le t\le K} d(z_t,x_i)$.

3. If all objects x_i do not change their cluster membership to be given by an initial partition of objects in clusters, then output the result and stop. Otherwise go to step 2.

The results of an object classification can be evaluated by a partition quality. In particular, the best partition X_{opt} is a solution of an optimization problem

$$J(X_{opt}) = \min \sum_{t=1}^{K} J(z_t,X_t) \tag{12}$$

where $J(z_t,X_t)$ is defined, for example, by formula (10).

Note that if a measure of multiset similarity (9) is used in a clustering procedure, then the condition $\min d(X_p,X_q)$ is to be replaced by $\max s(X_p,X_q)$.

The following approache to structurize a collection of objects can be useful for solving practical problems. At first, objects are classified with a hierarchical clustering, and several possible partitions of objects are to be selected. Then the set of partitions is analysed with a nonhierarchical technique, and the most suitable partition is searched.

4 Ordering multisets

There are many approaches to arrange objects in order. Some of them use a comparison of objects, others assume a certain value of object to be given. The binary relations between objects (order, quasiorder, equivalence or incongruence) depend on decision maker preferences and object properties determined by their attributes.

Consider a method of ranking qualitative objects that takes into account a "preferability" of object or an object relevance to information needs of decision maker. Define an information value of the object x as a relative measure of the multiset x

$$v(x) = \sum_{g\in G_x} w(g)n_x(g) \;/\; \sum_{g\in G_x} n_x(g) \qquad (13)$$

where w(g) is a so-called semantic weight of the attribute g, and G_x is a subset of attributes that belong to the object x. The semantic weight w(g) can be calculated in the following way.

Take out an arbitrary collection of objects X_0 from the general file X. All objects $x_i \in X_0$ are evaluated by a decision maker using a criterion of object preferability with an ordered scale of estimates $b_p(x)$. Therefore the "probe" file X_0 will contain objects x_i that have different degrees $b_p(x_i)$ of object correspondence with decision maker preferences. The file X_0 can be divide into two subfiles X_p and X_q. First file X_p consists of objects x_i with $b_p(x_i) \geq \theta_p > 0$ which is the most interesting for the decision maker. θ_p is a level of object preferability to be given by the decision maker. Second file X_q consists of objects x_i with $b_q(x_i)=1-b_p(x_i)$ which is the least interesting for the decision maker.

In other words, the multiset X_0 is the sum of multisets more and less preferable $X_0 = X_p + X_q$, where X_p and X_q are linear combinations of multisets x_i

$$X_t = \sum_{i\in I_t} b_t(x_i)x_i, \quad t=p,q.$$

If the files X_p and X_q are chosen to be considerably "unlike", then various attributes g will have different "informativeness" for each file. So the semantic weight w(g) of attribute g characterizes a difference between distributions of the attribute g in the multisets X_p and X_q, and can be determined as

$$w(g)=\begin{cases} [P(X_p|g) - P(X_q|g)]\cdot[\zeta(g,X_p)]^{\alpha} & \text{if } b_p(x_i)>b_q(x_i), \\ 0 & \text{otherwise.} \end{cases} \qquad (14)$$

Here

$$P(X_t|g) = \sum_{i\in I_t} b_t(x_i)\chi_i(g) \ / \ \sum_{i\in I_0}\chi_i(g) \qquad (15)$$

is a conditional probability of belonging an object with the attribute g to the file X_t, t=p,q;

$$\zeta(g,X_p) = P(g)/P(X_p) = \sum_{i\in I_0}\chi_i(g) \ / \ \sum_{i\in I_p} b_p(x_i) \qquad (16)$$

is a ratio of probabilities to sample an object with the attribute g and to sample an object belonging to a file X_p;

$$\chi_i(g) = \begin{cases} 1 & \text{if } g\in G_{x_i}, \\ 0 & \text{otherwise} \end{cases} \qquad (17)$$

is the characteristic function; α is an adjusting parameter. Note that w(g) is a continuous, monotonic and normalizated function of its arguments $P(X_t|g)$, t=p,g, if $0<\alpha<1$ [8]. If $\alpha=0$, then w(g) has a discontinuity of the first kind in zero point.

Hence the information value of the object is connected with its preferability for the decision maker and depends on the quantity of qulitative attributes and their semantic weights. So all objects from the general file X are ordered (quasiordered) according to their information values. The qualtity of object ranking can be evaluated by a criterion

$$R(\alpha) = \sum_{x\in X} |b_p(x) - \beta_p(x,\alpha)| \qquad (18)$$

where $\beta_p(x,\alpha)$ is an algorithmic degree of the object preferability that is defined as

$$\beta_p(x,\alpha) = b_p(x^o) \quad \Leftrightarrow \quad r^{\beta}(x) = r^b(x^o).$$

Here $r^{\beta}(x)$ is a rank of the object x into the file X ordered in accordance with the information value v(x), $r^b(x^o)$ is a rank of the object x^o into the file X_0 ordered in accordance with the degree of preferability $b_p(x^o)$. Minimization of a difference between "algorithmic" and "expert" ranking is provided with a variation of the adjusting parameter α in the function w(g).

The algorithm of multisets ranking by an object informativeness includes the following steps.

1. Select any "probe" submultiset of objects X_0.
2. Evaluate the degree of preferability $b_p(x^o)$ for

each object $x^o \in X_0$ by the decision maker. Range objects x^o by $b_p(x^o)$.

3. Calculate semantic weights w(g) of attributes g by formulae (14)-(17) and information values of objects v(x) for $x \in X$ by formula (13). Range objects x by v(x).

4. Find α_{opt} by solving the problem min $R(\alpha)$ (18). Go to step 3, recalculate w(g) and v(x), and arrange objects in the best order.

Note that the proposed algorithm of multisets ranking has a low sensitivity to a variation of the preferability degree $b_p(x^o)$. This algorithm can be used also to range the same collection of objects simultaneously in various ways by dividing the file of objects into several subfiles with different (in semantic sence) degrees of the object preferability for a decision maker.

5 Applications

Approaches to struturize qualitative objects presented as multisets can be fruitful for the development of new clustering and ordering techniques. An investigation of natural grouping and possible relations between the objects caused by their properties can help a decision maker formulate choice strategies and decision rules to be adequate with the reality, make his choice more substantial, reasonable, and sensible. Some of techniques discussed in this paper were applied to analyse the decisions made in elaborating the several state scientific-technological programs.

Ordering files of R&D projects in the program on biotechnology [6] were used to arrange a problem-oriented database of specific DSS. Project descriptions presented as multisets were sorted in accordance with information needs of the program manager who was responsible for the program coordination. Such DSS provided a quick retrieval of relevant textual documents intersting for the user to analyse them for various purposes. Projects with the highest degree of preferability came out first.

A peer review panel (jury) selected competitive R&D projects to be included in the program on hightemperature-superconductivity [4]. These projects were evaluted by experts with the following qualitative criteria: the project contribution to the achievement of program goals, the prospectivity of project, the novelty of the approach to solve the task, the level of team qualification and so on. Based on expert estimates the panel chose the best projects; or selected part of projects for the next round of competition; or divided projects into several groups: the projects adopted, rejected, and to be improved. The analysis of project distribution based on multisets clustering in the multicriterial space allowed to discover the natural groups of projects, and find the simple decision rules for projects selection.

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THE SENSITIVITY ANALYSIS OF "INEXACT" MULTICRITERIA DECISIONS

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Abstract. We elaborate a view of sensitivity analysis as a decision analysis activity. Our two main results concern: 1) a formal concept of "inexact" riskless preference; and 2) illustrative mathematical optimization problems for identifying "large" subsets of value function parameter values that preserve the preferability of a prescribed decision alternative.

Keywords. Sensitivity analysis, additive value function, inexact riskless preference, multiplicative value function, rectilinear convexity, stable rectangle

1 Introduction

Sensitivity analysis has received less formal attention than other modelling phases of decision analysis, perhaps contributing to a view that sensitivity analysis is more craft than science (e.g., [9]). We adopt the view that sensitivity analysis is itself a decision analysis activity, but applied to a "post-optimality" decision.

More specifically, we ordinarily assess only one model of the DM's preferences and identify the most-preferred action(s) under that model (e.g., [4]). However, suppose that the DM's preference discriminations are not indefinitely fine (i.e., "inexact"). Different samples of preference expressions may then "determine" different members of a parametric family of preference models that themselves prescribe different most-preferred action(s). In light of this possibility we consider whether to take an action recommended by the one actually assessed preference model (a "reference action"), to defer the decision (e.g., to refine "inexact" preferences) or to consider an alternative action (e.g., choose the reference action but also purchase "insurance" against the "preferability" of other choices). We report here on progress in addressing these ideas.

2 A View of Sensitivity Analysis

We are concerned with sensitivity analyses motivated by "inexactly" assigned preference model constituents (henceforth *inexact parameters*; e.g., a scaling constant w_j is in [0.2,0.4]). The view of sensitivity analysis-as-decision analysis suggests that the DM should iteratively:

- assess Ω^{ψ}, the set of ordered p-tuples in $\mathbb{R}^p$ that represents the parameter values "consistent" with the DM's "inexact" preference judgments. Ω^{ψ} is a subset of Ω, the set of parameter values that are feasible to the preference model;
- evaluate the consequences of inexactness (e.g., Is there more than one set of "best" action(s) consistent with the $w \in \Omega^{\psi}$? If so, which ones?); and
- decide what to do next (e.g., reassess Ω^{ψ}, trying to tighten inexact parameter assessments; find the "cost" of inexactness to be acceptable relative to the value of another iteration, and so end the process).

The recent interest in elements of this view (e.g., [5], [6]; also see [10]) has been in identifying a subset $\mathbb{I}$ of Ω that preserves the preferability (i.e., "optimality"). We call such a subset $\mathbb{I}$ a *stable subset.*

In another paper ([8]) we describe how to assess a range of parameter values that is "consistent" with the "range" of the DM's inexact preferences (e.g., $w_j \in [0.2,0.4]$). (These methods arise directly from the indifference methods used for single-valued parameter assessment, as in [4].) In this paper we examine one way of identifying stable subsets and illustrate it with respect to value function scaling constants (i.e., $w=[w_1,..,w_p]$, $w \in (0,1]^p$, $\mathbf{1}^{\mathrm{T}}w \in (1,p)$).

Stable subsets have a natural value as aids for guiding set assessments of inexactly-valued parameters, with the goal of obtaining a set $\Omega^{\psi} \in \mathbb{I}$ that preserves the preferability of the reference action. For this use of stable subsets we expect that some stable subsets should be more useful than others. We identify a "good" stable subset in this sense by solving a mathematical optimization model with a "goodness" criterion represented in the objective.

When the reference action is not optimal throughout Ω we focus on stable subsets that take the form of rectangles in $\mathbb{R}^p$. In particular, we define a *stable rectangle* in terms of one interval of the real line $I_j=[l_j,u_j]$ for each inexact parameter. In this form stable rectangles provide one interval to aid the "inexact assessment" of each parameter.

3 A Concept of "Inexact" Riskless Preference

We commonly assume that people do not make indefinitely fine preference discriminations. Yet we require such discriminating judgments in order to exactly specify our preference models (e.g., [4]). It thus appears that we consider sensitivity analysis to be important in order to consider the consequences of the violated requirement. However, in order to justify a particular sensitivity analysis procedure we require better articulated assumptions about what we mean by "inexact preference". Otherwise, a sensitivity analysis procedure may depend on a sense of "inexactness" that is not well-defensible from a behavioral perspective or that is not well-exploitable from a practical perspective.

In this work we assume the DM to be indifferent between actions that are identical with respect to all attributes, to strictly prefer an action a_R to an action a_S if a_R dominates a_S, and to satisfy the conditions for modelling preference with an additive or a multiplicative value function (e.g., [2], [4]). We call a DM's preferences *inexact* if there exist at least three actions a_R, a_S, and a_T that pairwise differ with respect to two (or more) attributes and that satisfy $a_S \sim a_R$ (indifference), $a_S \sim a_T$, and $a_T \succ a_R$ (strict preference). We thus define inexact preference with reference to intransitive preferences among actions whose comparison nominally entails "value tradeoffs".

Inexact preference so defined is important when we use preference comparisons involving value tradeoffs to numerically assess value function parameters (e.g., [2], [4]). Inexact preferences can then only be expected to inexactly determine parameter values.

4 "Goodness" Criteria for Stable Rectangles

Our rationale for stable rectangle goodness criteria is, loosely, that the DM wants to obtain $\Omega^{\psi} \in \mathbb{I}$ and the "larger" a stable rectangle is the "easier" it will be for her to do this (i.e., "larger targets are easier to hit"). "Size" is thus one sensible criterion for "good" stable rectangles. However, we should justify adopting any one definition of "size".

For our immediate purpose we assume that the DM has not yet assessed Ω^{ψ} and that she does this by assessing an interval I_j^{Ψ} of the real line for each inexact parameter. This set Ω^{ψ} is thus also a rectangle in $\mathbb{R}^p$, symbolized by $\mathbb{I}^{\Psi}$.

Now (loosely) suppose that the DM is determined to satisfy $I_j^{\Psi} \subseteq I_j$, $j=1,.., p$, and that she perceives the "effort" required to satisfy each such condition is decreasing in the length of I_j. If, in addition, the DM's preferences for stable rectangles satisfy the axioms of an additive value function (e.g., [4]) then a suitable objective has the form $max \sum_{j=1}^{p} c_j v_j(u_j - l_j)$, where the v_j are single-interval value functions, the c_j are positive scaling constants, and u_j (l_j) is the upper (lower) endpoint of I_j.

The DM may alternatively wish to maximize the probability of satisfying $I_j^{\Psi} \subseteq I_j$, $j=1,..,p$, in a single set assessment of the parameters. Loosely, if the DM believes that the probability of observing each such event to be an increasing function P_j of (u_j-l_j) and that these events depend only on the respective interval lengths, then an objective is (loosely) consistent with the form $max \prod_{j=1}^{p} P_j(u_j - l_j)$.

We recognize that in practice it may be neither worthwhile nor feasible to make "detailed" assessments of the v_j, P_j, or c_j because the DM will have few preferences or beliefs in these matters. However, we do wish to construct stable rectangles that will be useful as elicitation aids so the goodness criteria should be consistent with whatever beliefs or preferences can be identified economically and reliably.

5 Stable Rectangles for Additive Value Model Scaling Constants

Recall that an additive value function has the form $V(a_i) = \sum_{j=1}^{p} w_j v_j(a_{ij})$, where the w_j are positive scaling constants that calibrate the $v_j(a_{ij})$, single attribute valuations of the actions, to a common "unit of preferability" (e.g., [4]). Admissible choices of scaling convention determine $v_j(x_j^*)=1$ and $v_j(x_{j*})=0$, where x_j^* (x_{j*}) is the most (least) preferred level of the jth attribute. In our work we relabel the largest scaling constant to be w_1 (also arranging for it to be uniquely so), scale it to a value of one, and represent $w_2,..,w_p$ as positive linear functions of w_1 (e.g., $w_2=\lambda_2 w_1$, $\lambda_2>0$). This allows us to treat w_1 as exactly-assessed and $w_2,..,w_p$ as inexactly-assessed through the coefficients $\lambda_2,..,\lambda_p$ with values in $(0,1)$.

We have constructed stable rectangles for additive value models using a program, *AVALUE1*, written in the *GAMS* modelling language [1]. Using *AVALUE1* we first find the reference action and the set of nondominated, potentially optimal actions ([7]). We then construct a stable rectangle by solving a problem of the form

$$\begin{aligned} &\mathit{max}\ \langle \textit{objective function} \rangle \\ &\mathit{st.}\ \langle \textit{objective characterizing constraints} \rangle \\ &\quad\ \langle \textit{optimality constraints} \rangle \\ &\quad\ \langle \textit{feasibility constraints} \rangle . \end{aligned}$$

The feasibility constraints enforce the condition that $\mathbb{I} \subseteq [0,1]^p$ and that $l_j \le u_j$. We may also require that $\mathbb{I}$ contain w^0, the single-valued assessment of the scaling constants made in the decision analysis (i.e., $l_j \le w_j^0$; $u_j \ge w_j^0$).

The optimality constraints enforce the condition that the reference action be optimal at the vertices of the stable rectangle. Let O be the set of parameter values in $(0,1]^p$ that preserves the optimality of the reference action, its *preference set*. O is convex, indeed a polytope. The optimality constraints therefore ensure the optimality of the reference action throughout $\mathbb{I} \subseteq O$. The optimality constraints have the form

$$v_1(a_{11}) - v_1(a_{k1}) + \sum_{j=2}^{p} \beta_j \left[v_j(a_{1j}) - v_j(a_{kj}) \right] \ge -\gamma ,$$

where $\beta \in \{\{l_1,u_1\} \times .. \times \{l_p,u_p\}\}$ and a_k is a nondominated, potentially optimal action other than a_1, the reference action. We take $\gamma=0$ here but consider other values for γ below.

The objective characterizing constraints define the variables appearing in the objective function. We provide for two such variables, *size* and *minlength*. *Size* represents the size of the stable rectangle according to one of the two "size criteria" discussed above. We implement both in additive forms,

$size = \sum_{j=2}^{p} c_j (u_j - l_j)$ or $size = \sum_{j=2}^{p} c_j \, log(u_j - l_j)$.

We define *minlength* in two ways: $u_j - l_j \geq minlength; j = 1,..,n$ make *minlength* the length of the smallest interval defining $\mathbb{I}$; $u_j - w_j^0 \geq minlength$ and $w_j^0 - l_j \geq minlength$ define *minlength* analogously, but relative to semi-intervals of the form $[l_j, w_j^0]$ and $[w_j^0, u_j]$.

The objective (to be maximized) is $value = (w_{size} \times size) + (w_{minlength} \times minlength)$, where w_{size} and $w_{minlength}$ are nonnegative coefficients.

We have assembled these and other expressions to obtain stable rectangles for a three-attribute, three-action decision problem and for a ten-attribute, ten-action decision problem ([3], pp. 173-182).

In the ten-attribute, ten-action decision problem we have observed w^0 to lie on an edge of stable rectangles constructed under a variety of optimization problem formulations. This may not be a desirable outcome if we believe w^0 to be "centered" in $\mathbb{I}^{\Psi}$.

One way of "better centering" w^0 in $\mathbb{I}$ is to require only that $\mathbb{I}$ preserve, unordered, the $k \geq 2$ actions most highly valued under w^0. If the additive value function is measurable (e.g., [2]); the value function rank orders "preference differences") we can assign a "suitable" value to γ in the optimality constraints. This relaxation permits the reference action to be nonoptimal in subsets of a solution rectangle while constraining the "loss" to no more than γ.

6 Stable Rectangles for Multiplicative Value Models

We think it important to construct stable rectangles for nonadditive preference models and so are extending the *AVALUE1* class of optimization problems to the case of multiplicative value functions. There are obstacles to doing this uncritically because multiplicative value functions, with the form

$$V(a_i) = \frac{1}{W}\left(\prod_{j=1}^{p} [1 + W w_j v_j(a_{ij})] - 1\right); \; W = \prod_{j=1}^{p} [1 + W w_j] - 1; \; w_j \in (0,1], \; \sum_{j=1}^{p} w_j \neq 1$$

make *AVALUE1*-type optimization problems nonconvex, so their solutions are not guaranteed to be globally optimal. We also observe the preference sets (O) to be nonconvex. Constraining the vertices of $\mathbb{I}$ to be in O, as we do in the optimality constraints therefore does not guarantee $\mathbb{I} \subseteq O$. We have focused on this latter problem and have made some progress in addressing it in the form of a generalization of convexity that we call *rectilinear* convexity (r-convexity).

Let sets S,T satisfy $S \subseteq T \subseteq \mathbb{R}^p$; let $S^C = T\text{-}S$; and let s_1, s_2 be arbitrary points in

S that differ in only a single component. A set S is r-convex if and only if all convex combinations of s_1 and s_2 are also in S; we show that if the vertices of a rectangle are in an r-convex set the rectangle itself is in the set. We also formulate a class of (nonconvex) optimization problems whose solutions jointly classify O as being r-convex or not. In the case that O is r-convex we can, in principle, replace all additive value functions in an *AVALUE1*-type optimization problem with multiplicative value functions.

In a three-attribute test problem that we have been examining closely each of the three actions are nondominated and potentially optimal in $(0,1]^3$. The solutions to the diagnostic optimization problems lead us to "conclude" that each of the three possible preference sets is r-convex and that we have successfully constructed stable rectangles using a variant of *AVALUE1*.

7 Conclusions

David Ríos Insua and Simon French ([7]) enumerate several criteria for a sensitivity analysis tool. We add two criteria to these. A tool should:

- consider assessed inexactness in preference model constituents as well as how the valuations of the actions change over the induced parameter space. (Neither by itself can have a complete meaning for post-optimality decisions and both appear to be necessary for it);
- provide information that is interpretable with respect to making post-optimality analysis decisions.

Our view of sensitivity analysis appears to provide a promising way of thinking about sensitivity analysis in this regard. As we have conceived it, the view considers both assessed inexactness in the parameter values and how the valuation of the actions changes over the induced parameter space. It also provides information to the DM that is interpretable with respect to making post-optimality decisions. The DM's set assessment of the parameter space either admits disagreeing inferences about the most preferred action(s) or it does not. If it does we may be able to determine the consequences of inexactly-assigned parameters (e.g., how many and which "best" actions the set Ω^{ψ} admits). The DM can then use this information to help decide what to do next.

Not unexpectedly, we also think that rectangles may be a useful way of thinking about subsets of the parameter space, whether stable subsets or empirical subsets Ω^{ψ}. We alluded to some desirable properties of stable rectangles and continue to find others (e.g., if the value function is measurable then consider Von Winterfeldt and Edwards' flat maxima discussions [9] and assign a "reasonable" value to γ).

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Goal Programming and Multiple Criteria Decision Making: Some Reflections

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Abstract. Two issues are discussed in this paper. The first is whether an algorithmic approach can be a logical and practical way of selecting the best multi-criteria technique. The second issue relates to the correct election of a goal programming (GP) variant for a given decisional problem. With this purpose the preferential logic underlying the most widely used GP variants (lexicographic, weighted and MINMAX) are connected with the actual preferences of decision maker. From the analysis undertaken some insights and recommendations in choosing the best GP variant or mix of variants arise. Although the output of this paper cannot be considered completely new, it can, however, help analysts in building logically sound GP models which rightly reflect the reality being modelled.

Keywords: Goal programming, utility theory, decision theory.

1 Introduction

Multiple Criteria Decision Making (MCDM) has been perhaps the fastest growing area of Decision Analysis in the last twenty years. The undoubtable success of the MDCM paradigm has led to an impressive boom of innovative applications as well as technical developments. Nowadays when an analyst faces a real decision making problem his main difficulty does not only lie in show to isolate the relevant criteria in the corresponding situational context but also on how to select the most suitably MCDM technique. In fact the analyst is besieged by an enormous amount of seemingly sensible MCDM techniques.

The degree of complexity of many decisional problems involving several criteria, incomplete information as well as many decisional variables and constraints demands flexible and pragmatic MCDM methodologies. Perhaps, this is the reason for the enormous popularity of Goal Programming (GP). In fact GP is the most widely used technique in terms of practical applications. The number of cases as well as the wide range of fields where GP has been applied is actually impressive (see, for example, [17], [18], [20], [22], etc.). However, despite its popularity, at least in terms of applications, several seemingly theoretical weaknesses in GP plus many naive applications have generated among the MCDM profession the feeling that GP has had its time but is almost "dead" nowadays (see the paper by Ignizio [14] analysing this matter and making a sensible reflection on the present and future of GP).

These difficulties surrounding GP have increased lately with the publication of some papers which aim to devise algorithmic structures with the purpose of establishing something like a ranking of MCDM techniques according to corresponding advantages and disadvantages. In these rankings GP appears almost in the last position!

This paper has a twofold purpose: first, to critically comment and assess efforts undertaken to create algorithms to help in the selection of the best MCDM technique and second, to address some questions related to the selection of the best GP variant for a given decisional problem. It will be shown how an inquiry into the structure of preferences of the decision-maker (DM) can provide interesting insights to choose the most suitable GP variant or mix of GP variants (i.e. lexicographic, weighted or MINMAX).

The paper is organized as follows. After these introductory ideas a review of the main algorithmic attempts devised to choose the best MCDM technique is presented. These attempts are criticized from a logical as well as practical point of view. In the following section the choice of the best GP variant or mix of variants is addressed by connecting the formulation of the model with the actual structure of DM's preferences. The paper ends up by summarizing the main findings of this research.

2 An Algorithm to Choose the Best MCDM Technique ?

Gershon & Duckstein [10, 11], Ozernoy [15] (for discrete problems) and Tecle & Duckstein [21] among others address the following crucial problem in MCDM. "How to choose the best MCDM technique among the well-armed arsenal of options". The common idea of these authors is that the choice of a MCDM technique is actually a multi-criteria problem in itself. Based on this idea, they seek to develop something like an algorithmic structure capable of ordering a set of MCDM techniques. For instance Tecle & Duckstein [21] develop an algorithm capable of ranking 15 popular MCDM techniques. The algorithm is underpinned by the multi-criteria technique known as composite programming (CTP) which is actually an extension of compromise programming (CP).

As a result of their research CP, CTP, the method of the displaced ideal, and cooperative game theory are found to be the "best" techniques. On the contrary the STEM method, GP and the Surrogate Worth trade-off method occupy the bottom of the ranking. Although the authors satisfactorily check the robustness of the algorithm proposed as regards to the weights used, they do not however check this aspect as regards to other crucial parameters like: the DM's competence, complexity of the set of contraints, number on criteria under consideration, etc.

Under my view, this procedure -as happens with similar attempts- underlies a problem of logical circularity. Indeed, we can wonder why the authors choose CTP to rank the different techniques. Is the ranking obtained robust with respect to the technique chosen? This matter reminds us of the old philosophical problem of justifying induction by resorting to an inductive principle. In this case, Tecle & Duckstein [21] justify the superiority of CP and CTP (i.e. an extension of CP) by resorting to CTP! It is obvious that this kind of result is not exempt of a certain circular bias.

It is also interesting to point out some oddities generated by the commented algorithm. For instance, in terms of the criterion consistency of the results, ELECTRE is one of the best ranked techniques when many inconsistencies -at least with the first versions of ELECTRE- have been formally and empirically demonstrated (Arrow & Raynaud [2] see specially Annex 1). In the same way, it is important to wonder how GP, which is an approach based on a Simonian philosophy of satisficing, STEM, based on an interactive optimization philosophy with local preferences or multiattribute utility theory (MAUT), based on a utility philosophy with absolute preferences can be compared. In short, each MCDM technique assessed is based upon a philosophy and there is not a single right philosophy.

In summary, for the reasons commented above there are serious doubts on the possibility of building a robust algorithm effective enough to choose the best MCDM technique. It seems that this interesting problem cannot be tackled in a purely mechanistic way as was the case in the papers commented.

The comparison of MCDM methodologies has also been tackled by other researchers following a different approach. Thus, Buede & Maxwell [6] conduct a series of simulated experiments to compare the results obtained from the application of five MCDM methodologies. The basic purpose of this research was to measure the rank disagreement of all the methodologies with respect to an additive multiattribute utility model (MAUT). It is empirically shown how the disagreement between the rank generated by MAUT and the analytic hierarchy process (AHP) is small, while this disagreement is very important when the ranks generated by the MAUT model and by the other methods are compared. Buede & Maxwell recommend the use of MAUT or AHP and show concern on the use of the other methods.

Although the commented research is interesting and sound its main recommendation (the superiority of MAUT and AHP with respect to the other methods) seems misleading. In fact, Buede & Maxwell assume that the "best" technique is the additive version of MAUT (i.e. it plays the role of something like a CP ideal) finding that AHP is the technique nearest to the ideal, placing the other methods far from this ideal. It is obvious that there are not theoretical reasons to justify that an additive MAUT model is the best theoretical formulation for any decisional problem. The choice of the most suitable approach will depend upon many circumstances as will be commented below. In fact, in certain problems this kind of formulation can be considered the "best" while in others it is untenable (see next section).

Eventhough the efforts commented above are well articulated, they assume a philosophy which is not easy to accept: "To be optimistic about the possibility of building something like an algorithm to rank different MCDM techniques". It is difficult to accept that this interesting problem can be tackled in such a mechanistic way, without attaching decisive importance to the practical features of the decisional problem under consideration.

A different attitude or philosophy towards this problem consists in accepting that the relative advantages and disadvantages among different MCDM approaches will largely depend upon the characteristics of the problem situation. Within this philosophy GP appears again as a flexible and pragmatic MCDM methodology which is the most suitable to be applied to many decisional contexts. Indeed, for a decision making problem with many criteria, say, for example, seven and with a complex constraint set (several hundred restraints and decisional variables) it is only tractable by formulating a GP model. In fact, for a problem this size it is unrealistic even to try to obtain a good approximation of the efficient set through multi-objective programming techniques.

3 The Choice of a Goal Programming Variant

The overall purpose of GP is to minimize the deviations between the achievement of the goals and their targets; that is, to minimize the unwanted deviational variables. The minimization process can be accomplished with different methods. Each one leads to a different GP variant. The most widely used GP variants are the following: *weighted goal programming* (WGP) which attempts to minimize a composite objective function formed by a weighted sum of unwanted deviational variables, *lexicographic goal programming* (LGP) which attaches pre-emptive priorities to the different goals in order to minimize the unwanted deviational variables in a lexicographic order and *MINMAX goal programming* (MGP) which attempts to minimize the maximum of deviations (see for example [12], [13] for a technical description of these variants).

In most GP applications reported in the literature one of the above variants is chosen without justifying the reasons for this election. It seems as if the choice of the corresponding GP variant were a question related to the tastes of the analyst or that it depends upon the availability of the corresponding computer codes. However the election of the right GP variant or mix of GP variants is a crucial matter if we want the GP model to capture the essential features of the reality modelled. In fact, to each GP variant corresponds a different DM's philosphy about preferences. Hence the variant or mix of variants which best reflect the actual DM's preferences should be chosen. In what remains of this section some insights about this important topic will be provided.

Dyer [8] clearly shows how a WGP formulation implicitly assumes the existence of an additive separable utility function. Moreover if in a WGP model the targets have been set at their anchor values then a linear and additive utility function is actually maximized (e.g. [18], chap. 7). Hence it seems sensible to check these kind of structure of preferences before the decision problem is modelled through WGP. To achieve this purpose we can resort to some kind of technical tests like the *difference independence* condition suggested by Dyer & Sarin [9] or by a straightforward interpretation of the DM's wishes. Thus, a WGP formulation will be suitable when the DM is looking for a technological optimum or solution of maximum efficiency (see [4]). Roughly speaking a WGP model is adequated when the DM wishes to reach the maximum sum of achievements although the solution can be extremely biased towards some of the goals under consideration.

The non-compatibility between lexicographic orderings and the utility functions is well known (Debreu [7], pp. 72-73). That is a LGP model does not optimize the DM's utility function. In order to assess the effect of this property on the pragmatic value of LGP it is necessary to realize that the reason for this non-compatibility lies in the non-continuity of preferences inherent to lexicographic orderings. In fact an assumption of non-continuity of preferences implies the impossibility of ordering the DM's preferences by a monotonic numerical representation or utility function (e.g. Romero [18], pp.43-46).

Therefore within a LGP context, the worthwhile matter of discussion is not to disqualify or not the lexicographic approach because of the commented incompatibility but to investigate if the reality of the problem situation is or is not compatible with an assumption of continuity of preferences. There are many scenarios -chiefly within a natural resources management context- where the non-continuity of preferences seems plausible. For instance, let us assume a forestry planning problem where two attributes are considered: timber production and an index measuring the risk of biological collapse of the forest. It is obvious that in this context the acceptance of the continuity of preferences would be unrealistic. Indeed, the assumption of continuity would imply accepting that there is always an increment in the volume of timber produced which compensates an increase in the risk of biological collapse of the forest no matter how great the value of the index. In this kind of situation a non-compensatory lexicographic model can reflect with accuracy the reality modelled.

In conclusion, the potential problems associated with the use of LGP do not lie in the commented non-compatibility with utility functions but in the careless use of this approach in decisional contexts where the DM's preferences are cleary continuous and consequently a compensatory model should be used. Another practical problem does not lie in the use of lexicographic orderings but in its abuse creating an excessive and unjustified number of priority levels ([1], [19]). In short, problem situations with discontinuous preferences are possible and in some cases important but not general enough to justify the excessive use of LGP models as any survey of GP applications shows. Lexicographic orderings should be used cautiously and usually in combination with other GP variants as will be commented below.

With regards to MGP, is interesting to notice that to this GP variant underlies the use of the L_∞ metric and for this metric when all the targets are fixed at their anchor values the solution provided by the model represents a balanced allocation among the achievement of different goals ([3], [4]). Hence, the solution generated by a MGP model satisfies the following chain of equalities:

$$\mathbf{W_1\ [f_1(\underline{x})\ -\ b_1] = \ -\ -\ -\ W_i\ [f_i(\underline{x})\ -\ b_i] = \ -\ -\ -\ W_n[f_n(\underline{x})\ -\ b_n]}$$

being $\underline{x}$ the optimum vector of decisional variables, W_i and b_i the weighting factor and the target for the ith attribute respectively. The MGP variant is appealing because the consideracion of the above path seems compatible with the structure of many DM's preferences. For instance a household usually focuses on a well-balanced consumption basket in the following way: not too much clothing to the detriment of food, too many vacations in detriment of education, etc. It is sensible to assume other DM's, such as governments or public institutions, can behave similarly. It is also interesting to notice that MGP formulation implicitly assumes the existence of a Rawlsian [16] utility function, where all the attributes making the goals are considered as perfectly complementary goods.

It now seems as if we have arrived to a paradoxical result. The most appealing GP variant is MGP because its underlying equilibrating logic seems to fit the preferences of many DM while LGP can be very interesting in certain particular cases, it does not generally fit in the DM's preferences as MGP or WGP. However any survey of GP applications reveals the opposite result: LGP is the most widely used variant followed by WGP and MGP (e.g. [17], [18] chap. 8).

The ideas presented in this section lead to two important conclusions. First, the GP variant has been generally chosen in a mechanistic way without looking for a correspondence between the preferential logic underlying the variant chosen and the actual structure of the DM's preferences. Second, the reliance in a single GP variant is not justified in most of the problem situations. In fact, it is easy to accept that in many decisional problems the nature of some of the goals under consideration make advisable to model them within a lexicographic structure, while for the rest of the goals a weighted and/or MINMAX structure would seem more realistic. Perhaps the meaningful question for an analyst building GP models is not to choose the right GP variant but to choose the right mix of variants.

4 Summary and Conclusions

The analysis of some attempts undertaken in the literature to address the choice of the "best" MCDM technique as a MCDM problem in itself reveals that this kind of procedure presents important logical flaws. In fact, the choice of the "best" MCDM technique is completely dependant on the main features of the problem situation. In short, the most suitable technique to be used for a particular decisional problem is decisively conditioned by their own characteristics.

The intuitive connection between the preferential logic underlying each GP variant and the actual structure of DM's preferences seems useful to help the analyst in the election of the best GP variant or mix of variants. The reliance in a single GP variant -chiefly LGP- usually adopted in the applied literature does not seem justified. Although the output of this paper cannot be considered completely new, it aims to help analysts in building GP models which are logically sound and which accurately reflect the reality being analyzed.

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Stopping Rules in Collective Expert Procedures

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Collective expert interrogation procedures are widely used for group decision-making. There are some forms of the procedures, but in any case an expert procedure is a continuing in time process of group work, and a choice of a moment of its termination is one of important questions of examination. In literature two rules are mainly considered. The first one, the consent rule, speaks that termination is conducted at the reaching of consensus or essential proximity of expert judgments. First, this rule was formulated in the Delphi method. The rule was very popular and for a long time have remained the single stopping rule. Later in a few works there was a criticism of procedures supposing the trend of individual experts' judgments to yield to influence of authorities or "majority opinion". These works pay attention to a danger of conformistic trends and "groupthink" when experts refuse in fact their judgments for the sake of unanimity of a group. As a reaction to this criticism the stabilization rule appeared. In accordance with the rule termination is conducted at an achieving a high or complete stabilization (unvariability in time) of experts' judgments. First this rule appeared in [1], then in [2]. The problem connected with the stopping rule has not studied experimentally. The first effort in this direction devoted to rules of consent and stabilization is given in the work.

The experiment included test questions the answers on which were known to the researcher but unknown to respondents. All answers were numerical. There were some groups of experts. The interrogation was conducted by the method of individual feedback of Delphic type. The method assumes that at each round experts receive information from only one other expert (a new one each time) [3]. The interrogation comprised four rounds since estimates of all experts stabilized after the fourth round.

Data processing consist of the analysis of the relation between stopping rules and the resulting index of quality of expert inquiry. As this index we took the accuracy of expert estimates, their proximity to true answers. It was supposed that the rule is preferable which expresses more sharply at the moment of the highest accuracy. For this the dynamics of the accuracy was studied, the moment of the highest accuracy was fixed, and values of consent and stabilization at this moment were observed. Besides, the correlation analysis was conducted between the accuracy, on the one hand, and consent and stabilization, on the other hand.

The accuracy was described by two indicators, of individual and group errors. The indicator of the error of the current individual estimate of expert i (at some round, on some question) is defined as

$$t_i = \frac{|x_i - x^*|}{x^*},$$

where x_i is the estimate of expert i , x^* is the true value. In every session this measure was computed for all experts, questions, and rounds. We are interested in the round dynamics of this as well as other indicators. For each round, for each expert group this indicator was averaged over experts and questions. The indicator of the error of the current group-average estimate (at some round, on some question) is defined as

$$T = \frac{|\bar{x} - x^*|}{x^*},$$

where $\bar{x}$ is the group-average estimate. For each round and for each group the indicator was averaged over questions. As the indicator of consent of the expert group (at some round, on some question) we use the standard coefficient of variation. The indicator of stabilization is defined for the pair of neighbour rounds since second round. First, for each expert (at some round, on some questions) the difference d between his estimates on current and previous rounds was found. For the whole group the ratio of the total difference over all experts to the number of experts was found. Because of the diapason of the variance of estimates on various questions strongly differed, these ratios were normalized by the group-average estimate (of previous round). For some group and some round the indicator was averaged over questions.

The analysis of the data shows that the growth of accuracy as individual as group occurred from round to round but gradually slowed down, and to the last round almost stopped. One may suppose that new rounds would not bring any visible growth of accuracy. The indicator of stabilization behaved analogously. Stability of estimates was high from the beginning, however grew sharp at the fourth round in comparison with the third one. We can declare the significant stabilization at the fourth round which was the cause of the termination of the interrogation. What about the indicator of consent, it increased very sharp (the scatter decreased) between first and second rounds, decreased between second and third rounds, and slightly increased between third and fourth rounds. On the whole we traced constantly convergence of estimates however one can not say that consent reached the high level. We can suppose that new rounds could give some growth of consent but it would be induced by only psychological mechanisms of conformism and would not be accompanied by the growth of accuracy.

The relation between indicators of accuracy and indicators of consent and stabilization is studied also with the help of the correlative analysis. The analysis was conducted independently for each expert group and for each round, starting from the second because at the first the indicator of stabilization was absent. Coefficients of correlation were computed between indicators t and T, from one side, and of consent and stabilization, from other side. On the whole there was not significant correlation between chosen indicators.

The experiment showed that the end indicator of expert interrogation, accuracy, increased from round to round and to the last round practically stabilized. Behaviour of the indicator of stabilization was like, and its value was big at the last round. Consent increased too, however have not reach the high level to the last round, and perhaps would grow further at prolongation of the inquiry. Meanwhile the correlation analysis have not reveal the essential relation between accuracy and any indicators of termination. Thus, the experiment does not give a base for an obvious preference of one stopping rule. Resuming we can say that, after some previous works, this work ascertains informal arguments in favour of the rule of stabilization, however they did not get persuasive empirical evidence. The study of stopping rules needs in continuation.

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Multiple Criteria Discrete Dynamic Programming

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Abstract. Dynamic programming is classically concerned with maximization of the value assigned by a real-valued function defined over sequences of decisions. Multicriteria (multi- objective) dynamic programming extends the approach to a vector-valued criterion function. The main purpose of the paper is to give new definitions of separability and monotonicity which allow to extend the theory of discrete multiobjective dynamic programming. The vector principle of optimality and theorems applied in decomposition methods are formulated. Numerical algorithms for such problems are briefly described. The paper ends with examples which illustrate the numerical aspects of the procedures.

Keywords. Multiple criteria, separability, monotonicity, principle of optimality, dynamic programming.

1. Introduction

The earliest results for dynamic vector-valued models are given in Brown and Strauch [1]. Since then many papers describing vector principle of optimality, the role of separability and monotinicity, multiple objective shortest path problem, integer problems, computational aspects and methods of scalarization have appeared. Different dynamic programming methods with multiple criteria were applied in many areas as well. The extended literature review can be found in Lee and Haimes [6] and Trzaskalik [7].

When solving MCDP problems, assumptions of separability and monotonicity are required. Classical definitons are discussed in Yu and Seiford [9], Carraway and Morin [2], Carraway, Morin and Moskowitz [3], Li [4], Li and Haimes [5]. However, in many frequently encountered multicriteria problems, operators are not associative or vary from stage to stage. In this paper we will extend the existing definitions of separability and monotonicity. Applying these new definitions we will clasify many of multi-objective dynamic programming problems as backward or forward problems . The main purpose of this paper is to extend the theory of discrete MCDP. We will consider discrete backward and forward problems with finite number of periods, state variables and decision variables. Optimality equations are formulated in backward and forward case. Numerical procedures based on those equations are shortly described. Illustrative numerical examples are given. The proofs of all the theorems (presented first in Trzaskalik [8] and the full description of numerical procedures (in the form of pseudo-codes) can be found in Trzaskalik [7].

2. Assumptions and notation

We consider a *discrete decision process* which consists of T *periods*. Let Y_t be the set of all feasible *state variables* at the beginning of period t (t=1,...,T), and $X_t(y_t)$ - the set of all feasible *decision variables* for period t and state y_t. We define $D_t \cong \{d_t=(y_t,x_t): y_t \in Y_t, x_t \in X_t(y_t)\}$ as sets of all *realizations in period t*. $\Omega_t: D_t \rightarrow Y_{t+1}$ are given *transformations*. We assume that

$$\underset{\substack{t=1,\dots,T \\ y_{t+1}\in Y_{t+1}}}{\forall} \ \underset{\substack{y_t \in Y_t \\ x_t \in X_t(y_t)}}{\forall} \quad y_{t+1}=\Omega_t(y_t,x_t).$$

Let

$$D_t(y_t) \cong \{(y_t,x_t): x_t \in X_t(y_t)\}$$

and

$$\bar{D}_t(y_{t+1}) \cong \{(y_t,x_t): y_t \in Y_t, x_t \in X_t(y_t), y_{t+1}=\Omega_t(y_t,x_t)\}.$$

$d \cong (d_1,\dots,d_T)$ is said to be a *process realization*, if

$$\underset{t=1,\dots,T}{\forall} \quad y_{t+1}=\Omega_t(y_t,x_t).$$

Let D be the *set of all process realizations.*

Let d be a given realization. *Backward sequence* $\{d(y_t): t=T,\dots,1\}$ is defined in the following way:

$$d(y_T) \cong (y_T,x_T)$$

$$d(y_{T-1}) \cong (y_{T-1},x_{T-1},y_T,x_T)$$

$$\dots\dots\dots\dots\dots\dots$$

$$d(y_1) \cong (y_1,x_1,\dots,y_T,x_T).$$

Forward sequence $\{\bar{d}(y_{t+1}): t=1,\dots,T\}$ can be defined as follows:

$$\bar{d}(y_2) \cong (y_1,x_1)$$

$$\bar{d}(y_3) \cong (y_1,x_1,y_2,x_2)$$

$$\dots\dots\dots\dots\dots\dots$$

$$\bar{d}(y_{T+1}) = (y_1,x_1,\dots,y_T,x_T).$$

Discrete decision process **P** *is given*, if sets $Y_t, X_t(y_t)$ and functions Ω_t are identified. Process $\mathbf{P}(y_t)$ is a *backward partial process*, if it is a subprocess of **P** and it starts in y_t. *Process* $\bar{\mathbf{P}}(y_{t+1})$ is a *forward partial process*, if it is a subprocess of **P** and it ends in y_{t+1}.

In each period t there are defined K *period functions* $F_t^k: D_t \to R$ $(k=1,\dots,K)$. For a given realization d we obtain values:

$$F_1^1(y_1,x_1),\ F_2^1(y_2,x_2),\dots,F_T^1(y_T,x_T)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$F_1^K(y_1,x_1),\ F_2^K(y_2,x_2),\dots,F_T^K(y_T,x_T)$$

F is M-dimensional (M≥2) *criterion function* for the whole process with components

$$F^m \cong \Phi^m(F_1^1, F_2^1,\dots,F_T^1,\dots,F_1^K,\ F_2^K,\dots,F_T^K)$$

for m=1,..., M. Let $F \cong [F^1,\dots,F^M]'$.

A *multi-criteria discrete process* (**P**,F) *is given*, if there are defined: multi-period decision process **P** and vector-valued criterion function F. In the further considerations we assume, that a multi-criteria decision process (**P**,F) is given.

Assume that there are given two realizations: $\bar{d}$, $\tilde{d}$ and vectors

$$F(\bar{d}) \cong [F^1(\bar{d}),\dots,F^M(\bar{d})]'$$
$$F(\tilde{d}) \cong [F^1(\tilde{d}),\dots,F^L(\tilde{d})]'$$

Domination relation $\underline{>}$ is defined as follows:

$$F(\bar{d}) \underline{>} F(\tilde{d}) \iff \forall_{l=1,\dots,M}\ F^m(\bar{d}) \geq F^m(\tilde{d}) \text{ and } \exists_{i=1,\dots,M}\ F^i(\bar{d}) > F^i(\tilde{d})$$

If $F(\bar{d}) \underline{>} F(\tilde{d})$, $\bar{d}$ *dominates* $\tilde{d}$. $\overset{*}{d}$ is *efficient*, if

$$\sim \exists_{\tilde{d} \in D}\ F(\tilde{d}) \underline{>} F(\overset{*}{d})$$

$\overset{*}{D}$ is the *set of all efficient realizations.*

3. Separability

3.1 Backward separability

F^m is *backward separable* , if there exist functions $f_t^m(F_t^1,\dots,F_t^m)$ and operators $\overleftarrow{\square}_t^m$ such that condition

$$F^m=f_1^m\,\overleftarrow{\square}_1^m(f_2^m\,\overleftarrow{\square}_2^m(\dots(f_{T-1}^m\overleftarrow{\square}_{T-1}^m f_T^m)\dots))$$

is fulfilled. The last function from sequence $\{g_t^m\}$, where $g_T^m\cong f_T^m$ and $g_t^m\cong f_t^m\overleftarrow{\square}_t^m g_{t+1}^m$ fulfills condition $g_1^m=F^m$. F is *backward separable*, if all F^m are backward separable.

Let

$$F_t\cong[f_t^1,\dots,f_t^M]'$$

$$G_t\cong[g_t^1,\dots,g_t^M]'$$

$$\overleftarrow{\square}_t\cong[\overleftarrow{\square}_t^1,\dots,\overleftarrow{\square}_t^M]'$$

Then

$$F=F_1\overleftarrow{\square}_1(F_2\overleftarrow{\square}_2(\dots(F_{T-1}\overleftarrow{\square}_{T-1}F_T)\dots))$$

$$G_T=F_T,\quad G_t=F_t\overleftarrow{\square}_t G_{t+1},\quad G_1=F.$$

3.2 Forward separability

F^m is *forward separable*, if there exist functions $f_t^m(F_t^1,\dots,F_t^k)$ and operators $\overrightarrow{\square}_t^m$ such that condition

$$F^m=((\ \dots(f_1^m\overrightarrow{\square}_1^m f_2^m)\dots)\overrightarrow{\square}_{T-2}^m f_{T-1}^m)\overrightarrow{\square}_{T-1}^m f_T^m)$$

is fulfilled. The last function from sequence $\{\bar{g}_t^m\}$, where $\bar{g}_1\cong f_1^m$ and $\bar{g}_{t+1}^m\cong g_t^m\overrightarrow{\square}_t^m f_t^m$ fulfills condition $\bar{g}_T^m=F^m$. F is *forward separable*, if all F^m are forward separable.

Let

$$\bar{G}_t\cong[\bar{g}_t^1,\dots,\bar{g}_t^M]'.$$

Then

$$F=((\dots(F_1\overrightarrow{\square}_1F_2)\dots)\overrightarrow{\square}_{T-2}F_{T-1})\overrightarrow{\square}_{T-1}F_T,$$

$$\bar{G}_1=F_1,\quad \bar{G}_t=\bar{G}_{t-1}\overrightarrow{\square}_tF_t,\quad \bar{G}_T=F.$$

4. Monotonicity

4.1 Backward monotonicity

If B is a set of such elements, that $\forall b \in B$ operation $a \overset{\Leftarrow}{\square}_t b$ is feasible, then

$$a \overset{\Leftarrow}{\blacksquare}_t B \cong \{c: \underset{b \in B}{\exists} c = a \overset{\Leftarrow}{\square}_t b\}$$

and

$$\overset{\Leftarrow}{\blacksquare}_t \cong [\overset{\Leftarrow 1}{\blacksquare}_t, \ldots, \overset{\Leftarrow M}{\blacksquare}_t]'.$$

Assume that F is backward separable. *Attainable sets* $W(y_t)$ are defined as follows:

$$W(y_T) \cong \bigcup_{(y_T, x_T) \in D_T(y_T)} F_T(y_T, x_T)$$

$$W(y_t) \cong \bigcup_{(y_t, x_t) \in D_t(y_t)} F_t(y_t, x_t) \overset{\Leftarrow}{\blacksquare}_t W(\Omega_t(y_t, x_t))$$

F is *backward monotone*, if

$$\underset{t=1,\ldots,T-1}{\forall} \quad \underset{\substack{y_t \in Y_t \\ (y_t, x_t) \in D_t(y_t)}}{\forall} \quad \underset{w, \bar{w} \in W(\Omega_t(y_t, x_t))}{\forall} \quad w \geq \bar{w} \Rightarrow \left(F_t(y_t, x_t) \overset{\Leftarrow}{\square}_t w \geq F_t(y_t, x_t) \overset{\Leftarrow}{\square}_t \bar{w}\right)$$

If F is backward separable and monotone, multi-criteria process (**P**,F) is a *multi-criteria backward process* and each partial process (**P**(y_t),G_t) is a *multi-criteria backward partial process.*

4.2 Forward monotonicity

If A is a set of such elements, that $\forall a \in A$ operation $a \overset{\Rightarrow}{\square}_t b$ is feasible, then

$$A \overset{\Rightarrow}{\blacksquare}_t b \cong \{c: \underset{a \in A}{\exists} c = a \overset{\Rightarrow}{\square}_t b\}$$

and

$$\overset{\Rightarrow}{\blacksquare}_t \cong [\overset{\Rightarrow 1}{\blacksquare}_t, \ldots, \overset{\Rightarrow M}{\blacksquare}_t]'.$$

Assume that F is forward separable. *Attainable sets* $\bar{W}(y_t)$ are defined as follows:

$$\bar{W}(y_2) \cong \bigcup_{(y_1, x_1) \in \bar{D}_1(y_2)} F_1(y_1, x_1)$$

$$\bar{W}(y_{t+1}) \cong \bigcup_{(y_t,x_t)\in \bar{D}_t(y_{t+1})} \bar{W}(y_t)\vec{\blacksquare}_{t-1}F(y_t,x_t)$$

F is *forward monotone*, if

$$\forall_{t=2,\ldots,T}\ \ \forall_{\substack{y_{t+1}\in Y_{t+1}\\(y_t,x_t)\in \bar{D}_t(y_{t+1})}}\ \ \forall_{w,\bar{w}\in \bar{W}(y_{t+1}}\ \ w\geq\bar{w} \Rightarrow$$

$$\left(w\vec{\square}_{t-1}F_t(y_t,x_t)\geq\bar{w}\vec{\square}_{t-1}F_t(y_t,x_t)\right)$$

If F is forward separable and monotone, *multi-criteria* process (**P**,F) is a *multi-criteria forward process* and each partial process $(\bar{\mathbf{P}}(y_{t+1}),\bar{G}_t)$ is a *multi-criteria forward process.*

5. Optimality equations

5.1 Backward process

We consider a backward process (**P**,F). Let $D(y_t)$ denote the set of all realizations of the multi-criteria partial process $(\mathbf{P}(y_t),G_t)$.

Theorem 1. (backward Principle of Optimality)

If $\overset{*}{d}\cong(\overset{*}{y}_1,\overset{*}{x}_1,\ldots,\overset{*}{y}_T,\overset{*}{x}_T)\in\overset{*}{D}$, then condition

$$\sim \exists_{\substack{t=1,\ldots,T\\ \tilde{d}(\overset{*}{y}_t)\in D(\overset{*}{y}_t)}} G_t(\tilde{d}(\overset{*}{y}_t))\geq G_t(\overset{*}{d}(\overset{*}{y}_t))$$

is fulfilled.

It results from *Theorem 1*, that if $\overset{*}{d}$ is efficient, all elements of its backward sequence are efficient realizations of the appropriate partial backward processes.

For each subset $A\subset\mathbb{R}^M$ let

$$\text{"max"}A\cong\{\overset{*}{a}\in A:\sim\exists_{\tilde{a}\in A}\ \tilde{a}\geq\overset{*}{a}\}.$$

We define *Pareto-optimal multifunctions*

$$\overset{*}{G}_t(y_t)\cong\text{"max"}\{G_t(d(y_t)):d(y_t)\in D(y_t)\}$$

Those multifunctions connect each state y_t with the set of all Pareto-optimal vectors of the backward partial process $(\mathbf{P}(y_t), G_t)$. We denote

$$H(y_T) \cong \bigcup_{(y_T, x_T) \in D(y_T)} F_T(y_T, x_T)$$

Theorem 2. $\forall_{y_T \in Y_T} \overset{*}{G}_T(y_T) = \text{"max"} H(y_T)$.

Let

$$H(y_t) \cong \bigcup_{(y_t, x_t) \in D_t(y_t)} F_t(y_t, x_t) \overset{\Leftarrow}{\blacksquare}_t \overset{*}{G}_{t+1}(\Omega_t(y_t, x_t))$$

Theorem 3. $\forall_{t=T-1,\dots,1} \forall_{y_t \in Y_t} \overset{*}{G}_t(y_t) = \text{"max"} H(y_t)$.

Let $W \cong F(D)$, $\overset{*}{W}$ is the set of all Pareto-optimal vectors and

$$H \cong \bigcup_{y_1 \in Y_1} \overset{*}{G}_1(y_1)$$

Theorem 4. $\overset{*}{W} = \text{"max"} H$.

5.2 Forward process

We consider a forward process $(\mathbf{P}, F)$.

Let $\bar{D}(y_{t+1})$ denote the set of all realizations of the multi-criteria partial process $(\bar{\mathbf{P}}(y_{t+1}), \bar{G}_t)$.

Theorem 5. (Forward Principle of Optimality)

If $\overset{*}{d} \in \overset{*}{D}$, then condition

$$\sim \exists_{\substack{t=1,\dots,T \\ \tilde{d}(\overset{*}{y}_{t+1}) \in \bar{D}(\overset{*}{y}_{t+1})}} G_t(\tilde{d}(\overset{*}{y}_{t+1})) \geq G_t(\bar{\overset{*}{d}}(\overset{*}{y}_{t+1}))$$

is fulfilled.

It results from *Theorem 4*, that if $\overset{*}{d}$ is efficient, all elements of its forward sequence are efficient realizations of the appropriate partial backward processes.

We define *Pareto-optimal multifunctions*

$$\overset{*}{\bar{G}}_t(y_{t+1}) \cong \text{"max"}\{\bar{G}_t(\bar{d}(y_{t+1})) : \bar{d}(y_{t+1}) \in \bar{D}(y_{t+1})\}$$

That multifunctions connect each state y_{t+1} with the set of all Pareto-optimal vectors of the partial forward process $(\bar{P}(y_{t+1}), F)$. We denote

$$\bar{H}(y_2) \cong \bigcup_{(x_1,y_1)\in\bar{D}_1(y_2)} F_1(y_1,x_1)$$

Theorem 6. $\forall_{y_2\in Y_2}\ \overset{*}{\bar{G}}_1(y_2) = \text{"max"}\ \bar{H}(y_2)$.

Let

$$\bar{H}(y_{t+1}) \cong \bigcup_{(y_t,x_t)\in\bar{D}_t(y_{t+1})} \overset{*}{\bar{G}}_{t-1}(y_t) \vec{\blacksquare}_{t-1} F_t(y_t,x_t)$$

Theorem 7. $\forall_{t=2,\dots,T}\ \forall_{y_{t+1}\in Y_{t+1}}\ \overset{*}{\bar{G}}_t(y_{t+1}) = \text{"max"}\bar{H}(y_{t+1})$.

Let

$$\bar{H} \cong \bigcup_{y_{T+1}\in Y_{T+1}} \overset{*}{\bar{G}}_T(y_{T+1}).$$

Theorem 8. $\overset{*}{W} = \text{"max"}\bar{H}$.

6. Numerical procedures and illustrative example

The *backward procedure* based on Theorems 1-4 consists of two stages. In stage 1 the set of all Pareto-optimal values $\overset{*}{W}$ is found. This stage is divided into 3 steps. In step 1 we compute sets $\overset{*}{G}_T(y_T)$ for all states $y_T \in Y_T$. In step 2 for t=T-1,...,1 we compute sets $\overset{*}{G}_t(y_t)$ for all states $y_t \in Y_t$. In step 3 we find set $\overset{*}{W}$ = "max"H. In stage 2 the set of all efficient realizations $\overset{*}{D}$ is obtained.

The *forward procedure* based on Theorems 5-8 consists also of two stages. In stage 1 set $\overset{*}{W}$ is found. This stage is divided into 3 steps. In step 1 we compute sets $\overset{*}{\bar{G}}_1(y_1)$ for all states $y_1 \in Y_1$. In step 2 for t=2,...,T we compute sets $\bar{G}^*_t(y_t)$ for all states $y_t \in Y_t$. In step 3 we find set $\overset{*}{W}$ = "max"$\bar{H}$. In stage 2 the set of all efficient realizations $\overset{*}{D}$ is obtained.

Example

Let us consider a 3-period process. Assume, that $Y_1 = Y_2 = Y_3 = Y_4 = \{0,1\}$, $X_t(y_t) = \{0,1\}$, $\Omega_t(y_t,0) = 0$ and $\Omega_t(y_t,1) = 1$ for $t=1,2,3$, $y_t \in Y_t$. The values of period criteria functions are as follows:

$$
\begin{array}{llll}
F_1^1(0,0) = 7, & F_1^1(0,1) = 6, & F_1^1(1,0) = 3, & F_1^1(1,1) = 4 \\
F_1^2(0,0) = 5, & F_1^2(0,1) = 6, & F_1^2(1,0) = 9, & F_1^2(1,1) = 10 \\
F_2^1(0,0) = 6, & F_2^1(0,1) = 3, & F_2^1(1,0) = 7, & F_2^1(1,1) = 7 \\
F_2^2(0,0) = 8, & F_2^2(0,1) = 9, & F_2^2(1,0) = 8, & F_2^2(1,1) = 4 \\
F_3^1(0,0) = 9, & F_3^1(0,1) = 8, & F_3^1(1,0) = 2, & F_3^1(1,1) = 3 \\
F_3^1(0,0) = 7, & F_3^1(0,1) = 6, & F_3^1(1,0) = 3, & F_3^1(1,1) = 4
\end{array}
$$

The vector criterion function has the form $F = [F^1, F^2]'$ and its components F^1 and F^2 are defined as follows:

$$F^1 = \sum_{t=1}^{3} \left| F_t^1 - F_t^2 \right|, \qquad F^2 = \sum_{t=1}^{3} F_t^1 F_t^2$$

It is easy to show that function F is backward and forward separable. Function F is also forward and backward monotone. It means that we can apply both the backward and the forward procedure.

Applying the backward procedure we obtain:

t	$\overset{*}{G}$ (0)		$\overset{*}{G}$ (1)	
3	[5,36]'		[8,20]'	
2	[7,84]',	[14,47]	[6,92]',	[11,48]'
1	[9,119]',	[16,82]'	[13,111]',	[20,74]'
	[6,128]',	[11,84]'	[12,132]',	[17,88]'

$\overset{*}{W} = \{[13,111]', [20,74]', [12,132]', [17,88]'\}$

$\overset{*}{D} = \{(1,0,\ 0,0,\ 0,0),\ 1,0,\ 0,1,\ 1,0),\ (1,1,\ 1,0,\ 0,0),$
$(1,1,\ 1,1,\ 1,0)\}$

Applying the forward procedure we obtain:

t	$\bar{G}_t(0)$	$\bar{G}_t(1)$
1	[2,35]', [6,27]'	[6,40]'
2	[4,83]', [8,75]' [7,96]'	[8,62]', [12,54]' [9,68]'
3	[13,111]', [12,132]' [20,174]', [17,88]'	[9,107]', [13,99]' [12,120]'

7. Final remarks

The paper gives a view on an important class of MCDP problems, which can be defined as backward or forward processes. The definitions of these processes employ concepts of backward and forward separability and monotonicity, which have been defined in the paper. These definitions are more general than concepts given in previous works. On the basis of introduced notions and presented theorems it is possible to construct numerical algorithms which allows to generate the set $\overset{*}{W}$ of Pareto-optimal values of the criterion function and the set $\overset{*}{D}$ of efficient realizations of the considered process.

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MCDM and Models of Voting Decision Making[1]

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Abstract: A rational voter's behaviour can be analysed within the framework of a standard consumer behaviour pattern: voting for candidate(s), the voter is "buying" a policy represented by candidates and maximizing his "utility" subject to a "budget constraint" given by a used voting procedure. In this paper we formulate a model of an individual voter behaviour in the election of a committee, based on so called voting constraint and a utility function, defined over a set of feasible voting strategies. A class of utility functions is suggested on the basis of a multi-criteria optimization problem, in which the voter minimizes a distance between his ranking of the major political issues and the aggregate ranking of these issues by the candidates, generated by the voter's voting portfolio selected from the set of feasible voting strategies. Using goal programming techniques optimal voting portfolio can be calculated. A game theoretical analysis of the problems associated with voting decision making is suggested.

Keywords: goal programming, feasible voting strategy, multiple-vote systems, optimal voting strategy, portfolio voting, voting constraint, voting game, voting procedure

1 Introduction

By voting we usually mean the following pattern of collective choice: There is a set of alternatives and a group of individuals. Individual preferences over the alternatives are exogenously specified and are supposed to be orderings. The group is required to choose an alternative (or a subset of the set of alternatives) on the basis of the stating and aggregating of all individual preferences, or to produce a ranking of alternatives from the most preferred to the least preferred.

We should distinguish between a voting method and a counting method. While

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a voting method says how an individual choice is presented by the voter, a counting method says how the indiviual choices are to be aggregated. The selected voting method together with an appropriate counting method define a voting procedure.

There are basically two groups of voting methods: a) Ranking methods, when voting is done by an explicit or implicit presentation of a ranking of alternatives. b) Multiple-vote methods, when voting is done by distributing more votes among the candidates. The most commonly used single-vote method, by which voting is done by presenting a single vote for one and only one selected alternative, may be considered as a special case of a multiple-vote method.

While many different sophisticated counting methods can be employed together with ranking voting systems (Condorcet's voting procedures, Hare, Coombs, Dodgson, Copeland etc.), the counting method for various multiple-vote systems (such as approval voting, cumulative voting, Borda's voting, interval voting) is simple: The votes for different alternatives are summed up and a resulting ranking is given by the total number of votes for each alternative.

Properties of voting procedures based on ranking have been presented extensively in many books and papers on public choice theory, (see e.g. DUMMET [4], NURMI [8], STRAFFIN [10], in a framework of multicriteria-decision making HWANG and LIN [5]).

In this paper we focus our attention on multiple-vote procedures that provide the possibility to use optimization models of voting behaviour and to consider a multi-attribute character of alternatives leading to multi-criterial decision making of a rational voter. We are trying also to look for parallels between rational consumer choice theory and rational voter decision making.

2 Feasible Voting Strategies

Voting is considered to be multiple criteria decision-making whenever a voter casts votes to select candidates or alternative policies. A rational voter's behaviour can be viewed within the framework of a standard consumer behaviour pattern: voting for candidate(s), he is "buying" a policy represented by candidates maximizing his "utility" subject to a "budget constraint" given by the used voting procedure.

To formulate a model of a single voter's behaviour we shall start with a replica of the consumer's budget constraint, which we shall call a *voting constraint.*

Let m be the number of candidates (i = 1, 2, ..., m) and v be the number of votes the voter can cast. Let x_i be the number of votes the voter gives to the i-th candidate. In general x_i may be any non-negative number. A vector

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$$

permissible under a particular voting system will be called a voter's feasible strategy. In general, voting means to select a feasible strategy from the set

$$X = \{x \in \mathbb{R}_m, \sum_{i=1}^{m} x_i \leq v, x_i \geq 0, x \in D\}$$

where D is an additional requirement on the votes reflecting properties of a specific voting system. The algebraic system

$$\sum_{i=1}^{m} x_i \leq v, x_i \geq 0, x \in D$$

defining *feasible voting strategies* we shall term a voting constraint.

The following examples illustrate different voting constraints (feasible sets) for several well-defined voting procedures:

a) Single-vote plurality:

$$X_s = \{x \in \mathbb{R}_m, \sum_{i=1}^{m} x_i \leq 1, x_i \in \{0,1\}\}$$

is the set of a voter's feasible strategies under a single-vote plurality system (most frequently used standard voting system, the voter has only one vote no matter how many candidates offer themselves for his choice).

b) Approval voting:

$$X_A = \{x \in \mathbb{R}_m, \sum_{i=1}^{m} x_i \leq m, x \in \{0,1\}\}$$

is the set of a voter's feasible strategies under approval voting (the voter casts one vote for each candidate he approves, BRAMS and FISHBURN [2]).

c) Cumulative voting:

$$X_c = \{x \in \mathbb{R}_m, \sum_{i=1}^{m} x_i \leq k, x_i \geq 0, x_i - \textit{integer}\}$$

is the set of a voter's feasible strategies under cumulative voting, where k is the size of the committee (the voter has as many votes as the number of seats to be filled, and is allowed to divide them as he pleases, perhaps giving them all to one candidate or distributing them among more candidates, MERRILL [7]).

d) Borda voting:

$$X_B = \{x \in \mathbb{R}_m, \sum_{i=1}^{m} x_i = m^2 - \sum_{i=1}^{m} i, x \in \mathcal{P}\}$$

where $\mathcal{P}$ is the set of all permutations of

$$(m-1,m-2,...,1,0)$$

is the set of a voter's feasible strategies under Borda voting (each voter is permitted to assign weights - integers m-1, m-2, ..., 0 in a one-to-one fashion to the m candidates, BORDA [1]).

e) Interval voting:

$$X_I = \{x \in \mathbb{R}_m, 0 \le x_i \le M\}$$

where M is a positive constant, is the set of feasible strategies under interval voting (each voter is asked to rate candidates on a scale from 0 to M, e.g., from 0 to 100, see e.g., JOSLYN [6], RIKER [9]).

3 Optimal Voting Strategies

From the point of view of an individual voter, a voting decision (under some particular voting system) means to select and state or submit exactly one feasible voting strategy from the corresponding feasible set, given by the voting constraint.

We shall suppose that the voter is a rational agent that uses his resources (given by the voting constraint) in the "best" possible way: he selects one of his feasible strategies which is "most preferred" by him in some sense.

Usually it is supposed that each rational voter has a well-behaved preference relation on the set of candidates, which is at least complete, reflexive and transitive (weak ordering), sometimes antisymmetry is also required (strong ordering). However the process of discovering this preference relation remains unclear.

Clearly the candidates' qualifications may be judged by multiple criteria, such as trustworthiness and/or honesty, capabilities, general political stance (conservative, moderate, liberal), and positions on specific political issues, evaluated from the standpoint of the voter's interests. These criteria are summarized in the voter's mind, to produce a value (utility) function. Then the voter rates the candidates as to the first choice, second, third, ... etc., based upon the voter's utility function toward the candidates.

Multiple criteria decision making theory provides an appropriate methodology to describe and analyze this rather vague process of forming an individual voter's utility function.

In stating an individual voter's optimal choice problem as a multi-criteria optimization problem, we shall start with several trivial and, in a sense, simplifying assumptions:

a) **Multiple issues**. We shall suppose that there is a list of major political issues $\pi = (\pi_1, \pi_2, \ldots, \pi_t)$ where $t > 1$ and that an individual voter is able to rank the issues by their importance to him. Let us denote the voter's ranking over this list by $\mathbf{r} = (r_1, r_2, \ldots, r_t)$, where r_{ik} is an integer between 0 and t, expressing the position of the k-th issue in the voter's ordering, or a weight (say, between 0 and 100) expressing the relative importance of the issue to the voter, etc.

b) **Observability of candidates' position.** We suppose that each candidate publicly states his own ranking over the list of major political issues. Let us denote the j-th candidate ranking by $\mathbf{z}_j = (z_{j1}, z_{j2}, \ldots, z_{jt})$.

c) **Trustworthiness of the candidates.** We suppose that there is no uncertainty about the future position of the candidates (publicly declared rankings express the true position of the candidates).

Under these assumptions we can state the problem of an individual voter's optimal voting choice as the following multi-criteria mathematical programming problem.:

Let $x \in X$ be a feasible voting strategy (we shall also call it a selected portfolio of candidates), then

$$z_k(\boldsymbol{x}) = \sum_{i=1}^{m} z_{ik} \frac{x_i}{\sum_{j=1}^{m} x_j}$$

is an aggregate ranking of the k-th issue generated by the selected portfolio of candidates and

$$\boldsymbol{z}(\boldsymbol{x}) = \left(z_1(\boldsymbol{x}), z_2(\boldsymbol{x}), \ldots, z_t(\boldsymbol{x})\right)$$

is a corresponding aggregate ordering vector. Thus, it makes sense to expect that the rational voter will select his voting strategy in such a way that the aggregate ranking z(x), generated by selected voting strategy, will be "as close as possible" to his personal ranking vector **r**.

We received a standard multi-criteria optimization problem

$$\text{"}\boldsymbol{min}\text{"} \{\boldsymbol{z}(\boldsymbol{x}) - \boldsymbol{r} \,|\, \boldsymbol{x} \in X\}$$

where "min" means minimization in vector sense.

Using a distance function $d(\mathbf{z},\mathbf{r})$ as a measure of "closeness" we can define a voter's utility function over the set of feasible voting strategies

$$u(\boldsymbol{x}) = C - d(\boldsymbol{z}(\boldsymbol{x}), \boldsymbol{r})$$

where C is a constant (e.g., an ideal distance C = 0). The optimal voting strategy of an individual voter may be defined as a feasible voting strategy, maximizing the utility function u(x) with respect to the feasible set X.

Table 1

	A	B	C	voter
x	3	1	2	2
y	1	2	3	1
z	2	3	1	3

To illustrate the concept of an optimal voting choice let us consider a voting situation with 3 candidates A, B, C and 3 political issues x, y, z, characterized by Table 1 (rankings of the issues by the candidates and by the voter). We shall derive optimal voting strategies for different voting procedures using the absolute value distance function.

In Table 2 we compare aggregate ranking generated by different feasible voting strategies to the individual voter's ranking. Each row of the table gives a feasible voting strategy and the corresponding absolute value distance between aggregated ranking and individual voter's ranking.

The first four rows of Table 2 correspond to the feasible voting strategies under the single-vote procedure. We can see that in this case the optimal voting strategies are (1,0,0) and (0,1,0), i.e., the voter gives his single vote either to the candidate A or to candidate B. By this approach we can also get an individual voter's ordering of the candidates: $A \sim B$, $A \succ C$, $B \succ C$.

The first 8 rows of Table 2 correspond to the feasible voting strategies under the approval voting. The voter's optimal strategy is (1,1,0), i.e., he gives his votes to both candidates A and B.

The last six rows of Table 2 correspond to the feasible voting strategies under the Borda voting procedure. The voter's optimal strategy in this case is (1,2,0), i.e., for the voter it is optimal to give one of his votes vote to A and two votes to B.

Table 2

x	d(z(x)-r)
(0,0,0)	6
(1,0,0)	2
(0,1,0)	2
(0,0,1)	4
(1,1,0)	1
(1,0,1)	3
(0,1,1)	3
(1,1,1)	2
(2,1,0)	3
(1,2,0)	4/3
(0,2,1)	10/3
(0,1,2)	8/3
(2,0,1)	8/3
(1,0,2)	10/3

Table 2 gives the complete list of all discrete feasible strategies (14 strategies) under cumulative voting. The voter's optimal strategy under cumulative voting is in this case, the same as under approval voting: (1,1,0), i.e., for the voter it is optimal to give one vote to candidate A and one vote to candidate B.

4 Portfolio Voting

The optimal choice approach, introduced in the previous section, raises a lot of questions when applied to the choice of exactly one candidate. But it makes more sense when we use it as a model of selection of a committee.

Let us suppose the following proportional voting rule for electing a committee, called a "portfolio voting rule" (see TURNOVEC [11]):

Each voter chooses among party lists. Let n be the number of voters, m be the number of parties, and k be the number of seats in a committee. Each voter has k votes (as many votes as the number of seats). By x_{ij} let us denote a number of seats assigned by the i-th voter to the j-th party. For simplicity, we suppose that all voters take part in the election and use all their votes, then

$$\sum_{j=1}^{m} x_{ij} = k,\ p_{ij} = \frac{1}{k} x_{ij}$$

(for all i = 1,2,...,n and j = 1,2,...,m) and $\mathbf{p}_i = (p_{i1}, p_{i2}, \ldots, p_{im})$ is an i-th *invidual voting portfolio*, or, in our previous terminology, an i-th voter feasible voting strategy. Then $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ such that

$$t_j = \frac{x_j}{nk},\ x_j = \sum_{i=1}^{n} x_{ij}$$

is a *social voting portfolio* according to which the seats are to be distributed in a committee. Let us suppose further that there is a list of major political issues $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_t)$ and denote the i-th voter ranking over this list by $\mathbf{r}_i = (r_{i1}, r_{i2}, \ldots, r_{it})$. Each party declares its own ranking over the issues $\mathbf{z}_j = (z_{j1}, z_{j2}, \ldots, z_{jt})$.

The individual voter's rational behaviour can the be defined by an optimal solution to the following multi-criteria optimization problem:

minimize

$$d(\sum_{j=1}^{m} p_{ij} z_j, r_i) \tag{1}$$

subject to

$$\sum_{j=1}^{m} p_{ij} = 1,\ \ p_{ij} \geq 0 \tag{2}$$

where d is a distance between the i-th individual ranking and the aggregate committee ranking generated by the i-th voter's portfolio. An optimal solution

$$p_i^0 = (p_{i1}^0, p_{i2}^0, \ldots, p_{im}^0)$$

to the problem (1) - (2) we shall call an *i-th individual optimal voting portfolio*.

5 Optimal Voting Portfolio and Goal Programming

Optimal voting portfolio can be calculated using standard linear (or quadratic) programming optimization techniques. Using the absolute value distance

$$d(\sum_{j=1}^{m} p_{ij} z_j, r_i) = \sum_{k=1}^{t} abs(\sum_{j=1}^{m} p_{ij} z_{jk} - r_{ik})$$

the problem (1)-(2) (an i-th individual optimal voting portfolio) will take the form:

minimize

$$\sum_{k=1}^{t} abs\left(\sum_{j=1}^{m} p_{ij} z_{jk} - r_{ik}\right) \tag{3}$$

subject to

$$\sum_{j=1}^{m} p_{ij} = 1, \quad p_{ij} \geq 0 \tag{4}$$

The problem (3)-(4) is a standard goal programming problem (minimization of the sum of absolute value deviations of linear function values from some a priori values under linear constraints), that can be solved as the following linear programming problem (see e.g., CHARNES and COOPER [3]):

minimize

$$\sum_{k=1}^{t} (u_k + v_k) \tag{5}$$

subject to

$$\begin{aligned} \sum_{j=1}^{m} p_{ij} z_{jk} - u_k + v_k &= r_{ik} \\ \sum_{j=1}^{m} p_{ij} &= 1 \\ p_{ij},\ u_k,\ v_k &\geq 0 \end{aligned} \tag{6}$$

Due to the properties of the linear programming simplex method any optimal solution to the problem (5)-(6)

$$(p_{i1}^0, p_{i2}^0, \ldots, p_{im}^0, u_1^0, \ldots, u_t^0, v_1^0, \ldots, v_t^0)$$

gives us an optimal solution to the original minimization problem (3)-(4)

$$(p_{i1}^0, p_{i2}^0, \ldots, p_{im}^0)$$

and

$$\sum_{k=1}^{t} abs\left(\sum_{j=1}^{m} p_{ij}^0 z_{jk} - r_{ik}\right) = \sum_{k=1}^{t} (u_k^0 + v_k^0)$$

For the voting situation and the voter characterized by Table 1, we obtain the

following individual voting portfolio problem:
minimize

$$u_1 + u_2 + u_3 + v_1 + v_2 + v_3$$

subject to

$$3p_1 + p_2 + 2p_3 - u_1 + v_1 = 2$$

$$p_1 + 2p_2 + 3p_3 - u_2 + v_2 = 1$$

$$2p_1 + 3p_2 + p_3 - u_3 + v_3 = 3$$

$$p_1 + p_2 + p_3 = 1$$

$$p_1, p_2, p_3, u_1, u_2, u_3, v_1, v_2, v_3 \geq 0$$

Optimal solution:

$$p_1^0 = \frac{1}{2},\ p_2^0 = \frac{1}{2},\ p_3^0 = 0$$

$$u_1^0 = 0, u_2^0 = \frac{1}{2},\ u_3^0 = 0$$

$$v_1^0 = 0,\ v_2^0 = 0,\ v_3^0 = \frac{1}{2}$$

In this case, an optimal voting portfolio for the voter in a multiple-vote procedure is to distribute equally his votes between two candidates (parties) A and B.

6 Voting as a Game

An obvious criticism of our "consumer type" models of rational voting behaviour follows from the fact that (not like in the consumer utility maximization) the level of satisfaction of an individual voter with the results of voting depends not only on his decision, but also (and to a very great extent) on decisions of many other voters. It ignores strategical aspects of voting. A sophisticated voter is aware of the fact that his individual voting portfolio will certainly differ from a social voting portfolio and his satisfaction should be measured by a distance between his individual ranking and the aggregate committee ranking generated by the social voting portfolio, rather than by the individual voter's portfolio.

The distance between the k-th individual ranking and the aggregate committee ranking generated by a social voting portfolio **t**, where

$$t = \frac{1}{n}\sum_{i=1} p_i, \quad t_j = \frac{1}{n}\sum_{i=1}^{n} p_{ij}$$

can be measured by the k-th voter's distance function

$$\delta_k(p_k, p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n) = d\left(\frac{1}{n}\sum_{j=1}^{m}(p_{kj} + \sum_{i \neq k} p_{ij})z_j, r_k\right) \tag{7}$$

where only the individual portfolio variables p_{kj} are under the control of the k-th voter. Therefore, we can formulate a game of the n-voters with the pay-off functions (7) and the strategy sets

$$X^{(k)} = \{p_k \in \mathbf{R}_m, \sum_{j=1}^{m} p_{kj} = 1, p_{kj} \geq 0\} \tag{8}$$

A game theoretical analysis of the game (7)-(8) from the point of view of an information of the voters about the other voters' preferences and/or about a coalitional cooperation can be fruitful.

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DISTRIBUTED MULTIOBJECTIVE OPTIMIZATION PROBLEMS AND METHODS FOR THEIR SOLUTION.

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Abstract. Methods for solving distributed multiobjective problems for interrelated linear mathematical models with own linear criterion function are considered in the paper. A general procedure of constructing distributed methods of searching solutions for interrelated multicriterion mathematical programming based on solving a minimax problem formed with respect to a set of criterion function is discussed. The procedure is named distributed as it admits parallel and asynchronous execution of actions for seaching local solutions with next coordination to obtain a solution of the general problem.

Keywords. Multiobjective optimization problems , distributed methods, interrelated multicriterion linear programming.

1 Introduction

Investigation of general-purpose integrated computer-aided control systems necessitates solving the problems of integration of functional interaction between their structural units. For example, a corporation which amalgamates enterprises of different industries that are located in different parts of the world can be represented in the form of a multilevel system. The upper-level subsystem manages the corporation as a whole in the following lines: finances, laws, resources, manpower; these functions are performed by the president of the corporation. The next level can be represented by subsystems of managers representing the interests of the corporation in Japan, Europe, South America which exercise the control over the following lines: production, marketing, personnel, product distribution. Subsystems of the next level are represented by enterprise managers which exercise the control over the following: personnel, financial settlements, industrial relations, production planning and routine management, quality control, etc. Taking into account that in each subsystem for various sides of activities there is solved the problem of decision making on the choice of a particular control on the basis of formalized approaches, the need for decentralization of the decision-making process itself in such systems is apparent due to a large number of reasons, viz.: first, a large dimensionality of the problem as a whole, second, the physical space isolation of subsystems and their structural units, which results in dispersion of the means supporting the decision-making process for them, third, the inclusion of criteria interests of decision-making process for them, third, the inclusion of criteria interests of decision-making problems in individual subsystems and their structural units and opinions of individual specialists responsible for the quality of decisions made. All this generates the need for the development of algorithms for solving interrelated multiobjective optimization problems through the distributed obtaining of particular solutions of individual problems with their subsequent synchronization and coordination to obtain a final solution of the general problem.

2 Problem Statement

We consider the interrelated totality $I = \{1,\ldots,m\}$ of solutions choice problems in such a system ignoring their physical essence, i.e. what subsystems and functional departments they belong to for the choice of the interrelated solutions. We assume that solution choice problems are described by linear models. Denote the q-th problem by Z^q and describe it in the following way: it is necessary to choose the solution x^q in the q-th problem which maximizes the criterion function:

$$f^q\left(x^q\right) = c^q x^q \tag{1}$$

under constraints:

$$A^q x^q \le b^q \quad \ldots \quad (2),$$

$$x^q \in \prod\nolimits_X^q \quad (3),$$

$$G^q x^q \le P^q u^q + d^q \quad (4),$$

$$u_q \le S^q x^q \quad (5),$$

where $x^q = \left\{x_p^q, p = 1,n_q\right\}$ is a row vector of solution of the q-th problem of dimensionality n_q;
$u^q = \left\{u_k^q, \forall k \in I^q\right\} = \left\{u_k^{qr}, r = 1,\ldots,r_k^q, \forall k \in I^q\right\}$ is a column vector of connecting variables of the q-th problem with other problems on the set $I^q \in I$ (I^q is a set of indices of the problems whose solutions influence the q-th the problem due to changes of its domain of feasible solutions by constraints (4));

$u_q = \left\{u_q^j, \forall j \in I_q\right\} = \left\{u_q^{jr}, r = 1,\ldots,r_q^j\right\}$ is a column vector of the influence of the q-th problem on other problems of the set $I_q \in I$ (I_q is a set of problem indices which are influenced by solution of the q-th problem); A^q, G^q, P^q, S^q are matrices of corresponding dimensions,

$c^q = \left\{c_h^q, p = 1,\ldots,n_q\right\}$ is a row vector of coefficients of the criterion function of the q-th problem; b^q, d^q are column vectors of right-hand sides of constraints of the corresponding dimensions;

$\prod_X^q = \prod_{p=1}^{n_q}\left[d_{p(\imath)}^q, d_{p(\hat{a})}^q\right]$ is a hyperparallelepiped of the change of a local variable defining the solution of the q-th problem; $d_{p(\imath)}^q, d_{p(\hat{a})}^q$ are the lower and upper bounds of the change of the p-th component of solution x^q, respectively. Moreover, for any pair of interrelated problems $(k, q) \in I, k \neq q$, one of the following relations can be satisfied: $k \in I^q$ and $q \in I_k$ indicate the mutual influence of k-th and q-th problems; $k \in I^q$, $q \notin I_k$ indicates the influence of the k-th problem on the q-th one, and $q \in I_k, k \notin I^q$ indicates the influence of solution of the q-th problem on the k-th one. We denote the above mentioned subsystem interrelation by the binary relation of the $L^q = \{(k, j): \forall(k, j) \in I, k \neq j; k \in I_q,$ $j \in I_k \vee k \in I_q, j \notin I^k \vee j \in I_k, k \notin I^q\}$.

For simplicity we denote the domain described by relations (2),(3) by D_0^q which describes own possibilities of the solution-choice problem, and the domain described by relation (4) $D_{(X)}^q(u^q)$ which describes the domain of change of solutions of the q-th problem formed by the vector of the influence of other problems on the q-th problem; the domain described by relation (5) is denoted by $D_{(u)}^q(x^q)$, which describes the domain of changes of solutions of the q-th problem which form the vector of the influence of the set I_q on other problems; if $u_q = S^q x^q$ - relations (4) are considered as the rules of generation of the influence of the problem solution on problems of the set I_q related to it. The domain described by relations (2)-(5) is denoted by $D^q = D^q(x^q, u^q, u_q) = \left\{(x^q, u^q, u_q): x^q \in D_0^q, x^q \in D_{(x)}^q(u^q), u_q \in D_{(u)}^q(x^q)\right\}$ and is called the domain of locally feasible solutions of the q-th problem under the given values of the vector of connecting variables u^q.

By virtue of interrelation of domains D^q with respect to variables u^q and u_q, $\forall q \in I$, the problem of the choice of solutions in the above-described system of interrelated problems (1)-(5) is reduced to solution of the multiobjective optimization problem formulated on the basis of sets of criterion functions (1) of individual problems on the domain of feasible solutions defined by the domain $D_0 = \bigcap_{q \in I} D^q(x^q, u^q, u_q)$ with regard to specific features of vectors connecting the variables u_q and u^q . To solve the multiobjective optimization problems according to the approach described in [1] it is necessary to find the best and the worst values of all criteria $f^q, q \in I$ on the domain D_0 of feasible solutions. We shall not dwell here on the algorithms for solving such problems in the distributed formulation and refer the reader to references [2-3] but consider distributed algorithms for solving multiobjective optimization problems on the domain D_0 .

3 Methods for Solving Distributed Multiobjective Problems

Prior to considering the algorithm for solving the distributed multiobjective optimization problem, to a solution of which the decision-making problem is reduced in the computer-aided integrated systems is reduced,we consider the formulation of such problem for interrelated problems in non-distributed statement. In this case the multiobjective problem will be formulated as follows:

$$f = \left\{ f^q(x^q) = c^q x^q, \forall q \in I \right\}$$

under constraints:

$$x^q, u^q, u_q, (\forall q \in I) \in D_0 = \bigcap_{q \in I} D^q(x^q, u^q, u_q).$$

We assume that for all problems there are the worst and the best values of criterion functions on the domain D_0 , and in this case the solution-choice problem in systems of that class will be reduced to solving the minimax problem [1] of the following form:

$$\min_{x^q, (\forall q \in I) \in D_0} \max_{q \in I} \rho_q \frac{f_q^0 - c_q x^q}{f_q^0 - f_{q(min)}} . \tag{6}$$

Here f_q^0 and $f_{q(\min)}$ are, respectivelly, the maximum and the minimum values of criterion $f^q(x^q)$ attained on the domain $D_0; \rho_q$ - are weight coefficients which take into account the preference of criteria of individual problems defined in the expert manner $(\rho_q > 0, \forall q \in I, \sum_{q \in I} \rho_q = 1)$.

Solution of problem (6) can be obtained by using the approach given in [1] and problem (6) can be reduced to solution of the following problem:

$$\min_{x^q, \forall q \in I, k_0} k_0 \tag{7}$$

under constraints:

$$\begin{cases} x^q, u^q, u_q, (\forall q \in I) \in D_0, \\ c_q x^q \geq f_q^0 - \dfrac{k_0}{\rho_q}(f_q^0 - f_{q(min)}), \forall q \in I, \end{cases} \tag{8}$$

where k_0 - some parameter of the problem. As far as in the accepted description of the domain D_0 all relations have a linear character, criterion functions also linear, to solve the problem (7), (8) it is possible to apply the known methods of solving linear programming problems. However, in this case the decision-making problem in the integrated system is reduced to solution of a single problem simultaneously on the whole vector of solutions $x = \{x^q, \forall q \in I\}$. This requires the concentration of all information about models describing their domains of feasible solutions in one place and does not take into account essential physical isolations of individual decision-making problems despite their mutual influence. To substantiate the distributed approach to solutions of the problem (7), (8) we introduce some designations and definitions.

Formalization of the distributed optimization problem in each subsystem is based on the representation of connecting variables u^q and u_q in the disagreed form; by this is meant one and the same connecting variable, but defined from conditions of different problems. So, when considering the q-th subsystem, the connecting variable $u^q = \{u_k^q, \forall k \in I^q\}$ defined in the subsystem, proceeding from its domain of feasible solutions, characterised by relations (2) - (4), assuming u^q not to be specified from other subsystems, and its criterion function of the form (1), will be denoted by $v^q = \{v_k^q, \forall k \in I^q\}$ and interpreted as the "desired" value of the connecting variables for the q-th subsystem. The same variable, defined in each of q-th subsystems of set I^q, proceeding from their domains of feasible solutions at the desired values of connecting variables will be denoted by u_k^q interpeted as the"own" value connecting variables for the k-th subsystem. Thus relation (5), defined from constraints (2) - (4) at chosen variable x^q and u^q will always define the"own" values of output connecting variables $u_q = \{u_q^j, \forall j \in I^q\}$, and the value of the chosen here variables of the kind u^q will in this case define the desired for the q-th subsystem input variables from other subsystems $v^q = \left\{v_k^q, \forall k \in I^q\right\}$. Taking into account the above-mentioned, the domain of feasible solutions $\mathrm{D}^q = \mathrm{D}^q(x^q, v^q, u_q) = \left\{(x^q, v^q, u_q); x^q \in \mathrm{D}_0^q, x^q \in \mathrm{D}_x^q(v^q),\right.$ $\left. u_q \in \mathrm{D}_u^q(x^q)\right\}$ will be called the domain of disagreed feasible solutions of the q-th problem. We understand that one and the same vector of connecting variables in relation (4) assumes values $v^q = \left\{v_k^q, \forall k \in I^q\right\}$ such that $\exists r \in 1,\ldots,r_k^q, \exists k \in I^q, v_k^{qr} \neq u_k^{qr}$. We shall call the vector $y^q = (x^q, v^q, u_q) \in \mathrm{D}^q$ locally feasible disagreed solution for the q-th problem, we shall call the domain $\mathrm{D} = \prod_{q \in I} \mathrm{D}^q$ the disagreed domain of feasible solutions of decision-making problem in the system as a whole which will be denoted as the problem Z consisting of a totality of interrelated problems $Z^q, q \in I$ of the (1)-(5) form; the vector $y = \left\{y^q = (x^q, v^q, u_q), \forall q \in I\right\} \in \mathrm{D}$ will be called the disagreed feasible solution of the problem Z of search for solutions in the system as a whole.

Thus the problem $Z^q, \forall q \in I$, will be considered on the direct product of spaces of the vector of local solutions x^q,connecting desired variable v^q and connecting own variable u_q, i.e. in the space of vectors $Y^q = X^q \times V^q \times U_q$, and the problem Z is determined in the space of vectors $Y = \prod_{q \in I} Y^q = \prod_{q \in I} (X^q \times V^q \times U_q)$. Such representation of the models of problems $Z^q, \forall q \in I$, and problem Z permits completely to take into account physical isolation of subsystems to which these problems belong, and by that to disperse the information description of models describing the domains of feasible solutions and criteria, which are taken into account when choosing a solution in each subsystem; this makes the problem Z essentially distributed, with understanding that its information is dispersed in different places.

The domain of feasible solutions of the problem Z can be represented as :

$$\mathrm{D}_0 = \mathrm{D} \cap \mathrm{D}_c, \tag{9}$$

where D - is the disagreed domain of the problem Z, and D_c is the domain of coordination characterised by agreed connecting variables, i.e.

$D_c = \Big\{ Y = \big\{ y^q = (x^q, v^q, u_q), \forall q \in I \big\} : (v_k^{qr} = u_k^{qr}, \forall r = 1, \ldots, r_k^q, \forall (k,q) \in I,$ $k \neq q, \forall (k,q) \in L \Big\}$. Taking into account a structure of the domain D, the domain of the feasible solutions of the problem Z can be represented as follows $D_0 = \prod_{q \in I} D^q \cap D_c^q$, where D^q - is the disagreed domain of q-th problem and D_c^q - is the domain of the coordination of the problem, i.e.:

$D_c^q = \Big\{ y^q = (x^q, v^q, u_q), v_k^{qr} = u_k^{qr}, \forall r = 1, \ldots, r_k^q, k \in I^q, u_q^{kr} = v_q^{kr},$ $\forall r = 1, \ldots, r_q^k, \forall k \in I_q \Big\}$. Such representation of the domain D_0 of admissible solutions in the form (9) due to the artificially extended space Y ,enables to obtain the representation about ideological orientation of the distributed approach to solution of interrelated problems. By virtue of the absence of the direct interrelation of domains D^q of the q-th problem , $\forall q \in I$, in the disagreed domain D of the problem Z in each of the problems at some step (s) a local feasible solution $y^{q(s)}, \forall q \in I$, is searched for and a disagreed feasible solution of the problem Z . $Y^{(s)} = \big\{ y^{q(s)}, \forall q \in I \big\}$ is constructed on which it is possible to introduce some measure of disagreement $\delta(y^{(s)})$ characterising a "proximity", a "degree of belonging" of the vector $y^{(s)}$ to the domain of coordination D_c and consequently proximity to the domain of of feasible solutions D_0 of the problem Z. Here the continuous function $\delta(y)$ such that $\delta(y) = 0, \forall y \in D_c$, will be hereinafter called the function of disagreement of the problem Z. We shall construct now a procedure of coordination of problem solutions at each step so that to ensure the next disagreed solution of a problem Z at step $(s+1)$ with the smaller value of disagreement then at previous step s. The convergence of such procedure will be provided at the expense of the possibility of correction of problems, ensuring the obtaining of such solutions in them which would satisfy the conditions of convergence of procedure of coordinating interrelated variable.

Thus, if we assume that the problems $Z^q, \forall q \in I$, are consistent and the algorithms of their solution are specified by operators $P^q, \forall q \in I$, the scheme of such procedure can be as follows.

Step 0. Problems Z^q on the domains $D^q (q \in I)$ leaving aside their interrelation are solved with the help of the operator P^q:

$$P^q : \mathrm{D}^q \to y^{q(0)}, q \in I \ .$$

Step s $(s = 1,2,\ldots)$.

1. Exchange between interrelated subsystems of values of connecting variables obtained at the previous step of the algorithm is executed, i.e. the problem transfers vectors of connecting variables $v_k^q, u_q^j, \forall k \in I^q, \forall j \in I_q$ and receives the corresponlding vectors from other problems $u_k^q, \forall k \in I^q$, $v_q^j, \forall j \in I_q$.

2. Correction of the problems $Z^q, \forall q \in I$ with the help of the operator K^q :

$$K^q : Z^q \to Z^{q(s)}, q \in I \ ,$$

which can correct the problem either at the expense of narrowing the domain of feasible solutions of problems by introduction of additional constraints, refinement of parameters of constraints,etc. i.e. $\mathrm{D}^q \to \mathrm{D}^{q(s)}$ or at the expense of changing of objective functions of the problems.

3. Solution of problems $Z^{q(s)}, \forall q \in I$, by application of the operator P^q at the step s -th :

$$P^q : \mathrm{D}^{q(s)} \to \tilde{y}^{q(s)}, q \in I \ .$$

4. Coordination of the solutions $\tilde{y}^{q(s)}, q \in I$, of subproblems with the help of the operator C^q to provide the satisfaction of condition $\delta(y^{(s)}) < \delta(y^{(s-1)})$, where $y^{(s)} = (y^{q(s)}, \forall q \in I)$:

$$C^q : \tilde{y}^{q(s)} \to y^{q(s)}, q \in I \ .$$

We have considered the question of the general substantation of the suggested approach to the distributed solution of the interrelated problem $\boldsymbol{Z}$ on the basis of minimization of the disagreement function introduced in a special manner regardless of its particular form. The way of specifying the function of disagreement defines the choice of the way of realization of the operators of the problem solution $P^q, \forall q \in I$, operators of coordination C^q of the whole problem $Z^q, \forall q \in I$, and correction $K^q, \forall q \in I$, in the above-given scheme of distributed solution of the problem $\boldsymbol{Z}$.

Let designate a solving interrelated problems $\boldsymbol{Z}$ procedure by π . This procedure taking into account the above- mentioned, will have an iterative character at each step of which the change of the system state, connected with a change of the domain of disagreed feasible solution in one problem at least due to the local actions of the operator $P^q, \forall q \in I$. The procedure π will be called distributed , if it admits parallel and asynchronous execution of local actions at each step. By realization of the distributed procedure π will be meant:

- a set of local actions with concrete realization of operators $P^q, \forall q \in I$, executing them;

- concrete realization of the operators of coordination $C = \{C^q, \forall q \in I\}$ carrying out coordination of the results of execution of local actions through connecting variables;

- a set of concrete realizations of the operators of correction $K = \{K^q, q \in I\}$ of local actions;

- establishment of the order of execution of local actions, operations of the coordination and correction at each step of the procedure π.

As applied to problem (7), (8) we designate by $\mathrm{D}^q(k_0) = \left\{ y^q = (x^q, v^q, u_q) : y^q \in \mathrm{D}^q, c_q x^q \geq f_q^0 - \frac{k_0}{\rho_q}(f_q^0 - f_{q(\min)}) \right\}$ the domain of disagreement of feasible solutions of the q-th problem for the given value of parameter k_0 and by $\mathrm{D}(k_0) = \prod_{q \in I} \mathrm{D}^q(k_0)$ we designate the disagreed domain of feasible solutions of the problem Z for the given value of parameter k_0. Then the domain described by constraints (8) for the given value of parameter will be designated by $\mathrm{D}_0(k_0)$ and represented in the following form $\mathrm{D}_0(k_0) = \{ y = \{ y^q, q \in I \} : y^q \in \mathrm{D}^q(k_0), \forall q \in I, v_k^{qr} = u_k^{qr}$, $\forall r = 1, \ldots, r_k^q, \forall (k,q) \in I, k \neq q, (k,q) \in L \}$, or according to (9) and properties of the domain of coordination of connecting variables D_c this domain can be rewritten in the following form: $\mathrm{D}_0(k_0) = \mathrm{D}(k_0) \cap \mathrm{D}_c = \prod_{q \in I} \mathrm{D}^q(k_0) \cap \mathrm{D}_c^q$. A solution of problem (7), (8) can be formulated as a mapping, with the help of the procedure π, from the domain of feasible solutions D_0 into the set of feasible solutions $\mathrm{D}_0(k_0) = \mathrm{D}_0(k_0 = k_{0(\min)})$. With account for iterative character of the procedure π and its distribution , at the s-th step each operator acts in the following manner. The operator $P^{q(s)}$ executes the procedure of search for disagreed solution and parameter $k_0^{q(s)}$ when solving the following specially formulated problem $\widetilde{Z}^{q(s)}$ of the form:

$$\min_{k_0, y^q} k_0 \tag{10}$$

under constraints

$$y^q \in \mathrm{D}^q(k_0)$$
$$F^{q(s-1)}(y^q, k_0^s) \geq F^{q(s-1)}(y^{q(s-1)}, k_0^{(s-1)}), \tag{11}$$

where the last vector inequality describes the domains of change of connecting variables and parameters , defined from the condition of their coordination at the ($s-1$)-st step to obtain the solution of problem (7), (8), the function form is defined by a direct form of the operator coordination. We shall designate constraints described by relations (11) by $\tilde{D}^{q(s-1)}(k_0^{(s-1)}) = \left\{ y^q : y^q \in D^q(k_0^{(s-1)}), F^{q(s-1)}(y^q) \geq F^{q(s-1)}(y^{q(s-1)}) \right\}$ and call them the disagreed domain of feasible solutions of the problem for coordinated connecting variables, and parameter k_0 of the $(s-1)$-th step. We assume that such a procedure exists and permits to execute the following operations:

$$P^{q(s)} : \tilde{Z}^{q(s)}(\mathrm{D}^{q(s-1)}(k_0^{(s-1)})) \rightarrow \tilde{y}^{q(s)} = (\tilde{x}^{q(s)}, \tilde{v}^{q(s)}, \tilde{u}_q^s), k_0^{q(s)}.$$

The coordination operator $C^{(s)}$ executes a procedure of search for new values of disagreed values of feasible solutions $y^{q(s)}, \forall q \in I$, and parameter $k_0^{(s)}$ of the s-th step, which minimize parameter and value of the general disagreement of the problem determined by the function $\delta(y = \{y^q, \forall q \in I\}, \tilde{y}^{(s)} = \{\tilde{y}^{q(s)}, \forall q \in I\})$ according to the values obtained with the help of operator $P^{q(s)}$, i.e.

$$C^{(s)} : \tilde{y}^{(s)} = \{\tilde{y}^{q(s)}, \forall q \in I\}, k_0^{q(s)}, \forall q \in I \rightarrow y^{(s)} = \{y^{q(s)}, \forall q \in I\}, k_0^{(s)}$$

The correction operator $K^{(s)}$ executes the procedure of construction of new disagreed domain of feasible solutions $\mathrm{D}^{q(s)}(k_0^s)$ at the s-th step for new values of connecting variables $y^{q(s)}$ of the parameter $k_0^{(s)}$ and a specially constructed function $F^q(y^q, k_0)$ of additional constraints in relation (11):

$$K^{q(s)} : \tilde{\mathrm{D}}^{q(s-1)}(k_0^{(s-1)}) \rightarrow \tilde{\mathrm{D}}^{q(s)}(k_0^{(s)}) \ .$$

The procedure π will be convergent if the generated sequence $y^s = \{y^{q(s)}, \forall q \in I, k_0^s\}, s = 0,1,\ldots$ of solutions of the distributed problem Z of the form (7), (8) converges to solution $y \in D_0(k_0 = k_{0(\min)})$. General questions of convergence of the procedure π are studied in [4] and this is associated with the fact that the disagreement function for the constrained and closed domain should be continuos and such that $\delta(y) = 0, \forall y \in D_c$ and $\delta(y) > 0, \forall y \notin D_c$.

The basis of the distributed procedure π for solving problem (7),(8) is suggested to be of two classes of its realization for problems described by models of the form (2)-(5);

- evaluation of intervals of possible values of disagreed connecting variables using the methodology of sequential analysis and elimination of variants at each step of the iterative procedure of minimization of parameter k_0 [3];

- direct minimization by properly formulated disagreement function with respect to connecting variables and values of parameter k_0 [2,4].

It is clear that the considered schemes of distributed multiobjective problem solving methods can be easily adapted to conditions of their implementation in distributed computer-aided control systems.

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Decision Making: Some Experiences, Myths and Observations

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1 Introduction

Decision making in management is one of the most important, if not the most important, function of management. How decisions are made, and how managers make the best decisions possible are central issues in management. Managers make decisions, all day long, day in, day out, and many never think about the decisions, except to make them and get them out of the way.

There are many simple paradigms of decision making. Until recently about all that could be said was that you should list the alternatives, indicate the advantages and disadvantages, and then somehow choose an alternative. Unfortunately, many decision makers simply choose an alternative, based on instinct, intuition, or feel. Then they go on. Though that is fine in many cases, in some cases it is not. In this paper I consider some myths of decision making and explore them in light of some experiences that colleagues and I have had with entrepreneurs. I then propose some steps that we, as decision analysts and scientists, should consider taking to try to sensitize decision makers to thinking more about their decisions and making better decisions.

Since the development of management science, there has been great interest in making decisions based on mathematical models. The first management science models appeared around 70 years ago. The field of multiple criteria decision making (MCDM) is roughly 30 years old, although its roots may be found in much earlier places. It is the making of decisions taking into consideration multiple conflicting objectives. However, there were earlier developments. To get a flavor of a really early decision making "technique", perhaps the first in recorded history, consider what American statesman and inventor Benjamin Franklin did (see Parton [4]), approximately two hundred years ago. When faced with a proposal that he had to act upon, he folded a sheet of paper and kept it handy, and wrote arguments in support of the proposal on one side of the sheet, and wrote arguments against the proposal on the other side of the sheet. After a period of time, when he had written down all the arguments he could come up with, he then identified one or more arguments on one side of the issue of roughly the same importance and effect as one or more arguments on the other side. He then crossed off those arguments. He continued the process until all the arguments on one side of the sheet were eliminated. Because he had struck off arguments of equal importance and effect at each stage, the remaining arguments made the difference and determined the decision.

Franklin's approach may be considered a decision making method, an early one to say the least!

The purpose of this paper is to present some decision making myths, and talk about overcoming the problems associated with the myths. Before talking about the myths of decision making, I want to discuss some recent experiences that I had with the Center for Entrepreneurial Leadership at the School of Management at the State University of New York at Buffalo.

2 The Center for Entrepreneurial Leadership

During the 1994-1995 academic year, I served as faculty advisor to the Center for Entrepreneurial Leadership (CEL) at the State University of New York at Buffalo. That program has a group of between fifteen and twenty entrepreneurs that meets at least once a week for roughly a ten-month period. The purpose of the meetings is to have the entrepreneurs identify and come to grips with problems facing their companies. The companies range from very small (3 to 5 employees total) to moderate to large (some with hundreds of employees, and one that year with well over a thousand employees). During my year as faculty advisor, we had seventeen entrepreneurs. As you might guess, the group shares many experiences. The intent is for the group to be close, and have a lot of camaraderie. To assist the process there are several parties as well as an overnight retreat. At the end of the program there is a formal graduation.

Substantively, our CEL program includes a session devoted to an introduction to and an overview of the program, including a description of the process used in the program. Other sessions involve presentations by successful entrepreneurs, presentations by our faculty, as well as clinics and followup sessions for each of the entrepreneurs. Our presentations by faculty are introductions to relevant topics for the entrepreneurs, but they are presented in a pragmatic, rather than an academic, way.

Prior to the clinic session held for each entrepreneur (also called fellow), the entrepreneur is guided by his or her mentor (a successful entrepreneur, perhaps an alumnus or former fellow) to assemble data about their business, and look closely at what they are doing, and ask questions of themselves. (Henceforth I use the generic 'his'. During the 1994-95 year two of our fellows were women! Roughly ten percent of our entrepreneurs have been women historically.) At that time they also meet their (usually four) reactors: other successful entrepreneurs (many of whom are former fellows) and (usually one) faculty members (per fellow), who interact with them prior to and during the clinic session.

During a fellow's clinic session, the mentor introduces the fellow and presents a brief overview of the enterprise. Then the fellow makes a presentation about his company. He presents a story about his company, introduces salient points, and identifies problems that he has with his company. I like to think of the process of these sessions as soul-baring sessions. Sessions can be difficult for the fellows because some have serious problems, and the clinics involve having them confront their problems systematically, perhaps for the first time. During the presentation questions are raised by the reactors, as well as by the other fellows. The reactors ask most of the questions. After a round of questioning, the reactors make final recommendations to the fellow. A videotape of the session is made available to the fellow. In addition, I prepare notes and recommendations and meet one-on-one with each fellow about a week after the session. Towards the end of the year, each of the fellows present a short followup session to the other fellows, in which he updates the group on the progress made since the clinic. Additional points are raised by the fellows at that time.

I attended all but two of the clinic sessions; one of my colleagues attended those two sessions. For both sessions I missed, I participated in the feedback (though normally one-on-one, in those cases two-on-one) sessions with my colleague and the fellow. In the process of working with the entrepreneurs, I had a chance to see how they looked at decision making and decision-making processes. I also attended the followup sessions.

The clinic sessions and the entire CEL program is very well received by entrepreneurs in the Buffalo area, and we have an enthusiastic CEL alumni group that meets regularly and is actively involved with our school.

I presented a session on decision making. During that session, I introduced an example of a decision that I borrowed from my colleagues Professors Pekka Korhonen and Jyrki Wallenius of the Helsinki School of Economics, and then embellished, which I reproduce below.

3 The Example

You are managing a portfolio for a 70 year-old conservative relative. The relative currently has $100,000 to invest, and you are considering four possible investments for the next calendar year.

They are as follows:

Bond Fund (Fund 1) - This fund is conservative and gives a low return if the economy is stable or improves, and loses a small amount if the economy declines. See the precise rates in the table below.

Money Market Fund (Fund 2) - This fund is very conservative, and never loses money. It gives a small return if the economy is stable or declines, and breaks even if the economy improves. See the precise rates in the table below.

Aggressive Stock Fund (Fund 3) - This fund is aggressive, and does very well if the economy is stable or improves, but loses substantially if the economy declines. See the precise rates in the table below.

Contrarian Fund (Fund 4) - This fund is also aggressive, but contrarian, and does very well if the economy declines. It performs modestly in a stable economy, but loses substantially if the economy improves. See the precise rates in the table below.

Unfortunately, you have no idea of what the economy will do, but must decide how much of the $100,000 should be invested in each investment. There are no other investments available.

Please fill in the amounts to invest in the table. There are no right answers.

	STATE OF ECONOMY Annual Return			MUST TOTAL $100,000
Fund\Future	Declining	Stable	Improving	Amount Invested
Fund One	-2%	5%	3%	
Fund Two	4%	3%	0%	
Fund Three	-7%	9%	10%	
Fund Four	15%	4%	-8%	

The participants take a few minutes to make their determination. Then I ask who has invested in which funds. I then point out that funds one and two are dominated and should never be chosen (in spite of my statement that there are no right answers). Fund one is dominated by two-thirds of the same amount in fund three and one-third in fund four. The returns for such a blend are 1/3 %, 7.33 %, 4 %, which are greater than the corresponding returns for fund one no matter what. It can similarly be shown that fund two is dominated by one-half of the same amount in fund three and one-half in fund four. The returns for such a blend are 4 %, 6.5 %, 1 %, which are at least as great as the corresponding returns for fund two no matter what.

4 Some Comments on the Example

I had some good questions from the group in presenting the problem, such as does the relative have any additional assets, pensions, etc. from which to support himself? When I presented the problem, as was the experience of Korhonen and Wallenius, virtually always a substantial fraction of people chose some of asset 1 and some chose some of asset 2. Of course there are always a few who choose only assets 3 and 4. Even presenting this problem to mathematicians and other sophisticated technical people, I have found that there are still those who choose assets one and two! When people choose only assets three and four, I ask them why. Rarely are they able to satisfactorily explain why!

Clearly, the above problem is a decision problem and a simple one at that, and people doing it regard it as a decision problem. Yet even with such a simple problem, there are nonetheless decision makers who choose dominated (poor) solutions.

I used the above problem as a starting point, and then presented an introduction to decision making and multiple criteria decision making in particular.

In all the presentations by the fellows in our CEL, it was interesting to see how the entrepreneurs tended to think of decisions. Decisions were of course important, but generally the entrepreneurs did not think of a decision as a significant point in time. Rather something came up necessitating a decision, and the person made a decision, usually without thinking much about it. The idea of using some sort of a framework was rather foreign to them. One message I tried to convey as a result of what I learned from participating in the sessions that I incorporated in my presentation on decision making and MCDM was that we should try to sensitize decision makers to some of the pitfalls of quick decision making without using decision support tools. Without such tools we can easily choose dominated strategies, such as funds one and two in the above example. Let's now turn to the myths of decision making.

5 Ten Myths of Decision Making

When I was invited to this meeting, I was asked to talk about myths of MCDM/decision making that I presented earlier. Some time ago I came up with ten reasonable assumptions of decision making, which I presented and then argued were myths. I now briefly review the myths, with some brief comments on each. The following are the myths:

1. The myth of a decision. Though there is sometimes a well-defined decision and a decision point or time, often there is not. Usually, a decision just happens.

2. The myth of a decision maker. We think of a decision maker making a decision, like a wise man in isolation. What is a more accurate scenario is that often a group rather than an individual makes a decision. Even where there is not a group, many people normally influence the decision maker.

3. The myth of a fixed set of alternatives. We tend to think as the set of alternatives as being fixed. More likely the set is dynamic, and changes over time.

4. The myth of an optimal solution. Though in principle decision makers would like to make optimal decisions and have optimal solutions, they are probably not able to distinguish between good and optimal solutions. The emphasis is therefore on finding good solutions and sometimes only on finding solutions.

5. The myth of seeking nondominated solutions. As I found with the experiment using the example borrowed from Korhonen and Wallenius, most decision makers do not

concern themselves with nondominated solutions. They are happy to have a solution, and get on with other things.

6. The myth of a utility or value function. Though a utility or value function may be a useful construct for decision makers, it seems more important to analysts than to decision makers.

7. The myth of decisions being static. In decision analysis, we think about good and bad decisions and good and bad outcomes. Of course decision makers want to make good decisions. But when a good decision is made, and a bad outcome occurs, a good manager doesn't shrug his shoulders and say "Too bad." He proceeds to adjust the decision in some way, to negotiate and improve the bad outcome.

8. The myth that sophistication/complexity is good. From an analyst's point of view, we sometimes get the feeling that sophistication/complexity is good. However, from the perspective of the user, that is simply not true. Sophistication/complexity is not good. Simplicity where attainable is much preferred.

9. The myth that mathematical convergence is good. Here too, from an analyst's or a mathematician's point of view, mathematical convergence may be desirable. However, if we can help people think through problems, then even though a process may not converge mathematically, it may still be fine from a practical perspective. After all, isn't that what users want?

10. The myth that all technical model assumptions must be satisfied for a decision-making model to be useful in practice. Like the last two myths, we might expect that all technical model assumptions must hold in order for a model to be useful. However, several researchers, including Howard Raiffa, have stated that this need not be the case.

6 Overcoming the Problems of the Myths

What I found in working with the CEL entrepreneurs was consistent with the myths. Though the myths may be appropriate in theory, they are not necessarily true in practice; in fact they may be erroneous in practice. This is also consistent with work that I have done together with Eero Kasanen, Hannele Wallenius, and Jyrki Wallenius [2]. In that study we examined six real-world major strategic decisions. We examined the myths, and found only weak support for them. Many of the points we raised in our brief discussion of the myths were valid.

The two that were most strongly supported were :

1. We were able to identify a nominal decision maker in every instant. However, even though it was clear who was the nominal decision maker, there were others involved

either in influencing the decision maker or in making the decision. The nominal decision maker in some situations had relatively little influence on the decision.

2. We were also able to show that the decision could be isolated by the people making the decision. They could usually identify the point in time a decision was made.

The other myths were only weakly supported. There are several messages that we learned from the study, and we incorporate them into the discussion below.

7 Where Do We Go from Here?

What have we learned? First, our decision-making framework and our decision-making myths are only rough approximations of how decisions are made. There is no standard framework for decision making, and there are many different ways decisions are made. We should study actual decision making further to learn more about decision making. That is what we have attempted to do in Kasanen et al [2]. Of course, there is still more to do. However, we can do certain things to improve decision making and the role of MCDM methods in decision making. We shall consider some things we may do below.

Second, we have to make our models more desirable from the perspective of decision makers, and more usable and useful to decision makers.

Third, we have to abandon our algorithm first approach, whereby we develop algorithms and then try to peddle them, rather than work with decision makers and understand their decision-making process.

Finally, we have to sensitize decision makers and have them think more about decisions, rather than having them simply make a decision and move on. How can we do this? Here are a few ideas that may be useful:

1. We should try to sensitize decision makers to think about their decisions. When confronting a decision, they should think about the magnitude of the decision, and some idea of what an optimal decision is worth compared to a less-than-optimal decision. If the difference does not justify the time for analysis, then they should make a decision and go on, as many do now.

2. Decision makers should think about tradeoffs among criteria, that improving one criterion should worsen another.

3. Where analysis is worthwhile, decision makers should first do what I like to call a simple utility analysis. Rather than calibrate and use a formal utility function, they should try to decide whether any of the alternatives have a reasonable chance or probability of a truly unacceptable or disastrous outcome. If there are alternatives with a sufficiently small chance of such outcomes, then normally only those alternatives should be considered as an active set of alternatives for choice. It may happen that none of the alternatives have sufficiently small probabilities of unacceptable or

disastrous outcomes, or for other reasons, decision makers take decisions that risk unacceptable outcomes, but they should be aware of such risks. There are examples of companies that have made decisions where they ignored the chance of unacceptable outcomes, and disastrous outcomes occurred.

4. Where a utility function would be appropriate, one may be calibrated and used. However I believe that utility functions will be used only in certain very important situations.

5. Alternatives should be scanned, with the concept of dominance in mind. It makes sense to use a computer program for doing this, because it is quickly and easily done using a computer. A spreadsheet or a decision-making program is appropriate. Dominated alternatives should generally be eliminated, except for certain special instances.[1]

6. Too often when decision makers determine a set of alternatives, they then limit their choices to the original set. Some judgment must be made as to whether additional alternatives should be generated or not. A simple cost-benefit approach is appropriate, whereby the search is justified if the expected gain from the search is greater than the expected cost of the search. In our latest computer version of AIM (Lotfi et al., [3]), we have a built-in procedure to help generate new alternatives. Sometimes radical thinking can be useful in generating new alternatives. Involving others, and encouraging and asking dumb questions are ways of generating alternatives that are creative and perhaps valuable.

8 Conclusion

A lot of fine work has been done in the MCDM area. Our field has addressed many problems and come up with many solution approaches. Much of that work has been programmed for computer and used in various studies. What we need to do is to learn from what we have done, and use and peddle what is useful. There is a need for something simple, something perhaps that can be thought of a modern version of Ben Franklin's approach. An idea would be to use a spreadsheet with weights, or perhaps something a bit more involved. Just as the electronic spreadsheet has revolutionized the use of the computer for many tasks, a simple spreadsheet decision method could also be of great use to decision makers in making all sorts of decisions. This is the idea that Lotfi, Stewart, and I used in developing our Aspiration-level Interactive Method (AIM) for MCDM (Lotfi et al., [3]). (Another approach with a similar philosophy is V*I*S*A for Windows [1].) It involves specifying aspiration levels for

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[1]Dominated alternatives should be excluded from further consideration, except where some of the criteria are proxies for unknown underlying criteria (e. g., in buying a house, asking price may be a proxy for the unknown underlying criterion purchase price), and where there are some unformulated or unstated criteria.

each objective, and internally using those aspiration levels to generate weights for a proxy utility function to rank alternatives. We also use many ideas of others in trying to help the user choose an alternative. For example, we use the ideas of Wierzbicki [6] in creating our metric for ranking alternatives, and we use ideas of Roy et al. [5] in identifying similar solutions to ones under consideration. We are currently reconfiguring AIM to be a Windows program with a graphical user interface. We hope to have the new version ready in about six months. We encourage others to consider some of the issues we have been thinking about.

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Methods

2.2 Algorithms

An Interior Multiobjective Linear Programming Algorithm Using Aspirations

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Abstract. We describe in this paper a new multiple objective linear programming (MOLP) algorithm that is based on the single-objective path-following primal-dual linear programming algorithm, and combines it with aspiration levels and the use of achievement scalarizing functions.

Key words. Interior-point MOLP algorithms, aspirations, scalarizing functions.

1 Introduction

The single-objective path-following primal-dual linear programming algorithm, and the use of aspiration levels and achievement scalarizing functions [16-17] are combined to provide a new MOLP algorithm. The resulting algorithm falls in the class of interactive MOLP algorithms as it requires interacting with the Decision Maker (DM) during the iterative process to obtain statements of aspirations for levels of objectives of the MOLP problem. The interior-point algorithm is used to trace a path of iterates from a current (interior) solution, and approach as close as desired a nondominated solution corresponding to the optimum of the achievement scalarizing function. It is expected that an interior algorithm will speed up the overall process of searching and finding the most preferred MOLP solution by avoiding the need for numerous pivot operations and their corresponding interactive sessions inherent in simplex-based algorithms.

The algorithm introduced by N.K. Karmarkar in 1984 [7] provides a way for solving linear programming problems by moving through the *interior* of the constraints polytope. Following the original algorithm, a host of additional variants were developed (see, e.g., [1], [2], [4], [14]). An interior-point approach provides us with a different philosophy to approach MOLP problems. Instead of generating only nondominated solutions, the procedure generates a path of dominated solutions in such a way that the DM is motivated to look for and able to find better solutions for the current one. The final solution is nondominated and - hopefully - the most preferred one.

First attempts in using an interior-point linear programming algorithms for MOLP problems through the affine-scaling primal algorithm were reported in [1-2]. The use of achievement scalarizing function was first proposed by Wierzbicki [16-17], and led Korhonen and Laakso [9] to incorporate it into a MOLP algorithm which was further developed by Korhonen and Wallenius [11].

The paper is arranged as follows. In section 2 we provide a short summary of the path-following primal-dual algorithm. Section 3 reviews the fundamentals of achievement scalarizing functions. Section 4 describes our proposed approach and develops it into a MOLP algorithm. Section 5 illustrates it with an example, and section 6 provides a summary and some suggestions for future research.

2 The Path-Following Primal Dual Algorithm

The affine-scaling primal algorithm used in early attempts for addressing MOLP problems [1-2] is concerned with making movement through the interior of the constraints polytope by maintaining primal feasibility and considering cost-reduction only. In contrast, the primal-dual algorithm (see, e.g., [3],[8], [13]), summarized in this section, maintains both primal and dual feasibility and iterates to reduce duality gap.

We consider a *primal problem* **(P)** in standard form given through

$$\textbf{(P)}: \quad \textbf{\textit{minimize}} \quad c^{\mathrm{T}}x \quad \textit{subject to:} \quad \mathbf{A}x = b, \quad x \geq 0, \tag{2.1}$$

where $c, x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$ and $\mathbf{A}$ is an $m \times n$ constraints matrix. After adding the reduced-cost vector, z, the *dual problem* **(D)** is defined through

$$\textbf{(D)}: \quad \textbf{\textit{maximize}} \quad b^{\mathrm{T}}y \quad \textit{subject to:} \quad \mathbf{A}^{\mathrm{T}}y + z = c, \quad z \geq 0, \tag{2.2}$$

where $y \in \mathbf{R}^m$ and $z \in \mathbf{R}^n$. The non-negativity constraints on the primal vector, x, in (2.1) and on the reduced cost vector, z, in (2.2) can be eliminated by adding a barrier term to the objective functions of **(P)** and **(D)**. This results in the pair of primal and dual problems given, respectively, through

$$(\mathbf{P}_{\mu}): \quad \textbf{\textit{minimize}} \quad c^{\mathrm{T}}x - \mu \sum_{i=1}^{n} \ln(x_i) \quad \textit{subject to:} \quad \mathbf{A}x = b, \quad \mu > 0. \tag{2.3}$$

$$(\mathbf{D}_{\mu}): \quad \textbf{\textit{maximize}} \quad b^{\mathrm{T}}y + \mu \sum_{i=1}^{n} \ln(z_i) \quad \textit{subject to:} \quad \mathbf{A}^{\mathrm{T}}y + z = c, \quad \mu > 0 \tag{2.4}$$

With these formulations, in addition to pursuing cost reduction as in the affine-scaling primal algorithm, we also take steps that move us away from the walls and attempt to center us in the polytope. The barrier parameter μ serves to balance these two conflicting actions of centering and cost reduction. To solve the constrained optimization problems, $\mathbf{P}_\mu$ and $\mathbf{D}_\mu$, for a *given* barrier parameter $\mu > 0$, we construct their respective *Lagrangians* which are given through

$$L_p(x, y, \mu) = c^T x - \mu \sum_{i=1}^{n} ln(x_i) - y^T(b - \mathbf{A}x). \tag{2.5}$$

$$L_d(x, y, \mu) = b^T y + \mu \sum_{i=1}^{n} ln(z_i) - x^T(\mathbf{A}^T y + z - c), \tag{2.6}$$

Let e be the column vector of all *1's* and $\mathbf{X}$ and $\mathbf{Z}$ be $n \times n$ diagonal matrices defined by $\mathbf{X} = diag(x_1, x_2, \ldots, x_n)$ and $\mathbf{Z} = diag(z_1, z_2, \ldots, z_n)$ respectively, where the *i*-th diagonal element is the *i*-th component of the corresponding vector. Then, the first-order necessary conditions for the optimal solution of both the primal and dual problems are given by:

$$\mathbf{XZ}e = \mu e \tag{2.7}$$

$$\mathbf{A}x = b \tag{2.8}$$

$$\mathbf{A}^T y + z = c \tag{2.9}$$

Note that condition (2.8) maintains primal feasibility and (2.9) maintains dual feasibility. In addition, from condition (2.7) we have

$$x_i z_i = \mu, \qquad \text{for all} \quad 1 \le i \le n. \tag{2.10}$$

By fixing μ we can solve x, y, and z from (2.7) - (2.9). Since these vectors are dependent on the choice of the barrier parameter, μ, we get a family of solutions depending on the value of μ. The *central trajectory* r is defined as the set of all vectors $x(\mu)$, $y(\mu)$ and $z(\mu)$, satisfying (2.7) - (2.9).

Assuming that the $m \times n$ matrix $\mathbf{A}$ is of full row rank m and that starting feasible and interior vectors $x > 0$, y and $z > 0$ for the problems (**P**) and (**D**) in (2.1) and (2.2) are available, we have to iterate toward satisfying (2.10). Then, for a given $\mu > 0$, we have to find step direction vectors dx, dy and dz that take us from the current feasible solution and move us to a new iterate while satisfying the necessary conditions. Fixing the value of the barrier parameter μ to reflect the current duality gap through

$$\underline{\mu} = \sigma \frac{e^T \mathbf{XZ}e}{n}, \qquad \text{where } 0 < \sigma < 1. \tag{2.11}$$

and defining an auxiliary vector, $w(\bar{\mu}) \in \mathbf{R}^n$, through

$$w(\underline{\mu}) = \underline{\mu} e - \mathbf{XZ}e, \tag{2.12}$$

then, by applying Newton's method to the set of necessary conditions leads to the following expressions for the required step direction vectors $\{dx, dy, dz\}$

$$dy = -(\mathbf{AZ}^{-1}\mathbf{XA}^T)^{-1}\mathbf{AZ}^{-1} w(\underline{\mu}), \tag{2.13}$$

$$dz = -\mathbf{A}^T dy, \tag{2.14}$$

$$dx = \mathbf{Z}^{-1} w(\underline{\mu}) - \mathbf{Z}^{-1}\mathbf{X}dz. \tag{2.15}$$

The new iterates for the primal and dual problems of (2.3) and (2.4) are given as

$$x = x_0 + \rho\alpha_p dx, \qquad 0 < \rho < 1 \tag{2.16}$$

$$y = y_0 + \rho\alpha_d dy, \tag{2.17}$$

$$z = z_0 + \rho\alpha_d dz, \tag{2.18}$$

where the step sizes α_p and α_d for the primal and dual problem respectively, are found from the ratio tests given through

$$\alpha_p = min\left\{-\frac{x_i(k)}{dx_i(k)} : \forall\ dx_i(k) < 0\ ,\ \ 1 \leq i \leq n\right\}, \tag{2.19}$$

$$\alpha_d \equiv min\left\{-\frac{z_i(k)}{dz_i(k)} : \forall\ dz_i(k) < 0\ ,\ \ 1 \leq i \leq n\right\}, \tag{2.20}$$

3 Scalarizing Functions and Aspiration Levels

We consider the following multiple objective linear programming (MOLP) problem through the rest of this paper:

$$\text{"}max\text{"} \quad v = \mathbf{C}x$$
$$\text{subject to:} \quad x \in \mathbf{S} = \{x \mid \mathbf{A}x = b,\ x \geq 0\} \tag{3.1}$$

where $x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, the $m \times n$ constraint matrix $\mathbf{A}$ is of full rank m, and the objective matrix, $\mathbf{C}$, whose rows are the single objectives, is of dimension $q \times n$.

In (3.1), $x^* \in \mathbf{S}$ is termed an *efficient solution* iff there does not exist another $x \in \mathbf{S}$ such that $\mathbf{C}x \geq \mathbf{C}x^*$ and $\mathbf{C}x \neq \mathbf{C}x^*$. And $x^* \in \mathbf{S}$ is *weakly efficient* iff there does not exist another $x \in \mathbf{S}$ such that $\mathbf{C}x > \mathbf{C}x^*$. Let $\mathbf{V} = \{ v = \mathbf{C}x \mid x \in \mathbf{S}\}$ be the set of *feasible* criterion vectors (i.e., the feasible region in criterion space). Vectors $v \in \mathbf{V}$ corresponding to efficient points are called *nondominated* criterion vectors and vectors $v \in \mathbf{V}$ corresponding to weakly efficient points are called *weakly nondominated* criterion vectors. The set of all efficient points is called the *efficient set*, denoted $\mathbf{E}$, and the set of all nondominated criterion vectors is called the *nondominated set*, denoted $\mathbf{N}$. The sets of weakly efficient, and weakly nondominated solutions are denoted by $\mathbf{E}^W$ and $\mathbf{N}^W$ respectively.

The search in the set of nondominated solutions can be controlled by means of *an achievement (scalarizing) function* as suggested by Wierzbicki [16]. Such a function projects any given (feasible or infeasible) point $g \in \mathbf{R}^q$ onto the set of (weakly) nondominated solutions. The simplest form proposed by Wierzbicki is the following:

$$s(g,v,w) = max\,[\,(g_i - v_i)/w_i,\ \ i \in \{1,2,...,q\}], \tag{3.2}$$

where $w > 0$ is a q-vector of weights, $g \in \mathbf{R}^q$ a given point in objective space, the components of which are called *aspiration levels*, and $v \in \mathbf{V} = \{v = \mathbf{C}x \mid x \in \mathbf{S}\}$. By minimizing $s(g,v,w)$ subject to $v \in \mathbf{V}$, we obtain a solution v^*, which is weakly nondominated, $v^* \in \mathbf{N}^w$, because for all $v \in \mathbf{V}$, $s(g, v^*, w) \leq s(g,v,w)$ and, therefore,
$(g_i - v_i^*)/w_i \leq (g_i - v_i)/w_i$, for at least one $i \in \{1,2,...,q\}$, which implies that $v_i \leq v_i^*$, for at least one $i \in \{1,2,...,q\}$. Hence, it follows that there exists no other $v \in \mathbf{V}$ such that $v > v^*$ implying that v^* is weakly nondominated. If the given point $g \in \mathbf{R}^q$ is feasible, then for a solution $v^* \in \mathbf{N}^w$, we have $v^* \geq g$. If the solution is unique, $v^* \in \mathbf{N}$; otherwise it may be dominated. To generate only nondominated (instead of weakly nondominated) solutions, more complicated forms can be used to guarantee uniqueness (see, e.g. Wierzbicki [16]). For simplicity, we assume here that the solution v^* is always nondominated. The assumption is not very restrictive, because intermediate solutions are usually dominated, and to generate

the final solution the lexicographic formulation, e.g., can be used if the solution is not unique (see, e.g., [15, p. 445]).

Given $g \in \mathbf{R}^q$, the minimum of an achievement scalarizing function $s(g,v,w)$ is found by solving the following problem:

$$\begin{array}{ll} & \min\ \varepsilon \\ \text{subject to:} & x \in \mathbf{S} \\ & \varepsilon \geq (g_i - C_i\, x) / w_i\,,\ i = 1,2,\ldots,q \\ & x \geq 0, \end{array} \tag{3.3}$$

where C_i $(i = 1,2,\ldots, q)$, refers to the i-th row of the objective matrix $\mathbf{C}$. The problem (3.3) can further be written as:

$$\begin{array}{ll} & \min\ \varepsilon \\ \text{subject to:} & x \in \mathbf{S} \\ & \mathbf{C}x + \varepsilon w - z = g \\ & x, z \geq 0. \end{array} \tag{3.4}$$

The MOLP-problem is now reduced to a single-objective optimization problem. Without loss of generality, we can assume that $\varepsilon > 0$, because by replacing the aspiration level vector g by $g + \lambda w$, where $\lambda > 0$ is an arbitrary scalar, $\varepsilon \rightarrow \varepsilon + \lambda$, but the values of the decision variables remains the same. It means that the aspiration level vector g has to be specified in such a way that there exists no $v \in \mathbf{V}$ such that $v \geq g$. The DM can vary an aspiration level vector and the system will provide a nondominated solution for evaluation. Termination is reached when the DM is convinced that the *most preferred* solution is found. The problem we face now is that shown in (3.4) which can be written compactly as the augmented problem described by

$$\begin{array}{ll} & \min\ \bar{c}^T \bar{x} \\ \text{subject to:} & \bar{\mathbf{A}}\bar{x} = \bar{b} \\ & \bar{x} \geq 0. \end{array} \tag{3.5}$$

where $\bar{x} \in \mathbf{R}^{n+q+1}$ and $\bar{c} \in \mathbf{R}^{n+q+1}$ has a 1 in the (n+1) position, $\bar{b} \in \mathbf{R}^{n+q}$ the matrix $\bar{\mathbf{A}}$ is $(m+q) \times (n+q+1)$ and

$$\bar{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & 0 & \mathbf{O} \\ \mathbf{C} & w & -\mathbf{I}q \end{pmatrix} \qquad \bar{b} = \begin{pmatrix} b \\ g \end{pmatrix} \tag{3.6}$$

When an interactive approach is used to solve a MOLP-problem, a DM's intervention is usually required more than once before a final solution (that is, the most preferred) is found. The DM intervenes by specifying a new aspiration level vector g, and the problem (3.4) is resolved to finding a new solution for the DM's evaluation. When a simplex-based algorithm is used to solve the problem (3.4), the parametric programming provides us with a convenient way to find the second, third, etc. solution. The problem can be formulated as follows:

$$\begin{array}{ll} & \min\ \varepsilon \\ \text{subject to:} & x \in \mathbf{S} \\ & \mathbf{C}x + \varepsilon w - z = \mathbf{C}x^c + t(g^* - \mathbf{C}x^c) \\ & x, z \geq 0, \qquad 0 \leq t \leq 1 \end{array}$$

where x^c is the current solution point and g^* is a new aspiration level vector.

The use of parametric programming provides us with an obvious possibility to produce a (weakly) nondominated path from the current solution to the new solution. The solutions on the path are at least weakly nondominated, because it is an optimal solution of the problem (3.4) for some $t \in [0, 1]$. The path can be displayed in, for example, a visual form using computer graphics [9], which enables the DM to consider solutions in a very illustrative way, and see how the values of the objectives change, when the DM is traversing from one solution to another.

The use of aspiration levels does not require any assumptions concerning the DM's preference structure. The DM is free to make a search for as long as desired, and stops, when it is felt that the most preferred solution is found. No formal termination rules are used. If desired the DM may check the optimality conditions of the final solution as proposed in [6].

4 The Proposed Algorithm

We propose to use the path-following primal-dual algorithm to solve the linear programming problem shown in (3.5) by defining its primal and dual. The primal problem, denoted by **P**, is defined by augmenting the different vectors and matrices shown in (3.6). This results in the (primal) problem described through

$$(\mathbf{P}) \qquad \begin{aligned} &\mathit{min} \quad \bar{c}^{\mathrm{T}}\bar{x} \\ &\mathit{subject\ to:}\ \bar{\mathbf{A}}\bar{x} = \bar{b} \\ &\qquad\qquad\quad \bar{x} \geq 0, \end{aligned} \tag{4.1}$$

where $\bar{x} = [\,x\ \varepsilon\ z]^{\mathrm{T}}$, $\bar{\mathbf{A}}$ is an $(m+q)\times(n+q+1)$ matrix, $\bar{b} \in \mathbf{R}^{m+q}$ and are given by

$$\bar{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & 0 & \mathbf{O} \\ \mathbf{C} & w & -\mathrm{Iq} \end{pmatrix} \qquad \bar{b} = \begin{pmatrix} b \\ g \end{pmatrix} \tag{4.2}$$

and where the coefficient vector of the objective function vector is denoted by $\bar{c}$ and is all zeros except at the $(n+1)$ position where its coefficient is one. Adding slack variables, the dual problem, **D**, to the problem **P** is given by

$$(\mathbf{D}) \qquad \begin{aligned} &\mathit{max} \quad \bar{b}^{\mathrm{T}}\bar{y} \\ &\mathit{subject\ to:}\ \bar{\mathbf{A}}^{\mathrm{T}}\bar{y} + \bar{z} = \bar{c} \\ &\qquad\qquad\quad \bar{z} \geq 0, \end{aligned} \tag{4.3}$$

In the original reference point approach, for each aspiration level vector a nondominated solution is generated for the DM's evaluation. However, especially in the beginning of the search process it makes no sense to produce precise solutions for the DM, because in the beginning the main emphasis is to understand what is possible to achieve, and not to find the most preferred solution. The interior point approach provides a natural tool to control accuracy. Therefore we propose an approach in which rough estimates for a nondominated solution are produced as long as the DM will change aspiration levels. When the DM has ceases to change aspiration levels, the system moves the current interior iterate to the final nondominated solution with any required accuracy.

Traditional (interactive) simplex-based MOLP procedures enable the DM to make a search in the nondominated set. This is based on a common sense assumption that comparisons between dominated solutions may seem irrelevant from the DM's point of view. However, a search in a dominated (interior) set may be a reasonable, because no simplex-based algorithm is needed to move from facet to facet in the criterion space, and perhaps, the DM prefers to take a "shortcut" through the feasible region instead of following an exterior path. The path is relevant to the DM only if the whole path is displayed explicitly. If only the next solution is shown to the DM, then how the solution is found is more or less technical. Thus there are computational reasons to use an interior approach, but in some cases its use can be argued even with behavioral aspects.

Timing Changes in Aspirations

A distinctive feature of our MOLP algorithm is that the DM can let it run, undisturbed, with a given set of aspirations and let the (interior) solution path come as close as desired to the boundary of the constraints polytope. The DM can, of course, monitor the progress of the solution path while it makes its progress towards the boundary and interrupt it if it results in deterioration of certain key objectives. By specifying new aspirations, the DM can then steer the solution trajectory in a different direction and toward different, more desirable, values for the objectives. Using the primal-dual algorithm provides a good indicator about the progress of the algorithm since the value of the duality gap is reduced (to be exact, the duality gap is non-increasing) at each iteration. Therefore, by monitoring the duality gap (assuming, of course, that we started with a feasible primal and dual solutions) we can detect when we come close to the optimal solution for the given aspiration vector. Specifying a threshold tolerance allows us to time when to suggest to the DM to change the current aspirations. This is done whenever the duality gap falls below that threshold.

Summary of the Proposed Algorithm:

Step 1: Consider the problem formulation (4.1) and ask the DM to specify a starting aspiration level (e.g., the ideal point). Use this vector as the initial aspiration vector $\boldsymbol{g}$ in vector $\bar{\boldsymbol{b}}$ (see, 4.2).
Find a starting feasible, and strictly interior, primal solution vector, $\bar{x}_0$, and a feasible dual solution pair $\{\bar{y}_0, \bar{z}_0\}$to the LP-problem in (4.1). Specify a maximal number of iteration to run with each aspiration vector, $\bar{k}$, and set a step size factor, ρ. Set the iteration counter, k, at $k=0$. and denote $\bar{x}(k) = \bar{x}_0 = [x_0(k), \varepsilon(k), z(k)]^{\mathrm{T}} > \boldsymbol{0}$.

Step 2: Define the scaling matrices, $\bar{\mathbf{X}}$, and $\bar{\mathbf{Z}}$ having $\bar{x}(k)$ and $\bar{z}(k)$ as their diagonal, and fix the barrier parameter $\bar{\mu}$ and the auxiliary vector $w(\bar{\mu})$ through

$$\bar{\mu} = \sigma \frac{\boldsymbol{e}^{\mathrm{T}} \bar{\mathbf{X}} \bar{\mathbf{Z}} \boldsymbol{e}}{n}, \qquad 0 < \sigma < 1, \qquad w(\bar{\mu}) = \bar{\mu}\boldsymbol{e} - \bar{\mathbf{X}}\bar{\mathbf{Z}}\boldsymbol{e}.$$

Step 3: Solve for the step direction vectors $\bar{dx}$, $\bar{dy}$ and $\bar{dz}$ through the following systems of equations:

$\bar{dy} = -(\bar{\mathbf{A}}\bar{\mathbf{Z}}^{-1}\bar{\mathbf{X}}\bar{\mathbf{A}}^{\mathrm{T}})^{-1}\bar{\mathbf{A}}\bar{\mathbf{Z}}^{-1}w(\bar{\mu})$

$\bar{dz} = -\bar{\mathbf{A}}^{\mathrm{T}}\bar{dy}$

$\bar{dx} = \bar{\mathbf{Z}}^{-1}w(\bar{\mu}) - \bar{\mathbf{Z}}^{-1}\bar{\mathbf{X}}\bar{dz}$

Step 4: Evaluate the next primal iterate through

$\bar{x}(k+1) = \bar{x}(k) + \rho\alpha_p\bar{dx}(k)$,

and the next dual iterates through

$\bar{y}(k+1) = \bar{y}(k) + \rho\alpha_d\bar{dy}(k)$, $\quad \bar{z}(k+1) = \bar{z}(k) + \rho\alpha_d\bar{dz}(k)$,

where the step sizes, α_p and α_d are found from the ratio tests shown in (2.19)-2.20)

Step 5: Evaluate the relative duality gap:

$$gap = \frac{|\bar{c}^T\bar{x} - \bar{b}^T\bar{y}|}{1.0 + \|\bar{c}^T\bar{x}\|}$$

If the problem is primal feasible and the duality gap is small: ***GoTo Step 6***, **Else**, if $k \geq \bar{k}$: Set $k := 0$, ***GoTo Step 6***. **Else:** $k := k+1$. ***GoTo Step 2***.

Step 6: Ask the DM for an aspiration vector $\boldsymbol{g}$. If k > 0 and the DM refuses, ***Stop***; otherwise set $\varepsilon(k)>0$ and $z(k)>0$ such that $Cx(k)+\varepsilon(k)w-z(k)=g$ and re-specify $\bar{x}(k)$. ***GoTo Step 2***.

5 An Illustrative Example

To demonstrate our proposed approach, consider the following MOLP problem:

$$\max f_1(x) = x_1$$
$$\max f_2(x) = x_2, \qquad \textit{Subject to:}$$

$x_1 + 5x_1 \leq 64$	$x_1 + 4x_2 \geq 8$
$x_1 + x_2 \leq 10$	$6x_1 + x_2 \geq 6$
$4x_1 + x_2 \leq 56$	$-3x_1 + 2x_2 \leq 16$
$2x_1 - x_2 \leq 22$	$-x_1 + 2x_2 \leq 20$
$x_1 - x_2 \leq 10$	$x_1, x_2 \geq 0$

Converting this system of constraints to the form required by our proposed approach results in an augmented system given by

$$\begin{aligned} &\textit{minimize} && \bar{c}^{\mathrm{T}}\bar{x} \\ &\textit{subject to:} && \bar{\mathbf{A}}\bar{x} = \bar{b} \\ & && \bar{x} \geq \mathbf{0}, \end{aligned}$$

where: $\bar{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{O} \\ \mathbf{C} & w & -\mathbf{I} \end{pmatrix}$, $\bar{b} = \begin{pmatrix} b \\ g \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\bar{c}^{\mathrm{T}} = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]$.

and $\bar{x} = [x\ \ \varepsilon\ \ z]^{\mathrm{T}} \in \mathbf{R}^{14}$. For this augmented system one can find initial feasible primal-dual vectors that satisfy the constraints of:, $\bar{\mathbf{A}}\bar{x}_0 = \bar{b}$, and $\bar{\mathbf{A}}^{\mathrm{T}}\bar{y}_0 + \bar{c} = \bar{z}_0$.

$$\bar{x}_0 = [2\ \ 2\ \ 52\ \ 16\ \ 46\ \ 20\ \ 10\ \ 2\ \ 8\ \ 18\ \ 18\ \ 13\ \ 1\ \ 10]^{\mathrm{T}}$$
$$\bar{y}_0 = [-1\ \ -1\ \ -1\ \ -1\ \ -1\ \ 0.5\ \ 0.5\ \ -1\ \ -1\ \ 0.25\ \ 0.25]^{\mathrm{T}}$$
$$\bar{z}_0 = [1.25\ \ 6.25\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0.5\ \ 0.5\ \ 1\ \ 1\ \ 0.5\ \ 0.25\ \ 0.25]^{\mathrm{T}}$$

Assuming that preference of the DM is characterized by a utility function $u(x) = x_1 x_2$, this problem has a unique solution – expressed in terms of the original decision variables – given by $x = [10\ \ 10]^T$. Note that this solution is on a face of a polytope and not at an extreme point.

To test our algorithm, we set our initial aspiration level at g_1 and run the proposed algorithm with a step size factor of $\rho = 0.99$. As the solution trajectory approaches the optimal boundary point for the specified aspiration level, the duality gap decreases. At that point we ask the DM for a new aspiration level. In the example reported here, we ask the DM for three different aspiration levels. The specific aspiration levels used in our example are given below and the results of these runs are shown in Tables 1-3, and depicted in Fig. 2 below.

$$g_1 = [14.0\quad 5.0]^T,\quad g_2 = [\ 5.0\quad 14.0]^T,\quad g_3 = [\ 11.0\quad 11.0]^T$$

Table 1, Using g_1

k	x_1	x_2	*Gap*
0	2.0000	2.0000	—
1	12.0219	7.4522	76.1238
2	11.5988	6.8909	8.7644
3	12.4270	6.2650	2.0431
4	12.6738	5.2052	0.7393
5	12.9896	3.9998	0.1665
6	12.9981	4.0074	0.0111
7	12.9999	4.0000	0.0005

Table 2, Using g_2

k	x_1	x_2	*Gap*
0	12.9999	4.0000	—
1	12.9351	4.1381	14.7268
2	11.9317	8.0390	11.1328
3	12.0896	7.1414	8.3395
4	8.4752	11.0725	4.0562
5	8.4057	11.1185	1.4465
6	4.0244	11.9813	0.3693
7	3.9824	11.9909	0.0975

Table 3, Using g_3

k	x_1	x_2	*Gap*
0	3.9824	11.9909	—
1	4.0617	11.9630	6.8986
2	9.0125	10.9478	2.5009
3	9.0666	10.9330	1.1757
4	9.9447	10.0360	0.1976
5	9.9997	9.9977	0.0120
6	9.9999	10.0000	0.0006
7	10.0000	10.0000	$\sim 10^{-5}$

Note how, for each change of aspirations, the solution trajectory "kicks" away from the walls, performs a centering step and moves toward a boundary point closest to the specified aspiration vector, regardless of its feasibility (all aspiration levels used in the example were infeasible). The wide difference in the aspirations used in generating Fig. 2, was specifically chosen to illustrate the responsiveness of the algorithm to changes in aspirations.

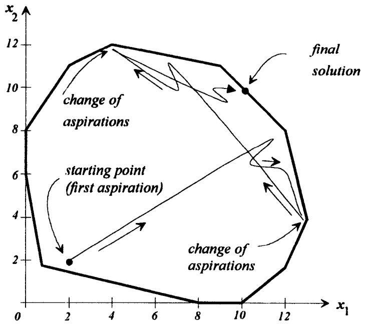

Fig. 2, solution trajectory

6 Summary

A new algorithm for addressing linear programming problems with multiple objectives was presented. The algorithm combines the path-following primal-dual algorithm with scalarizing functions. The resulting algorithm allows the decision maker to control the progress of the algorithm by specifying aspiration levels for the various objectives and allows modifying the internal solution trajectory to accommodate these changes in aspirations. Further efforts in this area should address such issues as starting the algorithm, timing and mechanism of the DM's intervention and the DM's satisfaction with the final solution.

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A FUZZY SOLUTION APPROACH TO A FUZZY LINEAR GOAL PROGRAMMING PROBLEM

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Abstract. In conventional multiobjective decision making problems, the estimation of the parameters of the model is often a problematic task. Normally they are either given by the decision maker (DM) who has imprecise information and/or expresses his considerations subjectively, or by statistical inference from the past data and their stability is doubtful. Therefore, it is reasonable to construct a model reflecting imprecise data or ambiguity in terms of fuzzy sets and a lot of fuzzy approaches to multiobjective programming have been developed.

Many decisors might follow a satisfaction criterion rather than the criterion of maximizing an objective function; and the satisfaction criterion leads to the concept of goal. Goal programming (GP) is an appropriate approach to the problem and when attributes and/or goals are in an imprecise environment and they cannot be stated with precision, we work in fuzzy goal programming.

This paper presents a fuzzy solution to a GP problem when all the parameters may be fuzzy numbers. The method relies on α-cuts of the fuzzy solution to generate the possibility distribution of the objective functions. Ideas are illustrated with a numerical example.

KEYWORDS: Goal Programming, decision making, fuzzy programming, possibility distribution, fuzzy number.

AMS CLASIFICATION (1991): $90C70, 90C29$.

1 Introduction

The fuzzy programming approach to multiobjective linear programming problems was first introduced by Zimmerman [12] and further developed by Hannan [5, 6], Tanaka [9] and other authors. In particular Buckley [3, 4] has introduced the possibility theory to solve fuzzy linear programming problems.

In [1] we discussed the general formulation of fuzzy linear goal programming problem where the aspiration level of DM for each objective value is represented by a fuzzy number as well as the coeficients of constraints of the model and also the inequalities of them are given in fuzzy terms.

The purpose of this paper is, on the one hand, to offer to the DM some information that permits him to stablish suitably target values, in fuzzy terms, to each goal and, on the other hand, to specify the possibility distribution for the objective function that allows him to estimate the degree in which the goals are reached.

In this paper imprecise goals of the decision maker and also uncertainty on the restrictions of the problem, are incorporated into standard goal programming formulation by considering the coefficients and target values as fuzzy triangular numbers, that we suppose are weakly noninteractive [11].

The first step in the formulation of a general GP model is to stablish a set of target values $\tilde{g}_r$, i.e., the achivement level desired for each attribute considered in the problem situation, that transforms the objective functions in goals. Usually the DM is the one who provides those numerical values and in order to help him determine an acceptable level of achivement for any of the attributes considered, we stablish the pay-off table for the most possible values of the fuzzy parameters, i.e., for the level of possibility $\alpha = 1$ and also we give to the DM the pay-off table at the possibility level $\alpha = 0$. Then the FP-LGP is constructed and takes the form:

Find the decision $x^* = (x_1^*, \ldots, x_n^*)^t \in \mathbb{R}^n$ such that:

$$\left.\begin{array}{ll} \tilde{G}_r \equiv \tilde{c}_r^1 x_1 + \tilde{c}_r^2 x_2 + \ldots + \tilde{c}_r^n x_n \geq \tilde{g}_r & r = 1, 2, \ldots, k \\ \tilde{A}x \leq \tilde{b} & \\ x \geq 0 & \end{array}\right\} \text{(FP-LGP)}$$

where the satisfying conditions are handled as an additional fuzzy constraints and where symbols with a 'tilde' on top are fuzzy parameters that we suppose are given by fuzzy numbers and where $\tilde{A}, \tilde{b}, \tilde{c}$ are considered as possibility distribution.

2 Characterizing and finding a solution for FP-LGP

Consider the crisp problem LGP$(A, b, c; g)$:

$$\left.\begin{array}{ll} G_r \equiv c_r^1 x_1 + c_r^2 x_2 + \ldots + c_r^n x_n \geq g_r & r = 1, 2, \ldots, k \\ Ax \leq b & \\ x \geq 0 & \end{array}\right\} \text{(LGP}(A, b, c; g))$$

Using the extension principle [2] and the joint possibility distribution of all parameters, we define $\Pi[\tilde{Z} = z]$, the possibility distribution of Pareto optimal solution (POS) of the objective function $\tilde{Z}$:

$$\Pi(\tilde{Z} = z) = sup_{A,b,c}\,\{\Pi(\tilde{A}, \tilde{b}, \tilde{c})/z \text{ is a POS of the LGP(A,b,c;g)}\}$$

where the joint possibility distribution $\Pi(\tilde{A}, \tilde{b}, \tilde{c})$ is just the minimum of $\mu_{\tilde{c}_{kj}}(c_{kj})$, $\mu_{\tilde{a}_{ij}}(a_{ij})$ and $\mu_{\tilde{b}_i}(b_i)$ over all $\tilde{c}_{kj}$, $\tilde{a}_{ij}$ and $\tilde{b}_i$.

We will construct the possibility distribution of $\tilde{Z}$ in terms of its α-cuts. Some definitions and results are needed.

2.1 Definition 1

$$\begin{aligned}\Omega(\alpha) = \quad & \{\ [z]\ /\ \forall z \in [z], z \text{ is a POS of the LGP}(A, b, c; g) \\ & \text{with} \quad A \in A_\alpha,\ b \in b_\alpha,\ c \in c_\alpha\}\end{aligned}$$

where A_α, b_α, c_α are the α-cuts of $\tilde{A}$, $\tilde{b}$, $\tilde{c}$.

2.2 Definition 2

For any (A, b, c) we define the function f such that

$$f(A, b, c) = [z] = \{\ z/z \text{ is a POS of the LGP}(A, b, c; g)\}$$

that we suppose non-empty. Then

$$\Omega(\alpha) = f\,(A_\alpha \times b_\alpha \times c_\alpha) = f\,\left([A^L_\alpha, A^R_\alpha] \times [b^L_\alpha, b^R_\alpha] \times [c^L_\alpha, c^R_\alpha]\right).$$

2.3 Theorem 1

f is upper semicontinuous (hemicontinuous) in its domain, $\mathcal{D}$.

2.3.1 Proof

Let any (A_0, b_0, c_0) of $\mathcal{D}$ and $[z_0] = f(A_0, b_0, c_0)$. It is possible to prove that $\forall \epsilon > 0\ \exists \delta > 0$ such that:

$$[z] = f(A, b, c) \subset B([z_0], \epsilon)$$

for all (A, b, c) in $\mathcal{B}((A_0, b_0, c_0), \delta)$ $\square$

2.4 Theorem 2

Let be $[z] = f(A, b, c)$ and $[z'] = f(A', b', c')$ where $c < c'$, $b < b'$ and $A' < A$ then:

$\forall z \in [z] \quad \exists z' \in [z'] \quad$ s.t. $\quad z < z' \quad$ and $\quad \forall z' \in [z'] \quad \nexists z \in [z] \quad$ s.t. $\quad z' < z$

We propose the method below: from a POS of the LGP$(A, b, c; g)$, $z = cx$, a POS of the LGP$(A', b', c'; g)$, z', will be obtain so verifying $z \leq z'$:

If $c'x$ is a POS of the LGP$(A', b', c'; g)$ then $z' = c'x$; if no we will applie the optimality test:

$$\begin{array}{l} \text{maximize} \displaystyle\sum_{r=1}^{k} \beta_r \\ \text{subjec to} \\ c'_r y - \beta_r = c'_r x \;\; r = 1, \ldots, k \\ y \in \mathcal{X}(A', b') \\ \beta_r \geq 0, \;\; r = 1, \ldots, k \end{array}$$

that produce a POS greater or equal than $c'x$.

2.5 Corollary 1

Let $[z_\alpha]^L = f(A^R_\alpha, b^L_\alpha, c^L_\alpha)$ $\quad [z_\alpha]^R = f(A^L_\alpha, b^R_\alpha, c^R_\alpha)$ then $\quad \forall [z'] \in \Omega(\alpha)$

$$(I) \quad \forall z \in [z_\alpha]^L, \quad \exists z' \in [z'] \quad \text{s.t.} \quad z \leq z'$$

$$(II) \quad \forall z' \in [z'] \quad \nexists z \in [z_\alpha]^L \quad \text{s.t.} \quad z' < z$$

Analogous result is obtained for the $[z_\alpha]^R$. □

2.6 Proposition 1

From theorem 2 and the α-cuts properties, we have that if $\alpha_1 < \alpha_2$ then

$$(I) \quad \forall z \in [z_{\alpha_1}]^L \quad \exists z' \in [z_{\alpha_2}]^L \text{ such that } \quad z < z'$$
$$(II) \quad \forall z \in [z_{\alpha_2}]^R \quad \exists z' \in [z_{\alpha_1}]^R \quad \text{such that} \quad z < z' \quad \square$$

2.7 Theorem 3

From theorem 1 it is possible to prove that

$$(I) \quad \text{For any} \quad [z] \in \Omega(\alpha) \quad \text{then} \quad [z] \subset Z_\alpha.$$
$$(II) \quad \text{If } z \in Z_\alpha \quad \text{then} \quad \exists [z] \in \Omega(\alpha) \quad \text{s.t.} \quad z \in [z].$$

2.8 Definition 3

Let

$$\begin{aligned}\Theta(\alpha) &= \{z/\ z \text{ is POS of the LGP}(A,b,c;g) \\ &\quad \text{for } A \in A_\alpha, b \in b_\alpha, c \in c_\alpha\}\end{aligned}$$

From Theorem 3 we have concluded that $\Theta(\alpha) = Z_\alpha$ and from Zorn's axiom, there is a function ϕ defined on $\mathcal{P}(\Theta(\alpha))$ that reaches an element in each subset of $\Theta(\alpha)$, in particular in those subsets formed for all POS of LGP$(A,b,c;g)$ for each (A,b,c), i.e., on the elements of $\Omega(\alpha)$. Let ϕ' be the restriction of ϕ to $\Omega(\alpha)$.

2.9 Lemma 1

(I) For any function ϕ we have that $\phi' \circ f$ is a continuos function.

(II) It is possible to find a function ϕ such that if we denote $\phi'([z_\alpha]^L) = z_\alpha^L$ then $\exists z_\alpha^R = \phi'([z_\alpha]^R$ such that $z_\alpha^L \leq z \leq z_\alpha^R \quad \forall z \in \phi'(\Omega(\alpha))$.

These results are consequences from theorem 1 and corollary 1, respectively.

2.10 Lemma 2

From lemma 1 we have

$$\phi' \circ f\,(A_\alpha \times b_\alpha \times c_\alpha) = \phi'\,(\Omega(\alpha))$$

is a connected set in $\mathbb{R}^k$. Broadly speaking, from the above lemmas we could consider:

$$\phi'(\Omega(\alpha)) = [z_\alpha^L, z_\alpha^R]$$

2.11 Theorem 4

For any $\alpha \in [0,1]$ we have that $p_r \circ \phi'\,(\Omega(\alpha)) \quad r = 1,\ldots,k$ is an interval.

2.12 Proposition 2

Let ϕ_0' the function corresponding to $\Omega(0)$ that verify the above lemmas.

If $\alpha_1 < \alpha_2$ and $\phi_0'(\Omega(\alpha_1)) = [z_{\alpha_1}^L, z_{\alpha_1}^R]$, $\phi_0'(\Omega(\alpha_2)) = [z_{\alpha_2}^L, z_{\alpha_2}^R]$ then:

$$(a) \quad z_{\alpha_1}^L < z_{\alpha_2}^L; \quad (b) \quad z_{\alpha_2}^R < z_{\alpha_1}^R$$

2.13 Main results

(I) For any z_0^L, POS for the LGP$(A_\alpha^R, b_\alpha^L, c_\alpha^L; g)$ with $\alpha = 0$ there exists an increasing chain $z_0^L < \ldots < z_\alpha^L < \ldots < z_1^L < z_1^R < \ldots < z_\alpha^R < \ldots < z_0^R$ of POS of the general problem.

(II) Based on that, we may set the possibility distribution of $\tilde{Z}$ considering the above elements as the α-cuts bounds of it.

(III) If we define $\Pi(\tilde{z}_r = z_r) = \Pi(\tilde{Z} = z)$ with $p_r(z) = z_r$ then we have proved that $\tilde{z}_r$ **is a fuzzy number.**

2.14 Numerical example

Consider the multiobjective linear program with fuzzy parameters:

$$\max \tilde{Z} = \tilde{C}x = (\tilde{c}_1 x, \tilde{c}_2 x, \tilde{c}_3 x)$$
$$\text{subject to}$$
$$\left.\begin{array}{l} \tilde{a}_{11}x_1 + 17x_2 \leq 1400 \\ 3x_1 + 9x_2 + \tilde{a}_{23}x_3 \leq 1000 \\ 10x_1 + \tilde{a}_{32}x_2 + 15x_3 \leq 1750 \\ \tilde{a}_{41}x_1 + 16x_3 \leq 1325 \\ \tilde{a}_{51}x_2 + 7x_3 \leq 900 \\ 9.5x_1 + \tilde{a}_{62}x_2 + 4x_3 \leq 1075 \\ x_j \geq 0,\ j = 1, \ldots, n \end{array}\right\} \equiv x \in \mathcal{X}(\tilde{A}, \tilde{b})$$

where $\tilde{c}_1 = (\tilde{c}_{11}, 100, 17.5)$, $\tilde{c}_2 = (92, \tilde{c}_{22}, 50)$ and $\tilde{c}_3 = (\tilde{c}_{31}, 100, 75)$.

We shall asume that all the fuzzy coefficients are characterized by the triangular possibility distributions shown in Table 1:

	$\tilde{c}_{11}$	$\tilde{c}_{22}$	$\tilde{c}_{31}$	$\tilde{a}_{11}$	$\tilde{a}_{23}$	$\tilde{a}_{32}$	$\tilde{a}_{41}$	$\tilde{a}_{51}$	$\tilde{a}_{62}$
$\mathbf{n_1}$	40	70	10	6	3	7	4	7	3.5
$\mathbf{n_2}$	50	75	25	12	8	13	6	12	9.5
$\mathbf{n_3}$	80	90	70	14	10	15	8	19	11.5

The first step is to obtain the pay-off tables for $\alpha = 1$ and for $\alpha = 0$ in the worst linear program possible and considering these values the DM specifies the target values for each objective function. The FP-MOLP then is converted into:

$$\begin{array}{l} \text{Find a POS } x = (x_1, x_2, x_3) \text{ s.t.} \\ \tilde{z}_1 = \tilde{c}_1 x = (40, 50, 80)x_1 + 100x_2 + 17.5x_3 \geq (6000, 7500, 7500) \\ \tilde{z}_2 = \tilde{c}_2 x = 92x_1 + (70, 75, 90)x_2 + 50x_3 \geq (7000, 10000, 10000) \\ \tilde{z}_3 = \tilde{c}_3 x = (10, 25, 70)x_1 + 100x_2 + 75x_3 \geq (6500, 9000, 9000) \\ x \in \mathcal{X}(\tilde{A}, \tilde{b}) \end{array}$$

The lowest optimum at a level of possibility α, z_α^L, can be determined by solving the worst linear program at the possibility level α, that corresponds to a classical linear program with the most restricting constraints and the lowest beneficial objective function. Conversely, the linear program with the most beneficial objective and the weakest constraints at possibility level α produces the highest optimum, z_α^R.

Tables 1 and 2 present the solutions obtained by solving the worst and the best linear programs at possibility level 0, 0.25, 0.5, 0.75 and 1, respectively.

α	z_1	z_2	z_3
0	6061.431624	9762.962963	7196.225071
0.25	6439.877690	9811.146851	7645.386586
0.5	6868.717458	9871.863220	8139.315194
0.75	7370.550913	9950.732487	8698.087537
1	7983.867055	10057.326541	9355.895197

Table 1. The worst solutions.

α	z_1	z_2	z_3
0	12324.702288	13984.424945	13880.209966
0.25	10896.591899	12658.120645	12353.614983
0.5	9716.326531	11870.561224	11102.806122
0.75	8827.692756	10680.000613	10448.123045
1	7983.867055	10057.326541	9355.895197

Table 5. The best solutions.

3 Choice of the decision vector

The possibility distribution of $\tilde{Z}$ provide a possibilistic assessment of the risk involved in a decision problem. The DM must achive a trade-off between optimal objective value and the decision vector x, corresponding to it. This choice should be based on the attitude towards risk of the DM.

It is neccesary, for each problem, to choose the decision vector x such that $\tilde{c}x$ be the best possible with respect to $\tilde{Z}$.

Then we need to compare fuzzy numbers and first of all it is neccesary to set up a criterion. The method [7] used here is based on a fuzzy preference relation $\mu(\tilde{A}, \tilde{B})$, that is a simplification of the Yuan method [10].

The degree in wich a fuzzy number $\tilde{A}$ is greater than $\tilde{B}$ is given by:

$$\mu(\tilde{A},\tilde{B}) = \begin{cases} 0 & E_2^A - E_1^B < 0 \\ \dfrac{E_2^A - E_1^B}{E_2^A - E_1^A + E_2^B - E_1^B}; & 0 \in [E_1^A - E_2^B, E_2^A - E_1^B] \\ 1 & E_1^A - E_2^B > 0 \end{cases}$$

Let $\mu(\tilde{A},\tilde{B}) = \lambda$, if $\lambda > \frac{1}{2}$ $\tilde{A}$ is prefered to $\tilde{B}$ with degree λ, if $\lambda < \frac{1}{2}$ $\tilde{B}$ is prefered to $\tilde{A}$ with degree $1-\lambda$ and if $\lambda = \frac{1}{2}$ $\tilde{A}$ and $\tilde{B}$ are indiferent.

According to above and to generate a candidate for the Pareto optimal solution which will be also satisfying, the DM is asked to specify the degree of aversion towards risk for each constraints h_i, $i = 1,\ldots,m$ and reference values $\bar{\mu}_r$, $r = 1,\ldots,k$. The FP-MOLP can now be transformed into:

$$\begin{aligned} &\text{maximize } (\mu(\tilde{c}_1 x, \tilde{z}_1),\ldots,\mu(\tilde{c}_k x, \tilde{z}_k)) \\ &\text{subjec to} \\ &\mu(\tilde{b}_i, \tilde{a}_i x) \geq h_i,\ i = 1,\ldots,m \\ &x \geq 0 \end{aligned}$$

Once the DM's attitude towards risk and reference values are specified, the corresponding Pareto optimal solution, which is in the minimax sense nearest to the reference values or better than them, is obtained. It is possible that the solution given by the "min" operator may not be a Pareto optimal solution and for circumventing the necessity to perform the optimality tests, we have considered reasonable to use augmented minimax problems instead of minimax problems:

$$\begin{aligned} &\text{minimize } \left\{ \max_{r=1,\ldots,k} \left(\bar{\mu}_r - \mu(\tilde{c}_r x, \tilde{z}_r) + \rho \sum_{r=1}^{k} (\bar{\mu}_r - \mu(\tilde{c}_r x, \tilde{z}_r)) \right) \right\} \\ &\text{subjec to} \\ &\mu(\tilde{b}_i, \tilde{a}_i x) \geq h_i,\ i = 1,\ldots,m \\ &x \geq 0 \end{aligned}$$

or equivalently

$$\begin{aligned} &\text{minimize } \lambda + \rho \sum_{r=1}^{k} (\bar{\mu}_r - \mu(\tilde{c}_r x, \tilde{z}_r)) \\ &\text{subjec to} \\ &\bar{\mu}_r - \mu(\tilde{c}_r x, \tilde{z}) \leq \lambda \\ &\mu(\tilde{b}_i, \tilde{a}_i x) \geq h_i,\ i = 1,\ldots,m \\ &x \geq 0 \end{aligned}$$

In our example, the DM sets the initial reference values $\bar{\mu}_r = 1$, $r = 1, \ldots, k$ and the degree of aversion towards risk $h_1 = h_2 = 1$, $h_3 = h_4 = 0.8$ and $h_5 = h_6 = 0.5$ and $\rho = 0.0001$. Working on the second formulation of the problem we obtain the solution above:

$$\begin{array}{ll} x_1 = 57.550036 & \mu(\tilde{c}_1 x, z_1) = 0.31093 \\ x_2 = 38.344090 & \mu(\tilde{c}_2 x, z_2) = 0.31094 \\ x_3 = 44.557176 & \mu(\tilde{c}_3 x, z_3) = 0.577824 \end{array}$$

Next, let $h_1 = 0.6$ and solve the problem again. The results are

$$\begin{array}{ll} x_1 = 56.907108 & \mu(\tilde{c}_1 x, z_1) = 0.384448 \\ x_2 = 41.325589 & \mu(\tilde{c}_2 x, z_2) = 0.384448 \\ x_3 = 41.028743 & \mu(\tilde{c}_3 x, z_3) = 0.598372 \end{array}$$

4 Summary and conclusions

We have shown a fuzzy solution of the FP-LGP whose α-cuts are connected and nested and whose bounds have been obtained solving the "worst" , LGP$(A^R_\alpha, b^L_\alpha, c^L_\alpha; g)$, and the "best", LGP$(A^L_\alpha, b^R_\alpha, c^R_\alpha; g)$, problems defined for each $\alpha \in [0, 1]$.

From the above solution and based on the attitude of the DM towards risk we have given a method of selecting a decision vector corresponding to that $\tilde{Z}$ applying a particular fuzzy preference relation between fuzzy numbers.

An advantage of the proposed approach is that the DM can participate fruitfully in the process because, the concept of goal and fuzzy sets suited very well to practical situations, where the process has been operating for some time and the DM has a fairly good idea about the process.

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COPE - ing with V•I•S•A - Integrated Support from Problem Structuring through to Alternative Evaluation

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Abstract

Work in the field of multiple criteria analysis has generally focused on evaluation procedures, taking as its starting point a well defined problem with specified alternatives and criteria. However, in reality problems are rarely so well structured; hence, in order to usefully support decision making in practice, multiple criteria analysts need to address the issue of problem structuring. In this paper we describe a study which sought to integrate SODA (Strategic Options Development and Analysis), an approach to problem structuring which uses the COPE software for cognitive mapping, with multiple criteria evaluation based on a multi-attribute value function using V•I•S•A . The study took the form of a two day action research workshop to explore the strategic direction of the Supplies and Commercial Services Department of a large UK NHS Hospital Trust and to develop an action plan consistent with the agreed direction.

Keywords Problem structuring, cognitive mapping, multiple criteria analysis

1 Introduction

Work in the field of multiple criteria analysis has generally focused on evaluation procedures, that is, it takes as its starting point a well defined problem with specified alternatives and a set of criteria against which these are to be evaluated. In practice, problems are rarely so well structured. Even if the alternatives are apparently clear, time has to be spent on defining objectives or relevant criteria. However, often an investigation starts with nothing more than a broad statement of an issue of concern, for example:

How can we reduce traffic congestion in the city?

What is the best future for the international MCDA community?

These issues clearly involve multiple, conflicting goals, and it is likely that there will be alternative courses of action, or strategies, to be considered - however, we are a long way from the well defined input required for any multiple criteria assessment tool. If multiple criteria analysis is to be accepted as an appropriate tool for problem solving in organisations far more attention must be paid to problem structuring, a view also put forward by Bouyssou et al[8] in their Manifesto for a New Era of MCDA and supported by others, for example, Belton[6], Bana e Costa[5], Buchanan and Henig[9]. In this respect there is much to be learned from the body of work stemming from the fields of Operational Research and Systems in the UK, collectively referred to as Problem Structuring Methods (see Rosenhead[20]). This umbrella term encompasses SODA - Strategic Options Development and Analysis developed by Eden et al[12,13], which uses the COPE software for cognitive mapping. Each of these problem structuring approaches pays attention to multiple objectives and multiple perspectives in a more or less formal way. Clearly there is potential to integrate multiple criteria analysis within these broader problem structuring methods, as has been advocated by a number of authors (Belton[6], Bana e Costa[5], Watson and Buede[21]). Whilst a theoretical consideration of the joining of different approaches allows one to speculate on potential benefits, we feel that most can be learned from an action research approach[14] - that is, from trying to integrate their use in practice with a client group.

In an earlier paper Ackermann and Belton[3] focus on common themes in the use of SODA and V•I•S•A to facilitate the management of corporate knowledge, in particular in the way in which computer technology is used to facilitate the acquisition, organisation and utilisation of knowledge. Important common features are the provision of interactive facilities and the visual display of information. This provides the opportunity to 'play' with the body of knowledge, or model, which is created, enables the exploration of issues or alternative viewpoints, the investigation of the implications of changed priorities, or, the introduction of new alternatives. The model acts as a 'sounding board', against which the group can confirm or challenge their intuitive beliefs. Furthermore it prompts discussion and debate, often acting as a catalyst for the generation of new ideas. This existing commonality in the way of working should serve to facilitate the integration of the use of the approaches in practice.

In this paper we describe briefly a two day action research workshop to investigate the integrated use of COPE and V•I•S•A to explore the strategic direction of the Supplies and Commercial Services Department of the United Leeds Teaching Hospitals (ULTH) Trust. The methodologies which underlie COPE and V•I•S•A will be outlined in the following sections before going on to describe the workshop in section 4. In section 5 we reflect on this experience and suggest fruitful areas for future research.

2 Introduction to SODA and COPE

SODA is based upon a theoretical framework focusing predominantly on Personal Construct Theory (Kelly[19]) and a technique called cognitive mapping (Eden[11]). Maps aim to represent the problem/issue as the decision maker (participant) perceives it, in the form of a means-ends network-like structure, and is usually generated during a interview. An alternative method of capturing material is through *structured* brainstorming sessions (Ackermann[1]). As an individual map may include between 50 to 100 ideas, a software tool - COPE - has been developed to capture the contents and associated structure (see figure 2). It provides support to groups as individual maps can be 'woven' together to form a group composite map reflecting the views of all of the group members. During a workshop, this model facilitates the conversation process which in turn enables group members to develop a common understanding of the issues and to use the model as a 'negotiative device'. As a result, participants develop a clear idea of their strategic direction and can agree upon a set of actions whilst the model can be used in the longer term as an 'organisational memory' or decision support tool.

3 Introduction to V·I·S·A

V•I•S•A is a multi-criteria decision support system which is based on a multi-attribute value function, a conceptually simple but theoretically well founded and empirically well validated approach to multiple criteria analysis (see, for example, Von Winterfeldt and Edwards[22]). The underlying model is V_i = SUM_j (w_j v_{ij}), where V_i is the overall evaluation of option I, w_j is the weight assigned to criterion j, v_{ij} is the score of option i on criterion j. Criteria are generally structured as a value tree, or criteria hierarchy. An important and distinctive feature of V•I•S•A is its extensive facility for visual interactive sensitivity analysis (hence the name), which enables decision makers to explore the implications of changing or differing priorities and values.

An important part of the process is the elicitation and structuring of criteria in a hierarchical form appropriate for analysis. There is no single, "best" way of doing this; different approaches are discussed by Von Winterfeldt and Edwards[22], Buede[10] and by Keeney[18]. It is this aspect of analysis which we anticipate will integrate naturally with the use of COPE. The detailed evaluation of alternatives calls for the acquisition of information reflecting the performance of the alternatives with respect to the identified criteria (scores in the above model) and capturing feelings about the relative significance, or importance, of the different criteria (weights in the above model). This information is then synthesized to indicate the preferred course of action and form the basis of an extensive investigation of the sensitivity, or robustness of preferences for the alternative courses of action. This part of the process helps to cement understanding of the model and develop a shared ownership of any recommended outcome. Such analysis would not normally be part of a SODA workshop.

4 Background to the Study

The United Leeds Teaching Hospital Trust was established in 1991 as one of the first wave of UK hospital trusts. The work described here was in collaboration with the Supplies and Commercial Services department which is part of the Facilities Directorate of the Trust.

The senior management team of the Facilities Directorate were aware of " ..the potential for significantly improving value for money in relation to the procurement and use of materials and in the purchasing of services by adoption of current good practice and by exploiting developing technology". They envisaged the establishment of an integrated supply chain as a key goal of such change. This would largely replace the numerous procurement and distribution routes which existed for different types of materials and services such as Supplies, Pharmacy, Catering, etc. However, it was recognized that to achieve such radical change, relationships between the Supplies function and their customers would have to be carefully managed to achieve a mutual understanding of the problems created by existing processes and the benefits which alternative ways of working could bring. The Trust recognized the potential contribution of external facilitation in this process and wished to explore various possibilities. The Management Science Department at Strathclyde University has established links with the Facilities Directorate of the Trust and senior managers were aware of work using COPE and V•I•S•A. This led to the agreement to carry out a two day action workshop investigating the use of COPE and V•I•S•A to explore the strategic direction of the Supplies and Commercial Services Department with a view to developing an action plan consistent with the agreed direction. Thus, the workshop was exploratory from both the viewpoint of the facilitators and the senior managers of the Facilities Directorate.

5 The Workshops

The workshops took place on two separate days within a two week period and involved twelve staff from ULTH and the two facilitators. The staff members included a number of managers from the Supplies function, senior managers in the Facilities Directorate and individuals representing Supplies' customers, namely Catering, Nursing , Pharmacy, Pathology, Hospital Services, Therapy services, Capital Planning and Estates. One of the facilitators was experienced with working with groups using COPE, the other with V•I•S•A; each had a basic knowledge of the other approach.

The first workshop commenced with a session dedicated to idea generation. Focusing upon a set question " *how can the supplies and distribution services of the Trust be improved"* and using a well established approach, participants were encouraged to write their ideas down onto ellipses (dominoes) which they then 'stuck' to a wall covered with flip chart paper. (see Ackermann[2] for detail s of the Domino Technique). Thus, each participant had the opportunity to develop the areas of concern to him or her, for example, the supplies managers were aware of potential high-tech developments whereas some customers were more conscious of issues at ward level, without being constrained by the need to provide others with 'air-time'. As they were placed onto the wall the ideas were numbered and moved into groups or clusters of similar material by the facilitator. This was to impose some structure on the emerging picture to help facilitate understanding. At the same time the ideas were entered directly into the computer using the COPE software so that a model could be produced and used later on during the day.

Once the process of idea generation had slowed down, the facilitator quickly reviewed the clusters of ideas generated by the participants to ensure that they were all familiar with the key issues which had emerged. The clusters, each corresponding to a key issue, were then examined in more detail to ensure that ideas were in the right cluster and to begin to develop a hierarchy of ideas within each cluster. This process of determining a hierarchical structure not only helps in producing a clear understanding of the key issue (the most superordinate idea) but also suggests which are the possible actions (the most subordinate ideas) which impact on that key issue, as illustrated in figure 1.

The process of agreeing which is the outcome and which is the option when examining a pair of ideas stimulates debate and generally yields further contributions as participants discover that they have attributed different meanings to the ideas written on the cards. This process of developing a sense of shared meaning is very powerful as it helps participants comprehend the different perspectives on the strategic issue being explored. The process and resultant benefits described here are very similar to the approach used to elicit criteria and structure a value tree within a multi-criteria analysis described by Belton[7,8]. Part of the COPE map is shown in figure 2.

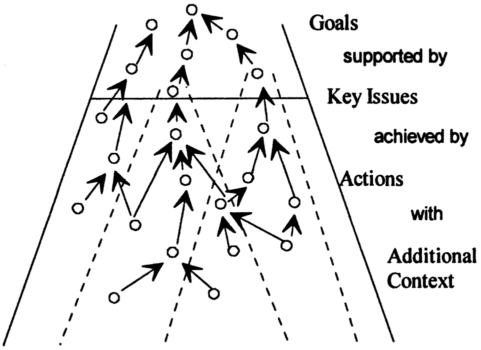

Figure 1 Hierarchical structure of a COPE map

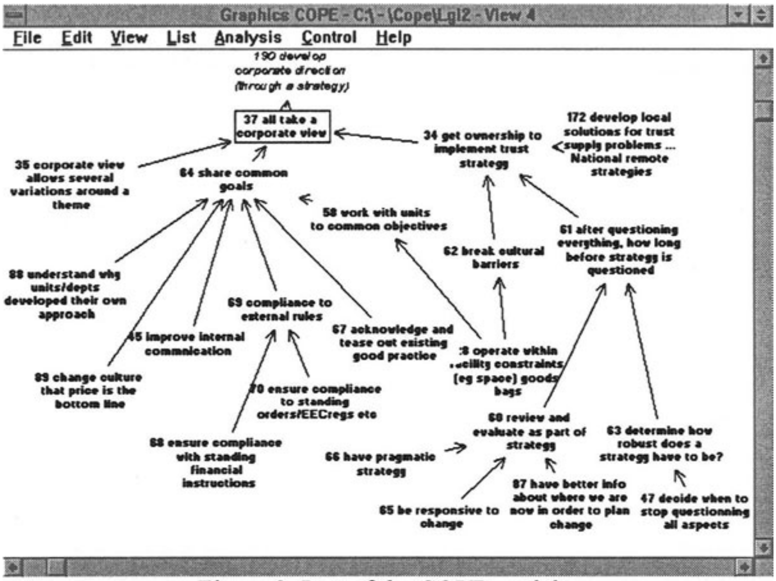

Figure 2 Part of the COPE model

In order to provide a sense of achievement and closure to the morning session and to give the facilitators some indication as to the general sense of priorities, a crude voting exercise was undertaken. Each participant was provided with a number of 'votes' (sticky dots) to be allocated to the key issues which they believed were the most important. They could place all of their votes on one key issue if they felt it to be of paramount importance or scatter them across the clusters. This exercise not only enabled the participants to discover which areas they felt were the most important but also gave the facilitators a clear indication of which key issues they should develop and elaborate.

After lunch participants were divided into four groups and asked to undertake a 'role think' exercise (see Eden and Huxham[14]). Each group was asked to take on a different role, for example, the Trust Board, and to consider how they would respond if ULTH was actually to implement a number of the actions generated during the morning session. In this way the actions could be tested to gain an initial idea of the possible reactions of various stakeholders and if necessary amended. Participants were also asked to explicate why (in their role) they had responded in that manner - ie what goals were informing their reactions. Valuable information concerning stakeholder aims and objectives could be extracted from these responses.

The day concluded with a session working with the computer model to elaborate and refine a key issue; attention was focused on *"get closer to the customer"* which earlier had gained the most votes. New material was entered directly into the model by the facilitator and linkages were added or revised as requested by the group. Working with the computer model firstly demonstrated that all of the material had been captured correctly - participants could see the ideas on the screen alongside the picture on the wall and if concerned could check one alongside the other. Secondly it facilitated the management of the growing number of relationships between the ideas which could not adequately be represented in two dimensional space. As seen in figure 3, different styles for key issues and context ensured that participants could easily distinguish between them. The day concluded with a recap of the key issues and how they related to one another.

Between the workshops the facilitators worked on tidying up the map and preparing an Overview which captured the key issues. This served as the starting point for the second workshop which began with a quick review of what had been achieved on the first day.

Time was then spent on further elaboration of the issue which had been identified as second priority, namely *to ensure goods get to the right place at the right time.* This allowed the participants to get back into the problem and reengage with the issues under discussion. Attention was then focused on the Overview with a view to identifying goals and how these are supported by key issues. The map developed by the group to represent the goals, key issues and relationships between them is shown in figure 3. This was to form the basis of the value tree to be used in a more detailed multi-criteria analysis.

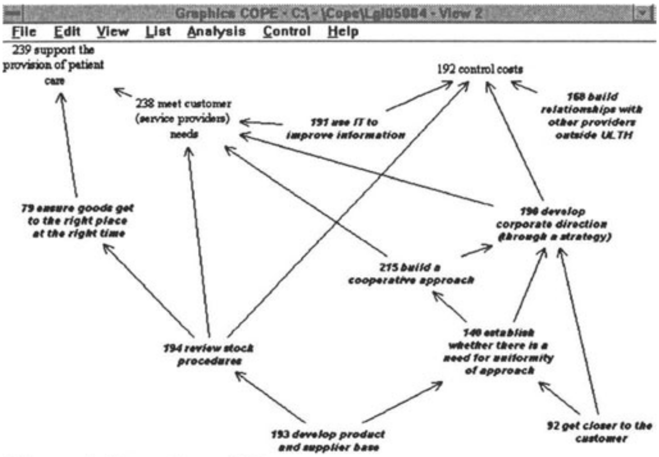

Figure 4 Overview of the map

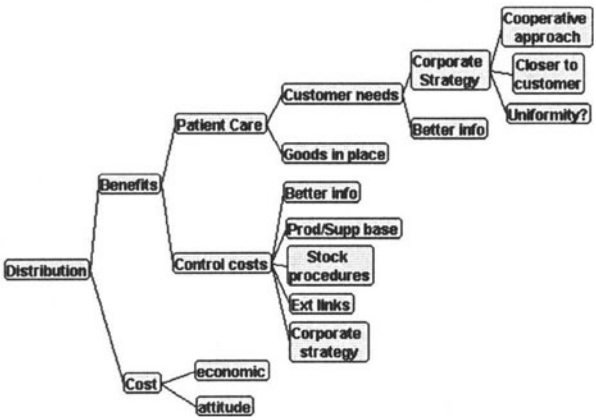

Figure 3 Value tree

The process of translating the overview map to a value tree, shown in figure 4 was done by the facilitators rather than directly involving the workshop participants. This is one of the key elements of linking the two approaches and will be discussed in greater detail in the final section. Whilst the facilitators were working on this the group was once again split into four groups; this time each group was tasked with identifying four specific activities which would allow the Supplies department to progress the goals which had been identified. A wide range of possible actions were identified, encompassing high-tech solutions, information gathering exercises and promotion exercises. Following the presentation of the actions to the whole group a voting process was used to select a subset (four) for more detailed evaluation.

The value tree was presented to and accepted by the group at this stage. Scoring the alternatives was done mostly using local scales - i.e. the highest and lowest rated alternative on each criterion are given scores of 100 and 0 respectively. Given the limited time available this was an appropriate means of speeding the process. A ranking of criteria weights within each family was established by asking "*if given the opportunity to move from an action which scores 0 on* ***all*** *criteria to one which scores 100 on any* ***one*** *criterion, which criterion would you choose?*". The process of elicitation was supported by the interactive use of V•I•S•A, using the thermometer scales for entering scores and bar charts for weights. The synthesis of information relating to the control of costs is shown in figure 5.

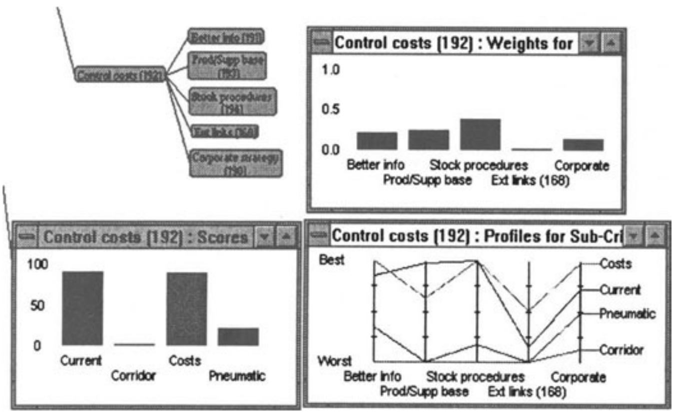

Figure 5 Part of the V•I•S•A analysis

Similar displays were explored for all other parts of the tree, confirming that the evaluation was in line with the feelings of the group.

6 Reflections

As described in the Introduction to the paper, the workshops were intended to be exploratory. As such, it was never anticipated that a clear action plan would emerge. Nevertheless, the workshops did enable the group to make progress towards a definition of the strategic direction and led to an increased understanding and awareness of the issues. From the facilitators' perspective the experience highlighted three issues pertinent to the integration of these approaches, as discussed below.

Working iteratively The action research design had anticipated the sequential use of the methodologies. However, it was clear from this intervention that a far more powerful outcome would stem from a more complete integration, cycling between the approaches and sometimes using them in parallel. This was particularly evident when V•I•S•A was being used to evaluate the options. The process of scoring and weighting generates extensive discussion about the criteria used. Normally this dialog is captured only by the numerical value assigned to represent a score or weight. However, the parallel use of COPE to elaborate the group map can, ensure that the information is captured for future reference and enrich the overall picture and understanding.

Role of facilitators The use of approaches in parallel as described above has implications for facilitation. We feel that from a practical standpoint it would be difficult for one facilitator to embrace both processes simultaneously, emphasising the need for two facilitators. In an integrated intervention it is more likely that the roles will be shared by the two facilitators and the balance will change throughout the workshop.

Translating the overview of the COPE map to a value tree This was carried out by the two facilitators working together (independently of the participants) due to their being embedded in the content and familiar with the two methodologies. Although it could be desirable to involve the group, it would not be easy to explain to them what was required or why. Furthermore, it was not a straightforward task, as the structure of a cognitive map is different to that of a value tree. Within the cognitive map there are both direct and indirect links between concepts; in addition a single concept can impact on and be supported by several others. These issues were handled by the facilitators in an informed, but ad-hoc way, suggesting a need for more in depth research.

In addition to these, which are of considerable significance for the integration of the approaches, other, less significant but relevant observations can be made.

- It is important that both facilitators are familiar with both methodologies..
- Using two independent pieces of software did not appear to have a detrimental effect on the process - however, integration at this technical level can easily be achieved
- It is important to be aware of issues pertaining to the environment (Huxham[27], Hickling[26]).

In conclusion, we feel that the intervention has shown that it is possible to successfully combine these two approaches to provide useful decision support in a workshop environment. Moreover, we feel that for some problems the insights and understanding arising from the combined use of the approaches would be greater than from either used in isolation. However, this work has highlighted a number of areas for consideration and further research, as outlined above.

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An Algorithmic Package for the Resolution and Analysis of Convex Multiple Objective Problems

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Abstract. The aim of this paper is to describe an algorithmic package which allows us to carry out a complete treatment of a general multiple objective convex problem. It includes the generalisation of many of the algorithms used in the linear case, as well as some others developed specially for our problem. This treatment can be divided into two main blocks:

- *Generation of efficient solutions:* both the weighting and the constraint method are developed, through an automatic generation of weights in the former, and of bounds in the latter.
- *Goal Programming:* We include two versions of the traditional lexicographic algorithms, adapted to the convex case under study, and we also allow the possibility to generate the set of solutions which are satisfying and efficient at the same time. Finally, we also carry out a post-optimal analysis on the target values, so as to find whether they can be improved or not. This analysis, which takes the form of an interactive method, can even lead to an efficient, as well as satisfying, solution for the original problem.

Some computational results are presented, which show the behaviour of the algorithms, in terms of C.P.U. time, on some test problems with different number of variables and constraints. These algorithms have been implemented in FORTRAN language, on a VAX 8530 computer, and with the aid of the NAG subroutine library, mark 15.

1 Introduction

Throughout this paper, we will deal with a general non-linear convex multiple objective problem, under the following formulation:

$$(P)\begin{cases} min\, \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}),\ldots,f_p(\mathbf{x})) \\ s.t. \quad \mathbf{g}(\mathbf{x}) \leq \mathbf{b} \end{cases}$$

where all the functions f_i, $i = 1,\ldots, p$, and g_j, $j = 1,\ldots, m$ are convex in the feasible set X of (P), and $\mathbf{x} \in R^n$.

Our aim will be the development of a family of algorithms, that let us treat the problem under study, that can be the base of a future software package which intends to be the complement of other packages existing for linear problems (see, for example, [12]). To this end, we adapt some of the most widely used techniques for linear MOP, namely the weighting and constraint methods, and the lexicographic goal programming approach. We will centre our attention in the modifications made so as to apply them to our problem, as well as in the implementation aspects and the computational results.

2 Generating techniques

2.1 Theoretical background

The fact of working with more than one objective function, brings in a natural way the necessity to compare and put vectors of R^p in order. This ordering is generally carried out through the definition of a preference structure given by some domination cone D (see [8]). Nevertheless, when no a priori information is given to the analyst about the decision maker's preferences, the former can only take for granted that one particular solution dominates other if it improves or takes the same value for all the objective values. This fact yields to the consideration of the domination cone R^{p+}, and the corresponding efficiency concept: Pareto optimality.

Apart from this concept, we will also use the weak Pareto optimality (corresponding to the cone $int(R^{p+})$, that is, dominance occurs when all the objective functions are strictly improved), and Kuhn-Tucker proper efficiency (which is equivalent to Geoffrion efficiency if a constraint qualification is satisfied, under our convexity hypothesis). If we denote by $E(P)$ the set of the efficient solutions for (P), $E^w(P)$ the set of weakly efficient solutions, and $K(P)$ the set of properly efficient solutions, then we have:

$$K(P) \subset E(P) \subset E^w(P).$$

The two generating techniques that are developed in this paper correspond to two different ways of characterising these sets, as solutions of problems obtained through a scalarisation of the original multiple objective problem ([11]).

- *Characterisation as solutions of weighting problems.* For each vector of weights $\mu \in R^p$, the following problem can be defined:

$$(P_\mu)\begin{cases} min & \mu'\mathbf{f}(\mathbf{x}) \\ s.t. & \mathbf{g}(\mathbf{x}) \leq \mathbf{b} \end{cases}$$

If we denote its set of optimal solutions by $S(P_\mu)$, then two sets can be defined:

$$D(f,g) = \bigcup_{\mu \in \overset{\circ}{\Re}^{p+}} S(P_\mu)$$

$$D'(f,g) = \bigcup_{\mu \in \Re^{p+}\setminus\{0\}} S(P_\mu)$$

That is, D is the set of solutions of weighting problems with all the weights strictly positive, while D' allows to use some zero weight (not all). then, it can be proved that, under our convexity hypothesis, D coincides with $K(P)$ and D' coincides with $E^w(P)$. Besides, if a point of D' is the unique solution of the corresponding weighting problem, then it can be affirmed that it is Pareto optimal.

- *Characterisation as solutions of constraint problems:* For each vector of bounds $\alpha \in R^p$, p constraint problems can be defined:

$$P_j(\alpha)\begin{cases} min & f_j(\mathbf{x}) \\ s.t. & \mathbf{g}(\mathbf{x}) \le b \\ & f_i(\mathbf{x}) \le \alpha_i \quad i = 1,\dots,p \quad i \ne j \end{cases}$$

  Then, it can be affirmed that $\mathbf{x}^*$ is Pareto optimal for (P) if and only if it is an optimal solutions for all the problems $P_j(\alpha)$, $j = 1,\dots, p$, with $\alpha = \mathbf{f}(\mathbf{x}^*)$. Besides, if $\mathbf{x}^*$ is an unique solution for some problem $P_j(\alpha)$, for arbitrary j and α, then it can be proved that $\mathbf{x}^*$ is Pareto optimal. If this uniqueness condition is not satisfied, then we can only affirm that $\mathbf{x}^*$ is weakly efficient.

2.2 Weighting algorithm

As it has been previously stated, one way of obtaining efficient points consist just in solving weighting problems for non negative weight vectors. Anyway, if we want to obtain a significant approximation of the efficient set, these vectors must be chosen in a logical way. To this end, the weighting algorithm has the following features:

- In order to avoid bias effects in the efficient solutions obtained, the weights must be normalised previous to the resolution of the corresponding problem. This has been done building the payoff matrix, and calculating the ideal and anti-ideal point for each objective function (f_i^*, and f_i^{max} respectively). Then, given a vector $\mu \geq \mathbf{0}$, we consider:

$$\omega_j = \frac{\mu_j}{f_j^{max} - f_j^*}, \quad j = 1,\dots,p$$

  This way, the points obtained have a closer relation to the weights used, so that if we use a series of weight vectors uniformly distributed in $[0, 1]^p$, the corresponding solutions will be uniformly distributed in the efficient set. Besides, the weighted function that is minimised in each problem is no longer the sum of quantities measured in different units.
- The scalar problem that has to be solved for each vector of weights is a non-linear convex problem, provided that the convexity hypothesis of the original problem hold, and $\mu \geq \mathbf{0}$. Thus, it can be solve using any of the algorithms existing to this end. We have used the quadratic sequential programming scheme described in [4] for the resolution of all the non-linear problems.

- It is clear that, if we want to obtain a good approximation to the efficient set, a sufficient number of efficient points must be generated. To this end, the algorithm provides the user with the possibility of generating automatically a series of weight vectors and solve the corresponding problems. Namely, given the number $q \in \mathbf{N}$, the process is as follows:

1. Using a series of nested loops, initial families of weights that sum up q are calculated, in the following way:

   Function 1: for $i_1 = q$ to $i_1 = 0$ (step -1)

   Function 2: for $i_2 = q - i_1$ to $i_2 = 0$

   Function 3: for $i_3 = q - i_1 - i_2$ to $i_3 = 0$

   $\ddots$

   Function p - 1: for $i_{p-1} = q - i_1 - \cdots - i_{p-2}$ to $i_{p-1} = 0$

   Function p: $i_p = q - i_1 - \cdots - i_{p-1}$.

2. From this family of values (i_1, i_2, ..., i_p), that verify $\sum_{j=1}^{p} i_j = q$, the corresponding weights are derived:

$$\omega_j = \frac{i_j}{q} \quad (j = 1,\ldots,p)$$

   which now verify obviously the relation $\sum_{i=1}^{p} \omega_i = 1$.

3. Finally, the weights are normalised following the previously described scheme.

- Along this procedure, the algorithm employees the optimal solution obtained in one problem as the initial estimation for the next. This way, the initial point is feasible, and, if the corresponding weight vectors are close to each other, it may be a good estimation of the optimal solution, and this reduces the computing time of the whole procedure.
- If a sudden change is observed in the results of two close weight vectors, the algorithm makes it possible to carry out a zoom of this zone, introducing, through the same scheme, a given number of weights between them.
- Finally, as it has been seen in section 3.1, whenever a zero weight is used, we can only affirm in principle that the solution obtained is weakly efficient, unless we know it is the unique optimal solution to the problem. But testing uniqueness can be very difficult in general non-linear problems. So, in this cases an efficiency test is carried out, which is based on the constraint method and, thus, developed in the next section.

The computational results obtained applying this algorithm to several test problems can be considered as good. It can be seen that the behaviour of this method, in terms of C.P.U. time, depends obviously on the number of variables and of non-linear constraints. On the other hand, the number of linear constraints does not make the computing time vary significantly. Thus, for example, problems with 100 variables have been solved in an average time of 14 seconds per problem, so that the whole generating process for a problem with 3 objective functions and q = 25 (351 problems) takes about 4764 seconds. For examples with 15 variables and 10 non-linear constraints, the average time per problem is about 5.6 seconds.

2.3 Constraint algorithm

As it was stated in section 3.1, this method obtains efficient solutions through the resolution of constraint problems, for given bound vectors $\alpha = (\alpha_1, ..., \alpha_p)$. As it was mentioned in section 3.2, the aim of the algorithm is to generate a significant number of efficient solutions, and this also brings the necessity to carefully choose the bound vectors α. To this end, our implementation has the following properties:

- Choosing the values α_i between the ideal and anti ideal values of the corresponding objective function assure us that the corresponding bounds will be effective, that is, the initial feasible set will actually be restricted. But still in this case it cannot be assured that the feasible set of the problem will not be empty. Trying to solve a problem with empty feasible set (when the number of variables is high, it is not easy to know this fact before hand) can cause heavy computational errors, due to the nature of the algorithm employed to do so. This is why a test to detect this eventuality is carried out previous to the resolution of each constraint problem. This test is based in the introduction of deviation variables, and is described in section 3.1.
- As it was done in the weighting algorithm, in order to generate efficient solutions, the possibility to build automatically constraint problems is allowed. In this case, the user supplies the number of bounds that wishes to introduce between the ideal and anti ideal values of each objective function. The algorithm combinatorially join these bounds, and solve the p resulting constraint problems for each bound vector obtained in this way (after testing whether the feasible set is empty or not).
- As it was stated in section 2.1, only when the optimal solution of a constraint problem is unique, it can be assured that it is Pareto optimal. This is why, in this case, as well as for the solutions of weighting problems with some zero weight, some efficiency test should be carried out. One way to do it is through the characterisation of efficiency that the constraint method itself gives us. Namely, it was stated that $\mathbf{x}^*$ is efficient if and only if it is an optimal solution for the constraint problems $P_i(\alpha)$, $i = 1,..., p$, with $\alpha = \mathbf{f}(\mathbf{x}^*)$. In practice, this is done solving the problems for $\alpha_i = f_i(\mathbf{x}^*) + \varepsilon$, for a given tolerance ε, in order to avoid computational troubles due to rounding errors, and allowing the corresponding tolerance in the optimal solutions of the constraint problems.

The average time per problem for the constraint algorithm is, in general, grater than for the weighting method, due to the fact that each problem has p-1 more non-linear constraints. This fact, together with the difficulties in relation with the election of α, makes it highly inadvisable as a generating method. Nevertheless, it can be used as an efficiency test, as it has been described before, for these cases when we are not sure whether a solution obtained with zero weights is efficient or not.

3 Goal Programming

Goal Programming (see [3], [7], [6]) has proved to be a very powerful technique for multiple objective programming, specially developed for linear problems. In this section, our aim is to adapt the algorithms existing for the linear case to our convex problem, as well as studying other aspects relative to this method.

3.1 Algorithm with deviation variables

Next, a Goal Programming algorithm is developed, which is based on the lexicographic method used in linear problems.

Once the decision maker is shown the payoff matrix of the problem, he is asked to introduce a target value for each objective function, placed between the ideal and anti ideal values. For our convex problem, and due to its initial formulation where all the objectives are minimised, all the goals will be supposed to take the form $f_i(\mathbf{x}) \le u_i$ $(i = 1, \ldots, p)$.

Besides, the goals must be classified in priority levels according to the decision maker's preferences among them. Thus, he can define s priority levels ($s \le p$) and indicate to which level does each goal belong. He can also assign weights $\mu_i \ge 0$ to the goals that share the same priority level.

Following the lexicographic scheme, the algorithm solves a problem for each priority level. The k-th problem takes the form:

$$(P_k)\begin{cases} \min & h_k(\mathbf{p}) \\ s.t. & \mathbf{g}(\mathbf{x}) \le \mathbf{b} \\ & f_i(\mathbf{x}) \le u_i' \quad (i \in N_1, \ldots, N_{k-1}) \\ & f_i(\mathbf{x}) - p_i \le u_i \quad (i \in N_k) \\ & p_i \ge 0 \quad (i \in N_k) \end{cases}$$

where

- Each goal has been built introducing positive deviation variables p_i, so that the following final expression is reached:

$$f_i(\mathbf{x}) - p_i \le u_i$$
$$p_i \ge 0$$

Let us observe that the suppression of the negative deviation variables (commonly used in linear problems) is compensated, maintaining the $\leq$ inequality, which preserves the convexity of the corresponding constraint.

- If the i-th objective function belongs to the k-th priority level, we will denote it by $i \in N_k$.
- The achievement function $h_k(\mathbf{p})$ is convex and strictly decreasing in each variable p_i, $i \in N_k$. In our algorithm, we have used functions of the form
$$h_k(\mathbf{p}) = \sum_{i \in N_k} \omega_i p_i^{\theta},$$
where $\theta \geq 1$ is an exponent chosen by the user, and ω_i is the weight assigned to the corresponding goal, normalised dividing by its target value: $\omega_i = \mu_i/u_i$.
- Let us denote by $(\mathbf{x}_k^*, \mathbf{p}^*)$ the optimal solution of the problem (P_k). After solving it, the target values are actualised, if necessary: if $p_i^* = 0$, the point $\mathbf{x}_k^*$ satisfies the i-th goal, and then we take $u_i' = u_i$. If, on the other hand, we have $p_i^* > 0$, then the corresponding goal is not satisfied, and we take the new value $u_i' = u_i + p_i^* + \varepsilon$, with ε being a small tolerance which prevents the subsequent feasible sets from being empty. This way, all the s problems are solved, even if for some level there do not exist satisfying solutions, in case is was possible to improve the values of the functions placed in latter levels, keeping, of course, the values achieved for the current one (which, although not satisfying, are the best that can be taken).
- Before solving the problem (P_k), it is tested whether the optimal point of the preceding level, $\mathbf{x}_{k-1}^*$ satisfies the current level goals. If so, the algorithm jumps this levels and goes to N_{k+1}. If not, then the point $(\mathbf{x}_{k-1}^*, \mathbf{p}^0)$ is taken as the initial estimate for problem (P_k), where:
$$p_i^0 = \begin{cases} 0 & \text{if} \quad f_i(\mathbf{x}) \leq u_i \\ f_i(\mathbf{x}_{k-1}^*) - u_i & \text{if} \quad f_i(\mathbf{x}) > u_i \end{cases} \quad (i \in N_k),$$
so that the feasibility of the initial estimate is assured.
- Previous to the application of the lexicographic algorithm itself, a problem similar to (P_k) can be solved, so as to test whether the original problem has got any feasible solution. This problem (which is also solved in the constraint method, as it was previously mentioned) takes the form:
$$(P_0) \begin{cases} min & h_0(\mathbf{p}) \\ s.t. & g_i(\mathbf{x}) - p_i \leq b_i \quad i = 1, \ldots, m \\ & p_i \geq 0 \quad i = 1, \ldots, m \end{cases}$$

After getting some computational results, it can be seen that the performance of the algorithm is good. For example, for a problem with 100 variables and 3 goals (each one in a different priority level), it takes an average overall time of about 62 seconds. Apart from the logical dependence of C.P.U. time on the dimensions of the problem, it can also be seen that, for large problems, it is quicker to solve problem (P_0) with a constant objective function, and just with the initial constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{b}$.

For the subsequent levels, the option $\theta = 1$ is slightly better for small problems, while the option $\theta = 2$ is clearly better for large problems, and, besides, it let us obtain more precise solutions. The reason of this behaviour is that the algorithm used to solve the problems is a sequential quadratic one, and, so, works better with quadratic functions. Finally, it is important to remark that the association of several functions in a single priority level does not necessary produce a better C.P.U. time, for each function implies the introduction of a new variable and a new non-linear constraint in the corresponding problem.

3.2 Satisfying and efficient solutions

After applying the lexicographic algorithm, a satisfying solution (if it exists) is obtained. But it cannot be assured that this solution is efficient as well. In fact, this is one of the most common criticisms to Goal Programming (see, for example, [13]). The possibility to obtain efficient solutions in Goal Programming problems has already been studied by several authors ([5], [10]...), specially for the linear problem. In [1], a complete study of the possibility to obtain satisfying and efficient solutions is carried out. It is proved that the set of such solutions can be approximated using the weighting method within the satisfying set, and so, the behaviour of this algorithm can be deduced from what has been stated in sections 2.2 and 3.1,

3.3 Algorithm without deviation variables

The previous considerations about the possibility to obtain satisfying and efficient solutions made us consider the construction of a lexicographic Goal Programming algorithm that got them in a straight way.

To this end, a straight method has been designed, where no deviation variables are used. This algorithm lets the existence of only one objective function in each priority level, and it just minimises the corresponding function at each step. Namely, the problem that is solved at the k-th level is:

$$(P_k)\begin{cases} min & f_k(\mathbf{x}) \\ s.t. & \mathbf{g}(\mathbf{x}) \le \mathbf{b} \\ & f_j(\mathbf{x}) \le u_j' \quad j = 1,\dots,k-1 \end{cases}$$

where the actualisation of the target values is now carried out according to the following scheme:

$$u_k' = \begin{cases} u_k & \text{if} \quad f_k(\mathbf{x}_k^*) \le u_k \\ f_k(\mathbf{x}_k^*) + \varepsilon & \text{if} \quad f_k(\mathbf{x}_k^*) > u_k \end{cases}$$

The reason why only one objective function is allowed in each priority level is that the consideration of a function of the type $\Sigma \omega_i f_i$ can lead to obtain a non satisfying solution for the corresponding goals, while such solutions exists, depending on the weights ω_i used.

The last problem that is solved along this procedure is a constraint problem, just like the ones used in section 2.3, and so, if its solution is unique, it can be affirmed that it is efficient. Nevertheless, it can be argued that the final solution obtained is, among all the satisfying ones, the one that takes the best possible value for the objective function f_p, placed in the last priority level, which is not logical at all from the point of view of the lexicographic order given by the decision maker. So, in the case when satisfying solutions exist (and only in this case), we can repeat the process, but with the opposite priority order, and this will assure us to obtain the solution which produces a the best possible value for f_1 within the satisfying set. When no satisfying solutions exist, the final solution of this algorithm is the same one that the method with deviation variables would obtain.

This algorithm has been tested with the same problems that were used in the previous one. From the comparison between the C.P.U. times obtained in both cases, it seems that the algorithm with deviation variables is faster for small problems, due to the fact that it just intends to find a satisfying solution for each level, and the search stops just when getting it, while the straight method keeps on until it minimises the current function. However, this behaviour does not take place in general with large problems. In most of them (although times can depend critically on the form of the objective functions), the straight method is faster, because the computational burden of considering more variables and constraints in the algorithm with deviation variables is now significant. Only when the objective function have a growth slower that linear (for example, logarithmic functions), the algorithm with deviation variables is faster, because it always minimises linear or quadratic functions.

3.4 G.S.I. algorithm

The straight method described above assures us, in most of the cases, to obtain an efficient, as well as satisfying solution, when such solutions exist. But it does so in rather a strict way, in the sense that it just minimises the most preferred function.

But the decision maker's preferences are not necessarily so strict. He might instead prefer to improve all the objective functions. It is then interesting to have the possibility to carry out a post-optimisation analysis, so as to test up to which level can each target value be improved so that there still exist satisfying solutions for the problem.

The G.S.I. algorithm (Goal Sequential Improvement) is an interactive method where the user can lexicographically improve his target values, through the resolution of problems similar to (P_k) of the previous straight method. Namely, the corresponding function is minimised in the satisfying set, actualising after each step the target values of the goals that belong to the current level.

Besides, the last problem that is solved is again a constraint problem, and so, it can be assured that the final solution (if it is unique) is also efficient. This way, an algorithm is obtained that harmonises the concepts of satisfying and efficient solution, in the sense that lets the user obtain an efficient solution in a lexicographic way, according to his expressed preferences.

Further details about the algorithm and its behaviour can be found in [2].

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From TRIMAP to SOMMIX - Building Effective Interactive MOLP Computational Tools

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Abstract. This paper aims at presenting a path of development of interactive computational tools devoted to provide decision aid in multiple objective linear programming (MOLP) problems, made by a research team at the University of Coimbra. After referring to the motivation of this research area, the main characteristics of the research work which has been carried out in the last years are described, with special emphasis on the MOLP interactive environments TRIMAP, TOMMIX and SOMMIX.

Keywords. Interactive environments, MOLP

1. Introduction

Fifteen years ago the first author of this paper dealt with an energy planning problem using a multiple objective linear programming model (MOLP), considering three objective functions [5]. The basic idea underlying the study consisted in using an algorithm for generating all efficient extreme points of the feasible polyhedron. The method developed by Zeleny [14] was used. Although the case study was based on data related to the portuguese reality, this was an academic work and the model then constructed had some limitations leading to a medium-sized problem. Even under these circumstances, it was possible to realize, in practice, the vulnerability of the algorithms to generate all efficient extreme points. Its number is usually too large to be presented to the decision maker (DM) as potential choice alternatives. Besides, a lot of these points have objective function values with only slight variations while in other cases these variations are very high. It became clear that the direct presentation to the DM of a set of alternatives with these characteristics was not adequate. On the other hand, the computation of all efficient extreme points involves a cumbersome computational effort for that type of case studies. Under these circumstances, we resort to interactive methods as a way to overcome the problems raised by the application of generating methods. For details on this stage see [6].

The application of two of the most well-known MOLP interactive methods, STEM [3] and Zionts-Wallenius [15], revealed the fragility of these type of classical methods in some real-world problems. Without entering into details, we concluded that for problems such our energy planning problem (in which the feasible region possesses sub-regions of the efficient frontier where the objective function values change very smoothly between adjacent extreme points, while among these regions there are efficient faces showing sharp variations of those values) it would be necessary to develop procedures aimed at performing a more strategic search. At least in an initial phase it would be useful to have procedures that could help to identify the efficient sub-regions containing the solutions more interesting to the DM, while the classical methods, based on a more local search, would be used in a second phase. The classical methods have revealed of little help to assist the DM when directly applied to the problem, without any prior knowledge about the shape of the efficient region. The conclusions drawn from these studies laid the foundations for the development of the TRIMAP interactive method. These conclusions were clearly illustrated in a paper comparing the use of STEM, Zionts-Wallenius and TRIMAP methods to the energy planning problem (see [8]).

2. The TRIMAP package

The TRIMAP interactive environment is aimed at assisting the DM in a progressive and selective search of efficient solutions, by using an user-friendly Human-computer interface environment, including graphical displays, instead of looking for the convergence to the optimum of any implicit utility function of the DM. TRIMAP aims to support the DM in the *learning* of the characteristics of the efficient region, helping him/her to identify and progress towards the solution or set of solutions which more closely correspond to his/her preferences. This is done in a rather natural manner, promoting a selective and progressive search based on experimentation, and avoiding the exhaustive search of the efficient region. TRIMAP is devoted to linear problems with three objective functions, because the idea of its development was born to overcome some difficulties raised by a three-objective case study.

Since the search strategy underlying the TRIMAP package is the cornerstone of further developments, we will make here a brief review (based on its block diagram; fig. 1) of the tools it offers to the DM as well as the the way we idealized for their use. For further details see [6], [7].

(a) In order to give the DM a first information of the range of the objective function values over the efficient region, some "well distributed" solutions are computed. TRIMAP begins by computing the basic efficient solutions which optimize each objective function individually. The efficient solution which minimizes a weighted Tchebycheff distance to the ideal solution is also computed.

(b) The user is offered the following graphical displays:

- weight space, filled with the indifference regions corresponding to the (basic) efficient solutions already computed. Direct limitations introduced on the variation of the weights or inferior bounds on the objective values translated onto the weight space are also displayed.

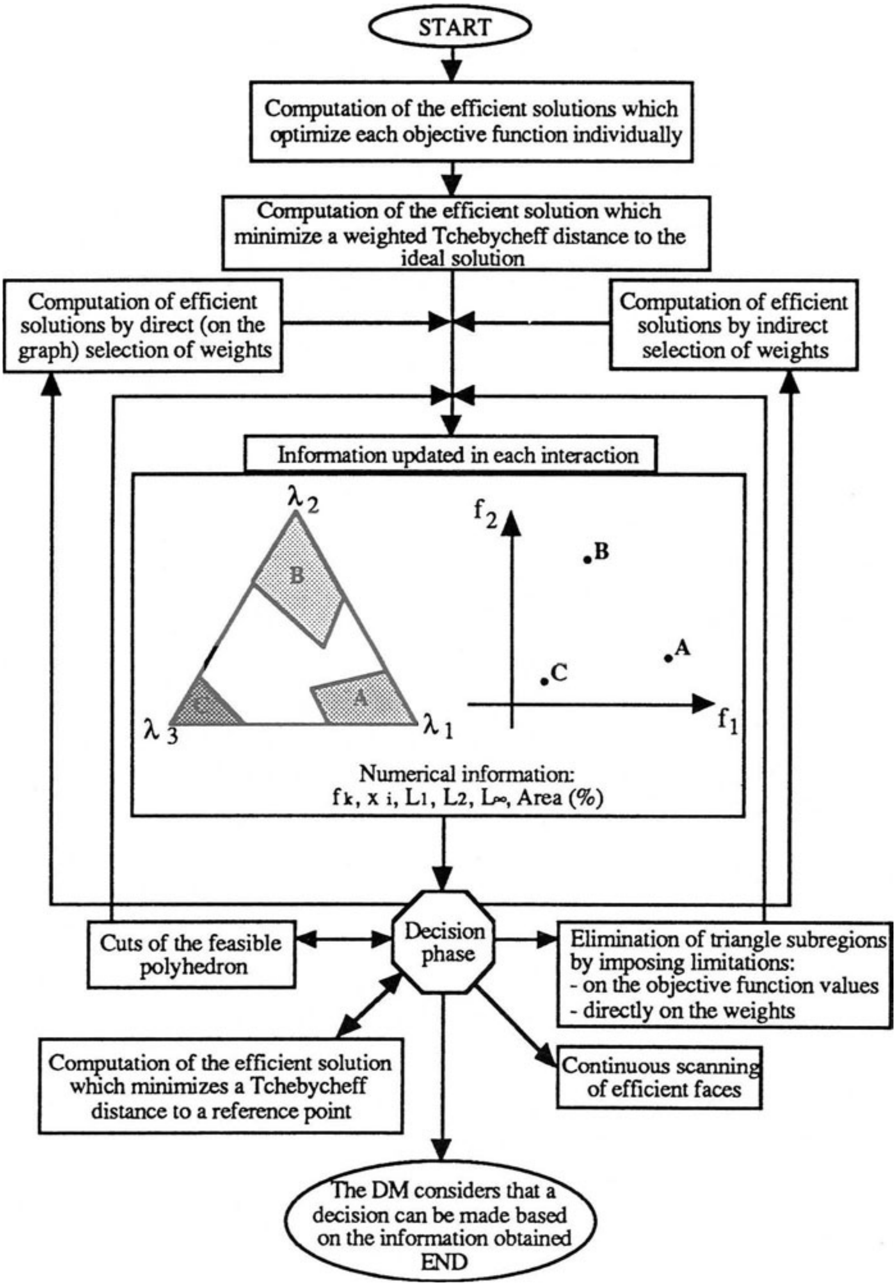


Fig. 1 - TRIMAP block diagram

- 2-dimensional projection of the objective function space, showing all efficient solutions already computed and enabling also to identify the efficient edges and faces known.

- Spider-web graph, showing for each solution the difference in each objective function between its value and a reference point established by the user (eventually the ideal solution).

These graphical displays promote the comprehension of the shape of the efficient region by simple visual inspection.

(c) To gather more knowledge about the shape of the efficient region, the user can continue the search in a selective and progressive manner by:

- selecting sets of weights corresponding to zones of the weight space not yet filled with indifference regions for which the DM thinks is is important to have some more knowledge about the problem. These sets of weights are used to construct weighted sums of the objective functions whose optimization yields a new basic efficient solution.

- Selecting three solutions already known, and the weighted-sum objective function whose gradient is normal to the plane passing through those solutions is optimized yielding a new basic efficient solution (whenever the weights are not all positive a perturbation of that plane is considered to ensure this condition).

(d) The surface scanning command enables to scan a nondominated surface between two extreme points previously computed. The program shows a line dynamically scanning any straight line between two selected extreme points on the surface at a selected speed. The values of the objective functions corresponding to the scanned points are dynamically displayed in bar graphs.

The dialogue with the DM is done in the objective function space, which empirical studies found to be the "most familiar" to the DM in the sense that less cognitive effort is required from him/her. By comparing the two graphs which are always on the screen (weight space and projection of the objective function space) it is possible, by simple visual inspection, to avoid the search of sub-regions of the weight space which are of no interest. Other possibilities of the package include cuts of the feasible region, which although are not interesting to deal with case studies, are useful for teaching purposes (see also [4]).

(e) A set of editing features are also available in a menu, which enable, using dialogue boxes, to edit the objective function and constraint coefficients, right-hand side of the constraints, eliminate and add new constraints, include an objective function as a constraint, etc.

The TRIMAP package has been the first step of a line of development of interactive procedures devoted to provide decision aid in MOLP problems, which can in our opinion overcome some limitations of the classical methods, namely by combining the theoretical knowledge on MOLP with modern techniques of computer science, in particular as far as Human-computer interaction is concerned. A key issue in the development of these computational tools is to enable supporting the DM in learning how to reduce the scope of the search and focus on the sub-region(s) of the efficient region which better suit his/her expectations. This approach generally begins by making a strategic search which leads essentially to eliminate progressively the zones which reveal of no interest and, in a second phase, to search in more detail the zones which the initial screening revealed as potentially interesting.

3. The TOMMIX package

The main ideas underlying the development of the TRIMAP package as well as the experiments performed using our energy planning case study led us to realize the advantages of combining in the same computer package - an integrated interactive method environment - one method of each type (feasible region reduction, weight space reduction, criterion cone contraction and directional scanning, according to the categorization inspired on [13]). In fact, experiments led to the conclusion that there is no method which is better than all the others in all aspects of evaluation. The comparative advantages and disadvantages depend on the characteristics of the problem under study (dimension, shape of the efficient region, etc.), the type and amount of information presented to and required from the DM, the adequacy to different phases of the search process (whether the study is being initiated or the DM already has some knowledge about the shape of the efficient region and wants to continue the search focussing it on a particular sub-region).

TOMMIX interactive environment has been developed mainly for three-objective problems. It includes the interactive methods STEM, Zionts-Wallenius, TRIMAP, Pareto Race [11] and Interval Criterion Weights [12], which are representative of distinct types of search for efficient solutions: reduction of the feasible region, reduction of the weight space, contraction of the criterion cone and directional scanning. For more details on this interactive MOLP environment and its computer implementation see [9] and [1]. The application of TOMMIX to energy planning and telecommunication planning is reported in [10] and [2].

It shall be remarked that, although TOMMIX is an integrated package in which the internal structure of the methods is fully respected, the key idea from a practical point of view (which is well documented in the papers referred above) is to provide the DM/analyst the possibility of using and combining in sucessive interactions the different techniques to compute efficient solutions, strategies to reduce the scope of the search, and ways of information communication, enabling to switch from a procedure to another one as this is judged convenient, being possible to collect and integrate the information gathered in previous interactions. Besides its use as a interactive method environment, which has been especially interesting for teaching purposes, we are interested in the set of procedures underlying each method and not exactly in the manner these are used within the operational framework of the methods.

The main idea is to extend the flexible environment offered by TRIMAP to the main procedures used in some representative interactive MOLP methods, by keeping the flexibility of search characteristic of TRIMAP in which any decision could be rescinded at any moment of the search process.

4. An evolutionary path towards SOMMIX

The fundamental idea underlying the development of the TOMMIX package is the possibility of beginning the study of a three-objective LP problem by using procedures of the same type as those provided by TRIMAP. In this first phase, basic efficient solutions are computed, using weighted sums of the objective functions, in which the weight sets are well dispersed within the weight space, either by direct choice of the weights graphically on the graph or eventually using a process of the type proposed in the ICW method to distribute sets of weights over the weight space. In a second phase, a progressive and selective reduction of the scope of the search is performed, either by reducing the objective function space when the DM considers that he/she knows enough about the problem to introduce inferior bounds on the objective function values (reservation levels), or imposing further constraints on the objective function space using a procedure like the one in STEM method, or contracting the cone of the objective function gradients. In order that the sucessive utilization of the various procedures may be taken into account in the subsequent steps of the search, it is necessary to have a process of keeping the information that is being supplied by the DM so that it can be later combined. In TOMMIX this has been accomplished by developing techniques that enable to translate into the weight space the limitations introduced on the objective function values and contractions of the cone of the objective function gradients. The use of local search procedures is only advised in later phases of the interactive process, either by using procedures such as the ones in Zionts-Wallenius method, or computing all efficient basic solutions which can still be reached, or computing the efficient extreme points which are adjacent to a given basic solution, or performing a directional search of the type used in Pareto Race method. For details see [9] and [1].

All these issues become more complex when the MOLP problem has more than three objective functions, namely because it is impossible to visualize the weight space in a way as clear as in the three-objective case in which the whole triangle can be displayed. However, a lot of problems demand the explicit consideration of more than three objective functions. So the next step consisted in investigating coherent manners to extend the environment developed for TOMMIX to the general case. This work gave rise to the SOMMIX interactive environment. Obviously, the utilization of the interactive means offered to the user by SOMMIX is more complex than in the three-objective case. Herein the practical functioning of SOMMIX is illustrated using a four-objective LP problem.

The tools to compute new efficient solutions as well as the dialogue processes with the DM enabling him/her to establish progressively his/her preferences throughout the sucessive interactions are of the same type of the ones available in TOMMIX (but where the method structure has been preserved). However, the graphical forms how information is generated and processed is quite different. In SOMMIX the interactive environment is based on a control panel which offers a large set of commands making the information available to the user (fig. 2). In general, tools based on control panels need some training to learn in a systemic and integrated manner the information available by means of their different components in a way that motivates the sucessive actions in the search process. Intuition is crucial to deal with the control panel rationally.

In what follows we will illustrate the use of the SOMMIX package to study a four-objective problem, by referring the available tools (and instruments within each tool) whenever they are necessary in the search process. This is not intended to be exhaustive in describing the package, but aims at showing its usefulness in assisting the DM throughout the interactive decision process. The search begins by computing basic efficient solutions using sets of weights "well distributed" in the weight space (which can be the corresponding to the optima of the objective functions individually, or sets of weights manually selected with the mouse, or automatically selected using the same technique as in ICW method). As in the three-objective case, the indifference regions in the weight space corresponding to each basic efficient solution already known are computed, but their visualization raises some questions. Two tools are aimed at keeping the information gathered in this first phase:

- the *solution basket*, where the objective function values for the basic efficient solutions already known are displayed using bar graphs (fig. 2). Each solution has associated a pattern or colour.

- The weight space is represented by scroll bars, one for each dimension (fig. 2). A small window at the upper left corner indicates whether the weights correspond to solutions not yet computed (there is no colour or pattern visible), or to solutions already known whenever the current set of weights belongs to the indifference region of that solution (the same pattern or colour used in the solution basket is displayed).

Other possibilities exist to exploit graphically the weight space at its current status. For this example, one weight is held constant (the checkbox at the right of the weight scroll bar of f_1 is checked; fig. 2), and the triangle corresponding to a cut of the 4-dimensional weight space for the particular value of that weight is displayed (fig. 3). This is accomplished by making a mouse click on the instrument represented by an icon at the upper right corner of the *Weights* tool, and fixing the value of that weight at the cut level. The behaviour of the other weights can be then investigated. If it is intended to continue the search on this triangle the indifference regions of these solutions can be later used in the four-dimensional weight space.

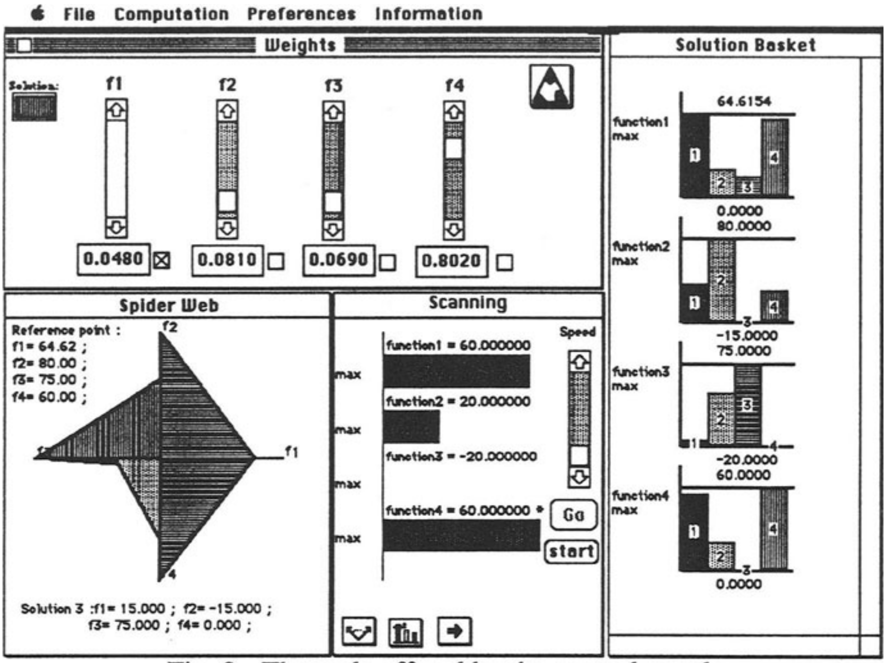


Fig. 2 - The tools offered by the control panel

The instruments represented by the icons in fig. 3

enables the user to select a set of weights by positioning and clicking the mouse on the desired point of the weight space;

enables the user to amplify the graph (making a zoom of the triangle);

this option (available for problems with more than 3 objective functions only) enables to visualize the dynamic evolution of the triangle for sucessive cuts following a same direction (scanning continuously within a given interval of the value of one weight). A scroll bar is also available to control the step between two consecutive cuts. This scanning may be stopped at the level it is found more interesting to proceed the search.

A third tool which is also available is the *spider-web* which displays a graph where in each axis (one for each objective function) it is represented the distance of each solution to the same component of the reference point. The user can select the nondominated extreme solution to be displayed on the frontmost plane of the spider-web graph. For the sake of an easier identification of the solutions in the different graphical displays (solution basket, weight space, spider-web) the same patterns or colours are used.

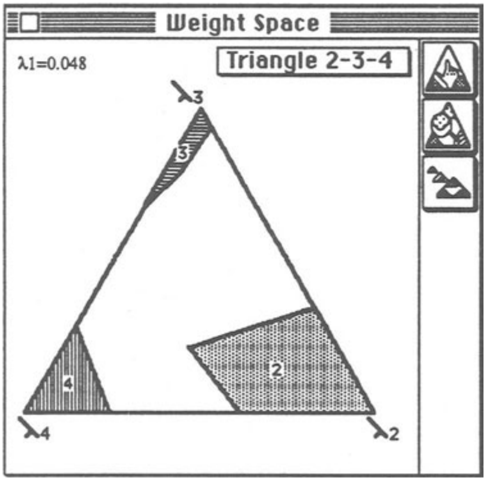


Fig. 3 - A cut of the four-dimensional weight space for λ_1=0.048

Based on the available information it is necessary to decide how to continue the search of the efficient region. In this phase the DM can opt for distinct strategies:

(a) searching for new efficient extreme points by selecting directly sets of weights in zones of the weight space not corresponding to any of the already computed solutions.

(b) Considering that he/she knows enough about the efficient solution set to establish reservation levels on the objective function values and to continue the search. Note that imposing these additional constraints on the objective function values depends essentially of having a strategic view of the shape of the efficient region. The information available in the spider-web graph may help in establishing those bounds.

(c) A procedure of the same type of STEM method can also be used having in mind, just after computing the efficient solution which minimizes a weighted distance to the ideal solution, to establish inferior bounds for the objective function values by specifying the relaxation quantities for one or more objective function values in order to improve the remaining ones with respect to the current solution. A new iteration as in STEM can be made using those relaxation quantities. This information can also be linked with the information already available in the spider-web graph to aid the specification of reservation levels on the objective functions. Note that the efficient solutions computed by this, or any other, process, which are not extreme points can also be kept in the solution basket and displayed in the spider-web graph.

(d) The DM can decide to make a contraction of the cone of objective function gradients as a way of reducing the scope of the search.

In all cases there are automatic procedures which translate the additional constraints into weight space reductions.

Let us suppose that the search is continued by imposing a reservation level on an objective function value. This information elicited from the DM can be operationalized in the weight space since a small box in the lower left corner indicates whether the current set of weights (specified by the scroll bars) satisfies that additional constraint (which is displayed by a dotted pattern; fig. 4). From now on the search is directed towards sets of weights which satisfy the additional constraint on the objective function value. It is important to refer that intermediate decisions and choices made by the DM throughout the interactive process are always rescindable. This is possible by means of "on-off" switches at the right of the weight space that enable to eliminate the constraints previously imposed on the objective space or the contraction of the cone of the objective function gradients (fig. 4).

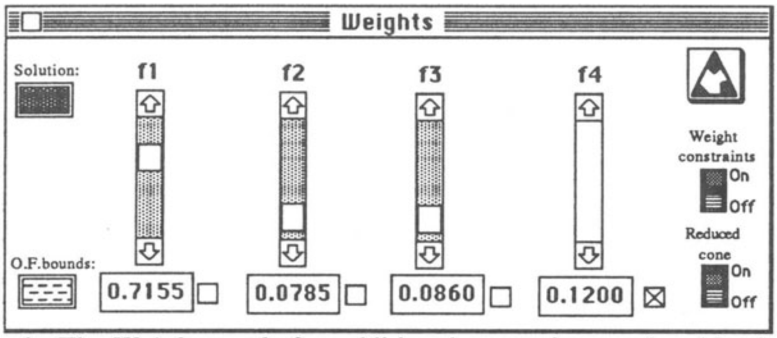


Fig. 4 - The *Weights* tool after additional constraints on the objectives

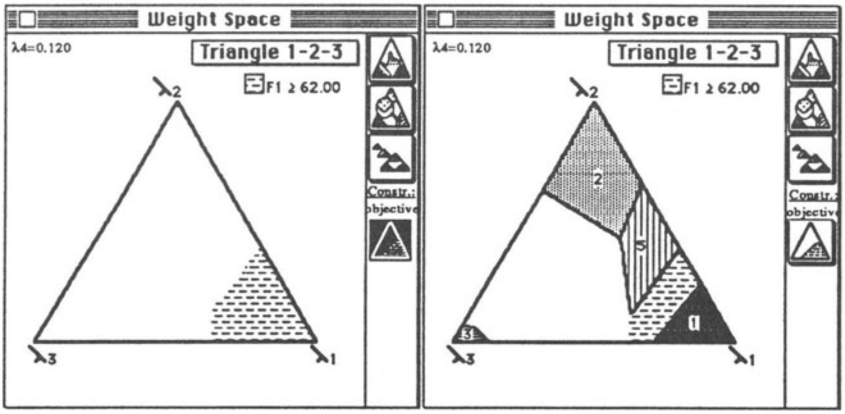


Fig. 5 - A cut of the weight space for λ_4=0.12, after a limitation on an objective has been introduced (this is a distinct cut regarding fig. 3)

Whenever a cut in the weight space is made, by fixing one weight, the corresponding cut of the weight space satisfying the additional constraint previously imposed is displayed on the triangle (fig. 5). The search may be continued on that cut level and new solutions may be computed, which are then considered in the original four-dimensional weight space. Until now we briefly described, based on an example, how the control panel can be used for making a progressive and selective search of the efficient region. Let us now admit that the DM focussed the search on a part of the efficient region to illustrate a "more exhaustive" search process in a second phase.

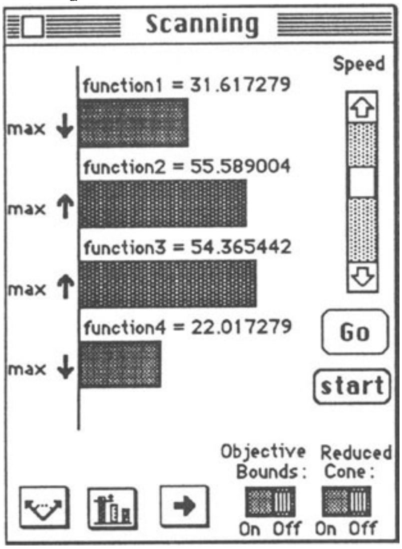


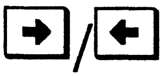
Fig. 6 - The scanning tool

In our example the restricted zone of the efficient region which is of interest to the DM has been defined by introducing a reservation level on an objective function value. In this phase the package offers the tools:

- The scanning window which enables the use of Pareto Race method, which consists in a local directional search, starting from an efficient point specified by the user. In the scanning tool some instruments represented by icons are available (fig. 6).


- It changes the current direction of motion in order to improve the objective function selected by the user.


- It enables to fix the current value of one objective as a lower bound and to relax it later.


- These switch between forward and backward on the direction of motion.

Besides these commands, "on-off" switches are available that enable to keep or rescind previous decisions that led to the introduction of reservation levels on the objective functions and/or contractions of the cone of objective function gradients.

This is the tool that both from analytical and graphical points of view seems more suited to make a free search in sub-regions of the efficient region previously identified. Other forms of computation of efficient solutions are available, which can be of interest in a final phase of search:

- VMA (“vector maximum algorithm”) is intended to compute all efficient extreme points. It may be useful whenever the information expressed by the DM have sufficiently reduced the feasible region so that the associated computational effort is not too demanding.

- Adjacent solutions to a current extreme point solution is of interest whenever a good compromise solution is found and the DM can have a locally complete knowledge of the efficient frontier.

5. Conclusions

In this paper an evolutionary path of development of interactive computational environments devoted to assist DMs in dealing with MOLP problems has been presented. In complex decision problems, it is important not just to have theoretical sound procedures, but also to offer interactive computer tools which can promote the exploration and comprehension of the problem as well as the DM's preferences.

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Pareto Simulated Annealing

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Abstract. The paper presents a multiple objective metaheuristic procedure - Pareto Simulated Annealing. The goal of the procedure is to find in a relatively short time a "good" approximation of the set of efficient solutions of a multiple objective combinatorial optimization problem. The procedure uses a sample of generating solutions. Each of the solutions explores its neighborhood in a way similar to that of classical simulated annealing. Weights of the objectives are set in each iteration in order to assure a tendency to approach the efficient solutions set while maintaining a uniform distribution of the generating solutions over this set.
Keywords. Multiple objective combinatorial optimization, metaheuristic procedures.

1. Introduction

Many real-life problems are combinatorial, i.e. they concern a choice of the best solution from a finite but large set of feasible solutions [18]. It is also a well known fact that the solutions of real-life problems are often evaluated from several points of view which may be described by different objectives [20], [26]. These facts were reasons for significant research efforts in the fields of *combinatorial optimization* (CO) and *multiple objective decision making* (MODM). The research resulted in many practical applications of methods developed in each of the fields. Nevertheless, surprisingly few theoretical works concern *multiple objective combinatorial optimization* (MOCO) problems [27]. Some exceptions are *multiple objective shortest path* problems [5] and *multiple objective project scheduling* problems [22], [24]. As a consequence, very few practical applications of MODM methods for combinatorial problems are reported in the OR literature [28].

In opinion of the authors, the relatively small number of applications of MOCO is not due to the fact that combinatorial problems rarely require multiple objectives but due to the notable difficulty of such problems. Indeed, different objectives are often used in particular classes of combinatorial problems. For example, in vehicle routing the typically used objectives are total cost, distance, number of vehicles, travel time [1], and in project scheduling the typical objectives are net present value, project completion time, mean weighted delay, number of delayed tasks, mean weighted flow time, total resource utilisation [22, [24]. The objectives, however, are usually used separately or they are combined into a single objective.

It is the authors belief that the MOCO problems create one of the most difficult classes in the field of OR. This difficulty results from the following two factors:

- solving a MOCO problem requires intensive co-operation with the decision maker (DM); this results in especially high requirements for robust tools used to generate efficient solutions,
- many combinatorial problems are hard even in single objective versions; their multiple objective versions are frequently more difficult.

A MODM problem is ill-posed from mathematical point of view, because, except of trivial cases, it has no unique solution. It is usually assumed that the DM should select one of the *efficient* solutions [19]. The goal of MODM methods is to find an efficient solution most consistent with the DM's preferences, i.e. the *best compromise*. Interactive procedures, which became especially popular in recent years, generate efficient solutions in computational phases alternating with phases of decision. Such procedures, however, can only be used if sufficiently robust tools for finding efficient solutions are available.

Furthermore, many of single objective combinatorial problems belong to the class of NP-hard problems. Generation of efficient solutions in a MOCO problem is, of course, not easier than finding solutions optimizing particular objectives and, in many cases, is even harder. For example, the single objective shortest path problem is one of the simplest combinatorial problems while corresponding multiple objective problem is NP-hard [8]. So, even problems for which relatively efficient single objective exact methods are known, may be difficult if multiple objectives have to be considered.

Tools used for generation of efficient solutions in MOCO, alike single objective optimization methods, may be classified into one of the following categories:

- exact procedures,
- specialized heuristic procedures,
- metaheuristic procedures.

The main disadvantage of exact algorithms consists in their high computational complexity. In result, only limited class of real-life MOCO problems may be solved exactly.

The main weak point of specialized procedures, both heuristic and exact, is their inflexibility. This factor seems to be especially important in the case of MOCO. For example, many MODM methods start by presenting to the DM the ideal point, which is found by independent optimization of particular objectives. If the objectives have different mathematical form, their optimization may require different specialized procedures. Furthermore, MODM methods use various tools for generating efficient solutions. Some of them apply the Geoffrion's theorem (see e.g. [26]), others use penalty functions [3], [13], [17], utility functions [14], ε -Constraints (see e.g. [26]), or achievement scalarizing functions [28]. Of course, each of the tools, may require individual specialized procedure. In result, solving a given MOCO problem may involve several specialized procedures and it may be difficult to change the formulation of the model, the family of objectives or even the MODM method used to solve the problem.

Consequently, this is not only opinion of the authors (compare e.g. [27]) that the most promising practical approach to MOCO consists in generating efficient solutions with metaheuristic procedures.

In recent years several metaheuristic methods for single objective combinatorial problems were proposed. They allow finding nearly optimal solutions for a wide class of combinatorial problems in a relatively short time. The methods have been successfully applied to many problems that cannot be solved exactly in the polynomial time. Single objective metaheuristics, however, might not be robust enough for generation of efficient solutions, especially, if interactive exploration of efficient solutions set is used.

The paper proposes a metaheuristic procedure, called Pareto Simulated Annealing (PSA), for MOCO problems. The goal of the procedure is to find a set of solutions being a "good" approximation of the efficient solutions set. The set of solutions obtained by the procedure may then be presented to the DM. If the set is small enough, the DM may select the best compromise from it. Otherwise, the DM may explore the set of solutions guided by an interactive procedure.

The paper is organized in the following way. In the next section a formal statement of the MOCO problem is presented. In the third section classical single objective metaheuristic procedures are considered as tools for generation of efficient solutions. Some previous proposals of multiple objective metaheuristic procedures and their weaknesses are described in the fourth section. In the fifth section the PSA procedure is presented. Evaluation of multiple objective metaheuristic procedures is discussed in the sixth section. Finally, main features of the procedure are summarized and some possible directions of further research are described in the seventh section.

2. Problem statement

The general MOCO problem is formulated as:

$$\max\{f_1(\mathbf{x}) = z_1, \ldots, f_J(\mathbf{x}) = z_J\}$$

s.t.

$$\mathbf{x} \in D,$$

where: *solution* $\mathbf{x} = [x_1, \ldots, x_I]$ is a vector of discrete *decision variables*, D is the set of feasible solutions.

Solution $\mathbf{x} \in D$ is *efficient* (*Pareto-optimal*) if there is no $\mathbf{x}' \in D$ such that $\forall_j f_j(\mathbf{x}') \geq f_j(\mathbf{x})$ and $f_j(\mathbf{x}') > f_j(\mathbf{x})$ for at least one j. The set of all efficient solutions is denoted by N.

3. The use of single objective metaheuristic procedures for generation of efficient solutions

Single objective metaheuristic procedures were proposed and became popular relatively recently. They include such methods as: simulated annealing (SA) [2] [15], [16], tabu search [9] and genetic algorithms [10]. The methods are called metaheuristics, because they define only a "skeleton" of the optimization procedure that have to be customized for particular applications. The popularity of the methods increases due to the following advantages:

- generality, i.e. they may be used for solving various classes of problems,
- robustness, i.e. they allow to find solutions close enough to the optimal one in acceptable time for many real-life problems (please note, however, that the efficiency strongly depends on the way of customization for a given application),
- flexibility, i.e. they are relatively insensitive to changes in the problems formulation,
- simplicity, i.e. they can be relatively easily implemented.

The above advantages make the procedures especially useful for OR practitioners.

As most of the MODM methods use single objective optimization as a tool for generating efficient solutions it seems rational to apply the classical single objective metaheuristic procedures in MOCO. Yet such procedures may appear too inefficient for practical applications. As was mentioned in the first section, finding the best compromise requires a co-operation with the DM and gathering preference information from him/her. Taking into account the moment of collecting preference information with respect to computational process, MODM methods can be classified into [12], [23]:

- methods with a priori articulation of preferences,
- interactive methods, in which preferences are expressed progressively during the search over the non-dominated set,
- methods with a posteriori articulation of preferences.

Methods from the first class start by collecting the preference information from the DM, which is then used to build a model of his preferences. The model is then exploited in order to find the best compromise. In many cases the preference model takes the form of a real-valued function, e.g. utility function [14], penalty function [3], [13], [17] or scalarizing function [28]. Exploitation of such a model consists in optimizing the function on the feasible set. So, single objective metaheuristic procedures can be used in this step.

Methods with a priori articulation of preferences may, however, be only used if the preferences of the DM are well established at the beginning of the solution process. Yet even in this case the DM may be unable or unwilling to deliver all the information required to build the model of his preferences. Furthermore, functional models do not allow incomparability of some solutions which is often observed in practice [20]. Some comparative experiments also indicate that methods of this class perform relatively poor in the case of multiple objective mathematical programming [4].

Interactive methods consist of computational phases alternating with phases of decision. In each computational phase a solution or a sample of solutions, usually efficient, is generated. Single objective optimization methods are usually used as tools for generating these solutions. This allows for using single objective metaheuristic procedures in the case of interactive procedures applied to MOCO problems.

Interactive methods, however, may only be used if the time needed for the computational phase is acceptable for the DM. Although, metaheuristic procedures are relatively robust tools for combinatorial optimization, they might not be robust enough for the use in interactive procedures, especially in the case of methods generating samples of solutions.

Methods with a posteriori articulation of preferences are often criticized for their high computational complexity. This is clearly the case if the whole set of efficient solutions is to be generated. To this class, however, belong also methods finding approximations of the set of efficient solutions N. This approach seems to be especially interesting in the case of MOCO problems, because of mentioned above prohibitive computational complexity of exact methods. Please note, however, that finding the whole efficient solutions set or its approximation does not always completes the solution procedure. This set may be too large for the DM to analyze it and to select the best compromise without further support. So, the DM may decide to use one of procedures for interactive exploration of a finite but large set of alternatives.

4. Review of existing multiple objective metaheuristic procedures

Several multiple objective metaheuristic procedures have already been proposed. The goal of the procedures is to find a sample of feasible solutions being a "good" approximation of the efficient solutions set. Such procedures may be used in the first phase of methods with a posteriori articulation of preferences or in computational phases of interactive procedures generating samples of solutions.

Schaffer [21] (see also [10]) and Srinivas and Kalyanmoy [25] proposed methods based on genetic algorithms Although, genetic algorithms may be used in the case of combinatorial problems the methods are designed for continuous case. Usefulness of these procedures in the case of MOCO is yet to be tested.

Fortemps, Teghem and Ulungu [7] have proposed an algorithm based on simulated annealing and suggested their use in MOCO. The algorithms of their proposals are very close to that of the single objective SA. The general scheme of the procedures is given below:

Select a starting solution $\mathbf{x} \in D$
Update set M of potentially efficient solutions with $\mathbf{x}$
$T := T_0$
<u>**repeat**</u>
  Construct $\mathbf{y} \in V(\mathbf{x})$
  Update set M of potentially efficient solutions with $\mathbf{y}$
  $\mathbf{x} := \mathbf{y}$ (accept $\mathbf{y}$) with probability $P(\mathbf{x}, \mathbf{y}, T, \Lambda)$

if the conditions of changing the temperature are fulfilled **then**
decrease T
until the stop conditions are fulfilled

where: $V(\mathbf{x}) \subseteq D$ is the neighborhood of solution $\mathbf{x}$, i.e. the set of feasible solutions that may be reached from $\mathbf{x}$ by making a simple move, T - temperature, $\Lambda = [\lambda_1, \ldots, \lambda_j]$ is a vector of weights.

Unlike single objective SA these procedures result not in a single solution but in a set of *potentially efficient solutions*, i.e. the set composed of solutions efficient with respect to all generated solutions. The other difference is in the way of calculating the probability of accepting a new solution, denoted by $P(\mathbf{x}, \mathbf{y}, T, \Lambda)$. In the case of single objective SA a new solution is accepted with probability equal to one if it is not worse than the current solution. Otherwise, it is accepted with probability less than one. In the case of multiple objectives one of the following three situations may appear while comparing a new solution $\mathbf{y}$ with the current one $\mathbf{x}$:

- $\mathbf{y}$ may dominate (weakly dominate) $\mathbf{x}$,
- $\mathbf{y}$ may be dominated by $\mathbf{x}$,
- $\mathbf{y}$ may be nondominated with respect to $\mathbf{x}$.

In the first situation the new solution may be considered as not worse than the current one and accepted with probability equal to one. In the second situation the new solution may be considered as worse than the current one and accepted with probability less than one. Fortemps, Teghem and Ulungu [7] have proposed several multiple objective rules for acceptance probability which in different way treat the third situation. Some characteristic rules are described below.

Rule C may be seen as a local aggregation of all objectives with an achievement scalarizing function based on the Chebyshev metric with the reference point $\mathbf{x}$. It is defined by the following expression:

$$P(\mathbf{x}, \mathbf{y}, T, \Lambda) = \min\left\{1, \exp\left(\max_j \left\{\lambda_j \left(f_j(\mathbf{x}) - f_j(\mathbf{y})\right) / T\right\}\right)\right\}.$$

Rule SL may be seen as a local aggregation of all objectives with a weighted sum of the objectives. It is defined by the following expression:

$$P(\mathbf{x}, \mathbf{y}, T, \Lambda) = \min\left\{1, \exp\left(\sum_{j=1}^{J} \lambda_j \left(f_j(\mathbf{x})_j - f_j(\mathbf{y}) / T\right)\right)\right\}.$$

Rule W is defined by the following expression:

$$P(\mathbf{x}, \mathbf{y}, T, \Lambda) = \min\left\{1, \exp\left(\min_j \left\{\lambda_j \left(f_j(\mathbf{x}) - f_j(\mathbf{y})\right) / T\right\}\right)\right\}.$$

Please note, that the higher is the weight associated with a given objective the lower is the probability of accepting moves that decrease the value on this objective and the greater is the probability of improving value on this objective. So, controlling the weights one can increase or decrease the probability of improving values of the particular objectives.

The authors have tested the above approach on some MOCO problems. It was observed that the procedure works good enough for relatively small problems only. In the case of larger problems the set of potentially efficient solutions may represent a small region of set N. This was the reason for developing the PSA procedure which tends to generate a "good" approximation of the whole set N even for relatively large MOCO problems.

5. Description of the Pareto Simulated Annealing

PSA uses some ideas known from two existing single objective metaheuristic procedures: genetic algorithms [10] and simulated annealing [15], [16]. These ideas and their origin may be summarized as follows:

- genetic algorithms
  - The use of a sample (population) of interacting solutions
- simulated annealing
  - The exploration of a neighborhood of the considered solution
  - The acceptance of a new solution with some probability depending on a parameter called the temperature
  - The scheme of decreasing the temperature

The general scheme of the PSA procedure may be summarized as follows:

Select a starting sample of generating solutions $S \subset D$
Update set M of potentially efficient solutions with S
$T := T_0$
<u>**repeat**</u>
  <u>**for each**</u> $\mathbf{x} \in S$ <u>**do**</u>
    Construct $\mathbf{y} \in V(\mathbf{x})$
    Update set M with $\mathbf{y}$
    Select solution $\mathbf{x}' \in S$ closest to $\mathbf{x}$ and nondominated with respect to $\mathbf{x}$
    <u>**if**</u> there is no such solution $\mathbf{x}'$ or it is the first iteration with $\mathbf{x}$ <u>**then**</u>
      Set random weights such that:
      $\forall_j \lambda_j \geq 0 \text{ i } \sum_j \lambda_j = 1$
    <u>**else**</u>
      <u>**for each**</u> objective f_j

$$\lambda_j = \begin{cases} \alpha \lambda_j^{\mathbf{x}} & ,\text{if} f_j(\mathbf{x}) \geq f_j(\mathbf{y}) \\ \lambda_j^{\mathbf{x}} / \alpha & ,\text{if} f_j(\mathbf{x}) < f_j(\mathbf{y}) \end{cases}$$

    $\mathbf{x} := \mathbf{y}$ (accept $\mathbf{y}$) with probability $P(\mathbf{x}, \mathbf{y}, T, \Lambda)$
  <u>**if**</u> the conditions of changing the temperature are fulfilled <u>**then**</u>
    decrease T
<u>**until**</u> the stop conditions are fulfilled

where: $\Lambda^{\mathbf{x}} = \left[\lambda_1^{\mathbf{x}},\ldots,\lambda_J^{\mathbf{x}}\right]$ is the weighting vector used in the previous iteration for solution $\mathbf{x}$, $\alpha > 1$ is a constant close to one (e.g. $\alpha = 1.05$), $P(\mathbf{x},\mathbf{y},T,\Lambda)$ is one of the multiple objective rules for acceptance probability described above.

In each iteration of the procedure a sample of solutions, called generating sample, is used. The main idea of PSA is to assure a tendency for approaching the set of efficient solutions as well as an inclination for dispersing the solution constituting the generating sample over the whole set N. In result each solution tends to investigate a specific region of set N.

The tendency for approaching the set of efficient solutions is assured by using one of the mentioned above multiple objective rules for acceptance probability. The inclination for dispersing the solutions from the generating sample over the whole set N is obtained by controlling the weights of particular objectives used in these rules. For a given solution $\mathbf{x} \in S$ the weights are changed in order to increase the probability of moving it away from its closest neighbor in S denoted by $\mathbf{x}'$. This is obtained by increasing weights of the objectives on which $\mathbf{x}$ is better than $\mathbf{x}'$ and decreasing weights of the objectives on which $\mathbf{x}$ is worse than $\mathbf{x}'$.

Please note that the algorithm of PSA is essentially parallel because calculations required for each point from S, i.e. construction of a new solution form its neighborhood, setting the weights and accepting the new solution, may be done on different processors.

It is also worth mentioning that one of the crucial points of the procedure from the point of view of its effectiveness is updating of the set of potentially efficient solutions. A data structure called Quad Tree allows for very effective implementation of this step [6], [11].

PSA is not a complete method for solving MOCO problems but just a tool for generating approximation of the set of efficient solutions. It is proposed to use the following three phase method for finding the best compromise of a MOCO problem:

Phase I. Finding an approximation M of set N with PSA
Phase 2. Selection of the best solution $\overline{\mathbf{x}}$ from the set M
Phase 3. Searching for a solution $\hat{\mathbf{x}}$ dominating $\overline{\mathbf{x}}$

The second phase is the only one that requires co-operation with the DM. In this phase the DM learns of the possible trade-offs as well as of his/her preferences and selects the best compromise. As was mentioned before, he/she may require some support in this phase. One of the interactive methods for multiple criteria problems with a large but finite set of alternatives may be used to support the DM.

In the third phase solution $\overline{\mathbf{x}}$ may be used as a reference point of a scalarizing function. This function can be optimized with a single objective metaheuristic procedure. This phase may result in a solution $\hat{\mathbf{x}}$ dominating $\overline{\mathbf{x}}$. So, this phase consists in an attempt at improving $\overline{\mathbf{x}}$ with respect to all objectives.

6. Evaluation of multiple objective metaheuristic procedures

Since the goal of multiple objective metaheuristic procedures is to find a "good" approximation of the set N, it is important to have some evaluation technique allowing for comparison of different approximations.

Solution obtained by a single objective optimization method may be evaluated by comparison with another solution obtained in a different way or with some reference solution, e.g. global optimum or the best solution known so far. Analogously in the multiple objective case a set of potentially efficient solutions may be compared with another such set obtained in a different way or compared with some reference set R of solutions, e.g. the set of efficient solutions or the best approximation known so far. The quality metric described below concern the latter case.

The proposed quality metric is defined as:

$$Dist = \max_{\mathbf{y} \in R} \left\{ \min_{\mathbf{x} \in M} \left\{ \max_{j=1,\dots,J} \left\{ 0, w_j \left(f_j(\mathbf{y}) - f_j(\mathbf{x}) \right) \right\} \right\} \right\}.$$

In other words, for each solution $\mathbf{y}$ from reference set R the closest solution from set M is found. The distance between two solutions is measured by the scalarizing function based on the weighted Chebyshev metric with the reference point $\mathbf{y}$. So, the closest solution to $\mathbf{y}$ is the one that minimizes the maximal weighted deviation from $\mathbf{y}$ on each objective. The weights used in the above expression are first set as:

$$w_j = \frac{1}{\Delta_j}$$

where: Δ_j is range of objective f_j in the reference set, and then normalized such that:

$$\sum_{j=1}^{J} w_j = 1.$$

Some results of computational experiments with the procedure will be described during the conference presentation. Application of the procedure to the multiple objective cell formation problem is presented in another paper presented at the Conference.

7. Summary and conclusions

A multiple objective metaheuristic procedure for combinatorial problems has been presented. The procedure tends to generate a "good" approximation of the efficient solutions set in a relatively short time. The main advantages of the procedure are as follows:

- it finds representation of the whole set of efficient solutions for relatively large problems,
- it naturally allows for a parallel implementation.

The following directions of further research may be considered:

- adaptive setting of the size of he generating sample which is a new parameter introduced in PSA,
- the use of concepts from other single objective metaheuristic procedures, e.g. tabu search.

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A Reference Direction Interactive Algorithm of the Multiple Objective Nonlinear Integer Programming*

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1. Introduction

The multiple objective convex nonlinear integer programming problem can be stated as follows:

$$\text{"max"} \{ f_k(x) \mid k \in K \} \tag{1}$$

subject to the constraints:

$$g_i(x) \le 0, \qquad i \in M \tag{2}$$

$$0 \le x_j \le d_j, \qquad j \in J \tag{3}$$

$$x_j - \text{integer}, \qquad j \in J \tag{4}$$

where $f_k(x)$, $k \in K$ are concave functions; $g_i(x)$, $i \in M$ are convex functions; $K = \{1,2,...,p\}, M = \{1,2,...,m\}, J = \{1,2,...,n\}$. The symbol "max" means that each objective function has to be maximized.

A few definitions are given preliminary to improve the clearness of the text:

Definition 1: The solution x is called efficient if there does not exist another solution $\bar{x} \neq x$, such that the inequalities

$$f_i(\bar{x}) \ge f_i(x) \text{ for each index } i \in K \text{ and}$$
$$f_i(\bar{x}) > f_i(x) \text{ for at least one index } i \in K$$

hold.

Definition 2: The solution x is weak efficient if and only if there does not exist another solution $\bar{x} \neq x$, such that

$$f_i(\bar{x}) > f_i(x) \text{ for each index } i \in K.$$

Definition 3: The p–dimensional vector $f(x)$ with components $f_i(x)$, $i \in K$; is called nondominated, if x is an efficient solution.

Definition 4: The preferred (weak) efficient solution is the (weak) efficient solution the decsion maker (DM) chooses as a best one at the current iteration of the proposed algorithm.

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Definition 5: The most preferred (weak) efficient solution is the (weak) efficient solution satisfying the preferences of the DM in a greatest degree. It is also the final solution in the algorithm.
Definition 6: The reference direction is defined by the difference between the reference point (aspiration levels) given by the DM and the nondominated solution, obtained at the previous iteration.

The problem (1–4) does not possess an analytically defined optimal solution. For this reason a selection among the set of efficient solutions is necessary. It is subjective and depends upon the decision maker (DM). The so called interactive algorithms (see, for example, [2], [18], [20], [21]) have large application in solving multiple objective integer programming problems. They offer a flexible way to help the DM in selecting a preferred efficient solution. The interactive algorithms consist of two alternate phases : 1. Interaction (dialogue) with the DM and 2. Generating of solutions. Very often an appropriate surrogate single objective integer problem is solved during the second phase. In the dialogue phase the DM indicates his/her preferences (for example aspiration levels for all objectives) by giving a reference point in the criterial space, i. e. he/she gives additional information so that a new single objective problem is created and solved or the algorithm terminates. In this way the solving of multiple objective integer problems by means of such an algorithm is reduced to solving a series of single objective integer problems. The nonconvex integer programming problems are algorithmically unsolvable in the general case. As shown in [1], [10], [11], no exact algorithms can be developed to solve nonconvex integer programming problems. From the viewpoint of computational complexity theory the single objective convex integer programming problems (linear and nonlinear) belong to the class of the NP-hard problems (see [3], [10], [16], [17]). There does not exist an exact algorithm, which can solve these problems in time depending polynomially on the problem input data length or on the problem size. For this reason approximate algorithms are applied when larger problems have to be solved. Different types approximate algorithms have been developed. An important place between them take the so called tabu search algorithms. Especially good results for their performance are obtained when an initial feasible integer solution is known, as shown in [6]. The difficulties, connected with single objective integer problems solving, increase for multiple objective integer problems, because many single objective integer problems have to be solved in an interactive algorithm. In spite of these difficulties, interactive algorithms are created for solving multiple objective integer optimization problems (see, for example, [15], [19], [20], [21]). They are still few especially in the nonlinear case. An interactive algorithm for solving convex multiple objective nonlinear integer programming problems is proposed here.

An interactive algorithm should be able to find efficient solutions comparatively quickly and should not place great demands on the DM in order to be practical and to achieve wide acceptability. One way to speed up the performance of an interactive algorithm is to minimize the number of single objective integer programming problems being solved in order to find a preferred efficient solution. Another way is the application of an approximate algorithm with polynomial computational complexity for solving the single objective integer problems. An exact algorithm for their solving may be used only if the DM considers that the approximate efficient solution, found at the previous iteration is very close to the most preferred efficient solution.

Both ways above mentioned are used in the proposed algorithm. It is based on the reference direction approach (see [13],[14]), which helps the DM to make decisions directing him/her quickly to the most preferred efficient solution. The number of iterations and of single objective problems solved is decreased in this way. The reference direction methods are based on the supposition that if more than one nondominated point is obtained during the current iteration, the DM can make better choice of a current preferred efficient solution (see for example [12]). At each iteration of the proposed algorithm the DM gives a number of points situated at the equal intervals along the reference direction, which are projected onto the efficient frontier. The obtained projections in the criterial space are nondominated solutions correspondingly. The original multiple objective problem is reduced to a series of single objective nonlinear integer problems. Each efficient solution can be evaluated immediately after it has been generated and the DM decides if he/she wants to change the reference direction. The proposed interactive algorithm is flexible in sense that the DM can project only one point along the reference direction if he/she wants. In this case the algorithm performs as a reference point algorithm. In addition the algorithm uses an approximate procedure for solving single objective integer problems which speeds up its performance.

2. The proposed algorithm

Let X denotes the set of solutions, satisfying constraints (2–4).

The following single objective problem is proposed to obtain a (weak) efficient solution:

Objective function

$$F(x) = \{ \min_{k \in K1} (f_k(x) - vf_k)/(af_k - vf_k) + \min_{k \in K2} (f_k(x) - af_k)/(vf_k - af_k) \} \tag{5}$$

maximized over the set of constraints:

$$f_k(x) \geq \min(af_k, vf_k), \quad k \in K \tag{6}$$

$$x \in X, \tag{7}$$

where af_k denotes the desired value (aspiration level) for the objective function $f_k(x)$. It is given by the DM during the dialogue phase. By vf_k is denoted the value of the same objective function for the solution found at the last iteration, and

$$K1 = \{ k \in K \mid af_k > vf_k \}, K2 = \{ k \in K \mid af_k < vf_k \},$$

$$K3 = \{ k \in K \mid af_k = vf_k \}, K = K1 \cup K2 \cup K3.$$

It should be noted that the problem (5–7) has always a feasible solution if the feasible set X is nonempty and has always an optimal solution if the objective functions $f_k(x)$, $k \in K$, are finite over X.

The optimal solution of (5–7) is a weak efficient solution for (1–4), see Theorem 1 in the Appendix.

The single objective problem (5–7) has the following properties: (*i*) its optimal solution is a (weak) efficient solution of (1–4); (*ii*) the solution found at the previous iteration is feasible for the next single objective problem; (*iii*) the feasible solutions for the problem (5–7) are situated close to the efficient frontier. Property (*i*) is desirable in order to show only (weak) nondominated solutions to the *DM*. The property (*ii*) is necessary because the solution, obtained at the last iteration is used as an initial solution for the next problem. The property (*iii*) is very suitable, because it justifies the use of an approximate single objective algorithm. It depends on the solution found at the previous iteration. The values minimum (af_k, vf_k) restricts the feasible domain of the current scalarizing problem (see the constraints (6)) and make it quite more tighten than the feasible domain of the original problem (1–4). For this reason the current approximate nondominated solution obtained will be close to the corresponding exact nondominated solution.

The optimal solution of (5–7) is a weak efficient solution for (1–4). However, if it is desired to obtain only efficient solutions, then the following single objective surrogate problem is used:

Objective function

$$F1(x) = \{F(x) + \beta\sum_{k\in K}(f_k(x) - vf_k)\}, \tag{8}$$

maximized over the set of constraints (6–7). Here β is a small positive number. This formulation is similar to those one used in [14]. As shown in [14] the optimal solution of this problem is an efficient solution for (1–4).

The DM estimates the distance between the current (weak) efficient solution and the most preferred efficient solution at each iteration. If it is large, the current single objective nonlinear integer problem is solved by means of an approximate polynomial algorithm. The solutions obtained by means of one approximate algorithm are approximate to the efficient solutions. The accuracy of these solutions depends on the accuracy estimate of the single objective algorithm used. This feature has to be taken in mind at the creation of interactive multiple objective integer algorithms. Its influence is not very great because the accuracy of the approximate efficient solution obtained at a given iteration does not depend usually on the approximate solution found at the previous iteration. When a given approximate efficient solution seems close to the most preferred solution, the DM can use an exact single objective algorithm to obtain the optimal solution of the current single objective problem. The last approximate efficient solution found is used as a starting point in the exact algorithm. The search procedure goes on until the most preferred solution is found. When multiple objective problems of a larger size are being solved, the DM can use only the approximate algorithm for solving the single objective nonlinear integer problems. The exact algorithm may be used for solving only the last single objective problem.

The steps of the proposed algorithm can be stated as follows :

Step 1 : Set $h = 0$. By means of an approximate single objective algorithm find an initial solution $(x^h, f_k(x^h), k \in K)$, solving the problem (5–7) with initial values $vf_k = 0$, $af_k = 1$, $k \in K$. Show the solution to the *DM*. If the *DM* is satisfied, stop; otherwise set $vf_k = f_k(x^h)$ and go to Step2.

Step 2 : Input the initial aspiration levels. Compute the reference direction q_k: $q_k = af_k - vf_k$; $k \in K$.

Step 3 : Ask the *DM* to specify t – the number of points along the reference direction the *DM* wants to project onto the efficient frontier. Set $r = 1$.

Step 4 : Ask the *DM* to choose *imeth* $= 1$ for applying exact algorithm to solve the single objective problem or *imeth* $= 0$ for applying approximate algorithm.

Step 5 : Ask the *DM* to choose *ieff* $= 1$ if the formulation (8) is used for the current single objective algorithm or *ieff* $= 0$ when the formulation (5–7) is used.

Step 6 : Compute the step $\alpha = r/t$ on the reference direction and the current aspiration levels $af_k = f_k(x^h) + \alpha q_k$; $k \in K$. Solve the current single objective problem, keeping in mind *ieff* and *imeth*. Show the solution to the *DM*. If the *DM* is satisfied, stop; otherwise set $r = r + 1$. If $r > t$ go to Step 7, else go to Step 4.

Step 7: Ask the *DM* if he/she wants to change the aspiration levels. If yes, set $vf_k = f_k(x^h)$, input the new aspiration levels, set $h = h + 1$, compute the reference direction q_k, $q_k = af_k - vf_k$; $k \in K$. Go to Step 3.

Remark **1:** Without initial values given for af_k and vf_k the algorithm can not start. The values $af_k = 1$ and $vf_k = 0$ are suitable for beginning the computations. If the constraints (6) are not satisfied with these initial values for af_k and vf_k, then the corresponding objective functions f_k may be modified slightly by addition of large enough constants.

Remark **2:** It is recommendable for the DM to project no more than 3 or 4 points at *Step 3* of the algorithm, in order not very long computational time to get lost for exploration of one reference direction.

The approximate algorithm for solving single objective nonlinear integer problems, used at Step 6 of the interactive algorithm is based on a tabu search strategy. It is designed to solve single objective convex nonlinear integer problems. The tabu search algorithms have very high effectiveness in some cases, as shown in [7], [22]. For integer optimization problems from given subclasses, the solutions found by means of an algorithm of this type are within 98–99% optimality even for problems with very large size (see [7]). In many cases the computational time of tabu algorithms is drastically less than the time necessary for other algorithms (at average one to four orders of magnitude)– see, for example, [8], [22]. The obtained encouraging results are a good reason to consider the applying of tabu search subprocedures in interactive multiple objective algorithms as very useful. The used tabu search algorithm, as considered in [5], [6], has polynomial computational complexity ($O(n^3m + n^2m^2)$ computations of values of nonlinear functions) and consists of four phases. In Phase 1 an integer feasible point is searched. In Phase 2 a better integer feasible point is searched. These both phases are based on a local search technique, where if S is defined as a set of n-dimensional 0–1 vectors s with one nonzero component: $S=\{ s \mid s_i = 1; s_j = 0, j \neq i \text{ and } j \in J \}$, the neighborhood $N(x)$ of each solution x is defined by $N(x) = \{ x' \mid x' = x + s; s \in S \}$. In Phase 1 the function $B = \sum_{i\in Q} f_i(x)$ is minimized, where $Q = \{ i \mid f_i(x) < \min(af_i, vf_i)\ i \in K; x \in X \}$. Phase 3 is designed to diversify the search process. Three different ways for computation of diversification search direction are used with the aim to direct the search in regions not explored yet. A learning strategy is applied also in order to use the information for the history of the search. In Phase 4 the forbidden "moves" stored in a tabu list are evaluated by means of aspiration criteria and if there is available "perspective move" the procedure continues with Phase 1 or Phase 2, otherwise the procedure terminates. The last solution found in the proposed interactive algorithm is a feasible solution for the single objective problem created at the next iteration and it may be used as an initial solution when the single objective procedure is started at Step 6. This feature is very convenient and speeds up the tabu search procedure used. It improves the quality of the dialogue phase because it is tedious for the DM if he/she must wait a long time (many hours) to obtain one nondominated solution in case the single objective integer problem being solved has a large size.

The proposed reference direction interactive algorithm generates a number of (approximate when such one single objective algorithm is chosen) (weak) efficient solutions along the direction defined by *DM*'s aspiration levels and the solution found at the last iteration. The *DM* controls the number of nondominated solutions he/she would like to see in the chosen direction. The specific features of the proposed algorithm are as follows: 1) The original problem is reduced to solving a series of single objective nonlinear integer problems; 2) A number of (approximate) (weak) efficient solutions are generated by projecting of points along the reference direction onto the efficient frontier and presented to the *DM* at each iteration, thus providing more information to the *DM*; 3) The dialogue with the *DM* is in the form of aspiration levels, thus avoiding excessive demands on the DM; 4) The solution of the current single objective problem is a feasible solution for the next single objective problem solved, thus reducing computational efforts and time; 5) For all the test examples used in [9] (which are convex nonlinear integer problems) the computational results of the applied approximate tabu search algorithm show deviation not greater than 1%, as shown in [6]. Hence the obtained solutions of the single objective integer problem, solved at certain iteration may lie on or very close to the efficient frontier of the multiple objective nonlinear integer problem (1–4).

3. A research decision support system based on the proposed algorithm

A research decision support system is created, which implements the interactive algorithm presented here. The system is built on a modular principle and has a simple structure. The system extensions and modifications are easy realizable. The system is written in C and FORTRAN77 languages and allocates the operating memory dynamically.

Three basic modules are used in the decision support system : one driver programming module and two optimization modules. The system includes also some auxiliary modules which are connected with the main software modules.

The driver module executes the steps of the presented interactive algorithm. A convenient dialogue with the user for evaluation of the solutions found at each iteration is included in it also. In the driver module, taking into account the *DM* preferences, the both optimization modules are started, designated correspondingly for approximate and for exact solving the single objective nonlinear integer problems.

During the running of the programming system all approximate and exact (weak) efficient solutions are stored in a protocol array in the driver module. The DM can view and analyze all the solutions obtained.

The both optimization modules are implementations corresponding to the described tabu search algorithm and to one exact algorithm (see [4]) based on explicitly enumeration technique. It should be noted that the exact algorithm is suitable only for integer problems with a small size (up to 15 – 20 variables), because of the exponential dependence of the running time on the problem size.

The decision support system is designed to run on IBM compatible personal computers, operating under MS-DOS.

4. An illustrative example

The decision support system is tested by number of test examples taken from [9]. The original test examples are modified to multiple objective ones.

The performance of the system is illustrated on the following simple example:

"max" $\{ f_1(x) = x_1^2; f_2(x) = x_2 \}$

subject to the constraints :

$$g_1(x) = x_1^2 + x_2^2 - 100 \le 0$$
$$0 \le x_1 \le 10$$
$$0 \le x_2 \le 10$$
$$xj - \text{integer}, \quad j \in J; \qquad J = \{1, 2\}$$

Iteration 0 : The starting point used for the problem is: $x = (2, 1)$. The initial solution is $(f_1, f_2) = (4, 1)$. The obtained approximate weak efficient solution is $x^0 = (3, 9)$, $(f_1, f_2)^0 = (9, 9)$.

Iteration 1 : The new aspiration levels are $(af_1, af_2) = (80, 3)$. Let the *DM* wants to see three solutions, i.e. $t=3$. The obtained approximate weak efficient solutions are: $x^{11} = (7, 7)$, $(f_1,f_2)^{11} = (49,7)$; $x^{12} = (8, 6)$,

$(f_1,f_2)^{12} = (64, 6)$; $x^{13} = (8,6)$, $(f_1,f_2)^{13} = (64 ,6)$.

Iteration 2 : The *DM* sets the following aspiration levels:

$(af_1, af_2) = (70, 4)$. Let $t=1$. The obtained approximate weak efficient solution is: $x^2 = (9, 4)$, $(f_1,f_2)^2 = (81, 4)$.

Iteration 3 : The new aspiration levels are $(af_1, af_2) = (50, 5)$. Let again $t = 1$. The obtained approximate weak efficient solution is: $x^3 = (8, 6)$, $(f_1,f_2)^3 = (64, 6)$. The *DM* wants to obtain an efficient solution. The obtained efficient solution is $x^3 = (8, 6)$, $(f_1,f_2)^3 = (64, 6)$. If this solution is acceptable to the *DM,* the process terminates; otherwise, it continues.

5. Concluding remarks

The proposed interactive algorithm is designed to solve multiple objective convex nonlinear integer problems. It is based on the reference direction approach and provides global information about the efficient set to the *DM* without increasing the computational efforts excessively. The dialogue phase does not put extra demands on the user. Two single objective algorithms, considered in [4], [5], [6] are used for solving the surrogate problems exactly and approximately. The approximate algorithm incorporates a tabu search technique. It speeds up the performance of the interactive algorithm without great decrease in the quality of the obtained solutions.

A research decision support system, based on the proposed algorithm is developed. It is written in C and FORTRAN 77 languages and allocates the operating memory dynamically. The user friendly interface and the presented interactive algorithm are a basis for solving real multiple objective nonlinear integer problems by means of this system.

6. Appendix

Theorem 1: The optimal solution of (5–7) is a weak efficient solution for (1–4).

Proof : Let x^* is the optimal solution of the problem (5–7). Then the following inequality holds:

$F(x^*) \geq F(x)$ **for each** $x \in X$. **(9)**

Let us assume that x^* is not a weak efficient solution for the problem (1–4). Therefore, there exists a point $\hat{x} \in X$, such that

$f_k(x^*) < f_k(\hat{x})$ **for** $k \in K$ **(10)**

The following relation

$$F(\hat{x}) = \min_{k\in K1}\{(f_k(\hat{x}) - vf_k)/(af_k - vf_k)\} + \min_{k\in K2}\{(f_k(\hat{x}) - af_k)/(vf_k - af_k)\} >$$
$$> \min_{k\in K1}\{(f_k(x^*) - vf_k)/(af_k - vf_k)\} + \min_{k\in K2}\{(f_k(x^*) - af_k)/(vf_k - af_k)\} = F(x^*) \quad (11)$$

is obtained after transformation of the objective function $F(x)$ of the problem (5–7), having the inequality (10) in mind. It follows from (11) that $F(\hat{x}) > F(x^*)$, which contradicts to (9). Hence x^* is a weak efficient solution for the problem (1–4).

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ROUGH SET APPROACH TO MULTI-ATTRIBUTE CHOICE AND RANKING PROBLEMS

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Abstract. We propose an original way of applying the rough set theory to the analysis of multi-attribute preference systems in the choice (Pα) and ranking (Pγ) decision problematics. From the viewpoint of rough set theory, this approach implies to consider a pairwise comparison table, i.e. an information table whose objects are pairs of actions instead of single actions, and whose entries are binary relations instead of attribute values. From the viewpoint of multi-attribute decision methodology, this approach allows both representation of decision maker's (DM's) preferences in terms of "if ...then..." rules and their use for recommendation in Pα and Pγ problematics, without assessing such preference parameters as importance weights and substitution rates. The rule representation of DM's preferences is alternative to traditionally decision support models. The rough set approach to (Pα) and (Pγ) is explained in detail and illustrated by a didactic example.

Keywords. Rough sets, Multiple attributes, Choice, Ranking, Preference modelling, Decision rules

1 Introduction

Every decision support model requires preferential information more or less explicitly related with their parameters. A typical form of preferential information is a set of pairwise comparisons of actions from which we can assess substitution rates in the functional model (MAUT approach) or importance weights in the relational model (outranking approach) (cf. [1], [3], [4]). This kind of preferential information seems to be close to the natural reasoning of the DM. He/she is typically more confident exercising his/her comparisons than explaining them. The transformation of this information into functional or relational models seems, however, less natural. According to Slovic [14], people make decisions by searching for *rules* which provide good justification of their choices. So, after getting the preferential information in terms of exemplary comparisons, it would be natural to build the preference model in terms of "if...then..." rules.

The induction of rules from examples is a typical approach of artificial intelligence. The rules represent the preferential attitude of the DM and enable his/her understanding of the reasons of his/her preference. The recognition of the rules by the DM justify their use for decision support. So, the preference model in the form of rules derived from examples, fulfils both representation and recommendation tasks (cf. [12]).

This explains our interest in the rough set theory ([5], [6]) which proved to be a useful tool for extracting a set of rules from analysis of vague description of decision situations ([8]). Another advantage of the rough set approach is that it can deal with a set of inconsistent examples. Moreover, it provides useful information about the role of particular attributes and their subsets, and prepares the ground for generation of rules involving relevant attributes.

Until now, however, the use of rough sets has been restricted to the sorting problematic (Pβ) ([15]). This use is straightforward because the set of sorting examples can be directly put in the information table analysed by the rough set approach. In the case of choice and ranking problematics (Pα and Pγ), this straightforward use is not possible because the information table in its original form does not allow the representation of preference orders between actions. The aim of this paper is to overcome this limitation by substitution of the original information table by a *pairwise comparison table* (PCT).

The paper is organized in the following way. In the next section, basic concepts of the rough set theory are recalled. In section 3, the PCT will be introduced. Then, in section 4, the rough set approach to the analysis of PCT in the case of choice and ranking problematics will be described. A didactic example illustrating the proposed approach will be given in section 5. Final section groups conclusions.

2 Introductory remarks about the rough set theory

2.1 The general idea

The rough set concept proposed by Pawlak ([5],[6]) is founded on the assumption that with every object of the universe of discourse there is associated some information (data, knowledge). For example, if objects are cars available on a market, their technical and economic characteristics form information (description) about the cars. Objects characterized by the same description are indiscernible (similar) in view of available information about them. The *indiscernibility relation* generated in this way is the mathematical basis of the rough set theory.

Any set of indiscernible objects is called elementary set and forms a basic granule (atom) of knowledge about the universe. Any subset Y of the universe can either be expressed precisely in terms of the granules or roughly only. In the latter case, subset Y can be characterized by two ordinary sets called *lower and upper approximations*. The two approximations define the rough set. The lower approximation of Y consists of all elementary sets included in Y, whereas the upper approximation of Y consists of all elementary sets having a non-empty intersection with Y. Obviously, the difference between the upper and the lower approximation constitutes the *boundary region* including objects which cannot be properly classified as belonging or not to Y, using the available information. Cardinality of the boundary region says, moreover, how exactly we can describe Y in terms of available information.

2.2 Information table

For algorithmic reasons, knowledge about objects is represented in the form of an information table. The rows of the table are labelled by *objects*, whereas columns are labelled by *attributes* and entries of the table are *attribute values*. In general, we will use the notion of attribute instead of criterion because the former is more general than the latter: the domain (scale) of a criterion has to be ordered according to decreasing or increasing preference, while the domain of an attribute does not have to be ordered. We will use the notion of criterion only when the preferential ordering of the attribute domain will be important in a given context.

2.3 Decision rules

An information table can be seen as *decision table* assuming that the set of attributes $Q=C\cup D$ and $C\cap D=\emptyset$, where set C contains so called *condition attributes*, and D, *decision attributes*.

From the decision table a set of *decision rules* can be derived and expressed as logical statement "if ...then ..." relating descriptions of condition and decision classes. The rules are *exact* or *approximate* depending whether a description of a condition class corresponds to a unique decision class or not. Different procedures for derivation of decision rules have been presented (e.g. [2], [16]).

3 Pairwise comparison table

Application of the rough set theory to multi-attribute decision analysis was in fact restricted to the sorting problematic ([8]). In this case, the preferential information consists of sorting examples, i.e. actions, with their description by attributes, assigned to particular categories. The examples constitute rows of the decision table. The rough set approach applied directly to this table gives the usual results: measures of the quality of approximation, reducts of attributes, the core and sorting rules. In the case of multi-attribute choice and ranking, the preferential information provided by the DM consists of comprehensive pairwise comparisons of actions. The representation of the pairwise comparisons is strongly related to the ordering properties of preference relations defined on particular attributes. Thus, the preferential information has to be represented by objects corresponding to pairs of actions being compared, described by binary relations defined on each condition attribute (eventually criterion) and by the result of the comprehensive comparison (decision attribute). Obviously, this representation cannot be given in the information table used in the rough set approach. We propose the definition of a *pairwise comparison table* (PCT) to enable this representation.

Let A be a finite set of actions (feasible or not), considered by the DM as a basis for exemplary pairwise comparisons. Let also C the set of (condition) attributes, characterising the actions. The attributes can mean criteria (cf. [11]) but they are not limited to.

For any $q \in C$ let V_q be its domain and T_q a finite set of binary relations such that $\forall\, v'_q, v''_q \in V_q$ exactly one binary relation $t \in T_q$ is verified. For interesting applications it should be card $(T_q) \geq 2$. Furthermore, let T_d be a set of binary relations defined on set A (comprehensive pairwise comparisons) such that at most one binary relation $t \in T_d$ is verified $\forall\, x,y \in A$.

The *pairwise comparison table* (PCT) is defined as information table $S_{PCT} = \langle B, C \cup \{d\}, T_C \cup T_d, g \rangle$ where $B \subseteq A \times A$ is a non-empty *sample of comparisons*, d is a decision attribute corresponding to the comprehensive pairwise comparison, $T_C = \bigcup_{q \in C} T_q$ and $g: B \times (C \cup \{d\}) \to T_C \cup T_d$ is a total function such that $g[(x,y),q] \in T_q$ $\forall (x,y) \in A \times A$ and $\forall q \in C$, and $g[(x,y),d] \in T_d$ $\forall (x,y) \in B$. It follows that for any pair of actions $(x,y) \in B$ there is verified one and only one binary relation $t \in T_d$. Thus, T_d induces a partition of B. In fact, information table S_{PCT} can be seen as decision table since the set of condition attributes C and decision attribute d are distinguished.

In this paper, we consider S_{PCT} related to $P\alpha$ and $P\gamma$ problematics and assume that the exemplary pairwise comparisons provided by the DM can be presented in terms of preference binary relations defined as follows

$$T_q = \{P_q^h,\ h \in [-l_q, r_q]\}$$

where h is a relative integer and l_q, r_q are positive integers $\forall q \in C$, and $\forall (x,y) \in B$:

- $xP_q^h y$, $h>0$, means that action x is preferred to action y by degree h with respect to attribute q,

- $xP_q^h y$, $h<0$, means that action x is not preferred to action y by degree h with respect to attribute q,

- $xP_q^0 y$ means that x is similar (asymmetrically indifferent) to y.

Of course, $xP_q^0 x$ $\forall x \in A$ and $\forall q \in C$, i.e. P_q^0 is reflexive, and

$$[xP_q^h y,\ h \geq 0] \Leftrightarrow [yP_q^k x,\ k \leq 0].$$

Therefore, $\forall (x,y), (w,z) \in B$ and $\forall q \in C$:

- if xP_q^hy and wP_q^kz, $k \geq h \geq 0$, then w is preferred to z not less than x is preferred to y,

- if xP_q^hy and wP_q^kz, $k \leq h \leq 0$, then w is not preferred to z not less than x is not preferred to y.

The set of binary relations T_d is defined analogously, however, xP_d^hy means that x is comprehensively preferred to y by degree h.

Specifically, if $q \in C$ is a criterion, i. e. there exists a function $c_q: A \to \Re$ such that $\forall x,y \in A$, $c_q(x) \geq c_q(y)$ means "x is at least as good as y with respect to q", then, in order to define the set of preference relations T_q, one can use a function $k_q: \Re^2 \to \Re$ satisfying the following properties $\forall$ x, y, z $\in A$:

$$c_q(x)>c_q(y) \Leftrightarrow k_q[c_q(x),c_q(z)]>k_q[c_q(y),c_q(z)],$$

$$c_q(x)>c_q(y) \Leftrightarrow k_q[c_q(z),c_q(x)]<k_q[c_q(z),c_q(y)],$$

$$c_q(x) = c_q(y) \Leftrightarrow k_q[c_q(x),c_q(y)]=0.$$

The function $k_q[c_q(x),c_q(y)]$ measures the strength of positive (when $c_q(x)>c_q(y)$) or negative (when $c_q(x)<c_q(y)$) preference of x over y with respect to q. Typical representatives of k_q are $k_q[c_q(x),c_q(y)] = c_q(x) - c_q(y)$ and, if $c_q(z)>0$ $\forall z \in A$, $k_q[c_q(x),c_q(y)] = c_q(x)/c_q(y) - 1$. The strength of preference represented by k_q is then transformed into a specific binary relation P_q^h using a set of thresholds $\{\Delta_q^h, h \in [-l_q,r_q]: \Delta_q^0=0, \Delta_q^{h-1}<\Delta_q^h\}$, in the following way:

$$k_q[c_q(x),c_q(y)] \in]\Delta_q^{h-1},\Delta_q^h] \Leftrightarrow xP_q^hy \text{ for } h \in [-l_q,0],$$

$$k_q[c_q(x),c_q(y)] \in [\Delta_q^h,\Delta_q^{h+1}[\Leftrightarrow xP_q^hy \text{ for } h \in [0,r_q]$$

where $\Delta_q^{l_q-1} = -\infty$ and $\Delta_q^{r_q+1} = +\infty$.

The above definition of preference relations is not the only one which could be considered for the S_{PCT}.. Any system of preference binary relations can be used to express the pairwise comparisons (cf. [9]).

4 Rough set analysis of PCT in view of Pα and Pγ problematics

Since PCT is an information table, we can adapt all the concepts of the rough set analysis to it. Let $S_{PCT}=\langle B, C\cup\{d\}, T_C\cup T_d, g\rangle$ be an information table associated with a given preferential information. Two ordered pairs of actions (x,y), $(w,z) \in B$ will be considered *indiscernible* by the set of attributes $P \subseteq C\cup\{d\}$ in S_{PCT} iff $g[(x,y),q]=g[(w,z),q]$ $\forall q \in P$. Thus, every $P \subseteq C\cup\{d\}$ generates a binary relation on B, which will be called *PCT-P-indiscernibility relation*, denoted by I_P^{PCT}. It has the same properties as I_P. Equivalence classes of I_P^{PCT} are called P-elementary sets in S_{PCT} and $I_P^{PCT}(x,y)$ denotes the P-elementary set containing $(x,y) \in B$.

A crucial problem related with PCT is approximation of the set of binary relations T_d by attributes from set C. Let $\mathcal{B} = \{B_1, B_2,...,B_n\}$ be the partition of B induced by T_d. Each class B_j (j=1,2,...,n) corresponds univocally to binary relation $t_j \in T_d$. Let $P \subseteq C$ and $B_j \in \mathcal{B}$ (j=1,...,n). The *P-lower*, the *P-upper approximations* and the *P-boundary* of set B_j are respectively defined as

$$\underline{P}B_j = \{(x,y) \in B: I_P^{PCT}(x,y) \subseteq B_j\},$$

$$\overline{P}B_j = \{(x,y) \in B: I_P^{PCT}(x,y) \cap B_j \neq \emptyset\}.$$

$$Bn_P(B_j) = \overline{P}B_j - \underline{P}B_j\,.$$

P-lower and P-upper approximations of $B_j \in \mathcal{B}$ (j=1,...,n) can be seen as a rough approximation of the set of relations $t_j \in T_d$, called, in short, *rough binary relations*. This definition of the rough binary relation generalizes that of Pawlak [7] because the indiscernibility relation I_P^{PCT} is defined directly on $B \subseteq A \times A$ and is not derived from indiscernibility defined on set A.

Using I_P^{PCT} and rough approximations based on it, one is able to maintain all basic concepts and properties of the rough set theory: approximation measures, reduction and dependency of attributes, reducts and the core, and decision rules.

Let us pass to exploitation of the rough set concepts in Pα and Pγ problematics. The rough set analysis of the PCT leads to a *set of decision rules* (exact and approximate) which, after discussion with the DM, become his/her preference model.

This model can then be used to establish sets of preference binary relations on the whole set $\mathcal{A}$ of potential actions. They are finally exploited in a way appropriate to the problematic at hand in order to work-out a recommendation (cf. [13], [20]).

Let us remark that, from the set of decision rules we can derive:

1) the set of lower approximations of relations $t_j \in T_d$ (j=1,...,n) on $\mathcal{A} \times \mathcal{A}$,
2) the set of upper approximations of relations $t_j \in T_d$ (j=1,...,n) on $\mathcal{A} \times \mathcal{A}$.

For a given j, the lower approximation of t_j is composed of all pairs $(x,y) \in \mathcal{A} \times \mathcal{A}$ matching exact rules indicating univocally decision class j. Similarly, for a given j, the upper approximation of t_j is composed of all pairs $(x,y) \in \mathcal{A} \times \mathcal{A}$ matching exact or approximate rules indicating decision class j, possibly not univocally. As to pairs $(x,y) \in \mathcal{A} \times \mathcal{A}$ which do not match any rule, one can either accept this fact or search for some "nearest" rules [17].

The lower approximations represent only certain relations t_j, whereas the upper approximations represent possible relations t_j among pairs of actions from $\mathcal{A}$. The boundary is composed of all doubtful relations t_j.

5 Illustrative example

Let us consider the well-known example of car selection ([3]). The preferential information obtained from DM is a ranking of a set A of cars relatively well-known by the DM. The ranking (*) according to the DM's subjective decreasing preferences and characteristics of the cars by six criteria ((1), maximum speed (km/h) (max); (2), consumption in town (lt/100 km) (min); (3), consumption at 120 km/h (min); (4), horse power (CV) (max); (5), space (m^2) (max); (6) price (francs) (min)) are given in Table 1.

The PCT derived from Table 1 consists of all possible 90 pairwise comparisons of the cars from the set A. We are using the following form of function k_q for each pair (x,y) of considered cars:

$$k_q[c_q(x),c_q(y)]=c_q(x) - c_q(y),\ q=1,...,6,$$

and the set of thresholds Δ_q^h shown in Table 2.

Table 1 Characteristics and ranking of the set A of cars

Action	Car	(*)	(1)	(2)	(3)	(4)	(5)	(6)
a_1	Peugeot 505 GR	1	173	11.4	10.01	10	7.88	49,500
a_2	Opel Record 2000 1.S	2	176	12.3	10.48	11	7.96	46,700
a_3	Citroën Visa Super E	3	142	8.2	7.30	5	5.65	32,100
a_4	VW Golf 1300 GLS	4	148	10.5	9.61	7	6.15	39,150
a_5	Citroën CX 2400	5	178	14.5	11.05	13	8.06	64,700
a_6	Mercedes 230	6	180	13.6	10.40	13	8.47	75,700
a_7	BMW 520	7	182	12.7	12.26	11	7.81	68,593
a_8	Volvo 244 DL	8	145	14.3	12.95	11	8.38	55,000
a_9	Peugeot 104 ZS	9	161	8.6	8.42	7	5.11	35,200
a_{10}	Citroën Dyane	10	117	7.2	6.75	3	5.81	24,800

Table 2 The set of thresholds Δ_q^h used for definition of the PCT

q | h	-2	-1	1	2	3
(1)	-30	-10	15	40	
(2)	5	0.8	-0.9	-6.4	
(3)	4	1	-1.2	-4.5	
(4)	-7	-2	2	7	
(5)	-2	-0.8	1	2.5	
(6)	25,000	5,000	-3,000	-10,000	-30,000

It is assumed that the comprehensive comparison of the pairs of actions has the following interpretation:

"If x is ranked better than y, then x is at least as good as (outranks) y (denotation xP_d^1y) and y is not at least as good as (does not outrank) x (denotation $yP_d^{-1}x$)."

In consequence, the intersection of the two relations xP_d^1y and $yP_d^{-1}x$ says that x is preferred to y in a strict or weak sense ([11]).

The above transformation of the PCT results in the S_{PCT}. The rough set analysis of the S_{PCT} gives the following results:

- there are no dependent criteria in the set $C=\{c_1, ..., c_6\}$; it means that there are no superfluous criteria, thus the core is equal to C,

- there are 69 C-elementary sets and the quality of approximation of the comprehensive comparisons by set C is equal to 1,

- the set of decision rules is composed of 31 exact rules, presented in Table 3 together with the information about the strength of each rule (number of exemplary comparisons which support the rule). The decision rules were derived using the RoughDAS system ([16]) in interaction with the DM who guided the derivation process and finally accepted the set of rules.

The set of rules has then been applied to set $\mathcal{A}$ of potential actions composed of ten cars from the set A and two new cars presented in Table 4. To perform this application we had to create a new PCT* and a new S^*_{PCT} composed of 42 pairwise comparisons for which the comprehensive decision is not known. The 42 items are composed of 2 mutual comparisons of the new cars and 40 comparisons of each new car with each old car.

From the application of the decision rules to each orderd pair of actions $(a,b)\in\mathcal{A}\times\mathcal{A}$ there may arise one of the following four states (cf. [18], [19]): 1) aP_d^1b and not $aP_d^{-1}b$, i.e. *true* outranking; 2) $aP_d^{-1}b$ and not aP_d^1b, i.e. *false* outranking; 3) aP_d^1b and $aP_d^{-1}b$, i.e. *contradictory* outranking; 4) not aP_d^1b and not $aP_d^{-1}b$, i.e. *unknown* outranking.

To represent the outranking relations obtained by the application of the decision rules to objects from S^*_{PCT}, we will use the concept of interval set [21]. Let X be a finite non-empty set, called the reference set, and 2^X be its power set. If $W_1\subseteq W_2\subseteq X$, the following subset W of 2^X

$$W=[W_1, W_2]=\{Y\in 2^X: W_1 \subseteq Y\subseteq W_2, W_1,W_2\in 2^X\}$$

is called a closed *interval set*. The set W_1 is called the *lower bound* of the interval set, W_2 the *upper bound* and $W_2\backslash W_1$ the *boundary*. In the same line, if $R_1\subseteq R_2\subseteq X\times X$, we define an *interval relation* R

$$R=[R_1, R_2]=\{J\in 2^{X\times X}: R_1\subseteq J\subseteq R_2, R_1,R_2\in 2^{X\times X}\}.$$

R_1, R_2 and $R_2\backslash R_1$ are, respectively, the lower bound, the upper bound and the boundary of the interval relation R.

Given an interval relation R defined on X, an *interval graph* is a triple (X, Ω_1, Ω_2) where X is the set of nodes or vertices, Ω_1 is the set of directed arcs corresponding to the lower bound of R (R_1) and Ω_2 is the set of directed arcs corresponding to the boundary of R ($R_2\backslash R_1$), i.e. Ω_1 is the set of "certainly established arcs" and Ω_2 is the set of "doubtful arcs", respectively. The graph obtained from the interval graph with only certain arcs, i.e. (X, Ω_1), is called the *lower bound of the graph.* The graph with both certain and doubtful arcs, i.e. (X, $\Omega_1 \cup \Omega_2$), is called the *upper bound of the graph.* The graph with only the doubtful arcs, i.e. (X, Ω_2) is called the *doubtful region* or *boundary of the graph.*

Table 3 The set of decision rules derived from the PCT

rule #	1)	2)	3)	4)	5)	6)	d)	+)	rule #	1)	2)	3)	4)	5)	6)	d)	+)
1			1	0			1	8	17		-1			1	-2	-1	4
2	0					-2	1	9	18					0	-2	-1	2
3	1					-1	1	9	19	-2				0		-1	6
4			-1	1	2		1	5	20	1		-1	2			-1	2
5	-2	1	1			3	1	2	21	0		-2				-1	1
6	2						1	6	22			2	-2			-1	2
7	-2			-1		3	1	2	23	0				-1		-1	1
8			-2	2			1	3	24	-2				-1	2	1	2
9		1	0			0	1	1	25					1	0	1	1
10			1			1	1	4	26	-1	0				1	1	1
11	-1	1					-1	8	27	-1				2		1	1
12			-1	0	0		-1	6	28		-2		-2			-1	3
13		-1			0	0	-1	1	29	0				1	-1	-1	1
14	-2				-2	2	-1	4	30	1				-2		-1	1
15	0		0			-1	-1	2	31	0	-1		1	0		-1	2
16			-1	-1			-1	4									

+) Strength of the rule

Table 4 Characteristics of new cars

Car	(1)	(2)	(3)	(4)	(5)	(6)
x	175	9.0	7	7	6.5	50,000
y	135	8.2	7.5	7	6.0	40,000

In result of the application of the decision rules to set $\mathcal{A}$, we obtain an *interval outranking relation S*, where the true outranking relation (S_1) is the lower bound, and the union of true, contradictory and unknown outranking relations (S_2) is the upper bound. The interval outranking relation *S* can then be represented by an interval graph ($\mathcal{A}$, S_1, $S_2 \backslash S_1$). If between a and b there is a true outranking relation, a certain arc links a and b. If between a and b there is an unknown or a contradictory outranking relation, a doubtful arc links a and b.

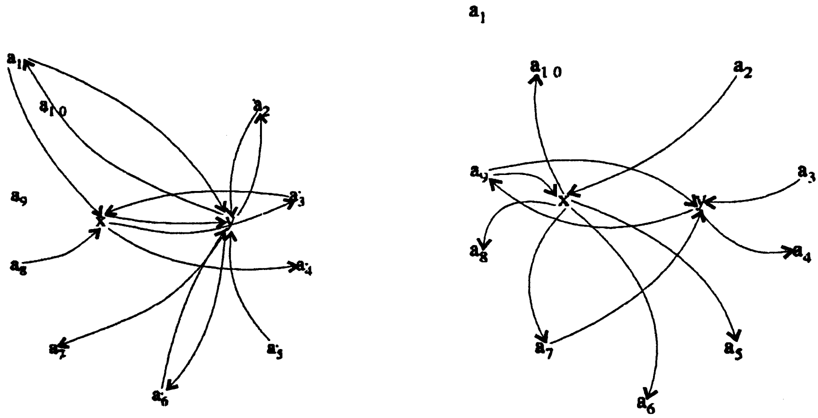
The lower bound and the boundary of the interval graph ($\mathcal{A}$, S_1, $S_2 \backslash S_1$) relative to the above example are presented in Fig. 1.

The *exploitation* of the interval outranking relation *S* for the decision *problem* $P\alpha$ is performed according to the main idea of ELECTRE I procedure ([10], [13]). This procedure looks for the kernel of the graph representing a crisp outranking relation. Our aim is then to define an *interval kernel N*, i.e. a kernel defined as an interval set. The lower bound and the upper bound of the interval kernel *N* are respectively defined as $N_1 = \underline{Z} \cap \overline{Z}$, and $N_2 = \underline{Z} \cup \overline{Z}$ where $\underline{Z}$ and $\overline{Z}$ are respectively the kernel of S_1 and S_2, after the reduction of the circuits.

Fig. 1 Interval graph of the outranking relation *S* on set $\mathcal{A}$.

a) lower bound (not including the arcs between actions from sample A)

b) boundary (doubtful region)



The set N_1 contains the actions of $\mathcal{A}$ which certainly belong to the interval kernel *N*, while the actions of $\mathcal{A}$ contained in N_2 possibly belong to the interval kernel *N*. In the considered example we have $\underline{Z} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, x, y\}$, $\overline{Z} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, x, y\}$, and then $N_1 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, x, y\}$, while $N_2 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, x, y\}$.

To exploit the interval outranking relation *S* for the decision *problem Pγ*, a crisp version of ELECTRE III procedure ([20]) is considered. This exploitation procedure takes into account the score of $a \in \mathcal{A}$ with respect to an outranking relation S, i.e. $B(a, S) = \text{card}\{c \in \mathcal{A}: aSc\} - \text{card}\{c \in \mathcal{A}: cSa\}$. The first class $\mathcal{A}_1$ of $\mathcal{A}$ in the preorder contains the elements of $\mathcal{A}$ with the highest score in S. Then the scores are recalculated in $\mathcal{A} \setminus \mathcal{A}_1$ and the procedure is repeated to obtain the other classes in the preorder. This procedure has been applied to the lower bound of *S*, i.e. S_1, and to the upper bound of *S*, i.e. S_2, giving respectively the following preorders:

$$a_1 \to a_2 \to a_3 \to (a_4, a_5) \to a_6 \to (a_7, a_8) \to (a_9, x) \to (a_{10}, y),$$

$$a_1 \to a_2 \to a_3 \to x \to (a_4, a_5) \to a_6 \to a_7 \to a_8 \to a_9 \to (a_{10}, y).$$

6 Conclusions

In this paper, application of rough sets to the analysis of preferential information in Pα and Pγ problematics was investigated. The key concept enabling this consideration is the PCT. Let us remark that this approach can be easily extended to the Pβ problematic, since the preferential information can be represented in a PCT also in this case. The main advantage of the proposed approach is the building of the preference model in terms of "if ...then ..." rules.

Another useful property of this approach is the 4-value logic used in different stages of the procedure and particularly suitable for presenting the recommendation to the DM. Furthermore, some more general or new concepts have been introduced: rough binary relation, interval binary relation, interval graph and interval kernel.

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Concepts of a Learning Object-Oriented Problem Solver (LOOPS)

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Abstract. This presentation discusses concepts of a learning object-oriented problem solver (LOOPS) which is on the one hand a new and general framework for a decision support system (DSS) and on the other hand answers some open or partially neglected questions in multiple criteria decision making (MCDM). These are, for instance: How should implicit knowledge about 'good alternatives' be processed? What method should be used? How should its parameters be adjusted?

The main methodological goals of LOOPS are 1) learning and 2) the integration of methods. These concepts are discussed and ways of their realization are suggested: Integration is achieved by providing several methods, by utilizing neural networks, and by developing a concept of generalized networks. Learning is realized by evolutionary algorithms. Essential to the implementation of these concepts within LOOPS is the object-oriented paradigm which is also discussed.

Keywords. Multiple criteria decision making, decision support systems, methodology, neural networks, learning, evolutionary algorithms, object-orientation

1 Introduction: Methodological Assumptions

A common assumption in multiple criteria decision making (MCDM) is that additional information on a given problem helps to reduce the set of efficient alternatives (= solution set without additional information) to a set of one or a few alternatives called the solution.

Many MCDM methods accumulate this kind of additional information, especially information on preferences. The mostly prefered approach to such a knowledge acquisition in MCDM is the interactive mode where a dialogue between the decision maker and the decision support system is established. We analyze an alternative way of utilizing knowledge within a multiple criteria decision support system (MCDSS): (Machine) learning from historical information.

The basic idea of this learning approach is that most decision situations repeat themselves. The recurrence of problems is a fundamental assumption in many DSS related works (see Rockart and DeLong [8], p. 17). In daily decision making non-recurring problems are rather seldom although these problems may be essential. However, most problems have to be solved again and again with slightly changed data (e.g. the set of alternatives). The main parameters (e.g. the preferences) often remain almost constant.

Considering the repetitiousness of decision problems it is worthwhile to use given data about historical decisions – method-based or human-based – or ex-post knowledge about decisions which would have been good in a past situation, in order to adapt parameters of a method to automate an appropriate decision making process.

Another basic assumption of our proposed MCDSS is based on methodological pluralism: In general (without considering a special problem situation), there are no strong reasons to prefer one special method over the many different approaches proposed in the MCDM literature. Therefore, different (appropriate) methods (from different schools of MCDM, e. g. reference point approaches, utility-based methods, outranking methods) should be available within an MCDSS. An appropriate method for a given MCDM problem can then be selected either interactively or using historical knowledge.

From these considerations we derive the demand for integrating different methods within a DSS. 'Integration' does not just simply mean that different methods are provided within the system although even this could be very helpful. Instead, we think of the linkage (combination), recombination, and generalization (by, for instance, neural networks) of methods.

An example for the linkage of MCDM methods: The decision analyst is in doubt which method or which parameters for a method should be used to solve a (MCDM) problem. Therefore, he or she decides to solve the problem with several method-parameter-combinations. The different resulting solutions, e. g. different scalar evaluations of alternatives, form criteria for a new choice problem, an MCDM problem of order 2. This problem can be solved by applying an MCDM method again. (Therefore, this procedure can be applied recursively.) The concept of applying several MCDM methods can be represented by a graph (MCDM network) the nodes of which are labeled by MCDM methods and the edges of which correspond to criteria or criteria evaluations.

2 Neural Networks, MCDM Methods, and Their Integration

In section one we have discussed two methodological preconceptions for a decision support system in the field of MCDM: Integration of different methods and learning. In this and the following section, we will discuss concepts how these goals can be achieved.

For the realization of integration of MCDM methods, we have already considered the idea of using (combining, linking, etc.) several methods. Such an approach can be described by a network-like structures. A special, well-established network approach can be used for MCDM in this way: The concept of (artificial) neural networks.

Neural networks have been successfully applied in fields like automatic control or pattern recognition (e.g. speech recognition). The flexibility of feed forward layered networks can be theoretically explained using a theorem by Kolmogorov [5] (see also Hecht-Nielsen [4]) which shows that any continuous function can be approximated with a superposition of special continuous functions (as done in the network).

Neural networks can be applied as MCDM methods. In [2], we have, for example, constructed neural networks which work like the simple additive weighting method or the conjunctive level approach. Malakooti and Zhou [7] have used neural networks which calculate multiple criteria utility functions. Neural networks can also be constructed for more complex approaches like outranking methods.

How can the relationships between MCDM methods and neural networks be interpreted? On the one hand neural networks can be regarded as a special, possibly more 'efficient' way (because of its inherent parallelism) of calculating (or simulating) what a method does. From this point of view the concepts of neural networks are on a different level (the implementational and computational level) than those of MCDM methods and, therefore, not relevant for developing new MCDM methods.

On the other hand, neural networks can be considered as an autonomous, new MCDM approach or even as a more general approach, going beyond traditional MCDM methods, because they can learn to behave like this or that formal method. Also human, not method-based MCDM processes can supply the data needed for neural network learning. So, neural networks can (on principle) learn to simulate a formalized problem solving behavior (i. e. a method) as well as an unformalized (i. e. human) decision process. In this respect, the neural network approach meets our first design criterion for a learning DSS.

Let us now consider the relevance of neural networks for the integration of methods. As discussed above, the application of several MCDM methods to a given MCDM problem can be represented by a graph. This similarity in structure indicates possibilities of integrating neural networks and MCDM methods by generalizing the meaning of the nodes of neural networks:

A single neuron calculates a weighted sum of its inputs, compares this with a threshold value (subtraction), and transforms the result for instance onto the interval [0,1]. This behavior can be interpreted as an MCDM method. (With a linear output function and a threshold value of 0, this corresponds immediately to the simple additive weighting approach). The behavior of nodes (neurons) can be generalized by using other 'MCDM functions' as the node (neuron) function.

The generalization of the meaning of nodes requires a closer look at the inputs, the outputs, and the parameters of a node and their treatment within a network: A 'normal' neuron receives several scalar inputs (a vector) and calculates a scalar output. It uses weights (with which the input edges may be labeled) and a threshold value (with which the nodes may be labeled) as parameters. MCDM methods, in general, may need more parameters. The input may correspond to the attributes of an alternative. This representation was used in Malakooti and Zhou [7] and Hanne [1], [2]. Then, the neural network calculates a value for a single alternative (scalarization). Given a finite set of alternatives (a multiple attribute decision making problem) the neural network can calculate the value of each alternative, so that an optimum can be chosen. For an infinite set of alternatives the neural network can be used by a global optimization method to find (or approximate) an optimum.

For the above representation is it crucial that the value of an alternative can be calculated without considering other alternatives and their criteria evaluations. Many MCDM methods are not compatible with this independence assumption (see also Zeleny [11], p. 135-148): For instance outranking approaches, reference point approaches, and the AHP with relative measurement. These approaches evaluate alternatives under the consideration of other alternatives' criteria data. For instance, an utopia point is constructed using the best criteria evaluations from the set of alternatives. Therefore, appropriate nodes for such MCDM methods require a complete representation of the MCDM problem. (Therefore, we assume a finite set of alternatives.)

Also the output of an MCDM method is, in general, more complex than a scalar neuron output. When an MCDM method gets a whole problem representation it is appropriate to evaluate all alternatives together (instead sequentially, as discussed above). Some methods do not just calculate a scalar value for each alternative but provide a more complex evaluation. For instance, PROMETHEE I evaluates alternatives with two criteria (the incoming and outgoing flow). Some filtering methods do not use special evaluations of an alternative but reduce the set of alternatives. For instance, we could first use a method which negatively selects dominated alternatives or alternatives which do not fulfill certain achievement levels. Then, a more selective approach for finding a final solution could be applied.

3 Learning: What and How?

In section one we have motivated the possibility of utilizing learning for MCDM. This section is intended to discuss algorithmic ways of learning in more detail. An assumption of learning is that in some form knowledge about 'good' decision making is available. This may come from former human decision making. If results of this decision making are considered appropriately the learning of this decision behavior helps to automate the decision making process so that the costs of decision making can be reduced while the quality is maintained. Analogously, results obtained by using an MCDM method can be used as learning data. For instance, a recurring problem is first solved several times with an interactive MCDM method, and then the resulting decision behaviour (e.g. evaluations of alternatives based on a utility function) is used as a reference result for the learning process. Later, similar problems can be solved without needing possibly expensive interactions with the decision maker.

Another source for the reference data: Some time after a problem situation occurred, information about 'good' alternatives becomes available. This information is not available at the time when the decision has to be made (otherwise there wouldn't be a real decision problem). Instead, this ex-post evaluation of alternatives is a result of a complex process, possibly involving uncertainties and some not available data (incomplete information). This process can possibly be (approximately) predicted by a human being and may also be learned by a machine (like a neural network).

In contrast to expert systems which try symbolically to simulate reasoning processes, this learning approach does not need that kind of knowledge. In connectionist approaches like neural networks only knowledge about the results of decision making (i.e. alternative evaluations) is necessary. This makes the learning process much easier and open for automation.

How can the desired functionality be learned? For different kinds of neural networks different learning algorithms have been proposed. Problems in finding a learning algorithms for (generalized) perceptron networks (Minsky and Papert [6]) caused a stagnation in developing this science in the 60s and 70s. Today the most common algorithm for the most common network architecture, the feed forward layered network, is the backpropagation algorithm (and its improvements). It is based on the minimization of the error between the calculated and the desired results (which is recursively propagated from the output nodes to the inner nodes of the network). This well-known algorithm has several disadvantages which are of special relevance for our attempt to generalize neural networks.

The backpropagation algorithm is a gradient hill climbing method which does not guarantee that a global optimum will be reached. Instead, the algorithm may be stuck in a local optimum. Secondly, the algorithm assumes a special architecture for the neural network and the (continuous) differentiability of the neuron function. Especially, the differentiability assumption is usually not fulfilled within generalized networks where the nodes calculate MCDM methods. These assumptions are not fulfilled also in many other neural network types, e. g. networks with neurons mapping their inputs onto $\{0,1\}$.

For these (and other) reasons we propose the application of 'universal' learning algorithms, especially those utilizing ideas from evolution processes. These evolutionary algorithms do not need the assumption that the optimized function has special properties. The neural network does not need to have a special architecture or special neuron functions. The algorithm is also quite robust when dealing with local optima. For evolution strategies which are important representatives of evolutionary algorithms, global convergence can be shown for a wide class of well-behaving functions (see Schwefel [10]). For genetic algorithms a schema theorem is valid (but they do not converge in their canonical formulation (Rudolph [9]).

For the Learning Object-Oriented Problem Solver (LOOPS) a relatively general type of evolutionary algorithm has been developed which encompasses evolution strategies and genetic algorithms. This provides a higher flexibility for the adaptation of the evolutionary algorithm to the given architecture.

Actually, these evolutionary algorithms cannot be used only for the learning within neural networks or their generalizations. They are a universal means for improving methods. Therefore, the optimization in a generalized network does not only change network-specific parameters but also parameters of single methods.

The generalization of the network and the universality of the evolutionary learning methods require an abstractization of the method (treating them as 'black boxes') which is achieved by object-oriented concepts which will be discussed in the following section.

Is it always possible to learn the desired functionality? About the effectiveness of learning little knowledge is available. Even if a learning algorithm (like special variants of the evolutionary algorithms) guarantees achieving a global optimum for certain kinds of problems, it is not clear whether this leads to satisfying results: The results may deviate too strongly from the desired solutions. Possibly, the (historical) input data is insufficient (or out of date etc.) so that the learned results are not valid for future applications or the method has just not learned to generalize correctly. In an anecdote, for instance, neural networks have once been used for recognizing tanks in a forest. In the learning set all pictures with tanks were made with a cloudy sky whereas those without tanks were taken during a sunny day. The network worked correctly with the sample set. But it just learned to differentiate sunny from cloudy weather. Therefore, it did not develop the ability to recognize tanks. The network did not learn to generalize.

As well-known in computational theory, methods cannot calculate arbitrary functions. For a given method, for instance a neural network of a given architecture, it is not clear whether such a network could produce a desired functionality. For instance, a single layered neural network cannot calculate the XOR function. Or the parameters of a simple additive weighting approach cannot cope with nonlinearities within a utility function.

4 Concepts of an Object-Oriented Implementation

The object-oriented concept is of special importance to the development of LOOPS. A more extensive description of the object-oriented paradigm and its application can be found in [1].

Object-orientation is based on the idea to put data and operations together to form objects. The state of an object (its data) can only be changed by sending a message to it. The object then executes a suitable operation (called method in object-oriented terminology). The decision which operation is executed is made at run-time by performing a method search (dynamic or late binding). This concept provides an encapsulation of objects.

Objects are always concrete instances of data and procedures. The information about the structure of their data and their procedures is defined in the object class. All objects of a class have the same data structure and process the same messages.

The key concept of object-orientation is inheritance. It allows the building of a derived class of a class which inherits all defined data and methods. Additional data and messages can be defined. Also inherited definitions can be overwritten ('spezialization').

The required abstraction of (MCDM) methods can be achieved by using the object-oriented concept: Methods can be represented by objects.

This is necessary in order to separate problems, methods, and learning methods. Therefore, this object-oriented implementation approach also provides advantages on the conceptional level. For instance, the relationships between these objects can be understood more clearly:

The object-oriented class concepts were used to implement different classes of problems. Also, inheritance is applied: A base class *problem* has been defined. From this, several classes are derived which serve to encapsulate different data structures representing problem types. Analogously, a base class *method* has been defined which allows the construction of derived classes for MCDM methods and neural networks. Finally, a base class *metamethod* has been implemented to facilitate learning.

In LOOPS different application scenarios are possible. In an application without learning, a (MCDM) *problem* is defined and also an appropriate *method*. The *problem* is linked with the *method* via a pointer (or reference) *solver*. To solve a problem a message *solve* is sent to the *problem*. The appropriate *solve* method is found by the dynamic binding mechanism (see above) and the *solve* method is executed. This procedure sends a message to the associated (via the *solver* pointer) *method* objects. The solution is an object of the same class as the original *problem* and replaces it.

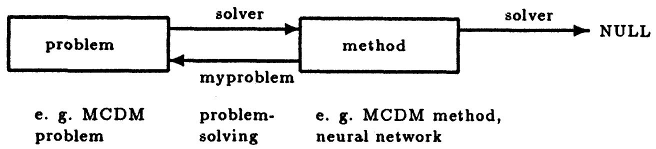

Figure 1: Problem solving

In the case of learning, a *metamethod* which provides the learning strategy has also to be defined. The *method* is linked to a *metamethod* via its pointer *solver*. Additionally, access to learning data is necessary. Thus, the method is linked to a reference problem (via a pointer *refproblem*) and its reference result (via a pointer *refresult*). To solve a problem, a learning process is performed first. A message *solve* is sent to the *method* object. This calls the *metamethod* which performs the learning process. The result of this process is an improved *method* which replaces the original *method*.

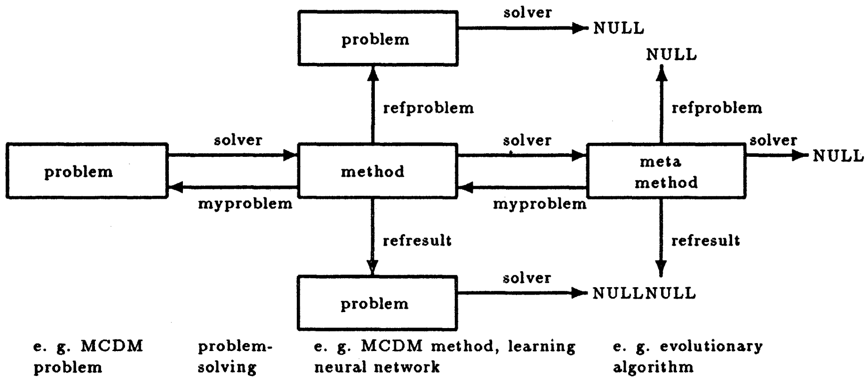

Figure 2: Problem solving and learning

If the *refresult* pointer is defined ($\neq$ NULL) the learning algorithm minimizes the difference between the solution of the reference problem and the reference result. If no reference result is defined, learning will minimize the norm of the achieved results.

More generally, for a given past date its past and future-related reference problems and results can be defined and linked with a *method*. For the learning process only the past related data are used. The future related data (which are also past data when the learning is performed) are used afterwards for judging the quality of learning. The success with these data which are not used for the learning process itself, indicates the ability of the method to generalize.

5 Conclusions

Although still in the process of implementation and development, the decision support system LOOPS which is based on the described concepts has been applied to test data in various ways.

Because of the universality of the developed concepts they are apparently useful also in other areas than MCDM. The main concepts – the object-oriented implementation (objectification) of methods, their class hierarchy, the relationship between problems, methods, and metamethods, the implemented evolutionary algorithms, the developed concept and implementation of neural networks and generalized networks – may be useful in many other areas as well. This universality is also relevant to the general discussion about decision support systems or DSS frameworks, respectively. The system has already been used for simulating the emergence of cooperative behavior by implementing classes for (finite two person) games, especially the (iterated) prisoner's dilemma, and strategies for these games, which are learned using evolutionary algorithms (see [3]).

Whereas the advantages of using learning in MCDM seem to be obvious, it is still quite unclear whether the integration approaches are useful or just an academic concept, too complex for practical application. Possibly, a traditional neural network approach would be sufficient for many multicriteria problems. Further research will consider possibilities of applying LOOPS to real world problems.

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Outranking-Driven Search Over a Nondominated Set

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Abstract. We consider the interactive exploration of implicitly or explicitly given large sets of alternatives. Upon review of classical interactive procedures, which usually assume a utility function preference model, we are distinguishing three typical operations used in various interactive procedures: contraction of the explored set, exploration of some neighbourhood of a current alternative, and reduction of a sample of the explored set. After pointing out some weak points of the traditional procedures, we describe three interactive procedures performing the three operations, respectively, using an outranking relation preference model. Due to proposed ways of building and exploiting the outranking relation, the weak points of traditional procedures can be overcome.
Keywords. Multi-criteria choice problems, interactive procedures, outranking relation.

1. Introduction

The choice problematic, whose aim is to select from a set of alternatives the one that best satisfies the decision maker (DM), is one of four reference problematics in multiple-criteria decision analysis (Roy, 1985; Vincke, 1992). A formal insight into multiple-criteria choice problems leads to distinction of two classes:

- problems with an explicitly known set of alternatives, e.g. in the form of a finite list,
- problems with the set of alternatives known implicitly, in the form of constraints defining the feasible alternatives.

The problems of the latter class are called *multiple objective mathematical programming* (MOMP) problems.

The distinction between multiple criteria choice problems with an explicitly and implicitly known set of alternatives is right from a formal point of view. In practice, however, a more important characteristic is ability or inability of the DM to perceive the whole set of alternatives. Indeed, if the set of alternatives is explicitly known but large, its review by the DM may pose the same problems as an exploration of the set of efficient alternatives in the MOMP. Of course, distinction between small and large is not sharp, but we will admit that the set is large if its review needs a special guidance, characteristic for interactive procedures. In this paper, we are just dealing with *multiple-criteria choice problems with large sets of alternatives* (MCCL), thus including MOMP but not limited to it*.

From mathematical point of view, a multiple criteria choice problem is ill-defined because, excluding trivial cases, it has no unique solution unless additional preference information is specified by the DM. It is usually assumed that the DM should prefer one of the *efficient* alternatives over non-efficient ones (Rosenthal, 1985). The goal of methods supporting the choice of the best alternative is to find an efficient alternative which would be most consistent with the DM's preferences, i.e. which would yield the *best compromise* among criteria creating a coherent family approved by the DM. Finding the best compromise requires, of course, the cooperation with the DM in order to get preference information from him/her.

Taking into account the moment of collecting the preference information with respect to the exploration process, methods for solving MCCL problems require either a priori or a posteriori, or progressive articulation of preferences (cf. Hwang et al., 1980; Słowiński, 1984). The progressive articulation gave birth to interactive methods, today commonly recognized as the most powerful in searching over the large set of alternatives for the best compromise (cf. Vanderpooten, 1990b).

Interactive procedures are characterized by phases of decision alternating with phases of computation. The preference information obtained from the DM in the decision phase contributes to construction of his/her *model of preferences*. The model is *comprehensive* if it allows finding the best compromise or a reduced subset of efficient alternatives containing the best compromise. The preference model is *local* if it allows finding the best compromise in some neighbourhood of a considered alternative. The model is *implicit* if the DM simply gives a consent to particular steps of a computational procedure trusting that it will allow him/her finding out the best alternative. An *explicit* model is a function or a system of relations. Each functional model has its corresponding relational model, while the inverse statement is not true (Słowiński, 1983; Vincke, 1992).

The traditional approach to preference modelling assumes that either *preference* or *indifference* relation is true while comparing any pair of alternatives. The corresponding functional model is the *utility function* (Keeney and Raiffa, 1976). A more realistic approach to preference modelling, developed mostly by, so called, European School of MCDA, takes into account possible *incomparability* of some alternatives, i.e. situation, in which preference information obtained from the DM is not sufficient to choose between indifference or preference (Roy, 1985; Roy and Bouyssou, 1993). Incomparability can be included in a relational model only. The most common model of this type is an *outranking relation* defined as grouping of indifference and preference

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relations. Incomparability is typical for problems in which some of the criteria are in strong conflict on the set of alternatives.

In interactive procedures, the preference model (implicit or explicit) is used either to find a reduced subset of alternatives including the best compromise or to guide the search of better alternatives in the next computation phase and, finally, decide if a currently analyzed alternative is the best compromise. Most of the existing interactive methods use a utility function preference model, i.e. they do not take into account possible incomparability of some pairs of alternatives. From the other side, interactive methods that do not use any explicit preference model do not support satisfactorily the DM in progressing towards the best compromise in successive iterations.

In this paper, several ways of using an outranking relation as DM's preference model in interactive methods are discussed. The outranking relation is used in order to:

- take into account possible incomparability of some alternatives presented to the DM in the decision phase,
- support the DM in progressive learning of his/her preferences and of the possible trade-offs.

The paper is organized in the following way. In the next section, some basic notation and definitions will be introduced. Section 3 presents the preference modelling dilemma: utility function or outranking relation. In section 4, three typical operations used in the interactive procedures for MCCL are characterized and some of their weaknesses are pointed out. In section 5, it is shown how these operations can be performed using an outranking relation, thus avoiding many drawbacks of existing interactive procedures. The last section groups conclusions.

2. Problem statement and basic definitions

The general multiple criteria choice (MCC) problem is formulated as:

$$\max\{f_1(a) = z_1, \ldots, f_J(a) = z_J\} \qquad \text{(P1)}$$
$$a \in A,$$

where A is a set of feasible alternatives a, $f_1, \ldots, f_J$ are *criterion functions* (*objectives*).

The general multiple objective mathematical programming (MOMP) problem is formulated as:

$$\max\{f_1(\mathbf{x}) = z_1, \ldots, f_J(\mathbf{x}) = z_J\} \qquad \text{(P2)}$$

$$\text{s.t.} \qquad g_l(\mathbf{x}) \begin{array}{c}\geq\\=\\\leq\end{array} 0, \qquad l = 1, \ldots, L,$$

where an alternative (*solution*) $\mathbf{x} = [x_1, \ldots, x_I]$ is a vector of I *decision variables* and $g_l(\mathbf{x}) \begin{array}{c}\geq\\=\\\leq\end{array} 0$, $l = 1, \ldots, L$, are constraints on the range of variation of $\mathbf{x}$.

An image of alternative a in the criterion space is a *point* $\mathbf{z}^a = \left[z_1^a, \ldots, z_J^a\right]$, such that $z_j^a = f_j(a)$, j=1,..,J. An image of set A in the criterion space is a set Z composed of points **z** being images of feasible alternatives.

Point $\mathbf{z}' \in Z$ is *non-dominated* if there is no $\mathbf{z} \in Z$ such that $z_j \geq z_j' \ \forall j$, and $z_i > z_i'$ for at least one i. The set composed of all non-dominated points constitutes the *non-dominated set* denoted by N.

The following set is a polyhedral *cone* in the criterion space:

$$C = \left\{ \mathbf{c} \in R^J | \mathbf{c} = \sum_{i=1}^{I} \alpha_i \mathbf{v}^i, \alpha_i \geq 0 \right\},$$

where $V = \{v^1, \ldots, v^I\}$ is a set of generators of the cone, $I \geq J$.

Another useful definition is that of the *achievement scalarizing function* in the objective space:

$$s\left(\mathbf{z}^{\mathrm{o}}, \mathbf{z}, \Lambda\right) = \max_j \left\{ \lambda_j \left(\mathbf{z}_j^{\mathrm{o}} - \mathbf{z}_j \right) \right\} + \rho \sum_{j=1}^{J} \lambda_j \left(\mathbf{z}_j^{\mathrm{o}} - \mathbf{z}_j \right)$$

where $\mathbf{z}^{\mathrm{o}}=[z_1^{\mathrm{o}}, \ldots, z_J^{\mathrm{o}}]$ is a reference point, $\Lambda = [\lambda_1, \ldots, \lambda_J]$ is a weighting vector, $\lambda_j \geq 0$, $\sum_{j=1}^{J} \lambda_j = 1$ and ρ is a sufficiently small positive number. The diagonal direction of function $s\left(\mathbf{z}^{\mathrm{o}}, \mathbf{z}, \Lambda\right)$ is defined by $-[\, 1/\lambda_1, \ldots, 1/\lambda_J \,]$.

3. Utility function or outranking relation - preference modelling dilemma

Classical decision theory assumes that only two basic situations: *indifference* and *strict preference*, are sufficient to represent DM's preferences on any pair of alternatives a and b. The two situations correspond to binary relations I and P, respectively, where I is reflexive, symmetric and transitive. In the case of a finite set of alternatives, the above assumption assure the existence of a utility function U such that:

$$a \, P \, b \Leftrightarrow U(\mathbf{z}^a) > U(\mathbf{z}^b),$$

$$a \, I \, b \Leftrightarrow U(\mathbf{z}^a) = U(\mathbf{z}^b).$$

In the case of an infinite set of alternatives some additional topological conditions, usually verified in practice, are necessary (Debreu, 1959).

The attractiveness of utility function, as a comprehensive preference model in MCCL, can be explained by its relatively easy exploitation. It consists in finding an alternative giving the highest utility. If it is used as local preference model, its exploitation consists in finding an alternative giving the highest utility in some neighbourhood of a considered alternative or in finding a direction of improvement of the utility in this neighbourhood. Assuming existence of a utility function (usually concave but not necessarily explicitly known) ensures mathematical convergence of an interactive procedure.

The ease of exploitation is, however, usually obtained at the price of some oversimplification. The most criticized property of this preference model is the transitivity of indifference relation. Indeed, if some differences between alternatives can be considered insignificant, transitivity of I may give trouble, because a sum of these differences, becoming significant, contradicts indifference (cf. Luce, 1956).

The existence of an indifference region can be caused by inability of the DM to perceive small differences on a criterion as well as by imprecision and uncertainty of measurement and/or of the mathematical model. This phenomenon, can be modelled with the indifference threshold q_j being, in general, a function of z_j.

Furthermore, experience indicates that, usually, there is no precise limit between the indifference and the strict preference, but there exists an intermediary region where the DM hesitates between both. This corresponds to situation of *weak preference* between a and b. It can be modelled with the preference threshold p_j being, in general, a function of z_j, such that:

$$p_j(z_j) \geq q_j(z_j) \geq 0$$

Another realistic situation which cannot be represented using utility function model is the situation of *incomparability* corresponding to the lack of preference judgment between a and b.

The four conflicting situations of indifference, strict preference, weak preference and incomparability are useful in establishing a realistic representation of DM's preferences. This claim has been formulated by Roy (1985) as the *axiom of limited comparability* on which the European School of MCDA is largely based. It further precises that the DM, or the analyst judging in his/her name, can develop a satisfactory preference model that assigns one or a grouping of two or three of these four basic situations to any pair of actions.

Indeed, the fact that two or three situations are considered possible by the DM for a given pair of alternatives is due to the fact that he/she cannot, does not wish to, or does not know how to decide upon the appropriate situation. A possible grouping of preference statements can be modelled by a consolidated system of preference relations (CSPR). The most popular are CSPR's that use the *outranking relation* S (cf. Roy, 1985). We say that a outranks b if a is considered to be at least as good as b. Thus, stating that $a\,S\,b$, the DM have not to decide whether this means the a is strictly preferred to, weakly preferred to, or indifferent to b. Notice that $a\,S\,b$ and $b\,S\,a$ implies indifference $a\,I\,b$, while $\neg a\,S\,b$ and $\neg b\,S\,a$ implies incomparability $a\,?\,b$. Thus, the comprehensive preference model based on outranking relation S is compatible with the axiom of limited comparability.

We should emphasize, however, that in practical decision aiding, the DM is **not** asked to judge explicitly the truth of relation S for each pair of alternatives. Instead, the outranking relation is constructed on the set of alternatives, using the available preference information, and then exploited in order to give answers pertinent to the decision problematic (cf. Roy and Bouyssou, 1993; Vanderpooten, 1990a). Notice that the exploitation is not as straightforward as in the case of utility function. In section 5, we are giving several examples of exploitation, useful in the MCCL context.

Construction of the outranking relation S is essentially based on two concepts called concordance and discordance tests. The goals of these tests are, respectively, to:

- characterize a group of criteria considered to be in concordance with the affirmation $a\, S\, b$ and asses the relative importance of this group compared with the reminder of criteria, and
- characterize, among the criteria which are not in concordance with the affirmation being studied, the ones whose opposition is strong enough to reduce the credibility of $a\, S\, b$ which would result from taking into account just concordance, and to calculate the possible reduction in it that would thereby result.

This way of construction prevents, in particular, that a major disadvantage on one criterion might be compensated by a number of minor advantages on other criteria.

When performing the concordance and discordance tests, the following preference information is usually taken into account:

- the *intra-criteria* information showing the discrimination power of the DM's preferences with respect to particular criteria z_j, i.e. indifference $q_j\,(z_j)$ and preference $p_j\,(z_j)$ thresholds,
- the *intra-criteria* information specifying the relative importance of criteria and value of discordance which gives criterion z_j the power to take all credibility away from the affirmation $a\, S\, b$, even when opposed to all the other criteria in concordance with this affirmation, i.e. the weights and the *veto* threshold $v_j\,(z_j)$.

4. Some typical operations performed in interactive procedures

A significant number of interactive procedures has been proposed for MCCL problems. Although each of them is a specific proposal, there are significant similarities between many of them. This observation resulted in several conceptions of classification of the methods (see e.g. Hwang et al., 1980; Słowiński, 1984; Vanderpooten 1990b; Shin and Ravindran, 1991). An interesting analysis of technical similarities between interactive methods for MOMP problems, especially in the way of generating efficient solutions, has been carried out by Gardiner and Steuer (1994).

A review of existing interactive methods leads to the conclusion that the exploration of the set of alternatives is organized using the following typical operations:

- contraction of a considered subset of the non-dominated set,
- definition and exploration of some neighbourhood of a current alternative (point in the criterion space),
- reduction of a representative sample of non-dominated points.

5. Implementation of the typical operations with the use of an outranking relation

In this section we will show how one can perform the three typical operations described in the previous section, using an outranking relation. Our claim is that it is not only possible to do this but that in the way it is done, one can avoid many drawbacks of existing interactive procedures.

As the methods described in this section operate in the criterion space, we will almost exclusively use the concept of a non-dominated point instead of an efficient alternative.

5.1. Contraction of a considered subset of the non-dominated set

This operation is used in the Cone Contraction method (Jaszkiewicz and Słowiński, 1992). The general scheme of this method can be summarized as follows:

Step 1. Generate a large sample of points representing the whole non-dominated set. In the case of MCCL problems with explicitly given set of alternatives, the sample is composed of all efficient alternatives.

Step 2. Filter the sample in order to obtain a smaller sample A composed of the most different non-dominated points.

Step 3. Construct a fuzzy outranking relation S in sample A.

Step 4. Construct two complete preorders $\overline{P}$ and $\underline{P}$ in sample A using descending and ascending distillations of S.

Step 5. Assess two achievement scalarizing functions $\overline{s}$ and $\underline{s}$ as compatible as possible with the preorders $\overline{P}$ and $\underline{P}$, respectively, using an ordinal regression method.

Step 6. Construct a polyhedral preference cone in the criterion space using diagonal directions of the scalarizing functions assessed in the step 5 as generators of the cone.

Step 7. Allow the DM to scan a non-dominated subset $\tilde{N} \subset N$ defined by the cone. Terminate the procedure at this step if the best compromise point has been found, or generate a new sample A from contracted subset $\tilde{N}$ of non-dominated points and return to step 3.

Notice that in order to perform the contraction operation the DM is not required to select the best point of the sample nor compare all the points in the sample. Instead, his/her preferences are assessed with a fuzzy outranking relation and exploited in a way which leads to contraction of the considered subset. The outranking relation is used as a comprehensive preference model taking into account incomparability of some points. Furthermore, the rate of contraction depends on the available preference information. More the preferences are vague, smaller is the contraction rate. It is typical that in the early stages of interaction, the DM prefers to avoid unjustified comprehensive preferences on set A. The difference between the two complete preorders, $\overline{P}$ and $\underline{P}$, shows the range of hesitation the DM would have when comparing directly the pairs of alternatives from sample A. The greater the difference is, the greater is the preference cone defining the contracted subset $\tilde{N}$. The contraction can yield a single point if the two preorders are the same, i.e. if there is no incomparability in the preference model.

5.2. Definition and exploration of some neighbourhood of the current alternative

This operation is used in the Light Beam Search method (Jaszkiewicz, Słowiński, 1994). The general scheme of this method can be summarized as follows:

Step 1. Ask the DM to specify a reference point or make the ideal point the first reference point.

Step 2. Find the starting middle point $\mathbf{z}^c$ by projecting the reference point onto set N.

Step 3. Ask the DM to specify the preferential information of inter- and intra-criteria type.

Step 4. Construct the outranking neighbourhood of the current middle point $\mathbf{z}^c$.

Step 5. Allow the DM to scan the outranking neighbourhood. Terminate the procedure at this step if the best compromise point has been found, or give the DM the possibility to move the middle point and return to step 3.

In the step 2, the projection of the reference point onto set N is made with the achievement scalarizing function.

The outranking neighbourhood of the middle point $\mathbf{z}^c$ is defined as a set of non-dominated points that are not worse that $\mathbf{z}^c$, i.e. outrank the middle point. Thus, the binary outranking relation S is a local preference model based on the preferential information obtained in step 3. The preferential information consists of the indifference thresholds q_j and, optionally, preference thresholds p_j and veto thresholds v_j. As it is related to the current middle point, it has a local character and the thresholds can be constant in a given iteration.

In step 5, the method offers to the DM two possibilities to move the neighbourhood over the non-dominated set. The first one consists in specifying a new reference point in the criterion space which is then projected onto set N giving a new middle point. The second possibility consists in moving the current middle point to a selected point in its neighbourhood. The two possibilities submit some analogy with projection of a focused beam of light from a spotlight at the reference point onto the non-dominated set. For this reason the procedure is called the Light Beam Search or, shortly, LBS.

5.3. Reduction of a sample of non-dominated points

The reduction operation is used in the method by Ferhat, Słowiński and M'Silti (1994), proposed originally to the linear MOMP problems. Its idea is general, however, and can be summarized as follows:

Step 1. Generate of a large sample of points representing the whole non-dominated set. In the case of MCCL problems with explicitly given sets of alternatives the sample is composed of all efficient alternatives.

Step 2. Filter the sample in order to obtain a smaller sample A composed of the most different non-dominated points.

Step 3. Construct a binary outranking relation in the sample.

Step 4. Calculate the kernel of the outranking relation.
Step 5. Allow the DM to scan interactively various profiles between points belonging to the kernel. Terminate the procedure at this step if the best compromise point has been found, or generate a new sample A composed of points selected by the DM during the scanning and return to step 3.

The crucial step is the exploitation of the outranking relation S which consists in extraction of kernel A^o of the graph corresponding to relation S (cf. Vanderpooten, 1990a). The kernel concept was originally introduced in Electre I (Roy, 1968). It has to fulfil the following general condition:

$$\forall \mathbf{z}^b \in A \setminus A^o, \qquad \exists \mathbf{z}^a \in A^o \text{ such that } \mathbf{z}^a S \mathbf{z}^b$$

In consequence, A^o is composed of indifference classes which are mutually incomparable. It is thus reasonable to search for the best compromise points in regions lying between the incomparable points from the kernel.

6. Conclusions

We have shown that the three typical operations of interactive procedures: contraction of the explored set, exploration of some neighbourhood of a current alternative, and reduction of a sample of the explored set, can be performed using an outranking relation preference model. The proposed ways of building and exploiting the outranking relation permit to overcome some weak points of traditional procedures assuming usually a utility function preference model. These weak points are:

- contraction defined either arbitrarily by the method, or directly by the DM,
- assumption that the DM is always able to compare the non-dominated points coming from any sample,
- impossibility of taking into account the vagueness of the DM's preferences typical for the early stages of interaction,
- possibility of eliminating interesting alternatives in result of contraction or filtering,
- risk of premature stopping if no requirement is imposed on the composition of the sample presented to the DM.

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An Interactive Method for Solving Multiple Objective Quadratic-Linear Programming Models

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Abstract. In this paper, we describe an interactive procedure for solving multiple criteria problems with one quadratic objective, several linear objectives, and a set of linear constraints. The procedure is based on the use of reference directions and weighted-sums. The reference directions for the linear functions, and the weighted-sums for combining the quadratic function with the linear ones are used as parameters to implement the free search of nondominated solutions. This idea leads to the parametric linear complementarity problem formulation. An approach to deal with this type of problems is given as well. The approach is illustrated with a numerical example.

Keywords: Multiple Objective Programming, Quadratic Programming, Achievement Scalarizing Function, Reference Direction, Linear Complementarity Model, Nondominated Criterion Values

1 Introduction

The field of multiobjective optimization has attracted a lot of attention for more than 20 years and several dozen procedures and computer implementations have been developed to address these problems (for surveys, see [2, 14]). In principle, many of the procedures can be used in multiple objective linear (MOLP) and nonlinear (MONLP) programming problems. Typical examples of such procedures are Surrogate Worth Trade-Off method [2], Geoffrion-Dyer-Feinberg method [5], Interactive Weighted Tchebycheff method [14], Reference Point method [14], and Satisficing Trade-Off method [11]. For linear problems, there also exist multiple criteria decision support systems (MCDSS): for example, ADBASE (Steuer [14]), DIDAS (Lewandowski and Grauer [10]), TRIMAP (Climaco and Antunes [3]), and VIG (Korhonen [7]) among others, which enable a decision maker (DM) to evaluate nondominated solutions interactively in a very flexible way and to control the search process. However, to generate nondominated solutions in nonlinear problems is - generally - much more complicated than in linear problems, which makes it hard to develop interactive approaches of practical significance for nonlinear problems.

Fortunately, not all nonlinear problems are difficult to solve. For instance, a multiple objective problem with one quadratic and several linear objectives which are to be optimized subject to linear constraints, can be solved almost as easily as a linear one. The problem is called a multiple objective quadratic-linear problems (MOQLP) by Rhode and Weber [12]. In literature, not many procedures exist to solve this problem. Rhode and Weber [12] developed an approach based on the weighted-sums of the criteria. The linear combination of the quadratic objective and linear objectives are optimized using a linear complementarity formulation. For each set of weights, the method produces a nondominated solution, but the method is not designed for interactive use. It does not provide the DM with the information how to find more nondominated solutions, which are somehow related to the DM's preference structure.

In this paper, we will describe an approach which enables the DM to freely search nondominated solutions in multiple quadratic-linear objective programming. The approach is based on the idea to combine two widely used principles scalarizing multiple objectives: a reference point approach (Wierzbicki [15] and weighted-sums. The reference point approach is applied to linear objectives. It results in a linear parametric optimization formulation with a single linear objective function. This linear objective function is combined with the quadratic function by using the weighted-sums. The resulting quadratic problem is transformed into a linear (parametric) complementarity problem which is the basic formulation for the proposed approach. By varying a parameter vector on the right-hand-side in the model, the DM can freely search the nondominated frontier. The approach is presented in more detail in Korhonen and Yu [9]. We will illustrate our approach by solving a numerical example by using the prototype of the decision support system we are developing.

The rest of the paper is organized as follows: In section 2, we discuss preliminary considerations. In section 3, we describe the approach, and the algorithm implementing our approach is given in section 4. In section 5, we illustrate the approach with a numerical example. We conclude the paper in section 6.

2 Preliminary Considerations

The multiple objective quadratic-linear programming problem is defined as follows:

$$
\begin{array}{lll}
\text{Min} & V(x) = \frac{1}{2}x'Dx & \\
\text{"Max"} & l(x) = Cx & \\
& & \\
\text{Subject to:} & A\,\mathbf{x} \le \mathbf{b} & (2.1) \\
& \mathbf{x} \ge \mathbf{0}, &
\end{array}
$$

where $D \in \Re^{n \times n}$ is a symmetric positive semi-definite matrix, $C \in \Re^{k \times n}$ is a matrix of coefficients of the linear objective functions. $A \in \Re^{m \times n}$ is a matrix (rank m) of coefficients of the constraints, $\mathbf{b} \in \Re^m$ is a vector of the rhs, and $\mathbf{x} \in \Re^n$ is a vector of the decision variables. By "Max" we mean that all linear objective functions have to be maximized simultaneously.

2.1 Basic Definitions and Notation

We will denote the objective function vector by $\mathbf{f}(\mathbf{x})$ and refer to its components as follows:

$$\mathbf{f}(\mathbf{x}) = (-V(\mathbf{x}), \mathbf{l}(\mathbf{x}))' = (f_0(\mathbf{x}), f_1(\mathbf{x}), ..., f_k(\mathbf{x}))'$$

We will use notation $K=\{0,1,..., k\}$ to refer to the index set of the objectives.
The problem can now be written as:

$$\text{"Max"} \quad \mathbf{f}(\mathbf{x})$$

$$\text{Subject to:} \qquad \mathbf{x} \in X = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \tag{2.2}$$

We define efficiency, weak efficiency, and proper efficiency for a point $\mathbf{x}^0 \in X$ in the usual manner:

Definition 1. $\mathbf{x}^0 \in X$ is *efficient* iff $\nexists$ $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) \geq \mathbf{f}(\mathbf{x}^0)$ and $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{x}^0)$.
Definition 2. $\mathbf{x}^0 \in X$ is *weakly efficient* iff $\nexists$ $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) > \mathbf{f}(\mathbf{x}^0)$.
Definition 3. $\mathbf{x}^0 \in X$ is *properly efficient* if it is efficient and if $\exists$ a scalar $M > 0$ such that for each $k \in K$, we have $\frac{f_k(\mathbf{x}) - f_k(\mathbf{x}^0)}{f_j(\mathbf{x}^0) - f_j(\mathbf{x})} \leq M$ for some j such that $f_j(\mathbf{x}^0) - f_j(\mathbf{x}) > 0$ whenever $\mathbf{x} \in X$ and $f_k(\mathbf{x}) - f_k(\mathbf{x}^0) > 0$ (Geoffrion [4]).
The objective function vectors $\mathbf{q} \in Q = \{\mathbf{q} \mid \mathbf{q} = \mathbf{f}(\mathbf{x}), \mathbf{x} \in X\}$ corresponding to (*weakly/properly*) *efficient* points are called (*weakly/properly*) *nondominated* solutions.

2.2 Generation of Efficient Solutions

A classic method to generate nondominated solutions is to use the weighted-sums of objective functions, i.e. to consider the solutions of the following linear program:

$$\max \{\lambda'\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in X\}. \tag{2.3}$$

Using the parameter set $\Lambda = \{\lambda \mid \lambda > \mathbf{0}$ in the weighted-sums linear program we can completely characterize the efficient set provided the constraint set is convex. However, Λ is an open set, which causes difficulties in a mathematical optimization problem. If we use $cl(\Lambda) = \{\lambda \mid \lambda \geq \mathbf{0}\}$ instead, the efficiency of $\mathbf{x}$ cannot be guaranteed anymore. It is surely weakly-efficient, and not necessarily efficient (see, e.g. Steuer [14, p. 215 and 221]). Another problem which appears when the weighted-sums are used in a multiple objective linear program (MOLP) problems is that the optimal solution corresponding to nonextreme points of X is never unique. The set of optimal solutions always consists of at least one extreme point, or the solution is unbounded. That disadvantage does not exist in the scalarizing function introduced by Wierzbicki [15].

Wierzbicki [15] proposed the use of an *achievement* (*scalarizing*) *function* to project any given (feasible or infeasible) point $\mathbf{g} \in \Re^{k+1}$ onto the set of nondominated solutions. Point $\mathbf{g}$ is called a *reference point*, and its components represent the desired values of the objective functions. These values are called *aspiration levels.*

The simplest form of achievement function is:

$$s(\mathbf{g}, \mathbf{q}, \mathbf{w}) = \max \left[\frac{g_k - q_k}{w_k}, k \in K \right] \tag{2.4}$$

where $\mathbf{w} \in \Re^{k+1}$, $\mathbf{w} > \mathbf{0}$, is a vector of weights, $\mathbf{g} \in \Re^{k+1}$, and $\mathbf{q} \in Q = \{\mathbf{f(x)} \mid \mathbf{x} \in X\}$. By minimizing $s(\mathbf{g}, \mathbf{q}, \mathbf{w})$ subject to $\mathbf{q} \in Q$, we find a weakly nondominated solution vector $\mathbf{q}^*$ (see, e.g. Wierzbicki [15, 16]). However, if the solution is unique for the problem, then $\mathbf{q}^*$ is nondominated. If $\mathbf{g} \in \Re^{k+1}$ is feasible, then $\mathbf{q}^* \in Q$, $\mathbf{q}^* \geq \mathbf{g}$.

Let us consider problem (2.1) by first ignoring the quadratic function:

$$\begin{aligned} &\text{"Max"} && \mathbf{l(x)} = \mathbf{Cx} \\ &\text{Subject to:} && \mathbf{x} \in X. \end{aligned} \tag{2.5}$$

Let now $\mathbf{g} \in \Re^k$ be an aspiration level vector for linear objective functions in (2.5). By applying the achievement scalarizing function (2.4), a weakly efficient solution corresponding to linear objective functions can be found as a solution of the LP-problem (see, e.g. Korhonen and Wallenius [8]). The search of the nondominated frontier can be implemented by parameterizing the aspiration level vector $\mathbf{g} \in \Re^k$ or the vector $\mathbf{w} \in \Re^k$, $\mathbf{w} > \mathbf{0}$. The parametrizing of $\mathbf{g}$ is more convenient, because it has a clear interpretation and its use is easier. Changes in $\mathbf{g}$ has an impact only on the right-hand side values of the model. The parametrization problem can be stated as follows:

$$\begin{aligned} &\text{Min} && \varepsilon \\ &\text{Subject to :} && \mathbf{Cx} + \varepsilon\mathbf{w} \geq \mathbf{g} + t\Delta\mathbf{g} \\ &&& \mathbf{x} \in X, \end{aligned} \tag{2.6}$$

where ε is a scalar variable, $\Delta\mathbf{g} \in \Re^k$ is a preferable change in direction starting from the current solution (Korhonen and Laakso [6]), and $t \geq 0$ is the corresponding parameter. Vector $\Delta\mathbf{g}$ is called a *reference direction.*

3 Development of the Approach

3.1 Formulation of the Basic Model

Based on the use of scalarizing function and the MOLP form of Korhonen and Wallenius [8], we may present the original problem as:

$$\begin{array}{lll} \text{Min} & V(\mathbf{x}) = \frac{1}{2}\mathbf{x}'D\mathbf{x} & \\ \text{Min} & \varepsilon & \\ \text{Subject to:} & C\mathbf{x} + \varepsilon\mathbf{w} \geq \mathbf{g} & (3.1) \\ & \mathbf{x} \in X. & \end{array}$$

In (3.1), we have transformed an original problem to the problem having one linear and one quadratic function. To solve the above two objective problem, we will use a weighted sum as a scalarizing function, and formulate the problem as follows:

$$\begin{array}{lll} \text{Min} & \mu \frac{1}{2}\mathbf{x}'D\mathbf{x} + (1-\mu)\varepsilon & \\ \text{Subject to:} & C\mathbf{x}+ \varepsilon\mathbf{w} \geq \mathbf{g} & (3.2) \\ & \mathbf{x} \in X & \\ & \mu \in (0, 1). & \end{array}$$

Theorem 1. $\mathbf{x}^* \in X$ is an optimal solution to (3.2) for some $\mu \in (0, 1)$ iff $\mathbf{x}^*$ is the properly efficient solution of problem (3.1).

Proof. See, e.g. Geoffrion [4, p.620 (Theorem 2)].

Theorem 2. $\mathbf{x}^* \in X$ is an optimal solution to (3.2) for some $\mathbf{g} \in \Re^k$ and $\mu \in [0, 1]$ iff $\mathbf{x}^*$ is a weakly efficient solution of problem (2.1).

Proof. See, Korhonen and Yu [9].

Because $\mu > 0$, we may divide the objective function of problem (3.2) by μ and write $\lambda = \frac{1}{\mu} - 1$. To search the nondominated frontier of problem (3.2), we parametrize λ and $\mathbf{g}$. Then we can present the problem in the following form:

$$\begin{array}{lll} \text{Min} & \frac{1}{2}\mathbf{x}'D\mathbf{x} + (\lambda + t\Delta\lambda)\varepsilon & \\ \text{Subject to:} & C\mathbf{x}+ \varepsilon\mathbf{w} \geq \mathbf{g} + t\Delta\mathbf{g} & (3.3) \\ & \mathbf{x} \in X & \\ & \lambda > 0. & \end{array}$$

where $(\lambda + t\Delta\lambda) > 0$. By varying $\Delta\lambda$ and $\Delta\mathbf{g}$ we may search a nondominated frontier.

3.2. Reduction to Linear Complementarity Problem

For each given t, $\Delta\lambda$, and $\Delta\mathbf{g} \in \Re^k$, we may use the *Karush-Kuhn-Tucker conditions* of quadratic programming, and reduce the problem to a linear complementarity problem [13]. Because in our model (3.3) ε is unrestricted and in a standard reduction all variables are nonnegative, we write $\varepsilon = \varepsilon^+ - \varepsilon^-$, $\varepsilon^+ \geq 0$, $\varepsilon^- \geq 0$, $\varepsilon^+\varepsilon^- = 0$. The linear complementarity formulation can now be given as:

$$
\begin{array}{llllll}
-\mathbf{Dx} & + \mathbf{C'y} - \mathbf{A'z} + \alpha & & & & = \mathbf{0} \\
& \mathbf{w'y} & + \beta^+ & & & = \lambda + t\Delta\lambda \\
& \mathbf{w'y} & & - \beta^- & & = \lambda + t\Delta\lambda \\
-\mathbf{Cx} - (\varepsilon^+ - \varepsilon^-)\mathbf{w} & & & & + \delta & = -\mathbf{g} - t\Delta\mathbf{g} \\
\mathbf{Ax} & & & & + \gamma & = \mathbf{b}
\end{array} \qquad (3.4)
$$

$$
\begin{array}{ll}
\mathbf{x}, \mathbf{y}, \mathbf{z}, \alpha, \varepsilon^+, \varepsilon^-, \beta^+, \beta^-, \delta, \gamma \geq \mathbf{0} & \\
\alpha_i x_i = 0, & i = 1,2, ..., n \\
\beta^+\varepsilon^+ = 0, & \\
\beta^-\varepsilon^- = 0 & \\
\delta_j y_j = 0, & j = 1,2, ..., k \\
\gamma_h z_h = 0, & h = 1,2, ..., m.
\end{array}
$$

In formulation (3.4), we have:

$$
\begin{array}{l}
\mathbf{w'y} \quad + \beta^+ = \lambda + t\Delta\lambda \\
\mathbf{w'y} \quad - \beta^- = \lambda + t\Delta\lambda \\
\beta^+, \beta^- \geq 0,
\end{array}
$$

implying that necessarily $\beta^+ = \beta^- = 0$. Thus the linear complementarity formulation can be given as following matrix form:

$$
\begin{pmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{pmatrix} - \begin{pmatrix} \mathbf{D} & 0 & -\mathbf{C'} & \mathbf{A'} \\ 0 & 0 & -\mathbf{w'} & 0 \\ \mathbf{C} & \mathbf{w} & 0 & 0 \\ -\mathbf{A} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \varepsilon \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \lambda \\ -\mathbf{g} \\ \mathbf{b} \end{pmatrix} + t \begin{pmatrix} \mathbf{0} \\ \Delta\lambda \\ -\Delta\mathbf{g} \\ \mathbf{0} \end{pmatrix} \qquad (3.5)
$$

$$
\begin{pmatrix} \alpha \\ \delta \\ \gamma \end{pmatrix} \geq \mathbf{0}, \quad \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} \geq \mathbf{0}, \quad \begin{pmatrix} \mathbf{x} \\ \varepsilon \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix}' \begin{pmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{pmatrix} = 0, \quad \beta = 0.
$$

To solve above *parametric linear complementarity problem*, We may use *The Parametric Principal Pivoting Method* (see. e.g. [13, p.293-296]) or *The Lemke's Complementary Pivoting Algorithm*. The idea is described in more detail in Korhonen and Yu [9].

4. The Interactive Procedure For Solving the Multiple Objective Quadratic-Linear Programming Problems

Step 1. Ask the DM to specify (subjective) lower and upper bounds (approximate ranges) for all objectives; let $\mathbf{g}_l$, $\mathbf{g}_u \in \Re^k$, denote these values for the linear objectives, and let d_l, $d_u \in \Re$, denote these values for the quadratic function. Define $\mathbf{w} := \mathbf{g}_u - \mathbf{g}_l$ (the vector of weights).

Set iteration counter h := 0 and λ^h := 0. Consider problem formulation (3.3):

$$\begin{array}{lll} \text{Min} & \frac{1}{2}\mathbf{x}'\mathrm{D}\mathbf{x} + (\lambda^h + t\Delta\lambda^{h+1})\varepsilon & \\ \text{Subject to:} & \mathrm{C}\mathbf{x} + \varepsilon\mathbf{w} \geq \mathbf{g}^h + t\Delta\mathbf{g}^{h+1} & (4.1) \\ & \mathrm{A}\mathbf{x} \quad \leq \mathbf{b} & \\ & \mathbf{x} \quad \geq \mathbf{0}, & \end{array}$$

where initially, t := 0, $\Delta\lambda^{h+1}$:= 0, $\mathbf{g}^h := \mathbf{g}_u$, and $\Delta\mathbf{g}^{h+1} := \mathbf{0}$.

Compute the minimum value for the quadratic function: $V_{min} := V(\mathbf{x}^h)$, and $\mathbf{q}^h := \mathrm{C}\mathbf{x}^h$. Define $\mathbf{g}^h := \mathbf{q}^h$, and redefine $d_l := V_{min}$. If $q_i^h < q_{li}$, then redefine $q_{li} := q_i^h$. The problem is solved by using the linear complementarity formulation (3.5). This formulation is also used in all subsequent considerations. Set h := h+1.

Step 2. Ask the DM to specify a new reference direction $\Delta\mathbf{g}^h$ (for more details, see Korhonen and Wallenius [8]) and $\Delta\lambda^h$. If the DM would like to reduce risk, $\Delta\lambda^h < 0$, and if he/she is willing to take more risk to have better values for linear objectives, $\Delta\lambda^h > 0$. The magnitude of $\Delta\lambda^h$ depends on the strength of the DM's preference. If the DM is not willing to give new parameters: $\Delta\lambda^h$ and $\Delta\mathbf{g}^h$, *Stop*; the most preferred values for the objective functions are obviously: $V(\mathbf{x}^{h-1})$ and $C(\mathbf{x}^{h-1})$.

Step 3. Determine the change $\Delta\mathbf{x}^h$ in $\mathbf{x}^{h-1}$ as the function of the parameters $\Delta\lambda^h$ and $\Delta\mathbf{g}^h$, and determine the range $[t_1, t_2]$ for t, which preserves the feasibility of the current basis. Construct the functions:

$$V(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h) = \frac{1}{2}(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h)'\mathrm{D}(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h)$$

$$= V(\mathbf{x}^{h-1}) + t\mathbf{x}^{h-1\prime}\mathrm{D}\Delta\mathbf{x}^h + \frac{1}{2}t^2(\Delta\mathbf{x}^h)'\mathrm{D}\Delta\mathbf{x}^h$$

$$C(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h) = \mathrm{C}\mathbf{x}^{h-1} + t\mathrm{C}\Delta\mathbf{x}^h.$$

Step 4. Provide the DM with a possibility to evaluate the values of the objective functions $V(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h)$ and $C(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h)$, when $t \in [t_1, t_2]$ using e.g. visual representation.

Step 5. Ask the DM to evaluate the criterion values and to choose the most preferred solution. Suppose the most preferred solution is found with $t = t^*$.

Step 6. Update: $\mathbf{x}^h := \mathbf{x}^{h-1} + t^*\Delta\mathbf{x}^h$, $\mathbf{g}^h := \mathbf{g}^{h-1} + t^*\Delta\mathbf{g}^h$, $\lambda^h := \lambda^{h-1} + t^*\Delta\lambda^h$, h := h + 1 and t := 0. If $t^* \in [t_1, t_2)$, then go to *Step 2*; otherwise perform a requisite basis change, and go to *Step 3.*

5. An Illustrative Example

We illustrate the proposed approach using the following modified example from the example in Rhode and Weber [1981, p.414]):

$$\begin{array}{lll}
\text{Min} & V(\mathbf{x}) = 0.25x_1^2 + 0.5x_2^2 \\
\text{Max} & l_1(\mathbf{x}) = 3x_1 + 7x_2 \\
\text{Max} & l_2(\mathbf{x}) = x_1 + x_2 \\
\text{Subject to:} & \\
& 0.5x_1 + x_2 \le 14 \\
& x_1 \le 14 \\
& x_2 \le 12 \\
& x_1 + x_2 \le 19 \\
& x_1, x_2 \ge 0.
\end{array}$$

Ask the DM to specify (subjective) lower and upper bounds (approximate ranges) for all objectives. Assume that the values $\mathbf{g}_l = (20, 5)'$ and $\mathbf{g}_u = (100, 20)'$ are given. Define $\mathbf{w} := \mathbf{g}_u - \mathbf{g}_l = (80, 15)'$. For the first iteration, we set $h := 0$, $\lambda^0 := 0$ and $\mathbf{g}^0 := \mathbf{g}_u$, $t = 0$, $\Delta\lambda^{h+1} := 0$, and $\Delta\mathbf{g}^{h+1} := \mathbf{0}$. As a result we get the initial solution $(x_1, x_2) = (0, 0)$ which gives $V_{min} := V(\mathbf{x}) = 0$ and $\mathbf{q}^0 := \mathbf{Cx} = \mathbf{0}$. We define $\mathbf{g}^0 := \mathbf{q}^0$, and redefine $d_l := V_{min}$ and $\mathbf{g}_l := (0, 0)'$. Set $h := h + 1$ (*Step 1*).

Ask the DM to specify a new reference direction. Assume the following aspiration levels are specified: $\mathbf{g}^h := (90, 12)'$. The reference direction is thus found as $\Delta\mathbf{g}^h = \mathbf{g}^h - \mathbf{q}^{h-1} = (90, 12)'$. Next, the following question: "*Are you willing to accept more or less risk, and how strongly?*" is presented to the decision maker. We assume that he/she is willing to take more risk in order to be able to improve linear objective functions, and we set $\Delta\lambda^h = 1$ (*Step 2*).

Using the values of $\Delta\mathbf{g}^h$ and $\Delta\lambda^h$ above, we specify the reference direction vector as shown in (3.5) and use the basis found in Step 1 as an initial basis. Then we start to increase parameter t: $0 \to \infty$. The solution remains feasible (optimal) until $t = \bar{t}$ (= 0.231). As described in *Step 3*, we construct the functions $V(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h)$ and $C(\mathbf{x}^{h-1}+t\Delta\mathbf{x}^h)$, which are defined when $t \in [0, \bar{t}\,]$. The values of the functions are displayed to the DM for evaluation e.g. using the visual representation described in Figure 1[1]. He/she can control the values of t by using function keys, and the values of objectives corresponding to the current value of t is shown in a numeric form and as bars. He/she is assumed to choose the most preferred solution when $t \in [0, \bar{t}\,]$ *(Step 5).*

Assume that the DM prefers the solution $V = 36.59$ and $C = (70.01, 13.68)'$ to the other solutions within the interval $t \in [0, \bar{t}\,]$. The solution corresponds to the values of the decision variables (6.43, 7.24) (*Step 5*). We update $\mathbf{x}^h$, $\mathbf{g}^h$, and λ^h as explained in *Step 5* and go to *Step 2* to ask the DM to specify a new aspiration levels for objectives. If the DM is willing to it, we continue from *Step 3*. If the DM is not willing to specify new aspiration levels for objective, we stop; the final

[1] Actually, Fig. 1 illustrates the interface of our decision support system we are developing to solve the problem described in this paper. The system will also make it possible to specify a reference direction in a dynamic way at any moment like in PARETO RACE (see, e.g. Korhonen and Wallenius [8]).

solution is found. We terminate our exposition of the example problem here, because it has served its illustrative purpose.

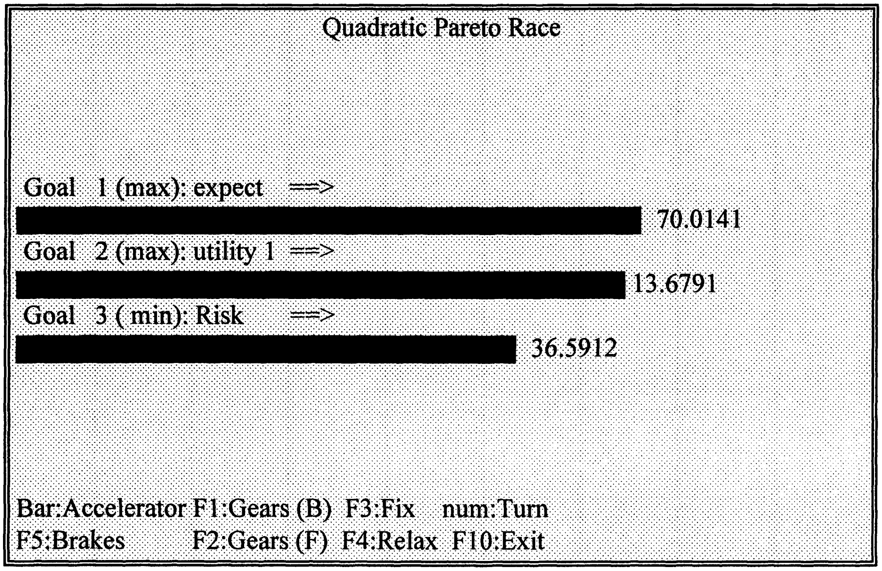

Figure 1: Illustrating the Search of Nondominated Solutions

6. Concluding Remarks

In this paper we have described an interactive method for solving multiple objective quadratic-linear programming problems. The method is based on the use of reference directions and weighted-sums. The parametric linear complementarity problem formulation will play a key role in the approach. The formulation is used to transform a quadratic problem into a linear form. Parametrizing the right hand values of certain rows, we may implement a continuous search on the nondominated frontier. The search direction is specified by the DM by means of reference directions. A visual representation is used to display various solutions to the DM. The DM thus can get a holistic perception of the problem at hand. Finally, a numerical example was given to illustrate the approach.

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An Approximation to the Value Efficient Set

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Abstract. In this paper we assume a decision making situation under certainty with incomplete information on the decision maker's preferences, by means of a vector value function defined on the consequence space. From that function we consider an additive value representation with partial information on the scaling constants defined by polyhedral cone and the concept of value efficient set. We introduce an approximation set whose generation may be easier than the one of the value efficient set. Nesting properties based on the interactive reduction of the uncertainty about the scaling constants, which provide more precise vector value functions, lead to better approximations. Finally, we propose an interactive algorithm based on the approximation set and two examples illustrating the method.

Keywords. Multicriteria decision making, efficient set, vector value function

1 Introduction

In the multicriteria decision making problems the set of efficient solutions plays a leading role in the solution process. The main property of this set is that for each solution outside the set (but still within the feasible region), there is an efficient solution for which all criterion (or objective) functions are unchanged or improved and at least one which is strictly improved. So, the first step of the multicriteria problem should consist of identifying the set of efficient solutions. There are several methods to help a Decision Maker (DM) to generate all or representatives subsets of efficient solutions (see, i.e., the books of Goicoechea *et al.*, 1982; Chankong and Haimes, 1983; Steuer, 1986). Most of these methods are specially useful in the case of linear problems where there exist some specific methods, but for nonlinear problems they may be hard to apply.

We shall introduce in this paper an approximation set to the value efficient set which intends to alleviate the solution process avoiding its generation. Our framework will be a multicriteria decision making problem under certainty with a set of alternatives $x \in X$, a finite set of n criteria defined by means of n real-valued functions z_i $(i = 1, ..., n)$, where $\mathbf{z} = (z_1, ..., z_n) : X \to \mathbf{R}^n$. We assume that we have to maximize each z_i, and there is partial information on DM's preferences in the objective space $Z = \mathbf{z}(X)$. This partial information is given through a vector value function $\mathbf{v} = (v_1, ..., v_p)$ (Roberts, 1972, 1979; Rietveld, 1980), which represents a strict partial order $\succ$ (irreflexive and transitive), such that

$$\mathbf{z} \succ \mathbf{z}' \Leftrightarrow \mathbf{v}(\mathbf{z}) \geq \mathbf{v}(\mathbf{z}')$$

where $\mathbf{v}(\mathbf{z}) \geq \mathbf{v}(\mathbf{z}')$ if and only if $v_i(\mathbf{z}) \geq v_i(\mathbf{z}')$ for all $i = 1, ..., p$ and there is $j \in \{1, ..., p\}$ such that $v_j(\mathbf{z}) > v_j(\mathbf{z}')$.

This vector function may be seen as way for lack of precision of the true and unknown DM's (scalar) value function. This framework leads to the vector maximum problem in Z, stated as $\max\{\mathbf{v}(\mathbf{z}) : z \in Z\}$, from which it arises the value efficient set

$$\mathcal{E}(Z, \mathbf{v}) = \{\mathbf{z} \in Z : \nexists \mathbf{z}' \in Z \text{ such that } \mathbf{v}(\mathbf{z}') \geq \mathbf{v}(\mathbf{z})\}$$

where the DM must do his choice. Because this set may be difficult to generate we consider an approximation which uses a new more precise vector value function based on partial information on scaling constants defined by a polyhedral cone, which intends to help a DM to come up with his most preferred solution.

The paper includes five more sections. In the second section we introduce some theory and concepts and we describe the approximation set for the case of two components in the vector value function. In the third section we extend that approximation to more than two components. In section four we consider some monotonicity properties for the approximation set. In section five an algorithm shows an interactive method based on the approximation set. We present two examples in section six and the final section provides some conclusions.

2 Basic concepts. Approximation for two components

Given our partial information problem with a vector value function $\mathbf{v} : Z \to \mathbf{R}^p$, it leads us to determine the value efficient set $\mathcal{E}(Z, \mathbf{v})$. Let us consider that the DM could reveal more information on his preferences in an interactive process as we shall state in what follows. Let be $K^0 \equiv \mathbf{R}^p_+$ which is a constant and convex cone.

Definition 1 *Let be $K \supseteq K^0$ a constant, convex, closed and acute cone which we call information cone, and K^P its positive polar. The set $K_* = K^P \cap S_p$ is called information set associated to K, where S_p is the simplex on $\mathbf{R}^p$.*

In what follows we shall assume that K^P is defined by a polyhedral cone, and it will be possible to determine its set of generators (Tamura, 1976), which normalized in S_p will be $\{\mathbf{k}^1, ..., \mathbf{k}^q\}$. Thus, we denote the information set $K_* = G\{\mathbf{k}^1, ..., \mathbf{k}^q\}$.

Definition 2 *Given a vector value function* $\mathbf{v}$ *and an information set* $K_* = G\{\mathbf{k}^1, ..., \mathbf{k}^q\}$, *the map* $\mathbf{v}^K : Z \to \mathbf{R}^q$ *defined as* $\mathbf{v}^K = \left(\mathbf{k}^1\mathbf{v}, ..., \mathbf{k}^q\mathbf{v}\right)$, *will be called vector value function associated to* K_*.

Note that, in particular, it will be $\mathbf{v} = \mathbf{v}^{K^0}$ for the *null information set* $K_*^0 = G\left\{(1, 0, ..., 0), ..., (0, ..., 0, 1)\right\}$. If K is a halfspace, then $K_* = \{\mathbf{k}\}$ and $\mathbf{v}^K = \mathbf{kv}$ will be a scalar value function.

Next, we consider the approximation set to value efficient set for the case where we have a vector value function with only two components (Mateos and Rios-Insua, 1994). Let $\mathbf{v} = (v_1, v_2)$ be such function defined on Z, and (v_1^*, v_2^*) the ideal point, which fulfills

$$\max_{\mathbf{z} \in Z} v_1(\mathbf{z}) = v_1(\mathbf{z}_1^*) = v_1^* \qquad \text{and} \qquad \max_{\mathbf{z} \in Z} v_2(\mathbf{z}) = v_2(\mathbf{z}_2^*) = v_2^*$$

and let $v_1(\mathbf{z}_2^*) = v_{1*}$ and $v_2(\mathbf{z}_1^*) = v_{2*}$ be the coordinates of the nadir point.

Definition 3 *Given a vector value function* $\mathbf{v} = (v_1, v_2)$, *the approximation set to* $\mathcal{E}(Z, \mathbf{v})$ *is defined as*

$$\mathcal{A}(Z, \mathbf{v}) = \{\mathbf{z} \in Z : v_1(\mathbf{z}) \geq v_{1*}\ ,\ v_2(\mathbf{z}) \geq v_{2*}\}\ . \tag{1}$$

It is verified that $\mathcal{E}(Z, \mathbf{v}) \subseteq \mathcal{A}(Z, \mathbf{v})$ and several nesting and convergence properties in terms of the vector function $\mathbf{v}^K$ (Mateos and Rios-Insua, 1994), that we shall not prove here. Our intention in this paper is to extend the above definition to more than two components in $\mathbf{v}$ and then, to show analogous properties.

3 Approximation for p components: The general case

Let $\mathbf{v}(\mathbf{z}) = (v_1(\mathbf{z}), ..., v_p(\mathbf{z}))$ be a vector value function, $\mathbf{z}_k^*$ the point where $\max_{\mathbf{z} \in Z} v_k(\mathbf{z}) = v_k(\mathbf{z}_k^*)$ and let us denote $v_j^k = v_j(\mathbf{z}_k^*)$, $j, k = 1, ..., p$. The vector $\mathbf{n} = (n_1, ..., n_p)$ where

$$n_j = \min_{k=1,\dots,p} \left\{v_j^k\right\}, \quad j = 1, ..., p$$

is the nadir point.

Now, we could think that the natural extension of the approximation set (1) for $p \geq 2$ components in $\mathbf{v}$ would be to define

$$\mathcal{A}(Z, \mathbf{v}) = \{\mathbf{z} \in Z :\ v_1(\mathbf{z}) \geq n_1, ..., v_p(\mathbf{z}) \geq n_p\}$$

But unfortunately, this set $\mathcal{A}(Z,\mathbf{v})$ usually does not contains the value efficient set as it would happen for two components.

Example In fact, let be $Z = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4\}$ and $\mathbf{v}(\mathbf{z}_1) = (1,2,3)$, $\mathbf{v}(\mathbf{z}_2) = (3,2,1)$, $\mathbf{v}(\mathbf{z}_3) = (1,3,1)$, $\mathbf{v}(\mathbf{z}_4) = (2,1,2)$. The nadir point is $(n_1, n_2, n_3) = (1,2,1)$. Hence

$$\mathcal{A}(Z,\mathbf{v}) = \{\mathbf{z} \in Z : \ v_1(\mathbf{z}) \geq 1, v_2(\mathbf{z}) \geq 2, v_3(\mathbf{z}) \geq 1\} = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$$

and $\mathcal{E}(Z,\mathbf{v}) = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4\}$. Thus, $\mathcal{E}(Z,\mathbf{v}) \not\subset \mathcal{A}(Z,\mathbf{v})$. □

So we cannot extend the approximation set in this way. One possibility to overcome this objection would be to relax the above conditions. Let us define

$$\begin{gathered}\mathcal{A}'(Z,\mathbf{v}) = \\ = \{\mathbf{z} \in Z : [v_1(\mathbf{z}) \geq n_1, v_2(\mathbf{z}) \geq n_2] \ \vee \ v_3(\mathbf{z}) \geq n_3 \ \vee \ \dots \ \vee \ v_p(\mathbf{z}) \geq n_p\} \cap \\ \{\mathbf{z} \in Z : [v_1(\mathbf{z}) \geq n_1, v_3(\mathbf{z}) \geq n_3] \ \vee \ v_2(\mathbf{z}) \geq n_2 \ \vee \ v_4(\mathbf{z}) \geq n_4 \ \vee \ \dots \ \vee \ v_p(\mathbf{z}) \geq n_p\} \\ \cap \dots \cap \{\mathbf{z} \in Z : [v_{p-1}(\mathbf{z}) \geq n_{p-1}, v_p(\mathbf{z}) \geq n_p] \ \vee \ v_1(\mathbf{z}) \geq n_1 \ \vee \ \dots \ \vee \ v_{p-2}(\mathbf{z}) \geq n_{p-2}\}\end{gathered}$$

which provides an approximation which always contains the value efficient set.

Proposition 1 *Let* $\mathbf{v} : Z \to \mathbf{R}^p$ *be a vector value function, then*

$$\mathcal{E}(Z,\mathbf{v}) \subseteq \mathcal{A}'(Z,\mathbf{v}) \ .$$

Proof. Let us assume that $\mathbf{z}^0 \in \mathcal{E}(Z,\mathbf{v})$ but $\mathbf{z}^0 \notin \mathcal{A}'(Z,\mathbf{v})$. Then, there is at least one i and j such that

$$\mathbf{z}^0 \notin \{\mathbf{z} \in Z : [v_i(\mathbf{z}) \geq n_i, v_j(\mathbf{z}) \geq n_j] \ \vee \ v_1(\mathbf{z}) \geq n_1 \ \vee \ \dots \ \vee \ v_p(\mathbf{z}) \geq n_p\}$$

and thus $v_1(\mathbf{z}^0) < n_1, ..., v_p(\mathbf{z}^0) < n_p$, being possible that it would not be one of the components i or j, but only one. Let j be the component which does not fulfill the above set. From this and from the definition of nadir point we obtain the following inequalities

$$v_1(\mathbf{z}^0) < n_1 \leq v_1(\mathbf{z}_j^*), ..., v_j(\mathbf{z}^0) \leq v_j(\mathbf{z}_j^*), ..., v_p(\mathbf{z}^0) < n_p \leq v_p(\mathbf{z}_j^*)$$

and thus $\mathbf{z}^0 \notin E(Z,\mathbf{v})$, which contradicts the hypothesis. □

If we suppose that it is possible to obtain more information over the DM's preferences, given as an information cone K such that $K_* = G\{\mathbf{k}^1, ..., \mathbf{k}^q\}$, then we shall have the new associated vector value function $\mathbf{v}^K = (\mathbf{k}^1\mathbf{v}, ..., \mathbf{k}^q\mathbf{v})$. Some ways to generate an information set and a resolution of inconsistency may be through pairwise comparisons which provide constraints over K_* (Zionts and Wallenius, 1983; Malakooti, 1985; Taner and Köksalan 1991), however we shall not consider them here.

Proposition 2 *Given a vector value function* $\mathbf{v} : Z \rightarrow \mathbf{R}^p$ *and an information cone* K *such that* $K_* = G\{\mathbf{k}^1, ..., \mathbf{k}^q\}$, *then*

$$\mathcal{E}(Z, \mathbf{v}^K) \subseteq \mathcal{A}'(Z, \mathbf{v}^K) \ .$$

Proof. It is similar to the above one if we note that the components of the vector value function are now $\mathbf{k}^j\mathbf{v}$ instead of v_j. □

However, this approximation may be improved. Let us consider the vector value function $\mathbf{v}= (v_1, v_2)$ and the nondominated set by $\mathbf{z}_i^*$, defined as

$$\begin{aligned}\mathcal{N}(\mathbf{z}_i^*) = \{\mathbf{z} \in Z : \ v_j(\mathbf{z}) > v_j(\mathbf{z}_i^*) \quad j \neq i, \ j = 1, 2\} \cup \\ \cup \{\mathbf{z} \in Z : v_j(\mathbf{z}) = v_j(\mathbf{z}_i^*) \ \forall j = 1, 2\}\end{aligned}$$

for $i = 1, 2$. Indeed, the approximation set that we considered in definition 3 for a vector value function with two components, may be seen as the intersection

$$\mathcal{A}''(Z, \mathbf{v}) = \mathcal{N}(\mathbf{z}_1^*) \cap \mathcal{N}(\mathbf{z}_2^*)$$

In an analogous way we may extend this idea to the case of a vector value function $\mathbf{v} = (v_1, ..., v_p)$, with $p \geq 2$, where

$$\begin{aligned}\mathcal{N}(\mathbf{z}_i^*) = \{\mathbf{z} \in Z : \ v_j(\mathbf{z}) > v_j(\mathbf{z}_i^*) \text{ for some } \ j \neq i, \ j = 1, .., p\} \cup \\ \cup \{\mathbf{z} \in Z : v_j(\mathbf{z}) = v_j(\mathbf{z}_i^*) \ \forall j = 1, ..., p\}\end{aligned}$$

for all $i = 1, ..., p$.

Definition 4 *We define the approximation set to the value efficient set* $\mathcal{E}(Z, \mathbf{v})$ *as*

$$\mathcal{A}''(Z, \mathbf{v}) = \bigcap_{i=1}^{p} \mathcal{N}(\mathbf{z}_i^*) \ .$$

This approximation set contains the value efficient set.

Proposition 3 *Let* $\mathbf{v} : Z \rightarrow \mathbf{R}^p$ *be a vector value function, then*

$$\mathcal{E}(Z, \mathbf{v}) \subseteq \mathcal{A}''(Z, \mathbf{v}) \ .$$

Proof. Let $\mathbf{z}^0 \in \mathcal{E}(Z, \mathbf{v})$ but $\mathbf{z}^0 \notin \mathcal{A}''(Z, \mathbf{v})$. Then, there is $\mathbf{z}_i^*$ for some i which dominates to $\mathbf{z}^0$, i.e., $\mathbf{v}(\mathbf{z}^0) \leq \mathbf{v}(\mathbf{z}_i^*)$. Thus, $\mathbf{z}^0 \notin \mathcal{E}(Z, \mathbf{v})$, which contradicts the hypothesis. □

Furthermore, this last approximation set is better than the first one introduced.

Proposition 4 *Let* $\mathbf{v} : Z \rightarrow \mathbf{R}^p$ *be a vector value function, then*

$$\mathcal{A}''(Z, \mathbf{v}) \subseteq \mathcal{A}'(Z, \mathbf{v}) \ .$$

Proof: Let us assume that $\mathbf{z}^0 \in \mathcal{A}''(Z, \mathbf{v})$ but $\mathbf{z}^0 \notin \mathcal{A}'(Z, \mathbf{v})$. Then, there is at least some i and j such that

$$\mathbf{z}^0 \notin \{\mathbf{z} \in Z : [v_i(\mathbf{z}) \geq n_i, v_j(\mathbf{z}) \geq n_j] \vee v_1(\mathbf{z}) \geq n_1 \vee \ldots \vee v_p(\mathbf{z}) \geq n_p\}$$

and thus $v_1(\mathbf{z}^0) < n_1, \ldots, v_p(\mathbf{z}^0) < n_p$, being possible that it would not be one of the components i or j, but only one. Let j be the component which does not fulfill the above inequalities. Then, from the definition of nadir point we obtain the following inequalities

$$v_1(\mathbf{z}^0) < n_1 \leq v_1(\mathbf{z}_j^*), \ldots, v_j(\mathbf{z}^0) \leq v_j(\mathbf{z}_j^*), \ldots, v_p(\mathbf{z}^0) < n_p \leq v_p(\mathbf{z}_j^*)$$

and thus $\mathbf{z}^0$ will be dominated by $\mathbf{z}_j^*$. Hence, $\mathbf{z}^0 \notin \mathcal{A}''(Z, \mathbf{v})$, which contradicts the hypothesis. □

However, $\mathcal{A}'(Z, \mathbf{v}) \not\subset \mathcal{A}''(Z, \mathbf{v})$ as we show with the next

Example Let $Z = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4\}$ and

$$\mathbf{v}(\mathbf{z}_1) = (9, 2, 1), \quad \mathbf{v}(\mathbf{z}_2) = (0, 8, 2), \quad \mathbf{v}(\mathbf{z}_3) = (2, 4, 7), \quad \mathbf{v}(\mathbf{z}_4) = (1, 3, 2)$$

The nadir point is $\mathbf{n} = (n_1, n_2, n_3) = (0, 2, 1)$, and we see that

$$\begin{aligned} v_1(\mathbf{z}_3^*) &= 2 > v_1(\mathbf{z}_4) = 1 > n_1 = 0 \\ v_2(\mathbf{z}_3^*) &= 4 > v_2(\mathbf{z}_4) = 3 > n_2 = 2 \\ v_3(\mathbf{z}_3^*) &= 7 > v_3(\mathbf{z}_4) = 2 > n_3 = 1 \end{aligned}$$

then $\mathbf{z}_4 \in \mathcal{A}'(Z, \mathbf{v})$ but $\mathbf{z}_4 \notin \mathcal{A}''(Z, \mathbf{v})$. □

Now, assume new information in terms of a more ample information cone K or, equivalently, of its corresponding information set K_*.

Proposition 5 *Given a vector value function* $\mathbf{v} : Z \to \mathbf{R}^p$ *and an information set* $K_* = G\{\mathbf{k}^1, \ldots, \mathbf{k}^q\}$, *then*

$$\mathcal{E}(Z, \mathbf{v}^K) \subseteq \mathcal{A}''(Z, \mathbf{v}^K).$$

Proof. Let us denote by $\mathbf{z}_{k^i}^*$ the points where the functions $(\mathbf{k}^i\mathbf{v})(\cdot)$, for all $i = 1, \ldots, q$, achieve their maxima, respectively. Thus, the proof is analogous to proposition 3. □

4 Monotonicity property of the approximation set

We show a monotonicity property for the set $\mathcal{A}''$. Assume that the DM reveals more information and leads him to a more reduced information set, we have

Theorem 1 *Given a vector value function* $\mathbf{v} : Z \to \mathbf{R}^p$ *and two information sets* K_*^1 *and* K_*^2, *such that* $K_*^2 \subseteq int(K_*^1)$, *then*

$$\mathcal{A}''(Z, \mathbf{v}^{K^2}) \subseteq \mathcal{A}''(Z, \mathbf{v}^{K^1}).$$

Proof. Let be $K_*^1 = G\{\mathbf{k}_1^1, ..., \mathbf{k}_r^1\}$ and $K_*^2 = G\{\mathbf{k}_1^2, ..., \mathbf{k}_s^2\}$, and denote by $\mathbf{z}_{k_i^1}^*$ y $\mathbf{z}_{k_j^2}^*$ the points where the functions $(\mathbf{k}_i^1\mathbf{v})(\cdot)$ y $(\mathbf{k}_j^2\mathbf{v})(\cdot)$, for all $i = 1, ..., r$ and $j = 1, ..., s$, achieve their maxima, respectively. We shall prove the theorem by induction over s. Let be $s = 1$. We have

$$\mathcal{A}''(Z, \mathbf{v}^{K^2}) = \mathcal{N}(\mathbf{z}_{k_1^2}^*) \qquad \text{and} \qquad \mathcal{A}''(Z, \mathbf{v}^{K^1}) = \bigcap_{i=1}^{r} \mathcal{N}(\mathbf{z}_{k_i^1}^*).$$

Suppose that $\mathbf{z}^0 \in \mathcal{A}''(Z, \mathbf{v}^{K^2})$, but $\mathbf{z}^0 \notin \mathcal{A}''(Z, \mathbf{v}^{K^1})$. In the first case

$$(\mathbf{k}_1^2\mathbf{v})(\mathbf{z}^0) = (\mathbf{k}_1^2\mathbf{v})(\mathbf{z}_{k_1^2}^*) \tag{2}$$

In the second case, there is some i such that $\mathbf{v}^{K^1}(\mathbf{z}^0) \leq \mathbf{v}^{K^1}(\mathbf{z}_{k_i^1}^*)$ and thus

$$(\mathbf{k}_1^2\mathbf{v})(\mathbf{z}^0) < (\mathbf{k}_1^2\mathbf{v})(\mathbf{z}_{k_i^1}^*) \tag{3}$$

From (2) and (3) we obtain that

$$(\mathbf{k}_1^2\mathbf{v})(\mathbf{z}_{k_1^2}^*) < (\mathbf{k}_1^2\mathbf{v})(\mathbf{z}_{k_i^1}^*)$$

which contradicts the definition of $\mathbf{z}_{k_1^2}^*$. Now, assume that the theorem is true for $s - 1$, and let be $K_*'^2 = G\{\mathbf{k}_1^2, ..., \mathbf{k}_{s-1}^2\}$. From the hypotheses we have

$$\mathcal{A}''(Z, \mathbf{v}^{K'^2}) \subseteq \mathcal{A}''(Z, \mathbf{v}^{K^1}) \tag{4}$$

and furthermore

$$\mathcal{A}''(Z, \mathbf{v}^{K^2}) = \bigcap_{j=1}^{s} \mathcal{N}(\mathbf{z}_{k_j^2}^*) \subseteq \mathcal{A}''(Z, \mathbf{v}^{K'^2}) = \bigcap_{j=1}^{s-1} \mathcal{N}(\mathbf{z}_{k_j^2}^*) \tag{5}$$

Thus, from (4) and (5), we have $\mathcal{A}''(Z, \mathbf{v}^{K^2}) \subseteq \mathcal{A}''(Z, \mathbf{v}^{K^1})$. □

In consequence we have the following

Corollary 1 *Given a vector value function* $\mathbf{v} : Z \to \mathbf{R}^p$ *and an information set* $K_* = G\{\mathbf{k}^1, ..., \mathbf{k}^q\}$, *such that fulfills* $K_* \subseteq int(K^0)$, *then* $\mathcal{A}''(Z, \mathbf{v}^K) \subseteq \mathcal{A}''(Z, \mathbf{v})$.

The approximation set $\mathcal{A}''(Z, \mathbf{v})$ may be rewritten as a function of the components of the vector value function as follows

$$
\begin{gathered}
\mathbf{z} \in Z \\
v_2(\mathbf{z}) > v_2(\mathbf{z}_1^*) - \alpha_2^1 M \\
\vdots \\
v_p(\mathbf{z}) > v_p(\mathbf{z}_1^*) - \alpha_p^1 M \\
\vdots \\
v_1(\mathbf{z}) > v_1(\mathbf{z}_p^*) - \alpha_1^p M \\
\vdots \\
v_{p-1}(\mathbf{z}) > v_{p-1}(\mathbf{z}_p^*) - \alpha_{p-1}^p M \\
\textstyle\sum_{i \neq j} \alpha_i^j \leq p-2, \; \left\{ \begin{array}{l} j = 1, ..., p \\ \alpha_i^j \in \{0, 1\} \end{array} \right.
\end{gathered}
$$

where we have to add the optimal solutions $\mathbf{z}_i^*$ and being M a enough big value. To have an idea about M we may compute the points $\mathbf{z}_i^*$ and $\mathbf{z}_{i*}$ which correspond to the ideal and antiideal of each component of the vector value function. Then, M may be any number which fulfills the inequality

$$M \geq \max \left\{ v_1(\mathbf{z}_1^*) - v_1(\mathbf{z}_{1*}), ..., v_p(\mathbf{z}_p^*) - v_p(\mathbf{z}_{p*}) \right\}$$

If the antiideal is not finite, the above bound would not be possible and we should have to look for another procedure.

5 An algorithm

We present in this section an algorithm which uses the approximation set to aid a DM to come with a solution. This procedure follows an interactive approach trying to incorporate the DM in the decision aid process (Vanderpooten, 1989; Vanderpooten and Vincke, 1989; Gardiner and Steuer, 1994).

Step 1. Let $h = 0$.

Step 2. Identify $\mathbf{v}^h = \mathbf{v}^{K^h}$ and calculate the ideal, antiideal and nadir points.

Step 3. Let $A^h = \mathcal{A}''(Z, \mathbf{v}^h)$, $\mathbf{v}^{h**} = \mathbf{v}^{h*} + \delta$ (where $\mathbf{v}^{h*} = \left(v_1^{h*}, ..., v_p^{h*}\right)$ and δ is a vector with small positive values) and $\lambda^h > \mathbf{0}$ is a vector of weights (i.e., $\lambda_i^h = \frac{\alpha_j}{\alpha_1 + \cdots + \alpha_p}$ where $\alpha_j = \frac{v_j^{h**} - n_j}{\max\left\{\left|v_j^{h**}\right|, |n_j|\right\}}$, $j = 1, ..., p$, see Vanderpooten and Vincke, 1989).

Step 4. Compute a compromise solution $\mathbf{z}_h$ by solving

$$\min_{\mathbf{z} \in A^h} s\left(\mathbf{v}^h(\mathbf{z}), \mathbf{v}^{h**}, \lambda^h\right)$$

where

$$s\left(\mathbf{v}^h(\mathbf{z}), \mathbf{v}^{h**}, \lambda^h\right) = \max_{i=1,\ldots,p}\left\{\lambda_i^h\left(v_i^{h**} - v_i^h(\mathbf{z})\right)\right\} - \sum_{i=1}^{p} p_i v_i^h(\mathbf{z})$$

is a function with p_i $(i = 1, ..., p)$ small positive values.

Step 5. $\mathbf{z}_h$ is presented to the DM

If he is satisfied with $\mathbf{z}_h$, stop.

Otherwise, ask him for more information about his preferences in terms of a polyhedral cone K^{h+1} such that $K^h \subset K^{h+1}$.

Step 6. Let $h = h + 1$ and go to step 2.

Comments

(a) The method is general enough to include discrete and continuous problems. We remark that we could use $\mathcal{A}''$ in discrete problems to make easier the generation of the value efficient set $\mathcal{E}$. Indeed, if we first obtain $\mathcal{A}''$, because this set may be an important reduction of Z and contains $\mathcal{E}$, there could be much less comparisons to obtain $\mathcal{E}$.

(b) It should be noted that in step 2 we require to solve $2p$ optimizations to compute the ideal, antiideal and nadir points.

(c) In step 4 we use an scalarizing function (augmented weighted Tchebychev norm) whose purpose is to make use of preference information in order to obtain a compromise solution $\mathbf{z}_h$.

(d) The main drawback may result from the difficulty in providing the preference information required in step 5. But there are some methods to overcome this and to the resolution of inconsistency as we noted in section 3.

6 Examples

We consider two examples to illustrate the above procedure. The first one is discrete and intends to show the underlying ideas. The second one shows the application to a continuous nonlinear problem.

A) Let us consider a DM who intends to buy an apartment. Each one is characterized in a three dimensional with elements $\mathbf{z} = (z_1, z_2, z_3)$, where z_1 =price ($\times 10^6$ ptas), z_2 =surface (m^2) and z_3 =distance from his work (km). The DM considers ten alternatives

$$\begin{array}{lll} \mathbf{z}_1 = (8, 75, 30), & \mathbf{z}_2 = (7.5, 70, 35), & \mathbf{z}_3 = (10, 100, 50), \\ \mathbf{z}_4 = (25, 110, 60), & \mathbf{z}_5 = (5, 40, 10), & \mathbf{z}_6 = (15, 90, 18), \\ \mathbf{z}_7 = (12, 65, 15), & \mathbf{z}_8 = (16, 71, 24), & \mathbf{z}_9 = (16, 94, 32), \\ \mathbf{z}_{10} = (11, 83, 33) & & \end{array}$$

and the maximization of the vector value function $\mathbf{v}^0(\mathbf{z}) = (-z_1, 1.5z_2 - z_3)$, which corresponds to the null information set K_*^0. The value efficient set is

$\mathcal{E}(Z, \mathbf{v}^0) = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_5, \mathbf{z}_6\}$. The optimal solutions for the components in $\mathbf{v}^0$ are, respectively (step 2)

$$\mathbf{v}^0(\mathbf{z}_5) = (-5, 50), \qquad \mathbf{v}^0(\mathbf{z}_6) = (-15, 117)$$

Thus, the approximation set is (step 3)

$$\mathcal{A}^0 = \{\mathbf{z} \in Z : -z_1 \geq -15,\ 1.5z_2 - z_3 \geq 50\} = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_5, \mathbf{z}_6, \mathbf{z}_7, \mathbf{z}_{10}\}$$

and the compromise solution (step 4) is $\mathbf{z}_6$. Suppose that the DM is not satisfied with this solution, then he should provide a new information set (step 5). Assume it is $K_*^1 = G\{(.9, .1), (.6, .4)\}$, thus the associated vector value function is

$$\mathbf{v}^1(\mathbf{z}) = (-.9z_1 + .15z_2 - .1z_3, -.6z_1 + .6z_2 - .4z_3)$$

and the optimal solutions for the components in $\mathbf{v}^1$ are, respectively

$$\mathbf{v}^1(\mathbf{z}_1) = (1.05, 28.2), \qquad \mathbf{v}^1(\mathbf{z}_6) = (-1.8, 37.8)$$

The approximation set is

$$\mathcal{A}^1 = \{\mathbf{z} \in Z : -.9z_1 + .15z_2 - .1z_3 \geq -1.8, -.6z_1 + .6z_2 - .4z_3 \geq 28.2\} = \\ = \{\mathbf{z}_1, \mathbf{z}_3, \mathbf{z}_6, \mathbf{z}_{10}\},$$

the value efficient set $\mathcal{E}(Z, \mathbf{v}^1) = \{\mathbf{z}_1, \mathbf{z}_3, \mathbf{z}_6\}$ and the compromise solution is $\mathbf{z}_3$. If the DM were satisfied with this solution the algorithm stops. Otherwise, the procedure would continue.

B) Now, let be the nonlinear multiobjetive problem with the vector value function

$$\mathbf{v}^0(\mathbf{z}) = (-.024 + 1.82\exp(-.2875z_1), 3z_2 + z_3, z_4)$$

and constraints

$$z_1 + 2z_2 + z_3 + 3z_4 \leq 25$$
$$0 \leq z_1 \leq 100,\ -3 \leq z_2 \leq 10$$
$$1 \leq z_3 \leq 5,\ -4 \leq z_4 \leq 20$$

For K_*^0 (step 1), by solving 6 optimization problems (step 2) we obtain $\mathbf{v}^{0*} = (1.844, 35, 10)$ and $\mathbf{v}_*^0 = (-.022, -8, -4)$. If we take $\delta = .001$, then $\mathbf{v}^{0**} = (1.845, 35.001, 10.001)$. Hence, we have (step 3) the weights are $\lambda_1 = .001$, $\lambda_2 = .551$, $\lambda_3 = .448$ and by solving the minimization problem in step 4 we obtain the compromise solution $\mathbf{z}_0 = (0, 10, 4.793, .071)$. Assume the DM were not satisfied with such solution, but he is ready to reveal a new information set (step 5), i.e., $K_*^1 = \{(.1, .1, .8), (.2, .3, .5)\}$. Thus, the associated vector value function is $\mathbf{v}^1 = \left(v_1^1, v_2^1\right)$, where

$$v_1^1(\mathbf{z}) = .182\exp(-.2875z_1) + .3z_2 + .1z_3 + .8z_4 - .0024$$
$$v_2^1(\mathbf{z}) = .364\exp(-.2875z_1) + .9z_2 + .3z_3 + .5z_4 - .0048$$

from which, following the above steps of the algorithm, lead us to obtain the compromise solution $\mathbf{z}_1 = (0, .10, 4.99, .003)$, which again would be presented to the DM to decide if he is satisfied or it would continue the process.

7 Conclusions

We have provided an approximation set to the value efficient set for the case of multicriteria decision making problem, where there is partial information on the DM's preferences through a vector value function which in an interactive process is transformed in a more precise one. This approximation arises from the difficulty of generating the efficient set, specially in continuous and nonlinear problems. To compute this set may be easy because it will be necessary to solve $2p$ optimization problems. We provide some properties of monotonicity and an interactive algorithm shows a method based in the approximation set which is applied to two examples.

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Choosing a Finite Set of Nondominated Points with Respect to a Finite Set of Reference Points

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[Extended Abstract]

The paper deals with the MOLP problem "max" $\{f(x) = z \mid x \in S \subset R^n\}$, where $f(x) \in R^m$, all $f_i(x)$ are linear functions, the set S is defined by linear constraints, and is bounded and closed. In order to obtain one nondominated point for a fixed reference point in tne criterion space R^m (as in the reference point method) we can formulate a linear programming problem based on the usage of the following function:

$$\psi(z^{**},z)=\max_{i=1,\dots,m}\left\{b_i\left|z_i^{**}-z_i\right|\right\}-\sum_{j=1}^{m}c_j z_j$$

Here b_i, $c_j > 0$ for all i,j; $z^{**} \in R^m$, z^{**} dominates the ideal point, z is an attainable point in R^m. The solution of this problem determines the searched nondominated point.

The following task is considered in the paper. Suppose that the MOLP problem is given. Suppose in addition, that a set A of p reference points is given, $A = \{r^u\}$, $r^u \in R^m$, $|A| = p$. We would like to find a set B_0 of q nondominated points for the given MOLP problem, where $B_0 = \{f(x^v)\}$, $v=1,\dots,q$, $q < p$ and the set B_0 is close to the set A. For this purpose we define a "distance" $g_u(B)$ between a reference point r^u and a set B of q attainable points in R^m as:

$$g_u(B)=\min_{v=1,\dots,q}\ \psi\left(r^u,f\left(x^v\right)\right)$$

We consider the functions $g_u(B)$ for all u as p new additional criteria and we would like to find a set B_0 that is efficient with respect to these criteria $g_u(B)$. Following again the basic idea of the reference point method for obtaining such B_0 we formulate a new linear mixed binary programming problem (P). The binary variables are used to determine the values $g_u(B)$ for an arbitrary B and for all u. The problem (P) contains constraints of the following type

$$g_u(B)\geq \psi\left(r^u,f\left(x^v\right)\right)+C(1-p_v^u) \quad \text{for all } u,\ v.$$

Here p_v^u are binary variables, $\sum_{v=1}^{q} p_v^u = 1, \forall u$. If $p_v^u = 1$ then $f(x^v)$ is the closest point to r^u. In addition $C > 0$ is chosen in such a way that these constraints are not substantial when $p_v^u = 0$.

In problem (P) an additional reference point $t \in R^p$ corresponding to the additional criteria is used, too. The reference point t allows to get different sets B_0 for a fixed set A.

The main results presented in the paper are:

1) The solution of the formulated linear mixed binary programming problem (P) determines a set $B = \{f(x^v)\}$, $v=1,\dots,q$, that is efficient, i.e. there is no other set B_1 of nondominated points for the initial MOLP problem ($|B_1| = q$), such that $g_u(B_1) \leq g_u(B)$ for all u and this inequality is strict for at least one u.

2) Each point x^v ($v=1,\dots,q$), determined by the same solution is efficient for the initial MOLP problem.

The same solution determines a partition of the set A which when computed seems acceptable.

Thus the paper proposes an instance of multiple criteria optimization over the efficient set of another multiobjective problem.

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A METHOD FOR SEARCHING RATIONALITY IN PAIRWISE CHOICES

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Abstract. In this paper an operational method is presented for searching rationality into paired comparisons of the alternatives. We base our method on the decomposition of a binary relation in terms of families of quasi orders. With the "mixture of maximal quasi orders", as the main concept, we attempt to use decision maker's inconsistencies as a kind of information for modelling his preferences.

Keywords. Preference modelling, Quasi order, Paired comparisons, Incoherences, Intransitivities.

1 Introduction

From a normative point of view, researchers do not study how people do make decisions, but how people should make decisions. Properties of preference models are based on some basic logical principles appeared in the field of economic theory. These principles define the rational decision maker (DM).

Traditionally, one important demand for a rational DM has been that, in a very special sense, he must be decisive. Thereby, in some normative theories, the binary relation approaching pairwise preferences has been modelled as a weak order, i.e. reflexive, transitive and complete.

However, some authors have mentioned the possibility of imprecisions in DM's judgments, [1], [2], [8]. From this observation, it is necessary to model DM's preference using a quasi order, i.e. a reflexive and transitive binary relation.

On the other hand, a group of papers have considered the transitivity as a questionable principle of rationality, [3] [5] [6].

Using these remarks, this paper proposes a new tool, based on the quasi order concept, for searching rationality in paired comparisons of alternatives. Evidently, DM does not need a rational behavior on the DM, i.e. strictly we do not consider a normative model. A DM could commit incoherences (intransitivities) and the analyst will not point him to these ones for changing them. In this way, the analyst avoids to slope DM's real attitudes. He could use incoherences as new source of information.

Our method has been motivated by the following assumption: Really, a rational DM does not exist. However, there is a closed relation between a real and a rational DM. Every real DM would like to be rational. We observed this phenomenon during our experimental studies about robust decision making, [7]. Thus, there is a potential rational attitude in all DMs, but many times such attitude is not real. The method we show in this paper considers this observation.

In Section 2 we introduce the mixture of maximal quasi orders (MMQO) as a tool for searching rationality.

Each quasi order of the mixture represents a "germ of rationality" included in the information about preferences available by the analyst. This mixture contains different rankings of alternatives. This fact leads us to remember a multicriteria decision making problem, but now criteria are not explicit for the DM. They are hidden conflicting criteria, included into DM subconscious mind, and now they are the source of incoherences as Roberts [9] considered them a source of incomparabilities.

In Section 3, we show a more efficient algorithm for searching the MMQO associated to a reflexive binary relation.

Finally, we offer some conclusions and open problems.

2 A New Model for Preferences in Paired Comparisons

Let A be a finite set of alternatives in a decision making problem. Let assume that the DM chooses among alternatives using a paired comparisons method. The pairwise preference approach introduces a relative evaluation of every pair of alternatives, modelled by a binary relation, S:

$$aSb \text{ if and only if } "a \text{ is at least as good as } b"$$

Representation results for special binary relations justify some rationality principles from a mathematical point of view.

For instance, if S is a weak order then there is a real function, v, over A such that

$$aSb \text{ if and only if } v(a) \geq v(b)$$

For a proof, see [10].

If S is a quasi order then there are a finite number of real value functions $v_1, ..., v_r$ such that

$$aSb \Leftrightarrow v_l(a) \geq v_l(b) \quad l = 1, .., r$$

for $a, b \in A$ and some $r \in \mathbf{N}$, see [8].

Then, we are going to consider a general reflexive binary relation S. The analysis of which parts of this binary relation can be represented analytically will be our primary aim. In this way, we try to allow the DM to choose among alternatives without using any coherence or rational principle.

The following result permit us to decompose a reflexive binary relation into representable parts. Before that, we show a definition useful for our purpose.

Definition 1 *Let A be a finite set and $\mathcal{F} = \{S_i\}_{i \in I}$ an ordered set of quasi orders included into A, using the inclusion, $\subset$, as order. $S^* \in \mathcal{F}$ is a maximal quasi order on A if there is not any $S \in \mathcal{F}$ such that $S^* \subset S$.*

The following is an extension of a result appeared in [5].

Proposition 1 *If S is a reflexive binary relation on a finite set A, there is a finite family, $\mathcal{F} = \{Q_k\}_{k=1}^{q}$, containing all maximal quasi orders included into S that fulfills*

$$a_i S a_j \Leftrightarrow \exists\, Q_s \; / \; a_i Q_s a_j$$

where $a_i, a_j \in A$.

Proof. We construct the family of maximal quasi orders.

If S is a quasi order, then $q = 1$ and $\mathcal{F} = \{S\}$.

If S is not a quasi order, S is not transitive. Then, there are $x, y, z \in A$ such that:

$$xSy,\; ySz \text{ and } \neg(xSz).$$

We construct the following binary relations S_1, S_2:

- $xS_1y,\; \neg(yS_1z)$
- $\neg(xS_2y),\; yS_2z$

- $\forall (u, w) \in A \times A - \{(x, y), (y, z)\} : \ uS_iw \Leftrightarrow uSw \ , i = 1, 2.$

If S_i is transitive, we stop the process and we consider two cases:

- If there is no $S^* \in \mathcal{F}$ such that $S_i \subset S^*$ then $S_i \in \mathcal{F}$.
- If there is any $S^* \in \mathcal{F}$ such that $S_i \subset S^*$ then $S_i \notin \mathcal{F}$.

If S_i is not transitive, there are $x_i, y_i, z_i \in A$ such that:

$$x_iS_iy_i, \ y_iS_iz_i \text{ and } \neg(x_iS_iz_i).$$

We use these relations for constructing S_{i1} and S_{i2} in the same way that S_1 and S_2 above. We repeat the same process sucesively.

We shall prove that $\mathcal{F}$ contains all maximals quasi orders included into S: Let Q be a quasi order such that $Q \subset S$. If Q is a maximal quasi order, it will be shown that $Q \in F$.

For all $(x, y) \in S$, the binary relation $Q \cup \{(x, y)\}$ has some intransitivity, because Q is maximal. Let (x, y) be an element of S and the binary relation $Q_1 = Q \cup \{(x, y)\}$. Let us consider the intransitivity $(x, y) \in Q_1$ $(y, z_1) \in Q_1$ and $(x, z_1) \notin Q_1$ (with any other form of the intransitivity the reasoning is similar).

- If this intransitivity belongs to S, that means that if $(x, y) \in S$ $(y, z_1) \in S$ and $(x, z_1) \notin S$, then, using this intransitivity for branching in the process described above, we obtain that $Q \in S$.
- If this intransitivity does not belong to S, that means that if $(x, z_1) \in S$, then we consider the binary relation $Q_2 = Q_1 \cup \{(x, z_1)\}$. Q_2 is not a quasi order, because Q is maximal. Then there is any intransitivity in Q_2. A similar reasoning as above will be used for this intransitivity.

As the process is bounded, because at most we would study a $Q_i = S$, we conclude that $Q \in \mathcal{F}$. Then $\mathcal{F}$ is unique.

$\diamond$.

The algorithm generated for proving this proposition is only for a theoretical use. In fact, its complexity is approximately 2^{n^2}, too much for computational purposes.

Example.

Assume S as the binary relation representing preferences in pairwise comparisons over $A = \{a, b, c, d\}$:

$$bSa, \ aSc, \ cSa, \ bSc, \ cSb, \ bSd, \ dSb, \ cSd, \ aSa, \ bSb, \ cSc, \ dSd$$

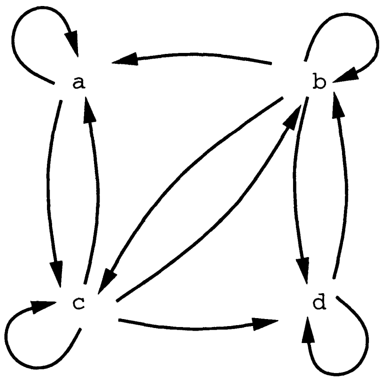

Figure 1: Binary relation S

The graph of the binary relation S is shown in figure 1.

This binary relation must be decomposed into the family of maximal quasi orders represented in the figure 2.

Definition 2 *We name this decomposition,* $\mathcal{F} = \{Q_k\}_{k=1}^{q}$, **mixture of maximal quasi orders (MMQO)** *associated to a binary relation* S.

Using the last proposition we obtain a representation result for a reflexive binary relation

Theorem 1 *Let A be a finite set and S be a reflexive binary relation on A. There is a finite set of maximal families of value functions $\mathcal{V} = \{V_i\}_{j=1}^{q}$ such that*

$$(x, y) \in S \Leftrightarrow \exists\, j \in \{1, ..., q\} \text{ such that } v(x) \geq v(y) \;\; \forall v \in V_j$$

Proof. From proposition 1 and the representation result for quasi orders. ◇

For this result we outline the following interpretation:
For choosing between two alternatives the DM has different criteria on his subconscious mind and in most cases they are in conflict with each other. This conflict between hidden criteria sometimes leads the DM to incomparability (or doubt attitude).For instance, in our example alternatives a and b are incomparable. Nevertheless, the doubt is an attitude that DM tries to avoid, either because he considers available to do pairwise comparisons or because it is very inconvenient for him to hesitate for psychological reasons, since people relate doubt with ignorance. We have observed the last fact in our experimental study described in [7]. A source of incoherences can be recognized in this kind of phenomena. In this way, the cycle $a \succ c \succeq b \succ a$

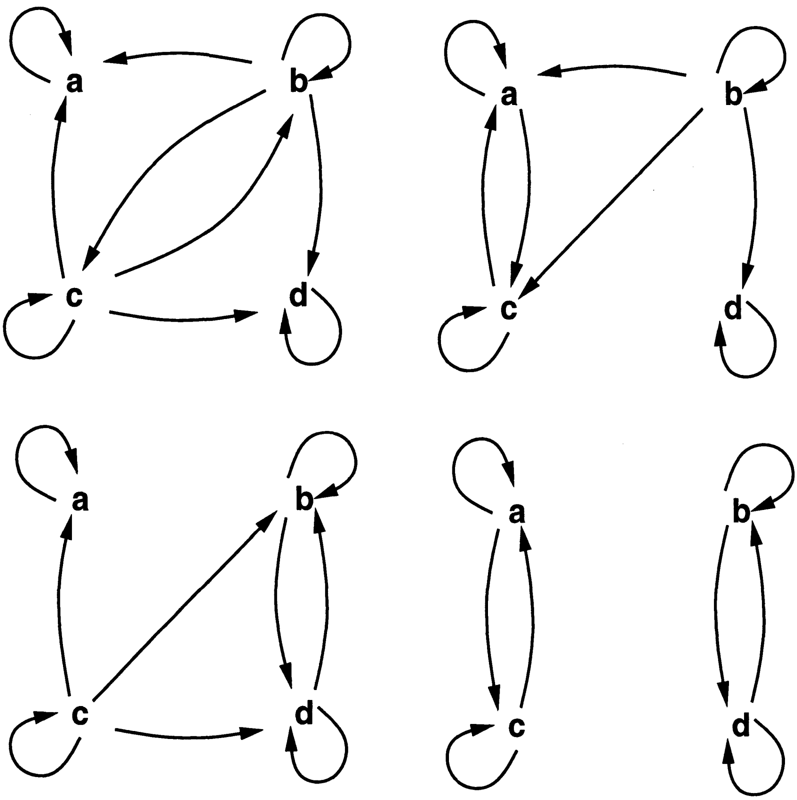

Figure 2: MMQO associated to S

can be consequence of the hesitation about some paired comparisons.

In theorem 1, different families in $\mathcal{V}$ represent hidden criteria in DM's subconscious mind (source of incoherences), whereas each family, V_i, represents the lack of precision in DM's judgments using the criterion represented by V_i (source of imprecision).

3 A More Efficient Algorithm for Computing a MMQO

The algorithm used in the proof of Proposition 1, for searching the MMQO, has sense only in a theoretical framework. A brief description of the procedure is the following: Two random elements of S, forming an intransitivity, are choose n. Crossing one of these elements off S and preserving the other one, alternatively, leads us to two new binary relations, S_1 and S_2. Now, choosing random intransitivities of S_1 and S_2, respectively, the process is repeated until no intransitivities are detected. In this way, we obtain a hypothetical tree named *intransitivities tree.* Each ramification of this tree, composed of two branches, is the consequence of a process as described above. Finally, the method makes an exhaustive exploration of all leafs of the intransitivities tree, trying to search for maximal quasi orders.

The algorithm presented in this Section is based on a new idea about the tree. Now, an element of S, (a, b), belonging to the greatest number of intransitivities, is chosen. Using this element, two binary relations are defined (ramification into two branches). The first one (left branch), represented by S' in the algorithm, is defined by elements S except (a, b). The second one (right branch), represented by S'' in the algorithm , includes (a, b) and all elements of S not defining intransitivities with (a, b). An analogous procedure will be used for separating both S' and S'' into two new binary relations. A pruning procedure has been devised, in order not to study twice the same pair, using the variable F. With F we force the algorithm to keep some pairs into the binary relation, because the elimination of these pairs has been studied in other part of the tree. As above, this procedure is repeated until no intransitivities are detected. On the other hand, the maximality of obtained quasi orders is taken into account during all the process with the variable Q.

A high level description (pseudo-code) of the proposed algorithm can be stated as follows:

```
FUNCTION Mixture (S, F, Q):Quasiorder
  If S is a quasiorder
  Then Q:=Insert-Keeping-Maximality(S,Q)
  Else Begin P:=choose-pair-in-more-intransitivities(S);
             S:=remove-pairs-forming-intransitivity
                 -with-an-F-pair(S,F);
             S':=S - {P};
             Q:=Mixture(S',F,Q);
             S'':=S;
             N:=set-of-intransitivity-pairs-forming
                 -intransitivity-with-pair(S,P);
             Prune:=False;
             While NOT (Prune) AND N ≠ ∅ do
                 begin
                   P':=first-pair (N);
                   N := N-{P'};
                   If P' ∈ F
```

```
                Then Prune:=True
                Else S'':=S'' - {P'}
              end;
          If NOT (Prune)
          Then Q:=Mixture(S'',F ∪ {P},Q)
      End;
Return Q
```

4 Conclusions

For constructing the MMQO model the following aspects were considered:

- The analyst has fixed the rationality concept in the model. Quasi orders are considered as ground of rationality.
- The maximality concept that appears in the MMQO model arises to allow the DM to keep as much of his original information as possible.

Some research problems arise from the MMQO study. Some of them are the following:

In [7] we observed that incomparabilities in paired comparisons are not as usual as incoherences. With the MMQO, incoherences appear as a new source of information about the DM preference structure. How to structure this information?

On the other hand, through the MMQO it is possible to describe conflicting ranks (may be objectives) in the DM subsconcious mind. How to use this concept with aggregation purpose?

Finally, a less complex algorithm for computing the MMQO associated to a binary relation is necessary.

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Zero-One Goal Programming Under Interdependence of Actions

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Abstract. An approach to modeling discrete multiple criteria problems under interdependence of actions is presented. The concept of interdependency of actions in multicriteria decision making is explored and its main characteristics discussed. The problem is formulated as a zero-one program; a goal programming technique can then be employed to obtain a suitable subset of efficient solutions.

Keywords. Interdependence of Actions, Multiple Criteria Zero-One Programming, Goal Programming

1 Introduction

Multiple Criteria Decision Making (MCDM) problems arise naturally in many situations, both strategic and routine. Many methods and theories have been developed during the past two decades for solving a wide range of multiple criteria decision problems in both continuous and discrete contexts.

The definition and generation of actions[1] is an important step in the process of MCDM, but one to which little research effort has been devoted (Vincke, 1992). Furthermore, the problem of selecting an alternative (i.e. a set of actions) from a discrete set of available actions have received very little attention in the multiple criteria literature. Moreover, all alternative selection models assume strict independence of actions, even though interdependence occurs in many real world subset selection problems such as project selection, location selection, and time stream decisions. For instance, nearby sites for waste disposal facilities are frequently interdependent with respect to environmental impact and/or infrastructure criteria. One can observe similar interdependence of actions in many location and project selection problems.

[1]We discriminate between *action* and *alternative*. An action is a specific individual choice, and an alternative is a subset of all available actions.

Gomes (1992) introduced the concept of interdependence between two actions in an urban transportation system. Assume that the probability of choosing highway project j is p_j. He defined the value according to a criterion of another project, i, interdependent with j on that criterion, to be

$$v_i = (v_{i|j})p_j + (v_{i|\sim j})(1 - p_j).$$

Through an example Gomes showed that the ranking of actions may be changed as a result of this kind of interdependency.

An analysis by Keeney and colleagues to select three out of five potential sites for nuclear waste disposal used multiattribute utility theory (see Merkhofer and Keeney (1987)). After reviewing their proposal, the U.S. Department of Energy selected a different subset. In analyzing this disparity, Keeney (1994) argued that the logic of the analysis involved individual evaluations of the five potential sites, rather than selection based on portfolio principles. He concluded that individual examination of the sites did not address some important considerations that affected all sites.

The main objective of this paper is to develop models and associated analytical techniques for MCDM subset selection problems in the presence of interdependent actions. The notion of interdependence of actions in MCDM is explored and a goal programming technique is used for the selection of a subset of efficient alternatives.

2 Interdependence of Actions

Let the set of available actions be $\mathbf{A} = \{a_1, a_2, \cdots, a_{|A|}\}$. Denote the *consequences* (outcomes) of $\mathbf{A}^* \subseteq \mathbf{A}$ with respect to criterion $p \in \mathbf{P}$ by $c_p(\mathbf{A}^*)$. We call two actions, a_i and a_j, *independent* if the selection of a_j has no effect on the evaluation of a_i according to every criterion, and vice versa. In symbols,

$$\begin{aligned} a_i \mathbf{I} a_j \quad \Longleftrightarrow \quad & c_p(\mathbf{A}^* \cup \{a_i\}) - c_p(\mathbf{A}^*) = c_p(\mathbf{A}^* \cup \{a_i, a_j\}) - c_p(\mathbf{A}^* \cup \{a_j\}), \\ & \forall p \in \mathbf{P}, \quad \textit{and} \quad \forall \mathbf{A}^* \subseteq \mathbf{A} - \{a_i, a_j\} \qquad (1) \end{aligned}$$

where $\mathbf{I}$ represents independence of actions. Relation (1) indicates that the amount by which action a_i increases the consequence on criterion p does not depend on whether a_j is also selected. Therefore, *consequence interdependence* of two actions is defined as follows:

Definition 1 *Actions a_i and a_j are interdependent if the consequence of one action according to at least one criterion changes with selection of the other.*

$$\begin{aligned} a_i \ \mathbf{ID} \ a_j \quad \textit{iff} \quad & \exists p \in \mathbf{P}, \quad \textit{and} \quad \exists \mathbf{A}^* \subseteq \mathbf{A} - \{a_i, a_j\}, \qquad (2) \\ c_p(\mathbf{A}^* \cup \{a_i\}) - c_p(\mathbf{A}^*) \quad & \neq \quad c_p(\mathbf{A}^* \cup \{a_i, a_j\}) - c_p(\mathbf{A}^* \cup \{a_j\}). \end{aligned}$$

In particular, under a linear value function,

$$a_i \ \mathbf{ID} \ a_j \quad \textit{iff} \quad \exists p \in \mathbf{P}, \quad \textit{and} \quad \exists \mathbf{A}^* \subseteq \mathbf{A} - \{a_i, a_j\}, \tag{3}$$
$$v_p(\mathbf{A}^* \cup \{a_i\}) - v_p(\mathbf{A}^*) \quad \neq \quad v_p(\mathbf{A}^* \cup \{a_i, a_j\}) - v_p(\mathbf{A}^* \cup \{a_j\}),$$

in which $v_p(\mathbf{A}^*)$ is the value of $\mathbf{A}^* \subseteq \mathbf{A}$ on ciriterion p, as evaluated by the Decision Maker (DM). In the above expressions **ID** represents value interdependence of actions. Note that $a_i \ \mathbf{I} \ a_j$ iff $a_j \ \mathbf{I} \ a_i$ and $a_i \ \mathbf{ID} \ a_j$ iff $a_j \ \mathbf{ID} \ a_i$.

Without loss of generality, we assume that $c_p(\mathbf{A}^*) > 0$. If $a_i \in \mathbf{A}$ define $c_p^i = c_p(\{a_i\})$. We are vitally interested in analyzing the consequences of not just actions, but alternatives (subsets of actions). It has been observed in practice that, usually, the consequences on any criterion $p \in \mathbf{P}$ are additive when more than one action is selected. We say that the consequences of a_i and a_j are *independent* when

$$c_p(\{a_i, a_j\}) = c_p^i + c_p^j. \tag{4}$$

Otherwise the consequences on criterion p of a_i and a_j are *interdependent.* In this case, the interdependency function $\gamma_p^{ij}(c_p^i, c_p^j)$ expresses the interdependence of the consequences of actions a_i and a_j on criterion p.

Note that the order of selection does not affect the value and consequence of a combination of actions. In particular,

$$\gamma_p^{ij} = \gamma_p^{ji}. \tag{5}$$

Generally, interdependence is not a transitive relation, i.e.

$$a_i \ \mathbf{ID} \ a_j \quad \textit{and} \quad a_j \ \mathbf{ID} \ a_k \quad \not\Longrightarrow a_i \ \mathbf{ID} \ a_k. \tag{6}$$

In simple cases, the consequence of combination of action a_i and a_j is as follows:

$$c_p(\{a_i, a_j\}) = c_p^i + c_p^j + \gamma_p^{ij}(c_p^i, c_p^j). \tag{7}$$

It is noteworthy that since γ can take both positive and negative values, in general, calculation of consequences of a subset of interdependent actions is very complex. One may need to apply some concepts of inclusion and exclusion to deal with this problem.

The interdependency notion can be generalized to any number of actions. The definition of interdependency among three actions (interdependency of degree 3) is presented next. The concept is similar for higher degrees of interdependency.

Definition 2 *Actions a_i, a_j, and a_k are called jointly interdependent if the consequence of one of them, according to at least one criterion, changes with selection of the other two. That is,*

$$a_i \ \mathbf{ID} \ \{a_j, a_k\} \quad \textit{iff} \quad \exists p \in \mathbf{P}, \textit{and} \ \ \exists \mathbf{A}^* \subseteq \mathbf{A} - \{a_i, a_j, a_k\} \tag{8}$$
$$c_p(\mathbf{A}^* \cup \{a_i\}) - c_p(\mathbf{A}^*) \ \neq \ c_p(\mathbf{A}^* \cup \{a_i, a_j, a_k\}) - c_p(\mathbf{A}^* \cup \{a_j, a_k\}).$$

It is noteworthy that, according to the above definition, it is possible for all *pairs* of a_i, a_j, and a_k to be independent and yet $\gamma_p^{ijk} \neq 0$. Also, it may happen that all pairs of actions are interdependent and $\gamma_p^{ijk} = 0$. Figure 1 shows three different interdependency situations that may arise if there are three actions.

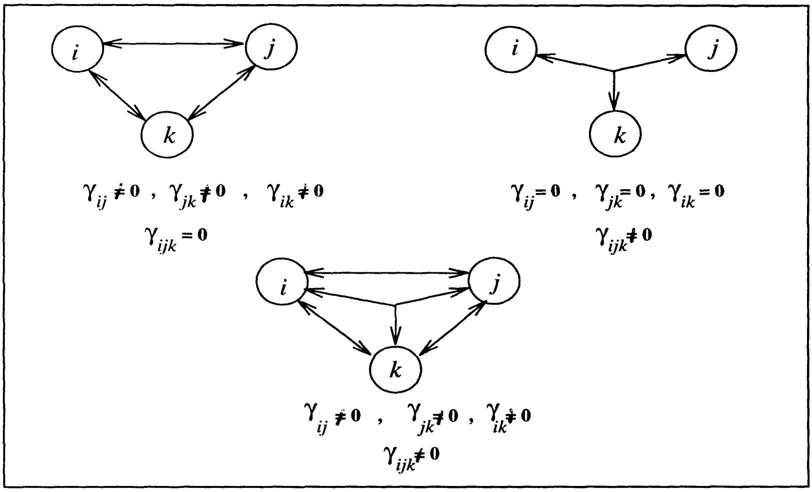

Figure 1: Three possible interdependencies involving three actions.

The following simple example shows that a subset selection problem in the presence of interdependent actions is quite different from conventional MCDM. In particular, it is shown that the selection of actions based on the *greedy algorithm* can be quite misleading.

Consider a Waste Disposal Location (WDL) problem in which one is looking for the two best among five potential sites. The criteria are *local population*, *infrastructure* (such as availability of roads, water, electricity and other facilities), and *environmental risk*. For all criteria, higher values are preferred. Assume that roads and/or other facilities near sites 4 and 5 are sufficient for just one of the sites; if sites 4 and 5 are both selected, an additional infrastructure investment is required. Therefore, the evaluation of site 5 (or 4), on the infrastructure criterion depends on whether site 4 (or 5) has already been selected. Hence, sites 4 and 5 have negative interdependency on the infrastructure criterion.

Additionally, assume that if sites 1 and 2 together are selected then a single power plant facility may be built for both, taking advantage of economies of scale. Therefore, actions 1 and 2 are positively interdependent on the infrastructure criterion.

It is also possible that the evaluation of one site on environmental risk may depend on whether another site is selected. Suppose that selection of both site 4 and 5 aggravates the risk of environmental damage in the region; hence, these two actions have a negative interdependence on the environmental risk criterion. Figure 2 shows the interdependent actions and their coefficients of dependency.

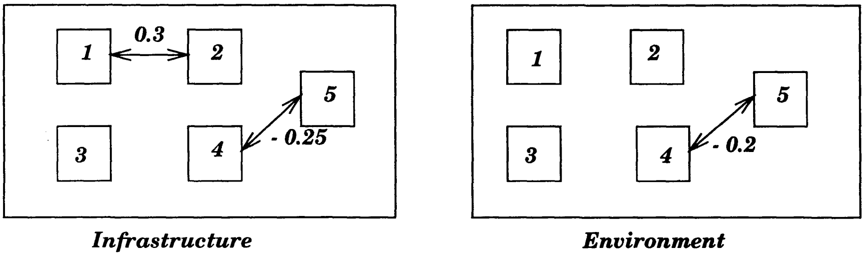

Figure 2: WDL Example: Interdependency of sites.

Table 1. WDL Example : Normalized values of five feasible sites.

Criteria	*Actions*					*Weights*
	a_1	a_2	a_3	a_4	a_5	
Population	0.45	0.45	1	0.55	0.84	0.23
Infrastructure	0.8	0.7	0.75	0.83	0.83	0.39
Env. Risk	0.6	0.87	0.5	0.75	0.6	0.38
Additive Value	0.6435	0.7070	0.7125	0.7350	0.7449	1

Table 1 lists the global importance of each criterion, and the individual consequences of each action according to each criterion. Assuming a *linear additive value* , action a_5 is best, and action a_4 is second best according to Table 1. Therefore, ignoring interdependency, the best pair of actions is $\mathbf{A}^* = \{a_5, a_4\}$.

Table 2 shows the reevaluated consequences of sites after selecting action a_5. Observe that the value of action a_4 is now 0.5031. Hence, the overall value of $\{a_5, a_4\}$ is $1.248 = .7440 + .5031$. However, when the first site is selected by the greedy algorithm (site 5), then the best second site, considering all interdependencies with site 5, is site 3. As Table 2 shows, $\mathbf{A}^* = \{a_5, a_3\}$ has $v^* = 1.4574 = .7449 + .7125$.

Table 2. WDL Example : Site consequences after selecting site 5.

Criteria	*Actions*				*Weight*
	a_1	a_2	a_3	a_4	
Population	0.45	0.45	1	0.55	0.23
Infrastructure	0.8	0.7	0.75	0.498	0.39
Environmental Risk	0.6	0.87	0.5	0.48	0.38
Additive Value	0.6435	0.7071	0.7125	0.5031	1

Nevertheless, by exhaustive examination, one finds that in this example the best subset of actions is $\{a_1, a_2\}$ with $v^* = 1.526$. Note that action a_2 was in the fourth position in Table 1, and action a_1 was individually dominated by both a_5 and a_4.

This simple example demonstrates that in the presence of interdependent actions in MCDM, one may need to examine all combinations of actions in order to find the best subset; and that selection according to the ranked list of individual actions may not yield the optimal solution. Moreover, this example illustrates that the selection of actions based on the *greedy algorithm* can be wrong. The next sections provide a model for a subset selection problem in MCDM in the presence of interdependent actions.

It would be quite useful to modify some promising recent MCDM techniques in order to enable them to take interdependence of actions into account. This paper describes how a general MCDM with interdependent actions can be modeled.

3 Formulation of the Problem

Recall that **A** is the set of actions, **P** is the set of criteria, and $c_p^i \geq 0$ is the consequence of action a_i according to criterion p. Define the binary variable x_i by

$$x_i = \begin{cases} 1 & \text{if } a_i \text{ is selected;} \\ 0 & \text{if } a_i \text{ is not selected.} \end{cases}$$

Now, let $\mathbf{S}_p^k$ denote the set of all actions with interdependence of degree $k > 1$ according to criterion p. In the WDL example in the previous section, we have: $\mathbf{S}_1^k = \emptyset \quad \forall k$; $\mathbf{S}_2^2 = \{\{a_4, a_5\}, \{a_1, a_2\}\}$; $\mathbf{S}_3^2 = \{\{a_4, a_5\}\}$; and $\mathbf{S}_p^k = \emptyset \quad \forall p \in \mathbf{P} \quad \textit{and} \quad \textit{for} \quad k \geq 3$.

According to our definition of interdependency, a general subset selection problem under interdependence of actions in the presence of resource constraints can be stated as a Multiple Criteria Zero-One (MCZO) problem as follows:

$$\begin{array}{lll} \textbf{Maximize } z_p = & \sum_{i=1}^{|\mathbf{A}|} c_p^i x_i \ + \sum_{k=2}^{|\mathbf{A}|} (-1)^{k+1} \sum_{S \in \mathbf{S}_p^k} \gamma_p^S \cdot (\prod_{a_i \in S} x_i); \ \forall p \in \mathbf{P} & \\ \textbf{Subject to}: & x \in \mathbf{X} \qquad\qquad\qquad\qquad \textbf{(D)} & \\ & x_i \in \{0, 1\}. & \end{array}$$

Without loss of generality, all criteria are to be maximized. In a simple case, where there exist interdependencies among at most three actions and γ represents only negative synergy, the objective functions of Problem **D** can be stated as

$$\textbf{Maximize } z_p = \sum_{i=1}^{|\mathbf{A}|} c_p^i x_i - \sum_{\{a_i,a_j\}\in \mathbf{S}_p^2} \gamma_p^{ij} x_i x_j + \sum_{\{a_i,a_j,a_k\}\in \mathbf{S}_p^3} \gamma_p^{ijk} x_i x_j x_k; \ \forall p \in \mathbf{P} \tag{9}$$

One could choose one of the following three general procedures for tackling this problem:

1. Assess the utility function of the DM, to aggregate all objectives into one; then solve the single objective problem.

2. Solve a vector optimization problem to find the set of efficient solutions.

3. Use a *Goal Programming* (GP) approach.

Each of the above procedures has its own strengths and weaknesses (see Stewart (1992)). Assessing the DM's value function is quite difficult and involves a great deal of subjectivity. In vector optimization there are two main difficulties. First, the set of efficient alternatives is usually large and, after using this method, the DM must still select the best solution. Second, due to the nonconvexity of the decision space, obtaining the set of *unsupported* efficient solutions in a MCZO problem may be quite difficult (see Steuer (1986)). In fact, most MCZO procedures are applicable only to small problems (Ulungu and Teghem, 1994). It should be noted that when one wishes to select a subset of actions, the individually dominated actions should not be removed first, since there is a possibility that under some value functions, a combination including some dominated actions may be the best alternative (see the WDL example). Clearly, in the presence of interdependence, this kind of occurrence becomes more likely, leading to large numbers of decision variables in the MCZO problem. The authors have developed some techniques for removing those dominated actions that cannot possibly be included in the best combination of actions under any value function (Rajabi *et al.*, 1995).

GP is recognized to be the most popular method in MCDM because of its combination of validity and acceptance by decision makers. It has been used widely in many different areas of application. White (1990) surveyed multiple criteria optimization publications and found that 280 out of 400 papers involve variations on GP techniques. GP is usually used to select the best alternative according to the target and priority of objectives specified by the DM. The next section provides a variation of zero-one GP suitable for problems with interdependence of actions.

4 Goal Programming Formulation

The multiple objective program introduced in last section can be restated as a lexicographic goal programming formulation as follows:

$$\textbf{Lex} \quad \textbf{Min } \hat{\mathbf{d}} = (d_1^-, \ldots, d_p^-, \ldots, d_{|\mathbf{P}|}^-) \qquad (\mathbf{D}')$$

$$\textbf{Subject to}:$$

$$\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{|\mathbf{A}|} (-1)^{k+1} \sum_{S \in \mathbf{S}_p^k} \gamma_p^S \cdot (\prod_{a_i \in S} x_i) + d_p^- - d_p^+ = G_p; \forall p \in \mathbf{P}$$

$$x \in X,$$

$$x_i \in \{0,1\},$$

where d_p^- and d_p^+ are negative and positive deviations, G_p is the DM's aspiration level for criterion p, and the rest of notation is as described for Problem $\mathbf{D}$. The nonlinearity in Problem $\mathbf{D}'$ can be removed easily. For each $S = \{i_1, i_2, \ldots, i_k\} \in \mathbf{S}_p^k$ for any k and p, define $Y_S = Y_{i_1 i_2 \ldots i_k} = x_{i_1}.x_{i_2} \cdots x_{i_k}$ and add the two following constraints:

$$\begin{aligned} x_{i_1} + x_{i_2} + \cdots + x_{i_k} - Y_S &\leq k-1, \\ -x_{i_1} - x_{i_2} - \cdots - x_{i_k} + kY_S &\leq 0. \end{aligned}$$

In this way, a subset selection problem in MCDM under interdependence of actions can be formulated as a *linear* zero-one goal program.

The *lexicographic* zero-one goal program can be solved using *sequential* zero-one goal programming. This procedure allows one to use any zero-one program routine so that models of the same size as single objective zero-one problems can be solved (Ignizio, 1976; Ignizio and Cavalier, 1994).

Solving Problem $\mathbf{D}'$ or its equivalent linear program usually leads to a unique solution that depends on the aspiration levels and goal priorities. This solution may be dominated (Zeleny, 1982). In some situations, when the DM faces a complex situation such as a subset selection problem under interdependence of actions, she or he is willing to have more than one solution in order to reexamine and select among them. This way other criteria that cannot be stated as mathematical functions can be added, so that the DM can select the best alternative according to both qualitative and quantitative criteria. On the other hand, as described in the previous section, presenting all efficient alternatives to the DM through vector optimization may not be useful.

A GP technique can be employed to obtain efficient alternatives that in some sense represent the set of all efficient solutions. One way to find a set of GP-efficient solutions is Hannan's formulation. (For the definition of GP-efficient solution see (Hannan, 1980).) If all goals are bounded, the following vector maximization program provides some GP-efficient solutions:

$$
\begin{aligned}
&\textbf{Maximize}(Z_1, \ldots, Z_p, \ldots, Z_{|\mathbf{P}|}) \\
&\textbf{Subject to:} \qquad\qquad\qquad\qquad (\mathbf{D}^{''}) \\
&\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i \ + \sum_{k=2}^{|\mathbf{A}|} (-1)^{k+1} \sum_{S \in \mathbf{S}_p^k} \gamma_p^S \cdot Y_S \geq G_p - d_p^{-*} \quad \forall p \in \mathbf{P} \\
&\qquad x \in X \\
&\qquad x_i \in \{0, 1\},
\end{aligned}
$$

where d_p^{-*} is the optimum solution of the problem $\mathbf{D}^{'}$ for specified aspiration levels and Y_S is the binary variable substituted for $(\prod_{a_i \in S} x_i)$. Note that if the DM aspires to a difficult-to-achieve target, there may be no new solution for the above program; the only optimal solution to $\mathbf{D}^{'}$ would be the efficient alternative in Problem $\mathbf{D}^{''}$.

Consider WDL example described in Section 2. Assume that $(2, 2, 2)$ are the aspiration levels for the first, second and third criteria, respectively, and the priority of the criteria is (d_2^-, d_3^-, d_1^-). Then the solution of the Problem $\mathbf{D}^{'}$ is (x_1, x_2) with $(1.10, .05, .53)$ as negative deviations. For this target, the only efficient solution of Problem $\mathbf{D}^{''}$ is (x_1, x_2). However, with $(1.5, 1.5, 1.5)$ as target with the same priority, the optimal solution of $\mathbf{D}^{'}$ is (x_2, x_4); then, using Problem $\mathbf{D}^{''}$, the GP-nondominated alternatives are $(x_1, x_5), (x_3, x_5)$ and (x_1, x_4). Note that, in the latter situation, alternative (x_1, x_2) is not feasible in Problem $\mathbf{D}^{''}$, due to its poor performance on the first criterion.

It is noteworthy that if more than one DM is involved in the process of decision making, each DM can assign his or her own aspiration levels and the above program can be used to identify desirable alternatives for each DM. In this way, a compromise solution involving all DMs may be obtained. Also, an individual DM can assign different targets and priority levels; in this case, solving problems $\mathbf{D}^{'}$ and $\mathbf{D}^{''}$ in sequence will generally lead to GP-efficient alternatives, if any exist.

5 Conclusion

In this paper we have introduced the notion of interdependent actions in MCDM. Different methods for expressing and working with interdependency have been proposed. To the best of the authors' knowledge, there is no MCDM tool that can handle interdependence of actions efficiently. It has been shown that a subset selection problem under interdependence of actions can be modeled as a MCZO program and GP provides a reasonable procedure to deal with this complex situation. Hannan's (1980) method can be used to generate a subset of GP-efficient alternatives. An initial application to the choice of future sources for the water supply of the Regional Municipality of Waterloo, Ontario, Canada, is under development.

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An Interactive Fuzzy Decomposition Method for Large-Scale Multiobjective Nonlinear Programming Problems

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Abstract: In this paper, we propose an interactive fuzzy decomposition method for large-scale multiobjective nonlinear programming problems with the block angular structure to obtain the satisficing solution for the decision maker (DM). In the proposed method, after eliciting the membership function for each of the objective functions, the satisficing solution for the DM can be obtained on the basis of the dual decomposition method from Pareto optimal solution set by updating the reference membership values interactively. An interaction processes for a numerical example under the hypothetical DM are illustrated to indicate the feasibility of the proposed method.

1. Introduction

Recently, several kinds of decomposition methods have been proposed for large-scale nonlinear programming problems (LS-NLPs) with block angular structure [2,3], in which the original problem is decomposed into small-size subproblems and the optimal solution of the original problem can be derived by solving small-size problems efficiently.

As an extension of such methods in a multiple-criteria and fuzzy environment [4], the authors [5,6,7] have proposed fuzzy decomposition methods for large-scale multiobjective nonlinear programming problems (LS-MONLPs) with block angular structure to obtain the compromise solution for the decision maker (DM). In these methods, it is assumed that the fuzzy goals [4] for each of the objective functions can be quantified by eliciting the membership function and all of the membership functions can be integrated through the fuzzy decision or the add-operator [4]. However, in general, the DM may not necessarily approve to adopt the fuzzy decision or the add-operator because it is very difficult to identify the DM's fuzzy preference function explicitly.

In this paper, by incorporating the interactive methods [4] from multiobjective programming problems into the above methods, an interactive fuzzy decomposition method for LS-MONLPs is proposed to obtain the satisficing solution for the DM. In the proposed method, after determining the membership functions for the objective functions, for the reference membership values specified by the DM in his/her subjective manner, the corresponding

Pareto optimal solution is obtained on the basis of the dual decomposition method [2,3]. By updating the reference membership values iteratively, the satisficing solution for the DM can be derived efficiently from Pareto optimal solution set. Based on the proposed algorithm, computer program is developed in FORTRAN and an illustrated numerical example is demonstrated to clarify the feasibility of the proposed method.

2. Problem Formulation and Fuzzy Decomposition Algorithm

Consider a large-scale multiobjective nonlinear programming problem (LS-MONLP) with the following block angular structure:

$$\left.\begin{array}{llllll} \min & f_1(x) & \triangleq & f_{11}(x_1) & +f_{12}(x_2) & +\cdots & +f_{1p}(x_p) \\ \min & f_2(x) & \triangleq & f_{21}(x_1) & +f_{22}(x_2) & +\cdots & +f_{2p}(x_p) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \min & f_k(x) & \triangleq & f_{k1}(x_1) & +f_{k2}(x_2) & +\cdots & +f_{kp}(x_p) \end{array}\right\} \tag{1a}$$

subject to

$$\left.\begin{array}{lllllll} g_1(x) & \triangleq g_{11}(x_1) & +g_{12}(x_2) & +\cdots & +g_{1p}(x_p) & \leq 0 \\ g_2(x) & \triangleq g_{21}(x_1) & +g_{22}(x_2) & +\cdots & +g_{2p}(x_p) & \leq 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_m(x) & \triangleq g_{m1}(x_1) & +g_{m2}(x_2) & +\cdots & +g_{mp}(x_p) & \leq 0 \\ & h_1(x_1) & & & & \leq 0 \\ & & h_2(x_2) & & & \leq 0 \\ & & & \ddots & & \vdots \\ & & & & h_p(x_p) & \leq 0 \end{array}\right\} \tag{1b}$$

where $f_i(x), i = 1, \cdots, k$ are k distinct objective functions, $g_j(x), j = 1, \cdots, m$ are m coupling constraint functions, $x = (x_1, \cdots, x_p) \in R^n$ is the decision vector, $h_\ell(x_\ell)$, $\ell = 1, \cdots, p$ are r_ℓ dimensional vector-valued constraint functions with respect to $x_\ell \in R^{n_\ell}$, and $f_{i\ell}(x_\ell)$, $g_{j\ell}(x_\ell)$ $i = 1, \cdots, k,\ j = 1, \cdots, m,\ \ell = 1, \cdots, p$ are nonlinear functions with respect to x_ℓ.

By considering the vague nature of human's subjective judgements, it is quite natural to assume that the DM may have a fuzzy goal for each of the objective functions [4]. These fuzzy goals can be quantified by eliciting the corresponding membership functions $\mu_i(f_i(x)), i = 1, \cdots, k$ through the interaction with the DM.

Throughout this paper, we make the following assumptions.

Assumption 1. The constraint functions $h_\ell(x_\ell), \ell = 1, \cdots, p$ are twice continuously differentiable and the subset S defined as follows is compact and convex.

$$S_\ell \triangleq \left\{ x_\ell \in R^{n_\ell} \mid h_\ell(x_\ell) \leq 0 \right\}, \quad S \triangleq \prod_{\ell=1,\cdots,p} S_\ell \tag{2}$$

where the notation $\prod$ means the direct product.

Assumption 2. The objective functions $f_i(x), i = 1, \cdots, k$ and the coupling constraint functions $g_j(x), j = 1, \cdots, m$ are twice continuously differentiable and convex on the subset S.

Assumption 3. The membership function $\mu_i(f_i(x))$ for each of the objective functions is strictly monotone decreasing, twice continuously differentiable and concave on the subset $f_i(X)$, where X is the feasible set of LS-MONLP:

$$X \triangleq \left\{ x \in S \mid g_j(x) \leq 0, j = 1, \cdots, m \right\} \tag{3}$$

After determining the membership functions $\mu_i(f_i(x))$, $i = 1, ..., k$ for each of the objective functions $f_i(x)$, $i = 1, ..., k$, the DM is asked to specify his/her reference membership values for all the membership functions. For the DM's reference membership values:

$$\bar{\mu} = (\bar{\mu}_1, \cdots, \bar{\mu}_k), \tag{4}$$

the corresponding Pareto optimal solution is obtained by solving the following minimax problem [4] :

$$\min_{x \in X} \left\{ \max_{i=1,\cdots,k} \left\{ \bar{\mu}_i - \mu_i(f_i(x)) \right\} \right\} \tag{5}$$

From Assumption 3, the minimax problem (5) can be equivalently transformed to the following nonlinear programming problem.

Primal problem : $P(\bar{\mu})$

$$\min \quad x_{p+1} \tag{6a}$$

subject to

$$\left\{ f_{i1}(x_1) + f_{i2}(x_2) + \cdots + f_{ip}(x_p) \right\} - \mu_i{}^{-1}(\bar{\mu}_i - x_{p+1}) \leqq 0, \quad i = 1, \cdots, k \tag{6b}$$

$$g_{j1}(x_1) + \cdots + g_{jp}(x_p) \leq 0, \quad j = 1, \cdots, m \tag{6c}$$

$$x_\ell \in S_\ell, \ \ell = 1, \cdots, p, \quad x_{p+1} \in S_{p+1}(\bar{\mu}) \tag{6d}$$

where $S_{p+1}(\bar{\mu})$ is the feasible set of the auxiliary variable $x_{p+1} \in R^1$, which is defined as follows:

$$S_{p+1}(\bar{\mu}) = [\max_{1,\cdots,k} \bar{\mu}_i - 1, \ \max_{1,\cdots,k} \bar{\mu}_i] \tag{7}$$

The relationships between the optimal solutions of the primal problem $P(\bar{\mu})$ and the Pareto optimal solution can be characterized by the following theorem [4].

Theorem 1. (1) If x^* is a unique optimal solution to $P(\bar{\mu})$ for some $\bar{\mu}_i$, $i = 1, ..., k$, then x^* is a Pareto optimal solution.

(2) If x^* is a Pareto optimal solution, then there exist $\bar{\mu}_i,\ i = 1, ..., k$, such that x^* is an optimal solution to $P(\bar{\mu})$.

Note that it is very difficult to directly solve $P(\bar{\mu})$ from the computational aspect, because we focus on LS-MONLPs in which the number of decision variables and/or constraints are very large. In order to overcome such high dimensionality of $P(\bar{\mu})$, we introduce the dual decomposition methods [2,3].

According to Lasdon[3], since the objective function in $P(\bar{\mu})$ is not strictly convex, we cannot directly apply the dual decomposition method to $P(\bar{\mu})$. In order circumvent such difficulty, instead of dealing with $P(\bar{\mu})$, we consider the following slightly modified version, where the term ρx_{p+1}^2 is added artificially at the objective function of $P(\bar{\mu})$, and $\rho > 0$ is a sufficiently small and positive constant number.

Modified primal problem : $P'(\bar{\mu})$

$$\min \quad x_{p+1} + \rho x_{p+1}^2 \tag{6a$'$}$$

subject to (6b), (6c), (6d)

It should be noted here that, for sufficiently small positive $\rho > 0$, the optimal solution of the modified primal problem $P'(\bar{\mu})$ is approximately equal to the optimal one of the primal problem $P(\bar{\mu})$.

For $P'(\bar{\mu})$, the Lagrangian function can be defined as the following form, where $\lambda = (\lambda_1, \cdots, \lambda_k)$, $\pi = (\pi_1, \cdots, \pi_m)$ are the vectors of the Lagrange multipliers corresponding to the constraints (6b) and (6c).

$$L(\bar{\mu} : x, x_{p+1}, \lambda, \pi) \triangleq \sum_{\ell=1}^{p} L_\ell(x_\ell, \lambda, \pi) + L_{p+1}(\bar{\mu} : x_{p+1}, \lambda) \tag{8}$$

where

$$L_\ell(x_\ell, \lambda, \pi) \triangleq \sum_{i=1}^{k} \lambda_i f_{i\ell}(x_\ell) + \sum_{j=1}^{m} \pi_j g_{j\ell}(x_\ell), \quad \ell = 1, \cdots, p \tag{9a}$$

$$L_{p+1}(\bar{\mu} : x_{p+1}, \lambda) \triangleq x_{p+1} + \rho x_{p+1}^2 - \sum_{i=1}^{k} \lambda_i {\mu_i}^{-1}(\bar{\mu}_i - x_{p+1}) \tag{9b}$$

Using the above Lagrangian function, the corresponding dual problem can be formulated as follows.

Dual problem : $D(\bar{\mu})$

$$\max_{(\lambda,\pi)\in U(\bar{\mu})} \quad w(\bar{\mu} : \lambda, \pi) \tag{10}$$

where the dual function $w(\bar{\mu} : \lambda, \pi)$ and the domain $U(\bar{\mu})$ are defined as bellow.

$$\left.\begin{array}{rcl} w(\bar{\mu}:\lambda,\pi) & \triangleq & \min \displaystyle\sum_{\ell=1}^{p} L_\ell(x_\ell,\lambda,\pi) + L_{p+1}(\bar{\mu}:x_{p+1},\lambda) \\ & & \text{subject to} \\ & & x_\ell \in S_\ell,\ \ \ell = 1,\cdots,p,\ \ x_{p+1} \in S_{p+1}(\bar{\mu}) \end{array}\right\} \tag{11}$$

$$U(\bar{\mu}) \triangleq \Big\{(\lambda,\pi) \in R^{k+m} \mid \lambda \geq 0, \pi \geq 0, \text{the dual function } w(\bar{\mu}:\lambda,\pi) \text{ exists.}\Big\} \tag{12}$$

The relationships between $P'(\bar{\mu})$ and the corresponding dual problem $D(\bar{\mu})$ are shown by the following theorems.

Theorem 2. Under Assumptions 1-3, the domain U of the dual function $w(\bar{\mu}:\lambda,\pi)$ is given as follows and and the dual function $w(\bar{\mu}:\lambda,\pi)$ is concave on U.

$$U = \Big\{(\lambda,\pi) \in R^{k+m} \ \mid \lambda \geq 0,\ \pi \geq 0\Big\}, \tag{13}$$

Theorem 3. Under Assumptions 1-3, the dual function $w(\bar{\mu}:\lambda,\pi)$ is differentiable with respect to $(\lambda,\pi) \in U$, and the partial differentiable coefficients are given by:

$$\frac{\partial w}{\partial \lambda_i} = \sum_{\ell=1}^{p} f_{i\ell}(x_\ell) - {\mu_i}^{-1}(\bar{\mu}_i - x_{p+1}), \quad i = 1,\cdots,k, \tag{14}$$

$$\frac{\partial w}{\partial \pi_j} = \sum_{\ell=1}^{p} g_{j\ell}(x_\ell), \quad j = 1,\cdots,m. \tag{15}$$

Theorem 4. Under Assumptions 1-3, the optimal solution of $P'(\bar{\mu})$ coincides with the optimal one of the corresponding dual problem $D(\bar{\mu})$.

From the above theorems, we can obtain the optimal solution of $P'(\bar{\mu})$ by solving the dual problem $D(\bar{\mu})$.

As shown in (8) and (9), the Lagrangian function $L(\cdot)$ is decomposable with respect to $(x_1,\cdots,x_p,\ x_{p+1})$ for some fixed Lagrange multipliers (λ,π). Therefore, the dual problem $D(\bar{\mu})$ can be decomposed into the following $(p+1)$ small-size problems for some fixed Lagrange multipliers (λ,π).

Subproblem $\mathrm{A}_\ell(\lambda,\pi)$ $\quad (0 \leq \ell \leq p)$

$$w_\ell(\lambda,\pi) \triangleq \min_{x_\ell \in S_\ell} \ L_\ell(x_\ell,\lambda,\pi) \tag{16}$$

Subproblem $\mathrm{A}_{p+1}(\bar{\mu}:\lambda)$

$$w_{p+1}(\bar{\mu}:\lambda) \triangleq \min_{x_{p+1} \in S_{p+1}(\bar{\mu})} \ L_{p+1}(\bar{\mu}:x_{p+1},\lambda) \tag{17}$$

[Algorithm 2]
Then, from the definition of the dual function (11), it holds that

$$w(\bar{\mu} : \lambda, \pi) = \sum_{\ell=1}^{p} w_\ell(\lambda, \pi) + w_{p+1}(\bar{\mu} : \lambda). \tag{18}$$

In order to improve the value of the dual function $w(\lambda, \pi)$, from Theorems 2 and 3, we can adopt the simple steepest descent method by using the partial differentiable coefficients (14) and (15).

From the above discussions, for some reference membership values $\bar{\mu} = (\bar{\mu}_1, \cdots, \bar{\mu}_k)$ specified by the DM, we can construct the fuzzy decomposition algorithm to obtain the optimal solution of $P'(\bar{\mu})$.

[Algorithm 1]

Step 1: Set the iteration index $t = 1$, and set the initial values of the Lagrange multipliers $(\lambda^t, \pi^t) \geqq 0$ appropriately.
Step 2: Solve the subproblems $A_\ell(\lambda^t, \pi^t), \ell = 1, \cdots, p$ and $A_{p+1}(\bar{\mu} : \lambda^t)$ and obtain the optimal solution $x_\ell(\bar{\mu} : \lambda^t, \pi^t)$, $\ell = 1, \cdots, p+1$. In the following, for simplicity, denote $x_\ell^t = x_\ell(\bar{\mu} : \lambda^t, \pi^t), \ell = 1, \cdots, p+1$.
Step 3: Obtain the dual function value $w(\bar{\mu} : \lambda^t, \pi^t)$ and compute the direction vectors according to the following formulae:

$$d_i^1 = \begin{cases} \dfrac{\partial w}{\partial \lambda_i^t} = \displaystyle\sum_{\ell=1}^{p} f_{i\ell}(x_\ell^t) - {\mu_i}^{-1}(\bar{\mu}_i - x_{p+1}^t) & ; \lambda_i^t > 0, \\ \max\left\{ 0, \dfrac{\partial w}{\partial \lambda_i^t} \right\} & ; \lambda_i^t = 0, \end{cases}$$

$$d_j^2 = \begin{cases} \dfrac{\partial w}{\partial \pi_j^t} = \displaystyle\sum_{\ell=1}^{p} g_{j\ell}(x_\ell^t) & ; \pi_j^t > 0, \\ \max\left\{ 0, \dfrac{\partial w}{\partial \pi_j^t} \right\} & ; \pi_j^t = 0, \end{cases}$$

where $i = 1, \cdots, k$ and $j = 1, \cdots, m$.
Step 4: For the given search direction vectors $D^1(\lambda^t) = (d_1^1, d_2^1, \cdots, d_k^1)$ and $D^2(\pi^t) = (d_1^2, d_2^2, \cdots, d_m^2)$, solve the following one-dimensional search problem to obtain the optimal step size α^t,

$$\begin{array}{l} \max \quad w(\bar{\mu} : \lambda^t + \alpha D^1(\lambda^t), \pi^t + \alpha D^2(\pi^t)) \\ \text{subject to} \\ \lambda^t + \alpha D^1(\lambda^t) \geq 0, \ \pi^t + \alpha D^2(\pi^t) \geq 0, \ \alpha \geq 0. \end{array}$$

Step 5: If $\alpha^t \approx 0$, then stop. Otherwise, set $\lambda^{t+1} = \lambda^t + \alpha^t D^1(\lambda^t), \pi^{t+1} = \pi^t + \alpha^t D^2(\pi^t)$, $t = t+1$, and return to Step 2.

It should be noted here that, from Theorem 1, the optimal solution of $P'(\bar{\mu})$ is not necessarily Pareto optimal, if the uniqueness of the optimal solution is not verified. Therefore, in order to guarantee the Pareto optimality of the optimal solution of $P'(\bar{\mu})$, we need to solve the Pareto optimality test problem:

Pareto optimality test problem : $T(x^*)$

$$\max\left\{\epsilon_1 + \cdots + \epsilon_k\right\} \tag{19a}$$

subject to

$$f_{i1}(x_1) + \cdots + f_{ip}(x_p) + \epsilon_i = f_i(x^*), \quad i = 1, \cdots, k, \tag{19b}$$

$$g_{j1}(x_1) + \cdots + g_{jp}(x_p) \leq 0, \quad i = 1, \cdots, m, \tag{19c}$$

$$x_\ell \in S_\ell, \ \ell = 1, \cdots, p, \quad \epsilon_i \geq 0, \ i = 1, \cdots, k. \tag{19d}$$

where $x*$ is the optimal solution of $P'(\bar{\mu})$. Similar to Algorithm 1, we can construct the algorithm based on the decomposition method to solve $T(x*)$. But, due to limitations of space, it is omitted.

3. Trade-off Rates and An Interactive Algorithm

The DM must now either be satisfied with the current Pareto optimal solution, or act on this solution by updating his/her reference membership values. In order to help the DM express his/her degree of preference, trade-off information between a standing membership function $\mu_1(f_1(x))$ and each of the other membership functions is very useful. Such a trade-off information between $\mu_1(f_1(x))$ and $\mu_i(f_i(x))$ for each $i = 2, ..., k$ is approximately obtainable since it is related to the strict positive Lagrange multipliers of $P'(\bar{\mu})$.

In the following, assume that the optimal solution (x^*, x^*_{p+1}) of $P'(\bar{\mu})$ satisfies the following conditions[1].

(1) (x^*, x^*_{p+1}) is a regular point of the constraints of $P'(\bar{\mu})$.

(2) the second-order sufficiency conditions of optimality are satisfied at (x^*, x^*_{p+1})

(3) there are no degenerate constraints at (x^*, x^*_{p+1}).

Moreover, assume all of the constraints (6b) are active, *i.e.*, $\lambda_i > 0, i = 1, \cdots, k$. Then, the following relation holds.

$$-\frac{\partial \mu_1(f_1(x^*))}{\partial \mu_i(f_i(x^*))} \approx \frac{\partial \mu_1(f_1(x^*))}{\partial f_1(x^*)} \frac{\lambda_i}{\lambda_1} \left\{ \frac{\partial \mu_i(f_i(x^*))}{\partial f_i(x^*)} \right\}^{-1}, \ i = 2, \ldots, k. \tag{20}$$

Now, we can construct the interactive algorithm to derive the satisficing solution for the DM from the Pareto optimal solution set, which are obtained by applying Algorithm 1.

Step 1: Elicit a membership function $\mu_i(f_i(x))$ from the DM for each of the objective functions.
Step 2: Set the initial reference membership values $\bar{\mu}_i^t = 1$, $i = 1, \ldots, k$, and set the iteration index $t = 1$.
Step 3: By applying Algorithm 1 based on the decomposition method, solve the modified primal problem $P'(\bar{\mu})$ and the corresponding Pareto optimality test problem. Then obtain the Pareto optimal solution and the approximate values of the trade-off rates between the membership functions.
Step 4: If the DM is satisfied with the current values of the membership functions, stop. Then the current Pareto optimal solution is the satisficing solution for the DM. Otherwise, ask the DM to update the current reference membership values $\bar{\mu}^t$ to the new reference membership values $\bar{\mu}^{t+1}$ by considering the current values of the membership functions and the approximate values of the trade-off rates between the membership functions. Set $t = t + 1$ and return to Step 3.

4. Numerical Example

Based on the proposed algorithm, we have developed a computer program in FORTRAN to solve the LS-MONLP. To show the feasibility and efficiency of both the proposed algorithm and the developed computer program, consider the following two-objective nonlinear programming problem with the block angular structure.

$$\begin{array}{llllll} \min & f_1(x) & \triangleq & f_{11}(x_1) & +f_{12}(x_2) & = (x_1-5)^2+(x_2-3)^2 \\ \min & f_2(x) & \triangleq & f_{21}(x_1) & +f_{22}(x_2) & = (x_1-2)^2+(x_2-4)^2 \end{array}$$

subject to

$$\begin{array}{llll} g_1(x) & \triangleq g_{11}(x_1)+ & g_{12}(x_2) & = (1/7)x_1+(1/12)x_2-1 \leq 0 \\ g_2(x) & \triangleq g_{21}(x_1)+ & g_{22}(x_2) & = (1/18)x_1+(1/6)x_2-1 \leq 0 \\ & 0 \leq x_1 \leq 7, & & 0 \leq x_2 \leq 6 \end{array}$$

For the above problem, suppose the hypothetical DM establishes the simple linear membership functions $\mu_1(f_1(x)) = 1-(1/6)f_1(x)$, $\mu_2(f_2(x)) = 1-(1/8)f_2(x)$ respectively. Then the modified primal problem $P'(\bar{\mu})$ is formulated as follows.

$$\min \quad -x_3 + \rho x_3^2$$

subject to

$$\begin{array}{ll} (x_1-5)^2+(x_2-3)^2-6(1-\bar{\mu}_1+x_3) & \leq 0 \\ (x_1-2)^2+(x_2-4)^2-8(1-\bar{\mu}_1+x_3) & \leq 0 \\ (1/7)x_1+(1/12)x_2-1 \leq & \leq 0 \\ (1/18)x_1+(1/6)x_2-1 & \leq 0 \\ 0 \leq x_1 \leq 7,\ 0 \leq x_2 \leq 6,\ x_3 \in S_{p+1}(\bar{\mu}) & \end{array}$$

where the parameter $\rho = 0.01$.

For the reference membership values $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2)$ specified by the DM subjectively, formulate the corresponding dual problem $D(\bar{\mu})$. $D(\bar{\mu})$ can be decomposed into the following three small-size subproblems.

Subproblem $A_1(\lambda_1, \lambda_2, \pi_1, \pi_2)$

$$\min_{0 \leq x_1 \leq 7} \left\{ \lambda_1(x_1 - 5)^2 + \lambda_2(x_1 - 2)^2 + \pi_1(x_1/7 - 1/2) + \pi_2(x_1/18 - 1/2) \right\}$$

Subproblem $A_2(\lambda_1, \lambda_2, \pi_1, \pi_2)$

$$\min_{0 \leq x_2 \leq 6} \left\{ \lambda_1(x_2 - 3)^2 + \lambda_2(x_2 - 4)^2 + \pi_1(x_2/12 - 1/2) + \pi_2(x_2/6 - 1/2) \right\}$$

Subproblem $A_3(\bar{\mu}_1, \bar{\mu}_2 : \lambda_1, \lambda_2)$

$$\min_{0 \leq x_3 \leq 1} \left\{ x_3 + 0.01x_3^2 - 6\lambda_1(1 - \bar{\mu}_1 + x_3) - 8\lambda_2(1 - \bar{\mu}_2 + x_3) \right\}$$

According to Step 2 of Algorithm 2, set the initial reference membership values $\bar{\mu}_i = 1, i = 1, 2$. At Step 3 of Algorithm 2, the optimal solution of $P'(\bar{\mu})$ is obtained by applying Algorithm 1. Table 1 shows the iterative processes in Algorithm 1, where the initial reference membership values are set as $\bar{\mu}_i = 1, i = 1, 2$. Since the hypothetical DM is not satisfied the current values of the membership functions, the DM updates his/her reference membership values as $\bar{\mu}_1 = 0.8, \bar{\mu}_2 = 0.6$ (Step 4 of Algorithm 2). In this example, at the 3rd iteration, the satisficing solution for the DM is derived. The whole interactive processes in Algorithm 2 are summarized in Table 2.

5. Conclusion

In this paper, an interactive fuzzy decomposition method for large-scale multiobjective nonlinear programming problems with the block angular structure has been proposed to obtain the satisficing solution for the DM. In the proposed method, assuming that the DM has a fuzzy goal for each of the objective functions, if the DM specifies the reference membership values for the membership functions subjectively, the corresponding Pareto optimal solution is obtained on the basis of the decomposition method. The satisficing solution for the DM can be derived efficiently from Pareto optimal solution set by updating the reference membership values iteratively. Based on the proposed algorithm, a computer program has been written in FORTRAN and an illustrative numerical example demonstrated the feasibility and efficiency of the proposed method. Applications of the proposed method will require further investigation.

Table 1. The processes of Algorithm 1 for $(\bar{\mu}_1, \bar{\mu}_2) = (1, 1)$

t	λ_1	λ_2	x_1	x_2	$w'(\bar{\mu}_1, \bar{\mu}_2 : \lambda_1, \lambda_2)$
1	0.100000	0.100000	3.5000	3.5000	0.11000
10	0.078704	0.066886	3.6218	3.4594	0.36024
50	0.078072	0.067355	3.6105	3.4632	0.36027
100	0.077932	0.067449	3.6082	3.4639	0.36027

Table 2. The interactive processes of Algorithm 2

t	$\bar{\mu}_1$	$\bar{\mu}_2$	$\mu_1(f_1(x))$	$\mu_2(f_2(x))$	$-\partial\mu_1/\partial\mu_2$
1	1.0	1.0	0.64126	0.64081	1.1540
2	0.8	0.6	0.74014	0.54226	0.87002
3	0.7	0.6	0.62985	0.59286	1.0029

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Basic Concepts in Derivation of Fuzzy Multiattribute Utility Functions

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Abstract. This paper concerned with basic concepts in fuzzy decision analysis. Conditions and techniques required in construction of the fuzzy utility function and its multiobjective extensions are discussed. Fuzzy lottery technique with fuzzy certainty equivalent based on possibility measure is presented in contrast with the classical probabilistic lottery technique. Fuzzy preference independence assumptions are examined for derivation of the fuzzy multiattribute utility functions.

Key words. Fuzzy utility function, Fuzzy multiattribute utility function, Possibility measure, Fuzzy preference independence.

1 Introduction

This paper intends to discuss the basic concepts in constructing the fuzzy utility function and its multiobjective extensions in fuzzy decision analysis.

Decision analysis has been constructed on the von Neumann-Morgenstern (v-N-M) expected utility (EU) theorem and the subjective probability theory, which opens the way for heuristic construction of the numerical utility function in the statistical decision theory (e.g., Pratt, Raiffa & Schlaifer [16], Raiffa [17], Schlaifer [20], DeGroot [2]). Decision problems in the real world, however, are including more and more complex and ambiguous properties. The presentation of human preferences is not unique but usually relative as "the problem of a degree." In addition, empirical researches have presented some contradictions to the EU hypothesis. The decision maker (DM) is alleged to be difficult to reveal his/her preferences precisely in the additive expected utility function form. For coping with these situations, studies on the nonadditive probability have early started (e.g. Davidson and Suppes[1], Fishburn [5][6]).

On the other hand, DM is facing the cognitive ambiguity in the real-world decision problems which can be treated properly with fuzzy set theory developed since Zadeh[23]. In this scientific environment, some people are pursuing the use of the fuzzy measures such as the fuzzy integral in the place of the probability measure in the EU theory(Mathiew-Nicot [12], Schmeidler [19]) often with no intention to construct the fuzzy numerical utility functions.

This paper discusses basic concepts in deriving the fuzzy utility function (FUF) and its multiattribute extensions. In the next section, some backgrounds are discussed and the use of the fuzzy integral in constructing the fuzzy utility function is examined. In Section 3, an alternative approach for constructing the FUF and the fuzzy multiattribute utility function (FMUF) based on the possibility measure is presented. A method for heuristic construction of the FUF and FMUF is discussed with a simple numerical example. The representation theorems of FMUF are presented. Finally some concluding remarks are provided for this research .

2 Aggregation With/Without Additivity: Some Backgrounds

2.1 Probability Measure and the Expected Utility Hypothesis

Let Ω be a set of universe, and Φ be a σ-field. Then in the measurable space (Ω, Φ), the probability measure $p. \Phi \rightarrow [0,1]$ is defined as a real-valued function. The probability space is defined with (Ω, Φ, P).

The EU theorem is constructed in the probability space. Let Ω be a universal set of the states of nature. An event $s_j \in \Omega$, j=1, $\cdots$, n, is an uncontrollable and unpredictable state with certainty. Let p_j be the probability measure for an event s_j, which we simply call the *probability*. The probability p_j is a number that represents a degree of belief for the occurrence of an event s_j. Let X be a set of objects to be evaluated by DM. Let x be a nonfuzzy set of the measures for elements in X which are called the *attribute*. Each element in x is treated as a random variable that is a measurable function mapping a set of all events Ω into a set of definite values. The x_j (s_j), $j = 1, \cdots, n$, shows a value of a consequence of an act taken by DM, i.e., a value of an attribute, when an event s_j occurs.

The EU hypothesis is defined on the probability space and described under the v-N-M system (e.g., Luce and Raiffa [11], Fishburn [6]) as

[I] $$x_s \succ x_r \Leftrightarrow u_s(x_s) \geq u_r(x_r) \quad \text{(Order preserving property)} \tag{1}$$

[II] $$Eu = \sum_{j=1}^{n} p_j(s_j)u(x_j) \quad \text{(Linearity)} \tag{2}$$

$$= u(\sum_{j=1}^{n} p_j(s_j)x_j) \tag{3}$$

The v-N-M type numerical utility function is order-preserved up to positive linear transformations.

The recognition for the numerical utility function to be derived with the reference lottery technique has led to construction of the heuristic method for the numerical utility function. The reduction principles suggest this method.

Reduction principle of preferences to probability: The evaluation of the preferences of DM is reduced to the evaluation of the probability.

Reduction Principle of a Lottery: Let ℓ be a lottery: Let $\hat{x}$ be a value of the certainty equivalent (CE) to which the lottery is indifferent. Let a lottery ℓ be evaluated with the EU value. Then the evaluation of the lottery is reduced to that of the corresponding CE. In particular, the EU value of the reference lottery ℓ expresses a utility value which corresponds to CE (Fig.1A).

While some challenges to the v-N-M EU hypothesis have been raised to its additivity, the recognition of the *ambiguity* in human decisions has led to the intention to mitigate the crisp property of the probability measure. In the next section, this approach is discussed.

2.2 Fuzzy Measure and Fuzzy Integral

Let μ be a function from a set $\wp(\Omega)$ of nonfuzzy subsets in Ω to [0,1]. Then μ is the fuzzy measure in the measurable space (Ω, Φ) when the following properties are satisfied (Dubois and Prade [4]).

[1] $\mu(\varnothing)=0,\ \ \mu(\Omega)=1$

[2] $\forall A_s, A_r \in \wp(\Omega),\quad A_s \subseteq A_r \Rightarrow \mu(A_s) \le \mu(A_r)$

[3] $\forall s \in \mathrm{N},\quad A_s \in \wp(\Omega)$ and $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ or

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \ \Rightarrow\ \lim_{s\to\infty}\mu(A_s) = \mu(\lim_{s\to\infty} A_s)$$

The property [1] is the normalization condition. The property [2] expresses the monotonicity and the property [3] shows the continuity. The fuzzy measure does not assume the additivity and thus is a generalization of the probability measure. The fuzzy measurable space is defined with (Ω, Φ, μ).

The extension of the probability measure to the fuzzy measure has two directions. One is based on the fuzzy set theory and is concerned with the conceptual ambiguity of the set for whose members the grades of the membership are assigned. Another one is concerned with its nonadditive property as an aggregation operator.

The Choquet integral is one of the fuzzy integral whose mathematical properties have been well-scrutinized (Murofushi and Sugeno[13], Grabisch [8]). The Choquet integral of a function $f(x)$ on $x \in \Omega$ with the fuzzy measure μ is represented as

$$\mathscr{C}_{\mu}(f(x_1), \ldots, f(x_n)) \underset{=}{\Delta} \sum_{j=1}^{n}(f(x_{\{j\}}) - f(x_{\{j-1\}}))\mu(A_{\{j\}}), \tag{4}$$

or in an continuous form,

$$\mathscr{C}_{\mu}(f(x)) \underset{=}{\Delta} (\mathrm{C})\int f d\mu \tag{5}$$

$$\underset{=}{\Delta} (\mathrm{C})\int_0^{\infty} \mu(\{x | f(x) \ge \alpha\}) d\alpha. \tag{6}$$

The Choquet integral is used in two ways. One is to use the Choquet integral in the place of the Lebesgue integral as an aggregation operator. Schmeidler (e.g.,[19]) first presented this approach. Under the assumptions of (i)weak order, (ii)comonotonic independence, (iii)continuity, (iv) monotonicity, and (v)nondegeneracy, the existence of the nonadditive probability ν and the affine real-valued function υ (unique up to positive linear transformations) is presented with the following form,

$$f \succ g \Leftrightarrow (\mathrm{C})\int_{s} \upsilon(f(\cdot))d\nu \geq (\mathrm{C})\int_{s} \upsilon(g(\cdot))d\nu \tag{7}$$

where $f(\cdot)$ denotes an act on a state. An act maps the states to outcomes. f and g are in the all Σ-measurable bounded set of acts. Eq.(7) defines a nonadditive EUF with the nonadditive measure ν in terms of the Choquet integral. In Eq.(7), the function υ on the consequence $x = f(\cdot)$ defines a utility function which is not fuzzified. Compare Eq.(7) with the additive EUF defined with the probability measure p in the continuous form under Savage's axioms ([18]) satisfied.

$$f \succ g \Leftrightarrow \int_{s} u(f(s))dP(s) \geq \int_{s} u(g(s))dP(s) \tag{8}$$

The fuzzy integral as the aggregation operator also is used to define a FMUF. The basic idea is to construct the following equation under the satisfaction of the idempotency, monotonicity and continuity properties (Grabisch [8]).

$$u(x_1,\ldots,x_n) = \mathscr{C}_{\mu}(x_1,\ldots,x_n) \tag{9}$$

where x_i $i = 1, \ldots, n$, is an attribute value. In this definition, the concern is with the fuzzification of the aggregation operator in assessing the multiplicity of attributes and thus the utility evaluation in fuzzy terms is lacked. In addition, the aggregation operator is not capable to articulate the value tradeoffs among attributes, although it is the core of multiobjective decision problems.

A device to evaluate a FUF for one attribute has been presented by Ponsard[15].The preference for an alternative is revealed with the membership function $\mu_X(x_i) \in [0,1]$ where x_i is an element of an attribute set X in Ω. With this concept, a FUF is defined as

$$u(x_i) = \mu_X(x_i) \tag{10}$$

As seen, the utility function is defined as the mapping from each attribute in the set X to an element of the interval [0,1]. While the preference order is assessed for the attributes $x_\bullet$, the concept of the utility function in Eq.(10) is constructed as an ordinal number with a degree of confidence.

In the next section, we will discuss an alternative approach for constructing FUF and its multiattribute extension based on the possibility measure.

3 Construction of the Fuzzy Numerical Utility Function and its Multiattribute Extension

3.1 Construction of the Fuzzy Numerical Utility Function

The possibility measure is one of the fuzzy measure in the fuzzy set theory (Zadeh[27]) and the use of the possibility measure for decision analysis has been suggested by Debois and Prade ([4]).

Let a subset $A_j \subset \Omega$ be nonfuzzy. A possibility measure Π is a function from $\wp(\Omega)$ to the real numbers [0,1] such that, for any $A_j \subset \Omega$,

[I] $\Pi(\emptyset) = 0$,

[II] $\Pi(\Omega) = 1$,

[III] $\Pi(\cup_j A_j) = \sup_j \Pi(A_j)$ for any collection of A_j.

The possibility measure Π is constructed from a possibility distribution function $\pi: \Omega \rightarrow [0,1]$ with $\sup_{u \in \Omega} \pi(u) = 1$ (normality condition) as

$$\forall A \subseteq \Omega \quad \Pi(A) = \sup_{u \in A} \pi_U(u) \tag{11}$$

where U is a defined set, or a proposition. The possibility distribution function $\pi_U(u)$ shows a degree of the possibility "$U = u$ is true" and is defined to be numerically equal to the membership function μ of a fuzzy set G that is defined as a restriction, or a predicate, for u.

$$\pi_U(u) \triangleq \mu_G(u). \tag{12}$$

A possibility set is represented with the possibility distribution Π_U on U which is expressed in terms of the fuzzy set.

$$\Pi_U = \mu_G(u_1) / u_1 + \mu_G(u_2) / u_2 + \cdots + \mu_G(u_n) / u_n \tag{13}$$

Note that Eq.(13) shows a possibility mixture of a utility (Fig.1B).

The *fuzzy number* is defined as a fuzzy subset of the real line R (Dubois and Prade [3]) and expressed with the possibility distribution Eq.(13). The fuzzy utility values are assessed with the fuzzy number having the possibility distribution Π_U. The possibility distribution function $\pi_U(u)$ expresses the possibility, or the degree of truth, for an assessed utility value u to be his/her intrinsic utility-value set U. From Eq.(12), the possibility for an assessed utility value u to be his/her "true" utility value is evaluated with the grade of the membership of u in the fuzzy set G of his/her utility value u.

The proposition for constructing a FUF with the assessed utility values is presented under the following assumptions.

[I] *Fuzzy weak order*: Fuzzy connectivity and fuzzy transitivity in the preference relation for an attribute hold.

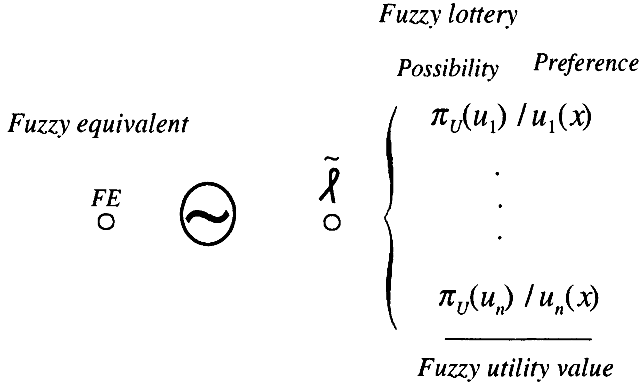
[II] *Independence rule of the possibility for utility*: The evaluation of the possibility for a utility does not depend on the evaluation of the utility value.

[III] *Monotonicity of the fuzzy preference*: This property is implied in the use of the fuzzy measure (Property[2]) for evaluation of the utility .

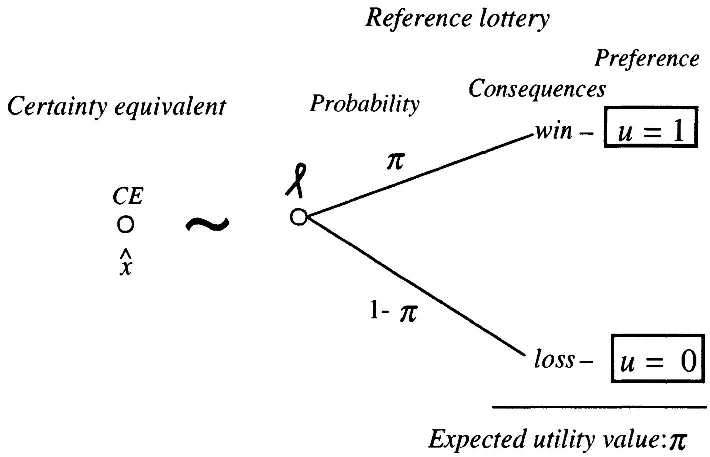
[IV] *Continuity of preference*: the continuity property of preferences is satisfied in the possibility set of the utility by the definition of the fuzzy measure (Property[3]).

Proposition for the construction of FUF. Define the attribute as a measure of performance levels of an object. Under the assumptions [I]-[IV], the possibility distribution Π_U of the utility values represents an FUF on an attribute.

The method for constructing FUF for an attribute can be discussed on the fuzzy lottery technique. Let a fuzzy utility value be expressed as a fuzzy



(B) Fuzzy lottery technique and fuzzy equivarence for derivation of the fuzzy utility function



Utility Value: $U(\hat{x}) \triangleq \pi$: a probability value

(A) Canonical lottery technique and certainty equivalence for derivstion of the utility function

Fig.1. Lottery techniques and equivalence experiments for derivation of utility functions

number with a possibility distribution. We call the possibility distribution for a fuzzy utility as a *possibility mixture*($\pi(u)$, u), or the *fuzzy lottery*, in the similitude of the probability mixture(p, u) or the classical reference lottery for the probabilistic utility. The *fuzzy equivalent* (*FE*) to the fuzzy lottery is assessed for deriving FUF. We propose the reduction principles for FUF.

Reduction principle of fuzzy preferences to possibility. The evaluation of the fuzzy preferences is reduced to the evaluation of the possibility distribution.

Reduction principle of the fuzzy lottery: Assume a fuzzy utility be assessed as a fuzzy number with a possibility distribution. Then evaluation of the fuzzy lottery for representing a fuzzy utility is reduced to the evaluation of a corresponding fuzzy equivalent (*FE*).

As a result, the FUF is derived with a set of the correspondence of a fuzzy lottery and an *FE*. Fig.2 shows the derivation of FUF based on the fuzzy lottery technique.

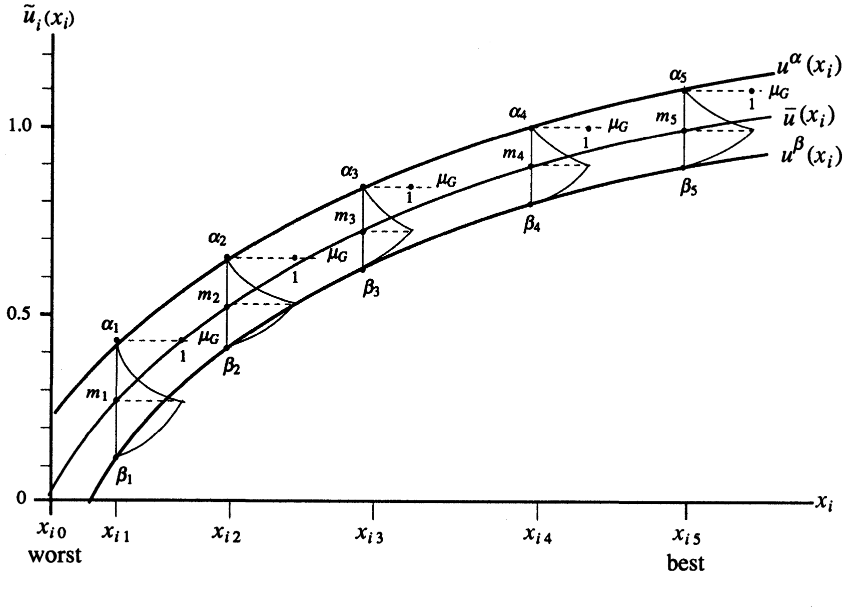


Fig. 2. Derivation of Fuzzy Utility Function: $\tilde{u}_i \stackrel{\Delta}{=} (m, \alpha, \beta)_i$

3.2 Requirements for Constructing the Fuzzy Multiattribute Utility Function and Its Representation Principle

To construct FMUF based on the FUF, some assumptions are required.

[1] *Fuzzy mutual preference Independence.* Let $\boldsymbol{U}$ be a universal set of utility values and Let $u_i \in \boldsymbol{U}$, $i = 1, ..., m$, be a utility value for a nonfuzzy attribute x_i. The fuzzy restriction $\boldsymbol{R}$ for a utility $u_i(x_i)$ to be a FUF is represented as a

fuzzy set $\mathbb{G}$ with $\mathbb{R}(u) = \mathbb{G}$. The utilities $u_1, ..., u_m \in \mathbb{U}$ are the noninteractive variables under the fuzzy restriction $\mathbb{R}(u_1, ..., u_m)$ if and only if $\mathbb{R}(u_1, ..., u_m)$ is separable (Zadeh[25]), i.e.,

$$\mathbb{R}(u_1, ..., u_m) = \mathbb{R}(u_1) \times \cdots \times \mathbb{R}(u_m). \tag{14}$$

The mutual noninteraction of the utility values generating $\mathbb{G}$ defines the fuzzy mutual preference independence for the attributes $x_i \in \mathbb{X}.$, $i = 1, ..., m$.

[2] *Mutual independence of the possibility evaluation.* The evaluation of the possibility distribution for a fuzzy utility $u_i(x_i)$ is mutually independent for $i = 1, ... m$..

[3] *Use of the L-R type fuzzy number for representing a value of FUF.* A value of FUF is assessed with the *L-R* type fuzzy number $\tilde{u} = (\bar{u}, u_\alpha, u_\beta)$ where $\bar{u}$ is the mean value of the fuzzy utility, and u_α and u_β are its left and right spreads respectively. the operation rules for the *L-R* type fuzzy number have been established by Dubois and Prade([3], [4]).

[4] *Articulation of the fuzzy tradeoffs among attributes.* Under the assumptions of the fuzzy mutual preference and possibility independence, fuzzy tradeoffs among every pair of attributes are assessed. In the two-attribute space, a value of one attribute x_j is assessed as indifferent to the multiple values of the other attribute x_i, which are represented with a possibility distribution Π_x and, as an approximation, with an *L-R* type fuzzy number $\tilde{x}_i = (\bar{x}, x^\alpha, x^\beta)_i$. With this one-to-many, fuzzy tradeoff experiment, the fuzzy scaling constants $\tilde{k}_i$, $i = 1, ..., m$, for constructing FMUF in the "aggregation" of FUF are derived as the *L-R* type fuzzy number $\tilde{k}_i = (\bar{k}, k^\alpha, k^\beta)_i$.

[5] *Articulation of the fuzzy preference order for the attributes.* The articulation of the fuzzy preference order for the selected attributes is performed with the fuzzy weak order. We proposed to use the fuzzy relation matrix for this purpose (Seo and Sakawa [21][22]) and demonstrated it in the IDSS environment. (Nishizaki and Seo[14])..

Representation principle of FMUF. When the assumptions [1]-[5] hold, FMUF is represented in the following forms.

Additive: $$\tilde{U}(x_1{}^o, x_2{}^o, \cdots, x_m{}^o) = \sum_{i=1}^{m} \tilde{k}_i \tilde{u}_i(x_i{}^o) \tag{15}$$

Multiplicative: $$\tilde{U}(x_1{}^o, x_2{}^o, \cdots, x_m{}^o) = \frac{1}{\tilde{K}}[\prod_{i=1}^{m}(\tilde{K}\tilde{k}_i\tilde{u}_i(x_i{}^o) + 1) - 1] \tag{16}$$

where $x_i{}^o$ is an assigned value of an attribute x_i.. The components of FMUF are represented as the crisp MUF values in Keeney-Raiffa sense ([9][10]).

The derivation of FMUF is performed on each parameter of the fuzzy utility value $\tilde{u}_i(x_i^o)$ as the fuzzy number $\tilde{u}_i = (\bar{u}, u^\alpha, u^\beta)_i$ assessed on the assigned value $x_i{}^o$. Corresponding to each parameter of the fuzzy scaling constants $\tilde{k}_i = (\bar{k}, k^\alpha, k^\beta)_i$, $i = 1, ..., m$, and $\tilde{K} = (\overline{K}, K^\alpha, K^\beta)$ derived with them, an FMUF value is calculated as a set of the crisp values derived with each parameter of $\tilde{u}_i(x_i^o) = k_s(\bar{u}_i, u_i^{\alpha}, u_i^{\beta})$ Note that the FMUF is a nonfuzzy set of the crisp values assessed on the three parameters of the FUF.

When the scaling constant k_s is assessed for the most preferred attribute x_S, the parameters $(\bar{k}$, k_α and $k_\beta)_j$ for the other fuzzy scaling constants $\tilde{k}_j$ are calculated from Eq.(17), the same as for the classical nonfuzzy case.

$$U^\bullet(x_s^o, x_r^o) = k_s^\bullet u_s^\bullet(x_s^o) + k_r^\bullet u_r^\bullet(x_r^o) + K^\bullet k_s^\bullet k_r^\bullet u_s^\bullet(x_s^o) u_r^\bullet(x_r^o)$$
$$= Constant \qquad (17)$$

3.3 A Numerical Example

For example, three FUF values $u_i(x_i^o)$ for assigned values (as an alternative) to three attributes x_i^0, i = 1,2,3 (say, housing price, design and environment in the housing sales) are assessed with the possibility distributions:

$$\Pi_U(u_1^o) = 0.3/0.2 + 0.5/0.4 + 0.8/0.5 + 1.0/0.7 + 0.7/0.8 + 0.5/0.9 + 0.2/1.0$$
$$\Pi_U(u_2^o) = 0.2/0.4 + 0.5/0.5 + 0.7/0.6 + 1.0/0.7 + 0.8/0.8 + 0.5/0.9 + 0.2/1.0 \quad (18)$$
$$\Pi_U(u_3^o) = 0.5/0.3 + 0.6/0.4 + 0.7/0.5 + 0.8/0.6 + 1.0/0.7 + 0.6/0.8 + 0.4/1.0,$$

from which the fuzzy numbers with three parameters for FUF are constructed by proper extrapolations.

$$\tilde{u}_1(x_1^o) = \{\bar{u}_1(x_1^o), u^{\alpha}{}_1(x_1^o), u^{\beta}{}_1(x_1^o)\} = \{0.10, 0.70, 1.05\}$$
$$\tilde{u}_2(x_2^o) = \{\bar{u}_2(x_2^o), u_2^{\alpha}(x_2^o), u_2^{\beta}(x_2^o)\} = \{0.34, 0.70, 1.05\} \quad (19)$$
$$\tilde{u}_3(x_3^o) = \{\bar{u}_3(x_3^o), u_3^{\alpha}(x_3^o), u_3^{\beta}(x_3^o)\} = \{0.25, 0.70, 1.08\}$$

Assessment of FUF values for several points of the attributes x_1, x_2, x_3 is performed with the fuzzy equivalent experiments and leads to the construction of three FUFs which correspond to the three attributes (see Fig.2). Each FUF is composed of three utility curves corresponding to the three parameters $\bar{u}_i(x_i), u_i^{\alpha}(x_i), u_i^{\beta}(x_i)$ of FUF $\tilde{u}_i(x_i)$, i = 1, 2,3 (Fig.2).

Each component value of FMUF is calculated respectively with each parameter, $\bar{k}_i, k_i^{\alpha}, k_i^{\beta}$ of the fuzzy scaling constants $\tilde{k}_i$ on an alternative set x_i^o, i =1, 2, 3, of the assigned attribute values. FMUF is constructed as a crisp set of these component, MUF values. In the multiplicative form,

$$\tilde{U}(x_1, x_2, \ldots, x_m) = \{\tilde{U}(x_1^o, x_2^o, \ldots, x_m^o)\} =$$
$$= \{[\frac{1}{\bar{K}}[\prod_{i=1}^{m}(\bar{K}\bar{k}_i\bar{u}_i(x_i^o)+1)-1], \frac{1}{\bar{K}}[\prod_{i=1}^{m}(\bar{K}\bar{k}_i u_i^{\alpha}(x_i^o)+1)-1], \frac{1}{\bar{K}}[\prod_{i=1}^{m}(\bar{K}\bar{k}_i u_i^{\beta}(x_i^o)+1)-1];$$
$$\frac{1}{K^{\alpha}}[\prod_{i=1}^{m}(K^{\alpha}k_i^{\alpha}\bar{u}_i(x_i^o)+1)-1], \frac{1}{K^{\alpha}}[\prod_{i=1}^{m}(K^{\alpha}k_i^{\alpha}u_i^{\alpha}(x_i^o)+1)-1], \frac{1}{K^{\alpha}}[\prod_{i=1}^{m}(K^{\alpha}k_i^{\alpha}u_i^{\beta}(x_i^o)+1)-1];$$
$$\frac{1}{K^{\beta}}[\prod_{i=1}^{m}(K^{\beta}k_i^{\beta}\bar{u}_i(x_i^o)+1)-1], \frac{1}{K^{\beta}}[\prod_{i=1}^{m}(K^{\beta}k_i^{\beta}u_i^{\alpha}(x_i^o)+1)-1], \frac{1}{K^{\beta}}[\prod_{i=1}^{m}(K^{\beta}k_i^{\beta}u_i^{\beta}(x_i^o)+1)-1]]\}$$

4 Concluding Remarks

There is another device for the fuzzy mixture operation by Mathieu-Nicot ([12]). His concept is based on the fuzzification of the possible act in a state and uses the probability for a state. His lottery is a stochastic mixture operation on which the fuzzy expected utility hypothesis is presented in the following form.

$$U[(\alpha\mu_{D_1}(s))+(1-\alpha)\mu_{D_2}(s)]=\alpha U(\mu_{D_1}(s))+(1-\alpha)U(\mu_{D_2}(s)) \quad (20)$$

where α is a probability and D_i denotes a fuzzy decision for an act. The act maps a state to a consequence. Although the utility function is defined on the fuzzy act, it is not fuzzified. On the contrary, we are concerned with the construction of FUF under the independency requirements. This device and its multiobjective extension come to the defuzzification of the ambiguous utility evaluation. The use of the *L-R* type fuzzy number makes it possible and leads to the numerical operations on its parameters as the crisp number and seems to be effective in practice.

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Feed-Forward Neural Networks for Approximating Pairwise Preference Structures

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Abstract. In this paper, we introduce a feed-forward neural network formulation that has the ability to learn and generalize Analytic Hierarchy Process (AHP)-style pairwise preference patterns in the training set, and can accurately approximate imprecise preference ratings based on pairwise preference judgments. A computational experiment verifies the robustness of the feed-forward neural network formulation.

Keywords. Multicriteria Decision Making, Analytic Hierarchy Process, Pairwise Preference Judgments, Artificial Neural Networks

1 The Analytic Hierarchy Process

In the AHP, the principal eigenvector of a matrix of pairwise comparisons is assumed to be an accurate measure of relative attractiveness, and the normalized principal eigenvector components are interpreted as preference ratings of the alternatives or criteria [11], [12]. The major numerical effort in the AHP involves computing the principal eigenvector and principal eigenvalue of each pairwise comparison matrix.

For simplicity of notation, in the remainder of this paper we will assume that the hierarchy of the decision problem has one level of criteria. This assumption does not represent a limitation to the methodology outlined below. In the case of multiple criterion levels, the sub-problems can be aggregated sequentially across the levels of the hierarchy.

Suppose that the decision problem has n alternatives. Denote the relative pairwise preference of alternative i over alternative j with respect to criterion m by a_{ijm}, and define the $n \times n$ matrix of pairwise comparisons with respect to criterion m by $\mathbf{A}_m = \{a_{ijm}\}$. To simplify the notation, we will omit the subscript m, so that $a_{ijm} = a_{ij}$, and $\mathbf{A}_m = \mathbf{A}$. In the AHP, the range of values of a_{ij} is limited to $[1/9,\ 9]$. The AHP assumes reciprocity, so that $a_{ij} = 1/a_{ji}$, for all i, j.

Throughout this paper, we will adopt Saaty's [12] interpretation that, *if* $\mathbf{A}$ indeed represents the true preference structure, the normalized principal eigenvector of $\mathbf{A}$, $\mathbf{r}$, can be interpreted as a measure of the true preference rating vector, and will refer to $\mathbf{r}$ as the "true" rating vector. However, in practice the a_{ij} will be approximate, so that $\mathbf{r}$ will only be a proxy for the true preference ratings. A number of researchers have studied the impact of imprecise judgments on variants of the AHP [1], [13], [14], [15]. The fact that elicited preferences are usually approximate is a major motivation for using a feed-forward artificial neural network (FFANN) approach to represent these structures.

2 Feed-Forward ANN

A FFANN consists of nodes, organized in layers, that are linked by directed arcs with connectivity weights that represent the strength of the connections. Nodes in a given layer are linked only to nodes in the next higher layer [10], [21]. Nodes in the input layer accept input from the outside world only, and do not serve any processing or computational function. Nodes in the output layer generate the output of the FFANN. The layers between the input and output layer are called hidden layers, and the nodes in these layers are the hidden nodes. Hidden layers enable a FFANN to represent complex mappings. Nodes in the input layer are connected with the first hidden layer, and nodes in the last hidden layer are connected to the output layer. Generally, nodes are connected only to the next higher layer in the network.

FFANNs have been applied to many non-trivial numerical tasks [4] and real world problems, notably to classification and pattern recognition problems [5], [24]. Recently, FFANNs have also been proposed for solving multicriteria decision problems [6], [20] [19]. Sanger [16] proposed an unsupervised learning algorithm for a single layer FFANN that is similar conceptually to algorithms in principal component and factor analysis.

A FFANN maps from the input space $\Re^{n_I}$ to the output space $\Re^{n_O}$, that is, for any given input vector $\mathbf{a} \in \Re^{n_I}$, the network computes an output vector $\mathbf{o} \in \Re^{n_O}$. The purpose of the FFANN training is to determine the values of the connectivity weights and node biases such that the network closely represents the unknown mapping. The network learns from patterns in the training set, which is a collection of paired input and output vectors observed from the unknown mapping. Once a FFANN has been trained, the knowledge of the unknown mapping is stored in the connectivity weights and node biases. A well-trained FFANN is able to generalize to patterns it has never seen before, and any new input vector presented to an appropriately trained network will yield an output similar to the one which would have been given by the actual mapping.

Given their generalizing capability, FFANNs are used to approximate the mapping from a reciprocal comparison matrix in the AHP to the associated preference ratings of the decision maker. The preference information provided by the decision maker through **A** serves as the input to the FFANN. Let **a** be the vector consisting of the elements of **A** above the main diagonal, *i.e.*, those a_{ij} for which $i<j$. Since **A** is reciprocal, **a** contains all of the relevant preference information in **A**, and is used as the input to the FFANN. Thus, the neural network will have $n_I = n(n-1)/2$ input nodes. The desired output vector consists of the n-dimensional true preference rating vector **r** of the decision maker, so that $n_O = n$. The compound vector $(\mathbf{a}^T, \mathbf{r}^T)$ represents one training pattern in the training set. In this paper, we limit ourselves to pairwise comparison matrices of sizes 4×4. A more extensive computational experiment which also includes 3×3, 5×5 and 6×6 matrices can be found in [18].

The NeuralWorks Professional II/Plus [7], [8] software was used to train the neural networks. In general, the number of hidden layers and nodes in each hidden layer depends on the complexity of the mapping. After experimenting with a number of alternative network configurations, training set sizes and network parameter settings, we found that for the 4×4 matrices a network architecture with 2 hidden layers and 20 and 8 hidden nodes in the first and second hidden layer, respectively, yields sufficient flexibility to learn the relevant preference information. The number of training patterns equaled $P = 1{,}000$. During the training process, the learning rate was gradually decreased from between 0.8–0.9 to 0.1, and the momentum factor from 0.6–0.7 to 0.05. Other network parameters were kept at the default values of the NeuralWorks software package.

The components of the network output vector **o** are normalized to sum to unity. Denoting training pattern t by $(\mathbf{a}_t^T, \mathbf{r}_t^T)$, and the corresponding normalized output of the FFANN by $\mathbf{o}_t$, in the training process the total network error of all P training patterns, measured by the sum of the squared component-wise errors, is minimized. The network error associated with training pattern t is measured by the root mean square error (RMSE_t) [8]. An equivalent network network error measure for Q patterns is $\overline{\mathrm{RMSE}} = 1/Q \times \sum_{t=1}^{Q} \mathrm{RMSE}_t$.

In order to avoid pattern memorization or network paralysis due to overtraining, the training progress was monitored by calculating $\overline{\mathrm{RMSE}}$ for $Q = 200$ randomly generated verification patterns that were *not* part of the training set. Each network was trained either until $\overline{\mathrm{RMSE}} < 0.001$ for these 200 patterns, or until it appeared that the $\overline{\mathrm{RMSE}}$ of the training set could not be improved further.

The computational effort of training the FFANNs is considerable, but not prohibitive. A typical training session using the NeuralWare software package [7], [8] on a 486 PC, 25 MHz, took several hours. This computational time is reasonable, since only a single network needs to be trained for the 4×4 matrix **A**. Once trained, this FFANN can be used to predict the ratings for *any* pairwise comparison matrix of that size, *i.e.*, it can be used for any application involving a 4×4 matrix **A**.

3 Experimental Results

In this section, we investigate whether (1) a FFANN is able to closely approximate the mapping from a pairwise comparison matrix to its principal eigenvector, and (2) a FFANN representation is more accurate than the power method in estimating the preference ratings if the preference information is imprecise. We address the former issue in Section 3.1 and the latter in Section 3.2.

3.1 Approximation of the Principal Eigenvector

Here, we only compare the output vector of a trained FFANN with the actual principal eigenvector obtained with the power method, assuming that the preference judgments exactly correspond with the true preference structure of the decision maker. In order to verify that FFANNs are indeed capable of yielding close approximations of the principal eigenvectors, we trained the FFANN using moderately inconsistent pairwise comparison matrices $\mathbf{A}$ [12]. As our goal was to verify whether the output of the trained network can approximate the principal eigenvector of $\mathbf{A}$, we used the normalized principal eigenvector of $\mathbf{A}$ obtained with the power method as the desired output in the training process.

Saaty [12] argues that few decision makers are fully consistent in their pairwise preference judgments, and inconsistency indices of less than 0.1 are reasonable in practice. Accordingly, we randomly generated the matrices of inconsistent judgments such that their inconsistency index did not exceed 0.1. We decided against including matrices with inconsistency indices exceeding 0.1, as such matrices may be fundamentally flawed in terms of the preference information elicited, in which case Saaty [12] recommends re-evaluating and double-checking the judgments, rather than proceeding with the preference analysis.

Given the desired output vector $\mathbf{r}_t$ and the predicted output vector $\mathbf{o}_t$ of pattern t, the root mean square error (RMSE_t), the angle between $\mathbf{r}_t$ and $\mathbf{o}_t$ (γ_t), and maximum bit error (MBE_t) were used to measure the accuracy of the predicted ratings of pattern t. Whereas we used the average RMSE ($\overline{\mathrm{RMSE}}$) to monitor the training progress, in the model validation phase we also measured the average angle $\overline{\gamma} = 1/P \times \sum_{t=1}^{P} \gamma_t$ and the average maximum bit error $\overline{\mathrm{MBE}} = 1/P \times \sum_{t=1}^{P} \mathrm{MBE}_t$ of all validation patterns, where P denotes the size of the training or validation sample. Obviously, smaller values of $\overline{\gamma}$ and $\overline{\mathrm{RMSE}}$ indicate evidence of a higher average degree of similarity, and thus of higher predictive accuracy of the FFANN models. Smaller values of the $\overline{\mathrm{MBE}}$ are indicative of a more favorable worst case accuracy of the neural network.

The actual principal eigenvectors were compared with the output vectors of the trained FFANNs using 1,000 independently generated moderately inconsistent validation patterns that were not in the training process. For our 4×4 matrices, we obtained the network performance measures $\overline{\text{RMSE}} = 0.01$ (0.006), $\overline{\text{MBE}} = 0.010$ (0.001) and $\overline{\gamma} = 1.919$ (1.151), where the figures within brackets are the standard deviations, across all 1,000 validation patterns. These accuracy measures are all small, indicating that the trained FFANN is indeed capable of closely approximating the true normalized principal eigenvector.

3.2 Predicted Ratings for Imprecise Preference Structures

In Section 3.1, we verified that the FFANN was indeed capable of representing the true principal eigenvector. The training data used in Section 3.1 were exact. We next study whether the FFANN can indeed generalize in the presence of imprecise preference patterns. To that purpose, we perturbed the components of perfectly consistent true pairwise comparison matrices, thus introducing impreciseness into the preference judgments. Impreciseness can be introduced in many different ways, but our experiment suffices to show the potential of FFANN in the presence of one particular type of impreciseness. Of course, further research should be conducted to confirm our conclusions in a more general setting.

We assume that the decision maker has a perfectly consistent and known *true* preference structure, but that the *revealed* preference information is moderately inconsistent, reflecting that the decision maker provides approximate rather than exact preference judgments. Thus, we first generated perfectly consistent pairwise comparison matrices $\mathbf{A}$ with normalized principal eigenvectors $\mathbf{r}$, and then perturbed each a_{ij} by the transformation $b_{ij} = a_{ij}(1 + \eta U)$, where U is uniformly distributed over the interval $[-1, \ +1]$, and $\eta = 0.8$ is a scalar. Hence, the mean disturbance equals zero, and the standard deviation equals about $0.46 a_{ij}$, representing a moderate disturbance. Thus, the matrices $\mathbf{B} = \{b_{ij}\}$ will be moderately inconsistent. Any matrix $\mathbf{B}$ with an inconsistency index exceeding 0.1 [12] was discarded, we ensured that all its elements were within the range of 1/9 to 9, and that the direction of the individual pairwise preference statements was not reversed. The average inconsistency of the $\mathbf{B}$ matrices in our experiments was about 0.02. Let $\mathbf{b}$ consist of the elements of $\mathbf{B}$ above the diagonal. Then, $(\mathbf{b}^{\mathrm{T}}, \mathbf{r}^{\mathrm{T}})$ is a training pattern in the training set.

The network training was conducted in the same manner as in Section 3.1. We used validation samples consisting of 1,000 independent replications, none of which were used in the training process. First, we generated pairwise comparison matrices $\mathbf{C} = \{c_{ij}\}$ in exactly the way the matrices $\mathbf{B}$ were generated, *i.e.*, $c_{ij} = a_{ij}(1 + \eta U)$. However, validating the results using $\mathbf{C}$ may bias the results in favor of the FFANN, because the network training sample had exactly the same characteristics. Therefore, we proceeded to generate doubly perturbed validation matrices $\mathbf{D} = \{d_{ij}\}$, by perturbing the $\mathbf{C}$ matrices once again, so that $d_{ij} = c_{ij} + \eta U a_{ij}$. Again, any matrix $\mathbf{D}$ with an inconsistency index exceeding 0.1, with elements outside the range from 1/9 to 9, or with reversals in preference direction, was excluded from the analysis. Since the FFANN was not trained to learn the type of impreciseness present in the **D** matrices, we can compare the predictive accuracy of the traditional AHP based on the power method and the FFANN on an equal basis.

Since in our experiment the vector of actual actual preference ratings, *i.e.*, the principal eigenvector of **A**, for each validation pattern is known, we can use $\overline{\text{RMSE}}$, $\overline{\text{MBE}}$ and $\overline{\gamma}$ to evaluate the prediction accuracy of the FFANN and the traditional AHP (power method) separately. A T-test of difference of means to determine which approach yields more accurate results, on average, for the 1,000 twice perturbed validation sample patterns, revealed that the FFANN predicted the true preference ratings more accurately than the power method. These results were statistically significant at the $\alpha = 0.01$ level, with t-values of 3.00, 3.32 and 3.16 in the case of γ, MBE and RMSE, respectively.

4 Conclusions

A number of research studies have found ANNs to be well-suited for pattern recognition [2], [3], [9]. A decision maker's preference structure may be viewed as a pattern. In the AHP, information about the preference structure of a decision maker is contained in pairwise comparison matrices. In this paper, we found that FFANN representations can provide robust approximations of a decision maker's preferences when the preference information at hand is imprecise. Hence, it appears that the FFANN formulation is a potentially powerful tool in determining preference ratings for effective decision support.

Of course, the analysis in this study is based on an experiment with 4×4 pairwise preference matrices only. Given the space limitations, we did not report on similar experiments for other size matrices in this paper, but a more comprehensive study reveals that the results obtained in the current experiment indeed extend to other matrix sizes and different matrix characteristics [18]. The attractiveness of using the FFANN representation is enhanced by the fact that only one network needs to be trained for each matrix size, and once trained the FFANNs can easily be embedded in any AHP software package. Thus, the user does not need to train a new network for every application.

Although in this paper the preference structure is based on the AHP [12], it would be interest to extend the current study to other preference structures, such as value functions, that may also be analyzed within an ANN framework. We defined impreciseness in one particular way, but this concept can be conceptualized and operationalized in numerous different ways, and it appears useful to evaluate the FFANN approach under various conditions. It may also be useful to explore ANN paradigms other than feed-forward networks.

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A Comparison Between Goal Programming and Regression Analysis for Portfolio Selection

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Abstract

The aim of this paper is to investigate the application of Goal Programming (GP) to portfolio evaluation and selection. The shares analysed are those in the British FTSE 100 index. A two stage model is proposed. The first stage predicts the sensitivity of the shares to specific factors using GP and regression analysis. The second stage of the model selects a portfolio using a GP model based on the decision maker's scenarios and preferences. A comparison between the sensitivities predicted by the first-stage of the GP model and that of the regression analysis is made.
Keywords: Goal Programming, Portfolio Selection, Regression Analysis

1 Overview of Modern Portfolio Theory

This paper illustrates the selection of a portfolio of shares from the FTSE 100 share index using Goal Programming and/or Regression Analysis. In the context of this paper, a portfolio refers only to a combination of shares in the British FTSE 100 share index.

Markowitz [9] suggests that a portfolio's risk and return is dependent on three variables; expected return, variance of the expected return and covariance of return of shares within the portfolio.

A simplification to portfolio theory was proposed by Sharpe [11]. He postulated that the returns of shares are interrelated only to some index (i.e common market response), thereby eliminating the need to compute all the covariances. However Sharpes model [11] makes some assumptions. The model assumes that the only common factor affecting all shares is the return of the market and ignores factors such as industry or economic influences. Another approach used for the portfolio selection problem is that of muticriteria analysis. This

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methodology is based on resolving the inherent multicriteria nature of the portfolio selection. Realistic models are built by not only considering the two basic criteria of return and risk (mean-variance model) but a number of important criteria such as marketability, dividends and other financial factors. A classification of this method is given in the paper by Zopounidis et al [16]. There have been several applications of GP to Portfolio Selection [4, 6]. The study by Lee and Chesser [7] presents a lexicographic GP model representing the investor's priorities. A similar model was discussed by Levary and Avery [8] which compared the use of linear programming to goal programming for the selection of optimal portfolios. Hallerbach and Spronk [1] in 1985 proposed an Interactive Multiple Goal Programme to construct a portfolio which best meets a given decision makers preferences. In this paper a two-stage model is presented which chooses a portfolio based on one or more economic scenarios. A sensitivity analysis is carried out for the first stage to determine the extent to which a particular share price is affected by changes in certain economic factors. Goal Programming and Regression analysis are both used as alternatives to calculate the predictive function and the results are compared.

The remainder of this paper is divided into five sections. Section 2 contains a review of GP. Section 3 gives a brief explanation of regression analysis. Section 4 presents the two-stage model. Details and analyses of the results are given in section 5. Finally, section 6 draws conclusions.

2 Goal Programming

Goal Programming is a branch of multi-objective decision making. It is based around the Simonan [12] concept of satisfaction of a number of goals rather than the optimisation of a single objective function as is the case in linear programming. In this paper we use weighted goal programming (WGP) which is algebraically expressed as:

$$MIN\ z = \sum_{i=1}^{k}(u_i n_i + v_i p_i)$$

Subject to,

$$f_i(x) + n_i - p_i = b_i \quad i = 1 \ldots k$$

$$x \in C_s$$

Where n_i, p_i represent the negative and positive deviations from the target value b_i for the $i'th$ objective. u_i, v_i represent the non-negative weights attached to these deviations, a zero weight being used for any deviation that the decision maker does not wish to minimise, such as positive deviation from a profit goal which represents the level of surplus profit. C_s is an optional set of 'hard' constraints which must be satisfied at the final solution point.

The reader is referred to recent textbooks by Romero [10] and Ignizio [2] for a comprehensive review of goal programming theory and modelling practice.

3 Brief Explanation of Regression Analysis

Regression analysis is used to assess the relationship between one dependent variable and several independent variables. This paper focuses on multiple regression as there exists a number of independent variables $x_1, x_2, \ldots x_k$. These independent variables are combined to predict the dependent variable for each case. The standard regression equation can be stated as:

$$Y = A + B_1x_1 + B_2x_2 + \ldots + B_kx_k$$

where,
Y' is the predicted value of the dependent variable,
A is the Y intercept,
x_j represent the various independent variables, $j = 1 \ldots k$
B_j represent the coefficients assigned to x_j during regression, $j = 1 \ldots k$

The regression coefficients B_j minimise the distance between the values predicted from the equation and the values obtained by measurement. This minimisation uses an L_2 distance metric (least squares) and also optimises the correlation between the set of dependent variables.

4 Two-Staged Approach For Selecting A Portfolio

4.1 First Stage: Sensitivity Analysis of the Shares

In the first stage a GP model is formulated which predicts the sensitivity of the shares to specific economic factors. Ignizio [2] states that a goal programming based regression approach can be used as an alternative to find the predictive function. Hence in our analysis both methods are used and the results compared.

The data used is taken from the FTSE 100 index. The share price for each share in the index is collected at six weekly intervals during the period 1988-1992. Only the data for 97 shares were available due to changes in the FTSE 100 index constituents. The data was cleaned by adjusting the share prices for any rights issues, script issues and dividends. The twelve economic factors used in the model are UK interest rate, US interest rate German interest rate, US inflation rate, German inflation rate, Dow Jones index Nikkei Average, Hang Sang index, Oil price, Gold price, House price and Sterling index.

4.2 Mathematical Representation of Sensitivity Analysis (GP Approach)

Sensitivity for each individual share for the 39 time periods is given by the solution to the following GP (Model A):

$$Min\ z = \sum_{i=1}^{39}(n_i + p_i)$$

subject to

$$\sum_{j=1}^{12} C_{ij}y_j + n_i - p_i = P_i \qquad i = 1, \ldots, 39$$

$$y_j\ free$$

Similarly sensitivity for FTSE index is given by (Model B):

$$Min\ z = \sum_{i=1}^{39}(n_i + p_i)$$

subject to

$$\sum_{j=1}^{12} C_{ij}y_j + n_i - p_i = F_i \qquad i = 1, \ldots, 39$$

$$y_j\ free$$

where
C_{ij}=change in factor j for period i
P_i=Share price movement in period i.
F_i=movement of FTSE in period i.
y_j=predicted sensitivity of share i or FTSE to factor j
Model A is solved for each of the 97 shares and model B is solved for FTSE index, using a goal programming package GPSYS [13]. This package is set up to find a pareto efficient solution to the GP problem [15]. For this investigation, pareto efficient solutions were obtained for the first and the second stage GP models.

Solving this series of models gives the sensitivity of each share and the FTSE index to the economic factors. Model A is used to predict the sensitivity of a given share to the economic factors. Thus the model must be processed 97 times. Each set of outcomes (y_j, $j = 1, \ldots 12$) is stored in a matrix which when complet forms the sensitivity matrix S_{ij} used in the second stage model. Model B is processed once to obtain the sensitivity of the FTSE against the economic factors. The outcome (y_j, $j = 1, \ldots 12$) represented by vector F_j used in the second stage model.

4.3 Multiple Regression Approach For Sensitivity Analysis

The model can be stated as:

$$Y = B + B_1x_1 + B_2x_2 + \dots B_{12}x_{12}$$

where Y represents share price,
x_j represents the $j'th$ factor, $j = 1 \dots 12$
B_j represents the coefficient factor for each specific factor, $j = 1 \dots 12$

Each of the 97 share prices are regressed against the corresponding factors to obtain the sensitivity between the price and factors. The regression coefficients are computed using MINITAB statistical package.

4.4 Second Stage: Selection of Portfolio

Using the results obtained from either of the two alternate methods of sensitivity analysis a subsequent weighted GP model is formulated to select a portfolio from amongst the given set of shares.

Mathematical Representation of the Model.

$$Min\ z = W_1 \sum_{j=1}^{12}(nfact_j + pfact_j) + W_2 \sum_{k=1}^{N}(nscen_k) + W_3 \sum_{i=1}^{97}(ppen_i) \quad (1)$$

subject to

$$x_i + npen_i - ppen_i = 0.05 \quad i = 1, \dots, 97 \quad (2)$$

$$\sum_{i=1}^{97} \frac{S_{ij}x_iL_{FTSE}}{F_JL_i} + nfact_j - pfact_j = 1 \quad j = 1, \dots, 12 \quad (3)$$

$$\sum_{j=1}^{12} M_{jk} \sum_{i=1}^{97} \frac{100 \times S_{ij}x_i}{L_i} + nscen_k - pscen_k = T_k \quad k = 1, \dots, N \quad (4)$$

$$\sum_{i=1}^{97} x_i = 1$$

$$sector\ constraints \leq 0.25 \quad (5)$$

$$0 \leq x_i \leq 0.09 \quad i = 1, \dots, 97$$

Where N is the number of scenarios. There is a strict upper bound of 9% of the portfolio that can be invested in any individual share. Also the set of objectives in (2) states that ideally this figure should not exceed 5% of the total funds available for investment. The extent to which shares take values up to 5%

rather than values in the range 5% to 9% is given by the weight placed on the minimisation of the positive deviations from these objectives in the achievement function, W_3. A further set of constraints (5) concerning unsystematic risk are that at most 25% of the portfolio can be invested in any one market sector [5].

The objective set (3) deals with the control of the systematic risk associated with the portfolio. Each objective in this set represents the specific risk of the portfolio to one of the economic factors included in the sensitivity analysis above. The individual sensitivity of a share to the factor divided by the FTSE's sensitivity to that factor gives a measure of the relative sensitivity of the share to the factor compared with the market. Due to the different magnitude of the share prices involved this coefficient is divided by a scaling factor of the ratio of the last price of the share against the last value of the FTSE. The sum of the percentage invested in each share multiplied by these coefficients then gives the sensitivity of the portfolio to the factor relative to that of the market. Thus, by setting the goal of these objectives to a value of one, the modeller is expressing the desire to follow the market as regards these factors. If the goal for a factor was set to zero this would indicate a wish to de-sensitise the portfolio towards that factor.

The second stage of the analysis of the model is based on the concept of a number of scenarios. A scenario is defined as a set of values assigned to the economic factors included in the sensitivity analysis. For each scenario the sensitivity for each share and the amount invested in the share is multiplied by the factor changes for the scenario and summed to give the total profit for the portfolio under that scenario. The target value expresses the modeller's required profit level for that particular scenario. This is represented by the objective set (4).

5 Results

Initially, the GP and regression models are solved for each of the eight time periods, during 1993 from February to December at six-weekly intervals as shown in table 1.

For each model a single scenario is used based on the actual values of the twelve factors referred to in section 4.1. For the first set of experiments, the scenario weight, W_2 in the achievement function (1) is assigned a value of 10. The percentage movements of the portfolios produced are given in table 1.

	Dec 92	Feb 93	Mar 93	May 93	June 93	Aug 93	Sep 93	Nov 93	Dec93
FTSE	2847.8	2851.6	2950.6	2869.5	2886.0	2823.9	3100.0	3188.3	3237.0
(a)		0.1	3.6	0.8	1.3	-0.8	8.9	12.0	13.7
(b)		-7.8	-2.9	1.6	6.6	7.6	20.5	14.3	15.3
(c)		-1.0	9.4	8.5	8.7	7.2	27.1	14.9	19.1

Table 1 : Percentage movement of portfolios. FTSE represents the value of the FTSE index.

(a) represents the percentage change in the FTSE from December 1992.
(b) represents the percentage gain for the GP model from December 1992.
(c) represents the percentage gain for the regression model from December 1992.

Table 1 is graphically summarised by figure 2.

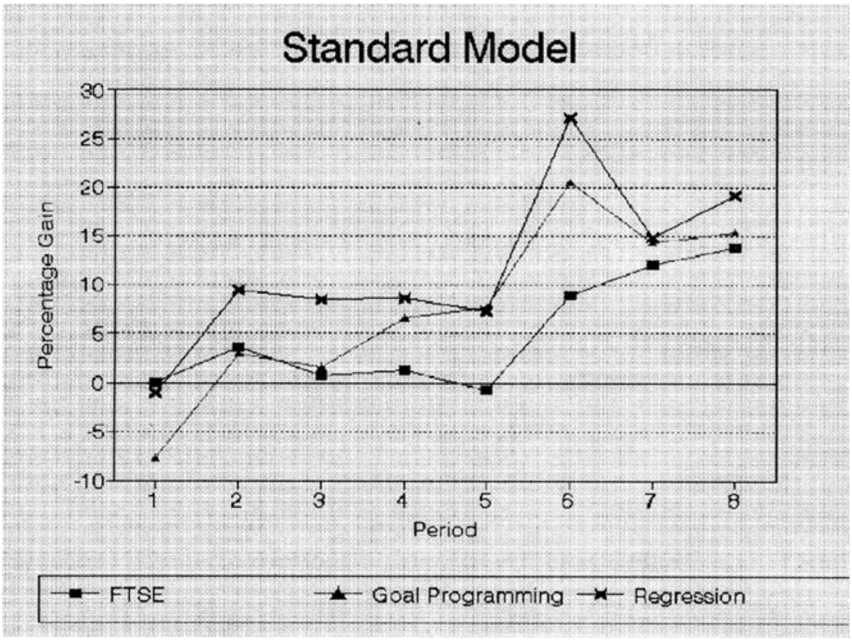

Figure 2 : Percentage movement of portfolios.

It can be seen that both the GP and Regression approaches do not outperform the FTSE for the first time period. Generally, the expected return is not as high as required. This relates to an under-emphasis on the scenario objective (4) in favour of the risk based objectives (3). In order to overcome this difficulty the weight associated with the scenario's negative deviational variable, W_2, is increased to 120. This value allows a trade-off of ten units against each of the 12 risk factors. The solution of the re-formulated models is given in Table 2.

	Dec 92	Feb 93	Mar 93	May 93	June 93	Aug 93	Sep 93	Nov 93	Dec93
FTSE	2847.8	2851.6	2950.6	2869.5	2886.0	2823.9	3100.0	3188.3	3237.0
(a)		0.1	3.6	0.8	1.3	-0.8	8.9	12.0	13.7
(b)		0.0	13.7	3.4	9.6	7.6	19.9	13.5	15.5
(c)		8.9	12.4	7.7	18.2	14.3	26.9	14.8	24.1

Table 2 : Percentage movement of portfolios with increased scenario weight

(a) represents the percentage change in the FTSE from December 1992.
(b) represents the percentage gain for the GP model from December 1992.

(c) represents the percentage gain for the regression model from December 1992.

Table 2 is graphically illustrated by figure 3.

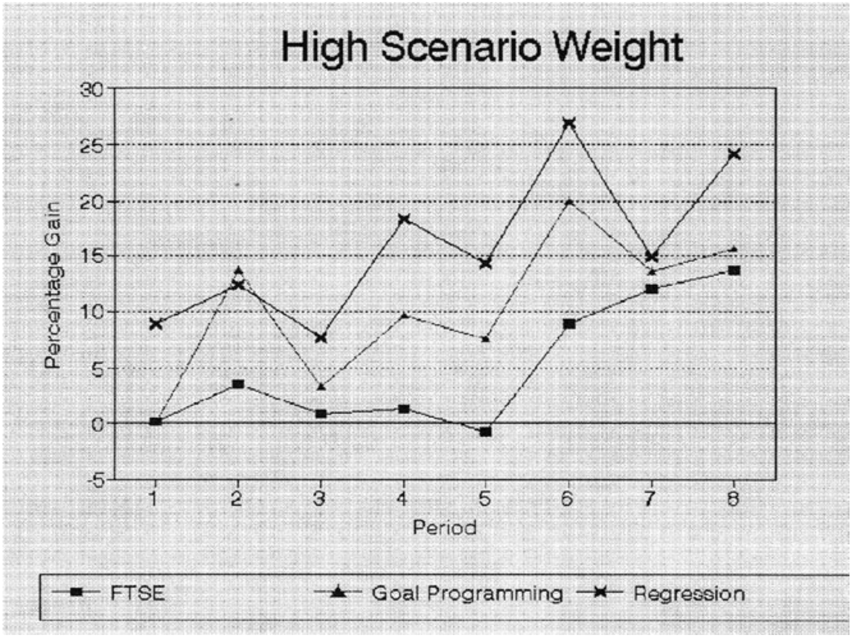

Figure 3 : Percentage movement of portfolios with increased scenario weight

The results in table 2 and figure 3 show a distinct improvement in the level of percentage gain for both the regression and GP approaches. This is particularly noticeable in the first period with both portfolios producing a non negative gain.

Regression still outperforms GP. However, when comparing the maximum level of risk for the portfolios against any one factor, regression is shown to have a considerably higher level of unsystematic risk than in GP. This is shown in table 3. In order to lower these unacceptably high levels of risk, a penalty (or preference) structure [3, 14] is used to limit the maximum level of risk for any one factor.

	Dec 92	Feb 93	Mar 93	May 93	June 93	Aug 93	Sep 93	Nov 93	Dec93
(a)		7.125	13.959	8.523	9.303	8.779	9.347	7.72	9.058
(b)		25.215	26.597	23.997	25.336	26.156	26.293	21.437	21.072

Table 3 : Maximum absolute risk levels for any factor. (a) represents maximum absolute risk levels for any one factor (GP)
(b) represents maximum absolute risk levels for any one factor (Regression).

5.1 Applying a Peference Structure to Lower Risk Level

To control the risk level, an increase in penalty is incorporated in conjunction with upper and lower objective bounds for each risk objective in the manner given by Jones and Tamiz [3, 14].

Any risk level in the interval [-1,1] is less sensitive than the FTSE to the factor, and is thus not penalised. Risk levels in the ranges [-5,-1) and (1,5] are penalised at the previous level of 10. Risk levels in the ranges [-12.5,-5) and (5,12.5] are considered much more undesirable and are therefore penalised at ten times the per-unit level of the lower risk ranges, giving an additional weight of 90. Risk levels with absolute magnitude greater than 12.5 are not permitted and are therefore modelled by means of strict objective bounds.

These considerations lead to the following algebraic reformulation of the objectives dealing with factor risk (3):

$$Min\ z = 10\sum_{j=1}^{12}(nrisk2_j + prisk3_j) + 90\sum_{j=1}^{12}(nrisk1_j + prisk4_j) \tag{6}$$

$$+W_1\sum_{k=1}^{N}(nscen_k) + W_2\sum_{i=1}^{97}(ppen_i)$$

subject to,

$$\sum_{i=1}^{97}\frac{S_{ij}x_iL_{FTSE}}{F_JL_i} + nrisk1_j - prisk1_j = -5 \quad j = 1,\ldots,12 \tag{7}$$

$$\sum_{i=1}^{97}\frac{S_{ij}x_iL_{FTSE}}{F_JL_i} + nrisk2_j - prisk2_j = -1 \quad j = 1,\ldots,12 \tag{8}$$

$$\sum_{i=1}^{97}\frac{S_{ij}x_iL_{FTSE}}{F_JL_i} + nrisk3_j - prisk3_j = 1 \quad j = 1,\ldots,12 \tag{9}$$

$$\sum_{i=1}^{97}\frac{S_{ij}x_iL_{FTSE}}{F_JL_i} + nrisk4_j - prisk4_j = 5 \quad j = 1,\ldots,12 \tag{10}$$

$$\sum_{i=1}^{97}\frac{S_{ij}x_iL_{FTSE}}{F_JL_i} \geq -12.5 \tag{11}$$

$$\sum_{i=1}^{97}\frac{S_{ij}x_iL_{FTSE}}{F_JL_i} \leq 12.5 \tag{12}$$

The results of the portfolio return with the new penalty structure is shown in Table 4 together with the new risk levels for both approaches.

	Dec 92	Feb 93	Mar 93	May 93	June 93	Aug 93	Sep 93	Nov 93	Dec93
FTSE	2847.8	2851.6	2950.6	2869.5	2886.0	2823.9	3100.0	3188.3	3237.0
(a)		0.1	3.6	0.8	1.3	-0.8	8.9	12.0	13.7
(b)		-2.1	3.5	7.1	8.2	8.2	17.5	12.1	17.2
(c)		-3.8	9.6	2.6	8.2	-0.1	15.2	8.7	13.9
(d)		4.125	6.731	5.814	8.287	5.318	7.278	7.212	8.435
(e)		8.736	12.5	12.5	12.5	12.5	12.5	12.5	12.5

Table 4 : Return and maximum risk levels with penalty structure
(a) represents the percentage change in the FTSE from December 1992.
(b) represents the percentage gain for the GP model from December 1992.
(c) represents the percentage gain for the regression model from December 1992.
(d) represents maximum absolute risk levels for any one factor (GP)
(e) represents maximum absolute risk levels for any one factor (Regression).
Table 4 is graphically illustrated by figure 5.

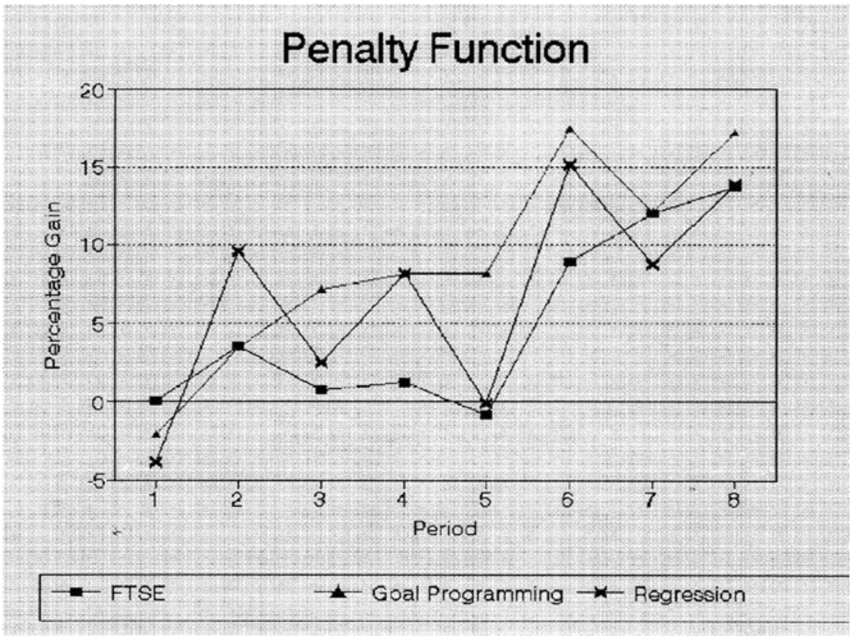

Figure 5 : Return and maximum risk levels with penalty structure

Both approaches still outperform the FTSE in all but the first time period. Although the general level of percentage return is reduced for the regression approach, the maximum levels of risk for both methods are now reduced - particularly for the regression approach - to a more satisfactory level.

6 Conclusions and Further Research

From the final results, GP for curtailed values of risk can be said to outperform regression analysis for the sensitivity stage. As suggested by Ignizio [2] this could be explained by the fact that GP, by its use of an L_1 distance metric, does not give overdue importance to outliers and dubious data points.

As regards the optimal time period for prediction, from our experimentation the best results appear to generally lie between periods 3 and 6. That giving a portfolio holding time of 18 to 36 weeks. Clearly, due to the negative percentage gains for the first period, 6 weeks is seen to be too short a period for prediction by this method.

Clearly further research is needed to determine the most suitable method of predicting the appropriate scenarios. We believe the use of neural networks for this purpose is one option which is worth considering.

A further idea would be to reduce the number of risk factors which should help ease the task of scenario building. This can be achieved by vigorous testing of the correlation between the factors. For this purpose principal component analysis may be used to identify the dominant factors.

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A General Purpose Interactive Goal Programming Algorithm

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Abstract

This paper reviews currently available goal programming (GP) and multi-objective programming (MOP) interactive algorithms and discusses design issues of interactive algorithms specifically for GP. Two factors which distinguish GP applications from most MOP application are the capability to incorporate a large number of objectives in the modelling and solution practice, and the possible use of a lexicographic priority structure. Therefore any interactive GP or adapted MOP algorithm should be capable of performing well under these conditions. In practice, many existing algorithms fail to operate efficiently under these conditions due to a large amount of information required from, or presented to, the decision maker or the time taken per interactive iteration. The use of the lexicographic structure also adds an extra degree of complexity and may cause problems with utility function based algorithms. Therefore, the remainder of this paper presents a general purpose GP interactive algorithm capable of handling both large scale, and lexicographic, goal programmes. The algorithm is designed to present a suitable amount of information to the decision maker and be efficient both in number of interactive iterations and in time taken per interactive iteration.

1 Introduction to Goal Programming

GP is a branch of multi-criteria decision analysis, belonging to the continuous case multi-objective programming sub-area. In standard goal programming all criteria of the model are identified, each defined by a linear function known as an objective and given target values by the decision maker to represent the desired level of the objective. The sum of unwanted deviations is then minimised in an achievement function, with the nature of the minimisation dependent on the type of GP used.

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The majority of GP applications use two distinct types of model. In the first type, known as weighted GP, all unwanted deviations are given numerical weights and minimised as a single weighted sum in the achievement function. The weighted GP has the following algebraic format:

$$Min \;\; z = \sum_{i=1}^{k}(u_i n_i + v_i p_i) \tag{1}$$

Subject to,

$$f_i(\mathbf{x}) + n_i - p_i = b_i \quad i = 1 \ldots k \tag{2}$$

$$x \in C_s$$

Where z represents the achievement function value. $n_1 \ldots n_k$ and $p_1 \ldots p_k$ represent the negative and positive deviational variables of the set of k objectives given by (2). u_i and v_i represent the non-negative weights given to the respective deviational variables. These take a strictly positive value when the corresponding deviational variable is penalised and the value zero otherwise. $f_i(\mathbf{x})$ is a function of the n decision variables which gives the achieved level of the i'th objective. b_i is the target level of the i'th objective. C_s is an optional set of m 'hard' constraints, which must to satisfied in order to make the solution feasible.

Alternately the unwanted deviations may be assigned to different priority levels according to the preferences of the decision maker, with deviations in a higher priority level considered to be of infinitely more importance to those at a lower priority level. This is known as lexicographic goal programming. The achievement function now takes the form of an ordered vector of priority levels, which is lexicographically minimised to give a solution. That is, on solving each priority level, the minimal value achieved at higher priority levels must not violated. The lexicographic goal programme has the following algebraic format:

$$Lex \;\; Min \;\; \mathbf{a} = [\sum_{i=1}^{k}(u_i^1 n_i + v_i^1 p_i), \ldots, \sum_{i=1}^{k}(u_i^l n_i + v_i^l p_i)] \tag{3}$$

Subject to,

$$f_i(\mathbf{x}) + n_i - p_i = b_i \quad i = 1 \ldots k \tag{4}$$

Where $\mathbf{a}$ represents the achievement function value. $n_1 \ldots n_k$ and $p_1 \ldots p_k$ represent the negative and positive deviational variables of the set of k objectives given by (4). For each of the l priority levels in the model, there exist a set of weights : u_i^j and v_i^j represent the non-negative inter-priority level weights given respectively to the negative and positive deviational variables in the j'th priority level. These take a strictly positive value when the corresponding deviational variable is penalised in the j'th priority level and the value zero otherwise. Any deviational variable can be penalised in at most one priority level, although the negative and positive deviational variables of an objective may be penalised in different priority levels. As in weighted GP, $f_i(\mathbf{x})$ is a function of the n decision variables which gives the achieved level of the i'th objective. b_i is the target level of the i'th objective.

The remainder of this paper is divided into four sections. Section two reviews GP interactive method literature. Section three discusses design issues involved for interactive GP. Section four presents a generalised interactive framework. Finally, section five draws conclusions.

2 Goal Programming and Interactive Algorithms

The standard GP process dictates that all objectives, target values, weights, and priority levels should be set before solution. The resulting goal programme is then optimised and the solution presented to the modeller. In other words, all parameters are set *a priori*, are unalterable, and there is no modeller involvement in the solution process.

These conditions lead to a certain rigidity in the GP process. The correct setting of parameters such as weights, target values, and priority levels is, in practice, a progressive rather than a single process [5]. Hence this leads to modellers solving the GP many times with small changes in parameters and altering the input-source on each occasion. This tends to lead to a heuristic process of weight determination on the part of the modeller, which in some cases is time-consuming or lacks the necessary analytical approach to arrive at a satisfactory solution.

In order to counteract these problems, interactive methods were developed which allow the involvement of the decision maker in the solution process. They give an analytical environment under which the modeller can progressively explore and set the goal programmes parameters in a logical manner.

The initial concept of an interactive method first appeared in the more general field of multi-objective programme with the introduction of the STEM method by Benayoun et al. [1] in 1971. Although this method has been superseded by more recent techniques, it is still widely known and acknowledged as a pioneering multi-objective interactive method [17]. The first interactive method specifically for goal programming is given by Dyer [3]. This method finds a point for which the overall utility value of the solution is improved and moves part-way towards the new solution from the old solution in a manner which maximises the utility. Other interactive goal programming techniques include the goal altering method of Masud and Hwang [10]; the method of Franz and Lee [4]; the augmented approach of Ignizio [7]; the objective classification method of Nakayama and Sawaragi [11]; the progressive target-value raising approach of Spronk [12]; the visual interactive method of Kohoren and Laasko [8]; and the modified version of the multi-objective Zionts/Wallenius method [19] presented by Lara and Romero [9]. In addition Weistroffer [18] presents a method specifically for the non-linear case. Reference will be made to these techniques where ideas have been drawn from them to form part of the interactive GP environment detailed in this paper.

3 Design Issues

There are several issues involved in the design of an interactive environment capable of processing a range of problems solvable by current GP systems. These are enumerated and discussed in this section.

3.1 Choice of Initial Solution

There are three approaches used to determine the starting solution.

(i) Infeasibly high (ideal) starting point : From this point the target values are progressively relaxed until feasibility is obtained . Its main disadvantage is, from the modellers point of view, the solutions at each interactive iteration appear to be getting worse rather than better. Thus the modeller may be unwilling to make the appropriate sacrifices.

(ii) Pareto inefficient (nadir) starting point : In this case the minimal, or low values for every objective are chosen. The modeller then progressively improves the solution in accordance with their preferences. This is the approach taken by, amongst others, Spronk [12] in his interactive goal setting approach. It has the advantage of allowing the modeller to 'build' the solution according to his/her preferences however for models with a larger number of objectives it is time consuming to have to raise each objective from a high to a low value, as this requires many interactive iterations.

(iii) Optimal starting point : This approach either asks the decision maker for an initial estimate of the parameters, or chooses them at random. The resulting goal programme is then solved to give an initial point. This is the approach undertaken by Ignizio [7]. It has the advantage of requiring fewer interactive iterations as it allows a modellers initial estimate. It has the disadvantage of psychologically narrowing the modellers focus to the area around the initial solution as they may be unwilling to make major changes to their initial estimate.

In practice, an interactive GP system is required to solve large-scale models, some of whom have hundreds or thousands of objectives [6]. Therefore approach (2) will take too long in terms of decision maker time to complete. Approach (1) may require a comparatively large number of interactive iterations as well as possessing the disadvantage of progressively worsening solutions. With approach (3), however efficiency is reached immediately, and therefore the modeller can make as few or as many iterations as necessary. The GP system outlined in section 4 will use approach (3).

3.2 Presentation of Solution

At each interactive iteration an amount of information regarding the solution at that point must be presented to the decision maker. This amount must be substantial enough to allow an informed decision to be made by the decision maker but not so large as to overwhelm them with information. The critical factor regarding the amount of information tends to be the number of objectives in the model, with most measures of information dependent upon this factor. Some popular types of information are presented below, together with the actual amount of data they generate.

(i) The target and achieved values for each objective : These are essential information for any interactive method, as they allow the decision maker to compare the actual and desired level of the objective at that point. Together they generate $2k$ pieces of information, with k being the number of objectives in the GP.

(ii) The ideal and nadir values for each objective : These are presented in some methods, e.g. Masud and Hwang [10], as they give reference points to the level of the target and achieved values. They remain fixed throughout the process but require a considerable amount of calculation prior to initial solution. Together they again generate $2k$ pieces of information.

(iii) The deviational variable weights and priority levels : These are necessary in methods which rely on changes in weights and priority level reordering, such as the method actually used in section 4. For representation as a list of objectives this generates $2k$ pieces of information for a weighted GP and $4k$ pieces of information for a lexicographic GP, as non-weighted deviational variables have to be represented by a zero.

(iv) Information regarding priority levels : The only information not regarding objectives may pertain to priority levels and could include such information as current value, pareto state, whether redundant or not. Each of these categories produces l pieces of information, where l is the number of priority levels.

As in practical models l is recommended to be less than 10 [7], this eliminates any problems concerning amount of information presented.

3.3 Terminating Conditions

The final solution obtained is not a clear-cut issue. The general principle of the interactive method is the improvement of the level of the satisfaction of the decision maker with the solution. Methods tend to lie somewhere on a spectrum between two extremes

(i) Search-Based : This approach tends to reduce the possible region in which the final solution lies at each iteration and is mathematically convergent. Thus the level of decision-maker satisfaction which the current solution should improve at each iteration and backtracking is rarely allowed.

(ii) Learning-Based : This approach is designed to allow the decision maker the means of exploring the feasible region in a logical manner. It allows for backtracking in a systematic manner and terminates when the decision maker has decided they are happy with the current solution or cannot improve upon a previous solution.

A third variant is the learning/search based approach which allows a period of exploratory 'learning based' iterations before embarking on more directed 'search based' iterations.

As the GP solver interactive method is required for general purpose investigation of changes in a small number of parameters as well as full scale interactive searches, the approach outlined in Section 6.3 leans towards the learning based approach, although a search-based approach is still possible by dis-regarding certain options.

3.4 Information Given by Decision Maker

Another factor of variation between methods is the way the decision maker is asked to eluicidate their preferences. One difference is the nature of the information required. This can be quantitative, i.e. 'By what amount do you wish to raise the target value?', requiring a numerical answer; or qualitative, i.e. 'Do you wish to improve the target value?'. Qualitative questions are generally easier to answer, especially for the decision maker who is unsure of their preference structure during optimisation, but require more intelligent analysis when using the information to reformulate the modified goal programme.

Some approaches, relate the questions asked, or ask for information, directly regarding the parameters, such as changes in weights,priority levels etc. The Spronk [12] and Masud/Hwang [10] fall into this category. Others, such as the Zionts/Wallenius method [19] rely on the answering on question about the decision makers like or dislike of certain scenarios. Again the latter approach may require less knowledge of the modeller but is more difficult to interpret when re-formulating the goal programme. This is particularly significant in computational effort with larger models.

The other issue is the amount of choice the decision maker is allowed when choosing the parameter for alteration. This varies from methods where the choice of parameter is automatically made by the method and suggested for approval to the decision maker, to those in which the decision maker has free choice from any parameter. The latter case is normally associated more with the learning-based methods given in section 3.3 whereas the first case is more correlated to the search-based approach. As the GP system requires a general purpose interactive environment, the emphasis on the method developed in Section 4. is very much toward free choice of parameter case.

4 An Interactive Environment

In this section the aspects of design discussed in section 3 are used to form the interactive method for the intelligent GP system. As the method used should be suitable for any type of interaction or sensitivity analysis required by the decision maker, on any size or type of GP, this presents certain required criteria that the method must fulfill.

The main criterion is that of generality, the method must be able to handle both large-scale goal programmes and those that have a lexicographic priority structure. The handling of a lexicographic structure is an under-discussed topic in previous interactive GP publications, as most of these methods have their origin in multi-objective programming, where no such structure exists. The fact that large-scale GP's will be handled by the method means information presented, interactive iterations required, and computational time, should all be kept to a reasonable level. As mentioned in section 3, the general purpose nature of the method leads towards a learning-based method with wide decision-maker choice over parameter selection rather than a directed search-based method.

Thus the method relies on successive alterations of the parameters by the decision maker, either on a quantitative or qualitative basis, and re-formulation and re-optimisation of the GP. The mechanics of the method are discussed below.

4.1 Information Display

Figure 1 gives the on-screen display of part of a solution report of an interactive iteration - being a portion of the objective set for a lexicographic GP test problems from the authors work on budget planning.

Figure 1 : Main Display for interactive process

```
No Name    Par P   Target       Achieved     NPL    NW     PPL    PW
39 OBJ39    E      642031.688   677164.16     3     1.5     4     1.2
40 OBJ40    E     1266380.     1267149.076    3     1.5     4     1.2
41 OBJ41    E P    681230.875   732591.594    3     1.5     4     1.2
42 OBJ42    E P   1181240.     1171006.859    3     1.5     4     1.
43>OBJ1P    E     2100670.     2193710.023    2     2.5     0     0.
44 OBJ2P    E     1104616.     1139195.126    2     2.5     0     0.
45 OBJ3P    E     1223059.     1325275.457    2     2.5     0     0.
46 OBJ4P    E      623961.      673051.352    2     2.5     0     0.
47 OBJ5P    E     1969925.     2053146.567    2     2.5     0     0.
48 OBJ6P    E      694133.      694133.       2     2.5     0     0.
49 OBJ7P    E     2656968.     2789356.273    2     2.5     0     0.
50 OBJ8P    E     1750533.     1811192.543    2     2.5     0     0.
51 OBJ9P    E     1464269.     1507597.022    2     2.5     0     0.
52 OBJ10P   E     2012468.     2118387.12     2     2.5     0     0.
53 OBJ11P   E     1903169.     1995717.139    2     2.5     0     0.
54 OBJ12P   E     1644178.     1752425.721    2     2.5     0     0.
55 OBJ13P   E     1099028.     1174388.453    2     2.5     0     0.
56 OBJ14P   E     2471830.     2615246.968    2     2.5     0     0.
57 OBJ15P   E     1024049.     1067157.464    2     2.5     0     0.
58 OBJ16P   E      983010.     1042086.791    2     2.5     0     0.
(I)nc (D)ec (C)hange weight,(R)aise (L)ower or (M)od target,(A)ch
(P)ro,(U)npro value,(F)orward,(B)ack,(N)ext la(S)t page e(X)it
```

The information is given as a list of objectives. The information displayed on each line thus relates to an individual objective and is given in column order from left to right as: the number of the objective (in the same order as read from input file); the current objective marker ('>'); the name of the objective (from input file); the pareto efficiency marker ('E' = efficient, 'I' = inefficient); the protection marker (discussed in section 4.2 : 'P'=protected); the target value; the achieved value; the negative deviational variable's priority level; the negative deviational variable's weight; the positive deviational variable's priority level; and the positive deviational variable's weight. As the marker only relates to a single objective, a maximum of $10k + 1$ pieces of information are presented to the decision maker at each interactive iteration, with a maximum of 201 being on-screen at any one time.

4.2 The Options

The last two lines of the interactive display screen gives the possible options for altering parameters of the current objective displayed (marked by a '>'). These options are discussed below:

(I)ncrease This option allows for an increase in the importance assigned to the objective by the decision maker. This option improves the objective by forcing the achieved value closer to the target value by means of improving the weight or, if necessary in lexicographic GP, priority level of the non-satisfied penalised deviational variable. Therefore this option is suitable for objectives in which the target value has not been achieved. In the cases of an achieved target value increasing the weight on the penalised deviational variable will have no effect as it already has the value zero. These cases are qualitatively improved by the (R)aise target option below.

As the (I)ncrease option works on a qualitative basis it has the principle of increasing the weight or priority level until the value of the objective is improved. The improvement is indicated by a non-degenerate simplex iteration. As every different weight or priority level structure tested requires the solution of a modified goal programme, in practice weights can only chosen at discrete points. Thus for weighted goal programmes a progressive squaring approach is employed, which continually squares a multiplier of the weight until a non-degenerate change in basis occurs. The algorithm requires a maximum squaring parameter, M_D, as continually squaring when no improvement is possible will eventually cause an arithmetic overflow. M_D should be set large enough to allow for difficult, unscaled case, but small enough not to cause arithmetic problems. As the (I)ncrease option exists alongside the (D)ecrease option for lowering weight in the case of overshoot, a multiplying factor U_f may exist, which is the weight at which the decision maker indicates the weight is too large, giving an upper bound for any further increase. Similarly the (I)ncrease option leaves a multiplying factor L_f at which the weight is too low to satisfy the decision maker on the objective in question, that being, O_f the last weight tested before the non-degenerate change in basis occurred. Letting O_w be the weight of the unmet penalised deviational variable and N_w be the new weight in the modified GP, the algorithm for progressive weight increase is given as :

Weight Increase Algorithm

```
If previous (I)ncrease or (D)ecrease took place THEN
   Use last C_f generated
ELSE
   LET C_f = 2
ENDIF
For i = 1 to M_D
   LET O_f = C_f
   IF U_f exists THEN
        C_f = (U_f - Cf)/2 ELSE
        C_f = C_f * C_f
   ENDIF
   LET N_w = O_w * C_f
   Form modified GP with penalised deviational variable having weight of N_w
   Optimise modified GP
   IF non-degenerate iteration took place THEN
        Store current C_f for future (D)ecrease's or (I)ncrease's
        LET L_f = O_f
        EXIT
   ENDIF
ENDFOR
```

This algorithm is sufficient for the weighted goal programme. That is, when M_D squarings have taken place with no effect this effectively guarantees no further improvement is possible. For lexicographic goal progammes the priority level structure must be taken into account. Upon unsuccessfully raising a weight in its current priority level, the deviational variable is then progressively placed in higher priority levels until an improvement is found. In order to maintain the integrity of the lexicographic structure, it is important that deviational variables which initially belonged to separate objectives in separate priority levels are not placed in the same priority level, as the decision maker may not wish direct numerical comparison between these objectives. Therefore the priority levels may be classified into two subsets, commensurable or incommensurable, to the objective being improved. If a priority level is commensurable the deviational variable is placed in that priority level and the weight increase algorithm is performed. If the priority level is incommensurable then the deviational variable is placed in a new priority level directly above the incommensurable one. These considerations lead to the lexicographic increase algorithm:

```
FOR l= CURPRI down to 2 DO
   IF l'th priority level is in commensurable subset THEN
        Place unmet deviational variable in l'th priority level
        Perform Weight Increase Algorithm
        IF non-degenerate iteration took place THEN
           EXIT
        ELSE
           reset C_f to one, erase U_f, L_f values
        ENDIF ELSE
        form new priority level directly above l'th priority level
        reoptimise new lexicographic goal programme
        IF non-degenerate iteration took place THEN EXIT
        ELSE
           remove added priority level
        ENDIF
   ENDIF
ENDFOR
```

Providing an increase in the objective can take place within the feasible region defined by the rigid constraint sets, the weight increases and lexicographic increase algorithms will bring the achieved value closer to the target value

(R)aise : This option allows for the qualitative improvement of an objective's value by raising the target value. It is suitable for use when the achieved value is equal to or greater than the target value. If the achieved value is below the target value then raising the target value will not affect the achieved value. Given that this is not the case and that only the negative deviational variable is penalised raising the target value is making it more optimistic and increasing the value of the objective. If only the positive deviational variable is penalised then raising the target value is making it more pessimistic and has the effect of relaxing the objective, and subsequently improving other objectives if pareto efficiency is maintained. As it is desired to raise the target in a systematic, qualitative manner a similar approach is used to that of the weight increase algorithm. However the target values set has a real interpretation to the model, whereas the weight set only has meaning in relation to the over weights in the model. Hence target values are more sensitive to change, both computationally and from the decision maker's point of view. Therefore the weight increase algorithm is replicated as the target increase algorithm, with the initial multiplication factor at a lower value (1.1 is used as default) to allow the generation of test GP's closer to the original. This algorithm can be used for both weighted and lexicographic models - as the target value is independent of priority level.

(L)ower : This is the opposite of the (R)aise option, which allows for the quantitative lowering of the target value. Following the argument used for the (R)aise option, this option is only appropriate when the achieved value is less then or equal to the target value. If no (R)aise has previously been requested, the multiplication factors must be less than 1, to allow generation of GP's with lower target values than the original. This is achieved by setting the initial multiplication factor to 0.909 (the reciprocal of 1.1).

(D)ecrease : This option is the opposite of the (I)mprove option, it allows for the qualitative relaxing of an objective's value by the decrease of the corresponding weight. It can be used to compensate for an overshoot when using (I)ncrease, or to improve other objectives in the case of pareto efficiency. The latter case is true as weights are relative to each other, lowering one weight increases the relative values of the others. The algorithm for weight decrease is the same as the algorithm for weight increase with two adjustments. Firstly the check for U_f is replaced by a check for L_f and if it exists the new multiplier is given by $C_f = (Cf - Lf)/2$. Secondly, if no (I)ncrease or (D)ecrease has taken place before, the initial multiplication factor is given by $C_f = 0.5$, to allow GP's with smaller weights than the original to be generated.

Options for quantitatively altering the parameters of an objective

(C)hange : This option allows for the quantitative changing of a deviational variables weight. The decision maker specifies the new weight and the a modified goal programme is re-optimised with the deviational variable having that weight. This option is appropriate when the modellers have various numerical weighting strategies they wish to try out.

(M)odify : This option performs similarly to the (C)hange option except it is for quantitative changing of the target values. It is appropriate when only certain target values are acceptable, these are known by the decision maker, or where several similar GP's are to be solved, each with differing, pre-determined target values.

Options for safeguarding the achieved value of an objective

(P)rotect : This option ensures that the current achieved value of an objective will not be denegrated by future interactive iterations. It is useful when for objectives on which any degregration will lead to an unsatisfactory solution from the decision maker's point of view. It can be used to protect the gains achieved by (I)ncrease, (R)aise, or (L)ower options. This option allows the concept given by Nakayama and Sawaragi [11] to be used, that is, the division of the objective set into three subsets: objectives which require improvement (the chosen objective in this case), objectives which are satisfactory but cannot be relaxed (protected objectives), and objectives which can be relaxed (all other objectives).

(U)nprotect : This is the opposite of the protect option in that it will allow a previously protected objective to take any value once again. This is achieved by removing the variable bounds and unfixing the variables fixed by (P)rotect. As this extra freedom may lead to a different optimal solution, the resulting goal programme is re-optimised.

4.3 Theoretical Considerations

As a general method, the interactive method described by this paper is largely learning-based with certain search-based aspects available. When considering any GP formulation, a weighting strategy is given. In weighted GP this consists of a set of as many weights as there are penalised deviational variables. In lexicographic GP this consists of the priority level structure and as many inter-priority level weights as there are penalised deviational variables. The solution to any GP weighting strategy will lie at some basic solution point to the model, this point will be pareto efficient if the target values have been correctly set or otherwise can be restored by the techniques given in Tamiz and Jones [14]. By basic GP theory, each efficient basic feasible solution will have at least one weighting strategy associated with it, and in practice will have a subset of the set of total weighting strategies that will result in it becoming the optimal solution.

The (I)ncrease and (D)ecrease options are designed to perturb the weighting strategy a sufficient amount as to leave the subset of weighting strategies pertaining to the current optimal solution and generate a different optimal solution. When starting from an extreme point, in order to improve an objective via change of weighting scheme, another objective must be degregated. Thus, after efficiency is reached, these options move only within the set of efficient extreme points.

In order to make the entire efficient solution region the target value changing options, (R)aise and (L)ower, should be utilised. When the desired efficient point lies between two efficient solutions, the optimal solution will alternate between these solutions as the decision maker uses (I)ncrease and (D)ecrease options to alter the weights. If the target values of the objectives are manipulated to bring the efficient solutions closer together, the region of the efficient boundary lying between the original efficient points may be reached.

The (P)rotect option has the effect of reducing the feasible region and thus cutting off part of the efficient solution set. The efficient solutions cut off correspond to those with the chosen objective below its current value. Thus, this option can be used in order to reduce parts of the feasible region with which the decision maker will not accept because the value of a single objective is too low. This allows for an amount of L_∞ (MINMAX) thinking on the part of the decision maker to be combined with the usual L_1 consideration of the objective levels as a whole. This option is particularly useful for larger problems as it allows piecewise shrinkage of the feasible set to a more manageable level. To ensure irrecoverability is not imposed, the cut-off portion of the efficient set can be restored by the (U)nprotect option, which removes the bound.

Thus the learning-based aspect of the process is given by the ability to move easily between points on, and to any point of the efficient boundary. This is supplemented by the ability to perturb the nature of the original model via target-value changes. The search-based aspect is expressed by the repeated use of (I)ncrease/(D)ecrease and (R)aise/(L)ower to converge on the required level of an objective and the use of (P)rotect to eliminate unattractive parts of the feasible region.

5 Conclusion

This paper has both reviewed the state-of-the-art in interactive goal programming and developed an interactive framework which does not exclude either the large-scale or lexicographic GP's. This interactive framework has been successfully integrated into GPSYS [13], an intelligent GP solver developed by the authors. This has allowed the demonstration that the method developed in section 3 is fully compatible with other GP analysis techniques. Particularly useful is the combination of the interactive framework with pareto efficiency detection and restoration techniques [14], which allows the generation of a pareto efficient solution at each interactive iteration. The interactive method is also compatible with normalisation and preference modelling techniques [15, 16].

The interactive framework developed in this paper allows the decision maker to effectively explore the feasible and efficient regions for solutions and should thus enhance the flexibility of goal programming and provide an aid for decision makers to explore the interaction of objectives and gain greater understanding of their individual models, particularly in the large-scale case. It can be used when several different versions of a model are required. Each having a slightly differing weighting scheme and/or set of target values, thus eliminating the need for the creation and storage of several models varying in only a few parameters. It is also useful for the more conventional task of applying sensitivity analysis and providing answers to 'What If?' type questions.

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A Tchebycheff Metric Approach to the Optimal Path Problem with Nonlinear Multiattribute Cost Functions

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Abstract. A global optimization problem of finding an optimal path in the network with multiple attributes on links and a nonlinear convex cost function is studied. It is shown that the modified weighted Tchebycheff metric scalarization can generate every nondominated path in the network. Two exact algorithms for solving the bi-attribute optimal path problem are presented and an illustrative example is enclosed.

Keywords. Networks, nondominated paths, multiattribute cost functions, weighted Tchebycheff metric

1. Introduction

A variety of problems arising in telecommunications, transportation, scheduling, and other areas of human activity can be modeled as networks. Some network programming problems have become classical problems in the operations research. One of them, the shortest path problem (SPP), in which a single attribute is assigned to each link of the network, has been attracting researchers for many years. There are many cases, however, in which assigning one attribute to a link of the network is not sufficient for a good representation of a real-life situation. Consider making a decision in the following circumstances: transporting hazardous material between two cities, transmitting information through a communication network in a country, evacuating people from a building on fire, scheduling large complex projects that consist of many activities, scheduling flights for an airline company. In each of these situations one can identify a network whose links carry several attributes like travel time, travel distance, travel cost, reliability, capacity, and others. Of course, the classical SPP can be solved with respect to each attribute separately and provide the fastest, shortest, least costly, or the most reliable path, respectively. Applying the concept of Pareto optimality to the set of all paths in the network, one can find the set of Pareto or nondominated paths with respect to the vector of attributes. The papers of Hansen [4], Climaco and Martins [2], Corley and Moon [3], Hartley [5], Brumbaugh-Smith and Shier [1], and many others have been devoted to the SPP with vector attributes on links, which in this paper is referred to as the Nondominated Path Problem (NPP). However, even if the entire set or a subset of all nondominated paths has been generated, the decision maker may still be left with the problem of choosing a single most preferred path within this set to use in his/her application. Introducing a utility or cost function of the multiple attributes being considered allows one to make this choice.

Consider then a network whose links carry a vector of attributes and in which path optimality is determined by a function of these attributes. The problem of choosing an optimal path in this network, referred to as the optimal path problem with a multiattribute cost function, is studied in this paper.

Although quite many research papers have been devoted to the optimal path problem with linear multiattribute cost functions, not much has been reported in the literature on the same problem with nonlinear functions. Henig [6] gave a comprehensive treatment of the nonlinear case for quasiconvex and quasiconcave cost functions and proposed several algorithms, among them two for the quasiconvex case. Mirchandani and Wiecek [8] studied the optimal path problem with multiattribute nonlinear cost functions, either concave or convex, and proposed a line-search algorithm for the convex function case. However, the two algorithms developed by Henig [6] as well the algorithm of Mirchandani and Wiecek [8] do not find an optimal path if this path is located in the interior of the convex hull of all nondominated paths of the network.

In this paper, the optimal path problem with nonlinear convex multiattribute cost functions, which is a global optimization problem over a discrete feasible space, is studied. We first develop theoretical relationships between the NPP and the Tchebycheff metric (TM) scalarization and then show that with this technique every nondominated path of the network can be found. This is an extension of the result developed by Steuer and Choo [9] for multiple objective programs over a discrete feasible set.

Section 2 of this paper includes problem formulation. In Section 3, the TM scalarization of the NPP is studied. In particular, properties of the modified weighted Tchebycheff metric (MWTM) introduced by Kaliszewski [7] are examined in the framework of multiattribute networks. Section 4 presents two methods for solving the bi-attribute optimal path problem (OPP). An illustrative example is included in Section 5. Finally, in Section 6 conclusions are drawn and several directions of further research are specified.

2. Problem Formulation

Consider a directed network consisting of a finite set of nodes N and a set of directed links $A \subset N \times N$. Specify the origin node O and the destination node D in the network and let P be the set of all feasible paths in the network from O to D. A path p from the node O to the node D is a sequence of links $p = \{ (i_0, i_1), (i_1, i_2), \ldots, (i_{d-1}, i_d)\}$, in which the initial node of each link is the same as the terminal node of the preceding link in the sequence, and $i_0, \ldots, i_d$ are all distinct nodes, where $i_0 = O$, $i_d = D$. With each link (i, j) of the network we associate a vector of attributes $c_{ij} = (c_{ij}^1, \ldots, c_{ij}^m)^T$. The vector of attributes corresponding to path p, denoted by z, is given by $z = \sum_{(i,j) \in p} c_{ij}$.

Let Z be the set of all attribute vectors z for all the paths in P and H(Z) be the convex hull of Z.

Definition 1. A path $p^i \in P$ is said to be nondominated if there does not exist any other path $p^j \in P$ such that $z^j \leq z^i$, with the inequality being strict for at least one component of z.

According to Definition 1, the NPP can be formulated as

$$\textbf{P1:} \quad \text{v-min} \quad z \qquad \text{s.t.} \quad z \in Z.$$

Let Z_N be the solution set of problem (P1), that is P_N is the set of all nondominated paths in P and Z_N is the set of corresponding nondominated attribute vectors.

Definition 2. A path $p \in P_N$ is said to be extreme nondominated if its attribute vector z is on the boundary of H(Z).

Let P_{EN} be the set of all extreme nondominated paths and Z_{EN} be the set of corresponding extreme nondominated attribute vectors. We observe that $P_{EN} \subseteq P_N \subseteq P$ as well as $Z_{EN} \subseteq Z_N \subseteq Z$.

In fact, the NPP can assume an equivalent formulation based on the linear program associated with the SPP. The following multiple objective linear program represents the NPP:

$$\textbf{P2:} \quad \text{v-min} \quad f(x) = [\, f_1(x), \ldots, f_m(x) \,] \qquad \text{s.t.} \quad x \in X,$$

where $f_k(x) = \sum_{(i,j) \in A} c_{ij}^k x_{ij}$, $k = 1, \ldots, m$, and X is the set of flows x_{ij} along the links in the network satisfying

$$\sum_{\{j:\,(i,j)\in A\}} x_{ij} - \sum_{\{j:\,(j,i)\in A\}} x_{ji} = \begin{cases} 1, & i = O \\ 0, & \text{for all } i \in N - \{O, D\} \\ -1, & i = D \end{cases}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A.$$

We can also observe that the set Z is the image of the feasible set X under the vector-valued mapping $[\, f_1(x), \ldots, f_m(x) \,]$. We refer to Z as the attribute (or objective) space and to X as the decision space of problems (P1) and (P2).

Let $F: R^m \to R$ be a real-valued cost function for evaluating each path in the set P, that is the cost associated with path p is F(z). The optimal path problem with a nonlinear multiattribute cost function is formulated as:

$$\textbf{P3:} \quad \text{minimize} \quad F(z) \qquad \text{s.t.} \quad z \in Z.$$

For F quasiconvex, Henig [6] showed that if a path $p^* \in P$ solves problem (P3), then it is nondominated, however the piecewise-linear surface $(\, (z^j, F(z^j)),\ z^j \in Z_N \,)$ is not unimodal. Therefore problem (P3) is a global optimization problem. The algorithms developed by Henig [6] as well as Mirchandani and Wiecek [8] solve problem (P3) in the set of extreme nondominated paths since they use the weighting scalarization to generate paths. Also in this case, the piecewise-linear surface

$((z^j, F(z^j)), z^j \in Z_{EN})$ is unimodal. These algorithms may not find an optimal solution of problem (P3) if it is nondominated but not extreme nondominated.

Throughout this paper we assume that the cost function F is continuous, monotone and convex over the attributes. Then the problem (P3) is equivalent to

P3': minimize $F(z)$

s.t. $z \in Z_N$.

Additionally, we also assume that F is differentiable.

3. Generating Nondominated Paths with The Tchebycheff Metric

In this section we first review the Tchebycheff metric scalarization technique and present its relationships with the NPP. Let u* be a perturbed utopia (ideal) vector in the objective space, that is

$$u_k^* = \min \{ z_k,\ z \in Z \} - \varepsilon_k,\ \varepsilon_k \geq 0,\ k = 1, \ldots, m. \qquad (1)$$

Given a point u*, a metric can be defined to measure the distance between u* and any $z \in Z$. Let $\| z - u^* \|^\lambda$ denote the weighted Tchebycheff metric, that is

$$\| z - u^* \|^\lambda = \max_k \{\lambda_k |z_k - u_k^*| \}, \qquad (2)$$

where $\lambda \in \Lambda = \{ \lambda \in R^m, \lambda_k > 0, \sum_{k=1}^{m} \lambda_k = 1 \}$, and $|y| = (|y_1|, \ldots, |y_m|)^T$.

The MWTM is defined as

$$\| z - u^* \|^\lambda = \max_k \{\lambda_k [|z_k - u_k^*| + \rho e^T |z - u^*|] \}, \qquad (3)$$

where ρ is a sufficiently small positive scalar and e^T is the vector of ones. By the definition of u*, $z - u^* > 0$ for all $z \in Z$, so the absolute value sign can be dropped in (2) and (3).

Definition 3. An element $v > u^*$ is said to define the isoquant of the MWTM and to be the vertex of this isoquant if and only if

$$\lambda_k = [1 / (v_k - u_k^* + \rho e^T(v - u^*))] [\sum_{j=1}^{m} 1 / (v_j - u_j^* + \rho e^T(v - u^*))]^{-1}, \qquad (4)$$

$$k = 1, \ldots, m.$$

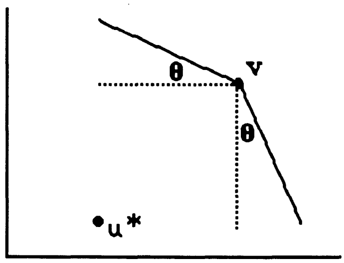

Fig. 1. The isoquant of the MWTM with vertex v.

Note that the vertex of the isoquant may not be an element of Z. Figure 1 shows the isoquant of the MWTM restricted to the set $\{y \in R^2: y_k \geq u_k^*, k=1, 2\}$, where v is the vertex of the isoquant and the angle $\Theta = \tan^{-1}(\rho/(1+\rho))$.

Given the NPP in the form of problem (P2), its MWTM scalarization is

P4: $$\min_{x \in X} \max_k \{\lambda_k [f_k(x) - u_k^* + \rho e^T (f(x) - u^*)] \},$$

which is equivalent to the following single objective program

P5: minimize α

s.t. $$\lambda_k [f_k(x) - u_k^* + \rho e^T (f(x) - u^*)] \leq \alpha, \quad k = 1, ..., m$$
$$x \in X.$$

Simplifying the inequality constraints in (P5) we get

P6: minimize α

s.t. $$\lambda_k [\sum_{(i,j) \in A} (c_{ij}^k + \rho \sum_{l=1}^m c_{ij}^l) x_{ij}] - \alpha \leq \lambda_k (u_k^* + \rho \sum_{l=1}^m u_l^*), \quad k = 1, ..., m$$
$$x \in X.$$

Now we state the main result that allows for algorithmic generation of any nondominated path in the network.

Theorem 1. Let
$$\rho < 1 / [(\max_k \max_{z, z' \in Z_N} | z_k - z_k' |) (m-1) - 1]. \tag{5}$$

A path $p^* \in P$ is nondominated if and only if there exists a $\lambda \in \Lambda$ such that the corresponding attribute vector z^* solves problem (P6).

Proof: Obviously, $p^* \in P_N$ if and only if $z^* \in Z_N$. Following Theorem 2 of Kaliszewski [7], given the parameter ρ as in (5), $z^* \in Z_N$ if and only if there exists a $\lambda \in \Lambda$ such that z^* solves problem (P6).

Without loss of generality we can assume that there exists exactly one nondominated attribute vector that minimizes objective z_k and this attribute vector does not minimize the other objective z_l, $l \neq k$, (otherwise the NPP would be trivial). This allows us to take $\varepsilon_k = 0$, $k = 1, 2$, in (1).

Next we decide upon a suitable value of the parameter ρ. Since $m = 2$, the right-hand-side expression of (5) can be easily found by solving the SPP with respect to each attribute separately. Let z^1 and z^2, be two distinct attribute vectors found by solving the SPP w.r.t. z_1 and z_2, respectively. Of course, all the other nondominated vectors will be located in the rectangular region R^{12} defined by z^1,

z^2, u^* and $u^{**} = (z_1^{\,2}, z_2^{\,1})$, see Figure 2. Using (4), we find the weights λ_k, k = 1, 2, such that u^{**} defines the isoquant of the MWTM.

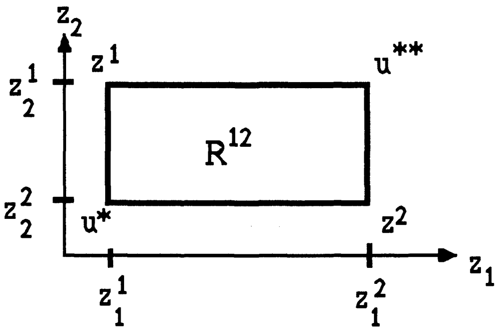

Fig. 2. Region R^{12}.

Consider now the behavior of the MWTM scalarization when problem (P6) is solved for the utopia point u^{rs} determined by any two vectors $z^r, z^s \in Z_N$, $z^r \neq z^s$ so that

$$u^{rs} = (z_1^{\,r}, z_2^{\,s}) \tag{6}$$

and u^{**} is given by

$$u^{sr} = (z_1^{\,s}, z_2^{\,r}). \tag{7}$$

Using (4) we find the weights λ_k, k = 1, 2, such that u^{sr} defines the isoquant of the MWTM. We get

$$\lambda_k = 1 / [\ (\ (u_k^{\,sr} - u_k^{\,rs} + \rho e^T(u^{sr} - u^{rs})) / (u_l^{\,sr} - u_l^{\,rs} + \rho e^T(u^{sr}\ u^{rs}))\ ) + 1\ ],$$

$$k, l = 1, 2,\ k \neq l.$$

Let $\Delta z^{rs} = u^{sr} - u^{rs}$, then

$$\lambda_k = (\rho \Delta z_k^{\,rs} + (1+\rho)\,\Delta z_l^{\,rs}) / (\,(1+2\rho)\,(\Delta z_k^{\,rs} + \Delta z_l^{\,rs})), \tag{8}$$

$$k, l = 1, 2,\ k \neq l.$$

Following the development above for the region R^{12}, consider now the region R^{rs} determined by the points z^r, z^s as well as the points u^{rs}, u^{sr} defined by (6) and (7). We investigate whether the region R^{rs} can contain another nondominated vector z^t. In some cases the size of the region may already imply that it may not contain a nondominated vector. Since every $z \in Z$ is integer, due to data integrality of the network, the following corollary results:

<u>Corollary 1.</u> **Size test**. Region R^{rs} may contain a nondominated vector z^t such that $z^t \neq z^r$ and $z^t \neq z^s$, if $\Delta z_k^{\,rs} > 1$ for both k (k=1,2).

If a region is large enough to accommodate a new nondominated vector we proceed to solving problem (P6) within this region.

Corollary 2. Let $z^r, z^s \in Z_N$, $z^r \neq z^s$. Let ρ be given as in (5), u^{rs} be the utopia point defined by (6), and λ be given as in (8) with $\Delta z^{rs} = u^{sr} - u^{rs} - e$. If z^* is an optimal solution of problem (P6), then the corresponding path p* is nondominated.

Problem (P6) solved according to Corollary 2, yields an optimal solution z^t (or multiple optimal solutions) located in the region R^{rs}. The following cases may be encountered:

i) z^t is a new nondominated solution or,

ii) $z^t = z^r$ or $z^t = z^s$,

Case ii) implies that there are no new nondominated solutions in the region R^{rs}. Hence, solving problem (P6) resolves whether any two nondominated vectors can accommodate a new one in the feasible rectangular region "between" them.

4. Solving the Optimal Path Problem

We propose two algorithms for solving the OPP that both use the MWTM scalarization and start with finding z^1, z^2 that define the region R^{12}. The algorithms generate new regions within the region R^{12} and examine them for nondominated solutions.

Each algorithm is equipped with a different technique of generating subsequent regions. Once the region R^{12} is specified, ALGORITHM N uses the MWTM scalarization to generate nondominated paths and creates regions nested in the regions previously found.

Since a variety of parametric procedures for enumeration of paths in Z_{EN} have been developed (Henig [6], Brumbaugh-Smith and Shier [1], and others), we also propose ALGORITHM EN that uses this approach. All the extreme nondominated paths are generated and the regions created by adjacent extreme nondominated attribute vectors are found. The MWTM scalarization is then used to search for other nondominated paths within those regions.

However, a region needs to be searched for a new solution only if it passes the size test (Corollary 1) and it is not *dominated* by another region. Definition 4 provides the test for dominated regions and Corollary 3 explains why dominated regions should be eliminated from further consideration .

Definition 4. **Dominance test.** Let R^{rs} and R^{tv} be two distinct regions determined by z^r, z^s, u^{rs}, u^{sr} and by z^t, z^v, u^{tv}, u^{vt}, respectively. If $F(u^{sr}) < F(u^{tv})$, then a region R^{tv} is said to be *dominated* by a region R^{rs}.

Corollary 3. If a region R^{tv} is dominated, then it does not contain an optimal solution of the OPP.

Proof: If R^{tv} is dominated by a region R^{rs}, then $F(u^{sr}) < F(u^{tv})$. Since F is differentiable, monotone and convex, then for any $z^1, z^2 \in Z$ such that $z^1 < z^2$, there is $F(z^1) < F(z^2)$. Therefore, $F(u^{tv}) < F(z)$ for any $z \in R^{tv}$ and $z > u^{tv}$. Hence there is no point $z^0 \in R^{tv}$ such that $F(z^0) \leq F(u^{sr})$.

Below we present both algorithms in generic form.

ALGORITHM N
begin
  optimal = ∞;
  p^1 := shortest path w.r.t objective z_1;
  p^2 := shortest path w.r.t objective z_2;
  $z^1 := (z_1^{\ 1}, z_2^{\ 1})$, where $z_2^{\ 1} := \sum_{(i,j) \in p^1} c_{ij}^{\ 2}$; Find $F(z^1)$;
  $z^2 := (z_1^{\ 2}, z_2^{\ 2})$, where $z_1^{\ 2} := \sum_{(i,j) \in p^2} c_{ij}^{\ 1}$; Find $F(z^2)$;

* ListR := $\{R^{12}\}$;
  while ListR $\neq \varnothing$ **do**
  begin
    select region R^{rs} from ListR;
    newpath := FALSE;
    if $\Delta z_k^{\ rs} > 1$, k=1,2 **then**
      if $F(u^{rs}) \leq F(u^{vt})$ for every R^{tv} in ListR **then**
        if z^t exists in R^{rs} **then**
          add R^{rt} to ListR;
          add R^{ts} to ListR;
          newpath = TRUE;
    if newpath = FALSE **then**
      if $F(z^r)$ < optimal **then**
        optimal := $F(z^r)$;
        path := p^r;
    remove region R^{rs} from ListR;
  end;
  if $F(z^2)$ < optimal **then**
    optimal := $F(z^2)$;
    path := p^2;
end;

ALGORITHM EN
follow ALGORITHM N except replace line * with the following:
  enumerate all other paths in Z_{EN};
  find R^{rs} for each pair of adjacent attribute vectors $z^r, z^s \in Z_{EN}$;
  add each R^{rs} to ListR;

Although solving some bi-attribute problems may still result in enumerating all paths in Z_N, we emphasize that the size test as well as the dominance test may significantly reduce complexity of the algorithms if the MWTM scalarization is applied not to every region of the attribute space. Furthermore, as the algorithms have the ability of generating nondominated paths *individually* in different regions, they are attractive for parallelization since multiple searches can be conducted simultaneously within different regions of the attribute space.

5. An Example

We illustrate ALGORITHM N with the following example. Consider the bi-attribute network depicted in Figure 3. The cost function is $F(z) = (z_1 - 4)^2 + (z_2 - 2)^2$ and $Z_N = \{ (1, 25), (4, 20), (5, 13), (8, 12), (13, 10), (17, 7) \}$.

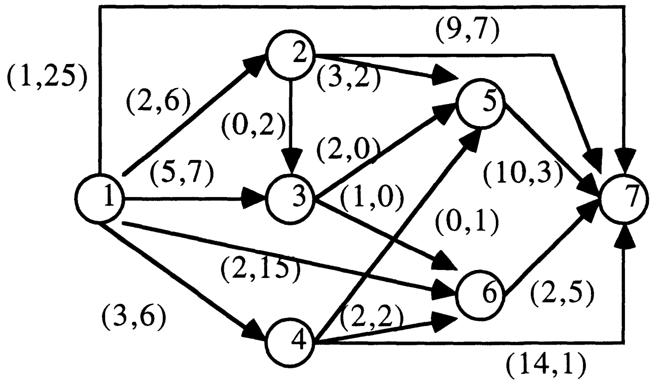

Fig. 3 A bi-attribute network.

The algorithm finds p^1 with $z^1 = (1, 25)$ and p^2 with $z^2 = (17, 7)$.
This creates region R^{12} and ListR = $\{ R^{12} \}$.
Then the following steps are executed:
Select R^{12} and find $z^3 = (5, 13)$. ListR = $\{ R^{13}, R^{32} \}$.
Select R^{13} and find $z^4 = (4, 20)$. ListR = $\{ R^{14}, R^{43}, R^{32} \}$.
Select R^{14} and $F(u^{14})>F(u^{34})$. Optimal = 538. path = p^1. ListR = $\{ R^{43}, R^{32}\}$.
Select R^{43} and $\Delta z_1{}^{43} = 1$. Optimal = 324. path = p^4. ListR = $\{ R^{32} \}$.
Select R^{32} and find $z^5 = (8, 12)$. ListR = $\{ R^{35}, R^{52} \}$.
Select R^{35} and $\Delta z_2{}^{35} = 1$. Optimal = 122. path = p^3. ListR = $\{ R^{52} \}$.
Select R^{52} and find $z^6 = (13, 10)$. ListR = $\{ R^{56}, R^{62} \}$.
Select R^{56} and find no path Optimal = 116. path = p^5. ListR = $\{ R^{62} \}$.
Select R^{62} and find no path. Optimal = 116. path = p^5. ListR = $\{\varnothing\}$.
Optimal path = p^5 = 1-3-6-7. Optimal value = 116.

6. Conclusions

This paper presents an application of the MWTM scalarization of Kaliszewski [7] to the optimal path problem with nonlinear convex multiattribute cost functions, that is a global optimization problem over a discrete feasible space. This problem has been studied in the literature and several algorithms producing an approximation of the optimal solution have been proposed.

It is shown that the TM approach yields an exact solution of the optimal path problem. Two algorithms are proposed for the bi-attribute case and presented in generic form. They both use the MWTM scalarization and generate nondominated paths in the network in order to find an optimal path.

Further studies have to be conducted on refinement of the proposed algorithms with regard to their complexity and implementation. Furthermore, the MWTM scalarization could be substituted by the augmented weighted TM scalarization of Steuer and Choo [9]. A final stage of research could include generalization of the methods to the multiattribute case, however, computational complexity would become quite involved. The authors believe that in this case parallelization of the methods would be indispensable.

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Part 3

Applications

3.1 Applications in Engineering

MULTICRITERIA OPTIMIZATION OF ABS CONTROL ALGORITHMS USING A QUASI-MONTE CARLO METHOD

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Abstract. The Quasi-Random Weighted Criteria method is proposed for multicriteria design optimization. This quasi-Monte Carlo method features increased computational efficiency and is particularly suitable for exploring alternative design configurations. A quasi-random sequence generates a set of candidate solutions representative of the range of available solutions for each design alternative. The method can be used recursively to produce more detailed Pareto surface descriptions near selected points.

In this paper the method is used to select between vehicle anti-lock brake system (ABS) control algorithm approaches and to optimize the parameters within each. An ABS system is highly nonlinear and therefore the control algorithms draw upon the methods of nonlinear control theory. Stochastic optimization was incorporated to ensure that the ABS system will perform well despite the uncertainties in the vehicle and in the environment. A variety of ABS design studies are presented.

KEY WORDS: ABS control, optimal design, multicriteria, optimization

1 INTRODUCTION

Computer assisted vehicle brake systems are commonly referred to as *anti-lock brake systems* (ABS). An anti-lock brake system makes adjustments to the fluid pressures that govern the four brakes on a vehicle. These adjustments modulate the outputs from the brakes to increase utilization of the road-to-tire adhesion.

The design of ABS control algorithms is a difficult task for the traditional approaches of intuitive design, test evaluation and analytical derivation. Many competing criteria must be considered. An intuitive understanding is difficult to develop for such a complex, nonlinear system that must operate with a wide range of stochastic conditions.

A multicriteria optimization method is proposed which is particularly suited to design problems with many competing criteria, with preferences between criteria difficult to decide a priori, and with alternative design strategies to compare, as is the case with the many approaches to non-linear control algorithm design.

Section 2 discusses the method, while Section 3 descibes ABS control algorithm design studies.

2 THE QUASI-RANDOM WEIGHTED CRITERIA METHOD

This section begins with a brief review of three similar methods. The proposed method is then described.

2.1 Similar Methods

Methods for problems with large numbers of criteria can carry large computational costs. In those described here the decision maker's iterative exploration of preference structure is no longer the focus of the solution effort. Rather, these methods attempt to provide many candidate solutions from which the decision maker can pick the most preferred.

2.1.1 Meisel's Method

Meisel proposed the use of a random input to direct the selection of weights for a weighted criteria formulation in repeated scalar substitute optimizations [11]. Meisel's method can generate large numbers of points distributed over the attainable set. Because the weights are randomly distributed the method is not computationally efficient. Also, Meisel's method does not capture Pareto optimal points corresponding to an even distribution across the ranges of possible criteria weight ratios, because the weight ratios are not evenly distributed.

2.1.2 Osyczka's Method

Osyczka [13] suggests the use of a Monte Carlo method to pick design variables over their estimated ranges. Criteria values are calculated for each set of design variables. This process generates a set of feasible points. In general many points will not be Pareto optimal and therefore the Pareto set must be extracted. There is no certainty that the set adequately represents the attainable Pareto set.

2.1.3 Statnikov's Method

Rather than utilizing randomness to pick design variables, R.B. Statnikov suggested a search strategy to generate a representative set of feasible points (22, 21, 19, 10]. A uniformly distributed scattering of feasible trial points is generated using quasi-random sequences (described below). Statnikov's method uses these sequences to deliver a uniform and efficient distribution for the design variables.

Because Statnikov's method is similar to Osyczka's in that it is based upon the generation of a large set of feasible points from which Pareto optimal points must be extracted, it too will require many calculations. Stadler and Dauer note that there is presently no assurance that this approximation converges in some fashion to the set of Pareto optimal solutions for a given multicriteria optimization problem [20].

2.2 Proposal of a New Multicriteria Optimization Method

The Quasi-Random Weighted Criteria (QRWC) Method [2] solves a series of weighted criteria scalar substitute problems with systematically adjusted scalar criteria weights. The method utilizes *quasi-random sequences* to generate the weights. Quasi-random sequences cover a hypervolume evenly and efficiently, while maintaining random-like relations between the components.

A distribution over the ranges of all criteria weight ratios is produced with the quasi-random sequences. The distribution is chosen to provide optimal solutions that can be considered *representative* of the potential combinations of criteria weights, making the method useful in quantitative comparisons between competing design strategies. Furthermore, points are spaced in the most efficient manner in that the weights for each additional scalar optimization are selected to provide a maximum amount of new information.

The method can be used recursively. After a set of candidate solutions is generated, a subset is selected by the decision maker and a new set of candidate solutions is generated within the neighborhood of the selection. The process can be repeated with increasingly smaller neighborhoods until the decision maker selects a final solution.

2.2.1 Generation of the Quasi-Random Sequence

In the QRWC Method, I - 1 components of the weight vector are calculated as independent quasi-random numbers from the *Hammersley point set.* This set is a finite sequence of points created to cover a hypercube uniformly and with low discrepancy. Discrepancy is a measure of the efficiency of a series in covering a volume. A low discrepancy sequence is one that is distributed such that each point is near a large amount of hypervolume that is not near other points. Niederreiter in his book on quasi-Monte Carlo methods [12] states that it is widely believed that the Hammersley point set with pairwise relatively prime bases attains the discrepancy of smallest possible magnitude.

Because the Hammersley point set was devised to cover a hypercube, not a simplex, a projection into a simplex must be made. It can be shown that the discrepancy of the projected series is lower than the discrepancy of the original series in the hypercube [2].

2.2.2 The Hammersley and the Shifted Hammersley Point Sets

There are a number of quasi-random series and point sets. The point set used in this work is termed the shifted Hammersley sequence, originally suggested by van der Corput [27] and developed by Halton [7] and Hammersley [8].

Shifting the first component is advantageous. Wozniakowski [28] shows that shifting the first component to $(l+t) / L$, with $0 \le l + t \le L$ reduces the average case error in numerical integration for one class of functions. Wozniakowski did not specify a value for t. This parameter was set to -0.5 in the studies reported in this paper. That value makes the first component of the series symmetric over the segment (0,1). The Hammersley sequences approach this symmetry asymptotically as L increases, but because the QRWC method uses small point sets the establishment of symmetry was judged desirable.

3 ANTI-LOCK BRAKE SYSTEM ALGORITHM DESIGN STUDY

In the present study the criterion vector **c** has six components and the parameter vector **b** has four components representing stochastic influences. The dimension of the design variable vector **x** varies with the algorithms, from two to five.

Variables related to ABS hardware are also included in some of the design studies. All design variables are unconstrained except for simple upper and lower bounds.

3.1 Criteria Functions

(1) Minimize ABS Inefficiency c_1 to reduce stopping distance.
(2) Minimize Average Tire Slip c_2 to improve steerability.
(3) Minimize Deceleration Variability c_3 to improve passenger comfort.
(4) Minimize Chatter (actuator reversals) c_4 to increase hardware life.
(5) Minimize Sensitivity c_5 of the ABS efficiency criterion to errors in the wheel deceleration measurement.
(6) Minimize Transition Response Inefficiency c_6 to improve response to a sudden change in tire-pavement adhesion.

3.2 Stochastic Parameters

The simulation includes four stochastic influences upon ABS performance: vehicle loading b_1, tire-pavement adhesion b_2, initial vehicle speed b_3, and brake output hysteresis b_4.

The relative importance of the stochastic influences in the ABS design studies presented in the next section was determined by accident data [27,26]. If accidents occur five times more often on dry pavement than on wet pavement, an improvement in dry pavement performance is given five times the importance of an improvement in wet pavement performance. Stochastic conditions are integrated into a single quantity for the objective function by expressing their relative importance in terms of accident probability:

$$\mathbf{c(x)} = \int_{\mathbf{b}_{low}}^{\mathbf{b}_{high}} (\text{Probability that } \mathbf{b} \text{ occurs in an accident})\ \mathbf{c(x,b)}\ \mathbf{db}. \qquad (1)$$

Here $\mathbf{c(x)}$ is evaluated over the range of stochastic parameters and is then used in an optimization conducted over all feasible $\mathbf{x}$. In numerical evaluation the integral is replaced by a summation of a discrete set of stochastic conditions.

3.3 ABS CONTROL ALGORITHMS

Three ABS control algorithms are compared in this study: the common *phase plane* and *sliding mode* methods, and a recent modification of sliding mode control. In phase plane control look-up tables determine a controller output for any combination of state variable values [4]. The sliding mode algorithm was based upon a description by Tan [23] that does not explicitly consider implementation with discrete controllers. Therefore, Tan's algorithm was adapted by defining four bands parallel to the sliding mode surface, as suggested by Gibson's "Dual Mode Switching" [6]. In the modified sliding mode scheme proposed by Tomizuka and Tan [24] the bounded or slow-varying uncertainties of the brake system are grouped together and treated as disturbances, estimated by a derivative feedback. The resultant system is locally linearized by a nonlinear prefilter and linear digital control theory is used to determine the controller design variables.

These three algorithms were adapted to a simulation controller capable of five outputs: fast and slow pressure release, fast and slow pressure apply, and hold pressure. The second and third algorithms utilize a surface identification estimation algorithm to update estimates of the tire slip level that corresponds to peak adhesion. A least squares estimation with a forgetting factor was used.

Theoretical control schemes must be modified when implemented on a vehicle system to accommodate system limitations. Modifications increase the difficulty for intuitive grasp or theoretical analysis of the control problem, and a systematic design optimization process becomes more advantageous.

3.4 THE SIMULATION

A quantity termed *wheel slip* is important in the analysis of braking. It is a measure of the difference between vehicle velocity and the velocity of the tire at its point of contact with the road:

$$\% \text{ Wheel slip} = \frac{V\text{-}wr}{V}\, 100\% \qquad (2)$$

where V is vehicle velocity, w is wheel angular velocity and r is wheel radius. The ABS controller is designed to maintain wheel slip near the value corresponding to peak adhesion. The basic elements of an ABS system are a wheel sensor, an anti-lock pressure modulating valve, and an anti-lock controller. A four-wheel ABS system would consist of four sets of these components. This study uses a simulation of one set, termed a one-quarter vehicle model.

The dynamic system is solved with a fourth order Runge Kutta method [9]. The state equations are

$$\dot{s} = v, \qquad \dot{v} = g\mu + 0.5\, C_d v^2, \qquad \dot{\omega} = (\mu F_z R - B)/I \qquad (3)$$

where s is distance covered by the vehicle, v is vehicle velocity, ω is wheel rotational velocity, g is the gravitational constant, μ is coefficient of tire-to-road adhesion, C_d is coefficient of aerodynamic drag, F_z is normal force between tire and road, R is tire radius, B is brake torque to retard the wheel, and I is wheel rotational inertia.

The simulation begins with a simulated driver input consisting of a fixed ramp pressure apply rate of 1900 N/m^2s lasting one second. Line pressure is then held constant until the ABS system is triggered. For simplicity, driver input is not considered after the ABS system has been triggered.

The following assumptions were made for the simulation:

1. The ABS system is a sampled data system receiving updated inputs every 8 milliseconds and responding at that same time, maintaining the response until the next update.
2. There is no delay in the reception of ABS input information.
3. There is no delay nor hysteresis in ABS actuator response.
4. ABS actuator response levels are exactly delivered.
5. Wheel speeds are accurately known.

6. Vehicle deceleration input contains noise. In hardware implementations vehicle deceleration is estimated with imprecision. In the simulation a normal distribution is used assuming that noise is generated by independent, additive factors. If independent, multiplicative factors were assumed a log-normal distribution would be more appropriate [1] The standard deviation of 10 m/s^2 was selected by engineering judgment because no published data were available. The same random sequence is generated for each simulation (a *common random series*) to provide fair comparison among simulations.
7. Wheel inertia is accurately known and is set at 1.1 kg m^2.
8. Aerodynamic drag is included, considered proportional to the square of vehicle velocity, $D=0.5C_dV^2$, where $C_d = 0.3$ is the coefficient of aerodynamic drag and V is vehicle velocity.
9. The ABS actuator controlling brake input pressure has two rates of change, 8,500 and 17,000 N/m^2s in both apply and release, and a pressure hold. Due to cost considerations, most ABS systems use discrete type actuators producing only a limited number of rates of pressure change.
10. Powertrain braking is negligible.
11. The brake input-to-output function is exactly known. In actual operation the controller modulates brake line pressure but actual brake torque output is unknown.
12. The brake input-to-output function is linear.
13. Wheel radius is accurately known and is set at 0.307 m.

3.5 RESULTS

The ABS models described in the previous sections generally result in criteria functions that have many local minima and that may be non-smooth. The controller works on a sampled system so controller input and output occur at intervals. Furthermore, a discrete ouput ABS actuator must call for one rate from a very limited set rather than provide a continuous adjustment of pressure levels.

Local gradient-based optimization algorithms will fail under such conditions. One alternative is to use a global optimization method. Another is to average the functions over runs repeated with stochastic variable variations. The averaging results in smoothing of the response surface because in effect it performs an integration of the function over the ranges of stochastic variables [3].

In the present study scalar optimizations were conducted using both a local optimizer, the sequential quadratic programming method NLPQL [17,16] and a global optimizer, the simulated annealing method *Hide and Seek* [14]. The local optimizer results were sensitive to starting point and consistently had a larger objective value than did the global optimizer solutions, even with stochastic variable integration. All subsequent optimizations therefore relied upon the global optimizer.

A number of ABS design studies are summarized in the following subsections. For details please refer to [2].

3.5.1 Comparison Between Algorithms

Solution sets of six points each were found for each of the three algorithms given above. Overall, sliding mode control did not match the capabilities of the other two control algorithms, see Fig 1. Note that for each algorithm there is little variation among the six solutions. The objective function criteria weights were varied between the optimizations to represent the ranges of possible relative weights, and so the limited variation indicates that these algorithms are not capable of much range in their performance trade-offs.

3.5. Inclusion of Control Hardware in MSM Controller Optimization

Other quantities can be designated as design variables in the optimizations. For example, the controller rates, previously fixed at 17000 N-m/sec and 8500 N-m/sec, can be allowed to vary. This was done with the MSM algorithm. The average improvement in the weighted criteria objective function was 7.9%. The results suggest that a slightly lower slow rate and a slightly higher fast rate would improve performance. Chatter was reduced 19%.

3.5.3 Inclusion of ABS Activation in MSM Controller Optimization

In both sliding mode and MSM algorithms the ABS system is activated when tire slip rises above the value estimated to correspond to peak adhesion. This activation was set intuitively. A pattern of optimization runs was conducted to optimize ABS activation in the MSM controller. An offset from the previous trigger value was added as design variable. Including the trigger offset brought a decrease in the average objective function value across six optimizations of 6.11%.

3.5.4 Impact of Sampling Rate

Optimizations were conducted with the MSM algorithm to examine the impact of slower (10 ms) and faster (6 ms) sampling rates. .Faster sampling brings a 12.5% decrease in the 6 objectives, while slower sampling brings a 5.83% increase. Comparing the 10 ms results with the 6 ms results, the faster sampling reduced average slip from 0.233 to 0.166, but actually increased average chatter and sensitivity.

4 CONCLUDING REMARKS

The simulation used in this article was adequate to demonstrate method capabilities, yet could be enlarged to better represent the complete ABS design problem. The studies are representative examples from many that could be conducted. In actual application further analysis would be undertaken to generate information pertinant to the selection of a final solution. The QRWC method can be focused upon a region of particular interest, and the focus can be narrowed with each set of evaluations until a solution is selected.

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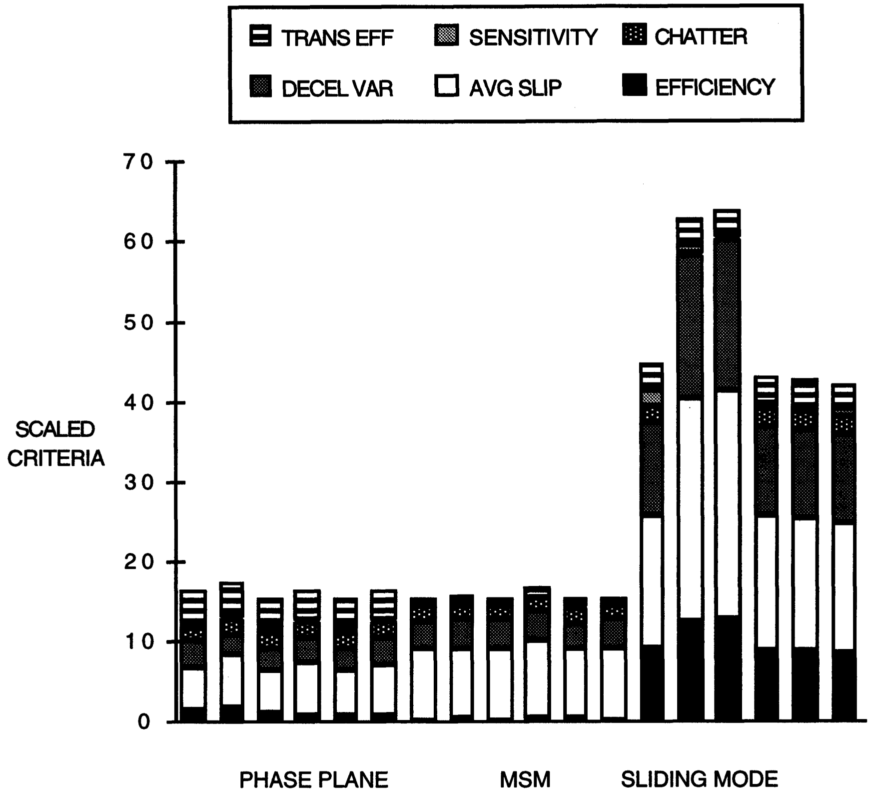

Figure 3.1 Comparison between the Three Algorithms

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Dynamic System Design via Multicriteria Optimization

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Abstract. Dynamic system design is formulated as a multicriteria optimization problem. The dynamic behavior of dynamic systems is described by nonlinear differential equations of motion found from the multibody system approach. Parameters of the multibody system serve as design variables in order to optimize the system with respect to its dynamic behavior which is evaluated by multiple integral type objective functions. Multicriteria optimization strategies are used to reduce the design problem to nonlinear programming problems. The gradients required for optimization are found from a semi–analytical sensitivity analysis. Problems resulting from an application of conventional optimization strategies to a vehicle design problem are discussed.

Keywords. Dynamic system design, multibody system approach, simulation, multicriteria optimization, sensitivity analysis.

1 Introduction

Design of dynamic systems is usually based on intuition and experience. Although industrial companies have started to switch from experimental studies to computational methods based on the multibody system approach, design of mechanical systems is still performed by intuitive changes of the design variables. This is due to the fact that the multibody system approach is well developed for analyzing the dynamic behavior of mechanical systems in a variety of disciplines [11], but there is a lack of tools for a systematic system design via optimization.

As far as optimization methods are used in dynamic system design, such applications are often restricted to single criterion optimization. Solutions obtained with such an approach are not fully satisfying for practical applications, since they take into consideration only a single aspect out of a couple of conflicting system requirements and specifications. In most cases, such 'optimal' solutions are even inferior to solutions obtained by common sense.

The multicriteria optimization approach seems to offer a promising way to change experimental system design to computer–aided design. Important, however, is that in the context of multibody dynamics criteria are highly nonlinear and their evaluation is rather time–consuming, since the evaluation process involves numerical integration of differential equations of motion. For this reason, many of the multicriteria optimization strategies developed for economical problems with rather simple criteria and constraint functions cannot directly be applied to dynamic system design. Preferable are strategies which require only a minor number of criterion evaluations by reducing the vector optimization problem to nonlinear programming problems. As mechanical engineers, the authors have the impression that there is a lack of multicriteria optimization algorithms of this type. In the paper, therefore, well known multicriteria optimization methods are applied to dynamic system design in order to show deficiencies and to encourage the optimization community to improve their methods for applications in mechanical engineering.

2 Design Concept

An integrated design approach for dynamical systems has to support all steps from problem formulation to problem solution by optimization, Fig. 1. Firstly, the technical system to be optimized has to be transformed to a mathematical model. The multibody system approach is widely used in vehicle dynamics, robotics, machine dynamics, biomechanics, and other disciplines. Then, design goals have to be defined which is often difficult, since technical requirements and human wishes are sometimes hard to be formulated as mathematical functions. Beside the complexity of the models this is maybe one of the reasons why even integrated design methods cannot substitute the design engineer, but support by software systems will help to make design better. In order to improve technical systems, design changes have to be made. Therefore, parameters of the model have to be classified either as design variables whose values can be varied for optimization purposes or as system constants whose values are fixed during optimization.

After these preparing steps, the optimization part of the design process may be started. In case of several criteria, a multicriteria approach has to be applied. Due to our experience, reduction of the design problem to one or a sequence of standard nonlinear programming problems has shown to be most efficient in the context of dynamic system design.

While realizing the final solution, but also in the steps of problem formulation and solution, contradictions can arise which enforce to change the model, the set of design variables or the performance criteria. In order to simplify such changes, both the modeling and the optimization part of the design process have to operate on the same data basis. This was the reason for developing an integrated design approach covering modeling as well as optimization. Numerical and symbolical methods are combined in the program package NEWOPT/AIMS (Analyzing and Improving Multibody Systems) [3] to support and highly automate the design process. Graphical user interfaces interactively lead through problem formulation and solution. Animation of the dynamic system helps to get a realistic impression of its dynamic behavior. In the following, several aspects of the design concept will be described in more detail.

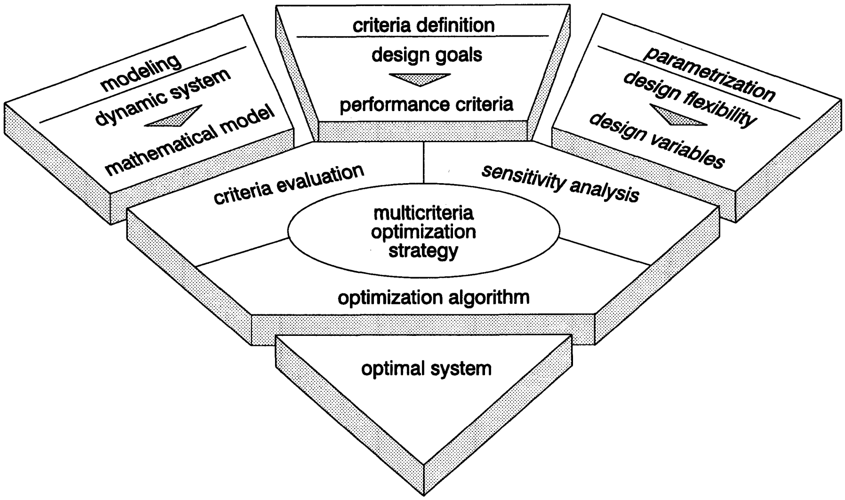

Fig. 1: Integrated modeling and design approach

3 Problem Formulation

The optimization problem is defined by three steps: modeling, parametrization and criterion definition.

3.1 Multibody System Modeling

Modeling requires several steps of mechanical and mathematical idealization. In the multibody system approach, deformations of the bodies are neglected and the bodies are considered as rigid. They are connected by ideal links and bearings allowing some well–defined motion of the bodies relative to each other. Coupling elements like springs, dampers and actively controlled elements determine the dynamics of the system. A spatial model of a vehicle is shown in Fig. 2.

The motion of a multibody system can be described by generalized coordinates $y \in I\!R^f$ and generalized velocities $z \in I\!R^f$ where f is the degree of freedom of the system. The dynamic behavior is then given by an initial value problem:

$$\begin{aligned} \dot{y} &= v(t,y,z), & y(t^0) &= y^0, \\ M(t,y)\,\dot{z} + k(t,y,z) &= q(t,y,z), & z(t^0) &= z^0. \end{aligned} \tag{1}$$

The differential equations of motion can be obtained from mechanical principles [1]. For technical applications, these equations are too complex to be generated by hand. Several computer codes have, therefore, been developed for an automatic generation. Especially in NEWOPT/AIMS, the code NEWEUL is used which generates the equations of motion in symbolical form [9].

Starting from initial conditions y^0, z^0 the equations of motion have to be solved by numerical integration. In the past, even a single analysis step involving such a numerical integration was such time–consuming that optimization seemed to be

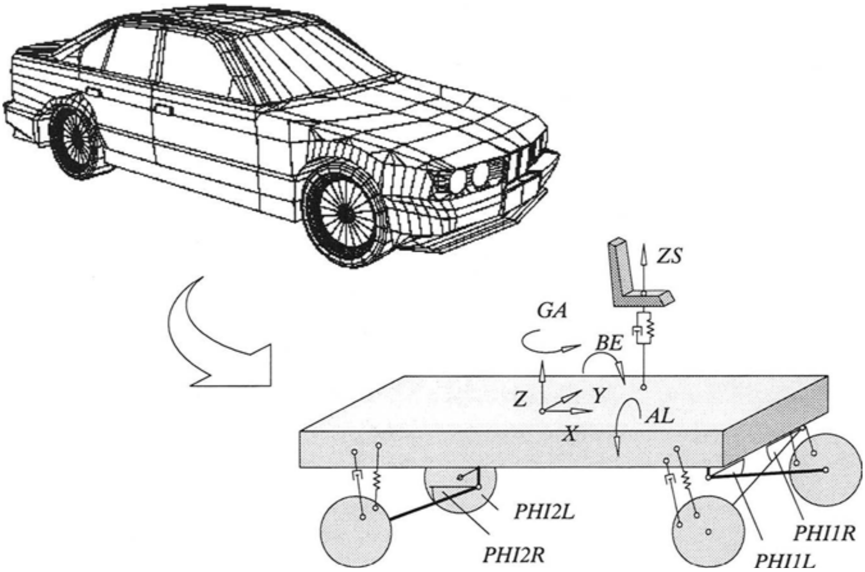

Fig. 2: Spatial multibody system model of a vehicle

utopian. Due to the remarkable improvements in computer technology, optimization of dynamic systems has become realizable, but an analysis step is still much more time–consuming than performing the optimization.

3.2 Design Variables

Model parameters like mass and moments of inertia of the individual bodies, damping and stiffness coefficients of coupling elements, and geometrical dimensions completely determine the dynamic behavior of multibody systems. Some of these parameters are certainly fixed due to technical restrictions and have to be considered as design constants. Others may be changed within given ranges for changing the dynamic behavior of the multibody system. These system parameters can be chosen as design variables and summarized in a vector $\boldsymbol{p} \in I\!R^h$. Such design variables directly influence the dynamic behavior of the multibody system via the kinematic relations $\boldsymbol{v}$, the mass matrix $\boldsymbol{M}$, the centrifugal and Coriolis forces $\boldsymbol{k}$, and the applied forces $\boldsymbol{q}$ in Eq. (1). If these dependences are taken into account, the equations of motion have to be rewritten as

$$\begin{aligned} \dot{\boldsymbol{y}} &= \boldsymbol{v}(t,\boldsymbol{y},\boldsymbol{z},\boldsymbol{p}), \\ \boldsymbol{M}(t,\boldsymbol{y},\boldsymbol{p})\,\dot{\boldsymbol{z}} + \boldsymbol{k}(t,\boldsymbol{y},\boldsymbol{z},\boldsymbol{p}) &= \boldsymbol{q}(t,\boldsymbol{y},\boldsymbol{z},\boldsymbol{p}). \end{aligned} \tag{2}$$

The initial conditions, too, can depend on the design variables. Since such dependences are not necessarily given as explicit functions, a more general formulation as implicit functions is chosen in NEWOPT/AIMS:

$$\boldsymbol{y}^0 : \boldsymbol{\Phi}^0(t^0,\boldsymbol{y}^0,\boldsymbol{p}) = \boldsymbol{0}, \qquad \boldsymbol{z}^0 : \dot{\boldsymbol{\Phi}}^0(t^0,\boldsymbol{y}^0,\boldsymbol{z}^0,\boldsymbol{p}) = \boldsymbol{0}. \tag{3}$$

If values are assigned to the design variables, the motion of the multibody system is completely determined by Eqs. (2) and (3).

3.3 Criteria

Use of numerical optimization methods requires that technical restrictions and design goals are formulated as mathematical functions. Due to historical reasons, such criterion functions are often classified either as objective functions or as constraints at the very beginning of the design process. With respect to multicriteria optimization, such a classification is already part of the optimization strategy and should be avoided. Within the context of optimizing multibody systems, a classification with respect to the way of computing gradients seems to be more appropriate.

Easy to handle are explicit criteria

$$\psi_i^E = \psi_i^E(\boldsymbol{p}), \quad i = 1(1)n_E, \tag{4}$$

which depend on the design variables only. The sensitivities of the values ψ_i^E on changes of the design variables can easily be found by direct differentiation:

$$\nabla\psi_i^E = \frac{d\psi_i^E}{d\boldsymbol{p}}, \quad i = 1(1)n_E. \tag{5}$$

In order to avoid computational errors, the differentiation should be supported by computer algebra packages like MAPLE [6] or Automatic Differentiation [5].

A second type of performance criterion evaluating the dynamic behavior of multibody systems can be formulated as an integral type performance function

$$\psi_i^I = G_i^1(t^1, \boldsymbol{y}^1, \boldsymbol{z}^1, \boldsymbol{p}) + \int_{t^0}^{t^1} F_i(t, \boldsymbol{y}, \boldsymbol{z}, \dot{\boldsymbol{z}}, \boldsymbol{p})\, dt, \qquad i = 1(1)n_I, \tag{6}$$

which is closely related to optimal control problems. The first term accounts for cases where special values for the final state $\boldsymbol{y}^1$, $\boldsymbol{z}^1$ or a minimum time t^1 must be achieved, the second term evaluates the dynamic behavior within an interesting time interval $[t^0, t^1]$. The final time t^1 may be fixed or given implicitly by the final state:

$$t^1 : \; H^1(t^1, \boldsymbol{y}^1, \boldsymbol{z}^1, \boldsymbol{p}) = 0. \tag{7}$$

Although the functions G_i^1 and F_i depend on the state variables and not on the design variables only, the functions ψ_i^I are determined entirely by the values of the design variables $\boldsymbol{p}$. Starting from given values for the design variables, the state trajectories and the final time can be found from the initial value problem (2) and (3), and the final condition (7). Based on this information the integral type criteria are evaluated. The whole process of evaluating the criteria can also be considered as black box where there exist unique relations between input $\boldsymbol{p}$ and output ψ_i^I, i.e. $\psi_i^I = \psi_i^I(\boldsymbol{p})$. Since these relations are not known analytically, the sensitivity functions cannot be found by direct differentiation.

Using finite differences has shown to be rather inefficient, inexact, and unreliable. Therefore, a semi–analytical approach was chosen for NEWOPT/AIMS. Especially, the adjoint variable method has proven to be very efficient and reliable for computing gradients of integral type performance criteria [2]. By introducing new variables, called adjoint variables, a problem invariant structure of additional equations can be found which are solved numerically, Fig. 3. Although the adjoint variable approach is much more efficient than the finite difference approach, it is about four to five times more time–consuming than the function evaluation itself. Optimization strategies applied to dynamic system design should, therefore, use more function evaluations than gradient computations.

If criteria do not fit into one of these two definitions, NEWOPT/AIMS allows to use fully user–defined criterion evaluations by simple interfaces. In such a case, the user also has to provide gradient information on his own.

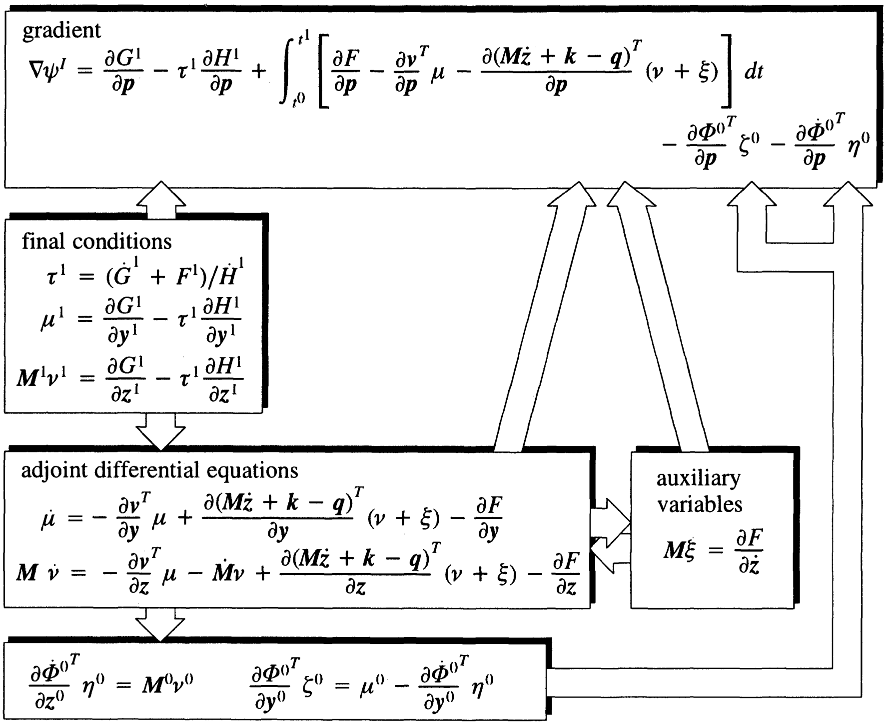

Fig. 3: Adjoint variable method for multibody systems

4 Multicriteria Optimization

In a classical optimization approach only a single objective function is minimized leading to a unique solution in general. Practical design problems do not look like this. Dynamic systems have to be optimized with respect to several conflicting specifications and several different designs are acceptable as optimal with respect to the same set of specifications. The theory of multicriteria optimization helps to avoid such discrepancies between theory and practise. Several criteria can be minimized simultaneously, and optimal compromise solutions can be defined if conflicts arise. In general, these optimal solutions are not unique but a whole set of solutions exists which are not comparable to each other. The designer can then decide on a final solution due to his intuition or additional information which has not been formulated mathematically.

In the first step of optimization, the criteria stated above have to be classified with respect to their purpose. Some criteria may be defined to exclude infeasible solutions. Such criteria are used either as equality constraints $g_i(\boldsymbol{p}) = 0$, $i = 1(1)l$, or inequality constraints $h_j(\boldsymbol{p}) \geq 0$, $j = 1(1)m$. Criteria whose values should be decreased may be used as objective functions. In general, several objective functions are left after such a classification leading to a multicriteria or vector optimization problem:

$$\underset{\boldsymbol{p}\in\mathcal{P}}{opt}\ \boldsymbol{f}(\boldsymbol{p}) \qquad where\ \ \mathcal{P} := \left\{\boldsymbol{p} \in \mathbb{R}^h \mid \boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{0},\ \boldsymbol{h}(\boldsymbol{p}) \geq \boldsymbol{0}\right\}. \tag{8}$$

Optimization of the vector criterion function $\boldsymbol{f}(\boldsymbol{p})$, $\boldsymbol{f} : \mathbb{R}^h \rightarrow \mathbb{R}^n$, means trying to minimize all objective functions individually. In general, a point where all criteria have their minimal values simultaneously is not feasible. Further, not all design points are comparable to each other, since the natural order on $\mathbb{R}^n$ applied to vector criteria leads only to a partial order [12].

On the basis of a partial order we cannot define a unique optimal solution but, at least, we can exclude non–optimal solutions: all design points $\boldsymbol{p} \in \mathcal{P}$ with $\boldsymbol{f}(\boldsymbol{p}) > \boldsymbol{f}(\boldsymbol{p}^1)$, i.e. $f_i(\boldsymbol{p}) \geq f_i(\boldsymbol{p}^1)\ \ \forall i \ \ \wedge\ \ \boldsymbol{f}(\boldsymbol{p}) \neq \boldsymbol{f}(\boldsymbol{p}^1)$, are not optimal where $\boldsymbol{p}^1 \in \mathcal{P}$ is a feasible point. The value of at least one criterion can be decreased without increasing the values of other criteria if switching from $\boldsymbol{p}$ to $\boldsymbol{p}^1$. Design points $\boldsymbol{p}^P \in \mathcal{P}$ which cannot be excluded are called Edgeworth–Pareto–optimal (EP–optimal) and summarized in the EP–optimal set $\mathcal{P}^P$:

$$\mathcal{P}^P := \{\boldsymbol{p}^P \in \mathcal{P} \mid \nexists \boldsymbol{p} \in \mathcal{P} :\ \boldsymbol{f}(\boldsymbol{p}) < \boldsymbol{f}(\boldsymbol{p}^P)\}\ . \tag{9}$$

EP–optimal points with different images are not comparable, all of them have to be considered as optimal. This offers on the one hand side possibilities to realize several different optima what corresponds to technical practice. On the other hand, the designer has the trouble of making the right choice.

In vehicle dynamics, for example, riding comfort and riding safety are two conflicting criteria. The designer's choice may then tend either to comfort leading to a comfortable limousine or to riding safety resulting in a sports car. The problem, however, is that in contrast to economical problems such technical criteria are hard to balance, since they are of totally different dimensions. The designer, therefore, has to be supported by the computer to get a feeling about the EP–optimal set.

In high–dimensional problems, the EP–optimal set cannot be visualized any more. Even computing a representative part of the EP–optimal set is already too time–consuming for dynamic problems. Therefore, a wealth of strategies based on stochastic sampling of EP–optimal points and multiple attribute decision making is not suitable in the context of dynamics. Much less time–consuming are strategies where the designer has to provide some preference information and only single EP–optimal points are computed. The strategies differ in the kind of preference information and in the stage when this information has to be provided. With respect to these criteria, some strategies are more appropriate in the context of dynamics than others.

Common to most of the strategies is the reduction of the vector optimization problem to nonlinear programming problems [7, 10]. This reduction is based on two fundamental principles: scalarization and hierarchization.

In the case of scalarization, Fig. 4, the objective functions are combined to a scalar utility function $u(\boldsymbol{f})$ which is optimized instead of the vector criterion. For all comparable designs, the utility function must have the property

$$\boldsymbol{f}^1 < \boldsymbol{f}^2 \;\Leftrightarrow\; u(\boldsymbol{f}^1) < u(\boldsymbol{f}^2). \qquad (10)$$

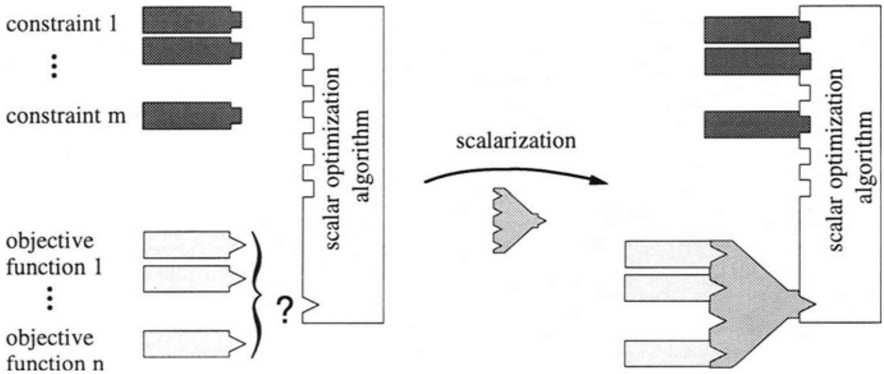

Fig. 4: Scalarization by introducing an utility function

Well known examples are the weighted objectives method

$$u(\boldsymbol{f}) = \sum_{i=1}^{n} w_i f_i, \quad w_i > 0, \qquad (11)$$

or distance methods

$$u(\boldsymbol{f}) = \left(\sum_{i=1}^{n} \left| f_i - \hat{f}_i \right|^{\varrho}\right)^{1/\varrho}, \quad 1 \le \varrho < \infty. \qquad (12)$$

Although the weighted objectives method is widely used, it does not seem to be very appropriate for dynamic problems. The weighting coefficients are hard to define and the result depends highly nonlinear on this choice. The distance methods are better to be handled, since the goals $\hat{f}_i$ have some physical meaning. However, before defining $\hat{\boldsymbol{f}}$ the designer should have an idea about the utopian solution. In case of a wrong choice, EP–optimality is not guaranteed.

For hierarchical methods the designer has to assign a level of importance to each objective function $f_i(\boldsymbol{p})$. The most important criterion will then be optimized first by neglecting the objectives on lower levels, Fig. 5. On the basis of this information a constraint on the criterion can be formulated and the next important criterion can be minimized with respect to this additional constraint. The levels of importance are described by a permutation vector $\boldsymbol{\sigma} \in \mathbb{R}^n$, where $\sigma_j = i$, if the criterion f_j is on level i. The optimization problem is then solved recursively by

$$\bar{f}_j = \min_{p \in \mathcal{P}^{\sigma_{j-1}}} f_j(p), \quad \sigma_j = 1(1)n,$$
$$\mathcal{P}^{\sigma_j} := \{p \in \mathcal{P}^{\sigma_{j-1}} | \, f_j(p) \le (1 + \varepsilon_j)\, \bar{f}_j\},$$
$$\mathcal{P}^0 := \mathcal{P}. \qquad (13)$$

The worsening factors ε_i have to be provided by the design engineer and have a direct influence on the EP–optimal point to be found. If $\varepsilon_j = 0 \; \forall j$, the method is called lexicographic optimization, otherwise it is called hierarchical optimization. Specifying the worsening factors seems to correspond closely to the human decision making process and offers good insight in the vector optimization problem. Another type of hierarchical strategy which has proven to be appropriate to dynamic problems is the goal programming method [4].

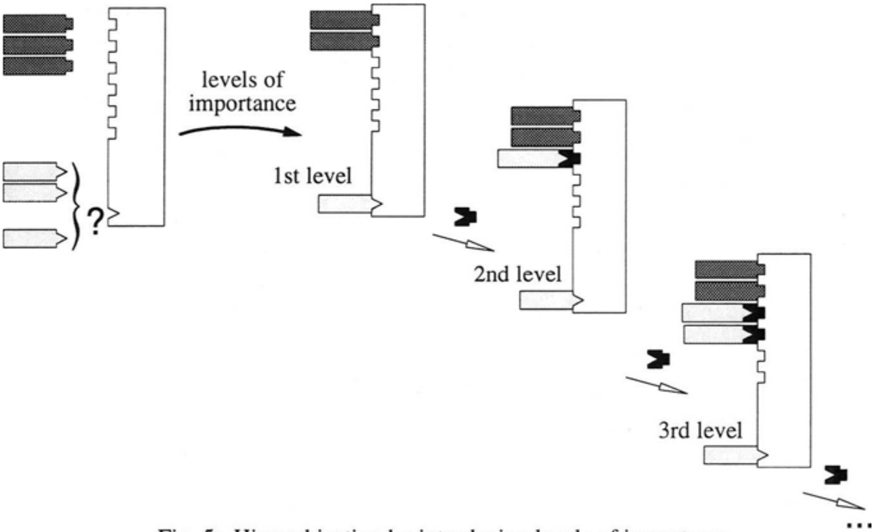

Fig. 5: Hierarchization by introducing levels of importance

In any case the designer has to provide additional information on the optimization problem and only a single EP–optimal design point will be found. For finding a subset representing the complete EP–optimal set, the optimization has to be performed interactively for different preferences. In NEWOPT/AIMS, the resulting nonlinear programming problems are solved with sequential quadratic programming (SQP) algorithms, respectively, which has proven to be very efficient for dynamic problems.

5 Application to Vehicle Dynamics

The chances of and the difficulties with multicriteria optimization in the context of dynamic system design can be seen from an application to a spatial vehicle model shown in Fig. 2. The model has $f = 11$ degrees of freedom, the generalized coordinates $y = [\ x_B, y_B, z_B, \alpha, \beta, \gamma, z_s, \phi_{1L}, \phi_{1R}, \phi_{2L}, \phi_{2R}\ ]^T$ describe the motion of the six rigid bodies: car body, driver, and the four wheel sets. The vehicle model is excited by a measured street profile.

For the investigations described in this paper, four design variables have been chosen for optimization: The spring and damping constants of the front struts, the distance between the center of gravity of the car body and the rear axle, and the distance between the left and the right wheels.

An important criterion is driving comfort which is closely related to vertical accelerations a_D of the driver. However, the human vibration perception depends on the frequency of the vibration as described in the ISO 2631 standard. Such a frequency weighting may be performed by a second order formfilter

$$\dot{v} = \begin{bmatrix} 0 & 1 \\ -1200 & -50 \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a_D, \qquad a_{Dfw} = 20\ [500\ 50]\ v. \tag{14}$$

The resulting weighted acceleration a_{Dfw} may then be used as a first criterion

$$\psi_1 = \int_{t^0}^{t^1} a_{Dfw}^2\ dt. \tag{15}$$

A second criterion is calculated from the pitch acceleration of the vehicle body resulting from rotations around the y–axis. Such kind of motion is enforced by crossways perpendicular to the driving direction. A suitable criterion function may then read as

$$\psi_2 = \int_{t^0}^{t^1} \ddot{\beta}^2\ dt \tag{16}$$

where no frequency weighting is performed. Since the two criteria should be optimized simultaneously, they are normalized to have a value of one for the initial design variables.

Using the weighted objectives method as a representative of the scalarization principle, we find for several different initial designs the solutions given in Fig. 6, respectively. Each single optimization run requires already about 20 minutes of CPU–time on a UNIX–workstation. Changes in the weighting coefficients have only minor influence on the resulting EP–optimal point. A problem which should not be neglected is the occurrence of local minima. Although EP–optimality is guaranteed by the multicriteria optimization strategy, the guarantee is lost if the scalar optimization algorithm is not able to find the global optimum of the utility function.

The result of the lexicographic strategy depends on the choice of the levels of importance. For $\sigma = [1, 2]^T$ the EP–optimal point $\psi = [0.39, 0.58\,]^T$ is found, for $\sigma = [2, 1]^T$ the other extreme point $\psi = [0.50, 0.47\,]^T$ is obtained. Using worsening factors, any EP–optimal point between these two points may be found. Although here the hierarchical strategy converged to EP–optimal points, usually the problem of local minima does also exist.

The results of both strategies do not really give a good impression of the whole EP–optimal set. In the present case of only two criteria, an overview of the EP–optimal set can be obtained by using a stochastic optimization procedure like ASA [8] in combination with the scalarization principle, see Fig. 7. The drawback, however, is the requirement of many criteria evaluations resulting in about 4 hours of CPU–time.

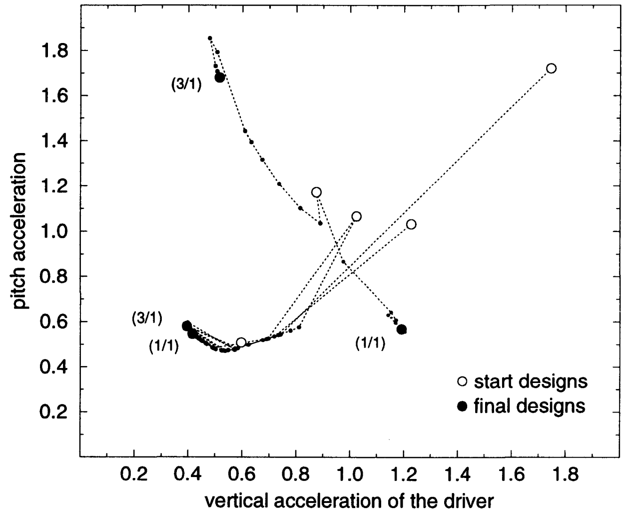

Fig. 6: Optimization results from the weighted objectives method

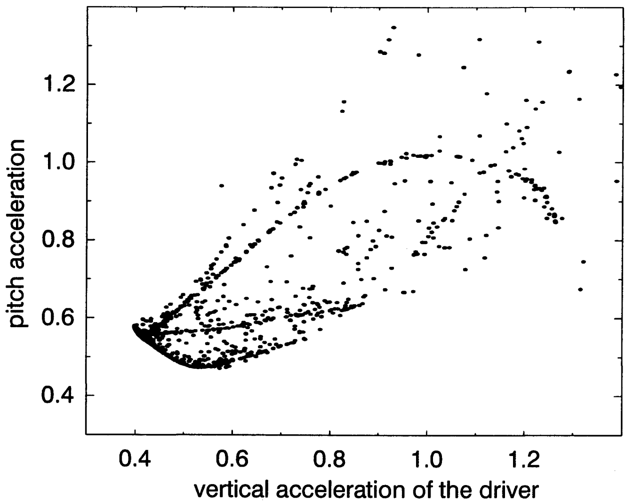

Fig. 7: Results from stochastic optimization

6 Concluding Remarks

Multicriteria optimization opens up new ways for designing dynamic systems. Existing methods, however, have to be adapted to the special requirements in dynamics, especially to the time requirements for criterion evaluations. Therefore, deterministic methods reducing the original problem to nonlinear programming problems in combination with efficient scalar optimization algorithms seem to be much more appropriate than stochastic methods. Major problems of this kind of approach are: (i) loss of guarantee for EP–optimality due to the convergence to local minima; (ii) impossibility of some strategies to compute the whole EP–optimal set in the case of non–convex problems; (iii) difficulty to get a clear and helpful impression on the whole EP–optimal set in any case.

Desirable are methods to move the design point along the EP–optimal set after a single EP–optimal point is found. Sensitivity information of all criteria with respect to movements of the design point within the EP–optimal set would help to perform well–aimed changes in order to iteratively improve one or a couple of criteria while remaining EP–optimal.

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Quality-Driven Decision Making in Digital System Design

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Abstract

Digital system design is a competitive business, with high complexity and a multi-aspect character. Together with nowadays increasing demand of application- and customer-specific embedded systems, suiting supporting design techniques and tools are needed. We are considering the concept of quality within digital system design in order to assure that the systems designed comply as much as possible with the requirements of their future customers. Aim of the paper is to provide and discuss backgrounds, ideas, and direction for future work in the field of quality-driven design decision making.

Vantage point is the definition adopted for quality of a purposive system: the quality of a purposive system is its total effectiveness and efficiency in solving the real-life problem. This however renders many practical difficulties: recognition of a design problem as well as the nature of a solution are subjective; design does not concern the reality as it is, but as it will possibly be realized; human rationality is limited; it is difficult to find relations between various aspects of the effectiveness and efficiency and to express them as one uniform measure; there is a trade-off between effectiveness and efficiency as well as among the different aspect; design problems are dynamic as is their perception. So, quality cannot be well-defined, however it can and should be modelled. It is hereby imprudent to meet the customer's requirements blindly; system requirements in combination with previous design knowledge should form the basis of a model of the required quality. Confrontation with the real world is hereby of primary importance. This way requirements, and other (partial) quality models are not sacred and inviolable anymore, but are subject to design and change. The design process is considered to be an evolutionary process, consisting basically of the steps of constructing the tentative quality models, and using them to derive solutions. Through analysis of the resulting solutions, quality models can be improved, and used again in order to obtain better solutions, and as important, better understand the problem. The design model being introduced contrasts with traditional design approaches, in such a way that the latter assume requirements to be given facts, and not part of the

design process. Design problem solving now is closely related to problem solving known from multicriteria decision making. Although existing design methods as Quality Function Deployment (QFD) support the task of modelling quality, and are gaining acceptance in design, the results obtained are not guaranteed to be meaningful. Proposed is therefore to systematically apply formal decision making methods in design. For example adopting multiattribute value functions and outranking methods. The decision situation should then conform with the assumptions put forward, in particular the axioma system adopted. This restricts the applicability of the decision model and makes it not that easy to handle, and less intuitive as QFD. Benefits now are that the search space for decision models is being limited, and most importantly the results obtained are being more meaningful. Decision methodology itself still provides marginal support for verification and refinement of decision models. These tasks however are crucial in design, since decision models have limited value and are tentative models. Proposed is therefore to facilitate systematic detection of what are called hot spots in the behavior of the decision model. Hot spots can be described as potential reasons of the model inconsistency in relation to the actual problem. They provide possible motives for the model refinement. Elementary concept used for verification and refinement of the decision model is providing explanation of the model's behavior. By given explanations the decision model should become more transparent to the designer so that he can actually relate the decision model with the real world. Obvious candidates to be hot spots are the design options considered best by the decision model, or the design options which are ranked differently by the various decision models and methods adopted. The designer analyses the explanation for a certain decision model and decides whether it is acceptable or not. If not that decision model should be refined. Explanations are constructed by considering the model parameters contributing the most for a certain behavior, or performing sensitivity analysis.

Although a quality model itself does not specify the solution, it plays a central role in design. It gives direction for synthesis of design options. It enables adopting the optimization mode with a priori articulation of preference information. It gives feedback whether the process is on the right track. It gives criteria by which the 'value' or 'quality' of the design options can be judged. Modelling quality is therefore considered to be the first and crucial step in performing quality-driven design. Adopting formal decision models together with rendering their transparency should facilitate this step.

Deriving A Maintenance Strategy Through The Application Of A Multiple Criteria Decision Making Methodology

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Abstract. This paper analyses the process of formulating and evaluating an appropriate maintenance strategy using the Analytic Hierarchy Process (AHP) methodology in a group decision making environment. The work presented is based on the process of formulating a maintenance strategy which involves several key decision makers in a leading automotive manufacturing company. The AHP methodology was developed to systematise the decision process. Whilst, the work is based on this application, the methodology used was constructed to provide a generic framework for the formulation of maintenance strategy. The effects on group decision making for such an application are discussed. The AHP has enabled subjective decisions to be made by a group through a vigorous and structured process involving sometimes conflicting personal objectives within a multiple criteria decision process.

Keywords: AHP, Maintenance, Strategy, Group decision making.

1 Introduction

Recent developments in competitive challenges have had a direct effect on perceiving the formulation of a maintenance strategy as one of the most complex tasks faced by manufacturing companies. These challenges include the existing global economic recession, the need for cost-effective production, and the need to maximise the availability of capital intensive machinery. Procedures for formulating comprehensive strategies [1], [2], and [3] have not been widely used by maintenance practitioners. Even manufacturing strategies [4], [5], and [6] have not been applied in the field of maintenance. The reason for this is the unique characteristics of maintenance in general and maintenance strategies in particular. Maintenance is a composite function, and hence, decisions are often complex and of a multiple criteria nature and usually involve conflicting objectives, such as the attitude towards preventive maintenance from both production and maintenance personnel.

Production personnel desire minimum preventive maintenance in an effort to increase production capability, while maintenance personnel desire to maximise preventive maintenance in an effort to reduce breakdowns.

In order to be more specific, the methodology in this paper is applied to a specific issue of major importance, namely to formulating a strategy based on equipment effectiveness. An actual validation of a simplified version of the model with different decision makers in a leading medium size automotive manufacturing company was carried out to test the performance of the Analytic Hierarchy Process (AHP) in this type of environment.

2. Equipment Effectiveness & Maintenance Strategies

It has been noticed that formulating a maintenance strategy guided by maintenance performance indices is a difficult task. The reason is that many indices have been published by a large number of authors (see, e.g., [7], [8], [9], [10], and [11]) each claiming that his particular index (or set of indices) is a measure of maintenance performance. Most of these indices are of limited value to the decision maker. They only indicate that some action may be necessary, but seldom, if ever, indicate what this action should be. To demonstrate the maintenance decision methodology, as developed by the authors, a key decision model known as overall equipment effectiveness (OEE) is used as an index for measuring maintenance performance. The OEE was chosen since its model is organised in a hierarchical structure that goes from the general to the specific. It starts with its upper level being of a general feature such as quality, productivity, and availability and the lowest level being specific details of maintenance losses such as set-up losses, and start-up losses as shown in Figure (1). Then by using AHP, maintenance strategies are formulated based on their effect on achieving the best OEE.

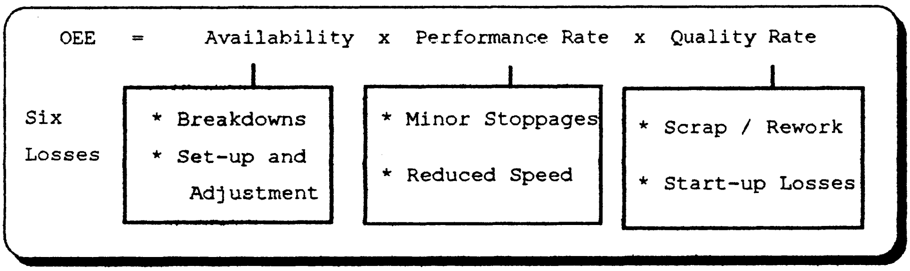

Figure 1. The O.E.E. model

2.1 Overall Equipment Effectiveness

Overall Equipment Effectiveness (OEE) is a measure of the value added to a product through equipment, and it is considered the benchmark for TPM (Total Productive Maintenance). In Japan companies compete to achieve maximum percentages of OEE, and accordingly, prestigious awards are given to best companies. The OEE is a percentage value and is increased as equipment availability, and productivity go up, and scrap rate goes down. OEE is an important and comprehensive measure of a company's performance. Equipment effectiveness is limited by the following six types of losses grouped as downtime, speed losses and defects. Reducing these six losses results in improving OEE. Among the six losses two are unavoidable such as set-up and adjustment losses, and start-up losses. The aim is to reduce them to a minimum. The other four can be reduced to zero. The objective is to choose the most appropriate strategy in order to maximise the overall equipment effectiveness taking into consideration the effect of the six losses.

2.2 Maintenance Strategies

The strategies presented here are examples of different aspects concerning formulation of maintenance strategies using the AHP. It is not the intention to present a comprehensive list of available strategies. Five strategies that affect the six losses are identified. A detailed outline of different characteristics of maintenance strategies is listed below. The features and requirements of each strategy are tabulated in Table (1).

Table 1. Maintenance strategies and their effect on losses

Strategy	Features & Requirements	Effect On Losses
1. Improve skill levels and multi skilling.	Investment in conducting training courses.	High effect on reducing breakdowns, and set-up and start-up times.
2. Invest in predictive maintenance and condition monitoring.	A high degree of investment in monitoring and diagnostic devices which tends to be of high cost.	Helps in preventing drastic breakdowns, and insures high quality products.
3. Invest in preventive maintenance and schedules.	Requires the stoppage of machines, and hence reduces productivity. There is a cost of changing spares, adding oil, ..etc.	Major benefits in reduction of breakdowns, set-up, and start-up times.
4. Improve machine design.	An expert level of skills, as well as multi disciplinary knowledge and communication. It requires the stoppage of machines, and hence reduces productivity.	Direct and major effects on all six losses are perceived especially speed, scrap rate, and breakdowns.
5 Invest in quality circles and small group activities.	Time spent on discussion is not considered production time.	Indirect effect on all six losses.

3. Structure Of The Hierarchy (Model)

A logically constructed hierarchy is a by-product of the entire AHP approach. In other words, AHP is not only a problem solving tool, but also a modelling tool of the problem concerned. Using the goal of increasing equipment effectiveness a hierarchy (see Figure 2) is developed. Following down from the apex of the hierarchy, the first level of the hierarchy deals with the perceived environmental scenarios. The second level identifies the decision makers, or the 'actors', who are related to maintenance. In this case the actors are maintenance managers, production cell managers, and quality managers. The third level is concerned with the six losses that affect the equipment effectiveness. The last level of the hierarchy involves the specific maintenance strategies. The following discussion, deals with the members of each one of the levels of this hierarchy in more detail.

3.1 Scenarios (level 1)

The first level of the hierarchy contains the environmental scenarios. According to Arbel and Orgler [12] environmental scenarios have two basic composites: i) economic conditions such as growth, capital investment, and employment, and ii) competition, which is affected by the type, size, number of competing firms and type of product. These composites form different operating environments, or scenarios, that can be classified as follows:

Scenario 1: *Expanding economy with little competition.*
Scenario 2: *Expanding economy with strong competition.*
Scenario 3: *Stable economy with little competition.*
Scenario 4: *Stable economy with strong competition.*

Naturally, additional or different scenarios may be identified depending on the specific environment faced by a particular company.

3.2 Actors (level 2)

An actor is an individual or a group which plays a significant role in responding to forces that shape current events and, therefore, future outcomes. The main 'actors' related to formulating maintenance strategy are maintenance, production and quality managers. Maintenance, production and quality managers are considered to be the most suitable decision making body interested in the formulation and the implementation of an appropriate maintenance strategy. This is especially true with respect to formulating the strategy based on performance indicators, since the majority of indices are usually under the direct control of those three managers. Other actors can be included in the hierarchy according to the structure of each company. The aim of this exercise is to present a methodology and a framework rather than a specific generic model. This shows that the concept of hierarchies is stable and flexible; stable in that small changes have small effect and flexible in that additions to a well-structured hierarchy do not disrupt the performance.

3.3 Objectives (level 3)

Formulating a maintenance strategy aims at reaching a multitude of objectives, some of which are conflicting, and others are related or complementary. Objectives are to: i) increase productivity, ii) increase quality, and to iii) increase availability.

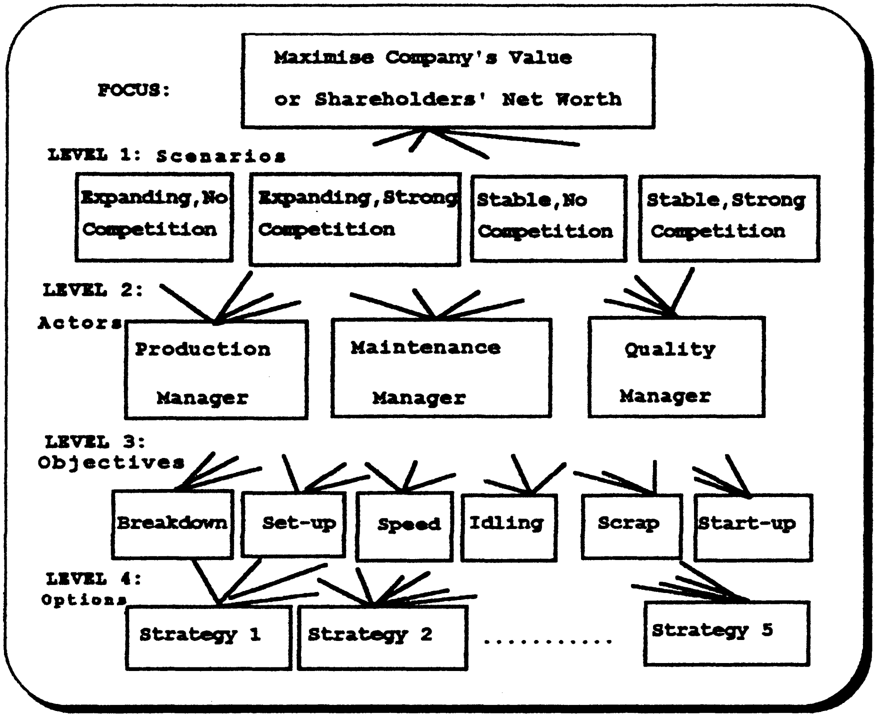

Figure 2. The hierarchy of model formulation

3.4 Options (level 4)

Finally, in any hierarchy, one usually considers the specific options, or strategies, that are applicable in formulating and implementing a maintenance strategy. An outline of different characteristics of maintenance strategies is detailed in section 4. The features and requirements of each strategy, in addition to their effect on the next level (level 4) are tabulated in Table (1).

4. A Detailed Analysis Through A Case Study

Decision applications of the AHP are carried out in two phases: hierarchic design and evaluation. In the previous section, the hierarchic design phase was considered. In this section the second phase; namely the evaluation phase is considered. The first step is to assess the relative likelihood of the four scenarios (see Figure 2).

The judgement reveals that the more likely scenarios are those involving an *expanding* economy, with the most likely being the one with *strong competition*. The evaluation is done through a pair-wise comparison by asking: "Which of the following two scenarios is most likely to occur with respect to a certain company ?". Table (2) summarises the answers to this question. The priority column in Table (2) suggests that the scenario labelled "*Expanding, strong competition*" is dominant in shaping the maintenance strategy for this particular company with a priority of 0.508, "*Expanding with no competition*" has a priority of 0.265, "*Stable, strong competition*" has a priority of 0.151, and "*Stable, no competition*" has a priority of 0.075. This outcome reflects the fact that *competition* is a dominating factor, and that the economy is expected to be growing.

Table 2. Relative likelihood of scenarios

Objective: Maximise Wealth	Expanding, No Competition	Expanding, Strong Competition	Stable, No Competition	Stable and Competition	Relative Likelihood
Expanding, No Competition	1	1/3	3	3	0.265
Expanding, Strong Competition	3	1	5	3	0.508
Stable, No Competition	1/3	1/5	1	1/3	0.075
Stable, and Competition	1/3	1/3	3	1	0.151
					CR = 0.075

In the next step the actors level of the hierarchy is considered. The analysis is based on the relative strength and influence of each actor in shaping and directing the maintenance strategy. One approach is to consider their influence with respect to the upper level in the hierarchy as in Figure (2). If the case of "*expansion*" is considered, production and maintenance managers will score high in their comparisons, while in the case of "*competition*" the emphasis will be on quality and maintenance managers. The next step is concerned with finding the priorities of the various actors under each of the four scenarios. In assessing, for example, the priorities of the actors with respect to "*Expanding, No Competition*", the following comparison matrix was obtained as shown in Table (3).

Table 3. Priorities of actors' level (level 2) with respect to scenario 1 (level 1)

Objective: Scenario 1	Production Manager	Maintenance Manager	Quality Manager	Relative Likelihood
Production Manager	1	4	7	0.71
Maintenance Manager	2	1	3	0.21
Quality Manager	1/7	1/3	1	0.08
				CR = 0.031

Continuing in the same fashion, the priorities of each objective under each scenario were derived; as shown in Table (4). The results demonstrate, for example, that the importance of the production manager is significant during *expansion* scenarios, while the importance of the quality manager is significant during high *competition* scenarios.

Table 4. Local Priorities of level 2 relative to level 1

	Expanding, No Competition	Expanding, Strong Competition	Stable, No Competition	Stable and Competition
Production Manager	0.71	0.54	0.11	0.09
Maintenance Manager	0.21	0.16	0.7	0.64
Quality Manager	0.08	0.3	0.19	0.27
Consistency	0.03	0.01	0.01	0.05

To derive the global priorities of the actors (i.e., how important these actors are to the overall goal and not just to each scenario) one must weight their relative (local) priorities (Table 2) by the priorities (likelihood) of the scenarios themselves (Table 4); this yields a vector (Table 5).

Table 5. Global priorities of actors

Actors	Priorities
Production Manager	0.48
Maintenance Manager	0.29
Quality Manager	0.23

When dealing with different actors, if no consensus is reached, then a geometric mean can be used as suggested by Saaty [13] to average the judgements. This completes the prioritisation of the first two levels, namely that of the scenarios and the actors. The actors' objectives are to minimise losses. For sake of brevity the judgements of different actors when considering the scenario of *"expanding economy, with competition"* are presented in Table (6).

Table 6. Local assessment of different actors' objectives r the scenario of *expansion with competition*

Expansion/ Competition	Breakdown	Set-up	Speed	Idling	Scrap	Start-up
Production Manager	0.39	0.08	0.22	0.07	0.19	0.05
Maintenance Manager	0.42	0.08	0.11	0.07	0.28	0.05
Quality Manager	0.11	0.05	0.07	0.05	0.46	0.26

Note that the priority figures of each actor in Table (6) are in the form of percentages, and hence the summation across the rows add up to unity, or 100 %. As shown in Table (6), out of his/her concern towards maximising productivity, the production manager prioritises his/her objectives to reduce breakdown and speed losses, while the maintenance manager prefers to maximise machines' availability and hence reducing breakdowns is the most important objective. Finally, the quality manager's concern is to minimise scrap and start-up losses. To proceed to the third and last level, that of the objectives and options, we start by identifying options that are relevant to each objective and then find the local priorities of the options with respect to each objective. In order to minimise space, the detailed comparison matrices will not be presented but a summary of the local priorities of various options, for the *expanding with competition* scenario, is provided in Table (7).

Table 7. A production manager's local priorities concerned with options of the scenario of *expansion with competition*

	Breakdown	Set-up	Speed	Idling	Scrap	Start-up
Strategy 1 (Skills)	0.16	0.35	0.1	0.17	0.05	0.15
Strategy 2 (Predictive)	0.23	0.05	0.05	0.06	0.5	0.06
Strategy 3 (Preventive)	0.15	0.34	0.07	0.39	0.05	0.16
Strategy 4 (M/C Design)	0.31	0.18	0.56	0.22	0.29	0.46
Strategy 5 (Q.C.)	0.15	0.09	0.22	0.16	0.13	0.17

The judgements are based on the information provided in Table (1). Note that although this information was given in a qualitative format, it was quite easy to transfer it into the model. The results indicate, for example, that the strategy of improving machine design is expected to contribute the most to every objective. This is understandable since all losses are expected to be minimised if the machine condition is improved. The local priorities are now converted into global priorities. One weights the local priorities of the options by the global priorities of the objectives and sums across all objectives; this yields Table (8) showing the global priorities of options.

Table 8. Global priorities of options (strategies)

Options	Priorities
Strategy 1 (Skills)	0.13
Strategy 2 (Predictive)	0.23
Strategy 3 (Preventive)	0.14
Strategy 4 (M/C Design)	0.34
Strategy 5 (Q.C.)	0.15

The global priorities of the options shown above represent the overall desirability of those options in satisfying the various objectives under the four scenarios and for the three actors. Note that choosing the most appropriate strategy can either be based on the decision 'one out of many', or on a 'portfolio' type of decision where a certain budget is allocated and all strategies are implemented according to their relative priority. Although introducing predictive maintenance and condition monitoring, and improving machine design were the most beneficial options, the net outcome can be different if one considers their accompanied cost and risk. The following section deals with an important aspect of AHP that is related to the formulation of an appropriate maintenance strategy as presented in the above case study. This aspect is the issue of consistency.

4. Inconsistency

The measure of inconsistency is useful in identifying possible errors in expressing judgements as well as actual inconsistencies in the judgements themselves. In formulating maintenance strategy the main reason for inconsistency is believed to be due to the lack of information. Most computerised maintenance systems measure quantity and duration of sporadic breakdowns, but chronic, hidden losses are not measured and seldom analysed and hence remain hidden.

Perfect consistency should not be expected in working with AHP. The issue really is, how much inconsistency is acceptable or tolerable in the expression of preferences ? AHP provides a method called the inconsistency ratio that calculates the degree of inconsistency of judgements. The ratio is based on a comparison with simulations of random judgements. Saaty [13] recommends that as a rule of thumb, if the consistency ratio (CR) is greater than about 0.10 (10 percent of what would result from random judgements), one should investigate and try to ascertain the possible cause of the inconsistency. If each of the possible causes discussed above is eliminated, then it is reasonable to proceed even though the inconsistency ratio is slightly greater than the 10 percent rule-of-thumb value. In a perfect consistent matrix ($CR < 0.1$) it is proven mathematically [13] that for every element in the comparison matrix. This formula suggests several useful applications. One can estimate the most inconsistent element by applying the above formula on all elements and comparing the result with the actual left hand side value of the above equation. The element with the highest variance is the most inconsistent. Another useful application is that one can estimate a missing element in an incomplete hierarchy, by solving the above equation with one unknown. Although this method assures a consistent solution, it does not necessarily mean the desired solution. An alternative is to use the author's proposed approach [14] by learning the pattern of previous comparisons through using an artificial neural network. It is believed that inconsistency in judgements, and conflicts in group decision making are facts of life. The talent is to devise a process that can tolerate and can deal with these features.

5. Summary & Conclusion

The main conclusion of this research work is that applying multiple criteria decision making through using AHP to formulate maintenance strategy has proven to be a comprehensive, practical, and powerful tool; a simple tool that can solve complex problems The requirements of appropriateness in terms of flexibility, systemisation, structure, and ability to deal with non-quantifiable and multiple objectives have all been met by using the AHP as a methodology for formulating maintenance strategy. Since the hierarchy usually starts from the general (top level) to the specific (bottom level), it is most likely that the focus and the first levels of the hierarchy can be considered generic for any company. However, the bottom levels of the hierarchy that include actors, and alternative strategies may be adapted according to the specific nature of a company. Finally, it has been shown in this work that using the OEE model, fig. (1), as a guide to perform multiple criteria decision making is a suitable choice. This is due to the hierarchical structure of OEE and its organised nature which starts with general upper level and proceeds to six specific maintenance losses. Future research is intended to analyse the logistics aspect of maintenance. This is planned to be performed by using resource allocation, cost / benefit, risk, and optimisation analysis.

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Ring Network Design: an MCDM Approach

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Abstract. Telecommunication networks, are optimised so to reduce the impact of fixed costs and to take advantage of system economy of scale. Satisfactory protection target expressed in terms of availability, survivability and soft failing performances should be guaranteed by the designer. In ring networks with self-healing capability such figures are related to the topological features of the network. To rationalise the comparison of different design alternatives and to drive automatic tools balancing all the conflicting criteria involved in the evaluation, the design problem can be formulated as an MCDM process. In the paper, an interactive MCDM procedure based on dynamic aspiration levels and achievement scalarising functions, is proposed to solve the multiple ring network design problem where both cost and protection are taken as simultaneous objectives.

1 Introduction

Telecommunication networks are economically optimised so to reduce the impact of fixed costs and to take advantage of system economy of scale. To reduce operational costs advanced networks are equipped with flexible systems capable to perform direct add-drop multiplexing in ring architectures with intrinsic self-healing capability. Protection target, i.e. availability, survivability and soft failing performances, are related to the topological properties of the network (number of nodes in a ring, number of rings in a point-to-point connection, number of nodes performing inter-ring communications), to the physical diversification of demand routing and to stand-by provisioning [3, 2]. Ring networks provide full single failures protection and short restoration time. However, the method may not react satisfactorily in case of multiple failures. The probability by which these events can occur becomes greater as ring dimensions increase. The specification of an *a priori* protection target can be considered unpracticable from a user viewpoint without any impact evaluation on network economy, and it can be convenient to formulate the ring network design problem as an MCDM process so to rationalise the comparison of different design alternatives and to develop automatic tools driven by the DM in a way to balance the conflicting criteria involved in the evaluation. Hereinafter, in section 2, we first formulate the problem of ring network design at minimum cost under availability constraints; in the following

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section 3 we move to the formulation of a bi-objective MCDM problem representing the ring network design problem in case economy and availability are the two objectives of the design. In section 4, we describe an MCDM interactive procedure able to assist in the determination of the "best compromise" solution and report some applications of the method to some case studies.

2 Ring network optimisation

The ring network design optimisation problem involves the determination of many design variables (nodes belonging to each ring, rings layout, location of inter-ring communication nodes, demand routings) under different kind of constraints concerning demand to satisfy, ring capacity, ring dimensions and routing diversification. Whether the network infrastructure and cables are already available, the optimisation criterion mostly requires to minimise total cost of multiplexers needed to build the ring networks. Cost of the multiplexers can be considered depending on their maximal capacity, while the cost to transit through a node interconnecting a pair of adjacent rings, can be taken proportional to the demand size. Even in such assumptions, the network design process is decomposed in steps and rings selection, demand routing and physical dimensioning are separately performed [4]. In other cases, the clustering of ring nodes is performed maximising traffic affinity among the nodes belonging to each ring and minimising ring size [7]. Along this line, another modelling suggests to put a constraint on the minimum amount of total demand satisfied as "internal demand", i.e. demand routed on all the rings without inter-ring transit [1]. To simplify the problem, in [5] the node clustering problem is treated considering only the cost of nodes and limiting the number of inter-ring transits used to route a demand (path length).

To solve the node clustering and the ring selection problems, we have considered a reduced complexity formulation in which a pre-processing of input data is performed so to lead to a set of "candidate rings" to be further processed in the optimisation [1]. In this case, instead of introducing binary decision variables to decide if a node belongs to a ring, we introduce an integer decision variable to decide if a candidate ring appears in the optimal solution and the number of integer variables is significantly reduced. As to the network protection, in a previous analysis [2] we have shown that point to point demand availability is mostly in relation to the ring size (number of nodes per ring) and demand survivability is in relation to the number of inter-ring transits performed in the demand routing. In the present paper, for discussion purposes, we assume the same cost function adopted in [5] and that protection could be evaluated by two parameters respectively representing the maximum and the average number of nodes per ring.

With such assumptions, the minimum cost ring network design problem can be formulated in the following way:

$$\textbf{min} \quad \sum_r \left(\sum_i E_{ir}\right) a_r X_r + c\sum_{i<j} \left(D_{ij} - \sum_r f_{ijrr}\right) \tag{1}$$

$$\textbf{s. t.} \quad \sum_{r,s \mid i\in r,\, j\in s} f_{ijrs} = D_{ij} \quad \forall (i,j) \mid i<j\ D_{ij}>0 \tag{2}$$

$$\sum_{i<j \mid D_{ij}>0} \left[\sum_{s \mid i\in s,\, j\in r,\, s\neq r} f_{ijsr} + \sum_{s \mid i\in r,\, j\in s} f_{ijrs}\right] - u_r X_r \le 0 \quad \forall r\in R \tag{3}$$

$$f_{ijrr} - D_{ij}X_r \le 0 \quad \forall (i, j) \mid i<j\ D_{ij}>0,\ i\in r,\ j\in r \tag{4}$$
$$f_{ijrs} - D_{ij}X_r \le 0$$
$$f_{ijrs} - D_{ij}X_s \le 0 \quad \forall (i, j) \mid i<j\ D_{ij}>0,\ i\in r,\ j\in s,\ r, s \text{ adjacent}$$
$$\textstyle\sum_r X_r \le N_{rmax} \tag{5}$$
$$(\textstyle\sum_i E_{ir} - N_{nmax})\, X_r \le 0 \quad \forall r\in R \tag{6}$$
$$\textstyle\sum_r [(\sum_i E_{ir}) - N_{navg}]X_r \le 0 \tag{7}$$
$$f_{ijrs} \ge 0 \quad \forall (i, j) \mid i<j\ D_{ij}>0,\ i\in r,\ j\in s,\ r, s \text{ adjacent} \tag{8}$$
$$X_r \ge 0 \text{ and integer } \forall r\in R \tag{9}$$

where:

N is the set of nodes,
R is the set of candidate rings,
$[D_{ij}]$ is the $|N|\times|N|$ symmetric point-to-point demand matrix,
$[E_{ir}]$ is the $|N|\times|R|$ node-ring incidence matrix ($E_{ir}=1 \Leftrightarrow i\in r$),
u_r is the maximal capacity of the Add-Drop Multiplexers (ADM) used to build the ring r,
a_r is the fixed cost of the ADMs,
c is the interconnection cost per unitary inter-ring traffic,
N_{rmax} is the maximal number of usable rings,
N_{nmax} is the maximal number of nodes per ring,
N_{navg} is the maximal average number of nodes per ring,
f_{ijrs} is the traffic between nodes i and j transferred from ring r to ring s,
X_r is the multiplicity of the ring r in the optimal solution.

The objective function (1) is given by the total cost of the rings, each one having two cost components respectively equal to the fixed cost of each node of the ring (taken with its multiplicity) and to a cost proportional to the total demand routed externally to the ring. Constraints (2) are the demand constraints; constraints (3) are the ring capacity constraints; constraints (4) are the flow-demand mapping constraints on the selected rings; constraints (5), (6) and (7) are protection constraints; constraints (8) are the non negativity constraints on the flow variables; constraints (9) are the integrality constraints on the decision variables.

Now we analyse the impact of variations of parameters u, N_{nmax}, N_{navg} on the solution considering the ten nodes network analysed in [1]. The demand matrix has 26 positive items with a total demand size equal to 291 demand units in the range 0-20. As illustrated in [1], a set of candidate rings is generated according to both topological and demand constraints. As a result, a set of 27 candidate rings is selected, each one having a number of nodes in the range 3-10.

In a first phase, the CPLEX 3.0[1] optimization package has been used to maximise the total demand D_{tot} satisfied in the network under a constraint of a preassigned maximal number of rings. The results are reported in fig. 2.1, where the number of rings is varied from 3 to 7 (7 is the minimum number of rings needed to achieve a feasible solution with all the demands satisfied).

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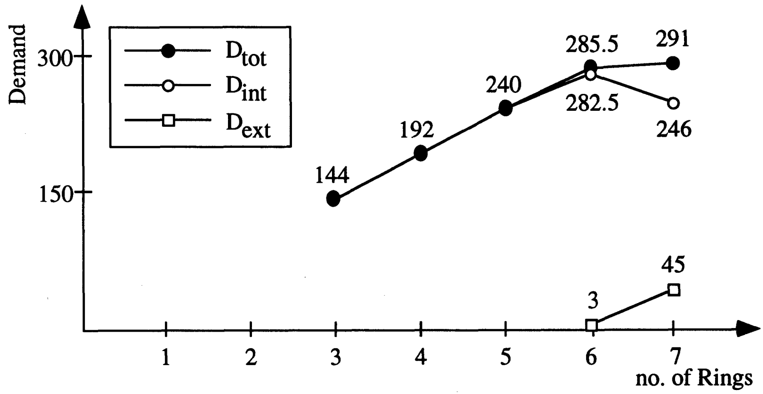

Figure 2.1. Max. satisfied demand vs. ring number

It can be seen from the figure that D_{tot} increases regularly up to 6 rings and almost coincides with the internal demand of the rings, D_{int}. Before reaching the feasibility, the external demand jumps from 3 to 45 units, so that resource utilisation is optimal when all the demands are satisfied as internal demands; a positive external demand, D_{ext}, can become significant only in a maximally packed feasible solution.

In a second phase, CPLEX has been used to solve the constrained minimum cost problem previously envisaged both varying the ring capacity in the range 48-312 units and relaxing the constraints on N_{nmax}, N_{navg}, N_{rmax}. The results are shown in table 2.1. It can be seen that with high ring capacity all the demands can be met by a single ring connecting all the nodes.

For lower ring capacities the minimal number of nodes per ring decreases down to 3 and also the maximal number of nodes per ring decreases down to 4. In all the optimal solutions a small amount of demand is satisfied as external demand, but in general its size can vary in an unforeseeable way, due to the discrete nature of the optimisation problem.

In further runs, the ring capacity has been fixed to 276 units and the minimum cost solutions have been constrained by different values of N_{navg}. The results are reported in table 2.2. It can be seen that the number of rings in the minimal cost solutions can vary in the range 2-10. When N_{navg} is set quite near to the minimal value, also the maximal number of nodes per ring is indirectly limited; on the opposite for higher values of N_{navg} a limitation on the maximal ring size can be achieved only by directly constraining N_{nmax}.

3 Ring network design as an MCDM problem

Using the notations of the previous section, the ring network design problem can be formulated as a bi-objective problem like in the following:

$$\textbf{min} \quad z_1=\Sigma_r (\Sigma_i E_{ir}) a_r X_r + c\Sigma_{i<j} (D_{ij} - \Sigma_r f_{ijrr}) \qquad (1)$$

Table 2.1. Sensitivity of the solutions w.r.t. ring capacity variations

Ring Capacity	Cost	External Demand	Min. no. of nodes/ring	Max. no. of nodes/ring	No. of Rings
48	25.9	38	3	4	7
96	18.9	38	3	5	4
168	13.5	20	6	8	2
276	10.5	-	3	10	2
312	10.0	-	10	10	1

Table 2.2. Sensitivity of the solutions w.r.t. N_{navg} variations

Avg. no. of nodes/ring	Cost	External Demand	Min. no. of nodes/ring	Max. no. of nodes/ring	No. of Rings
3.1	33.7	55	3	4	10
3.13	27.7	55	3	4	8
3.14	25.0	61	3	4	7
3.33	22.7	55	3	4	6
3.6	19.9	38	3	5	5
5.5	13.4	49	5	6	2
6.5	13.0	-	3	10	2

$$\textbf{min} \quad z_2=\Sigma_r\,[(\Sigma_i\, E_{ir})\, X_r\,]\,/\,\Sigma_r X_r \tag{1'}$$

s. t. constraints (2), (3), (4), (8) and (9).

We can notice the non linearity of the objective z_2. Nevertheless, if we bound the selection variables X_r, it is possible to linearize z_2 by adding some new variables and constraints. With regards to the simplest case of a binary X_r, for example, we can set:
$Y_r = X_r z_2 \;\; r=1,2,..., |R|$
and then impose the supplementary constraints:

$$\Sigma_r\,[(\Sigma_i\, E_{ir})\, X_r\,]=\Sigma_r\, Y_r \tag{10}$$
$$Y_r \le MX_r$$
$$Y_r \le z_2$$
$$Y_r \ge z_2 - M(1-X_r) \;\; r=1,2,..., |R|$$

where M is a sufficiently high positive constant.

To solve the problem we report some definitions and key results available from the literature [8]. It is well known that in an MCDM problem, under the assumption of monotonicity of preferences, the search for the "best compromise" solution can be restricted to the set of nondominated solutions or Pareto set, where a solution (criterion vector) $\mathbf{z}^* \in \mathbf{Z}_f$ (feasibility region) is nondominated (weakly nondominated) if there does not exist another feasible solution $\mathbf{z}'$ such that $\mathbf{z}' \le \mathbf{z}^*$ ($\mathbf{z}' < \mathbf{z}^*$) and $\mathbf{z}' \neq \mathbf{z}^*$. Let $\mathbf{Z}^N$ be the nondominated set, and $\mathbf{Z}^{\geq}$ be the convex hull of

$[\mathbf{Z}^N \oplus \{\mathbf{z} \in \mathbf{R}^p | \mathbf{z} \geq \mathbf{0}\}]$, where $\oplus$ means set addition and p is the number of objectives. If $\mathbf{z} \in \mathbf{Z}^N$ is on the boundary of $\mathbf{Z}^{\geq}$ then $\mathbf{z}$ is a supported nondominated criterion vector, otherwise, it is unsupported nondominated. Supported nondominated solutions can be easily generated using the so called weighted sums program, that is by minimising a linear combination of the criteria with strictly positive weights. The drawback of this program is the inability to generate unsupported members of $\mathbf{Z}^N$, because they are dominated by some convex combination of other nondominated criterion vectors. This shortcoming does not appear whether the objective function is given either by a weighted Tchebycheff norm or by an achievement scalarising function [10, 11]. In the last case, the controlling parameter is given by the so-called reference point, which represents the DM aspiration levels on the objectives. As it is well-known, a classic achievement scalarising function is the one derivable from the Tchebycheff norm:

$$S(\mathbf{z}, \mathbf{z}^*, \lambda) = \max_j \{\lambda_j(z_j - z_j^*)\} \qquad j=1,2,...,p$$

where $\mathbf{z}^*$ is the reference point and $\lambda_j>0$ is the weighting factor for the deviation of the j-th objective from the reference point.

For the selected function, every optimal solution $\mathbf{z}'$ of the problem

$$\min_{\mathbf{z} \in \mathbf{Z}_f} S(\mathbf{z}, \mathbf{z}^*, \lambda) \tag{11}$$

is weakly nondominated. Moreover, if $\mathbf{z}'$ is unique then it results nondominated, otherwise, at least one of the optimal solutions is nondominated.

If we use the following modified scalarising function:

$$S(\mathbf{z}, \mathbf{z}^*, \lambda) = \max_j \{\lambda_j(z_j - z_j^*)\} + \Sigma_j \rho_j(z_j - z_j^*)$$

where ρ_j are small positive values, every optimal solution $\mathbf{z}'$ of the problem (11) is nondominated. Moreover, if $\mathbf{z}'$ is nondominated, then by solving the above program assuming $\mathbf{z}^*=\mathbf{z}'$, the minimum is attained at $\mathbf{z}'$ and is equal to zero.

Such properties and definitions will be used in the MCDM strategy adopted to solve the ring network design problem.

4 Description of the adopted strategy

In this section, we present an interactive procedure that can be used to aid the DM to find out the most preferred or "best compromise" solution of the bi-objective ring network design problem formulated in section 3. Due to the significant computational burden associated with the generation of nondominated solutions for the problem in hand, it is not advisable to generate the entire Pareto set and later search for the most preferred solution. Merging these two phases together in an interactive procedure allows to limit the number of generated solutions by exploiting the DM indications to reduce progressively the exploration region in the criterion plane [6, 9]. We assume an achievement scalarising function, whose minimisation yields nondominated solutions. The relevant subproblem to be solved at iteration h can be formulated as:

Problem P^h

$$\textbf{min} \quad S(\mathbf{z}, \mathbf{z}^{*h}, \lambda^h) = \max_j \{\lambda_j^h(z_j - z_j^{*h})\} + \Sigma_j \rho_j z_j \qquad j=1,2$$

$$\textbf{s.t.} \quad z \in Z_f^{h-1}$$

where:

$\mathbf{z}^{*h}$, current reference point,

Z_f^{h-1}, current (rectangular) feasibility region in the criterion plane,

λ^h, weighting vector for the distances of the objectives from the reference point.

At each interaction the feasibility region of the problem is progressively reduced until either no other different nondominated solution can be found or the DM considers satisfying the last generated proposal. It does not seem suitable to directly ask the DM for a reference point, but to adopt as preference parameter required to lead the process, the choice of an "interval of interest" located between two previously generated nondominated solutions in the criterion plane. The presence of only two criteria, in fact, makes possible to easily visualise all the solutions on the criterion plane and this kind of information results particularly easy to deliver. To each DM indication corresponds a new rectangular feasibility subregion to be investigated. The new current reference point can be immediately obtained as a vertex ($\mathbf{z}_I^h$ = h-th ideal point) of such rectangular feasibility region, and the weights λ_j^h in the objective function are given by the slope of the diagonal of the rectangle (the line joining $\mathbf{z}_I^h$ to $\mathbf{z}_N^h$, h-th nadir point). More formally, the procedure can be articulated in the following general steps:

"Initial Step"

Let h=0, $\mathbf{Z}_f^0 = \mathbf{Z}_f$, L={∅}.

Calculate the two nondominated points $\mathbf{z}^{0R}$, $\mathbf{z}^{0L}$ corresponding to the independent minimisation of z_1 and z_2 on $\mathbf{Z}_f^0$.

Let L={$\mathbf{z}^{0R}$, $\mathbf{z}^{0L}$} and h=1.

"Current Step"

Let $\mathbf{z}_I^h = (z_1^{h-1R}, z_2^{h-1L})$, $\mathbf{z}_N^h = (z_1^{h-1L}, z_2^{h-1R})$.

Solve the problem P^h with $\mathbf{z}^{*h} = \mathbf{z}_I^h$ and λ^h derived from:

$$\lambda_1^h (z_{N1}^h - z_{I1}^h) = \lambda_2^h (z_{N2}^h - z_{I2}^h)$$
$$\lambda_1^h + \lambda_2^h = 1$$

if a new current solution $\mathbf{z}^h$ is generated **then** insert it in the list L between solutions $\mathbf{z}^{h-1R}$, $\mathbf{z}^{h-1L}$ (L={..., $\mathbf{z}^{h-1R}$, $\mathbf{z}^h$, $\mathbf{z}^{h-1L}$,...})

else STOP.

"Interaction Step"

if the DM is satisfied of $\mathbf{z}^h$ **then** STOP

else let h=h+1 and let the DM choose a new interval of interest defined by a pair of adjacent solutions $\mathbf{z}^{h-1R}$ and $\mathbf{z}^{h-1L}$ in the list L.
Define the new feasibility region $\mathbf{Z}_f^{h-1}$ by setting the constraints:

$$z_1 \leq z_1^{h-1L}$$
$$z_2 \leq z_2^{h-1R}$$

Return to the *"Current Step"*.

Let us now consider again the case study network examined in section 2. For this network, we have generated all the nondominated set $\mathbf{Z}^N$, under the assumption of ring capacity equal to 276 units and no maximal ring size restriction. Such set consists of 12 solutions that are displayed in the criterion plane in fig. 4.1. Some considerations can be derived from this figure:

- the nondominated solutions are not evenly distributed on the plane;
- the plane can be subdivided in 3 regions respectively characterised by an almost constant value of one of the objectives ($3.75 \leq z_2 \leq 6.5$; $23 \leq z_1 \leq 34$) (extreme regions) and an intermediate region ($z_1 \leq 23$; $z_2 \leq 3.75$) where both the two objectives vary rapidly;
- among the nondominated solutions there are both supported and unsupported solutions.

From these observations, it is evident that the preferred solutions belong to the intermediate region. For illustrative purpose, we have applied the suggested MCDM procedure to this example. The results are summarised in fig. 4.2 and in tab. 4.1. Initially, the two objectives have been separately minimised, obtaining the two "extreme solutions" $\mathbf{z}^a=\mathbf{z}^{0R}=(13, 6.5)$ and $\mathbf{z}^b=\mathbf{z}^{0L}=(33.75, 3.1)$ which limit a first region of interest containing the entire set $\mathbf{Z}^N$. The corresponding nadir and reference points (ideal and reference points coincide in the procedure) are then $\mathbf{z}_N{}^1=(33.75, 6.5)$ and $\mathbf{z}^{*1}=(13, 3.1)$ from which the first couple of weights (expressed in %) can be derived. We can notice a strong difference between λ_1 and λ_2, but this is due to the different dynamics of the objectives and the consequent normalisation adopted in the weights generation.

By solving the problem P^1 it has been yielded a first compromise solution $\mathbf{z}^c=\mathbf{z}^1=(16.9, 3.75)$. Later on, it has been decided to explore the interval ($\mathbf{z}^c$, $\mathbf{z}^b$), so that a new feasibility region $\mathbf{Z}_f^1=\{\mathbf{z}\in Z_f \mid 16.9 \leq z_1 \leq 33.75; 3.1 \leq z_2 \leq 3.75\}$ and consequently a new couple reference-nadir have been determined.

Carrying on with the procedure, the sequence $\mathbf{z}^d$, $\mathbf{z}^e$, $\mathbf{z}^f$ has progressively been generated, converging to $\mathbf{z}^f$ as final solution after 4 iterations, so avoiding the generation of all the elements of $\mathbf{Z}^N$. The saving in the number of generated solutions can be much higher if we consider larger networks with an higher number of candidate rings. As an example, we report the results achieved in a network of 15 nodes with 49 symmetric demands. The node clustering package used in the pre-processing phase has generated 52 candidate rings with sizes varying from 2 to 10 nodes per ring. In this case, the application of the MCDM procedure has generated 8 nondominated solutions, converging after 6 iterations.

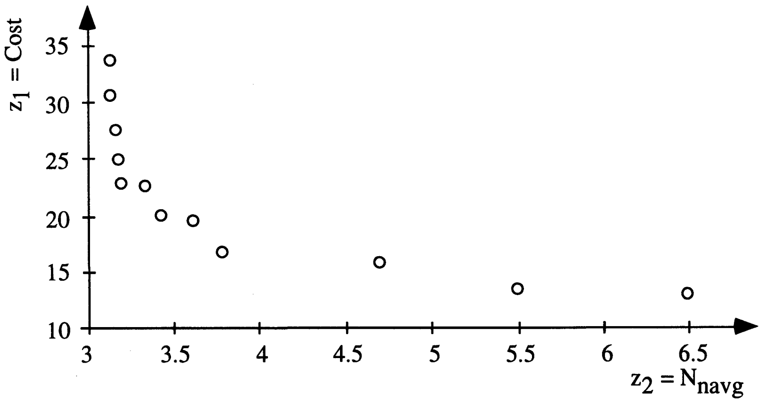

Figure 4.1. Nondominated solutions on the criterion plane

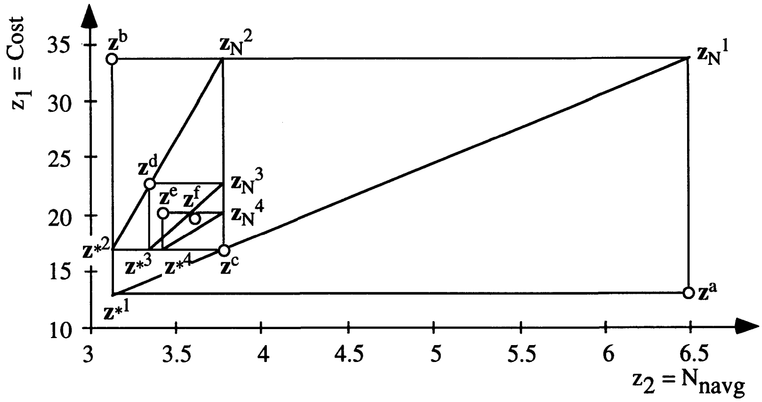

Figure 4.2. Feasibility regions and solutions generated by the MCDM procedure

Table 4.1. Feasibility region and current solutions

	Nadir Point		Reference Point		Weights		Current Solution	
Iter.	z_1	z_2	z_1	z_2	λ_1	λ_2	z_1	z_2
1	33.7	6.5	13.0	3.1	14.1	85.9	16.9	3.7
2	33.7	3.7	16.9	3.1	3.7	96.3	22.7	3.3
3	22.7	3.7	16.9	3.3	6.7	93.3	20.0	3.4
4	20.0	3.7	16.9	3.4	10.0	90.0	19.9	3.6

5 Conclusions

The problem of ring network design has been formulated as a NP-complete cost optimisation problem. In the paper, some characteristics of the problem have been analysed using CPLEX, simplifying the formulation in order to perform a sensitivity analysis with respect to the most important design variables, like ring capacity, ring structure and ring interconnections. As the ring networks are expected to provide very restrictive protection target in the relevant applications, the design can be formulated as an MCDM problem where both cost and availability of the network are the two objectives to be balanced.

However, the set of nondominated solutions in the criterion plane does not meet the convexity property, so we have investigated an interactive procedure in which at each iteration an achievement scalarising function is minimised in a reduced feasibility region determined by the DM. The scalarising function is represented by the maximal weighted deviation of the current values of the objectives from a reference point. Both the weights and the reference points are automatically updated as to the DM choices. Such procedure results an efficient supporting tool for the network designer provided that an efficient algorithm is developed to solve the current optimisation problem envisaged in the MCDM procedure.

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A Construction Accuracy Control System of Cable Stayed Bridge Using a Multi-objective Programming Technique

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Abstract: Cable-stayed bridges are gaining much popularity in Japan due to their beautiful shape. During and after construction, this kind of bridge needs to have the cable length adjusted in order to attain errors of cable tension and camber within some allowable range. Since the cable length is affected by the change of temperature, the operation of acuracy control has to be completed whithin a short period, say 2:00 to 8:00 in early morning. So far, this construction accuracy control has been performed by judgment based on the experiences of experts, or goal programming. In either case, however, many trial and errors are usually made in order to attain a satisfactory solution, and it is difficult to complete the operation within the time limit. To this aim, the autors developed a construction accuracy control system using the satisficing trade-off method which has been observed to be effective to many kinds of multi-objective programming problems. The system has a user-friendly human interface of graphic input-output, and hence is simple and easy to implement. The results of applications of the system to real bridges are also reported in this paper.

Key Words: multi-objective programming, aspiration level approach, applications, cabel-stayed bridge

1 Introduction

Cable stayed bridges are gaining much popularity in Japan due to their beautiful shape. After and during construction, this kind of bridges need to adjust the cable length in order to meet the cable tension and the configuration of girder of bridge (camber) with the design values. For this adjustment, taking into account of erection accuracy and operationability, the following criteria are considered (see, e.g., [2]):

i) residual error in each cable tension,

ii) residual error in camber at each node,

iii) amount of shim adjustment for each cable,

iv) number of cables to be adjusted.

Since the change of cable rigidity is small enough to be neglected with respect to shim adjustment, both the residual error in each cable tension and that in each camber are linear functions of the amount of shim adjustment. Let us define n as the number of cable in use, $\Delta T_i \quad (i = 1, \ldots, n)$ as the difference between the designed tension values and the measured ones, and x_{ik} as the tension change of i-th cable caused from the change of the k-th cable length by a unit. The residual error in cable tension caused by the shim adjustment is given by

$$p_i = |\Delta T_i - \sum_{k=1}^{n} x_{ik} \Delta l_k| \qquad (i = 1, \ldots, n) \tag{1.1}$$

Let m be the number of nodes, $\Delta Z_j \quad (j = 1, \ldots, m)$ the difference between the designed camber values and the measured ones, and y_{jk} the camber change at j-th node caused from the change of the k-th cable length by a unit. Then the residual error in the camber caused by the shim adjustments of $\Delta l_1, \ldots, \Delta l_n$ is given by

$$q_j = |\Delta Z_j - \sum_{k=1}^{n} y_{jk} \Delta l_k| \qquad (j = 1, \ldots, m) \tag{1.2}$$

In addition, the amount of shim adjustment can be treated also as objective functions of

$$r_i = |\Delta l_i| \qquad (i = 1, \ldots, n) \tag{1.3}$$

And the upper and lower bounds of shim adjustment inherent in the structure of the cable anchorage are as follows;

$$\Delta l_{Li} \leq \Delta l_i \leq \Delta l_{Ui} \qquad (i = 1, \ldots, n). \tag{1.4}$$

Now we have a multi-objective optimization problem in which $(p_1, \ldots, p_n, q_1, \ldots, q_m, r_1, \ldots, r_n)$ is to be minimized under the constraint (1.4). For this multi-objective optimization, engineers in bridge construction have tried to apply the goal programming (see, e.g., [1]): They try to get a desirable solution by adjusting weights imposed on critera. However, it has been pointed out in literatures (see, e.g., [5]) that this task is very difficult even in simple problems. In addition, the shim adjustment is usually to be done during a relatively short period (say, 2:00 am to 8:00 am) with a stable temperature, because the cable length is much affected by the change of temperature. Therefore, the decision of shim adjustment is to be made very quickly. Also, due to this reason, the goal programming is not satisfactory for practical use in our problem.

On the other hand, one of the authors developed an interactive multi-objective programming techniue, called the satisficing trade-off method (STOM) (see, e.g., [3]-[6]). The method is one of aspiration level approaches to multi-objective optimization, which are observed to be effective in many practical problems because they are very simple and easy to implement and do not require any mathematical consistency of decision makers' judgment, and in addition take aspiration levels of decision makers as a probe rather than weights imposed on criteria. Since aspiration levels usually reflect the wish of DM very well, the aspiration level approach can provide efficient human interfaces. We developed a software for interactive construction accuracy control of cable stayed bridge using STOM, and applied to several real bridges. In the following, the system and the result by using it will be shown in more detail.

2 Development of an Interactive Construction Accuracy Control System of Cable Stayed Bridge

One of biggest problems in construction accuracy control of cable stayed bridge is the time limitation. As stated in the preceding section, the operation must be finished within a short period. To this end, the system is required to be simple and easy to use, and in addition fast to obtain the final solution.

It has been observed through our experiences in several other applications that STOM has a property of fast convergence to a satisfactory solution. Therefore, we developed a graphical input-output system so that the construction accuracy control system may be simple and easy to implement: On the graph which shows the observed data of errection errors through a measurement for the bridge, the decision maker (an engineer in this case) inputs his aspiration level by a mouse. After setting aspiration levels, the user can select the calculation mode. Then, the computer solves the auxiliary min-max problem and shows a Pareto solution nearest to the given aspiration level. If the obtained solution is not satisfactory, the decision maker is required to trade-off among criteria, because the shown solution is Pareto optimal and therefore there is no other solution improving all criteria simultaneously. Taking the total balance of all criteria into account, the decision maker inputs his updated aspiration level. Again, the computer solves the auxiliary min-max problem, which provides an updated Pareto solution. The procedure is continued until the decision maker can obtain a satisfactory Pareto solution.

The graphical input/output system can improve the efficiency of operation very much. In addition, changing the mode into the numerical one, the decision maker can see the numerical data in more precise, and input his aspiration level as numerical values, if necessary. Fig.2.1 shows one phase of construction accuracy control system of cable stayed bridge using STOM.

Since the auxiliary min-max problem is of linear programming, it usually does not take so much time for computation. Therefore, for bridges with not so large scale, personal computers can work so well for our purpose. In fact, in cases of Swan Bridge and Chikuho Harp Bridge, we used a personal computer at the field.

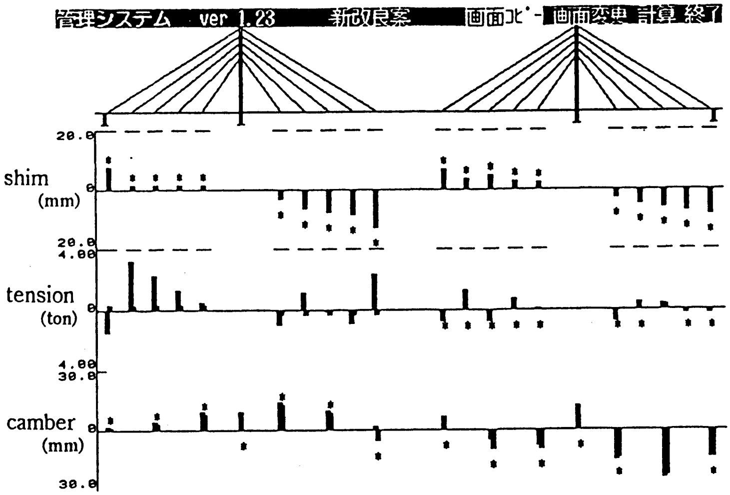

Fig. 2.1 One phase of construction accuracy control system for cable stayed bridges

3 Applications

We have applied the construction accuracy control system to several real bridges. Representative examples are the Swan Bridge in Ube city and the Chikuho Harp Bridge in Fukuoka.

3.1 Outline of Applied Bridges

The Swan Bridge, located in Ube City, in the west part of Japan, was constructed as a part of promenade in the beautiful "Tokiwa Park" in 1992 (Fig. 3.1)[8]. The park is famous of many swans, and the bridge was planned to image a swan. From this reason, the bridge was finally decided to be of a kind of cable stayed bridge. It has the total length of 156m, 2 pylons and 40 cables, and is now familiar to citizens of Ube City due to its beatiful shape. The project outlines of the superstructure are as follows:

- Total length: 156m

- Effective width: 3.5m
- Structure type: 3 span continuous cable stayed bridge
- Main material used: SS400, S10T, SWPR7A(JIS)
- Total weight of steel: 300.067tf

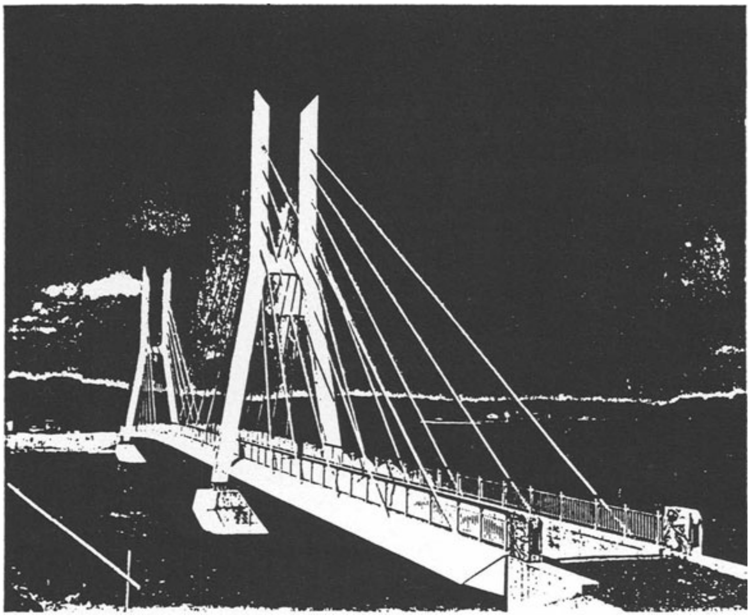

Fig. 3.1 Swan Bridge

The Chikuho Harp Bridge, located in Fukuoka, in the west part of Japan, was constructed as a part of promenade in Chikuho Green Park in 1992 (Fig. 3.2). The park is used for multiple purposes such as sports, recreation, playground for small children and so on. The Harp Bridge has 1 pylon and 16 cables, and its special feature is that the girder has the shape of "S". The project outlines of the superstructure are as follows:

- Total length: 146.45m
- Effective width: 2m+2m(green zone)+2m
- Structure type: 2 span continuous cable stayed bridge

- Main material used: SS400, S10T, SWPR7B(JIS)
- Total weight of steel: 421.374tf

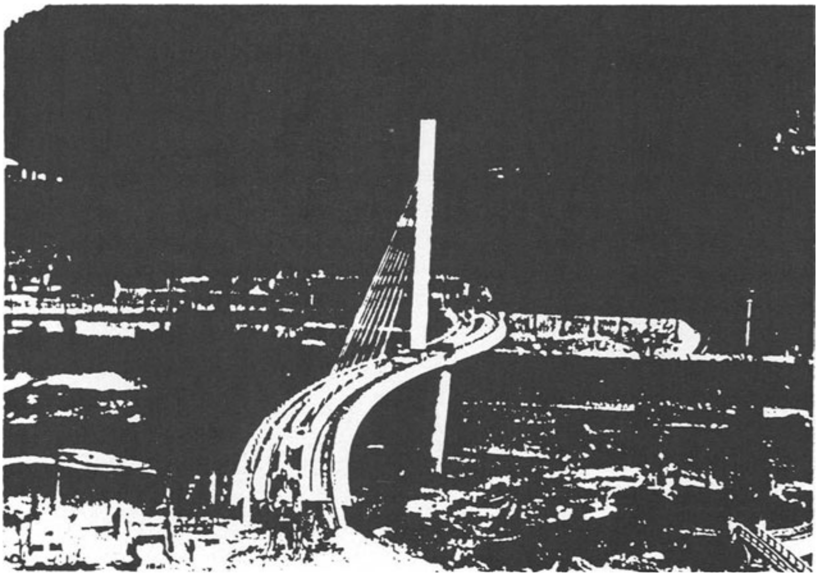

Fig. 3.2 Chikuho Harp Bridge

3.2 Adjustment of Cable Tension by Using the Construction Accuracy Control System

For the Swan Bridge, the construction accuracy control was executed 4 times, namely at the stage of which 8, 20, 32 and final 40 cables had been erected. The result of final adjustment is shown in Fig. 3.3. Table 3.1 shows the allowable range of construction accuracy. In case of this bridge, as the stiffness of girder is very large compared with that of cables, it is difficult to improve the configuration of girder by adjustment of cable tension. Therefore, the erection of girder was managed exactly at the stage 2.

Since the errors in configuration of girder were trivial in this case, the errors in cable tension were only targetted for improvement. As is readily seen in Fig. 3.3, the cables before adjustment were of uneven tensions, and they had 0%-30% of errors. However, after the operation of adjustment, the errors in tension were improved to the allowable range. The final results of

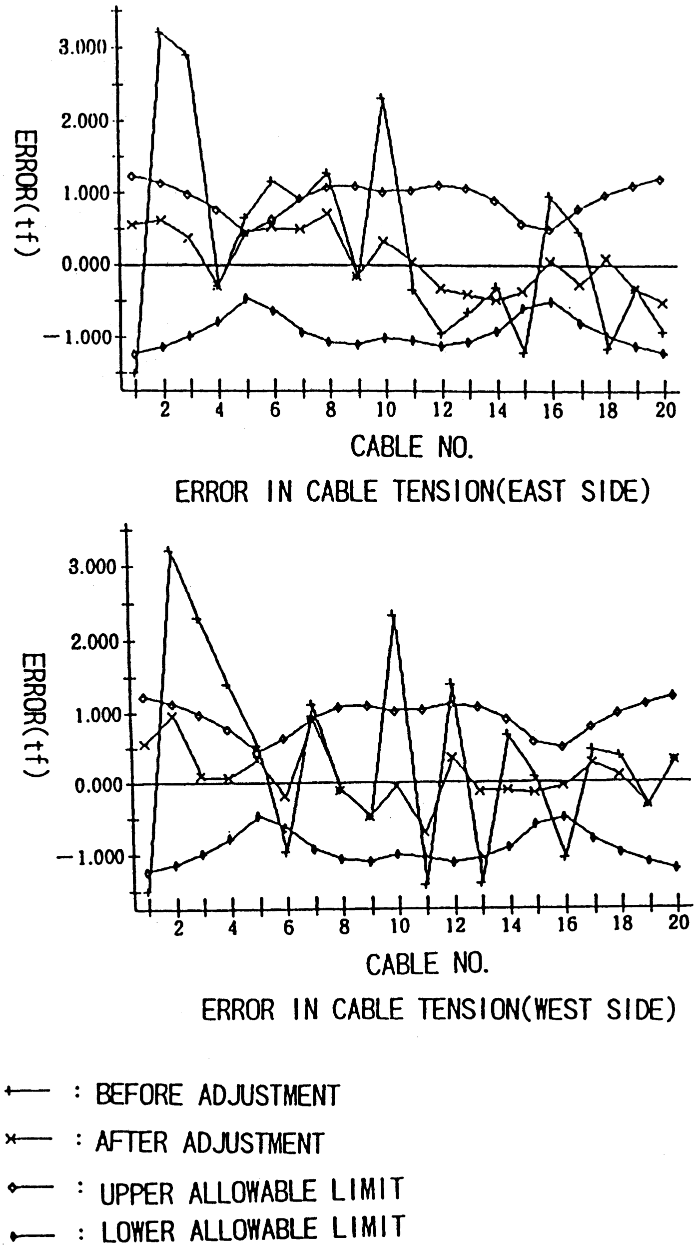

Fig. 3.3 Final result for Swan Bridge

maximum and average errors were 9.9% and 3.8%, respectively. The final error in configuration of girder was distributed from 6mm to -27mm.

For the Chikuho Harp Bridge, only once accuracy control yielded a satisfactory result. Since the "S" shape of bridge tends to cause errors in erection, we usually need to control the construction accuracy carefully in such a bridge. Our result shows that our construction accuracy control system can be applied effctively even to such a difficult shape of bridge.

Table 3.1 Allowable range of construction accuracy

Design Values	Allowable Range
cable tension	±10%
camber of girder (side span)	±25mm
camber of girder (center span)	±70mm
camber of pylon	±25mm

5 Concluding Remarks

It has been observed that the developed system using STOM works very well for construction accuracy control of cable stayed bridge. In particular, the human interface with graphic input-output makes the system simple and easy to use, and no feeling of difficulty for users.

As a future subject of this system, the followings are pointed out:

1) To avoid human error in input/output operation, we should develop the interface such as data on-line systems.

2) For more accurate calculation of sensitivities of design variables, the structural analysis should be carried out by three-dimensional nonlinear analysis, especially in case of large scale structures.

3) It is of interest to apply this system to other structures (e.g., Nielsen Bridges) in which cable members are used.

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MCDM in Water Resources Investment Planning

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Abstract. A state agency faces the problem of allocating its available funds to the high evaluated projects. The projects are evaluated in terms of different criteria, expressing economic, environmental, and social impacts. The criteria are defined for specific project groups: multipurpose reservoirs, water supply systems, irrigation systems, wastewater treatment plants. The data are provided by designers. Investment planning is considered as a dynamic process. The multicriteria decision making procedure is developed, based on the Compromise Ranking Method. As an illustrative example, the water resource projects evaluation and investment planning by Water Resources Fund of Serbia is presented.

Keywords. Investment, Water Resources, Multicriteria Ranking

1. Introduction

Each large water resources project may by considered as an important infrastructure, making a part of the nation's capital stock and playing a vital role in encouraging a more productive and competitive national economy. In the document [3] the use of performance measures, investment budgeting, and other devices designed to improve the quality of public investment are considered. The role of consistent investment analysis is to encourage the funding of the most worthy projects. Some federal (U.S.A.) programs require grant recipients to show the expected source of funds to build and maintain the system, improving the fiscal discipline. The project lists are limited in magnitude by the funds that can be demonstrated to be available; in contrast to "wish lists".

Many types of analysis could be used to help decision makers; evaluating projects investment and developing investment strategies and plans. Benefit - cost analysis is used by water resource agencies; but, it presents some shortcomings for infrastructure investments, because it is suited to well–defined projects. Cost-effectiveness analysis includes qualitative factors that can complement a benefit - cost analysis [3].

Economists have dealt with this decision–making problem by arbitrary assigning to decision makers the task of maximizing a social welfare function. But, a social welfare function is much more a theoretical construction than a representation suitable for empirical work [1]. The task of describing how public choices are made has been left to social scientists. A micro-oriented procedure of modeling public authorithy's objectives is the multicriteria analysis. The decision makers provide the set of objectives and the projects are evaluated in terms of accepted criteria. The multicriteria methods allow many degrees of freedom in defining the criteria weights, but this should be "under control" in order to avoid misuse of the method.

The execution of projects will have economic, environmental, and social impacts on the society. So, water resources projects should be evaluated in terms of more than one criterion. The multicriteria decision making (MCDM) analysis is developed and used (mostly) because of our inability to place monetary values on all relevant impacts of water resources [1], [4].

MCDM in water resources may be considered as a complex and dynamic process in which one managerial level and one engineering level can be distinguished. The managerial level defines the goals, and make "the final decision". "The decision makers" are public officials having the power to accept or reject projects or plans provided by the other level. The engineering level defines (designing) alternatives and points out consequences of choosing any one of them from the viewpoint of various criteria.

At the engineering level the optimization methods are applied in designing alternatives, these methods are presented in the book by Djordjević [2]. Many papers present the applications of MCDM methods in water resources [8] and in engineering [7]. An integration of Stochastic Dynamic Programming and Integer Goal Programming modeling framework is proposed in the paper by Sutardi et al. [8] to handle problems of multicriteria sequential decision-making under budgetary and socio-technical uncertainties inherent in water resources investment planning. The difficulty in applying this stochastic approach is within subjective probabilities assignment.

The planning of large water resources projects is a cooperative venture among national, provincial and local authorities. In the document [4] it is proposed to establish various advisory groups which can monitor and assist the engineering team. At decision-making level the following groups may be established:

- "an advisory team" of experts,
- a "steering committee" primarily from the national level,
- a "political linking committee".

In Serbia "a political linking committee" is the Assembly of Water Resources Fund of Serbia. The members provide linking to the decision-makers at the national and regional levels. The "Steering committee" is the Fund Board. The members judge whether or not the proposed project (from engineering level) are related with national priorities and objectives. The "team of experts" provide the informations and ranking-list of the projects for decision making. The informations and data for project applying for Fund's support are provided by the engineering (local) level.

In Serbia, the Water Resources Fund Board is the authority responsible for allocation of water resources budget among projects for all periods. Solving the dynamic investment problem, the Fund Board would like to reduce the problems of project rescheduling, postponement, and cancellation due to budgetary fluctuations.

The purpose of research presented in this paper was to provide the procedure to the Fund Board that may be used as decision–making aid in investment planning. The developed method for ranking investments according to multiple criteria is based on application of "Compromise ranking method" [6].

2. Dynamic Investment Planning

In this paper we assume the existance of a set of techniques for finding the optimal solution for an individual project [2], and address ourselves to the problem of investment planning in the presence of budget constraints. If there is a doubt at the decision–making level that the proposed solution is not optimal, the project would be returned to the engineering level for redesign.

The problem with budget constraints is considered as one of choosing which projects from the available set are to be constructed (today). Planning is not "one shot", so the investor's problem is to determine the schedule of building projects. The importance of the dynamic aspect of investment planning increase with the durability of building projects. Water resources projects are durable, so we are faced with dynamic investment planning [5].

The dynamic investment planning process is performed in three stages. In the first stage we consider the planning horizon of three periods; the period one of five years duration, the period two ("immediate future") of the next five years, and the period three ("distant future") of the next ten (or five) years. This planning stage would differentiate projects in terms of "present", "immediate future", or "distant future" investment. In the second planning stage detailed investment plan would be established only for the projects assigned to the "present". But, the investment plan could by modified by annual budget allocation if unexpected budgetary changes would occur. In the unstable economy the annual budget allocation would be the third planning stage.

This planning process is very complex because the systems are complex, many resources constraints exist, and investment planning is under budgetary and socio-technical uncertainties. The approach in the paper by Sutardi et al. [8] is based on subjective probabilities assignment. We found that decision makers in water resources planning are "risk averse", and they do not like to take part in subjective probabilities assignment. Also, the planning process could be more complex, because, besides the trade–off in the criteria space, the trade–off between preference structure and confidence structure would be difficult (confidence that the criterion value would occur with assign probability).

Instead of stochastic method, we are developing the method of "if–then" scenarios. Scenarios of future budget availability are defined by the team of experts. Two scenarios are "boundary": "minimum budget" and "maximum budget" scenarios.

3. Water Resource Projects Evaluation

Evaluating a large project implies handling huge amount of data. All data are handled by the engineering team performing the design project. It is not practical to list all data in the report for decision makers. Therefore, the agregated set of data is prepared. It is necessary that characteristic quantities are given in their natural units. The particular importance of the data is judged at decision–making level.

In Serbia, the data for each project applying for funding are submited by the engineering teams (local) on the forms established by the Fund Board. The water resource projects are classified in the seven groups (W.R. sectors) as following:

1. Multipurpose large reservoirs.
2. Large irrigation systems.
3. Flood control and river regulation.
4. Water supply systems.
5. Wastewater treatment plants.
6. Large land–drainage systems.
7. Erosion control works and systems.

In addition to characteristic data, the data expressing economic, environmental, and social impacts should be submited. For projects evaluation the aggregated data are prepared. For example, multipurpose reservoirs are evaluated in terms of the following criteria:

1. Total investment costs ($).
2. Funds (needed) from the Water Resource Fund ($).
3. Number of people gaining from the project.
4. Local water sources exploitation percentage (used from available) (%).
5. Area irrigated from the reservoir (ha).
6. Reduced flood damages ($).
7. Ratio of downstream low flows after and before the project.
8. Water for industrial supply (l/sec).
9. Hydroelectrical energy from the project [GWh/yr].

10. Benefit-cost ratio.

11. Importance (grade by experts).

12. Urgency (grade by experts).

The criteria for all project groups are similar to the criteria mentioned above, the first two, and the last three are the same, others are specific expressing the project characteristics.

4. Multicriteria Ranking

There are two approaches in multicriteria ranking. The first approach is based on an "aggregated" function $U(f_1, ..., f_n)$ representing total utility, or $D(f_1, ..., f_n)$ as "dissatisfaction" (like L_p metric in compromise programming). The second approach is based on preference relation B (like ELECTRE method). If the preference relation $a_j B a_k$ is confirmed, then the alternative a_j is better than a_k in multicriteria sense. It seems that these two approaches are very different, but comparing the compromise programming and ELECTRE method (as representatives) the similarity was found out by Opricović [6].

Relying to ideas of compromise programming the compromise ranking method is developed. It is assumed that each alternative a_j is evaluated according to each criterion function.

The input data is a matrix $||f_{ij}||_{nJ}$, where f_{ij} is the value of i-th criterion function for the j-th alternative; n is the number of criteria, and J is the number of alternatives.

The compromise ranking algorithm has the following steps:

1. Determine the best f_i^+ and the worst value f_i^- of the criterion functions, $i = 1, ..., n$

2. Compute the value S_j and R_j by the relations:

$$S_j = \sum_{i=1}^{n} w_i(f_i^+ - f_{ij})/(f_i^+ - f_i^-); \quad j = 1, ..., J$$

$$R_j = \max_i \left[w_i \left(f_i^+ - f_{ij}\right) / (f_i^+ - f_i^-)\right]; \quad j = 1, ..., J$$

where w_i, $i = 1, ..., n$, are the criteria weights.

3. Compute the values Q_j, $j = 1, ..., J$, by the following relation

$$Q_j = v(S_j - S^+)/(S^- - S^+) + (1 - v)(R_j - R^+)/(R^- - R^+)$$

where: $S^+ = \min_j S_j$, $S^- = \max_j S_j$,

$R^+ = \min_j R_j$, $R^- = \max_j R_j$,

v is the strategy weight ("group utility"),

4. Rank the alternatives, sorting by the values Q_j, $j = 1, ..., J$.

In this case study the weights w are proposed by the expert team; and $v = 0.6$.

Iterative procedure may be introduced as the following: Performing the ranking (the first run), the best and the worst alternative have the rank 1 and J (the position on the ranking list). The ranking is performed as the second run without the best and the worst alternative from the privious run. In the second run the best and the worst alternative will have positions 2 and $J - 1$, rspectively. So, the iterative procedure is performed until all alternatives are ranked. By such iterative procedure it is avoided the influence of the best and worst alternative on the ranks of the others.

The Compromise Ranking Method may be applied under following assumptions:

- The values of all criterion functions are given for all alternatives.
- It is not necessary for all alternatives to be noninferior.
- The relation between the utility (utility in general sense) and criterion function is linear.

5. Decision Making Procedure

Water resources planning and public decision making are costly. The evaluation and choice of projects adds to these costs by requiring information, skill and time. But, the appraisal of projects helps in making consistent decisions and in avoiding wastages of public resources. The evaluation of investments may not be useful if not completed "in time", but a fast planning may be misleading. The decision makers need the most relevant information for the final decisions. Too little information is detrimental to decision-making, but to much information can be useless if not organized in an efficient way. The rational of standard evaluation techniques, their shortcomings, and the revisions suggested by the literature are described in the book by Boeri [1]. One of the suggestions could be to define simplifications according to specific characteristics of projects, decision makers goals, and the conditions in which the proposed investments have to take place; taking into account that the approximations could cause errors.

We have been developing a multicriteria decision making procedure with the main purpose to help the Fund Board to allocate funds among the water resources projects. The Fund Assembly establish the application regulations and the public (state) share of investment. The general rule is to impose local cost shares in proportion to benefits, for the specific group of projects. The Assembly divides costs between state agency (Fund) and water district (local).

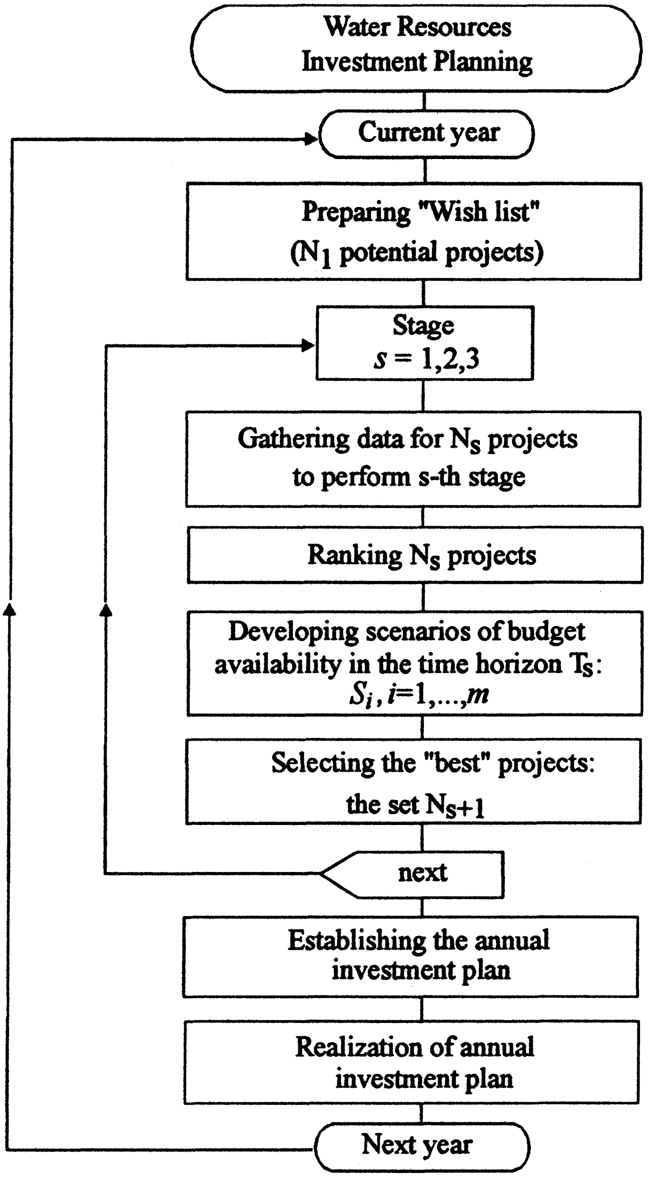

Fig. 1. The Investment Planning Procedure

The investments planning procedure, based on the ideas presented in this paper, is shown in the figure 1. The procedure starts with preparing (or updating) the "wish list" with the maximum number of potential projects (applying for the Fund funds). The computational procedure is performed in three stages for each group of projects, separately, and for "all projects". The number of projects is reduced from the "maximum wish" in the first stage to the "selected for investment" in the third stage. The time horizon (T_1) in the first stage is the 15-year interval, devided in three periods. In the second stage the time horizon is 5-year interval, and in the third stage the horizon is one year (current year). In the second stage the funds are allocated among the "best" projects from the first stage. In the third stage the ranking–list of projects is determined for investment in the current year. The annual investment plan has to be established by Water Resources Fund Assembly allocating the available funds among the sectors (groups of projects) and among the projects. If the proposed plan is not accepted by the Assembly, the decision making is renewed with thé informations and suggestions made by the Assembly members. The established annual investments plan should be realized in the current year. If the plan could not be completely realized, this would be a particular problem in the investment planning for the next year (in all stages).

The duration of the decision making procedure should be less than three months. With a good data base gathering data could be more efficient. Improving the water resource information system, this decision making procedure could become an efficient decision support system.

6. Conclusion

Water resource projects are evaluated in terms of economic, environmental, and social criteria. Multicriteria decision–making procedure is developed to help the Fund Board to allocate the funds among the projects. Water resource investment planning is a dynamic process; and the computational procedure is 3-stage iterative algorithm. The investment time horizon is a 15-year interval, but in the current year only the annual investment plan is realized.

Acknowledgements

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OPTIMAL AND ROBUST SHAPES OF A PIPE CONVEYING FLUID

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Abstract. An optimal and robust shape is determined for a pipe conveying a fluid. The problem is formulated as a structural optimization problem with the radius of the circular cross section as design variable. The critical velocity of the fluid flow in the pipe serves as the first criterion. The critical velocity is related to the upper limit of the nonconservative fluid force. It is furthermore desired that the critical velocity be insensitive to perturbations in the pipe shape and we thus consider the design's robustness with respect to such perturbations as the second criterion. The two criteria are ordered in accordance with their importance with the maximum of the follower force as the primary criterion. The optimal design for the force is considered first and this design is then modified to improve the robustness. The results are illustrated for a silicon rubber pipe conveying water at constant velocity.

Keywords. Structural optimization, Shape determination, Nonconservative system.

1 Introduction

A pipe conveying fluid is an important problem in the pressure and vessel piping, since it is a typical nonconservative system showing the unstable vibration for the internal fluid flow faster than the critical velocity[2], [4]. The primal concern is the improvement of the stability[1], [7]. The shape determination is an approach to this end successively applied to the Beck's column, a column with nonconservative follower force[3],[5], but it is also known that the critical force is quite sensitive to the small change of the shape at the optimal[6]. This study discusses the stability of the pipe conveying fluid and the optimal and robust shape in the context of the structural optimization.

2 Analysis of Pipe Conveying Fluid

2.1 Equation of Motion

Figure 1 shows the flexible pipe under consideration. It consists of a tubular cantilever conveying a stream of incompressible fluid with a constant mean flow

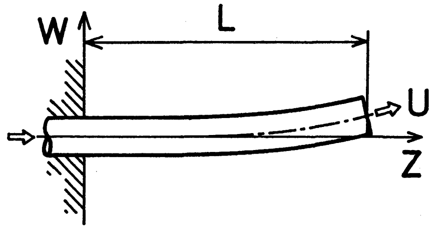


Fig. 1.1. Cantilever pipe conveying fluid

velocity. By taking the internal and external damping effect into account, the nondimensional system of the equation of small motions by Gregory and Paidoussis[2] is extended as

$$\{(m^2-1)w''\}''+\beta\{(m^2-1)\dot{w}''\}''$$
$$+\mu(u^2w''+2u\dot{w}'+\ddot{w})+\gamma\dot{w}+(m-1)\ddot{w}=0 \quad \text{in } 0\leq x\leq 1 \qquad (1)$$

in terms of the dimensionless variables defined as

$$x=z/L, \quad w=W/L, \quad t=(1/L^2)(EI_0/\rho_pA_0)^{1/2}\tau,$$
$$\beta=L^2/(\rho_pA_0EI_0)^{1/2}\zeta, \quad \gamma=(1/L^2)(EI_0/\rho_pA_0)^{1/2}\eta,$$
$$\mu=\rho_f/\rho_p, \quad u=UL(\rho_pA_0/EI_0)^{1/2}. \qquad (2)$$

where ( )' and ($\dot{}$) denote the partial derivatives with respect to x and t. The notation used is summarized as follows: z:longitudinal coordinate, τ:time, L:length of the pipe, W:lateral deflection of the pipe, ρ_f, ρ_p:material densities of fluid and pipe, E:elastic modulus of the pipe, m(z):nondimensional parameter of cross sectional size, $I(z)=\{m(z)^2-1\}I_0$:moment of inertia of the pipe, A_0:internal cross sectional area of the pipe, $A(z)=\{m(z)-1\}A_0$:cross sectional area of the pipe, ζ, η:internal and external damping coefficients, and U:velocity of fluid flow. The boundary and initial conditions are written as

$$w=w'=0 \qquad \text{at } x=0, \qquad (3)$$
$$\{(m^2-1)w''\}'+\beta\{(m^2-1)\dot{w}''\}'=0,$$
$$(m^2-1)w''+\beta(m^2-1)\dot{w}''=0 \qquad \text{at } x=1, \qquad (4)$$
$$w=\dot{w}=0 \qquad \text{at } t=0. \qquad (5)$$

By introducing a variable y(x, t) adjoint to the deflection w(x, t), the Lagrangian function L(w, y) is defined as a generalized energy functional as

$$L(w, y)=\int_0^T\Big[\int_0^1\Big[(m^2-1)y''w''+\beta(m^2-1)y''\dot{w}''$$
$$-\mu u^2y'w'+2\mu uy\dot{w}'+\gamma y\dot{w}+(\mu+m-1)y\ddot{w}\Big]dx+\mu u^2yw'\,\Big|_{x=1}\Big]dt \qquad (6)$$

and the equation of motion is transformed to the adjoint variatinal problem as

stationary L(w, y) with respect to w and y
subject to $w=w'=0,$
$y=y'=0,$ at $x=0$
$w=\dot{w}=0,$ at $t=0$
$y=\dot{y}=0,$ at $t=T.$ (7)

by means of the adjoint variational principle.

2.2 Stability Analysis

For the finite element analysis, the pipe is discretized by the beam elements of the cubic approximation for the priaml and adjoint variables w(x, t) and y(x, t) in the coordinate x. The size of cross section m(x) is also approximated by the linear function of x in each elemet. Thus, the functional L(w, y) is discretized as

$$\hat{L}(\mathbf{a}, \mathbf{b})=\int_0^T[\mathbf{b}^T(\mathbf{S}+u^2\mathbf{Q})\mathbf{a}+\mathbf{b}^T(u\mathbf{R}+\mathbf{C})\dot{\mathbf{a}}+\mathbf{b}^T\mathbf{M}\ddot{\mathbf{a}}]dt \tag{8}$$

where the real symmetric matrices **S**, **C** and **M** give the stiffness, damping and mass coefficients, respectively, and the unsymmetric load matrices **Q** and **R** are coming from the internal flow. The vectors **a** and **b** denote the nodal values and its derivatives of the primal and adjoint variables w and y, respectively.

The stationary condition of the discretized functional $\hat{L}$with respect to the vectors **a** and **b** yields the finite element equation of the pipe as

$$(\mathbf{S}+u^2\mathbf{Q})\mathbf{a}+(u\mathbf{R}+\mathbf{C})\dot{\mathbf{a}}+\mathbf{M}\ddot{\mathbf{a}}=\mathbf{0} \tag{9}$$
$$\mathbf{b}^T(\mathbf{S}+u^2\mathbf{Q})-\dot{\mathbf{b}}^T(u\mathbf{R}+\mathbf{C})+\ddot{\mathbf{b}}^T\mathbf{M}=\mathbf{0}. \tag{10}$$

Assuming the modal behavior for the nodal vectors **a** and **b** as

$$\mathbf{a}(t)=\mathbf{p}\exp(\lambda t) \tag{11}$$
$$\mathbf{b}(t)=\mathbf{q}\exp(\sigma t) \tag{12}$$

the finite element equations (9) and (10) reduce to the eigenvalue problem

$$[\mathbf{S}+u^2\mathbf{Q}+\lambda(u\mathbf{R}+\mathbf{C})+\lambda^2\mathbf{M}]\mathbf{p}=\mathbf{0} \tag{13}$$
$$\mathbf{q}^T[\mathbf{S}+u^2\mathbf{Q}-\sigma(u\mathbf{R}+\mathbf{C})+\sigma^2\mathbf{M}]=\mathbf{0} \tag{14}$$

and the symbols λ and σ give the eigenvalue of the primal and adjoint systems, respectively. Given the flow velocity u, the eigenvalues are evaluated for the pipe shape, that is specified by means of the vector $\mathbf{m}=[m_1, \cdots, m_{N+1}]^T$ consisting of the dimensionless pipe size m_j at the finite element nodes.

The stability of the pipe is determined by paying attention to the eigenvalues λ_i and eivenvectors $\mathbf{p}_i$ of the primal system. That is, the positive value of the real part of λ_i means that the amplitude of vibration mode $\mathbf{p}_i$ increases in time, and the negative real part means that the amplitude decreases. Thus, the purely imaginal eigenvalue is the critical for the stability of the pipe. The vibration mode $\mathbf{p}_i$ may become critical as to the increase of the internal flow velocity. Thus, for each vibration mode $\mathbf{p}_i$, the critical velocity $u^{(i)}$ is defined as the fluid velocity at which the real part of the corresponding eignevalue λ_i vanishes. The minimum critical velocity

$$u_{cr}=\text{minimum } \{u^{(1)}, u^{(2)}, ..., u^{(2N)}\} \tag{15}$$

is the most important giving the stability limit of the pipe from the engineering viewpoint. It should be noted here that the critical velocity u_{cr} depends on the pipe shape $\mathbf{m}$.

3 Shape Determination Problem

3.1 Maximization of Critical Velocity

The larger is the critical velocity of the internal flow, the larger the stable region of the pipe conveying fluid becomes. The shape determination problem of the pipe is considered by means of the maximization problem of the critical velocity. By taking the discretized dimensionless pipe size $\mathbf{m}$ as the design variable, the shape determination problem is formulated as

$$\begin{aligned} &\text{Maximize } \{\text{minimum } u^{(i)}, i=1, 2, \cdots, 2N\} \text{ with respect to } \mathbf{m} \\ &\text{subject to } V-V_0=0, \\ &\qquad -m_j+\underline{m}\le 0, \; j=1,2,\cdots, N+1. \end{aligned} \tag{16}$$

This maximization problem has the constancy constraint of the pipe volume and the lower limit constraint of the cross sectional size. The dimensionless size m(x) assumed to be linear in each element gives the overall pipe volume V as

$$V=\sum_{i=1}^{N} l_i(m_i+m_{i+1})/2 \tag{17}$$

with the element lengths l_i.

The formulated max–min optimization problem (16) is rewritten as a conventional minimization problem

$$\begin{aligned} &\text{Minimize } -Z \text{ with respect to } \mathbf{m} \text{ and } Z \\ &\text{subject to } Z-u^{(i)}\le 0, i=1, 2, ..., 2N \\ &\qquad V-V_0=0, \\ &\qquad -m_j+\underline{m}\le 0, j=1,2,\cdots, N+1 \end{aligned} \tag{18}$$

by introducing an artificial variable Z. The gradient projection method is employed to deal with this minimization problem numerically referring to the past successful application to the shape determination of Beck's column seeking the maximum critical load, during which two or more vibration modes relate to the critical load simultaneously[6].

3.2 Optimal Pipe Shape

The shape determination is examined for the silicon rubber tube conveying water. The case parameters used is as follows: number of finite elements: N=10, inner radius: $\sqrt{m_i}=1$, initial outer radius: $\sqrt{m_j}=2$, lower limit of outer radius: $\sqrt{\underline{m}}=\sqrt{2}$, internal damping coefficient: $\beta=0.01$, external damping coefficient: $\gamma=1.00$, density ratio of fluid & pipe material: $\rho=0.870$. The uniform cross sectional shape along the longitudinal coordinate x is used as the initial condition of the iteration in the gradient projection method. It is noted that the dimensionless pipe size m(x) corresponds to the cross sectional area, and its square root represents the radius of the pipe.

Figure 3.1 shows the pipe shape giving the maximum critical velocity. The pipe shape obtained as the solution of the minimization problem (18) is referred to as the optimal shape in the following. Figure 3.2(a) gives the dimensionless complex frequencies of the three lowest modes of the uniform cantilever pipe, and Fig. 3.2(b) does those of the optimal shape. As the increase of the velocity of the internal flow from zero, the first and the second eigenfrequencies move toward the negative direction of the real axis and then turn back to the positive direction of the real axis. Further increase in the flow velocity drives these complex frequencies to cross over the imaginary axis and leads the cantilever pipe unstable

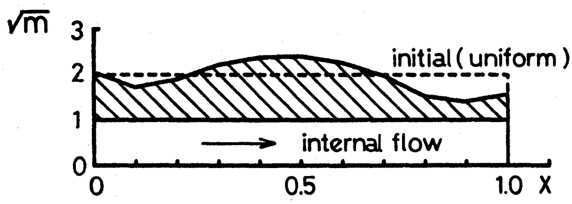
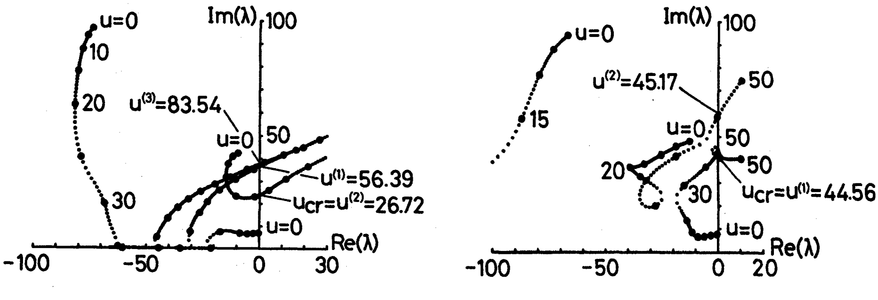


Fig. 3.1 Optimal shape of cantilever pipe



(a) Uniform shape (b)Optimal shape

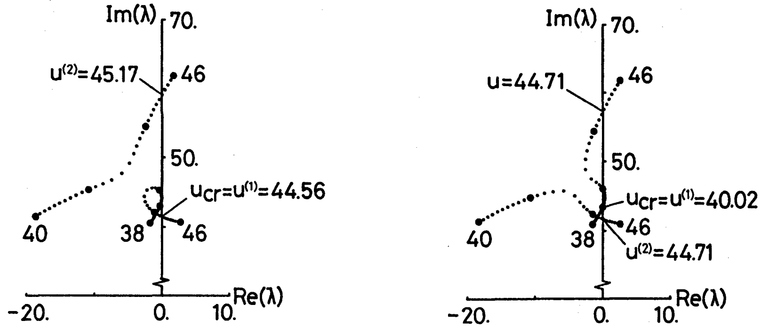
Fig. 3.2 Complex frequencies of cantilever pipe ($\beta=0.01, \gamma=1.00$)

state of flutter. This qualitative finding known to the uniform pipe is also maintained for the pipe with the optimal shape, although the detailed behavior of the locus is significantly different between the pipes of uniform and optimal shape.

The dimensionless critical velocity of the internal flow increases from u_{cr}=26.72 of the uniform pipe to u_{cr}=44.56 of the pipe with optimal shape with the same pipe volume. The fluid velocity $u^{(2)}$=45.17 at which the second complex frequency arrives at the imaginary axis is very close to the critical velocity $u_{cr}=u^{(1)}$=44.56. In fact, the first and the second modes are considered simultaneously during the last stage of the iteration process for the shape determination.

Table 1 Dimensionless pipe size and critical velocity

Position	Radius $\sqrt{m}$			
x	Uniform	Optimal	Before convergence	Robust M_g'=0.04
0.0	2.0000	2.0610	2.0482	2.0630
0.1	2.0000	1.7367	1.7421	1.7377
0.2	2.0000	1.9095	1.8880	1.9145
0.3	2.0000	2.2241	2.2363	2.2295
0.4	2.0000	2.3882	2.3956	2.3927
0.5	2.0000	2.4124	2.4018	2.4157
0.6	2.0000	2.2711	2.2475	2.2717
0.7	2.0000	1.9955	1.9991	1.9909
0.8	2.0000	1.5389	1.5483	1.5135
0.9	2.0000	1.4142	1.4142	1.4142
1.0	2.0000	1.5724	1.6413	1.5752
u_{cr}	26.72	44.56	40.02	44.06



(a) Optimal shape (b) Shape before convergence

Fig. 3.3 Complex frequencies of cantilever pipe (vicinity of the imaginary axis)

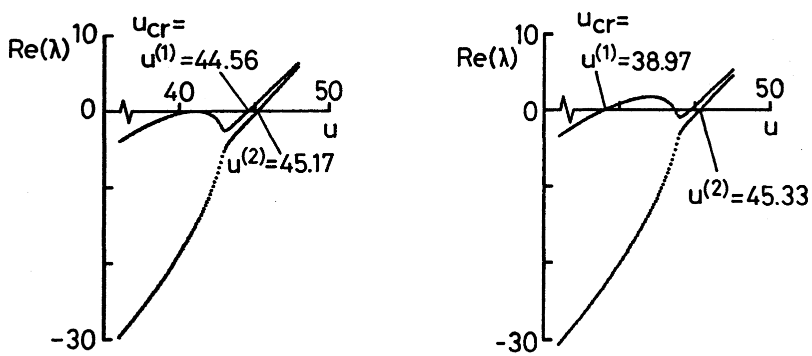
Figure 3.3(a) enlarges the vicinity of the critical point in Fig. 3.2(b). The complex frequency almost arrives upon the imaginary axis around u=40, the lower velocity than the critical value. In the case of the pipe with the shape very similar to the optimal one obtained during the iteration for the shape determination, the complex frequency reaches the imaginary axis at this moment with the velocity u=40.02 as is shown in Fig. 3.3(b). The detailed dimensionless pipe size of the optimal shape and the shape just before the convergence are given in Table 1. In spite of the almost negligible small difference in these pipe shapes, the critical velocity shows the meaningful difference of about 10 %. That is, the critical velocity of the cantilever pipe is very sensitive to the pipe size around the optimal shape giving the maximum critical velocity.

4 Improving Robustness of Optimal Shape

Although the optimal shape of the cantilever pipe shows the remarkable increase of the critical velocity, the critical velocity is quite sensitive to a small change of the pipe size at the optimal shape. That is, the optimal shape of the cantilever pipe lacks the robustness. This section discusses a way to improve the robustness of the optimal pipe shape.

4.1 Robustness Index

The relation between the complex frequency λ and the dimensionless velocity u of the internal flow shown in Fig. 3.3(b) is redrawn in the relation of the real part of the complex frequency Re(λ) to the dimensionless velocity u as shown in Fig. 4.1(a). The real part of the complex frequency of the first mode increases as the increase of the velocity but does not monotonously. This non-monotonicity is the reason of the unexpected high sensitivity of the critical velocity. Thus, the



(a) Optimal shape (b) Perturbed from optimal

Fig. 4.1 Fluid velocity and complex frequency absolute value of Re(λ) at the local maximum within the velocity range smaller than the critical velocity gives an index of the robustness

$$M_g=-\mathrm{Re}(\lambda_A) \tag{19}$$

where the subscript $(\ )_A$ denotes the value at the local maximum. When there are several local maxima, the index M_g is defined at the maximum among them.

The sensitivity analysis enables us to convert the small variation in the dimensionless pipe size **m** to the variation of the complex frequency λ as

$$\delta\mathrm{Re}(\lambda_A)=\mathrm{Re}(\lambda_A),_{\mathbf{m}}\,\delta\mathbf{m} \tag{20}$$

where $(\ ),_{\mathbf{m}}$ denotes the derivative with respect to the dimensionless pipe size **m**. When the variation in the complex frequency is equal to the robustness index M_g, the critical velocity jumps down to the velocity at the local maximum. The robustness index is also measured in the distance κ between the current shape and the boundary corresponding to Re(λ)=0 in the pipe size space as shown in Fig. 4.2, since the variation $\delta\mathbf{m}$ proportional to the derivatives of Re(λ_A)

$$\delta\mathbf{m}=\kappa\,\mathrm{Re}(\lambda_A),_{\mathbf{m}} \tag{21}$$

gives the smallest variation among those resulting such a jump in the critical velocity. This scalar κ is the l_2 norm of the variation $\delta\mathbf{m}$ when the gradient $\mathrm{Re}(\lambda_A),_{\mathbf{m}}$ is normalized to the unity. When the distance is measured in the infinity norm, the index becomes

$$M_g'=\kappa\,\|\,\mathrm{Re}(\lambda),_{\mathbf{m}}\,\|_\infty=\kappa\,\max[\,|\,\mathrm{Re}(\lambda),_{mj}\,|\,] \tag{22}$$

and gives the possible maximum change in the pipe size m_j.

4.2 Shape Modification to Improve Robustness

When the optimal shape is obtained by the maximization of the critical velocity and its robustness index M_g' is smaller than the desired value $M_g^{*\prime}$, the pipe is

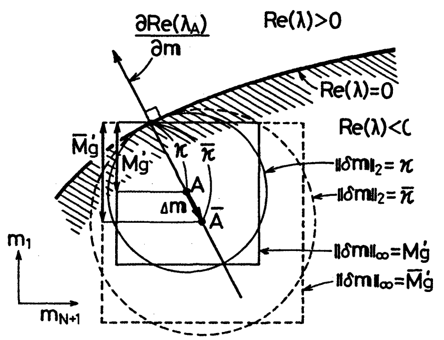


Fig. 4.2 Robustness index in design variables

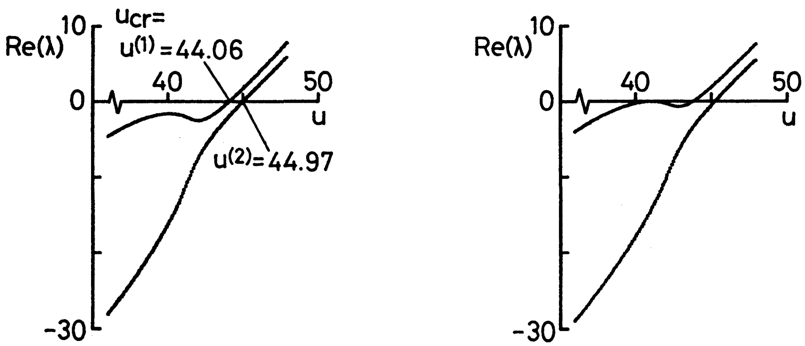
modified to increase the robustness index. The correction of the pipe shape $\Delta \mathbf{m}$ proportional to the gradient $Re(\lambda)_{,\mathbf{m}}$ brings the smallest change in the pipe shape. The expected change of the pipe size

$$\Delta \mathbf{m}=(\kappa^*-\kappa)[-Re(\lambda_A)_{,\mathbf{m}}]=-\frac{M^{*\prime}_g-M_g'}{\| Re(\lambda_A)_{,\mathbf{m}} \|_\infty} Re(\lambda_A)_{,\mathbf{m}} \tag{23}$$

modifies the pipe shape as $\mathbf{m}+\Delta \mathbf{m}$. The constancy of the volume is maintained by the proportional adjustment of the modified pipe shape with the lower limit of the size. This modification is a result of the linearization, and is used in repetitive manner until the specified robustness index is attained.

4.3 Optimal and Robust Shape

The robustness of the optimal shape of the cantilever pipe is improved by the proposed scheme. When the perturbation $\delta \mathbf{m}$ equivalent to the $M_g'=0.04$ is enforced to the optimal shape of Fig. 3.1, the flow velocity vs. complex frequency relation in Fig. 4.1(a) changes as shown in Fig. 4.1(b) and the critical velocity decreases from $u_{cr}=44.56$ to $u_{cr}=38.97$ discontinuously. By imposing the target value $M^{*\prime}_g=0.04$ on the robustness index, the optimal pipe shape is modified as is shown in Table 1. Fig. 4.3(a) gives the flow velocity vs. complex frequency curve of the modified pipe shape. As the result of the shape modification, the meaningful distance is kept at the local maximum of the real part of the complex frequency in the range of velocity smaller than the critical value. In the case of the pipe of modified shape, the perturbation in shape corresponding to $M_g<0.04'$ does not brings the discontinuous decrease of the critical velocity as is shown in Fig. 4.3(b). In this sense, the robust pipe shape is obtained with only 1.1 % decrease of the critical velocity from that of the optimal.



(a) Robust & optimal shape (b) Perturbed($\| \delta \mathbf{m} \|_\infty=0.04$)

Figure 4.3 Fluid velocity and complex frequency ($\| \mathbf{m} \|_\infty=4$ for uniform shape)

5 Concludions

The optimum structural shape is studied for the pipe conveying fluid. The stability of the pipe behavior is analyzed in the context of the eigenvalue problem by using the finite element method. For the first stage of shape determination, the critical velocity that is the minimum of the fluid velocity making the vibration modes unstable is used as the stability index of the pipe under concern. The critical velocity is reasonably increased as the result of the shape determination. The increase of the critical velocity of the cantilever pipe is remarkable but the velocity attained with the optimal shape is very sensitive to the small change in the pipe shape. That is, the optimal shape of the cantilever pipe lacks the robustness.

In order to improve the robustness of the optimal shape of the cantilever pipe, the shape modification is employed as the second stage of the shape determination problem. The robustness index defined is the minimal distance of the eigenvalue locus from the imaginary axis in the stable range of the internal fluid velocity. The necessary modification from the optimal shape is evaluated by using the specified robustness index and the sensitivity. The procedure works effectively and the optimal and robust shape is obtained for the cantilever pipe. The authors whould like to thank late Professor Y. Seguchi for his encouragement in the structural optimization study.

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Multi-Objective Modeling for Engineering Applications in Decision Support

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Abstract. In engineering computer-aided design, the final choice of the design might be supported by multicriteria optimization; we show here, however, that multicriteria optimization can be also used as a tool of helping in a flexible analysis of various design options or various modeling and simulation variants, even from the beginning stages of model construction. Various formats of defining linear and nonlinear models are discussed together with related problems of inverse and softly constrained multi-objective simulation. Such techniques are illustrated by engineering applications of a software package DIDASN++ in mechanics, automatic control and ship navigation.

Keywords: Multi-objective optimization, modeling, decision support, computer-aided design

1 Introduction

We consider here a specific type of decision process which is typical for engineering design. We assume that the decision maker – in this case, an engineering designer called a **modeler** – develops, modifies and uses **substantive models** specific for his profession. In the decision process, the modeler might have to specify at least partly her/his preferences and thus define a **preferential model**. However, we assume that the modeler preserves the right to change these preferences and the preferential model is represented by a partial order similar to the Pareto order.

Such decision process might consist of the following phases:

1. The problem recognition and formulation, data gathering and substantive model selection.

2. Substantive model formulation and initial analysis, including model validation.

3. Specification of a partial preferential model, detailed analysis of the substantive model, generation of scenarios or design options.

4. Final selection of a scenario or a design, implementation, feedback from practice.

Phases **1** and **4**, though very important, will not be considered here in detail. We concentrate here on multi-objective optimization techniques that might be used to support phases **2** and **3** – often very time-consuming for the modeler.

The application of such techniques in these phases might be called **multi-objective modeling and simulation**.

As a specific methodology of multi-objective analysis and optimization, we rely on the **aspiration-based techniques** which use the maximization of an **order-consistent achievement function** as a method of aggregating multiple objectives and of interacting with the modeler.

2 Some basic relations in aspiration-based methodology of multi-objective optimization

We recall here some basic concepts related to the aspiration-based methodology (see *e.g.* [10]) and the decision process considered. Assume that the general form of the substantive model is:

$$\mathbf{y} = \mathbf{f}(\mathbf{x}, \mathbf{z}); \;\; \mathbf{x} \in X_0; \;\; \mathbf{z} \in Z_0 \tag{1}$$

where $\mathbf{x} \in \boldsymbol{R}^n$ is a vector of **decision variables**, $\mathbf{z}$ is a **parameter vector**, X_0 is a **set of admissible decisions** often defined implicitly by additional inequalities or equations called **constraints**, $\mathbf{y} \in \boldsymbol{R}^m$ is a vector of **model outputs** or **decision outcomes** and includes various intermediary variables used when formulating the model. The function $\mathbf{f} : \boldsymbol{R}^n \times Z_0 \to \boldsymbol{R}^m$ which defines model outputs might be also defined implicitly or by a complicated model structure; we shall often write $\mathbf{y} = \mathbf{f}(\mathbf{x})$ suppressing its dependence on parameters $\mathbf{z}$. $Y_0 = \mathbf{f}(X_0)$ is called the set of **attainable outcomes**.

The modeler might specify several model outputs as especially interesting – we call them **objectives** or **criteria** and denote by $q_i = y_j$, forming an **objective vector** $\mathbf{q} \in \boldsymbol{R}^k$. While this vector and space might change during the decision process, we denote the relation between decisions and their outcomes by $\mathbf{q} = \mathbf{F}(\mathbf{x}, \mathbf{z})$ or $\mathbf{q} = \mathbf{F}(\mathbf{x})$. $Q_0 = \mathbf{F}(X_0)$ is called the set of **attainable objectives**.

We assume that the modeler simply states which objectives q_i shall be **maximized**, which ones shall be **minimized**, and which of them shall be **stabilized**, that is, kept close to a given **reference** or **aspiration level** $\bar{q}_i$. Aspiration levels are specified also for maximized and minimized objectives; the **aspiration point** $\bar{\mathbf{q}}$ is used as the main interaction parameter by which the modeler controls the selection of decisions and their outcomes. Beside the aspiration point, the modeler might also use a **reservation point** $\bar{\bar{\mathbf{q}}}$.

When aggregating many objectives which have often various units of measurement, the **ranges of objectives** q_i must be at least approximately known:

$$q_{ilo} \leq q_i \leq q_{iup} \;\; \forall i = 1, \ldots, k \tag{2}$$

The usual way of estimating these ranges is to compute the **ideal** or **utopia point** by optimizing separately each objective and to estimate its counterpart – the **nadir point** (which is difficult to compute exactly, but only approximate bounds of this type are needed). We assume that the aspiration (and reservation) levels are strictly contained in these ranges.

A way of aggregating the objectives into an **order-consistent achievement function** (see *e.g.* [16]) consists in specifying **partial achievement functions** $\sigma_i(q_i, \bar{q}_i)$ or $\sigma_i(q_i, \bar{q}_i, \bar{\bar{q}}_i)$ which should:

a) **be strictly monotone** consistently with the specified partial order (increasing for maximized objectives, decreasing for minimized ones, increasing below $\bar{q}_i$ and decreasing above $\bar{q}_i$ for stabilized ones);

b) **assume value 0** if $q_i = \bar{q}_i \;\; \forall i = 1, \ldots, k$ and aspiration levels are used alone – or **assume value 0** if $q_i = \bar{\bar{q}}_i \;\; \forall i = 1, \ldots, k$ **and assume value 1** if $q_i = \bar{q}_i \;\; \forall i = 1, \ldots, k$, if both aspiration and reservation levels are used.

If aspiration levels are used alone, it is useful to define partial achievement functions with a slope that is larger if the aspiration levels are closer to their extreme levels:

$$\begin{aligned} \sigma_i(q_i, \bar{q}_i) &= (q_i - \bar{q}_i)/(q_{iup} - \bar{q}_i) \text{ (max)} \\ \sigma_i(q_i, \bar{q}_i) &= (\bar{q}_i - q_i)/(\bar{q}_i - q_{ilo}) \text{ (min)} \\ \sigma_i(q_i, \bar{q}_i) &= \left\{ \begin{array}{ll} (\bar{q}_i - q_i)/(q_{iup} - \bar{q}_i), & \text{if } q_i > \bar{q}_i \\ (q_i - \bar{q}_i)/(\bar{q}_i - q_{ilo}), & \text{if } q_i \leq \bar{q}_i \end{array} \right\} \text{ (stab)} \end{aligned} \tag{3}$$

If both aspiration and reservation levels are used, it is more useful to define the partial achievement functions $\sigma_i(q_i, \bar{q}_i, \bar{\bar{q}}_i)$ as piece-wise linear functions of various forms, see e.g. [4].

If the values of $\sigma_i(q_i, \bar{q}_i)$ would be restricted to the interval [0;1], then they could be interpreted as fuzzy membership functions $\mu_i(q_i, \bar{q}_i)$ (see *e.g.* [19], [12], [22]) which express the degree of satisfaction of the modeler with the value of the objective q_i. The corresponding overall membership function would then have the form:

$$\mu(\mathbf{q}, \bar{\mathbf{q}}) = \bigwedge_{1 \leq i \leq k} \mu_i(q_i, \bar{q}_i) = \min_{1 \leq i \leq k} \mu_i(q_i, \bar{q}_i) \tag{4}$$

Such an interpretation can be used in graphic interaction with the modeler; however, membership functions $\mu_i(q_i, \bar{q}_i)$ and $\mu(\mathbf{q}, \bar{\mathbf{q}})$ are not strictly monotone if they are equal to 0 or 1. Therefore, inside a multi-objective optimization system, a slightly different overall achievement function must be used, with values not restricted to the interval [0;1]:

$$\sigma(\mathbf{q}, \bar{\mathbf{q}}) = (\min_{1 \leq i \leq k} \sigma_i(q_i, \bar{q}_i) + \varepsilon \sum_{i=1}^{k} \sigma_i(q_i, \bar{q}_i))/(1 + k\varepsilon) \tag{5}$$

where $\varepsilon > 0$ is a coefficient resulting in **proper efficiency** of maximal points of this scalarizing function. Proper efficiency (with a prior bound) of $\hat{\mathbf{q}} \in Q_0 = \mathbf{F}(X_0)$ means here that the objective vector is nondominated and has a joint upper bound $M = 1 + 1/\varepsilon$ on corresponding trade-off coefficients.

The order-consistent achievement function (6) is related to some basic concepts of multi-objective optimization (see *e.g.* [15], [10], [16], [13], [17], [18]).

The achievement function $\sigma(\mathbf{q}, \bar{\mathbf{q}})$ is nondifferentiable; in case of linear models this does not matter since the function is concave and its maximization can be equivalently expressed as a linear programming problem. For nonlinear models, however, optimization algorithms for smooth functions are more robust. Therefore, a useful modification of the achievement function is its smooth approximation, which can be defined e.g. by using an l_p norm (with $p > 2$; usually $p = 4 \ldots 8$ suffices):

$$\sigma(\mathbf{q}, \bar{\mathbf{q}}) = 1 - \frac{1}{k}\left(\sum_{i=1}^{k}(1 - \sigma_i(q_i, \bar{q}_i))^p\right)^{1/p} \tag{6}$$

Note that $\sigma(\mathbf{q}, \bar{\mathbf{q}})$ **is not**, in general, **a norm of the difference** $\mathbf{q} - \bar{\mathbf{q}}$ and is equivalent to such a norm only if all objectives are stabilized. A norm would not preserve needed properties of monotonicity for maximized or minimized objectives. It is known that approaches using a norm as a scalarizing function must either severely restrict the choice of aspiration points (as in displaced ideal approaches, see *e.g.* [21]) or might result in optimized solutions that are not efficient (as in goal programming approaches, see *e.g.* [1]). Therefore, the use of order-consistent achievement functions can be interpreted as **generalized goal programming**, improved by providing efficient solutions for arbitrary aspiration points and equivalent to goal programming in the specific case of all stabilized objectives.

3 Standards of defining large-scale linear models and issues of multi-objective modeling

Linear models are less often used in engineering than in economics, but they provide a starting point in modeling. In the case of large-scale models, a practical way to develop a model is to prepare first a linear version and then augment it by necessary nonlinear parts.

In a textbook, a standard form of multi-objective linear programming problem is usually presented as:

$$\text{"}\underset{\mathbf{x} \in X_0}{\text{maximize}}\text{"}(\mathbf{q} = \mathbf{C}\mathbf{x} \in \mathbb{R}^k); \tag{7}$$

$$X_0 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b} \in \mathbb{R}^m, \ \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\} \tag{8}$$

where the vector "maximization" is understood in the sense of a predefined partial order. Much research has been done on the specification of efficient decisions and objectives for linear models - see *e.g.* [2], [13] - but even more attention should be paid to the practical aspects of using multiobjective linear models. Note that the standard form refers to the equality form of constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ when defining X_0 and is achieved for the purpose of theoretical compactness by introducing dummy variables as additional components of vector $\mathbf{x}$. However, in the practice of linear programming it is known - see *e.g.* [3] - that the standard form is rather unfriendly to the modeler. Thus, specific formats of writing linear models have been proposed, such as MPS or LP-DIT format, see *e.g.* [9]. Without going into details, we note that such formats correspond to writing the set X_0 in the form:

$$X_0 = \{\mathbf{x} \in \boldsymbol{R}^n : \mathbf{b} \le \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{W}\mathbf{y} \le \mathbf{b} + \mathbf{r} \in \boldsymbol{R}^m, \ \mathbf{l} \le \mathbf{x} \le \mathbf{u}\} \tag{9}$$

where the vector $\mathbf{x}$ denotes rather actual decisions than dummy variables, thus $\mathbf{x}, m, n$ denote different variables than in standard textbook form. The model output $\mathbf{y}$ is composed of various intermediary variables (hence it depends implicitly on itself). Essential for the modeler is the freedom of choosing any of outputs y_j or decisions x_i as an objective variable q_i.

Even more complicated formats of linear models are necessary if we consider the case of dynamic models:

$$\begin{aligned} X_0 = \{\mathbf{x} \in \boldsymbol{R}^n &: \mathbf{w}_{t+1} = \mathbf{A}_t\mathbf{w}_t + \mathbf{B}_t\mathbf{x}_t; \ \mathbf{l}_t \le \mathbf{x}_t \le \mathbf{u}_t; \\ &\mathbf{b}_T \le \mathbf{y}_t = \mathbf{C}_t\mathbf{w}_t + \mathbf{D}_t\mathbf{x}_t \le \mathbf{b}_t + \mathbf{r}_t \in \boldsymbol{R}^m, \ \ t = 1, \dots T\} \end{aligned} \tag{10}$$

where $\mathbf{w}_t$ is called the **dynamic state** of the model (the initial condition $\mathbf{w}_1$ must be given), the index t has usually the interpretation of (discrete) time, and $\mathbf{x} = (\mathbf{x}_1, \dots \mathbf{x}_T)$ is a **decision trajectory** (called also control trajectory). Similarly, $\mathbf{w} = (\mathbf{w}_1, \dots \mathbf{w}_{T+1})$ is a **state trajectory** while $\mathbf{y} = (\mathbf{y}_1, \dots \mathbf{y}_T)$ is the **output trajectory**. Other similarly complicated forms of linear models result *e.g.* from stochastic optimization.

A modeler that has developed or modified a complicated large scale linear model should first validate it by simple simulation with some common sense decisions. Because of multiplicity of constraints in large-scale models it might, however, happen that the common sense decisions are not admissible (in the model); thus, simple simulation of large-scale linear models is actually difficult.

An important help for the modeler can be **inverse simulation**, in which some desired model outcomes $\bar{\mathbf{y}}$ are assumed and we check - as in classical goal programming - whether there exist admissible decisions which result in these outcomes. **Generalized inverse simulation** consists in specifying also some reference decision $\bar{\mathbf{x}}$ and in testing, whether this reference decision could result in the desired outcomes $\bar{\mathbf{y}}$. This can be written in the format of norm minimization, while it is useful to apply the augmented Chebyshev norm:

$$\begin{aligned} -\sigma(\mathbf{y}, \bar{\mathbf{y}}, \mathbf{x}, \bar{\mathbf{x}}) = (1-\rho)(\max\nolimits_{1 \le i \le n} |x_i - \bar{x}_i| + \varepsilon \textstyle\sum_{i=1}^{n} |x_i - \bar{x}_i|) + \\ + \rho(\max\nolimits_{1 \le j \le m} |y_j - \bar{y}_j| + \varepsilon \textstyle\sum_{j=1}^{m} |y_j - \bar{y}_j|) \end{aligned} \tag{11}$$

where the sign "-" results from the convention that achievement functions are usually maximized while norms are minimized and the coefficient $\rho \in [0;1]$ indicates the weight given to achieving the desired output versus reference decision; it is assumed for simplicity that all variables are standardized, measured in common units. An aspiration-based multi-objective optimization system can help in such inverse simulation, in which case we stabilize all outcomes and decisions of interest and use partial achievement functions $\sigma_i(q_i, \bar{q}_i)$ for such stabilized objectives to define an overall achievement function:

$$\begin{aligned} \sigma(\mathbf{y}, \bar{\mathbf{y}}, \mathbf{x}, \bar{\mathbf{x}}) = (1-\rho)(\min\nolimits_{1 \le i \le n} \sigma_i(x_i, \bar{x}_i) + \varepsilon \textstyle\sum_{i=1}^{n} \sigma_i(x_i, \bar{x}_i) + \\ + \rho(\min\nolimits_{1 \le j \le m} \sigma_j(y_j, \bar{y}_j) + \varepsilon \textstyle\sum_{j=1}^{m} \sigma_j(y_j, \bar{y}_j)) \end{aligned} \tag{12}$$

A further generalization of the above system function might be called simulation with elastic constraints, shortly **elastic simulation** or **softly constrained simulation**, which includes a distinction between **hard constraints** that can never be violated – such as physical laws, balance equations – and **soft constraints** which in fact represent some desired relations and are better represented as additional objectives with given aspiration levels.

Another possibility is the issue of testing scenarios related to various values of parameters and thus testing elements of model uncertainty. Similar techniques of multi-objective optimization can be used for this purpose, see [20], where a description and applications (for agricultural economics) of a multi-objective linear optimization system MENTAT is presented.

4 Standards of defining nonlinear models and the multi-objective optimization system DIDAS-N++

Even less developed than user-friendly standards of defining linear models are such standards for nonlinear models, necessary for engineering applications. While there exist some standards for nonlinear optimization systems such as GAMS or LANCELOT, they are devised more for optimization purposes than for practical multi-objective modeling. A useful standard was developed in the multi-objective nonlinear optimization system DIDAS-N (see e.g. [7]). Briefly, it consists in defining subsequent nonlinear model output relations:

$$\begin{aligned} y_1 &= f_1(\mathbf{x}, \mathbf{z}); \\ \ldots &= \ldots \\ y_{j+1} &= f_{j+1}(\mathbf{x}, \mathbf{z}, y_1, \ldots y_j), \quad j = 1, \ldots m-2; \\ \ldots &= \ldots \\ y_m &= f_m(\mathbf{x}, \mathbf{z}, y_1, \ldots y_{m-1}) \end{aligned} \tag{13}$$

together with bounds for decision variables and outputs:

$$x_{ilo} \leq x_i \leq x_{iup}, \quad i = 1, \ldots n; \quad y_{jlo} \leq y_j \leq y_{jup}, \quad j = 1, \ldots m \tag{14}$$

(bounds for model parameters $\mathbf{z}$ are less essential). This way, a directly computable (explicit, except for bounds) nonlinear model is defined; implicit models can be defined by specifying $y_{jlo} = y_{jup}$ for some j, which is then taken into account and resolved during optimization. Any variable y_j (and x_i, if needed) can be specified as maximized, minimized or stabilized objective.

The model equations and bounds are specified using a spreadsheet. The DIDAS-N system includes advanced automatic (algebraic) functions of model differentiation: it presents to the modeler all required partial and full derivatives and prepares an economical way of computing numerically the derivatives of an overall achievement function in a smooth form similar to Eq. (7). As a robust optimization solver, a shifted penalty (augmented Lagrangean) algorithm for nonlinear output constraints and a projected conjugate direction algorithm for linear box-like decision constraints were implemented in the system.

However, DIDAS-N is a closed, nonmodular system written in PASCAL, difficult for working with larger models, particularly when including large-scale linear model parts. Therefore, a new system called DIDAS-N++ was recently developed, see [5]. In DIDAS-N++, the nonlinear part can be linked with a linear part, which is indicated by the general format:

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_c\mathbf{x}_c, \\ \mathbf{y}_2 &= \mathbf{f}(\mathbf{x}_2, \mathbf{x}_c, \mathbf{z}, \mathbf{y}_1, \mathbf{y}_2) \end{aligned} \tag{15}$$

where $\mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_1, \mathbf{x}_2$ denote the vectors of model outputs and decision variables specific for the linear and nonlinear parts, while $\mathbf{x}_c$ is the vector of decision variables common for both parts.

Further, more difficult class are dynamic nonlinear models (particularly with time-delays), even if specified in discrete time; for a more detailed discussion of related issues see [6], [17].

5 Applications of multi-objective simulation and optimization for engineering design

Three cases of engineering design are selected to illustrate the somewhat abstract reasoning of previous sections. The first case concerns a classical problem in mechanical design - the design of a spur gear transmission unit, see [11]. The mechanical outlay of this unit is shown in Fig. 1.

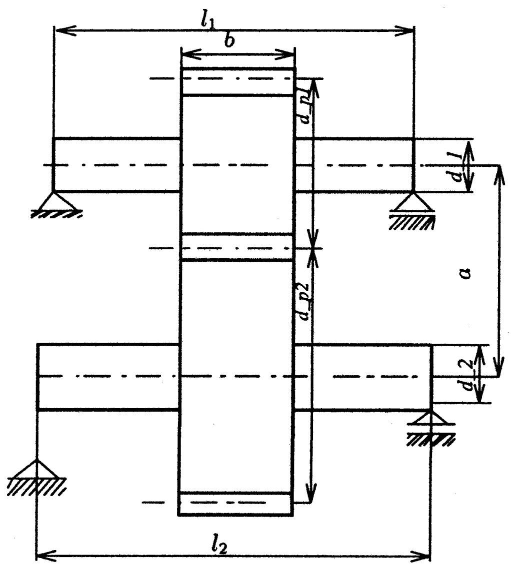

Figure 1: A diagram of the spur gear unit

The problem might be specified in a textbook format by defining three objective functions $f_i(\mathbf{x})$ and 14 constraints $g_j(\mathbf{x})$, some nonlinear and some expressing simple bounds, to describe the set of admissible decisions by $X_0 = \{\mathbf{x} \in R_n : \mathbf{g}(\mathbf{x}) \leq 0\}$. However, the specification in DIDASN++ format is more convenient for the modeler.

Because of its length, we cannot present here the model in detail. We note only that the decision variables are: the width of the toothed wheel rim b – which is also an objective – the diameters d_1 and d_2 of the input and output shafts, the number of teeth of the pinion wheel $\tilde{z}$ and the pitch of gear teeth $\tilde{m}$ (the last two variables are actually discrete). Other objectives might be: the volume of the gear unit $q_1 = f_1$ and the distance between the axes $q_2 = f_2$. Because of the rather complicated (actually – not convex) form of the model, its simulation without a good experience in mechanical design would produce unacceptable results. Fig. 2 shows the results of an inverse simulation of the model (with two model outcomes - objectives q_1 and q_3 denoted respectively by f_1 and f_3 - and two decision variables denoted by d_1 and d_2, all stabilized) while aspiration levels were arbitrarily selected. The optimization of a corresponding achievement function shows that such arbitrary aspiration levels cannot be realized in this model. The contours indicated in Fig. 2 represent the values of membership functions $\mu_i(q_i, \bar{q}_i, \bar{\bar{q}}_i)$ and the circles on these contours indicate the attained levels of objectives. Values 0 of these membership functions at circled points indicate that the requirements of the modeler cannot be satisfied.

In order to find results that are admissible for the model, other aspiration levels must be selected using the experience of a designer, see Fig. 3 where the aspirations were set according to data given in [11]. Improvement of both (or even all three) objectives considered can be obtained by switching to softly constrained simulation, as shown in Fig. 4, where the soft constraints on decision variables were relaxed in such a way as to obtain efficient results for the problem of minimizing both selected objectives. In Fig. 4, the improvement of objective values is shown by line segments leading to circles that indicate the attained values.

The second case shows the usefulness of including dynamic formats of models in multi-objective optimization systems. This case concerns ship navigation support (see [14]): the problem is to control the course of a ship in such a way as to maximize the minimal distance from possible collision objects while minimizing the deviations from the initial course of the ship, see Fig. 5. This is a dynamic problem, with the equations of the model described initially by a set of differential equations for $t \in [0; T]$:

$$\begin{aligned} \dot{w}_1(t) &= v_1 \sin x(t) \\ \dot{w}_2(t) &= v_1 \cos x(t) \\ \dot{w}_{1j}(t) &= v_j \sin \psi_j, \;\; j = 2, \ldots \check{n} \\ \dot{w}_{2j}(t) &= v_j \cos \psi_j, \;\; j = 2, \ldots \check{n} \end{aligned} \qquad (16)$$

with initial values of ship positions given as the vector $\mathbf{w}(0)$; between other

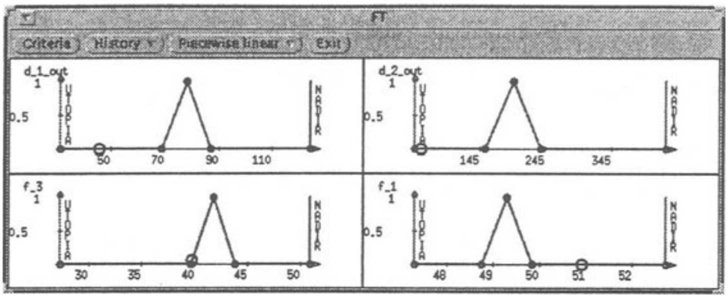

Figure 2: Interaction screen of DIDAS-N++ in the inverse simulation case, arbitrary aspiration levels

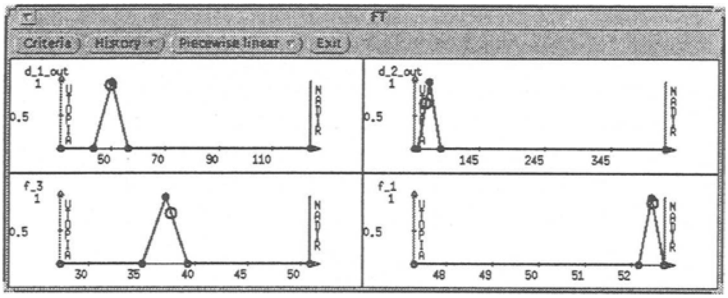

Figure 3: Interaction screen of DIDAS-N++ in the inverse simulation case, aspiration levels based on mechanical experience

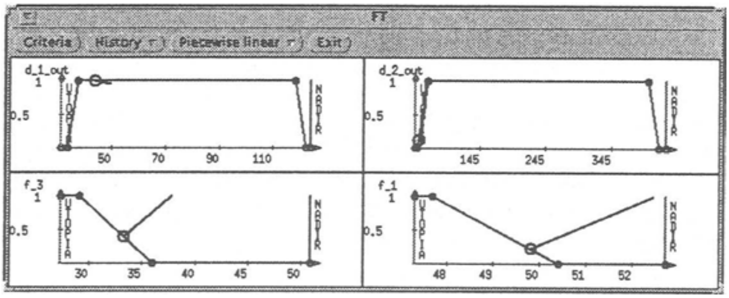

Figure 4: Interaction screen of DIDAS-N++ in the softly constrained simulation case, improvements of both objectives

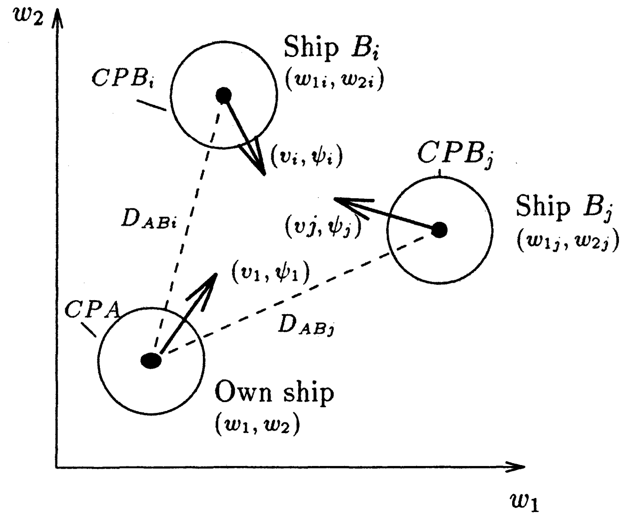

Figure 5: A diagram of ship collision control situation (CPA - safe zone for ship A)

model outcomes, the objectives can be modeled as:

$$\begin{aligned} q_1 &= \min_{t\in[0;T]} \min_{j=2,\ldots \tilde{n}} ((w_1(t) - w_{1j}(t))^2 + (w_2(t) - w_{2j}(t))^2) \\ q_2 &= \int_0^T (x(t) - \psi_1)^2 dt \end{aligned} \tag{17}$$

To be used in DIDAS-N++ system, this model was simply discretized in time, with the resulting model form similar to Eqns. (11) or (15). Experiments similar to described in the first case show the usefulness of algebraic model differentiation and model compiling, of multi-objective modeling and inverse or softly constrained simulation for the modeler.

The third case is an example of a dynamic model with time delays. It concerns a basic problem in automatic control: the choice of parameters of a PID (proportional-integral-differential) controller as to obtain good properties of the control system. The model can be described in various terms; for a control engineer, a useful model description is a block-diagram of the control system, see Fig. 6, where the transfer functions of the assumed model of control plant (inertial with delay) and of the PID controller are indicated.

The decision variables are the parameters K, T_i and T_d of the controller; various model outputs can be defined as possible objectives, e.g. the overshoot κ and the control time ω:

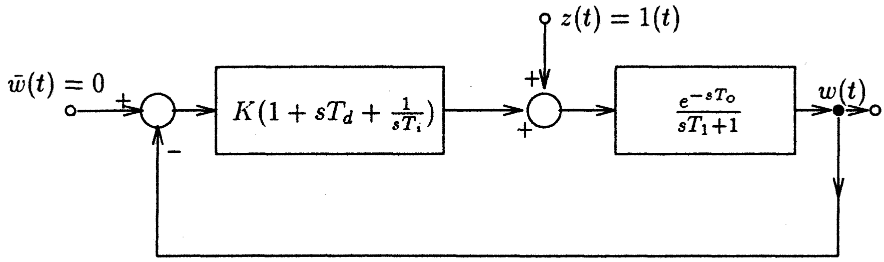

Figure 6: A block-diagram of a control system with an inertial plant with delay and a PID controller

$$\begin{aligned} \kappa &= |\min_{t\in[0;T]} w(t)| / w_{max}, \quad w_{max} = \max_{t\in[0;T]} w(t) \\ \omega &= \min\{t : |w(\vartheta)| \le 0.05 w_{max} \ \forall \vartheta \ge t\} \end{aligned} \tag{18}$$

or various integrals of control error over time:

$$I_1 = \int_0^T |w(t)|\,dt; \quad I_2 = \int_0^T |w(t)|^2\,dt; \quad I_4 = \int_0^T |w(t)|^4\,dt \tag{19}$$

The model, to get a form similar to Eqn. (15), must be converted to the form of difference-differential equations and discretized over time. The model, though complicated, is directly computable: it can be simply simulated for given values of K, T_i, T_d, or inverse or softly constrained simulated, or optimized multi-objectively. Various experiments with the model show again the usefulness of multi-objective modeling tools in model development and analysis.

6 Conclusions

While multi-objective optimization is usually applied to support the selection of efficient alternatives for a final decision, its techniques – when suitably extended – might be applied also for supporting more preliminary model analysis. Such preliminary stages of modeling require usually more effort than final model analysis. Thus, the extension of applications of multi-objective optimization to multi-objective modeling – including such techniques as inverse or softly constrained simulation – provides useful tools for a modeler.

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Applications

3.2 Applications in Environment

Multi-Criteria Decision Making to Rank The Jordan-Yarmouk Basin Co-riparians Water Allocations According to The Helsinki and ILC Rules

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The Jordan River system which is shared by Jordan, the Palestinians in the occupied territories, Syria, Lebanon and Israel is a major issue in the current Middle East peace negotiations. This study identifies the water resources of the system, review of the development plans in the basin , and the relevant legal riparian issues and practices related to the basin. The main conclusion is that **the application of the Helsinki and ILC international rules separately to allocate shared water resources in the basin between the different countries is not suitable to a region which is in conflict in every issue, since all relevant rules are not considered together and a conclusion is not reached on the basis of the whole rules**; this necessitates the need for applying the Multi-Criteria Decision Aid (MCDA) as a mathematical tool to help to solve the problem based on recognised water rights and sharing the water in a neighbourly way to avoid any future major conflict. The PROMETHEE method [1] as an MCDA method was applied. The results of the relative ranking of the Jordan-Yarmouk co-riparians indicated that Jordan (first) and the Palestinian in occupied territories (second) ranked high in this analysis (compared to Israel , Syria, and Lebanon) after selecting a complete and comprehensive set of rules adapted from the existing international laws with their relative importance, and measurement scale. In addition, sensitivity analysis was carried out to check the ranges of stability of the results. Furthermore, analysis regarding possible allocations of future investments in the basin subjected to different constraint was demonstrated using the new version of PROMETHEE V.

States, in their competing claims on international rivers have pressed four theoretical positions (see for example, [3], [4]) : Absolute Sovereignty (or Harmon Doctrine), Territorial Integrity, Community of Co-riparian States, and Limited Territorial Sovereignty. The first two theories have been generally rejected both analytically and in state practice. The last two theories, particularly the latter one, are supported by the preponderant weight of state practice. Underlying these two theories are two principles of international law which are embedded in the various drafts, resolutions and proposals attempting to "codify" the international law and can be said to consolidate the legal framework to guide the conduct of states. The two principles are **Prohibition Against Appreciable Harm and Equitable Utilisation** (this practice and its interpretation is covered in article IV and V of Helsinki Rules (developed by International Law Association, 1966), and the ILC rules covered in articles 5, 6 and 7 (developed by a UN affiliated body, 1992). It is noted that neither the Helsinki nor the ILC rules are legally binding, but the principle of equitable utilisation of international river has become the most widely advocated by the international legal community [3]. Applying the above rules to rank the Jordan-Yarmouk co-riperians is limiting (i.e. if we apply the rules which stress water contribution, Jordan and Israel will receive smaller amounts of water so for these countries it is important to apply the rules which emphasise the degree of dependence on the Jordan-Yarmouk water) .

The relevant set of the fundamental objectives to prioritise the co-riperian were elicited from the spirit of the Helsinki and the ILC rules, The set , measurement scale , the relative importance , and input data (according to pre 1976 war conditions) presenting the evaluation of the position of each riparian on each objective and the relevant threshold parameters, as defined in the set of PROMETHEE generalised criteria were identified as inputs to PROMETHEE . Priorities of objectives were calculated and became input to PROMETHEE. In order to set priorities the JAS software (Judgmental Analysis System) [2] was used to reduce subjectivity and to validate consistency. A group of decision makers were asked to provide their subjective value judgement in a pairwise comparison matrix. The elements of the overall pairwise comparison matrix were obtained by taking the arithmetic average of the corresponding element, and then the vector of priorities for the objectives was calculated.

Conclusion

The application of PROMETHEE has potentially a decisive positive contribution to the process of solving the riparian right conflict in the Jordan river basin, in addition, the geometric representation (GAIA software in PROMETHEE) presents a powerful tool and a valuable help in solving the conflict.

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Multi-Criteria Decision Support System for Water Strategic Planning in Jordan

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Abstract. The Nominal Group Technique as a structured group decision support process and PROMETHEE as a multi-criteria decision aid software are utilised to develop a future water sector strategy , and a system methodology to rank and select water development projects in the presence of conflicting objectives and constraints. The task described is novel in that it integrates the various decision analytical management techniques in order to increase the flexibility and efficiency of the current decision making process.

Key Words. Multi-Criteria Decision Aid, PROMETHEE, strategic planning, SWOT analysis, performance measurement.

1. Introduction

Water resources issues in the Middle East are a major complicating factor in the socio-political problems of the region. In addition, the lack of comprehensivenes and efficiency in managing water resources has become one of the serious problems facing countries in the region. Jordan already suffers one of the lowest levels of water resources per capita, as well as one of the highest rates in population growth in the Middle East. Water scarcity is becoming a significant constraint to development and the need to develop a national water strategy is urgent. There is no comprehensive framework for analysing policies and prioritising options to guide decisions about managing water resources. The ranking and selection of water development projects is dominated by single criterion analysis (usually financial feasibility). Non-optimal selection of projects has resulted in knock-on effects such as waste of resources, environmental degradation, loss of opportunities for external support and inadequate community support for water development projects. In this paper a strategic planning model to guide water resources decisions ; that is suitable for Jordan's needs; is described. Due to space limitations, components and results of the model will be briefly described, and sufficient attention will be given to describe the general methodology adopted.

2. The Model

The components of the model are shown in Figure 1 (steps 1-13). It is a prescriptive requisite analysis to guide the evolution of decision makers

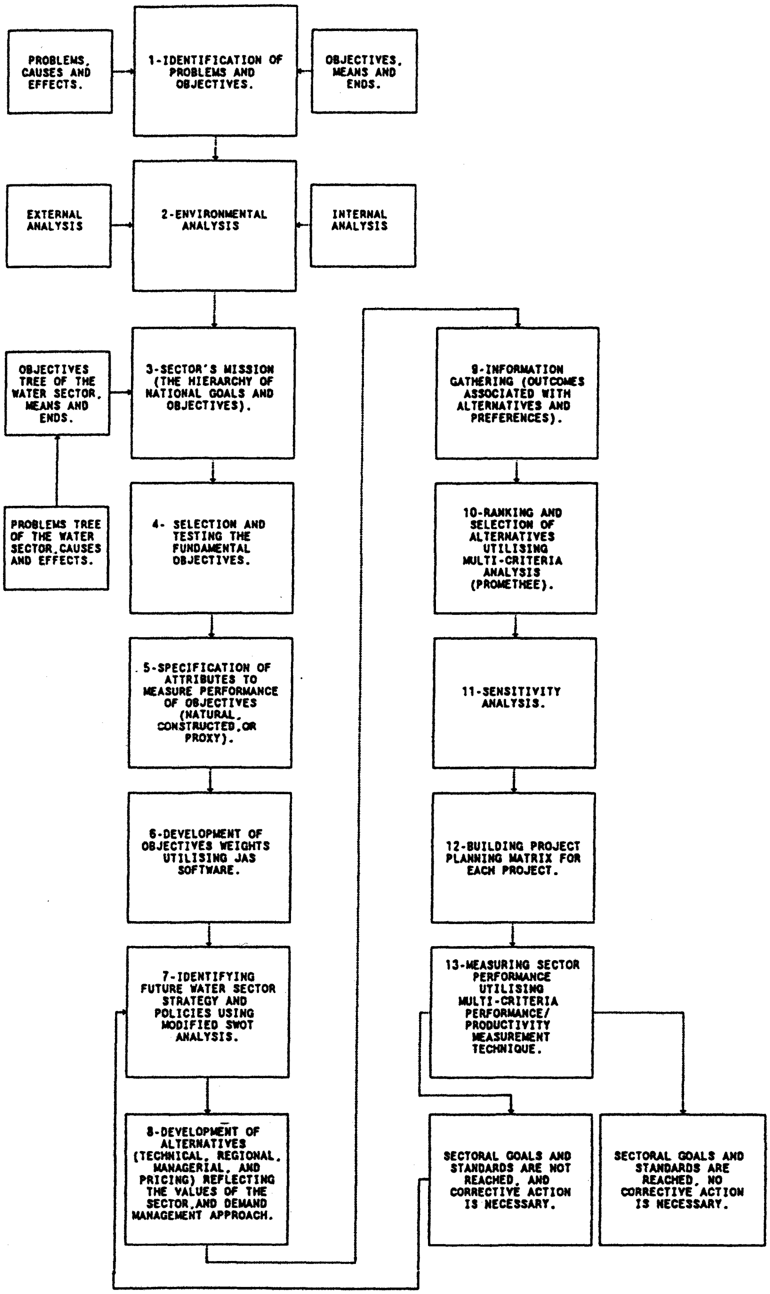

FIG 1. GENERAL OVERVIEW OF THE ADOPTED MODEL FOR PLANNING OF WATER PROJECTS AND RANKING

perceptions in a dynamic, and cyclic process (see for example, [4], [10]). The decision maker's beliefs and preferences are assessed, modelled, and explored to make insights and revision of judgements until no intuitions emerge about the problem. Components of the model are described in the following sections.

2.1 Identification of Problems and Objectives of the Water Sector

This is a necessary step to understand and define the core problem (causes-effects relationship) and objectives (means-ends relationship).

2.2 Environmental Analysis (External and Internal Analysis)

The importance of this step is to develop and assess the water sector profile and structure to reflect it's internal conditions and capabilities (strengths and weaknesses) and external environment (opportunities and threats) utilising Environmental Scanning technique [2]. Information is reviewed at various levels of the sector environment. In addition, environmental forecasting [2] is utilised as a necessary tool to predict the future characteristics (i.e. water demands).

2.2.1 Internal Analysis

A number of government reports, technical articles and relevant documents on the Jordan water sector were examined and reviewed. As a result an internal water sector profile study was carried out. A clear picture of the sector **strengths** and **weaknesses** was obtained.

2.2.2 External Analysis

Due to the complexity of the problem many external factors affect the water situation in Jordan. A study was carried out for the sector. A clear picture of the sector **opportunities** and **threats** was obtained .

2.3 The Sector's Mission, Selection and Testing of the Fundamental Objectives, Specification of Attributes , and Weights

This is a statement about purpose, philosophy and goals to reflect values and priorities of the decision makers. It is a necessary step to ensure unity of purpose, develop basis for allocating resources, serve as a focal point, and to facilitate the translation of objectives into a work structure in such a way that cost, time, and performance parameters can be assessed and controlled.
In order to establish the water sector mission, an organised brainstorming workshop was carried out with the help of a group of decision makers and a facilitator (co-author) in Jordan to identify the **objectives**, the **hierarchy**, the **fundamental objectives, attributes** and **weights.** A **problem tree** describing the

core problem, causes-effects relationship and an **objective tree** describing means-ends relationship for the water sector were constructed. The **fundamental objective set** satisfying certain necessary properties to be used later in the analysis (see, for example, [3], [7], [14]) was **selected** and **tested** against these properties. The following ideas were utilised to complete above tasks:

(i) Nominal group technique (NGT) as a creative process to identify objectives.

(ii) Value focused thinking to establish fundamental objectives.

(iii) Demand management philosophy to overcome the shortcomings of the existing supply augmentation approach.

(iv) Management by objectives to create options with the aim to best achieve the values specified for the decision situation.

The set of the fundamental objectives, their types, relative importance, and attributes were identified. In order to set **priorities** the JAS software (Judgmental Analysis System) was used to reduce subjectivity and to validate consistency to overcome some of the shortcoming's of other methods (like AHP) in defining the meaning of consistency, ratio scales, and the necessary data for analysis [6].

2.4 Identifying Future Strategy and Policies

Development of strategies should be compatible with policy options to achieve the sector mission. In addition, analysis of the different sector strategy options and alternatives should be screened through the criteria of the sector mission to identify the most desirable option and alternative. Two methods are usually utilised for focusing environmental analysis on strategy formulation [9]: critical questions, and **SWOT** analysis (Strengths, **W**eaknesses, **O**pportunities, and **T**hreats). In order to analyse the relative position of each organisation and the relationships among all the organisations' businesses, two methods are mentioned in the literature: the growth share matrix and the multi-factor portfolio matrix [9]. The first matrix shows the linkages between the business growth rate and the relative competitive position of the business (market share). This approach has been criticised as being too simplistic and the growth rate criterion has been considered insufficient for the evaluation of attractiveness [13]. Similarly, the market share for estimating competitive position may be inadequate. The second matrix consists of two sets of variables: business strengths and industry attractiveness. Each variable is divided into low, medium, and high rating, resulting in a nine-cell grid. Both matrices appear to give insufficient attention to the threats and constraints. They lack a conceptual framework for systematic analysis and resource allocation, and are not well defined for practical use.

2.4.1 An Overview of the SWOT Analysis, and its Drawbacks

SWOT analysis is used to aid strategy analysis. Key external opportunities and threats are compared with internal strengths and weaknesses. The objective is identification of one of the four distinct patterns in the match between the sector's

internal and external situations . These patterns are represented by four adjacent cells. Cell one is the most favourable situation, the sector faces several environmental opportunities and has numerous strengths that encourage pursuit of these opportunities. This situation suggests growth oriented strategies to exploit the favourable match. Cell four is the least favourable situation with the sector facing major environmental threats from a position of relative weakness. In cell two a sector with key strengths faces an unfavourable environment with major threats. Accordingly, strategies would use current strengths to build long term opportunities. Finally, a sector in cell three faces impressive opportunities but it is constrained by internal weaknesses, so the adopted strategy for such a sector is to eliminate the internal weaknesses so as to more effectively pursue the opportunities. The four cells can be represented in two perpendicular axes.
Untilnow, **the only modification to the SWOT basic method has been the introduction of the SWOT matrix concept** in order to systematically identify relationships between the different factors and to match them in a systematic fashion [13]. The procedure starts with preparation of a sector profile, internal and external environment, developing of alternatives, making strategic choice, and preparing contingency plans. A framework for identifying relationships is achieved by building a pairwise comparison matrix between the different components to find the best matches between the different components (WO, ST, WT, SO) . **This procedure may become very complex when many factors are being identified and it does not consider weights of these elements. The aim is simply to recognise promising strategies.**
Although SWOT analysis highlights the role of internal and external analysis in identifying sound strategies, it does not explain how strategic planners identify the different factors (internal strengths, weaknesses, opportunities, and threats), the sector mission, or how to assign relative importance for each of the factors. In addition, **an important point which is neglected in SWOT is that some factors could act as strengths and opportunities or weaknesses and threats (the information is fuzzy) and the decision maker is very seldom sure**. There are zones of uncertainty, half belief, or hesitation, so there is a need to help the decision maker to reflect this hesitation and to describe the actual relationship between these factors which reflect the **real shape of the four distinct patterns** described in the original SWOT method.

2.4.2 The Modified SWOT

In order to overcome the above drawbacks the following modifications were implemented :

a. Application of the PROMETHEE method as a Multi-Criteria Decision Aid tool to solve the problem of selecting a sound strategy, to allocate resources, and to rank future investments according to the sector mission .
b. Development of the sector mission utilising value focused thinking , demand management, and nominal group technique.

c. Management by objectives to create options with the aim to best achieve the values specified for the decision situation .
d. Computation of priorities using the JAS software to reduce subjectivity and validate consistency.

2.5 Development of Alternatives

A survey was carried out to identify possible feasible alternatives reflecting the values developed in the brainstorming workshop. The alternatives were classified into different groups: technical, regional, managerial and regulatory. The options were deduced from different available documents (including the new investment plan).

2.6 Information Gathering, Ranking and Selection of Alternatives Utilising the Multi- Criteria Analysis, and Sensitivity Analysis

The necessary information to carry out complete analysis utilising Multi-Criteria analysis was collected. To define the necessary inputs a brief description about the Multi-Criteria Analysis and PROMETHEE is given.

2.6.1 Multi-criteria Analysis, and PROMETHEE method

To aid the decision maker in solving problems and to introduce the value judgements and trade offs, three basic types of multi-criteria techniques and decision-making methods have been developed [5]:
(i) Outranking methods.
(ii) Trade-off methods which utilise utility functions.
(iii) Various interactive methods.
A tendency has been observed that the outranking methods are the most successful because of their adaptability to real problems and the fact that they are more easily comprehended by decision makers [8]. The PROMETHEE method as one of the outranking method is very powerful and user-friendly. Thus it is adopted for use in this study [1] . The PROMETHEE method includes the following steps:
(i) Enrichment of the preference structure by introducing generalised criteria to remove scaling effects.
(ii) Enrichment of the dominance relation by building:
- A multi-criteria preference index to express to which degree an option is preferred to another.
- An associated outranking graph and outranking flow to express how each option relates to the other options (strength and weakness of the option).

(iii) Exploitation for decision aid. PROMETHEE I provides a partial ranking, including possible incomparabilities. PROMETHEE II shows complete ranking of options. PROMETHEE V extends the application of the

PROMETHEE II method to the problem of selection of several options given a set of constraints.

(iv) The GAIA program (Geometrical Analysis for Interactive Aid) to provide a geometrical presentation of results obtained by PROMETHEE . GAIA is based on reducing the multi-dimensional criteria space to a two-dimensional criteria plane to allow direct visual presentation. It gives an effective presentation of results and prediction of (what if?) situations.

2.7 Building the Project Planning Matrix for Each Project

Planning at the strategic level to manage projects is a new and better way to assure successful projects as well as concentrating on the process of implementation which is equally vital. In order to **improve the quality of planning**, different planning steps were followed in a narrower sense. The Objective Oriented Project Planning method (**ZOPP**) was utilised (acronym for the German term **Z**ielorientierte **P**rojekt-**p**lanung (GTZ, Germany) . ZOPP planning was utilised to ensure a consistent train of thought, procedures, and uniform understanding, to facilitate communication and co-operation between all parties involved in any project. The method consists of inter-supportive elements based on the approach, the team work, visualisation, rules of application and project management.

2.8 Measuring the Water Strategy Performance Utilising Multi-Criteria Performance / Productivity Measurement Technique

An important step which needs to be carried out effectively is strategic control through measuring performance. The actual performance is compared to standards, and a corrective action (if necessary) is carried out to ensure that the planned event actually materialises. A computerised constantly monitored system for performance evaluation of the strategy was built, utilising productivity as a performance measure, to monitor the changes in efficiency, effectiveness and quality. The MCP/PMT method [12] was utilised because it:

- Links measurement and improvement to customer needs.
- Shows how effectively we use our resources.
- Provides a database for setting goals and assessing progress.
- Identifies opportunities for performance improvement through quality improvement.
- Provides feedback on how we are doing and what needs attention.
- Enhances motivation, so when quality is measured, people tend to focus their energies to produce superior performance.

3. Results and Discussions

3.1 Identification of Internal and External Environment Factors

The different factors which constitute the strengths, weaknesses, opportunities and threats, and their evaluation on a subjective measurement scale were identified.

3.2 Application of PROMETHEE in Modified SWOT to Select Future Water Strategy

The four objectives in SWOT : maximise strengths, opportunities, minimise weaknesses and threats , their relative importance, and the performance of the different factors were inputs to PROMETHEE.

Figure 2 shows a geometric representation of the result. According to the weights associated with the PROMETHEE, the decision axis π (this is the optimal direction based on multi-criteria analysis) is oriented to cell C3 (confined between the axis C2 (Weakness), and C4 (Threats)). This result represents a situation where a sector faces impressive market opportunities but is constrained by internal weaknesses.

The focus of strategy for such a sector is to eliminate the internal weaknesses so as to more effectively pursue market opportunities. To try to eliminate all weaknesses at the same time is difficult . Some minor weaknesses may be caused by others. The process of eliminating weaknesses should start by identifying the

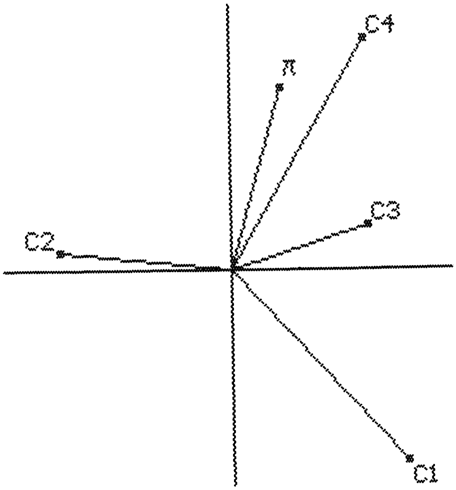

Fig. 2 Geometric Representation of the GAIA Plane in PROMETHEE (Modified SWOT)

major weaknesses. **A problem analysis** (constructing a problem tree which describes causes-effects relationships) and an objectives analysis (constructing an objective tree which describes means-ends relationships) were formulated for the water sector in Jordan . Such analysis would form a basis for identifying proper sectoral policies that will eliminate weaknesses. **In addition, selecting the set of the fundamental objectives (extracted from the objective tree) to reflect the**

future strategy is an important step and a necessary feedback to carry out analysis.

3.3 Application Of PROMETHEE to Rank and Select Projects

Inputs to PROMETHEE include : the generalised criteria, associated weights, options, types of preference functions for each criterion and their defined parameters, evaluations of each option according to objectives selected, the problem type (minimisation or maximisation), and constraints (initial cost, operation and maintenance cost, and regional development) .The output of PROMETHEE includes: partial and complete ranking , general sensitivity analysis, geometrical representation of results and solution of the problem subject to different constraints. A complete analysis and outputs for the above cases are presented in . A complete analysis for the water projects was carried out . Partial and complete ranking was demonstrated and sensitivity analysis was undertaken by changing the weights of the criteria and observing the changes in the ranking of the actions. Complete ranking of projects was presented graphically. The detailed results of the above analysis were presented for the following cases:

Case 1: ranking all water resources actions without introducing constraints.

Case 2: ranking all water resources actions after introducing capital, operational and maintenance costs constraints.

Case 3: ranking each group of actions (technical, regional, managerial, pricing and regulatory) independently before and after introducing capital, operational and maintenance costs constraints.

Case 4: ranking all possible current projects (this is the current situation case where execution some projects is not possible due to different reasons (political and others)) before and after introducing capital, operational and maintenance costs constraints.

Case 5: ranking current technical options based on the current situation (possible projects) and introducing regional development, and cost constraints (it is assumed that the technical options must accrue in the approximate proportion to the population of the four regions in the country while other categories of options mostly are beneficial to all regions).

3.4 Application of ZOPP to Construct a Project planning Matrix

A Project Planning Matrix (PPM) was constructed to develop a description of the project . This includes: an overall goal of the project, objectives, results/outputs, the necessary activities to implement the project, the important assumptions for success of the project, indicators to monitor the success, and verification/sources.
Analysis was carried out to check how relevant the assumptions are, and what risks they entail, incorporating this into the project concept. A check is also carried out so that the project management can guarantee the results/outputs for each

project. Finally specification of quantities and the necessary inputs for each activity were determined.

3.5 Application of MCP/PMT to Measure Performance of the Strategy

Different criteria against which performance measurement is to be evaluated were elicited utilising the Nominal Group Technique method (NGT). These measures represent the set of the fundamental objectives, support the strategy mission, measure the performance, consider interdependencies, and stress short and long term plans. The Productivity **MAP** software (**PMAP**) (a computerised decision support system for implementing MCP/PMT) was used [11]. For each measure a long term goal and a minimum acceptable level having significant meaning were specified, these two values are converted into a 10 point scale. Aggregating measures in an integrating fashion was achieved by developing a multi-attribute value function for all measures after converting all measures into a common scale, and introducing weights to measure the relative importance of each productivity measure. Finally, an integrated value function incorporating criteria weighting to allow the development of one indicator which indicates the overall performance of the strategy was built up. The total weighted performance of the strategy and the actual progress of the relevant measures during the period 1985-1992 were calculated. The total strategy is improving except in year 1992 where some decrease is observed. Monitoring is necessary to make sure that this decrease is temporary and no corrective action is necessary.

4. Conclusions

The PROMETHEE method and the modified SWOT analysis were utilised after establishing the objectives and weights in a systematic procedure to rank water projects that would best satisfy the national objectives. In Jordan, the socio-economic environmental dimension and the new approaches to the water sector strategic planning are extremely important. Without fundamental changes regarding conservation, efficiency, demand management and the adoption of a multi-objective basis in dealing with the existing problems they will escalate.

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An Application of MCDM in Local Water Resources Management [1]

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Abstract. This paper presents a practical application of MCDM in local water resources management problems. A stochastic dynamic programming model with a fuzzy criterion is proposed to monthly reservoir operations. A series of goal programming models are built for water supply and allocation by different planning and operating levels. DP–GP models fulfill the optimal operation tasks for Qinhuangdao water resources management.
Keywords. water resources management, fuzzy criterion, dynamic programming, goal programming

1 Background

Since the 1980s, Qinhuangdao city in North of China has endured serious water shortage. The local government invested greatly in constructing tunnels to divert water from Qinglonghe River to supplement the city's water resources and connect all water plants by pipelines. Figure 1 shows a simplified schematic lay-out of the water resources system, whose management is fulfilled by a Qinhuangdao HydroEngineering Management Information and Decision Support System (QHEMIDSS)(Zhang, et al.,1993). Featured by telemetric, telecommunication, telecontrol and 'teleregulate' capacities, QHEMIDSS consisted of three components: the first is data collection and processing, system monitoring and prewarning; the second is a dynamic graphical simulation of water resources system; and the last but most important is an optimal operation and flood management system (OOAFM).

Through many dialogues with local managers of various levels and on the spot investigations, it was determined that the main concerns of local managers were: (1) water supply and allocation issues of Figure 1's system; (2) Yanghe Reservoir system flood control issues; which are therefore the principal tasks of QHEMIDSS.

2 Model Hierarchy of QHEMIDSS

A set of models was constructed for Qinhuangdao water resources management, and integrated into OOAFM subsystem with forecast → reservoir oper-

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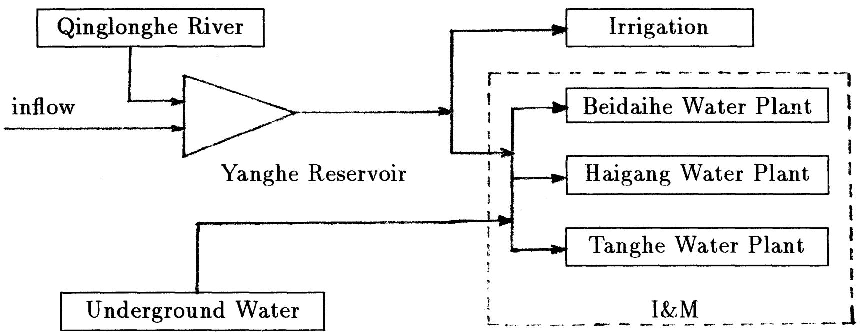

Figure 1. Simplification of Qinhuangdao Water Resources System

ations → water allocation, which effectively fulfilled the two tasks supported by other subsystems (Gu, et al.,1993). Figure 2 is the connection of the models for optimal operation within a 12-month water supply horizon. GPY denotes yearly planning module based on a chance-constrained goal programming model (CCGP), GPMP denotes monthly planning module based on a CCGP model; both GPY and GPMP are for planning while GPMOP denotes monthly operating module based on a linear goal programming model. The coordinating unit involves empirical analysis and learning algorithms to balance the results of GPY and GPMP, and give an initial feasible diverted water policy for DPSI, the module for reservoir operation based an independent stochastic dynamic programming model. The top modules are for planning and the lower ones are for operation. Both managers and operators accept such a model schedule.

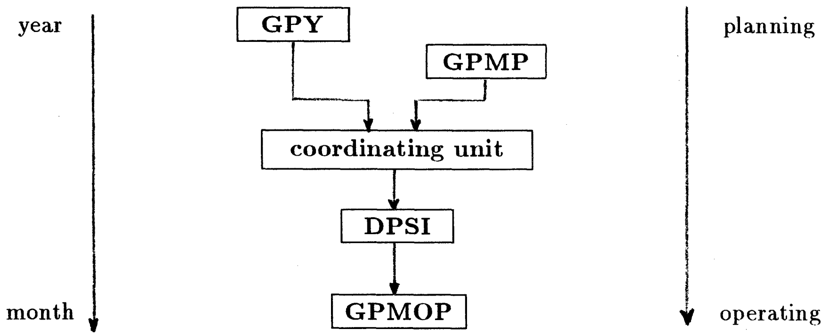

Figure 2. Hierarchy of Modules in OOAFM

3 Reservoir Operation Model

Due to insufficient historic data, administrative intervention and incommensurable objectives about conservancy purposes (industrial and municipal–I&M uses, irrigation, hydroelectric energy generation, etc.) and flood control, it is difficult to apply economic criterion to the calculation of return objective function. Moreover, local managers questioned the usual economic measures about operating returns' calculation. The reservoir operating goals are to keep the reservoir operating as normally as possible, and to manage to meet requirements as close as possible, i.e., if the reservoir storage is within the normal range, neither too small which may lead to drought and nor too large which may lead to flood. A satisfaction degree function, a kind of desirability functions (Gu, 1985) is introduced to depict the decision makers's attitudes towards reservoir storage (Liu and Gu,1993). A fuzzy criterion is proposed to deal with such a conflicting and complicated measures of effect on reservoir operations. Here a SDP model based on fuzzy criterion is applied to Yanghe Reservoir monthly operations.

3.1 Fuzzy Criterion Model and SDP Model

Suppose x is the reservoir level at the end of the nth period. Define $\pi_n(x)$: $R \to [0,1]$ to be the satisfaction degree with the level x. If $\pi_n(x) = 0$, the decision makers feel the most dissatisfied; if $\pi_n(x) = 1$, the decision makers are fully satisfied. Suppose the level x_F inclined to flood, level x_D be a critical level inclined to drought, and x^* is the most satisfactory level, then

$$\pi_n(x) = \begin{cases} \exp\left(-\left(\frac{x^* - x}{x^* - x_D}\right)^a\right), & \text{if } x < x^* \\ 1, & \text{if } x = x^* \\ \exp\left(-\left(\frac{x - x^*}{x_F - x^*}\right)^b\right), & \text{if } x > x^* \end{cases} \tag{1}$$

where a and b are indexes whose values affect the satisfactory function curves around x_D and x_F, respectively (Figure 3).

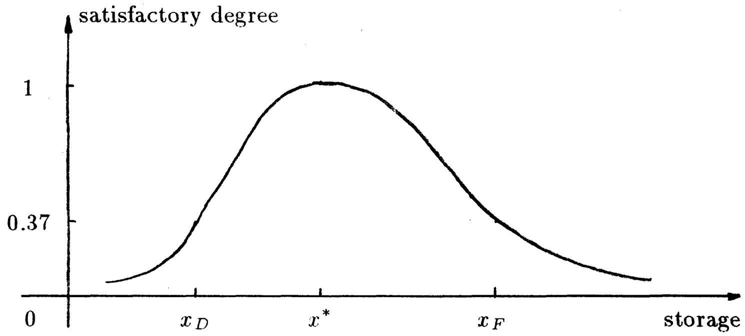

Figure 3. Satisfactory degree function

The recursive equation of stochastic independent dynamic programming with a fuzzy criterion to derive optimum operating policies can be expressed as

$$\begin{cases} f_N(x_N) & = & \sup\limits_{d_N \in D_N} R_N(x_N, d_N) \\ f_n(x_n) & = & \sup\limits_{d_n \in D_n} \{R_n(x_n, d_n) \\ & & +\theta \int\limits_{R^+} f_{n+1}(x_n + \xi_n - d_n - e_n)\phi(\xi_n)d\xi_n\} \\ n & \leq & N-1 \end{cases} \tag{2}$$

subject to

$$x_{n+1} = x_n + \xi_n - d_n - e_n \tag{3}$$

$$R_n(x_n, d_n) = \int_{R+} \pi_n(x_n + \xi_n - d_n - e_n)\phi_n(\xi_n)d\xi_n \tag{4}$$

where

- n: the periods of the optimum horizon, $n = 1, 2, \cdots, N$
- x_n: the initial storage of the nth period
- d_n: the release during nth period
- e_n: the evaporation loss of storage
- θ: the discount, $0 \leq \theta \leq 1$
- ξ_n: the stochastic streamflow during the nth period
- $\phi_n(\cdot)$: the conditional probability density of ξ_n
- D: the release set, $d \in D$ and $Q \geq d \geq q$, Q is the maximum release the spillway could bear, and q is the minimum release to meet the basic demands.

Comments. Eqn.(2) is not the usual stochastic fuzzy dynamic programming model proposed by Bellman and Zadeh (1970). The goal of Eqn.(2) is not to maximize the intersection of two membership values, but to maximize the sum of satisfaction of the whole stages, i.e. the return calculated by Eqn.(2)~(4) is not within $[0, 1]$ but may be greater than 1.0 (or $\in [0, +\infty)$). Strictly speaking, Eqn.(2) then should not be called as a fuzzy criterion model.

The range of release d is decided by the physical characteristic of the reservoir and the users' demands. Follows are the practical constraints or release policies.

$$S_{\min} \leq x_n + \xi_n - d_n - e_n \leq S_{\max} \tag{5}$$

$$\text{For} \quad q \leq x_n + \xi_n - e_n - S_{\min} < Q, \qquad q \leq d_n \leq Q. \tag{6}$$

$$\text{For} \quad x_n + \xi_n - e_n - S_{\min} < q, \quad d_n = x_n + \xi_n - e_n - S_{\min}. \tag{7}$$

$$\text{For} \quad x_n + \xi_n - e_n - S_{\min} \geq Q, \qquad d_n = Q. \tag{8}$$

Eqn.(5) reflects that the level should be no more than the reservoir ultimate capacity $S_{\max}$, and no less than the dead storage $S_{\min}$ at any time. Eqn.(6) ensures that all the requirements can be reached and will not exceed the reservoir physical limit. Eqn.(7) means when insufficient water is available, maximum possible volume is released for users. Eqn.(8) indicates that if the inflow exceeds that required to fill the reservoir and supply the target level, the excess is assumed to spill. Eqn.(2)~(4) are defined as satisfaction criterion model for reservoir operation, denoted as SCM.

3.2 Yanghe Reservoir Monthly Operation – SCM

Assume that the monthly inflows to Yanghe Reservoir fit the Pearson Type III distribution. Within a confidence level of 0.20, those distributions fit well with actual data by Kolmogorov-Smirnov test. And it is proved that the degree of satisfaction is not sensitive to the probability density. Then SCM Eqn.(2)~ (4) are applied to Yanghe Reservoir monthly operation, and implemented into module DPSI.

The parameters in the degree of satisfaction for each month are essential for SCM formulation and reflect the local managers' experience. By continual dialogue with local managers at different levels and much calculation, we identified the possible values of satisfaction degree functions for different types corresponding to different hydrological conditions. Table 1 lists the 12-month satisfaction degree parameters of Type IV, i.e. the inflow frequency to Yanghe Reservoir is 90%, and Qinglonghe's flow frequency is 50%.

Table 1. Parameters for Yanghe Reservoir monthly satisfaction degree functions (Type IV)

	satisfactory storage ($10^6 m^3$)	drought-inclined storage ($10^6 m^3$)	flood-inclined storage ($10^6 m^3$)	drought-inclined index	flood-inclined index
July	80.	50.	120.	2.0	2.0
Aug.	100.	80.	150.	2.0	3.0
Sept.	120.	80.	200.	2.0	4.0
Oct.	130.	85.	210.	2.0	5.0
Nov.	140.	90.	200.	2.0	5.0
Dec.	150.	85.	190.	2.0	6.0
Jan.	148.	80.	180.	2.0	6.0
Feb.	140.	80.	170.	2.0	5.0
Mar.	130.	75.	160.	2.0	5.0
Apr.	120.	70.	150.	2.0	4.0
May	100.	55.	120.	2.0	3.0
June	70.	40.	95.	2.0	2.0

In OOAFM, the module DPSI has two functions:

(1) to decide the actual amount of water diverted from Qinglonghe River;

(2) to decide the actual release each month.

At the beginning of every month, the module should be run once with the actual initial level of that month and revised horizon (decreased by one) for optimization. Thus in the course of an optimization procedure, DPSI will be solved 12 times.

DPSI is performed in 5 steps.

Step I. Beginning. Specify the type of inflows pair. Set $N = 1$. Specify values of optimization horizon length (periods number) and storage state number.

Step II. Input or modification of data and parameters. Give or modify the reservoir monthly data for optimization within the regulation horizon, such as storage loss, satisfactory degree parameters, even the initial storage of the current period, etc. Set all values for optimization.

Step III. Optimization. Solve the SCM formulated with above parameters backward, and get the initial results of the reservoir release and water amount diverted from Qinglonghe River.

Step IV. If release is greater than the actual demand, output the release and diverting water policy, and exit. Else continue.

Step V. If it is possible to increase drawing water from Qinglonghe River, increase water diverted at each month by one unit of flow as it is allowed within the horizon, go to Step III. Else set release equal to the actual demand, output. Stop.

Table 2 is the result with real data at Yinqing Management Bureau used to test module DPSI.

Table 2. Yanghe Reservoir monthly operations

(July, 1993 - June, 1994, Unit: $10^6 m^3$)

month	initial storage	users demands	reservoir release	water diverted	real diverted	tested release
July	40.10	21.200	21.200	19.28	21.01	21.200
Aug.	53.10	24.195	24.195	19.28	21.01	24.195
Sept.	83.20	17.100	17.100	18.66	21.01	17.100
Oct.	106.94	5.820	5.820	19.28	21.01	5.820
Nov.	130.45	4.200	4.200	14.00	15.84	4.200
Dec.	142.87	4.310	4.310	12.05	15.10	13.640
Jan.	137.45	4.495	4.495	0.0	7.24	4.495
Feb.	133.43	4.065	4.065	0.0	0.0	4.065
Mar.	131.49	4.495	4.495	0.0	0.0	4.495
Apr.	130.45	4.350	4.350	0.0	0.0	4.350
May	124.38	22.985	22.985	0.0	0.0	22.985
June	98.48	26.430	41.725	0.0	0.0	41.725

The results fit quite well with reality. Increasing the diverted water may induce an increase in draft from the reservoir. Columns 6 and 7 list the result of the current diversion policy used by local managers with the same parameters. It is seen that the tested release in December in Col.7 is greater than the calculated results in Col.4, because more water was diverted in that month. Currently water diverting is of less cost; the most important for regulation is to provide **sufficient** water for demands. If the level is high at the start of the flood period, spill the excess storage for flood control. The excess discharge can be used for hydroelectric energy generation, which is another kind of benefit for the reservoir managers. So diverting more water within allowed amount is preferred than saving diverting water, i.e. to draw water as much as possible. Then **the current diverting-water policy is still practicable**, which is quite important for DPSI and OOAFM. For the local managers feel satisfied with DPSI capacity, especially with the practicability of the present diverting water policies demonstrated by DPSI; hence increases the utilization of DPSI.

No matter how much increasing the diverted water, the parameters of degree of satisfaction function play the most important role in SCM. Actually, those parameters could be changed in order that the calculated results be applicable to the possible regulating policies.

4 Series GP models for water supply and allocation

As to water allocation problems in Figure 1, series goal programming models including chance-constrained goal programming for yearly planning, deterministic and probabilistic goal programming for monthly planning during water-supplying period, and deterministic goal programming for real-time monthly water allocation operation are applied. Together with SCM for Yanghe Reservoir operations, the models are key to Yanghe Reservoir system regulation and operation. Actual data are applied and the results fit well with reality.

4.1 Formulation of GP and CCGP

4.1.1 Yearly Planning GP Model (GPY)

Yearly planning GP model is a CCGP model. While making one-year water allocation plan, the local managers often consider two uses, I&M and agriculture. Some rules of water resources management should be noted:

(1) avoid using underground water as much as possible;

(2) divert water from Qinglonghe River as much as possible;

(3) provide enough water for I&M and agriculture uses;

(4) if I&M and agriculture uses are in conflict, I&M has a priority;

(5) among I&M users, Beidaihe water plant has a priority in the tourism seasons (June, July and August);

(6) provide water for irrigation in the spring (April, May and June) as much as possible.

(5) and (6) must be considered in monthly GP models.

The streamflows to Yanghe Reservoir and Qinglonghe River are both random variables, while the later is a controllable one, since we could divert only a portion of water from Qinglonghe River. So the diverted amount of water can be estimated according to streamflow forecast of Qinglonghe River and Yanghe catchment runoff forecast. It was also confirmed that under a confidence level of 0.20, the yearly inflows to Yanghe Reservoir fits P-III distribution. Hence the random characteristics of yearly and monthly inflows to Yanghe Reservoir are described by their respective P-III distributions or empirical distributions.

The constraints and the goals considered during model formulation are as follows:

(1) deterministic goal for I&M water demands,

$$a_{11}x_{11} + a_{12}x_{12} + a_{13}x_{13} \geq b_1, \tag{9}$$

(2) deterministic goal for agriculture water demands,

$$a_{21}x_{21} + a_{22}x_{22} \geq b_2, \tag{10}$$

where x_{11} = the water for I&M uses directly from Yanghe Reservoir, x_{12} = the water for I&M uses from Qinglonghe River, x_{13} = underground water for I&M, x_{21} = the water for agriculture uses from Yanghe Reservoir, x_{22} = the water for agriculture from Qinglonghe River, b_1 = the I&M water demand, b_2 = the agriculture demand, a_{ij} = ratio of water except the losses, $i = 1, 2; j = 1, 2, 3$. Here $a_{11} = a_{21} = 1 - 0.12 = 0.88$, $a_{12} = a_{22} = 1 - 0.30 = 0.70$, $a_{13} = 1 - 0.10 = 0.90$.

(3) Probabilistic goal for Yanghe Reservoir streamflow,

$$P\{(x_{11} + x_{21}) \geq b_3\} \geq \alpha, \tag{11}$$

where b_3 is inflow to Yanghe Reservoir under the specified probability level α.

(4) Dependent probabilistic goal for Qinglonghe,

$$x_{12} + x_{22} \leq b_4, \tag{12}$$

$$P\{y_Q \geq B_4\} \geq \beta, \tag{13}$$

where β is the probability level to be specified, B_4 is the streamflow of Qinglonghe River under the probability level of β, b_4 = water diverted from Qinglonghe River under a probability level, and $b_4 < B_4$, y_Q is the random variable for Qinglonghe River streamflow.

(5) Deterministic constraint on underground water,

$$x_{13} \leq b_5 \tag{14}$$

where b_5 = the available underground water.

Now under the probabilistic goals (11)$\sim$(13), the GP model is as follows,

$$\min P_1(d_5^+) + P_2(d_3^+ + d_4^+ + d_1^- + d_2^-) + P_3(d_3^-)$$

s.t.

$$\begin{cases} a_{11}x_{11} + a_{12}x_{12} + a_{13}x_{13} + d_1^- - d_1^+ & = & b_1 \\ a_{21}x_{21} + a_{22}x_{22} + d_2^- - d_2^+ & = & b_2 \\ x_{11} + x_{21} + d_3^- - d_3^+ & = & b_3 \\ x_{12} + x_{22} + d_4^- - d_4^+ & = & b_4 \\ x_{13} + d_5^- - d_5^+ & = & b_5 \end{cases} \tag{15}$$

where

d_i^+=positive deviation from the target of goal i,
d_i^-=negative deviation from the target of goal i,
$i = 1, 2, 3, 4, 5$.

4.1.2 Monthly Planning GP Model (GPMP)

The formulation of the monthly planning GP model is similar to the yearly planning GP except that the data should be monthly, and the I&M is divided into 3 water plants, i.e. Beidaihe, Haigang and Tanghe. So there are 4 users: three water plants and irrigation. The deterministic constraints for water supply increase to 3. And the weights and priorities in the model will change by the above-mentioned respective constraints for water sources or users in each month.

Two ways are provided to obtain the deterministic target data from monthly inflow to Yanghe Reservoir: (i) by the monthly ratio to the yearly amount, (ii) by the independent monthly P-III probability density function for each month. So the monthly planning GP is still a CCGP model.

4.1.3 Monthly Operation GP Model (GPMOP)

The monthly operation GP model is a deterministic linear GP model for real-time operation for water allocation. There are only two sources: release from Yanghe Reservoir which includes the diverted water from Qinglonghe River, and underground water. The users are the same as the monthly planning GP model.

The formulation of the monthly operation GP model for each month is quite similar to that of the monthly planning GP model. Since there is no diversion of water, the constraint for Qinglonghe River is not included.

In order to protect underground water and use it as little as possible, the positive deviation of the underground water is the highest and the original supply amount equals to zero. If there is a shortage, it means that underground water or release from the Yanghe Reservoir may be needed to meet the demands.

4.2 Solution Procedure

After conversion of the probabilistic goals to their respective deterministic equivalents, the solution to the yearly planning GP and the monthly planning GP are transferred to deterministic linear GP model, which could be solved by the reflected P-Space goal programming algorithm (Xuan and Fang, 1987).

There are three modules based on above models: module GPY for yearly planning, GPMP for monthly planning and GPMOP for monthly operation, which use the same reflected P-Space GP algorithm.

Next is only given a real example for yearly water supply for 1993–1994 period (for details, see Tang, 1995). The unit of water is $10^6 m^3$.

Input Table of **GPY**(1993–1994)

Sources: 3				
Users: 2				
users\sources	(Yanghe)	(Qinglonghe)	(Underground Water)	(demand)
(I&M)	1.0	1.0	1.0	57.00
(Agriculture)	1.0	1.0	0.0	88.00
(supply)	0.0	0.0	0.0	
(loss ratio)	0.12	0.30	0.10	
Priorities: 3				
(weights and priorities of the negative deviation variables)				
1.0	1.0	1.0	0.0	0.0
2	2	3	0	0
(weights and priorities of the positive deviation variables)				
0.0	0.0	1.0	1.0	1.0
0	0	2	2	1

Output Table of **GPY**

	Water Allocation Table during 1993–1994(Unit: $10^6 m^3$)					
Sources: 3						
Users: 2						
users	(Yanghe)	(Qingl.)	(Underg.)	(demand)	(allocated)	(deficiency)
(I&M)	0.00	57.00	0.00	57.00	57.00	0.00
(agri.)	34.69	53.31	0.00	88.00	88.00	0.00
(net sup.)	34.69	110.31	0.00		145.00	
(loss rt.)	0.12	0.30	0.10			
(raw sup.)	39.43	157.58	0.00		197.01	
(origin.)	39.43	0.00	0.00		39.43	
(add.)	0.00	157.58	0.00		157.58	

5 Conclusions

It lasted almost four years to undertake the Qinhuangdao project. During 1991–1994, the system underwent the severe drought in the summer of 1993 and flood in the summer of 1994. Through models' verification and validation, and continual practice, DP-GP models were all accepted by users, especially when the senior manager showed more concerns in computerized decision support during the severe drought in 1993, we took the opportunity and succeeded in having the senior manager understand what the degree of satisfaction is and then identifying the critical reservoir levels inclined to drought and flood by his experiences through dialogues on the spot. Comprehensive communication between different levels of local managers and us improved their understanding of models' functions and our understanding of the actual situation, which was the basis of models' successful application.

Here are some comments for GP models. Liu and Gu (1993b) suggested a DCGP model for water allocation. It took into account the real supply to each user under a reliability level at the basis of CCGP model. A DCGP model for yearly planning problem addressed above is as follows,

$$\min P_1(d_5^+) + P_2(d_3^+ + d_4^+ + d_1^- + d_2^-) + P_3(d_3^-)$$

s.t.

$$\begin{cases} a_{11}x_{11} + a_{12}x_{12} + a_{13}x_{13} + d_1^- - d_1^+ & = b_1 \mapsto \gamma_1 \\ a_{21}x_{21} + a_{22}x_{22} + d_2^- - d_2^+ & = b_2 \mapsto \gamma_2 \\ x_{11} + x_{21} + d_3^- - d_3^+ & = c_1 \\ x_{12} + x_{22} + d_4^- - d_4^+ & = c_2 \\ x_{13} + d_5^- - d_5^+ & = c_3 \\ (c_1, c_2, c_3) \sim \Phi \end{cases} \tag{16}$$

where γ_1, γ_2 are the reliability levels of goal 1 and 2 respectively; $\mapsto$: a sign pointer to the reliability level of the current goal; $c_i (i = 1, 2, 3)$ is the maximum supply quantity offered by the i-th source; Φ is the joint cumulative distribution function of (c_1, c_2, c_3); and the others are of same denotations as those above.

If the cumulative distribution function Φ is a single-point distribution, i.e.(c_1, c_2, c_3) is a deterministic vector, Eqn.(16) is a common GP model where all reliability levels γ_i equal to 1. If c_1, c_2 and c_3 are distributed independently, DCGP becomes CCGP.

In Eqn.(16), c_3 denotes the amount of underground water, which is a constant, i.e. b_5 in Eqn.(15). It is difficult to find a joint cumulative distribution function of Yanghe Reservoir inflow and Qinglonghe River flow. Actually the source Qinglonghe River is not regarded as a stochastic variable. So Eqn.(16) changes to Eqn.(15).

There is a question about the meaning of γ_i used in Eqn.(16). γ_i refers to *volumetric reliability*, instead of the usual reliability level in water resources management relevant to the probability of failure (see McMahon and Mein, 1978, Chapter 2). So here γ_i is equivalent to the inflow frequency. For Eqn.(15), it corresponds to α. If γ_i really denotes to reliability levels instead of volumetric reliability, the solution procedure of Eqn.(15) would be different from that in the preceding section. Factually it is a reliability programming model. In the actual cases, the CCGP model is used to deal with water supply and allocation planning issues with one stochastic water source.

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Application of ELECTRE III for the Integrated Management of Municipal Solid Wastes in the Greater Athens Area

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Abstract. The present work demonstrates the applicability of multicriterial aid for decisions (MCDA) in the area of municipal solid waste management in Greece by means of a case study for household wastes in the Greater Athens Area. For the case study area, a concise family of 24 evaluation criteria is proposed. Through these, five selectively composed alternatives for the integrated management of household waste are compared and ranked in a partial pre-order by means of the ELECTRE III MCDA method. First results favour integrated systems emphasizing on separate collection at the source, whereas the need for careful analysis of their sensitivity to inter- and intra-criteria information is demonstrated by a conducted robustness analysis.

Keywords. Integrated Solid Waste Management, ELECTRE methods, Athens.

1 Defining the field of enquiry

National planning for solid waste management is necessary for the development of every long-term, reliable action on this subject. Regional planning is a kind of national planning's 'localized specialization' in the scale of larger agglomerations. Such a National Planning, when compared to West European standards, is still at a premature stage in Greece, where 40% of the country's population of 10 million live in the Greater Athens Area (Fig. 1). Despite this, the Association of Municipalities and Communities of the Attiki Region (ESDKNA), which is responsible by law for the regional planning of solid waste management in the Greater Athens Area, has composed a medium-term program, consisting of the following topics (quoting from Vokas et al. [6]):

- Construction and operation of three facilities for integrated solid waste management around the Athens Basin, within which sanitary landfills and material recovery facilities will be in operation, aiming at (i) the reduction of the landfilled waste volume, and (ii) the limitation of this volume's risk potential, by stabilizing its compostable ingredients.

- Construction and operation of six transfer stations, with possible use as drop-off centres.
- Gradual promotion of separate collection at the source.
- Development and full implementation of automatized waste collection, by the modernization of the municipalities' and communities' cleaning services under the consulting of ESDKNA.
- Immediate extinction of uncontrolled landfills in Attiki, by shutting them down and rapidly restoring them; waste that would have been previously disposed in them, will then be sent to the new integrated facilities by means of local transfer stations.
- Temporary operation of local transfer stations in extremely remote municipalities and communities, until the completion of the central network.
- Experimental implementation of the combined system for automatized collection and transfer in the municipality of Chalandri, aiming to further develop it into a transfer station for the north suburbs.
- Final restoration of the shut-down landfill of Schistos and the still operating but rapidly filled-up landfill of Ano Liossia.
- Modernization of the management system for infectious hospital wastes with two parallel goals:
  - Modernization of internal collection, temporary storage and transport.
  - Construction of a second incinerator for hospital wastes.
- Management of construction and demolition wastes with the priority of restoring closed quarries.

It should be noted that, although regional management has distinct advantages, the communities called for participation are usually reluctant to have wastes of another part of the region disposed in their area and are, therefore, unwilling to host waste-processing facilities. Main aim of ESDKNA's medium-term program is the drastic reduction of Attiki's waste quantity being landfilled and its risk potential, a goal that calls for measures and decisions which have to be taken, however, in the framework of a national planning, introduced by the national government; ESDKNA has published an integrated proposal for the latter since 1989.

Figure 1: Areas on Solid Waste Management in the Greater Athens Area.

❶ Operating 'wild' landfills (23; the Vari, Pallini and Spata 'wild' landfills have been shut down recently but still present strong leachate problems, due to lack of restoration works);

❷ Large, currently operating landfill in Ano Liossia;

❸ Schistos shut-down landfill;

❹ Proposed sites for 2 of the 3 new landfills (Avlona and Trikerato-Mandra);

❺ Operating Transfer Stations (7: Schistos by the shut-down landfill, N. Smirni, N. Makri, local transfer stations in Ilioupoli, Argiroupoli, Kallithea, Chalandri);

❻ Proposed site for incineration plant (Phili);

❼ Communities involved since Feb. 1994 in pilot-programme for paper recycling;

❽ Communities having participated in 1986 pilot-programme for paper recycling;

❾ Communities collecting fees from ESDKNA, being close to Ano Liossia landfill and Schistos transfer station;

❿ Communities having proposed to form an inter-municipal corporation.

2 Objectives of the research

Scope and purpose of the research report is:

- to provide a concise overview and thorough analysis of the various alternatives by means of which the integrated management of household municipal solid wastes can be efficiently modelled, taking into account the strong multicriterial character of the issue. Discussion on the issues of modelling is conducted in the framework of a multidisciplinary approach, considering the needs of the analysis and various posed limitations.
- to compose a concise family of criteria to evaluate integrated systems for the management of household municipal wastes; describe the methodology to be applied for the quantification of the performances for various alternatives on them.
- to demonstrate (in the context of a case study) the applicability in Greece of multicriterial aid for decisions in the field of integrated regional management of household municipal solid wastes.
- to perform a ranking of *a priori* selected alternatives for the integrated management of household municipal solid wastes in the Greater Athens Area, as an application of a selected method of multicriterial aid for decisions, by means of a concise family of evaluation criteria and under boundary conditions and fixed variables determined by the case study area.

Thus, the ranking of five integrated systems for the Municipal Solid Waste Management is being described in this report by the use of the ELECTRE III method for multicriterial aid for decisions [4]. The evaluation of each alternative or potential action is conducted by means of 24 criteria, composed by literature sources ([1], [2], [5]) into a family, that can be considered concise and non-redundant, at least for the particular case study.

3 Employed methodology

Solid waste management is especially difficult and costly today, due to the increasing volumes of produced waste and the need to control (what are now recognised as) potential serious environmental and health effects of disposal. For the integrated management of household municipal solid wastes in the Greater Athens Area, the following alternatives have been considered [3]:

1) <u>'Sanitary' landfilling in the landfill of Ano Liossia</u> (present case); this potential action is fictive, in the sense that it is *a priori* excluded for the future, since the site's shut-down and restoration has already been scheduled, together with the restoration of the already shut-down landfill in Schistos (required investments of 15 and 11 million ECU respectively). However, a comparison with other potential actions (particularly in view of the expected increase of the disposal

costs for new landfills) was considered interesting and, therefore, this 'potential' action has been included in the present analysis.

2) Sanitary landfilling in three new landfills (two proposed, in Avlona and Trikerato-Mandra, together with a third one in the Mesogea-Lavrio area), according to ESDKNA plans that also received lately some political support, which may decisively facilitate their short-term implementation. ESDKNA has been supervising environmental impact studies for these and other sites; these studies assigned the highest ranking to Ritsona, but it was discharged in the process due to 'local' problems. Other candidates, like Varnavas or Laka-Maskari were found problematic by these studies (the former for landslide problems and the latter because of a high underground water level, unsuitable geology and as the area was found to be a deer crossing). Another area also considered (Milies-Grammatiko) received the same ranking as Avlona, but was discharged due to 'local' problems quite like Ritsona. Avlona gave some mild 'local' problems, whereas Trikerato-Mandra showed neither technical nor 'local' problems; the only raised issue referred to the identity of the site's manager, i.e. the municipality of Elefsina asked for it but met the opposition of both ESDKNA and the Ministry of the Environment. All three landfills will probably be constructed with composite-liner and landfill-gas utilization systems; the overall project requires an investment of about 84 million ECU, according to ESDKNA calculations.

3) Sanitary landfilling in three new landfills (cf. 2), a material recovery facility in each one and separate collection of paper in all municipalities and communities, according to medium-term plans of ESDKNA. The investment for landfills and material recovery facilities has been calculated to be in the order of 150 million ECU. The material recovery facilities might utilize the over-3-year experience gained by the pilot facility in Ano Liossia. Alternatively, if the co-composting of sewage sludge (300 t/d in winter and 170 t/d in summer) from the waste-water treatment plant in Psitalia is also considered, then some help might be found from gained experience by a similar plant, currently under construction in Peloponnese, which is expected to be delivered within 1994.

4) Sanitary landfilling in three new landfills (cf. 2) and separate collection of paper, glass, aluminium and fermentables. Extension of separate collection to other recoverable waste materials beyond paper is an ESDKNA vision, whereas it has already been proposed, particularly for the organic fraction, by some Municipalities of the area. This scenario could be considered competitive to (3), in the sense that it does not require sophisticated material recovery facilities. However, the separate collection of fermentables (compostables, putrescibles, fines) still requires the existence of a treatment facility, since the unavoidable presence of non-compostable material in the collected fraction should be taken for granted and, in any case, compostable materials require space and time for their maturing and stabilization. Separate collection may,

however, highly increase the efficiency of the process and the quality of the final product; therefore it is highly recommended today.

5) Incineration in one facility (100% of the area's remaining solid waste) and separate collection of paper, glass and aluminium. Investment costs for the incinerator are considered to be in the order of 470-670 million ECU, whereas the municipality of Phili has already proposed to let such a plant be constructed in its grounds. ESDKNA, together with ecological movements like Greenpeace, have openly expressed their emphatic opposition against incinerators in Attiki, due to concerns about their local technical reliability (mostly related to fears of possible PCDD/PCDF emissions) and the already particularly high air pollution and smog problems of the area (Phili is in Thriasio Plain, a major industrial area in Attiki). However, since the multicriterial software applied here specifically allows for such eventual 'strict' requirements (in the form of veto thresholds and/or coefficients of importance), it was considered useful to include this potential action in the analysis too, particularly since it is the only one including thermal treatment of solid waste. Separate collection of paper was preserved in the scenario, because it is already happening and expanding. Incidentally, such programs should not be abandoned after their initiation, particularly on purpose like here, because of the (almost irreversible) bad impression to the population. Glass and aluminium are included in this scenario, too, in view of the fact that their absence is profitable for the incinerator (heating value increase of the remaining waste, decrease of heavy metal content in both the flue gas and solid residues), since they are non-combustibles.

The family of criteria that was applied here has been a result of synthesis among a number of concise candidates proposed by Skordilis [5], Caruso et al. [1] and Hokkanen & Salminen [2]; it is consisting of the following 24 family members (their hierarchy is presented in Fig. 2):

1.1 Increase of the degree of implementation of the environmental legislation. This relates to national planning and policy, as prescribed by the authorised ministry (currently facilitating construction of new sanitary landfills, shutting-down of the existing 'wild' landfills and separate collection of recoverable waste materials), also considering that some European directives on solid waste management have not been adopted into national legislation (e.g., incineration directives).

1.2. Better use and application of environmental legislation; this refers to the applicability of the legislation (effectiveness and efficiency of the state's or the local government's controlling mechanisms).

1.3. Reduction of unemployment; it is important here for a special reason: According to the present situation, the municipal cleaning services are generally over-populated, but operate rather inefficiently. A reorganization of the solid-waste collection system may cause a reaction from their union, if they feel that their interests are threatened.

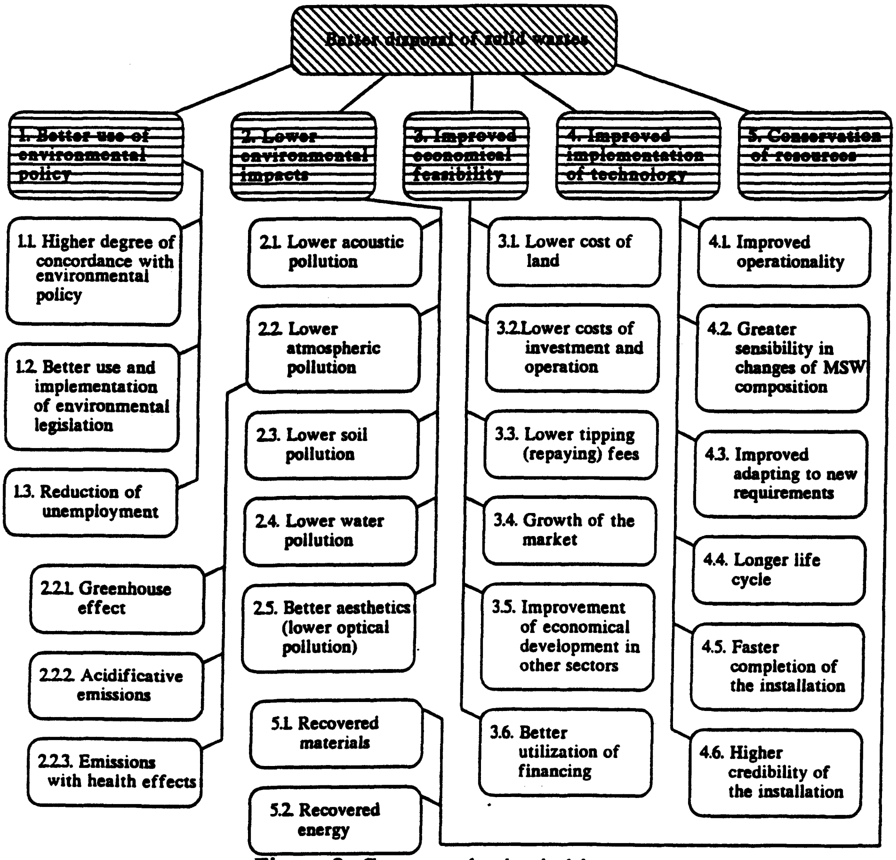

Figure 2: Composed criteria hierarchy.

2.1. Lower noise pollution; main relevant noise sources are (i) the waste treatment/disposal plant itself and (ii) traffic associated with it.

2.2. Lower air pollution; it is particularly important (among environmental criteria) in Athens, because of the already high air pollution levels. This criterion was quantified in terms of emissions (since, measuring actual impacts is an unclear and premature field yet); due to its above explained importance and nature, it was further decomposed into:

2.2.1. Emissions related with the greenhouse effect (CO_2, CH_4).
2.2.2. Acidificative emissions (SO_2, NO_x).
2.2.3. Emissions with health effects (Cd, Pb, PCDD/PCDF).

2.3. Lower soil pollution; this relates to the various kinds of depositions of air and water pollutants at the ground.

2.4. Lower water pollution, both for underground and surface water bodies. Greater Athens Area has been facing for many years problems concerning drink-water supply, particularly in the summer. Although these problems are expected to be solved in the near future by securing the supply of the Athens Basin with water from rather remote water bodies (e.g., Evinos), this criterion remains important due to the multi-lateral uses of water.

2.5. Lower optical pollution; this relates to aesthetics, i.e. sight of wastes, gulls, fleet of trucks, etc.

3.1. Lower cost of land; it depends on land ownership (public, municipal or private).

3.2. Lower investment/operating cost. It was decided to exclude investment costs from the criteria, since it is highly probable that any treatment/disposal plants built in Attiki will be financed (to the major part, at least) by the European Communities. Therefore, the operating cost does not include depreciation, although it can be easily introduced in the present simulation-evaluation model (approx. figures are already available).

3.3. Lower repaying (tipping) fees; they are secured by the citizens' taxation on waste and are a direct function of running costs but also depend on political decisions, therefore being non-redundant due to the operating-cost criterion.

3.4. Market reinforcement for recycled materials (also subject to the supply-and-demand law).

3.5. Better financial development in other sectors (e.g., heating of greenhouses with landfill gas).

3.6. Better utilisation of financing; substantial funds are expected to be available for Greece in the years 1993-99 from the European Communities. Previously available funds were not utilized to the maximum possible extent, mostly due to unavailability of an adequate number and size of reliable proposals for financing.

4.1. Better operationality; it was chosen to be measured in an ordinal scale. It describes, more or less, the prospects of a certain method to work effectively, considering all existing conditions in the place of application.

4.2. Better sensitivity to a change in waste composition; e.g., the compostable fraction of municipal solid waste is decreasing continuously in Greece in the last 10 years (specifically for the Greater Athens Area by about 8% according to ESDKNA's calculations), as result of the increasing consumption of packaged goods.

4.3. Better adaptability to new demands; these refer mostly to legislation of the European Communities that is converted into national Greek laws; e.g., if the long-discussed guideline for the recycling of packaging material is completed and made operational in Greece, this would certainly pose some considerably higher recycling requirements than those that are achieved or achievable today.

4.4. Longer operational life; finding a site for new treatment/disposal plants has become almost an unsolvable problem due to the NIMBY (Not In My Back Yard) syndrom; therefore, a maximum lifetime of facilities that are either operating or under construction, should be pursued.

4.5. Quick completion of installation; most politicians that start a project would like to 'be still around to collect' when it is finished and made operational.

4.6. Better reliability of installation; this refers to previous local experience with relevant technology.

5.1. Quantity of recovered materials. This does not refer to the profit from selling recovered material, but to the profit for the next generations, due to the preservation of scarce resources like oil, metals and land (ethical criterion).

5.2. Quantity of recovered energy. The logic behind this criterion remains the same as in the last one.

4 Conclusions

A modelling approach of solid waste management at regional level may be attempted equally efficiently by treating it in two principal ways, i.e. either as a continuous or as a discrete one, depending mostly on the origin and background of the research group (mathematical programming or decision analysis); however, the combination of both modelling ways as well as their individual use in parallel should by no means be excluded.

In the former case (mathematical programming), the analyst may have more mathematical advantages, being able to keep a relative distance from the problem and remaining flexible to switch between objectives (favoured tendencies) and goals (pursued, concrete results); this is achievable by relaxing the objectives, therefore turning them into goals and reducing the calculations necessary in the optimisation model. In the second case (decision analysis), the analyst must get rather more close to the problem in a sense, because he has to compose alternatives from the beginning (in order to perform comparisons) and not simply 'wait' for the results of his optimisation model to show him the 'optimum' combination (i.e. a particular integrated system for solid waste management).

The potential action ranking highest in the case of both applied methods, is No (4). An interesting feature, despite the qualitative determination of most criteria, is that No (4) seems to dominate Nr (3), which suggests that a system emphasizing on separate collection might be more advantageous than one relying on 'convenient' material recovery facilities. However, such comparative conclusions of technical nature may be elicited only after a more thorough determination of all parameters than the one conducted here mainly for reasons of demonstration. The multicriterial analyst can play the role of mediator very effectively in issues concerning solid waste management, because his equipment allows him taking into account a variety of interests and points of view for his consulting services.

Concerning further research, a first priority following from the present work is the more accurate determination of criteria scores, without excluding the possible alteration of the proposed family of criteria (composing and synthesizing a concise family of criteria alone is a by no means negligible task; quite on the contrary). Once the criteria performances have been determined (or, better, approximated) more accurately further sensitivity analyses should be conducted to determine the robustness and stability of the results. Further aspirations on the analysis surely include considering more alternatives in the discrete case and experimenting with the continuous approach of modelling, either in parallel or individually. At present, there are numerous available algorithms in the field of decision analysis (i.e. discrete case) that could also be applied beyond ELECTRE; this would be also an opportunity to compare the algorithms through the results as well as the results themselves.

Another task of interest would be to try and determine the effect of integrated systems in the scores of a participating functional element on some criteria, by comparing them with the scores of the same functional element when considered alone. E.g., what is the change on the market price of compost if a programme for separate collection of batteries and lamps is implemented, thereby reducing the heavy-metal content of waste sent to material recovery facilities? Issues like these are strongly related to consequence (what-if) analysis and it would be very helpful to be able to answer them in an efficient way, e.g., by suitable software also combining or allowing for MCDA. It is important to always keep in mind that, already in the face of data collection and interface with the decision maker, many important aspects of the problem are being cleared out. Interactive software, simply asking the right questions as input data, would greatly facilitate getting a clear view of the entire system with all its parameters from the very beginning of the planning process.

An additional final point, which gains continuously in popularity and operationality, is the use of geographic information systems in solid waste management, particularly when tackling location-allocation problems, but also on issues of relatively local and spatially limited nature, e.g., the evolvement and spatial expansion of a certain landfill during its operation.

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Applications

3.3 Applications in Management

A MULTIDIMENSIONAL FRAMEWORK FOR STRATEGIC DECISIONS

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Abstract. Literature offers a wide variety of approaches for solving strategic decisions in the firm. Within firms one can often observe a clear gap between the strategic and financial evaluations of major decisions. In this paper we describe how different types of evaluations are being made, describe the gap between the approaches and give some reason for its existence. Since we are convinced that a more integrated approach to solving decisions is desirable, we present a synthetic framework.
Given the nature of the decision problems involved, the framework allows for both quantitative and qualitative information. In addition, the framework is quite naturally one which encompasses a multiplicity of goals, constraints and viewpoints. In this way the multicriteria decision methodology offers a tool for learning and communication, bringing together disciplines and approaches which would otherwise be hardly capable of communicating. The framework is illustrated by a real-life example of a new product decision in a Finnish drug company.

Keywords. Strategic decisions, critical success factors, performance evaluation

1. Introduction

The focus of strategic management (SM)literature and its practical implementation is clearly changing [7], [14], [15]. Earlier approaches to SM concentrated on the use (or lack of use) of analytically rational decision processes and pictured the task of intelligent management as that of failure to act rationally [11], [12]. Strongly associated with the concept of intelligent management is that of rational choice, i.e. the specification of well-defined objectives and pursuit of those objectives by means of gathering information to assess alternatives in terms of their possible future consequences and choosing actions expected to fulfill objectives (in finance we had and have the expected utility approach towards this type of decisions). In other words, normative decision models have been successfully applied to **well-structured and well-specified decision situations** [21], but have proven ineffective for strategic decision-making processes, because strategic decisions are far from well-structured and the decision situations are far from well specified [8], [2], [3], [4], [5], [6].

The key question in a firm's strategic decision is: *how can future performance of an activity be improved*, i.e. how to estimate the effects on performance when changes in an activity either through decisions taken by the company or as a result of third party actions? What still makes things more complicated is that even if a decision is taken over a specific action it is highly uncertain what ex post performance will actually result from that activity. In this paper we adopt the view that part of the uncertainty is caused by a number of external forces which cannot be influenced by a decision maker, and that many of the key elements central to strategic decision-making are implicit. It is therefore difficult for the decision maker to explicitly describe them and at times even to be aware of their existence. It is therefore not so strange that these elements are left out of decision-making models. To illustrate this problem we will describe a case from the real-life business world, where the task was to develop a decision support system (DSS) for a Finnish drug company to support their strategic management process. The conceptual framework was based on Porter's competitive framework [16], [17]. We defined strategic management as **the process of creating a sustainable competitive advantage** (SCA). The source of a firm's SCA was it ability to manage its critical success factors (CSF). Hence, our case company, called C-3, **had to manage their CSFs better than their competitors** [10]. There were six CSFs and eight competitors. Hence, The CSFs were the criteria by which an SCA was determined.

This may seem like an easy task, however, only one of the CSFs was quantitative, the other five were qualitative and as we will describe in section two this was a source for considerable difficulty.

In this paper we present a framework for strategic decisions which consists of two main parts: One is concerned with a performance evaluation which results in the identification of firm characteristics - both quantitative and qualitative - that affect the firm's performance, taking account of the influence of external sources of risk that are beyond the control of management. The second part of the framework is directed towards the strategic management of the firm characteristics, again taking account of uncertainties. We are in this paper using firm characteristics and CSFs as synonymes. This framework still only on the conceptual level and the real-life case described here illustrates the necessity for dealing with this issue. In future work the authors intend to test the framework in order to aquire a proper validation of the framework.

The performance evaluation is based on an approach by Spronk and Vermeulen [19], in which in the first step the firm's perfomance is partly explained by a series of firm characterisitics. The unexplained part is then explained as the impact of external risk factors beyond the control of the firm. This is done by measuring the firm's sensitivity for unexpected changes in the risk factors. The firm's sensitivities can themselves be viewed as firm characteristics, but in the approach they are related to the firm charcteristics that were mentioned in the first place. This approach will be described in some more detail in section 3. This framework was originally developed to support decisions in finance and this paper explores the possibility of using the same approach in a strategic management context. Hence, the approach that has thusfar included relatively 'hard' and measurable firm characteristics only, will now be extended as to include qualitative charactersitics.

The performance evaluation yields insight not only into the relative strengths and weaknesses of the firm but also into the determinants of these strengths and weaknesses: being the firm's sensitivities for external risk factors. In this way, strategic management obtains a series of instruments through which the firm's performance can be manipulated. This issue will also be discussed in more detail in section 3.

2. Managing a successful drug release: feedback from reality

In the early nineties, a Finnish drug company, here called C-3, developed within three years from a minor player to the leading company in the antidepressant drug market. The study was done during the years 1990 thru 1993. In 1990 C-3 had less than 5% of the market share, in 1993 they held nearly 40% of the antidepressant market. A key to this rapid change was their release of a new antidepressant that held superior qualities compared to the competitors' products. But this was not the only explanation. The new drug possessed qualities fundamentally different from the existing ones, in terms of efficacy and tolerance, that it altered the whole market. Furthermore, the company had superior knowledge in marketing. This conclusion comes from the fact that they lacked funding to support the relaese through heavy advertising campaigns, instead they had to use other marketing and sales promotion methods such as personal selling. The use of sales personnel can be even more successful, but it requires entirely different skills than advertising. Sales forces create a push effect whereas, advertising create pull effects, thus they function from a different perspective, yet their objective is the same - to promote a product. Common to both is that the result of the actions taken may not show immediately (advertising is likely to show sooner) in terms of revenue, but will show immediately in terms of cost. When there is an increase in performance it may be difficult to determine which action actually caused the effect on performance.

Although the drug was successful, and still is (today they have over 50% of the market), we all know that there will be a time of saturation (cf. product life cycle) on the market. What is more threatning for a drug company and likely to set in before market saturation is the expiring of patents, both will have a declining impact on sales. As a consequence, the company faces decisions over whether to invest more in marketing efforts or to harvest and focus its resources on the next success drug coming in the pipe-line.

In the discussions with the chief executive officer (CEO) of C-3 it was determined that in the pursuit of a sustainable competitive advantage (SCA), which was the objective of the strategic management process, C-3 had to manage certain critical success factors (CSFs) better than their competitors. These CSFs were determined as **product quality, good reputation, R&D, wide product range, new products, marketing resources, and marketing knowledge.** We soon found that with the exception of marketing resources all others were descriptions of some competensies, that nobody seemed to have a clear idea on how to quantify them but it was considered necessary and good to quantify them. Marketing resources refered directly to the amount of money available for marketing, not number of marketing personnel or the skill and competence level of the personnel. Hence, in the strategic management process, the decisions involved taking into account multiple criteria, in terms of CSFs for our firm, but at the same time to consider the performance of the competitors vis à vis these CSFs. Once this was agreed on it quite soon became clear that some CSFs were more important than others, i.e. they would have a greater impact on the performance, and naturally, that some of the eight competitors were stronger than the others. Therefore, our first task was to aquire a ranking of the CSFs and then a ranking of how the CEO's perception of how the competitors managed their CSFs compared with C-3.

At first the CEO was asked to rank the CSFs according to their importance and then he was to rank C-3s level of performance with referens to the most important competitors which were eight, using a semantic scale known as a competitor strenght grid (CSG) [1], [10]. This did not prove successful as basis for quantifying simply because the CSG was too vague. After allowing the manager to do the same ranking using Expert Choice, based on the AHP [18], we finally acquired a good ranking. Now we had weights for the CSFs and for the competitors. However, reasoning in terms of weights was new to the CEO. After further discussions we came up with a different suggestion for the CEO.

We suggested that each CSF had some maximum and minimum value on sales and the CEO could well estimate these values for a specific product. Thus, each CSF when added together gave the total market share value of this drug. The CEO could also estimate these values for the competitors' products and he could even estimate what impact any action by a competitor would have on his own products. And he could do this several years into the future. The negative value effects were estimated according to a worst-case scenario. The AHP ranking was consisitent with the maximum value estimates (the CSF with the highest weight also had the highest possible impact in sales).

In our work with C-3 the through out the project the objective had been to develop a DSS that would enable the user to model both quantitative and qualitative elements. With the system, MOCK (Many Options on Complex Knowledge), the CEO could conveniently monitor the impact of each CSF on the performance, and he would know the value of the CSF. In Fig. 2 one display of the system is shown. Here the CEO can choose a specific year (1994), whereas the estimated sales for that year (37500) is displayed. He can then choose a CSF (provided that something has taken place on the market) and indicate this change (either negative or positive) by scrolling the positive or negative bars accordingly. When he has made the necessary changes the changes will **update** the market share value (sales) in a different display with the financial data. hence, the CEO can immediately assess the impact of changes in CSFs and is thus better prepared to react to changes in the overall business environment.

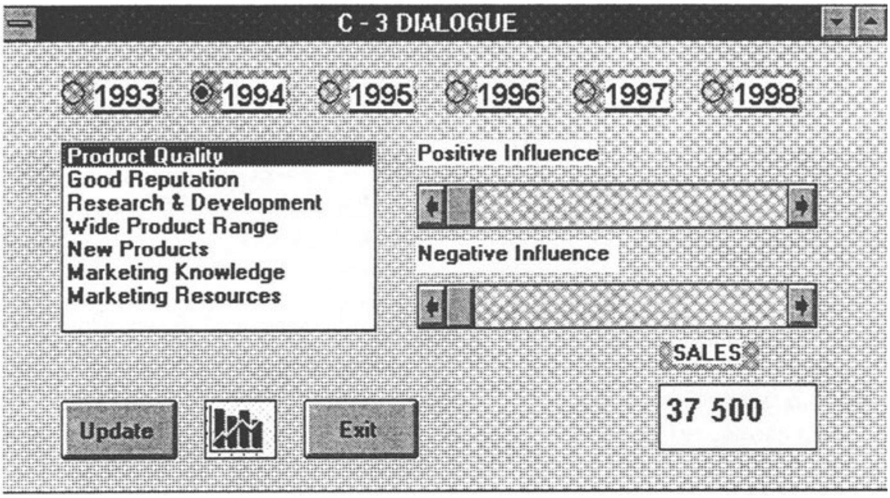

Fig. 2: Quantifying CSFs

3 The framework

In this section we will first describe the performance evaluation (PE) part of our framework for strategic decisions. Immediately after that, the second part of the framework, directed towards the strategic management (SM) of the firm characteristics, will be described. The purpose of the part of the framework dealing with PE is to identify firm characterisitics - both quantitative and qualitative - that affect the firm's performance, taking account of the influence of external sources of risk that are beyond the control of management.

The PE is based on an approach by Spronk and Vermeulen [19] in which in the first step the firm's performance (assume for the moment that we can define a single performance measure P_t, being the performance over the period (t-1,t)) is partly explained by a series of N firm characterisitics ($C_{1,t-1,\ldots,}C_{N,t-1}$) at time t-1[1]. The unexplained part of performance is then explained as the impact of M external risk factors ($f_1,\ldots,f_M$) beyond the control of the firm. This is done by measuring the firm's sensitivity ($b_1,\ldots,b_M$) for unexpected changes in these risk factors. The firm's sensitivities can themselves be viewed as firm characteristics, but in the approach they are related to the firm characteristics that were mentioned in the first place [22]. Thus, $b_m = b_m(C_{1,t-1,\ldots,}C_{N,t-1})$ or in other words, the characteristics of the firm

[1] For simplicity, we assume the firm characteristics to be given and fixed during the period (t-1,t). As shown in Spronk and Vermeulen [19] more realistic assumptions can be handled in straightforward manner.

do determine how the firm react on its environment. Two questions arise almost automatically: One, which are the firm characteristics contributing to the firm's performance and secondly, what are the external factors influencing this performance? Especially with respect to the first question, many answers - both theoretical and empirical - can be found in the literature. Yet, there is by no means a single and clearcut answer. On the contrary, many answers are contradictory and even in case an answer seems to be true at one time it may be rejected again within one year. Part of an answer to this is the fact that today's business world is increasingly fast changing, e.g. because of technological developments. The firm characteristics that contributed (or seemed to contribute) to firm performance may fail to do so tomorrow.

The second question is hardly dealt with in the literature, because it involves many qualitative factors that are difficult to model. We believe that by investigating and answering the second question (even if only partly), the answer to the first question also comes nearer[2]. One argument is that characteristics which help to explain the firm's reaction to unexpected changes in the environment and thus indirectly explain unexpected changes in the firm's performance, are also good candidates for explaining the expected value of the firm's performance.

The drug industry can provide us with a an example of the role of unexpected changes in the environment and how these can effect performance. Let us recall Fig. 2 with the CSFs, where one of the CSFs is **good reputation**. When the CEO for C-3 estimated the maximum positive and negative impact on company performance (which here was sales) the impacts were FIM 5 million positive and FIM 30 negative. This was explained by a worst case scenario. Suppose the company was involved in an environmental accident where the company for example polluted a river. This would have effects on company reputation and would affect sales of all drugs, not just one specific. Yet, in case the company would introduce and announce some environment-friendly production methods, the effects were estimated to be not any higher than FIM 5 million (good news having less effect on market value than bad news). Nevertheless these estimates have not been verified, because 'good reputation' is tacit and indeed difficult to grasp - yet strategically very important and some estimation is needed. Another example - not included in MOCK - is the impact of government regulations that can have a huge impact on profits, and where the short-term and long-term effects may be quite different from one another. Here we must stress that government regulations normally also affects the competitors.

Thus, the empirical questions to be answered are the following:

- What are the external risk factors influencing the firm's performance?
- How sensitive is the firm for each of these risk factors?
- How do these sensitivities depend on the firm's characteristics?

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[2] Vermeulen, [22], tries to explain part of the change in performance by changes of the firm characteristics by the firm during time t-1 and t.

Some factor candidates are not hard to be identified. In fact, inspection of the cash flow statement of a firm does already yield a number of factors. For instance, energy prices and wage rates will influence the cash outflows of any airline. In these examples, it may even be relatively easy to measure the sensitivities to unexpected factor changes. Most airlines know exactly what a one percent change in the oil price would mean for the size of its cash flow. In other cases, in which it may be more complicated to measure the sensitivities, the approach by Vermeulen, [22], and Spronk and Vermeulen, [19],. may be useful. First, the sensitivities b_m are assumed to be linear functions of the firm's characteristics:

$$b_m = b_m(C_{1,t-1,\ldots,}C_{N,t-1}) = \sum_{n=1}^{N} b_{mn}C_{n,t-1}, \text{ for m} = 1,\ldots, \text{M}. \qquad (1)$$

Where b_{mn} denotes the contribution of firm characteristics n to the sensitivity for external factor m. Next it is assumed that, at the beginning of the period over which performance is measured, the external risk factors can be identified and that the expected end of period values of these factors can be assessed/estimated. However, it is also assumed that only systematic risk factors (persistent influences like dollar risk or interest risk) can be identified and that there are always lots of idiosyncratic risk factors that together cause deviations from expected performance. Performance at time t becomes thus a stochastic variable, which can be written as

$$P_t = E_{t-1}(P_t) + UE_{t-1}(P_t) \qquad (2)$$

with E_{t-1} (P_t) being the end of period performance expected at the beginning of the period and UE_{t-1} (P_t) being the part of end of period performance which was not expected at the beginning of the period. The expected performance can be thought of as a function of the firm characteristics and the expected values of the systematic risk factors:

$$E_{t-1}(P_t) = E_{t-1}(C_{1,t-1,\ldots,}C_{N,t-1,}E_{t-1}[f_1],\ldots,E_{t-1}[f_M]) \qquad (3)$$

The unexpected part of performance becomes

$$UE_{t-1}(P_t) = \sum_{m=1}^{M}\left\{UE_{t-1}[f_m]\cdot\sum_{n=1}^{N} b_{mn}\cdot C_{n,t-1}\right\} + \varepsilon \qquad (4)$$

with $UE_{t-1}[f_m]$ being the unexpected risk factor change and ε the total (unexpected) effect of all idiosyncratic factors. Spronk and Vermeulen [19] use relations (2), (3), and (4) to estimate the reaction of firm characteristics to unexpected factor changes (b_{mn}) and then use (1) to estimate the reaction of individual firms to unexpected factor changes $(b_m)^j$ with J = 1,...,J being the index of the firm in the sample[3].

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[3] By estimating the firm sensitivities via the sensitivities of firm characteristics it becomes possible to use panel techniques instead of straightforward cross-section techniques. In

Thusfar, empirical studies with this model include 'hard' firm characterisitics only (examples are the firm's debt ratio, sales in different product groups, number of personnel, etc.). Within the framework for strategic decision, it is clearly opportune to include some qualitative firm characteristics in the analysis, as has been discussed in section 2 and in this section. The result of the analysis can be summarised as risk profiles of every individual firm in the sample. A risk profile is the vector showing the firm's sensitivities for the risk factors concerned. In many cases, the risk profile will also include the expected future performance of the firm. Looking at the CSFs of C-3 we realise that all but one is purely quantitative. **Marketing resources** is the only that is quantitative; number of people involved and amount of money available. The other factors are qualitative by nature, yet the CEO was able to estimate the maximum positive and negative effect of each CSF on sales. What is important to realise with the CSFs is that when the CEO of C-3 made his estimates he was considering his own firm's performance and the competitors' performance, based on the knowledge he had on them.

4. Conclusion: Towards strategic management

Because the firm's sensitivities are measured by estimating the sensitivities of the firm characteristics, the PE yields insight not only in the relative strengths and weaknesses of the firm as a whole, but also in the determinants of these strengths and weaknesses; being the firm's characteristics, both hard and soft. There are several types of instruments which can be used to change the firm's characteristics. Vermeulen et al.,[23], mention *level changing instruments* and *flexibility instruments.* In addition, the firm may have instruments at its disposal, labeled *transformation instruments,* that transform the unexpected change of the risk factor. One example of level changing instruments is the possibility to change the firm's amount of debt. Another is its possibility to change the number of employees. By changing its amount of debt, the firm changes its sensitivity for changes (both expected and unexpected) in the interest rate. By changing the number of employees, the sensitivity for changes in the wage rate is manipulated. A flexibility instruments is an instrument which gives the firm the *possibity* to change the sensitivity, depending on the realization of the change in the risk factor. One example is the possibility to call temporary workers when and if the production volume increases over a certain level. Another example is the possibilty to change the product range offered depending on changes in the market. An important difference between level changing and flexibility instruments is that the latter can be used to react to unexpected factor changes whereas the former type of instruments require a decision on the sensitivity level before knowing the realized factor changes. However, in many cases, flexibility instruments have to be acquired and or developed before they can actually be used. For example, considering an expedition through the Sahara desert, most people would prefer to buy a decent toolbox before

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addition, it may be safer to assume that the sensitivity of firm characteristics for some risk factor remains approximately the same for some periods than assuming that the firm sensitivity for the same factor remains the same.

the trip starts and not to wait until they might actually need to use that toolbox. By definition, the value of the risk factors are beyond the control of management. Nevertheless, transformation instruments may help to limit the effect of these risk factors by passing part of the risk on to other parties. These instruments are generally packed as contracts that are made between two parties or which are traded in (often financial) markets. An example of the first type is a fixed price-fixed volume-fixed quality contract for the delivery of goods. Another example is a bank loan with a cap on the otherwise flexible interest rate. Examples of the second type are financial futures and swaps. Hardly any instrument comes free. The 'price' of an instrument may be directly monetary, but other possibilities do exist: for instance by changing a labour intensive process for a capital intensive process, there is not only the direct effect on the cash flows but also there is a change in the risk profile becoming less sensitive for the wage rate and more for the interest rate.

Every strategy considered by a given firm can be viewed as a particular constellation of instruments, yielding a particular portfolio of firm characteristics. The portfolio determines both expected future performance (cf. (3)) and unexpected future performance (cf. (4)) of the firm. The choice between alternative strategies thus boils down to choosing between different risk profiles (which in this case also include expected future performance given the strategy concerned). In doing so, trade-offs are not only concerned with preferences, but also - probably implicitly - with the assessment of the evaluation of the uncertainty surrounding the respective risk factors. For example, different strategies may yield different sensitivities for the unexpected change in the energy price. Now the trade-off between energy price and e.g. wage rate risk will be different when management believes that energy prices will become very volatile than in the case management believes energy price will remain stable.

The proposed framework for strategic management is explicitly related to performance evaluation. It is multidimensional, both with respect to perceptions and with respect to preferences. It can include both hard and soft firm characteristics. The reactions of the CEO's confronted with this framework were unanimously positive. The issue for future research is now to validate the framework by performing several test with data from several different firms. The case described in section two is only a partial verification that this framework originally intended to support financial decisions also is applicable in ill-structured and ill-defined strategic management contexts.

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The Multiobjective Metaheuristic Approach for Optimization of Complex Manufacturing Systems

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The group technology approach is based on the idea that parts which require similar operations and machines should as much as possible be grouped together into part families and machine cells. Such an approach, reduces usually the flow of parts and tools, set-up times, throughput times and work-in process inventory. In the ideal case, each part family should be processed by its own machine cell. However, in practice, some parts need to be processed by different cells. It results in some bottlenecks.

The cell formation problem could be described as follows. Let $M=\{M_1,M_2,\ldots,M_m\}$, $P=\{P_1,P_2,\ldots,P_p\}$, $A=\{a_{ij}\}$ be a set of machines available in the considered enterprise, a set of parts which should be processed by machines and a matrix which contains the technological information defining which parts are performed on which machines (a_{ij}=1 means that part P_j should be processed by machine M_i, a_{ij}=0 otherwise), respectively.

The goal is to create a number of cells, denoted by $C=\{C_1,C_2,\ldots,C_n\}$. To each cell some machines and parts are assigned. Let x_{ik}=1 denotes that machine M_i is in cell C_k and y_{jk}=1 denotes that part P_j is assigned to cell C_k.

To evaluate the quality of each assignment many objective function could be used. The managers are usually interested in minimization of differences of the intracell workload. It could be defined as a function which minimizes the deviation of the intracell workload with respect to the average intracell workload:

$$f_1=\max_{k=1,\ldots,n}\left|\sum_{i:M_i\in C_k}\sum_{j:P_j\in C_k}g_{ij}-\frac{1}{n}\sum_{k=1}^{n}\sum_{i:M_i\in C_k}\sum_{j:P_j\in C_k}g_{ij}\right| \tag{1}$$

where g_{ij} denotes the processing time of part P_j by machine M_i.
Another objective which should be taken into account in order to evaluate the quality of cells is minimization of intercell workload balancing. In fact, the decision maker wants to minimize the number and cost of bottleneck operation. A bottleneck operation is a pair (M_i,P_j) for which a_{ij}=1 and M_i and P_j are assigned to different cells. If the coefficient h_{ij} stands for the material handling cost

associated with the intercell move corresponding to processing part P_j by machine M_i located in another cell, the function below represents the intercell load:

$$f_2 = \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{p} h_{ij}\left[x_{ik}\left(1-y_{jk}\right)+y_{jk}\left(1-x_{ik}\right)\right] \tag{2}$$

Instead of the above objective the function which directly minimizes the number of bottleneck operations is often applied. It could be defined as follows:

$$f_3 = \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{p} a_{ij}\left[x_{ik}\left(1-y_{jk}\right)+y_{jk}\left(1-x_{ik}\right)\right] \tag{3}$$

In practice, it seems to be also interesting and useful to minimize the number of bottleneck machines and bottleneck parts separately:

$$f_4 = \sum_{k=1}^{n}\sum_{i=1}^{m} \min\left\{1, \sum_{j=1}^{p} a_{ij}x_{ik}\left(1-y_{jk}\right)\right\} \quad f_5 = \sum_{k=1}^{n}\sum_{j=1}^{p} \min\left\{1, \sum_{i=1}^{m} a_{ij}y_{jk}\left(1-x_{ik}\right)\right\} \tag{4-5}$$

The above formulation of objectives allows for analysis of duplication costs of machines and/or material handling or subcontracting costs.

Finally, the cell formation problem could be defined as multiobjective combinatorial one as follows: minimize all or a subset of objectives (1)-(5) subject to demand defined by technological matrix A and additional constraints given by the DM on e.g. minimal and maximal number of machines and parts in one cell.

In order to solve the problem, a metaheuristic procedure, called the Pareto Simulated Annealing (PSA), is used [1]. The goal of the procedure is to find in a relatively short time a good approximation of the set of efficient solutions for a multiple objective combinatorial optimization problem. The procedure uses a sample of generating solutions. Each solution explores its neighborhood in a way similar to that of the classical simulated annealing. Weights of the objectives, used for their local aggregation, are tuned in each iteration in order to assure a tendency for approaching the efficient solutions set while maintaining a uniform distribution of the generating solutions over this set.

As PSA is a metaheuristic method it defines a general scheme of the calculation procedure only. In order to customize it to a given problem a neighborhood a given solution should be defined. A solution of this problem is defined by the subsets of machines and parts assigned to each cell. The neighborhood solution can be obtained from the current one by selecting any machine or part from one cell, say C_a, and by transferring it to other cell, say C_b. The assumption is made that $C_a \neq C_b$. Moreover, in the cell formation problems some additional restrictions, which reduce the feasible neighborhood are usually applied, e.g. on maximum and minimum number of machines in each cell.

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Portfolio Selection Using the Idea of Reference Solution

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Abstract. This paper proposes a decision aiding procedure for security selection. The presented procedure uses a multicriteria discrete analysis method based on the idea of reference solution. Application of the decision aiding procedure is demonstrated through the numerical example based on real world data from the Warsaw Stock Exchange.

Keywords. portfolio selection, reference solution, multicriteria decision aid methods

1 Introduction

There are three main approaches to stock evaluation and selection such as fundamental analysis, technical analysis and portfolio models. Traditionally, portfolio selection is conceived as a problem of finding an efficient set of portfolios, where the probabilistic estimates of expected returns and risks are used [3]. The investor chooses one portfolio out of an efficient set of portfolios. It should be pointed out that individual goals and investor's preferences cannot be incorporated into these models. Thus the portfolio selection is often considered as a multiple criteria decision making problem. In the multicriteria approach various methods of portfolio selection are applied e.g.: goal programming [2], ELECTRE methods [8], [9], interactive procedures [4], fuzzy programming [6], [7].

The portfolio selection process consists of two major activities: security selection (finding the most attractive stocks that are suggested for inclusion into the portfolio) and the determination of their proportions in the portfolio. This paper concerns decision aiding for the first of the subproblems mentioned above that is security selection. The next section describes security selection as a multicriteria discrete decision problem and presents the methodological framework of decision aiding procedure for security selection which uses the idea of reference solution. A real world data example is presented in section 3. The decision aiding procedure is applied to a sample of 47 stocks from the Warsaw

Stock Exchange. Finally, the proposed approach is summed up and some concluding remarks are made.

2 The methodological framework

2.1 Security selection as a multicriteria discrete decision problem

The security selection problem is a complex decision process. It consists of selecting a universe of stocks for consideration; choosing the characteristics desired for the evaluation of each stock; searching the set of stocks and finally selecting the subset of attractive stocks that are suggested for inclusion in the portfolio [4]. The problem of security selection can be stated as follows: given a finite number of stocks select the subset of stocks that are suggested to be included in the investment portfolio. Thus, the security selection problem is the multicriteria discrete decision problem [5], [6].

There are some important issues that should be taken into consideration. First, although modern finance theory has developed useful tools for evaluating the financial standing of public corporations, there is no deterministic relationship between them and the future stock prices. That is why it is difficult to determine the appropriate set of evaluation criteria and their importance. On the other hand, there is a possibility of comparing stocks ex post using a single criterion: price appreciation (or more generally the rate of return: price appreciation plus dividend). Thus it is easy to find out the extremely good and extremely bad stocks at a given time (see: [5])

We suggest searching the universe of stocks to find extremely good and extremely bad securities in the past (that are called stock market winners and stock market losers, respectively) and use them in comparing the currently evaluated stocks.

2.2 Decision aiding procedure

The proposed decision aiding procedure uses the BIPOLAR method [1]. BIPOLAR is a multicriteria decision support method of aiding discrete decision problems which uses the idea of reference solution. The fundamental assumption is that a decision maker (DM) according to his/her experience and knowledge is able to determine the bipolar reference system in the form of two sets of objects: desirable and non-desirable ones. Evaluated objects are compared to each other regarding their position in relation to reference system. The BIPOLAR method is based on the idea of concordance and veto threshold of ELECTRE I.

We propose using the reference sets constructed on the basis of the DM's preferences and historical data. The set of extremely good objects will be called the stock market winners set and the set of extremely bad objects will be called the stock market losers set. Thus the reference objects come from the past.

The way in which the reference sets are constructed suggests using a distribution analysis of considered criteria within the reference sets to compute suggested weights of the criteria and veto levels.

We assume that the decision maker (DM) is the portfolio manager interested in selecting the most attractive stocks into the portfolio. The aim of the decision aiding procedure is to help him to evaluate stocks. We assume that by looking back the DM is able to define categories of extremely good and extremely bad stocks. The DM is asked to define the stock market winners and losers categories. These categories will be used while determining the reference sets. The decision aiding procedure consists of four main steps:

Step 1: The determination of the reference sets: market winners and losers
Step 2: The construction of the reference sets using past data
Step 3: The detetermination the procedure parameters by distribution analysis in the reference sets.
Step 4: The BIPOLAR method: ranking and sorting stocks.

2.2.1 Step 1: The determination of the reference sets

First step of procedure concerns the determination of the reference sets. DM is asked to define two categories of objects: stock market winners and stock market losers. Regarding historical data DM describes which stocks would be the ideal investments and which ones would be extremely bad in a given past time. According to our earlier considerations DM describes minimal desired proce appreciation in the given time (winners) and a maximal accepted price depreciation (loseres) . If DM considers it useful, it is also possible to take into consideration more criteria while determining the reference sets, with regards to his/her individual preferences (e.g.: stock market capitalization, liquidity etc.).

2.2.2 Step 2: The construction of the reference sets

After the categories of stock market winners and stock market losers are determined by DM, the reference sets are created. Using the historical data the set of universe stocks is examined and all stocks that meet required criteria are included to the proper reference set. For each stock found a hypothetical buying date is determined. Then values of all considered criteria are settled.

2.2.3 Step 3: The determination of procedure parameters

The way in which reference sets were constructed suggests using a distribution analysis of considered criteria within the reference sets to determine the procedure parameters. They are then used to compute the weights of criteria and non-acceptance (veto) levels. To simplify this presenation we assume that all criteria are minimised. First, let us assume the following notations [1]:

$O = \{o_i\}$ - set of evaluated objects (stocks), $i=1,\ldots, n$
$K = \{k_j\}$ - set of considered criteria $k_j(o_i)$
$W = \{w_i\}$ - set of extremely good stocks (market winners)
$L = \{ l_i\}$ - set of extremely bad stocks (market losers)
$R=\{r_i\}$ - the reference system $R = \{ W \cup L \}$ and $W \cap L = \varnothing$

Furthermore, we assume that:

$$\forall w_i \in W \quad \forall l_j \in L \quad \sim l_j > r_i$$

where > denotes dominance relationship.

p_j- denotes weight of j criterion
v_j- denotes veto level of j criterion
s - accordance level, $\min p_j \leq s \leq 1$

For each criterion decile distribution is computed within both reference sets separately. The obtained distributions are presented to DM. Let $d^j_{i,S}$ denotes the value of i-th decile in the S set, where S=W or S= L (j-denotes the number of criterion).Let

$$f^j_S(x) = \{ i: d^j_{i-1,S} \leq x \leq d^j_{i,S} \}, \quad S=W \text{ or } S=L$$

denotes the number of fraction in which belongs value x. Moreover let us assume that

$$\text{if } x \leq \min k^j_S \text{ then } f^j_S(x) = 0$$

and

$$\text{if } x \geq \max k^j_S \text{ then } f^j_S(x) = 10$$

Let

$$P^j=10-f^j_L(d^j_{i,W}),$$

Value i = 1,...,9 describes decile number and is arbitrarily determined by the DM, but should be the same for each criterion. For example, if i equals 9 then the value of P^j shows the number of fraction in set L that includes value of the ninth decile within set W. Let p_j denote weight of j criterion. The weights of all considered criteria will be settled as follows:

$$p_i = \frac{P_i}{\sum_{j=1}^{m} P_j}$$

Veto levels are also determined on the basis of the reference sets as follows: $v_j=d^j_{k,S}$ where k=1,...,9 S=W or S=L. Value k is arbitrarily settled by the DM.

2.2.4 Step 4: The BIPOLAR method

BIPOLAR method [1] is a multicriteria decision support method for aiding discrete decision problems. It is based on the idea of reference solution. The evaluated objects are compared with each other regarding their position in relation to two reference sets: the set of extremely good objects and the set of extremely bad ones. The procedure consists of three phases. In the first phase the evaluated objects are compared with the elements of the reference system. The aim of this phase is to determine the model of the DM's preferences on O x R. In this phase each object is compared with all objects of reference sets. In this phase, in order to determine the preference structure, for each (o_i, r_t), the predominance ratios are computed according to the general rules valid for methods of the ELECTRE group (see: [1], [8], [9]). Next, the position of considered objects in relation to the reference system is settled. In this phase each object is compared to the whole reference sets. The issue is the evaluation of each object by describing its position in relation to the reference system. In the last phase we come to the conclusion about the relations within the set of considered objects. The aim of this phase is to qualify objects to the considered categories (ranking objects) and to sort the objects (ordering).

3. An example of the use of security selection support procedure

The Warsaw Stock Exchange was founded in April 1991. During the last four years the number of traded stocks risen from 5 to 60. The supervising body on the Polish stock market is the KPW (the equivalent of the American Security Exchange Commission). The KPW admits stocks to the public market and defines the info duties for public companies. They have to publish income statements (quarterly), balance sheets and cash flow statements (every two quarters) and simplified income statements (monthly). These info duties are rather tough, but they enable investors directly to evaluate the financial standing of each public corporation.

3.1 The reference sets determination

Let us suppose that a DM, taking into consideration his aspiration level, the planned investment horizon and/or additional factors (e.g.: the inflation rate, the risk free rate etc.), defines the categories of stock market winners and stock market losers using the only criterion; i.e. monthly average price appreciation as follows: a stock whose monthly average price has been at least doubled during the preceding six months is considered to be an extremely good stock (market winner). A stock whose price has been depreciated at least 20% within half a year will be considered to be an extremely bad object (market loser).

Furthermore, let us assume that the DM has decided to carry on an analysis based on monthly data. Then values of all the criteria are computed using average monthly prices (if the stock price is necessary to determine the value of the criterion). We assume that DM has decided to search for winners and losers within the set of all traded stocks of the commercial sector over the April 1991 - March 1995 period.

3.2 The reference system construction

Using the definition of extremely good and bad stocks (i.e. market winners and losers) the universe of all traded stocks within the given period is examined to find all firms that fulfil the required criterion. The firms found are included in the appropriate reference set. For each stock belonging to the reference system a hypothetical buying signal is determined. For market winners this is the last month preceding the price appreciation. By analogy, for market losers "anti-optimal" buying signals are set up i.e.: the last month before price depreciation. We found 30 market winners and 45 stocks classified to market losers category. Four winners and 10 losers were classified at least twice to the appropriate category.

3.2.1 The criteria evaluation

For the purpose of this example we assume that the DM analyses seven criteria. We divide them into three groups: Valuation Measures (P/BV - price / book value ratio (MIN), P/E - price / earning ratio (MIN)), Fundamental Variables (PMR - profit margin ratio (MAX), CQP - changes in quartely net profits (MAX)), Technical Indicators (ROC - rate of change (MIN), RSI - relative strength index (MIN), APP - price appreciation during last 3 months (MIN)). Values of the technical indicators and the valuation measures were computed using an average monthly price in the buy-month and earnings and sales for last four quarters. The values of the fundamental variables were computed on the basis of the last available income statements and balance sheets.

3.3 Procedure parameters evaluation

Being the objects of the reference sets, settled ex-post hypothetical buy-months and the set of concerned criteria, then the values of all criteria are computed for each object of the reference system. Because of missing data 6 stock market winners and 15 stock market losers were excluded from further computations. For each criterion distributions within the reference sets are settled separately. Tables 3.1 and 3.2 show the decile distributions in both sets. Those distributions are presented to the DM. Let us suppose that the DM chooses $i=8$ value to set up the importance of criteria and $k=7$ and $S=W$ to determine non-acceptable levels. The weights and the veto levels are computed according to the formulas presented in section 2.2.3. Moreover, suppose that the accordance level is equal to 0.75. Table 3.3 displays the procedure parameters.

Table 3.1. Distribution of criteria in the stock market winners set

criterion	1st	2nd	3th	4th	5th	6th	7th	8th	9th
P/BV	0.29	0.30	0.40	0.72	0.80	0.88	1.55	2.48	3.88
P/E	1.79	2.26	2.50	2.76	3.50	7.94	15.51	27.72	42.11
PMR	0.03	0.04	0.05	0.06	0.10	0.10	0.12	0.20	0.24
CQP	-0.43	-0.15	-0.01	0.15	0.31	0.94	1.14	1.35	4.39
ROC	-0.09	-0.06	-0.04	-0.03	-0.01	0.00	0.00	0.05	0.07
RSI	0.39	0.40	0.41	0.44	0.45	0.48	0.50	0.50	0.53
APP	-0.17	-0.11	-0.09	-0.04	-0.03	-0.01	0.00	0.01	0.08

Table 3.2. Distribution of criteria in the stock market losers set

criterio	1st	2nd	3th	4th	5th	6th	7th	8th	9th
P/BV	2.06	2.70	3.34	3.90	4.23	5.30	6.69	8.40	9.55
P/E	14.30	21.60	27.20	33.90	42.40	43.90	54.95	69.50	111.30
PMR	0.02	0.03	0.04	0.06	0.07	0.09	0.12	0.15	0.19
CQP	-0.16	0.44	0.79	0.96	1.16	1.45	1.65	2.72	4.42
ROC	0.00	0.00	0.03	0.10	0.15	0.19	0.22	0.27	0.40
RSI	0.49	0.50	0.50	0.50	0.50	0.52	0.54	0.60	0.62
APP	0.00	0.00	0.18	0.23	0.33	0.40	0.69	0.75	1.13

Table 3.3. Parameters of procedure

criterion	weigh	veto level
P/BV	0.20	1.55
P/E	0.15	15.51
PMR	0.10	0.05
CQP	0.10	-0.01
ROC	0.15	0.00
RSI	0.12	0.50
APP	0.18	0.00

3.4 Computations and results

First, for currently evaluated stocks, the values of all the considered criteria are settled. Table 3.4 shows the values of criteria in the universe of stocks.

3.4.1 Mono ranking and mono sorting

Two independent mono rankings are presented with regard to the market winners and market losers sets. There are three categories of objects in each case.. Looking at the object position with regard to the stock market winners we have:

class 1 ("overgood"); class 2 (other); class 3 (non-comparable). When we consider object position regarding the stock market losers we obtain: class 1 (other); class 2 ("underbad"); class 3 (non-comparable). Mono ordering (sorting) is based on mono-grouping and on values of degree reaching success and avoiding failure, respectively. (see: [1]). Obtained mono rankings and sortings are shown in Tables 3.5 and 3.6.

Table 3.4. Values of criteria in the universe of evaluated stocks

Stock	P/BV	P/E	PMR	CQP	ROC	RSI	APP
S1	2.2	15.6	0.27	-0.34	0.00	0.52	0.02
S2	1.4	5.6	0.40	0.08	-0.02	0.60	0.00
S3	1.0	8.0	0.41	0.44	0.03	0.65	-0.03
S4	2.0	11.1	0.26	-0.43	-0.03	0.46	-0.17
S5	1.4	8.9	0.45	-0.31	0.03	0.54	0.15
S6	0.9	10.0	0.12	10.96	-0.01	0.59	-0.07
S7	1.7	14.6	0.18	-0.09	-0.05	0.52	-0.11
S8	1.3	8.2	0.19	0.20	0.02	0.54	0.17
S9	0.8	15.0	0.49	-0.99	0.00	0.62	-0.10
S10	1.5	11.5	0.14	-0.92	-0.04	0.51	-0.12
S11	1.0	16.9	0.33	1.79	0.01	0.48	-0.06
S12	3.0	25.9	0.20	0.61	-0.01	0.51	0.01
S13	0.9	18.7	0.30	1.77	-0.02	0.54	-0.04
S14	1.1	8.3	0.30	0.50	-0.04	0.57	-0.07
S15	2.7	58.4	0.49	-0.47	-0.05	0.59	-0.17
S16	1.9	9.6	0.15	2.13	-0.01	0.57	-0.01
S17	1.6	12.8	0.11	2.17	-0.06	0.56	-0.12
S18	1.4	8.9	0.16	6.11	0.02	0.59	0.07
S19	1.9	8.1	0.39	0.00	-0.04	0.48	-0.16
S20	1.4	6.6	0.25	3.91	-0.01	0.61	-0.23
S21	1.3	11.1	0.33	-0.66	0.02	0.48	-0.05
S22	0.9	6.5	0.43	0.79	0.02	0.54	0.04
S23	0.8	14.2	0.14	2.33	-0.05	0.55	-0.17
S24	0.9	5.2	0.34	-0.08	-0.05	0.54	-0.11
S25	0.9	13.8	0.29	0.31	-0.03	0.60	-0.10
S26	2.3	9.6	0.55	0.73	-0.05	0.52	-0.12
S27	1.3	15.1	0.22	-0.53	-0.05	0.58	-0.14
S28	1.4	8.4	0.15	0.58	-0.01	0.50	-0.02
S29	3.1	11.5	0.08	0.40	-0.03	0.38	-0.30
S30	1.2	8.7	0.33	1.17	-0.05	0.50	-0.16
S31	2.2	21.0	0.33	0.17	-0.01	0.53	-0.09
S32	2.5	7.6	0.39	4.00	0.01	0.60	0.02
S33	1.4	6.7	0.45	-0.10	-0.04	0.62	0.02
S34	1.1	8.9	0.05	0.32	-0.04	0.53	-0.11
S35	1.1	14.1	0.50	-0.61	0.00	0.50	0.00
S36	1.5	10.7	0.34	1.31	-0.05	0.51	-0.19
S37	0.8	8.3	0.13	1.18	-0.04	0.56	-0.12
S38	2.7	8.0	0.60	0.05	-0.04	0.57	-0.09
S39	1.5	9.5	0.13	0.62	-0.06	0.49	-0.14

Table 3.5. Stocks ranking and ordering with regard to stock market winners set

class	stock
1. overgood	S6, S11, S13, S14, S17, S23, S25, S30, S34, S36, S37, S39 > S4, S7, S10, S19, S21, S24, S27, S28 > S3, S16, S20, S26 > S29 > S2 > S31, S38 > S9 > S35 > S18, S32
2. other	S1, S5 > S15 > S8 > S12, S22, S33
3.non-comparable	no objects

Table 3.6. Stocks ranking and sorting with regard to stock market losers set

class	stock
1. other	S8, S33 > S1, S5 > S3, S6, S18, S20, S20, S22 > S13, S14, S15, S16, S23, S24, S25, S30, S36 > S2, S4, S11, S12, S19, S29 > S7, S9, S10, S17, S21, S26, S27, S28, S31, S32, S34, S35, S37, S38, S39
2. underbad	no objects
3. non-comarable	no objects

3.4.2 Bipolar ranking and bipolar sorting

Bipolar ranking is based on mono grouping intersection. Non-comparable objects are not considered. So we consider three categories (classes) of objects. The first class 1 consists of objects that are better than good objects (overgood) and that are better than bad objects (other). Class 2 is composed of objects that are worse than good objects (other) and that are better then bad objects (other). Class 3 consists of objects that are worse than good objects (other) and that are worse than bad objects (underbad) (see: [1]). Table 3.7 displays final bipolar ranking and bipolar ordering.

Table 3.7. Bipolar ranking and ordering

class	stock
class 1	S6 > S13, S14, S23, S25, S30, S36 > S11 S17, S34, S37, S39 > S24 > S3, S20 > S16, S4, S19 > S7, S10, S21, S27, S28 > S26 > S29 > S2 > S31, S38> S9, S18 > S35 > S32 > S15
class 2	S1, S5 > S8 > S12 > S22 > S33
class 3	no objects

4. Concluding remarks

The portfolio selection problem consists in finding the most attractive stocks and the determination of the portfolio structure. This paper proposes the decision aiding procedure for security selection which is applied on the Warsaw Stock Exchange. The presented procedure uses the idea of reference solution. The Bipolar procedure is applied [1]. The reference sets of stock market winners and stock market losers are constructed using historical data and the decision maker's assumptions. The importance of the criteria and non-acceptance levels are settled by distribution analysis within the reference sets. Proposed approach enables decision maker to take into consideration all the relevant criteria. The decision maker can choose the stocks based on the results coming from above procedure. We do not take into consideration risk measures in security selection. Risk may be applied while determining the proportions of securities in the portfolio. The portfolio structure can be determined using some of multicriteria methods e.g.: quadratic programming [3] or goal programming [2]. Finally the proposed procedure can be applied to the evaluation of the currently existing portfolio.

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Model "Inflation - Non payment - Production - Loans" and its implementation in Russia

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Introduction

Inflation in Russia was over 20% per month in 1992, but now it is in range 5-8% per month. The financial decision in such conditions differs from ones under inflation 3-5% per year. Now Russian government sets the problem of inflation decrease to 1-2% per month as the main goal of its work.

The problem is to choose the decision under conditions of high inflation. In this paper one approach is discussed to help Russian top managers to make financial and production development decisions in current inflation situation.

Under conditions of a high inflation rate (over 10% per month) it is necessary to increase the working capital.

A simple analytical model, combining the production level, income, purchase of material, debtors and creditors, loans and their return, technological and financial cycle, was developed. It covers ideas of budgeting and cash flow analysis.

Top managers should find the real direction of the development of their company in multiple criteria environment:

- sales,
- price required,
- production decrease,
- loans,
- maximum permissible credit rate,
- financial stability,
- increase of creditors debt (risk of bankruptcy).

The combined effect of all factors is considered.

The influence of the set of control parameters / factors is taken into account. The set of measures of organizational, financial, educational, and technological types is considered, their effects on financial success of a company are discussed.

The inverse problem is analytically solved in linear approximation. The requirements on each parameters are set (sufficient condition).

The scheme of the multiple criteria selection solution is proposed. The approach of abatement is used.

An example of specific Russian companies is discussed.

As an alternative example of the proposed approach, it is shown that the evaluation of the business plan in hard currency (without inflation) or high inflating local currency results in deferent solutions.

This paper is devoted the problem of dependency of main financial factors on the production: volume of production, income, loans required under high inflation.

The analytical model allows to combine the principal indices, and the formal conditions could be established due to the linear model.

The necessity of this investigation is vital because of non-payments and production reduction in Russia after price liberation in 1992. The approach used seems to be appropriate for other countries with high inflation rate.

The estimates of the index considered are given for typical parameters.

1. Goals of development

The process of changes goes very fast in Russia. There are many firms that were in governmental property, but now joint stock companies with large proportion of private capital.

We could formulate early the main goal of typical Russian enterprise (see Spronk [4] as a case): to find its own place in market.

But now the goal changes in many cases and it is to survive financially.

This principal goal can be subdivided into specific goals. For example, one Ural enterprise to establish the following goals.

1.1. Market goals

- to have the certain share of the market
- to have market mechanism

1.2. Financial goals

- to leave the zone of bankruptcy
- to have defined sales, profit, dividend per share

1.3. Social goals

- to keep personnel
- to keep major part of social support at a company
- to improve social support

1.4. Manufacturing goals

- output
- assortment renewal
- quality
- to improve customer property

So top managers should work in a multicriteria space and make decisions under uncertain conditions.

2. Problem formulation

A manufacturing enterprise with a certain financial cycle is considered. We assume that this enterprise produces a Sales)roduct with some flow of intensity u.

Let us consider the simplest financial cycle of production.

$$\begin{tikzcd} \text{Purchase} \arrow[r, "T_D"'] & \text{Manufacturing} \arrow[r, "T_I"'] & \text{Sales} \arrow[ll, bend right] \end{tikzcd}$$

The product made is delivered to the ware stock, after sales it is paid with delay T_I.

The main expenses are raw materials, wages and fees, overheads, and taxes.

The raw materials are paid before using in the manufacturing by time T_D which includes the pre-payment, delivery time and the technological cycle.

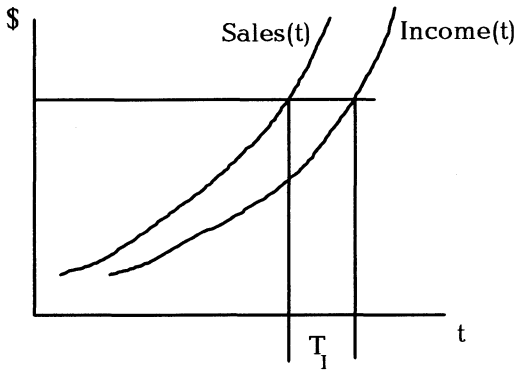
The wages, overheads, taxes and fees are paid in the current period or in short time after it. They are accounted as creditors.

Notation

t	- period index, used by default
u	- intensity of production
i_u	- rate of production decrease (%)
p	- current price per unit
i_p	- product price growth (inflation rate) (%)
m	- cost of raw material per unit
i_m	- material price growth (%)
S	- final product
M	- raw material expenditure

N - current payments
C - total expenditures of the company
Pr - net income
I - income
E - expenses
T_I - average delay in receipts
T_D - average time of pre-payment for raw material
L - loan
R - loan return
T_R - average duration of loan
i_R - loan rate (%)
r - ratio of net profit on the final product
i_L - rate of ratio change of credit to sales

3. Mathematical model



The total income I(t) includes sales from realization of goods and loans. Sales income is received with some delay.

Let us use the average time of delay of financial receipts T_I, defined by equation $I(t)=S(t-T_I)$.

Receipts are defined by equation $I(t) = \int a(\tau)S(t-\tau)d\tau$, where $a(\tau)$ is the part of payments with delay τ, and $\int a(\tau)d\tau = 1$ is the condition of the norm. The integration is assumed to be from $-\infty$ to $+\infty$.

Let i_S be the rate of changing of S: $S(t-T_I) = S(t)(1-i_S T_I)$. The law of changing is expressed by formulas:

$$I(t) = \int a(\tau)S(t-\tau)d\tau = S(t-T_I) = S(t)(1-i_S T_I)$$

$$I(t) = \int a(\tau)S(t)(1-i_S\tau)d\tau = S(t) \int a(\tau)(1-i_S\tau)d\tau = S(t)\ (1 - i_S\int a(\tau)\tau d\tau\ )$$

Thus, for the linear approach we have the following

$$T_I = \int a(\tau)d\tau$$

Income I includes sales $S(t-T_I)$ and loans L

$$I = S(t-T_I) + L$$

Expenditures E are paid for raw materials $M(t+T_D)$, current payments N, and loan return R

$E = M(t+T_D) + N + R$

Index N includes the payment for wages and salaries and related fees, overheads, and taxes. Let us include the required expenditures from profit after taxes, for example, for social sphere. These compensations are required according to the Russian tax law.

4. Main inequality

The balance of finance provide us the equation

Income + Initial_Cash = Expenses + Final_Cash

At a steady flow, the enterprise cannot pay more than it has. Hence the expenditures are not larger than incomes.

Expenses $\leq$ Income.

Using corresponding terms, we obtain the following:

$0 \leq S(t-T_I) - M(t+T_D) - N + L - R$

Let us add and subtract the term S-M, and rearrange the inequality

$0 \leq S(t-T_I) - S + M - M(t + T_D) + S - M - N + L - R$

It is known, that sales are equal to expenses and the net income

$S = C + Pr = M + N + Pr$, and we have that

$$0 \leq S(t-T_I) - S + M - M(t+T_D) + Pr + L - R \qquad (1)$$

Here Pr is the net income of the company, it is equal to the sum of depreciation and net profit after taxes and other practically indispensable expenditures, for example, for social sphere.

5. Non-payment analysis

The sales are paid with a delay, and the unreceived part is fixed as debtors D.

$S = I + D - D(t-1) = I + \Delta D$

The raw materials delivery is paid with a delay, and the unpaid part is fixed as creditors G.

$\Delta G = M - M(t + T_D)$

Rewrite the inequality (1), using the terms of debtors, creditors and loans

$$0 \leq - \Delta D + \Delta G + Pr + \Delta L \qquad (2)$$

From the (2) we have the condition for the increase of indices:

a) the debtors should not growth faster than the sources:

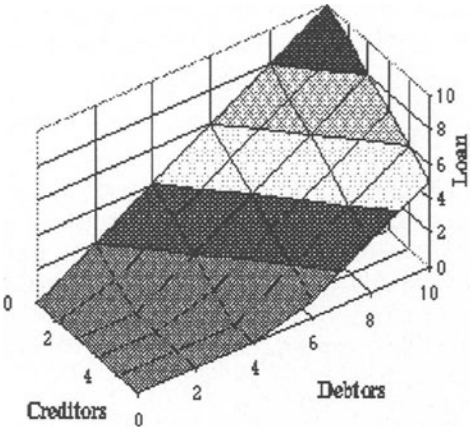
$\Delta D \leq \Delta G + Pr + \Delta L$

b) the creditors growth is covered by debtors, profit and loans:
$\Delta G \geq \Delta D$ - Pr - ΔL

c) the loan growth is required for nonpayment coverage:
$\Delta L \geq \Delta D$ - Pr - ΔG

In the space of creditors, debtors and loans the hypersurface of zero-balance value is the simplex. Some example is shown on the figure. This scheme is close to ones developed by Bushenkov and Lotov [1], Lotov [3].

But this information is too general to make concrete decisions, and we should develop more concrete model.



6. Linear system

Let the complexes i_pT_I, i_pT_I, i_uT_D, i_mT_D, i_LT_R, i_RT_R be substantially less than 1.

Let us make the system linear by using the linear laws of index changing

$u(t+\tau)= u(t)\ (1 - i_u\ \tau)$

Here, u is the intensity of the production flow (natural unit per period), i_u is the rate of fall of the production in per cents, and it is positive when the production reduces.

$S(t+\tau)= u(t)\ (1-i_u\ \tau)\ p(t)\ (1+i_p\ \tau) = S(t)\ (1-i_u\ \tau)\ (1+i_p\ \tau)$

$M(t+\tau) = u(t)\ (1-i_u\ \tau)\ m(t)\ (1+i_m\ \tau)= M(t)\ (1-i_u\ \tau)\ (1+i_m\ \tau)$

$L(t+\tau) = L(t)\ (1+i_L\ \tau)$

$R(t) = L(t-T_R)\ (1+i_RT_R) = L(t)\ (1-i_LT_R)\ (1+i_RT_R)$

The insertion of the linear expression in the main inequality gives

$0 \leq S(t)\ (i_uT_I-i_pT_I-i_uT_Ii_pT_I) + M(t)\ (i_uT_D-i_mT_D+i_uT_Di_mT_D) +$
$L(t)\ (i_LT_R-i_RT_R+i_LT_Ri_RT_R) + Pr$

Let us neglect the terms of $i_ui_pT_IT_I$, $i_ui_mT_DT_D$, $i_Li_RT_RT_R$. Thus

$0 \leq S(t)\ (i_u-i_p)T_I + M(t)\ (i_u-i_m)T_D + L(t)\ (i_L-i_R)T_R + Pr$

Let us divide this inequality by S(t), and take into account that r=Pr/S(t), m(t)/p(t)=M(t)/S(t)

$$\boxed{0 \le r - i_p T_I - m/p i_m T_D + i_u (T_I + m/p T_D) + L T_R (i_L - i_R)/S} \quad (3)$$

Inequality (3) is useful and basic for following analysis.

The complex i*T is in range of 0.05÷0.3 for Russia in 1993-1994 under the characteristic period of payments $T_{(I,D,R)} \approx 0.5 \div 1$ month, and the inflation (price increase) $i_{(u,p,m,L,R)} \approx 10 \div 30\%$ per month. This figure shows the accuracy of the approach (≈10÷30%).

7. Reverse task

The reverse task is solving in linear approximation analytically (see analogous approach in Leontiev [2]). The requirements for each parameters are established (sufficient condition).

Let us consider one variant of solving of the reverse task, when the decision maker what to maintain the manufacturing level.

Under steady production level $i_u = 0$, and inequality (3) is following

$0 \le r - i_p T_I - m/p\ i_m T_D + L T_R (i_L - i_R)/S$

Then the requirements for loans are formulated

$i_p T_I + m/p\ i_m T_D - r \le L T_R (i_L - i_R)/S$

Example.

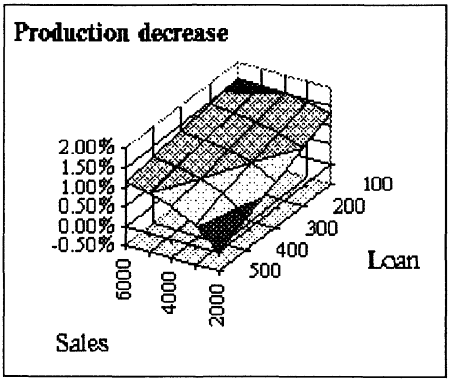
Index	r	m/p	$i_p = i_m$	$T_D = T_I$	T_R	S/L	i_L	Result
Value	0.10	0.5	0.2	1	3	3	0.2	0.2S≤(L-R)
Value	0.25	0.5	0.2	1	3	3	0.2	0.05S≤(L-R)

Conclusions:

1) The loans are required for maintaining the level of production.

2) The loan return should not be the difficulty for company, what why the rate should be low, may be privilege.

Another example of constraint surface is shown on the figure.



8. Abatement procedure

To find the applicable solution the abatement procedure is used.

The idea is to diminish the relevant range per each parameters, and the following procedure could be used.

Steps of procedure.

1) To find the range of each parameters change (from minimum till maximum).
2) To establish the accuracy enough for the solution.
3) To find full range of parameters change.
4) To calculate and observe the area allowed.
5) To decrease the relevant range per one / many parameters.
6) Repeat step 3 until the satisfactory able decision will not received with applicable accuracy.

The proof of coincidence is based on the idea of compressing reflections.

9. Model implementation

Two different approaches are used for implementation of this model into practice. There are statistical and dynamic ones.

Static approach

The static approach is built due to specific inequalities which concludes from inequalities 3 during the procedure of reverse tasks formulation.

Multifactor approach is used for the analysis of current situation. The prototype of software information procedure was used. This approach have a very limited value at current moment and it is not really used in practical applications.

Main reason is that the language of model is one of scientists, and it is understandable and convenient to users.

Dynamic approach

Dynamic approach is really used, when a set of parameters are chosen in its enrollment due to cash-flow analysis.

Top managers can

a) to change the volume planned,
b) to estimate the volume of loan required,
c) to reduce the expenses,
d) to shift the expenses later,
e) to keep the market,
f) to correlate with goals of an enterprise

Conclusions

The proposed model allows to show the qualitative and to sufficiently estimate the quantitative indices of production fall under high inflation.
It is shown that under high inflation the following circumstances occur:

- production decrease;
- non-payments increase (debtors and creditors);
- the loan are required for working capital

The limitation of essential indices can be calculated for credit rate, inflation, and other elements.

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Equity and MCDA in the Event of a Nuclear Accident

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Abstract. Decision making on countermeasures in the event of a nuclear accident such as Chernobyl is complex. Many criteria are involved. Aside from obvious ones related to the direct effects of radiation, there are issues relating to psychological stress, public acceptability and the need to consider the longer term economics of the affected regions. Thus there are many ways in which multi-criteria decision analysis (MCDA) can provide insights and support to the decision makers. Decisions on countermeasures will be made, at least initially, in the face of considerable uncertainty and the interplay between equity and uncertainty in the evaluation of different countermeasure strategies is far from straightforward. Apparently reasonable approaches, which appear to treat different population groups equitably, can, on closer examination, have unreasonable effects. The paper describes the MCDA components in relation to equity judgements of a decision support system (RODOS) for such emergencies being built by a consortium of institutes within the European Union, Eastern Europe and the former Soviet Union.

Keywords. dispersive equity, *ex ante* equity, *ex post* equity, multi-criteria decision analysis, nuclear accident, risk, RODOS, state dependent attribute modelling.

1 Introduction

Major nuclear accidents such as those at Three Mile Island and Chernobyl have directed attention to the need for tools to support coherent decision making on countermeasures. This paper reports part of the thinking that is shaping the design of RODOS, a decision support system for nuclear emergency management being built by a consortium of European Union, Eastern European and Former Soviet Union institutes. Decision making on countermeasures is complex and involves many criteria. Aside from obvious ones related to the direct health effects of radiation, there are issues relating to psychological stress, public acceptability and the need to consider the longer term economics of the affected regions. Figure 1.1 gives two attribute hierarchies that have been developed in case studies on relocation decisions after an accident. Thus there are many ways in which multi-criteria decision analysis (MCDA) can provide insights and support to the decision makers

- Normal Living
  - Resources
  - Effects
    - Health
      - Radiation Related
        - Fatal Cancers
        - Hereditary
      - Stress Related
    - Public Acceptability
      - Affected Region
      - Rest of USSR

From the International Chernobyl Study [3], [10]

- Overall
  - Cost
  - Effects
    - Health
      - Collective Dose
      - Individual Dose
      - Psychological
    - Social/Political Acceptability
    - Flexibility

From the BER-3 relocation exercise [3], [6]

Figure 1.1: Attribute hierarchies developed in two multi-criteria decision analyses on countermeasure strategies

[3]. In this paper we focus on issues related to equity. The public have a right to equal and fair treatment, but what exactly does that mean? And how do we model it in a decision analysis?

The next section provides some background on emergency management. Subsequent sections discuss equity issues which may arise in emergency management, previous work on equity and risk, framing issues related to equity and a concluding section which asks rather more questions than it answers. Further background on nuclear emergency management in general and RODOS in particular may be found in [3], [5], [9], [11] and *Radiation Protection Dosimetry* (1993) **50**, issues 2-4.

2 The Context of Emergency Management

The issues faced in nuclear emergency management are difficult and the political processes involved are complicated. In the course of an incident, responsibility passes between several different groups of decision makers of differing technical and political sophistication and, needless to say, the process varies from country to country and from culture to culture. Thus the thumbnail sketch below should be taken 'with a pinch of salt'. But we hope that it does enough to set the scene.

During the building and running of nuclear plants, many plans are made to deal with potential emergencies. Databases of demographic, agricultural, economic and geographic data are established. Evacuation routes and procedures are planned. Emergency exercises are held regularly to practise for different accident scenarios. But no accident ever goes 'as planned'. Moreover, the public are seldom, if ever, exercised. National and international guidance for each type of countermeasure (e.g. sheltering, evacuation or food bans) are given in the form of *intervention levels*. In general, these are lower and upper levels on the predicted doses. Below

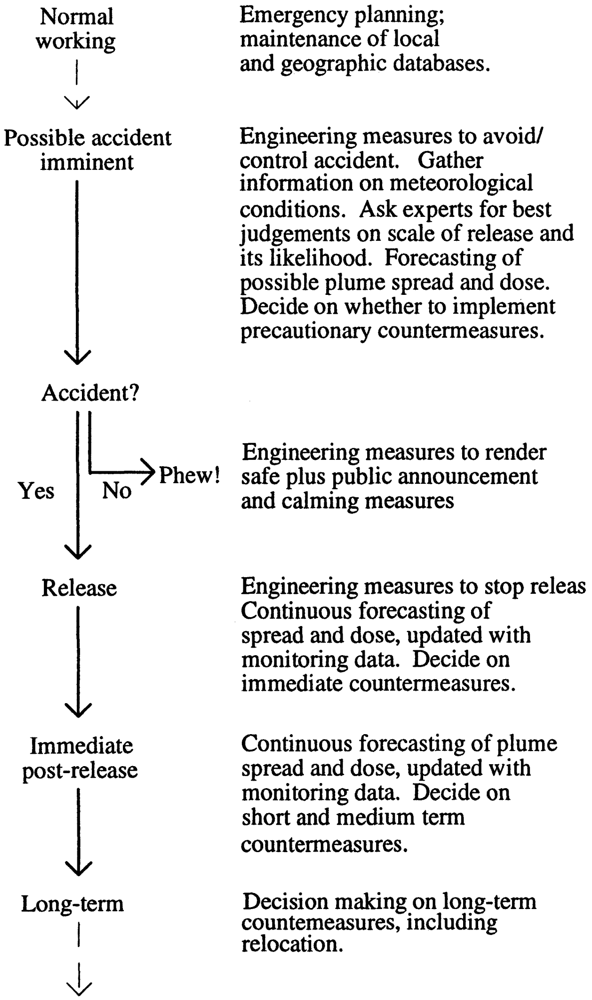

Figure 2.1: Simplified chronology of decision making on countermeasures during and accident

the lower level the advice is that intervention is unnecessary, whereas above the upper level it is required. Between the lower and upper levels, the action to be taken is left to the discretion of the authorities concerned in the light of the particular circumstances of the accident. Thus, in the event of an accident there are many decisions to be made. Emergency management is not simply the implementation of some predetermined rules.

When there is a serious and imminent risk of an accident, a number of actions might be taken. Plant engineers would take appropriate engineering actions to avoid or reduce the risk of a release. Decisions would be needed on precautionary measures such as: warning the public; distribution of stable iodine tablets; and the evacuation of some areas. If a release occurred, decisions would be needed on countermeasures such as: (issue and) uptake of stable iodine; sheltering; and evacuation. In the following days further decisions will be needed on, for example: food bans; decontamination of livestock and agricultural produce; decontamination of properties; restriction on activities; and restriction on access to the region. After several days or maybe weeks, decisions will be needed on longer term countermeasures, e.g. permanent relocation and changes to agricultural practice and industry.

Throughout, there will be a need to address value trade-offs between the 'harder' attributes such as cost and predicted radiation related health effects and the 'softer' ones such as political acceptability and psychological stress. Among factors affecting political acceptability are those relating to equity. However, before addressing these, it will be useful to understand a little about how the direct radiation related health effects are assessed.

Very briefly: current medical understanding of the health effects resulting from exposure to radioactivity is as follows. If the exposure is sufficiently high, deterministic health effects may result. A few tens of workers received such high doses during the Chernobyl Accident. Our concern will not be with such exposures: rather we consider the health effects arising from the 'low level' increase over background radiation which would be suffered by the population affected by an accident. Such effects are stochastic. Exposed individuals bear a higher risk of cancer and of genetic effects in their children. For an individual in a particular set of circumstances for a particular time period, it is possible to calculate from predictions of the spread of contamination a quantity known as the *individual effective dose equivalent*, often shortened to *individual dose*. For low level exposures, current medical belief supported by many studies is that the health risks to the individual are linearly related to the individual dose. This is known as the *linearity hypothesis*.

Current practice in radiation protection decision making focuses on the *collective effective dose* (or *collective dose*). This is the sum of the individual doses over a population. Under the linear hypothesis the expected number of health effects arising from the exposure to a population is proportional to the collective dose. Thus minimising collective dose is an important criterion, arguably the most important criterion, in such decision making. It should be noted that typically decision makers examine the collective dose for different subpopulations: e.g. infants under 1 year, children, women of childbearing age, and other adults. Also, they usually distinguish between public and worker populations and may apply some constraints to the maximum acceptable individual dose. In relation to the discussion here, we would remark that there is a general belief in the radiation protection community that, because collective dose is formed from an unweighted summation over (sub-)populations, it is an equitable measure with respect to those (sub-)populations. The possible fallacy in this belief is that there is no single concept of equity and that collective dose may be equitable in some senses, but inequitable or simply fail to include an evaluation of equity in other senses.

3 Equity in Emergency Management

The interplay of equity and risk in the evaluation of different countermeasure strategies is far from straightforward. Apparently, reasonable approaches can have unreasonable effects. Consider the following – admittedly contrived – example.

There are two villages, A and B, near a nuclear plant that are identical in all relevant ways: same population, same age distribution, same economic significance to the region, same type of housing, same type of agriculture: same everything. Thus in the event of an accidental release which put both villages equally at risk from equal collective doses, it is reasonable to assume that decision makers would be indifferent between:

i) evacuating village A and averting all their dose, while sheltering village B and averting half their dose;
ii) evacuating village B and averting all their dose, while sheltering village A and averting half their dose.

Suppose that there is an alarm at the plant. There is certain to be a release, but it is unclear whether it will be in 1 hour or 3 hours time. There are only enough buses to evacuate one of the villages within one hour, but if there are three hours warning both villages can be evacuated, because the buses can come back for the other village. Consider three possible strategies:

Strategy I: send all the buses to Village A; if the release does not occur for three hours, return for village B, otherwise shelter village B.

Strategy I': send all the buses to Village B; if the release does not occur for three hours, return for village A, otherwise shelter village A.

Strategy II: send half the buses to each village, evacuate half the population, shelter the rest; if the release does not occur for three hours, return for the rest of the villagers.

Clearly, the decision makers should be indifferent between strategies I and I'; but they may well prefer strategy II to either of the strategies I and I'. Strategy II seems more 'equitable'. Whatever the probabilities are, the expected averted doses are the same for all three strategies.[1] Thus collective dose alone will not discriminate between strategies on grounds of this interpretation of 'equity'. A theoretical justification of this conclusion may be found in [7]. Note also that equity is not just a matter of defining a fair distribution of risk over a population of individuals. The fair distribution of risks over subpopulations (here villages) also needs to be considered. For this we shall need concepts of dispersive equity, which will be defined below.

There is another aspect of equity which can be seen from this example. Suppose the decision makers anticipate this scenario before any accident is in the offing. They realise that if they make a binding agreement, that in the event of such a scenario arising they will toss a fair coin to choose between Strategies I and I'. This compound Strategy II' gives exactly the same probabilities of successful evacuation to the inhabitants of A and B as strategy II. *Ex ante*, i.e. before the accident, it seems precisely as equitable as strategy II. Of course, once an accident has happened and the coin is tossed, it is not equitable *ex post*.

A further complication is that equity is, in some sense, a property of the decision process as well as the outcomes; and the acceptability of a decision process designed to achieve equity may be context dependent. National lotteries allocate

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[1] We should admit that, in practice, a variation on Strategy II would be adopted in which the women and children from both villages would be evacuated first and then, if there were time, the coaches would return for the remaining men. The points that we shall draw from this example would be masked, but not obviated by this modification.

fortunes on the basis of randomising devices such as the drawing of numbered balls from an urn. The public accept – indeed demand – such 'fair' chance processes. The public also accept treatment in medical trials based upon a hidden application of randomisation. But would they accept emergency management decisions based upon the 'toss of a coin' as in Strategy II'?

4 Approaches to Equity and Public Risk

There is a considerable literature on equity and public risk. We focus on the strand begun by Keeney in the early 1980's. A recent survey of this is provided by [1]. Three issues in addition to the undesirability of health effects (fatalities within this literature) are explored as driving factors in decision making concerning risks to public health.

- *Ex ante equity* – the equity of the process and risks which eventually lead to the health effects;
- *Ex post equity* – the equity associated with the health effects that actually occur;
- *Dispersive equity* – the equity of the distribution of risk over *groups* within the population. One might also discuss *ex ante* and *ex post* dispersive equity.

One of the distinctions between *ex ante* and *ex post* equity is that the former will usually have to take account of complex correlation structures since individuals living close together, say, will tend to be either both exposed to the contamination or both unaffected. Equity issues will be of concern in all the decision making indicated in Figure 2.1; but in the early phase when decisions will focus on precautionary measures they are likely to be the most difficult, primarily because one moves from *ex ante* to *ex post* equity considerations and it is this move which seems to most discomfort decision makers.

However, we believe that, although the discussion in this literature sensitise us to issues vital to the design of RODOS, the utility and value models that are developed may not be directly applicable to the emergency management context. Firstly, in these discussions of equity there is an explicit assumption that the more uniformly a risk is shared the more equitable and, therefore, the more preferable it is. In emergency management this is not entirely true[2]. Consider the application of barrier nursing and enforced isolation to communities afflicted by acute, virulent epidemics, such as the Ebola epidemic which struck parts of Zaire in May 1995. This epidemic was treated by isolating the affected villages and thus protecting the rest of Zaire from infection. Those uninfected in the isolated communities bore a much higher risk of catching the disease for the greater good of the rest of the population of Zaire. Similar considerations can arise in nuclear accident emergency management. For instance, evacuation, while protecting from exposure,

[2] The Russian Roulette example in [2 ,p30 and pp372-74] is a relevant mind experiment to illustrate this point further.

carries its own risks. There may be road accidents. Evacuation is stressful and can cause stress-related health effects: e.g. coronaries. Suppose that there is a small village very near to a plant and a very large conurbation some kilometres away. Emergency managers might argue when there is a threat (but far from a certainty) of an imminent accidental release that to evacuate the village, thus announcing the threat publicly, may lead to panic in the nearby city, leading to much greater risks in total than those that threaten the village.

Secondly, in the literature on equity cited above the consequences of an event such as a nuclear accident are modelled by either a vector of 0's and 1's or a vector of probabilities. If there are N individuals in the population at risk then a consequence is modelled by $(x_1, x_2, ..., x_N)$, where $x_i = 0$ or 1 as the i^{th} individual lives or dies, or by $(p_1, p_2, ..., p_N)$ where p_i is the probability that he or she dies. This is a very concise way of representing the consequences which may not provide the decision makers with sufficient ways in which to articulate their preferences. There is a framing issue here and the pressure to produce concise, elegant mathematical models may have led to an oversimplification of the context. Consider the attribute hierarchies in Figure 1.1. In those, the attributes political and social/political acceptability pick up some of the issues to do with equity and they are articulated quite separately from the health effects attributes, which are based upon collective dose and hence, by the linearity hypothesis, on the probabilities of fatalities. Moreover, the Chernobyl attribute tree explicitly introduced acceptability to the rest of the USSR, an attribute which reflected the decision makers' judgements on the trade-offs between benefits to the affected community and the opportunity cost of countermeasures to the other parts of the USSR [10]. In other words, there is a dispersive equity consideration here between those bearing the risk and those who may have to bear increased risks from other sources (e.g. by foregoing development of improved medical facilities) to pay for the countermeasures. Real-life is far more complex than some of the discussions of equity have acknowledged to date.

Continuing the last point: models of *ex ante* equity preferences in, e.g., [8] have sought to model preferences between the processes that lead to the risks through the marginal probabilities to the affected individuals. This rationalistic assumption runs counter to empirical studies of public acceptability of risks. Risks from radioactivity are perceived as less acceptable than equal risks arising from other sources (see, e.g., [11], Appendix IV). This suggests again that, whatever the means used to introduce equity considerations into analysis, the consequences must be modelled more subtly than in the theoretical literature to date.

In modelling longer term decisions, attributes such as those related to public acceptability in Figure 1.1 seem to pick up equity issues and to do so to the satisfaction of the decision makers. However, in early decision making, particularly in relation to precautionary measures, this may not be so. The fact that *ex ante* equity issues will be discussed with hindsight in the event that a release does not occur whereas *ex post* equity issues will replace them in the event that it does has led us to suggest the use of event dependent attribute modelling. There are other reasons

- Overall
  - Health
    - Averted Dose
    - Stress
  - Public Acceptability
    - *Ex post* equity
  - Cost

Attribute tree in the event of a release

- Overall
  - Health
    - Stress
  - Public Acceptability
    - *Ex ante* equity
  - Cost

Attribute tree in the event of no release.

Figure 4.1: Possible event conditional attribute trees for decisions on precautionary measures

for this such as the legal requirement in several countries to conduct the analysis in terms of averted dose: when there is no accident there is no dose to avert. Thus we are investigating the use of two distinct attribute hierarchies and distinct multi-attribute utility models built upon them, one to apply to those consequences that may arise in the event that a release happens and the other to those in the event that it does not: see Figure 4.1 [4]. Note that the attributes in each hierarchy will differ in quality. For instance, stress in the event of a release will be long term and related to real radiation risks; in the event there is no release, stress will be short(er) term.

5 Elicitation Exercises

The above discussion raises many issues, but it does not solve them. Indeed, it is not for us as scientists and engineers to solve them. Their solution requires an expression of value judgements. That requires input at least from the political decision makers to articulate societal views and possibly from surveys of the public. The RODOS project team are running some emergency exercises designed to elicit such value judgements from the regional politicians and emergency managers. These exercises are being run in several countries and concern scenarios which extend from several hours before a potential accident to a day or so after its occurrence. Thus the issues of *ex ante* and *ex post* equity are confronted head on. Uncertainties in the scenarios allow for the possibilities that the plume may pass over relatively low population density rural regions or over densely populated urban regions. Since the feasibility of evacuating these different regions varies considerably issues of dispersive equity are also brought to the fore.

It is our expectation that these exercises will help us shape the design and knowledge base of RODOS in two ways. Firstly, we hope to build outline attribute hierarchies similar to those in Figure 1.1 and in Figure 4.1. Secondly, we also expect that some of the arguments relating to equity will be reflected not in the attribute hierarchy but in constraints applied to screen out countermeasure strategies which are so clearly unacceptable that further evaluation is unnecessary.

It is too early to draw any preliminary conclusions. Some exercises are still to be run; and those that have been run are still being analysed. Moreover, there are confidentiality matters to be resolved. What can be said now is that issues of equity are clearly very important in the thinking of the decision makers. We have seen debates between those who believe that equity means that all population groups exposed to the same level of risk should be protected by identical countermeasures and those who believe that all exposed to the same level of risk have a right to expect that all appropriate and feasible countermeasures will be applied to protect them. The former interpretation of equity implies that, if an urban region cannot be evacuated because of the numbers involved, then a village exposed to the same level of risk should not be evacuated either. The latter interpretation implies the reverse. In other words, the decision makers have debated how the concept of dispersive equity should be moderated by the feasibility of the available countermeasures.

6 Concluding Remarks

Nuclear accidents such as those at Chernobyl and Three Mile Island have given a practical focus to some of the more theoretical thinking that had been going on concerning equity and multi-criteria decision analysis. Within the RODOS project we are well aware of the complexity of these issues and also that the value judgements which are required are the responsibility of the politicians who will make the decisions and the societies that they represent. Thus we are organising some practical exercises to identify appropriate modelling frameworks in which to articulate such value judgements.

We recognise that the issues we have raised require wide discussion and we welcome comments and suggestions to help us shape RODOS further.

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THE EVOLVING ROLE OF MCDM IN RISK MANAGEMENT

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1 Introduction

The fundamental task that faces risk managers at all levels of decision making is the need, ability, and sometimes the courage, to make the required trade-offs among risks of various nature and among their associated costs and benefits. Quantifying multifarious risks and determining the acceptability level of each risk and its associated costs and benefits are the critical challenges at the heart of the Multiple Criteria Decision Making (MCDM) field.

The first challenges are to identify all sources of risk and uncertainty, to understand their interconnectedness and dependency on one another, and to distinguish among the multiple factors that characterize the nature and extent of what we generally term "risk" and "reliability." For example, the failure of a distributed system is commonly caused by the initial failure of one or two components, which then may trigger the failure of the system as a whole. Most available models assume independence in the failure of these components; a classic example of this is fault tree analysis (see for example Vesely et al. 1981). Yet, it is clear that the failure of a subsystem or a system's component may cause an unacceptable load on the remaining part of the system that ultimately might lead to overall failure.

Another challenge is to quantify these objectives. The extent of such quantification will largely determine the applicability of appropriate MCDM methodologies. Indeed, a systemic evaluation of these nonommensurate objectives and their associated trade-offs, and the subsequent formulation of meaningful and responsible policy options by managers and decision makers, depend on the successful quantification of the risks, costs, and benefits.

To better appreciate the evolving role of MCDM in risk assessment and management, we recall Einstein's statement that "so far as the theorems of mathematics are about reality, they are not certain; so far as they are certain, they are not about reality." Here we follow Einstein's thinking and adapt Heisenberg's principle to the two fields of risk and MCDM:

"To the extent risk assessment is precise, it is not real; to the extent risk assessment is real it is not precise."

"To the extent MCDM is precise, it is not real; to the extent MCDM is real, it is not precise."

For the purpose of this paper, the world of MCDM is divided into two separate, albeit somehow overlapping worlds -- the mathematical world and the decision-making world. The mathematical world, within which most MCDM analysts live -- is pure, precise, quantitative, compact, well-defined, and measured with cardinal metrics. The decision-making world is "less pure and more corrupt, uncertain, qualitative, fuzzy, ill-defined, and very difficult to measure with cardinal metrics; indeed, an ordinal scale is often used to communicate with decision makers. An attempt was made at bringing these two worlds together in the following risk based case studies, which are summarized in this paper:

1. Evaluation of Automobile Safety Features in a Multiobjective Risk Framework
2. Reliability Modeling for Water Distribution Rehabilitation
3. Reliability-Based Management of the Navigation System in the Upper Mississippi River
4. Holistic Risk Management for Software Acquisition

In general, however, MCDM analysts continue to and devote much of their time and energy to the safer mathematical world and still expect decision makers to use their results.

Acknowledging the existence of these two worlds, the need for the distinction between them, and the role that each one plays in the complex decision-making process, requires a change within the culture of the MCDM community. This call for a paradigm shift is based on the premise that both worlds are important -- they supplement, complement, and synergize each other. This mutual dependency is particularly significant in decision making under risk and uncertainty (the focus of this paper), since the probabilistic nature of most, if not all, real-world problem, requires the expertise of the analyst to quantify the risks involved.

This paradigm shift in culture also calls for educators in the MCDM field to reverse the emphasis placed in the classroom: from the current dominance of optimization and model solution to more emphasis on modeling and on synthesis and analysis. Shifting the focus from model optimization to systems modeling is justified on at least two important counts. First, the set of Pareto optimal solutions associated with a model can significantly change -- shrink or expand -- with the deletion or addition of even one objective function. Thus, the painstaking effort of the analyst during the optimization phase in the quest to determine all Pareto optimal solutions with precision is often wasteful. Second, under certain circumstances, mathematically inferior (dominated) solutions may be deemed desirable options by the decision makers. This is because no multiobjective model can truly represent the essence of the system being modeled. Of course, this argument becomes even stronger with multiple decision makers and with complex socio-economic systems.

One may also trace the evolving role of MCDM in risk management through the new developments that have taken place in risk-based methodologies. Decision trees [Raiffa 1968], which have been extensively used in the past as a potent methodological approach for decisions under risk (but with a single objective function) have been extended to handle multiple objectives and without commensurating all objectives into one utility [Haimes et al. 1990]. The same is true for extending single-objective dynamic programming to multiple-objectives dynamic programming [Chankong, Haimes and Gemperline 1981, and Li and Haimes 1989], and for extending single-objective discrete dynamic systems to multiobjective discrete dynamic systems [Li and Haimes 1990a, 1990b, 1991].

At this stage it is important to introduce the bridging element -- the shadow decision makers -- that often connect and harmonize the two worlds of MCDM and make interaction and working relations between them possible. Shadow decision makers invariably constitute the technical support staff of the decision makers. Although they are an integral part of the decision-making world, they are commonly versed with the quantitative, more precise dimension of the mathematical world of MCDM. They include the engineers, scientists, economists, political scientists, physicians, and social scientist, and others.

In general, the role of the shadow decision makers is more central and critical when complex and technical matters are dominant. In such cases, the real decision makers not only rely more heavily on the advice and the interpretations of their staffs, they often delegate more authority to them. This places a heavier burden, albeit a rewarding one, on the MCDM analyst for better communication to convince the shadow decision makers of the efficacy, resiliency, and sustainability of the proposed Pareto optimal policies from which preferred options are to be selected.

2 Evaluation of Automobile Safety Features in a Multiobjective Risk Framework

With over 168 million licensed drivers and 140 million registered automobiles on the roads in the United States today, the investigation of factors affecting automobile safety has taken on great significance [Federal Highway Administration, 1992]. In fact, the frequency and severity of motor vehicle accidents are of such consequence that they comprise the sixth leading cause of fatality in the United States and the number one cause of death due to injury [Federal Highway Traffic Safety Administration, (FHTSA) 1990].

The statistics in the FHTSA and in Blincoe and Falgan [1990] point to the necessity of focusing research energies on development and subsequent adoption of new automotive technologies to improve safety on highways in the United States and elsewhere. Before this can be done, however, the causes of vehicle accidents must be identified. Here we describe a framework for analyzing the interacting factors that contribute to automobile accident occurrence within a multiobjective framework.

The objective of this study (supported by the National Science Foundation and General Motors) was to examine the functional factors that contribute to automobile accident occurrence and to model the causation structure in the form of a fault-tree. A fault-tree model provides an intuitive qualitative framework as well as a quantitative framework for decomposing possible pathways to accident occurrence. Fault-tree analysis also provides a statistical representation of how interacting driver, vehicle, and environmental factors contribute to the likelihood of automobile accident occurrence. The application of this model facilitates pinpointing those factors that most contribute to accident causation and subsequently enables the identification and comparison of potential crash avoidance technologies [Kuzminski et al. 1995 and Eisele et al. 1995].

The frequency of highway accidents and their severities are influenced by a variety of factors, including a vehicle's dynamics and crashworthiness, environmental conditions, and driver factors. All of these interactive causal factors ultimately influence the onset and outcome of an accident. Improving highway safety through the introduction of crash-avoidance measures in vehicle design requires a thorough investigation of causation-consequence relationships, the availability of appropriate multiobjective decision-making methodologies, and optimal resource allocation. Such methodologies can evaluate the trade-offs between reducing the risk level of vehicle accidents and bearing the associated cost. Indeed, the ultimate quantification of the costs, risks, and benefits associated with the vehicle's design, reliability, and response to environmental and driver factors is a multiobjective optimization problem in nature, and is at the heart of the challenge facing the automotive industry today.

Using fault-tree logic, an extensive network of events leading to the occurrence of an automobile accident was developed. Environmental, driver, and vehicle failures and their interactions were all brought together to describe accident causation [Joshua and Garber 1992, Kuzminski et al. 1995, Eisele et al. 1995, Eisele 1994, Chowdhury 1994].

The probabilities of the basic events, and their logical relationships to an accident, were estimated and assessed using the 1991 Virginia Accident Database and exposure statistics gathered from expert assessment and various literature sources, including 1991 Department of Transportation data.

Select vehicle design modifications are evaluated in terms of their cost and driver-safety trade-offs using multiobjective decision analysis. For each prospective combination of redesign options, a framework is used which examines the minimization of four noncommensurate measures of harm and the minimization of vehicle production costs.

A general multiobjective decisionmaking framework utilizing the preferences of design engineers and other potential decision makers is used to evaluate a selected set of crash avoidance design modifications. These designs are evaluated for their mitigating influences on the contribution of driver factors to accident causation. Further, consideration of trade-offs with crashworthiness are considered, where crashworthiness is reported in terms of an expected accident injury severity distribution, an expected distribution of work days lost, a total body injury distribution, and an expected hospitalization stay distribution. The multiobjective framework used is based on the surrogate worth trade-off (SWT) method (Haimes and Hall 1974, and Chankong and Haimes 1983).

3 Reliability Modeling for Water Distribution Rehabilitation

The deterioration of the United States' water distribution systems has become a focal issue within the nation's overall infrastructure problem. At the same time, realizing the goal of maintaining large-scale water distribution systems at optimum standards is an extremely difficult and complex task. The complexity associated with the selection of the best replacement/repair strategy is due to the dynamic evolution of the failure modes of water mains, budget and other resource constraints, the large scale and intricacy of the problem, and competing and often conflicting demands placed by the various constituencies.

The overall goal of this project (supported by the National Science Foundation) was the development of effective operational management methodologies and procedures for large-scale public facilities -- facilities that are characterized by their hierarchical structure, their multiple objectives, and their elements of risk and uncertainty. In particular, special attention was paid to optimal maintenance-related decision making for deteriorating water distribution systems.

The various needs of systems engineering and operations research to improve the maintenance and rehabilitation processes of large-scale infrastructure in different fields were examined. A general framework of risk management [Haimes and Li 1991] was proposed which addressed the following research questions:

(a) How can we incorporate the hierarchical and multiobjective aspects of large scale public facilities into existing network-based methods?

(b) How can we incorporate the risk and uncertainty aspects of large-scale public facilities into operational management decisions through the use of conditional expectation and the statistics of extremes -- focusing on extreme and catastrophic events?

(c) How can we incorporate the impacts of present decisions upon the overall system at subsequent stages within a multiobjective risk-based decision-making framework?

A water distribution system often consists of a large number of components. However, in general resources are not available to completely update all the deteriorating water pipes and other system components. Limited resources have to be optimally distributed among the system's components in order to achieve the highest systems availability.

Deciding how to allocate limited resources among various system components (or subsystems) in order to yield maximum availability is generally a difficult task. The formulated nonlinear programming problem is hard to solve as a whole because of its high dimension. The mathematical model of this problem is also nonseparable in the sense of multilevel decomposition. This nonseparability is due to the equations of the overall system's reliability (availability), which must account for the reliabilities (availabilities) of all subsystems.

Theory and solution methodology were developed for large-scale nonseparable optimization problems using a three-level decomposition, where multiobjective optimization serves as a separation strategy [Li and Haimes 1990a, 1990b]. The method was applied both to problems of optimization of network reliability [Li and Haimes 1992a, 1992b, 1992c] and to optimization of network availability [Li, Dolezal, and Haimes 1993].

The nonseparable resource allocation problem for a deteriorating water distribution system was embedded into a separable, albeit multiobjective, resource allocation problem of a type easy to solve by a decomposition method. For the problem of the availability network, the optimal solution of the original nonseparable resource allocation problem was proven to be in the set of noninferior solutions of the corresponding multiobjective problem. One important product of this three-level decomposition approach is that it gives an importance measure for each system component, which provides the priority rank for the maintenance of various system components.

Impact analysis is an integral part of the decision-making process for the operational management of large-scale infrastructure. Short-term objectives should be balanced with long-term objectives.

In our research [Leach and Haimes 1987, Li and Haimes 1987 and Haimes and Li 1991], impact analysis was performed by way of a multiobjective multistage optimization model. Sensitivity of long-term objectives with respect to short-term constraints was derived to measure the future impact of current decisions. By providing useful information to decision makers, this methodology can be instrumental in avoiding adverse and irreversible consequences that might result from an unexamined preferred current decision.

The rehabilitation of an infrastructure system often leads to a sequential decision-making problem. The decision-tree method is a well-known approach that is suitable for solving such problems. The analysis of a decision tree aims at evaluating quantitatively and iteratively how well each of its decision sequences achieves its objective.

The multiobjective decision-tree method was developed in our research, which extends the traditional decision-tree method so that it incorporates multiple objectives and various risk measures [Haimes, Li, and Tulsiani 1990, and Li et al. 1992]. The significance of this extension for infrastructure systems is that the manager can be provided with details of the trade-off between different design alternatives. A second important benefit is that the impact of the adopted design option on different levels of risk can be separately evaluated.

In summary, the multiobjective reliability model developed for water distribution rehabilitation was responsive to and incorporated the following characteristics:

a. the water distribution network is complex b. there are various ways the system can "fail" c. each failure mode has an associated "reliability model" d. trade-off rehabilitation options and costs are at the heart of multiobjective analysis. More specifically, the reliability models determine the type and probability of failure and feed into a multiobjective decision-making model. The trade-offs are made among system unreliability and the costs of repair and rehabilitation.

4 Channel Reliability of the Navigation System in the Upper Mississippi River

The Upper Mississippi River originates out of Lake Itasca in Minnesota and flows generally southward, fed by several tributaries such as the Minnesota, St. Croix, Wisconsin, Rock, Des Moines, and Illinois Rivers. Just above St. Louis, the UMR meets the Missouri and then joins the Ohio River near Cairo, Illinois [Tweet 1983]. The first five hundred miles of the river downstream from Lake Itasca to Minneapolis is not navigable. The stretch from Minneapolis to Cairo, a distance of over eight hundred miles, forms the Upper Mississippi River Navigation System. The distance from St. Louis to Minnesota is navigable by means of a series of low navigation dams and associated locks which form a sequence of pools in the river. The stretch from St. Louis downstream forms the open river and is navigable without requiring any locks or dams.

The Corps of Engineers has been responsible for the construction, operation, and maintenance of the Upper Mississippi River navigation system for more than a century. The corps developed the original navigation system by making the Des Moines and the Rock Island rapids navigable. Moreover, the corps was in charge of the construction of the 4 1/2-foot channel, 6-foot channel, and 9-foot channel projects. Currently, the navigation channel on the Upper Mississippi River is mandated by the U.S. Congress to be 300 feet wide and nine feet deep.

Sedimentation to the navigation channel reduces the depth available for navigation. The Corps of Engineers maintains the required navigation standard through the use of structural measures, such as wing dams and closing dams, as well as through the use of maintenance dredging; however, these measures have an associated cost. In addition, there are several environmental concerns associated with the disposal of dredged material. Aside from these concerns, the physical deterioration of the various structures, including wing and closing dams, can also impact the need for dredging of the channel.

This summary shows that a multiobjective framework is necessary for an overall assessment that can eventually include uncertainties about the ecological impacts of navigation activities. Quantification of the navigation channel reliability is the first step in the development of a systematic framework for the management of the river navigation system that eventually includes examination of the trade-offs among costs, benefits, and reliability.

In order to develop this framework, we must first ask several questions: What exactly is the navigation channel reliability? Can we quantify it, that is, construct a reliability function for it? The answers to these questions will allow us to manage the navigation system effectively.

An important purpose of the reliability portion of the Upper Mississippi River navigation study (supported by the US Army Corps of Engineers) is to project general funding requirements to maintain the navigation system in the future. The objective was not to decide exactly which projects should be built -- a role which remains in the domain of professional engineering judgment, personal maintenance experience, and models of the physical processes involved -- nor to give accurate forecasts of needed resources in the short term. Rather, the role of the models described was to provide foundations upon which to quantify the benefits of increased rehabilitation funding for wing dams and closing dams on a system-wide basis and over a period of many years. Two reliability models for the navigation channel were developed -- one associated with the need for dredging of the *pool* and the other with the dredging of the *reach*. These models are complementary approaches to demonstrate the reduced-dredging benefits associated with rehabilitation of channelization structures [Tulsiani et al. 1995].

4.1 Dredge-Capacity Reliability Model

The dredge-capacity reliability model generates a probabilistic description of the annual dredge need for a given pool based on an assumed relationship between dredging and underlying features of the hydrograph. Underlying features that are considered include the total annual discharge through the pool, the number of hydrograph dips, and the number of flow peaks, where the hydrograph features are modeled as random variables. The dredge-capacity model also estimates a probability distribution of annual dredge need for the pool that is expected if *significant* rehabilitation is performed pool-wide. The two probability density functions, one of the unrehabilitated and one of the rehabilitated pool, are useful to characterize the variable cost of dredging the pool, or a system of pools that are similarly evaluated. A function relating the pool-dredging amount to the cost of dredging is required for this purpose.

In principle, one can use the probability function of dredge-need for the pool to evaluate the dredge-capacity reliability in the following two step process: (1) define the annual dredge capacity for the given pool; and (2) calculate the probability that the dredge need exceeds the capacity. In this study, a capacity-based approach was used to allow economists to distinguish and characterize a failure of the system as an exceedance of the normal operating budget for dredging.

An intermediate result from this model gives insight into the need for dredging on the pool-wide scale: it relates the annual dredge amount to the average daily discharge for the year (or versus some other extremal-oriented hydrograph feature). From this curve it is useful to study the impact of pool-wide rehabilitation on the relationship between dredging amount and the hydrograph reading in the year preceding dredging.

4.2 Reach Reliability Model

As a complement to the pool model described above, the evaluation of channel reliability can be extended down to the level of individual reaches to better understand and characterize the impact of channel sedimentation associated with individual structural rehabilitations. Thus, a reliability model that uses data on individual reaches as the statistical basis for an ideal characterization of sedimentation in the channel has been developed. This reach model can be used to generate a chart of the Upper Mississippi River on which the estimates of channel reliability are provided for all reaches together with the potential for improving the reach reliability by rehabilitating structures. Application of the reach model yields a general picture of the benefits of rehabilitation -- based on the identified significance of the parameters affecting sedimentation at the reach level -- but the model is not able to recommend projects at specific reaches. The amount of dredging is not considered in this model because it is assumed that set-up costs (between dredging events) dominate the cost differences attributable to dredging volume for a particular reach.

The inter-dredge reliability model describes the probability that in some time interval no dredging is required in a particular reach. The inter-dredge model also estimates the improvement in inter-dredge reliability to be expected if the reach is rehabilitated. It assumes that a reach can be characterized by a small set of parameters representing the channel morphology. A weighted sum of the parameter values, with weighting coefficients estimated from the real system, gives both the estimate of reliability and the expected improvement in reliability from rehabilitation. This ideal-process model of inter-dredge reliability generates the frequency of the need to dredge expected from an idealized reach-by-reach model. It is important to distinguish this from the observations of the real system, which are dredging records influenced by dredging policy shifts and other factors not related to the need to dredge.

4.3 Multiobjective Problem Formulation

The navigation channel management process is formulated as a multiobjective optimization problem. The objective is to maximize the reliability of the navigation channel while minimizing the total costs of maintaining and rehabilitating the channel. As previously noted, we also have the constraint of minimizing the environmental impacts of the disposal of the dredged material.

The reliability of the navigation channel can be improved in several alternative ways, such as [Tulsiani, 1995]:

(1) The infrastructure (wing dams and closing dams) can be improved. This can lead to reduced dredging requirements at the improvement locations, thereby changing the dredge demand distribution.

(2) The dredging capacity can be increased by buying more dredges or contracting for more private dredging, thereby changing the dredge capacity distribution.
(3) If environmentally safe disposal is a limiting constraint, then additional sites for safe disposal can be developed.

All three of these options have associated costs. They also have an associated risk in terms of the potential improvement in the channel reliability. The potential improvement in the dredge-capacity reliability is measured by evaluating various options for reducing the dredge volume and/or increasing the dredge capacity using the capacity-demand model. The potential improvement in the dredge frequency reliability is measured through a process of evaluating various options for rehabilitation of navigation structures and is represented by the dredge frequency model.

In summary, the multiobjective optimization problem is:

Objectives:	minimize cost
	maximize benefits
	maximize reliability
Decision options:	structural rehabilitation (reduces demand)
	construction of new structures (reduces demand)
	enhanced dredging capacity (adds capacity)
	add dredge disposal capacity (adds capacity)
Constraints:	resources (funding level)
	dredge availability
	environmental concerns

5 Holistic Risk Management for Software Acquisition

The ability to quantify risk is essential to the processes of budgeting and scheduling. During the process of hiring to complete specified tasks, customers must be able to verify contractor estimates and to make sound judgments on the risks of cost overruns and time delays. The following questions are central: Do developers with little experience over-estimate or underestimate the complexity of the task because of their past experience, the assumptions they make, the models they select, and how they define the model parameters? What are the sources of risk associated with project cost estimation? How can such risk be quantified? To address these questions in relation to software development, this discussion proposes a systematic acquisition process that is aimed at assessing and managing the risks of associated cost overruns and time delays.

A proposed acquisition process is grounded on four basic premises: a) Any single-value estimate is inadequate to capture and represent the variability and uncertainty associated with cost or completion time. One way to achieve probabilistic quantification is through the use of the fractile method and triangular distribution. b) The common expected value is inadequate when used as a measure of risk ; further, if used as the sole measure of risk, it may lead to inaccurate results. The conditional expected value of risk of extreme events is adopted to supplement and complement the common unconditional expected value [Asabeck and Haimes 1984]. c) Probing the sources of risks and uncertainties associated with cost overruns and time delays in software development is essential for the ultimate management of technical and nontechnical risks. d) To evaluate the trade-offs among all the risks, costs, and benefits, one must adhere to the principles of multiobjective optimization.

A methodological framework for selecting a contractor that can assist the customer in minimizing the risks of project cost overruns and schedule delays was developed [Haimes and Chittister 1995]. Although factors other than the selection of contractor(s) may decisively affect both software technical and nontechnical risks, they are treated here only as a general background; the interested reader is referred to Chittister and Haimes [1993, 1994] for a more in-depth discussion of these factors.

The process of selecting contractors is by itself quite complex; it is driven by legal, organizational, technical, financial, and other considerations--all of which serve as sources of risk. Because the world within which software engineering is developed is non-deterministic, and because the central tendency measure of random events (i.e., the expected value of software nontechnical risk) conceals vital and critical information about these random events, special attention is focused on the variance of these events and on their extremes. Two approaches--the fractile method and triangular distribution--are adopted in this study to quantify the probabilities of project cost overrun and delay in schedule completion. To capture the range of variation and the extremes of these probabilities, conditional expected values of extreme events are calculated using the partitioned multiobjective risk method (PMRM) [Asbeck & Haimes 1984] to supplement the common expected value of software nontechnical risk.

The ultimate objective of the methodological approach is to minimize the following three objectives or indices of performance:

$$\text{minimize} \left\{ \begin{array}{l} \text{Risk of project cost overrun} \\ \text{Risk of project completion time delay} \\ \text{Risk of not meeting performance criteria} \end{array} \right\}$$

Multiobjective trade off analysis, using, for example, the surrogate worth trade off (SWT) method, is conducted where all costs and risks are kept and traded off in their own units.

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Integer Goal Programming Model for Nursing Scheduling: A Case Study

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Abstract. No shortage of one nurse allowed in any shift since there is no nursing agent in Taiwan. The reallocation of one nurse in some unit to another unit is not usual. Furthermore, nurses are used to request a certain shift or a day-off on some certain days. The workload for each shift is determined by the hospital and the Department of Health. The remaining job for a head nurse is to assign a specific shift assignment to the nurses in her/his own unit. A model is proposed to help a head nurse for the above nursing scheduling problem. Firstly, the nurses in a unit split into several groups. Secondly, the nurses are assigned their specific shift assignment/day-off in a two-week palnning horizon by an integer goal programming (GP) model.

Keywords. Integer goal programming, heuristic, nursing scheduling.

I. Introduction

Since there is no nursing agent in Taiwan, no shortage of one nurse is allowed in any shift of a unit of a hospital. The reallocation of one nurse in some unit to another unit is unusual due to the particular specialization of the nurses on a unit. Hence, there is no such a pool of float nurses to account for insufficient manpower due to high turnover rate of nurses in Taiwan. And, nurses are used to request a certain shift or a day-off on some certain days. According to the regulations of Department of Health in Taiwan, the minimum daily manpower is determined for each unit in a hospital. The minimum daily manpower is divided among three shifts in a unit by the rules of each hospital. The remaining job for a head nurse is to assign a specific shift assignment to all nurses to satisfy the required workload with the available nurses in her/his own unit. Most hospitals in Taiwan have a head nurse to schedule the nurses manually. For a staff size of 20 nurses, it takes the head nurse 6 to 8 working hours to schedule the nurses manually for a 2-week planning period in the studied Christian Hospital, which is a local teaching hospital with about 400 beds. In order to save the expensive labor cost of head nurses and to improve the satisfaction of nurses about their shift assignments, an algorithm is proposed to help the head nurses to solve the nursing scheduling problem efficiently.

This paper formulates the problem and solves it in two stages. Firstly, the nurses in a unit split into several sub-groups with certain workload for each sub-group. Secondly, the nurses get their specific shift assignments/ day-off using an integer GP model for a two-week planning period.

2. Literature Review

There are three main approaches in scheduling nursing personnel: (1) cyclical schedules, (2) use of mathematical programming models, and (3) other approaches.

Cyclical scheduling was discussed by Howell [3], Smith [13], Megeath [8], and Hung [4].

Warner and Prawda [17] use a mixed-integer quadratic programming problem to specify the number of nursing staff of each skill class to be assigned among the units and nursing shifts of the planning period. Liebman et al [7] allocate the nursing staff in an extended care facility using 0-1 integer programming (IP) models. Miller et al [9] schedules days on and days off for all nurses on a unit for a given shift for a two-week to eight-week planning horizon using mathematical programming model. Trivedi and Warner [15] propose a branch and bound algorithm for optimum allocation of float nurses. Warner [16] describes a nurse scheduling system as a large multiple-choice program-ming problem with the consideration of individual nurses' preferences. Arthur and Ravindran [1] propose a 0-1 GP model to determine the days on and days off for all nurses with four goals and determine the shift assignment for each individual nurse by a heuristic rule. Bailey [2] develops an integrated days off and shift personnel scheduling model and presents a heuristic rule to allocate required shifts to the days-off patterns. Ozkarahan and Bailey [10] modify Bailey's [2] work as a GP model to solve the nursing scheduling problem for St. Luke Medi-Medical Center. Kostreva and Jennings [6] solve the large scale IP problem by generating limited number of possible schedules.

Beside the cyclical scheduling and mathematical programming, following are some other approaches for nursing scheduling. Smith and Wiggins [14] adopt a heuristic approach utilizing list processing and problem oriented data structures to develop a computer-based nurse scheduling system. Rosenbloom and Goertzen [12] pre- sent an IP model for scheduling days off and days on problem cyclically. Kostreva and Genevier [5] advocate nurse scheduling with considerations of circadian rhythms. Randhawa and Sitompul [11] build a heuristic-based decision support system for developing weekly work and shift patterns and for combining these patterns into nurse schedules.

3. An Integer Goal Programming Model

There is no regulation by law to limit the working pattern for nurses in Taiwan. The following model is developed in collaboration with the studied Christian Hospital. Therefore, certain policies of this hospital are included in the model. The regular salaries of nurses are paid monthly; the annually cumulative overworking days will be paid to the nurses at the end of each year. The workload requirements of each unit must be satisfied by the available nurses in the unit. The scheduling rules of this hospital are stated as follows:

(1) The planning period is two weeks.
(2) Consecutive working days must not exceed 6 days.
(3) The request of Saturday, Sunday or legal holiday is not allowed. They should take the weekends and holidays by turn.
(4) At least one nurse of long experience should be on duty for each shift everyday; no substitution between nurses of long experience and the other nurses; no substitution among experienced nurses, nurses under training (the training period for a new nurse is one month) and nurse aides.
(5) A nurse at most works one shift everyday.
(6) If a nurse chooses a fixed shift of evening shift or night shift, she/he is not allowed to request another shift in the planning period.
(7) If a nurse works on a night shift, she/he must not take a day shift or an evening shift in the consecutive day. If a nurse works on an evening shift, she/he prefer not to take a day shift in the next day.
(8) The nurse assigned to be on duty for Nursing Department is still required to share the workload of evening and night shifts.
(9) If there are several nurses requesting the same working shift at the same day, the nurse with fewer cumulative number of requests has higher priority.
(10) Those nurses who overwork have higher priority to have more days off (not including weekends and holiday). If there is no othernurse to request the same day off, the request of days off could be approved for those nurses who overtake holidays.
(11) All the workloads should be shared by those nurses without a fixed shif.
(12) The standard number of holidays is three for a two-week period. Those nurses who overwork have lower priority to overwork and higher priority to get more than three days off during a two-week planning period; those nurses who overtake holidays should have higher priority to take less than three days off.

The three goals of the nurse scheduling of this hospital include: Priority (1): If a nurse works on an evening shift, the nurse prefer not to take a day shift in the next day. Priority (2): Minimize the number of staff working daily. Priority (3): Satisfy the rules (9), (10), and (12) simultaneously.

3.1 Sub-grouping

Let $X_{ijt} = 1$ if nurse i takes the shift j on day t,
$\quad\quad\quad\;\; 0$ otherwise.

where

$i = 1, 2, \ldots, m$, m denotes the number of nurses in the sub-group,
$j = D, E, N, R$, D, E, N, and R denote the working shift day, evening, night and off,
$t = 1, 2, \ldots, 14$, there are 14 days in a 2-week planning horizon.

Fifty-six 0-1 variables are defined for each nurse. There are usually 15 to 25 nurses in a unit in the studied Christian Hospital. The integer GP model will be a quite large problem with decision variables ranging from 840 to 1400 zero-one variables. It will take excessive time to schedule a unit with 15 to 25 nurses. Hence, the size of integer GP model is unacceptable. Because the scheduling rule (10), the nurses can be separated into several sub-groups by allocating the workload for each sub-group as evenly as possible.

The sub-grouping is made by a straightforward heuristic procedure. The size of a sub-group may range from 6 to 8. For example, the total number of nurses in a unit is 15, there will be two sub-groups, one with 7 nurses and the other with 8 nurses. Those nurses with long experience will be assigned to the same group as much as possible to satisfy the scheduling rule (4). Assume the normal working days for the nurses with a fixed shift is 9 to 11 days for a two-week planning period. Subtract the total working days of nurses with a fixed shift from total workload of the unit, the result is the subtotal working days for nurses without a fixed shift. Then, the subtotal workload is divided by nurses without a fixed shift. Then, the workload for each sub-group is the sum of workload of each nurse in the sub-group.

3.2 The Integer GP Scheduling Model

Notation

D_t (E_t, N_t) = manpower required for day (evening, night) shift on day t
I_1 (I_2) = a set of nurses who work on an evening (night) shift only
I_3 = a set of nurses who are on duty for Nursing Department
I_4 = a set of nurses who are not in any set of I_1, I_2, or I_3 in the sub-group
I_a = a set of registered nurses (RN) where every RN is equivalent to one manpower
I_b = a set of RN's under training where every RN is equivalent to half manpower
I_c = a set of nurse aides (NA) where every NA is equivalent to half manpower
T_1 = a set of days on which the nurse is on duty for Nursing Department.
S_{J1k} (S_{J2k}) = the minimum (maximum) working days on shift J (J= D, E, N) for the nurses in I_k (k=1, ..., 4)

An integer goal programming model is presented to find a set of assignments for all the nurses in a sub-group. There are five types of strict constraints.

1. According to rule (2): $\sum_{t=k}^{k+6} x_{iRt} \geq 1$ for $k = 1, \ldots, 8$ and $i = 1, \ldots, m$
2. According to rules (5) and (6): $X_{ijt} = 1$ for $i \in I_3$ and $j = E$ or N and $t \in T_1$

$$X_{iEt} + X_{iRt} = 1 \qquad \text{for } i \in I_1 \text{ and } \forall t$$
$$X_{iNt} + X_{iRt} = 1 \qquad \text{for } i \in I_2 \text{ and } \forall t$$
$$X_{iDt} + X_{iEt} + X_{iNt} + X_{iRt} = 1 \qquad \text{for } (i \in I_4 \text{ and } \forall t) \text{ and } (i \in I_3 \text{ and } t \notin T_1)$$

3. According to rule (7): $X_{iNt} + X_{iD(t+1)} + X_{iE(t+1)} \leq 1$ for $i \in I_3$ & I_4 and $\forall t$

4. According to rule (11): $\sum_{t=1}^{14} X_{iJt} \leq S_{J2k}$ and $\sum_{t=1}^{14} X_{iJt} \geq S_{J1k}$ for $i \in I_1, I_2, I_3,$ & $I_4 \cap (I_a \cup I_c)$ and $k = 1, \ldots, 4$ and $J = D, E,$ and N

5. On the 1st and 2nd training week, nurses under training only work day shift. At the 3rd week, the evening shift is included. And, at the 4th week, the night shift is included. The minimum working days on evening (night) shift is 3 days at the 3rd (4th) training week. For example, if the planning period is on the 3rd and 4th training week.

$$\sum_{t=1}^{7} X_{iEt} \leq 4 \text{ and } \sum_{t=1}^{7} X_{iEt} \geq 3 \text{ and } \sum_{t=8}^{14} X_{iEt} \leq 4 \text{ and } \sum_{t=8}^{14} X_{iNt} \geq 3 \qquad \text{for } i \in I_b$$

There are three types of goal constraints.

6. According to the goal with the first priority:

$$X_{iEt} + X_{iD(t+1)} + d_{it}^- - d_{it}^+ = 1 \text{ for } i \in I_3 \text{ and } I_4 \text{ and } \forall t$$

7. Since the shortage of nurses in any shift of any day is not allowed in Taiwan, the negative deviations are not added in the following second and third types of soft-constraints. According to the goal with the second priority, the manpower of those nurses in set I_b and I_c is 0.5 full time equivalent (FTE).

$$\sum_{i \in Ia} X_{iDt} + 0.5 \sum_{i \in Ib} X_{iDt} + 0.5 \sum_{i \in Ic} X_{iDt} = Dt \qquad \text{for } \forall t$$

$$\sum_{i \in Ia} X_{iEt} + 0.5 \sum_{i \in Ib} X_{iEt} + 0.5 \sum_{i \in Ic} X_{iEt} - de_t^+ = Et \qquad \text{for } \forall t$$

$$\sum_{i \in Ia} X_{iNt} + 0.5 \sum_{i \in Ib} X_{iNt} + 0.5 \sum_{i \in Ic} X_{iNt} - dn_t^+ = Nt \qquad \text{for } \forall t$$

8. According to the third goal: $\sum_{t=1}^{14} X_{iRt} + dr_i^- - dr_i^+ = 3$ for $i = 1, \ldots, m$

Let O_i^+ denote the number of overworked days of nurse i, and O_i^- denote the number of overtaken holidays of nurse i. Let $H_i < 0$ and $H_i' < 0$ denote the coefficient for nurse i to decide the days off on Saturday and holiday (including Sunday). Let $S_i < 0$ denote the coefficient for nurse i to decide the acceptance of special requests on certain shift on some certain days. Let $MO^- = \text{maximum}\{O_i^-\}$. Let $Q_i = -O_i^+ - MO^- - 1$ for nurse i overworked and $Q_i = O_i^- - MO^- - 1$ for nurse i under-worked. Then, $Q_i < 0$ denote the coefficient to decide the acceptance of the day-off by request on weekdays. Considering the three goals, the objective function is to minimize the total cost:

$$\text{Min } P_1 \sum_{i=1}^{m} \sum_{t=1}^{13} d_{it}^{+} + P_2\left(\sum_{t=1}^{14} d_{et}^{+} + \sum_{t=1}^{14} d_{nt}^{+}\right) + P_3\left(\sum_{i=1}^{m} O_i dr_i^{+} + \sum_{i=1}^{m} -O_i^{+}(dr_i^{+} - dr_i^{-}) +\right.$$

$$\left.\sum_{i=1}^{m} \sum_{t=1}^{14} Q_i X_{iRt} + \sum_{i=1}^{m} \left(H_i(X_{iR7} + X_{iR14}) + H_i'(X_{iR1} + X_{iR8})\right) + \sum_{i=1}^{m} \sum_{t=1}^{14} S_i(X_{iDt} + X_{iEt} + X_{iNt})\right)$$

4. An Example

Consider the problem of scheduling 15 RN's including one head nurse and eight of RN's with long experience, 5 RN's at their 3rd and 4th training week in this planning period, and 3 NA's in a specific ward. The head nurse has a fixed working pattern with day shift from Monday to Friday of 8 hours and Saturday of 4 hours. The manpower required for the remaining nurses is: 5 FTE of day shift, 4 FTE of evening shift, and 3 FTE of night shift on each day, except 5.5 FTE of day shift on Saturday. Hence, $D_t = 5$ for t = 1, ..., 6 and t = 8, ..., 13, $D_t = 5.5$ for t = 7 and 14, $E_t = 4$ for all t, and $N_t = 3$ for all t. Notice that, the second Monday is a holiday for this case. I_1 = {#5 for t = 1, ..., 7 and #8 for t = 8, ..., 14}. The integer GP model is initially designed for a fixed shift on a two-week planning period. But, this real case complicates the modeling since nurses #5 and #8 work on a fixed evening shift only for one week. I_2 = {#2, #6, #9, #20}. I_3 = {#3}. I_4 = {#1, #4, #7, #10, #11, ..., #19, #21, #22}. I_a = {#1, #2, ..., #14}. I_b = {#15, #16, ..., #19}. I_c = {#20, #21, #22}. Hence, the total number of nurses, m, is 22. Nurse #1 to #8 are nurses with long experience. Nurse #3 is required to take evening shift on t = 4 and night shift on t = 1, 13, and 14. Hence, T_1 = {1, 3, 13, 14}.

The values of S_{N12} and S_{N22} in a two-week planning period for the nurses in set I_2 are 9 and 11. Hence, only two constraints of type (5) are required for each nurse in set I_2. For the nurse #3 in set I_3, the values of S_{D13}, S_{D23}, S_{E13}, S_{E23}, S_{N13}, and S_{N23} in a two-week planning period are 2, 4, 2, 4, 3, and 4. The values of S_{D14}, S_{D24}, S_{E14}, S_{E24}, S_{N14}, and S_{N24} in a two-week planning period for the nurses in set $I_4 \cap (I_a \cup I_c)$ are 5, 7, 3, 5, 0, and 1. For the nurse #5 in set I_1, the values of S_{E11} and S_{E21} in the first week are 4 and 6; the values of S_{D11}, S_{D21}, S_{E11}, S_{E21}, S_{N11}, and S_{N21} in the second week are 3, 4, 0, 1, 0, and 1. For the nurse #8 in set I_1, the values of S_{D11}, S_{D21}, S_{E11}, S_{E21}, S_{N11}, and S_{N21} in the first week are 3, 4, 0, 1, 0, and 1; the values of S_{E11} and S_{E21} in the second week are 4 and 6. For the nurses in set $I_4 \cap I_b$, the values of S_{D14} and S_{D24} in a two-week planning period are 2 and 4; the values of S_{E14} and S_{E24} in the first week are 3 and 4 (the variables of X_{iNt} are deleted for $i \in I_4 \cap I_b$ and t = 1, ..., 7); the values of S_{E14}, S_{E24}, S_{N14}, and S_{N24} in the second week are 0, 3, 3, and 4. And, the values of - O_i^+, O_i^-, H_i, H_i', S_i, and Q_i for each nurse # i are listed in Table 1.

At the sub-grouping stage, the 22 nurses split into 3 sub-groups, say Group A, B, and C. Group A contains nurse #1, #2, ..., #7. Group B contains nurse #8, #9, ..., #14. Group C contains nurse #15, #16, ..., #22. The required manpower on each shift for each sub-group is listed in Table 2. The group A is used for illustrating the integer GP model.

1. $\sum_{t=k}^{k+6} X_{iRt} \geq 1$ for k = 1, ..., 8 and i = 1, ..., 7
2. $X_{iNt} + X_{iRt} = 1$ for i = 2, 6 and $\forall$t
   $X_{iEt} + X_{iRt} = 1$ for i = 5 and t = 1, ..., 7
   $X_{iDt}+X_{iEt}+X_{iNt}+X_{iRt} = 1$ for (i = 5 and t = 8, ..., 14) & (i = 1, 4, 7 and $\forall$t) & (i = 3 and t = 2, 3, 5, 6, ..., 12)
   $X_{3N1} = 1, X_{3E4} = 1, X_{3N13} = 1, X_{3N14} = 1$
3. $X_{iNt} + X_{iD(t+1)} + X_{iE(t+1)} \leq 1$ for (i = 1, 3, 4, 7 and $\forall$t) & (i = 5 and t = 7, ..., 13)
4. $5 \leq \sum_{t=1}^{14} X_{iDt} \leq 7,\ 3 \leq \sum_{t=1}^{14} X_{iEt} \leq 5,\ \sum_{t=1}^{14} X_{iNt} \leq 1$ for i = 1, 4, 7

   $9 \leq \sum_{t=1}^{14} X_{iNt} \leq 11$ for i = 2, 6

   $2 \leq \sum_{t=1}^{14} X_{iDt} \leq 4,\ 2 \leq \sum_{t=1}^{14} X_{iEt} \leq 4,\ 3 \leq \sum_{t=1}^{14} X_{iNt} \leq 4$ for i = 3

   $4 \leq \sum_{t=1}^{7} X_{iEt} \leq 6,\ 3 \leq \sum_{t=8}^{14} X_{iDt} \leq 4,\ \sum_{t=8}^{14} X_{iEt} \leq 1,\ \sum_{t=8}^{14} X_{iNt} \leq 1$ for i = 5

6. $X_{iEt} + X_{iD(t+1)} + d_{it}^- - d_{it}^+ = 1$ for (i = 1, 3, 4, 7 and t = 1, ..., 13) & (i = 5 and t = 7, ..., 13)

Since there is no nurses under training in sub-group A, the variables d_e^+ and d_n^+ in constraint (7) are dropped.

$\sum_{i=1,3,4,7} X_{iDt}$ = 1 (= 2) for t = 1, 12, 13 (for t = 2, 3, ..., 11, 14)

$\sum_{i=1,3,4,5,7} X_{iEt}$ = 1 (= 2) for t = 1, 2, 3, 5, 6, 7, 11, 12, 13, 14 (for t = 4, 8, 9, 10)

$\sum_{i=1,2,3,4,6,7} X_{iNt}$ = 3 (= 2; = 1) for t = 1 (for t = 2, 3, ..., 6; for t = 7)

$$\sum_{i=1}^{7} X_{iNt} = 1\ (= 2;\ = 3) \quad \text{for } t= 8,9,10,11 \text{ (for } t= 12, 14; \text{ for } t = 13)$$

8. $\sum_{t=1}^{14} X_{iRt} + dr_i^- - dr_i^+ = 3 \qquad \text{for } i = 1, \ldots, 7$

Table 1 List of values of some cost coefficients.

Nurse#	$-O_i^+ / O_i^-$	H_i	H_i'	S_i	Q_i	Nurse#	$-O_i^+ / O_i^-$	H_i	H_i'	S_i	Q_i
#1	2.31	-5	-5	-4	-32.19	#2	8.43	-4	-4	-3	-26.07
#3	2.13	-6	-4	-6	-32.37	#4	10.44	-5	-6	-4	-24.06
#5	3.06	-4	-5	-1	-31.44	#6	8.25	-6	-6	-4	-26.25
#7	1.00	-4	-6	-1	-33.5	#8	8.44	-5	-3	-2	-26.06
#9	6.94	-4	-4	-6	-27.56	#10	2.81	-4	-4	-2	-31.69
#11	2.13	-5	-4	-5	-32.37	#12	-0.25	-6	-6	-4	-34.75
#13	5.38	-6	-5	-3	-29.12	#14	5.38	-4	-4	-3	-29.12
#15	1.13	-6	-3	-6	-33.37	#16	1.13	-6	-3	-6	-33.37
#17	2.13	-6	-2	-6	-32.37	#18	-2.13	-6	-2	-6	-36.63
#19	1.38	-6	-4	-6	-33.12	#20	33.5	-4	-5	-3	-1
#21	8.88	-5	-5	-5	-25.62	#22	2.63	-5	-4	-6	-31.87

Table 2 Manpower requirements for each sub-group.

Grp	Sun	Mon	Tue	Wen	Thu	Fri	Sat	Sun	Mon	Tue	Wen	Thu	Fri	Sat
A/D[a]	1	2	2	2	2	2	2	2	2	2	2	1	1	2
A/E	1	1	1	2[b]	1	1	1	2	2	2	1	1	1	1
A/N	3[b]	2	2	2	2	2	1	1	1	1	1	2	3[b]	2[b]
B/D	3	2	2	2	2	2	3	2	2	2	2	3	3	3
B/E	1	2	2	2	2	1	1	1	2	2	2	2	2	2
B/N	1	1	1	1	1	1	2	1	1	1	1	0	0	0
C/D	1	1	1	1	1	1	0.5	1	1	1	1	1	1	0.5
C/E	2	1	1	1	1	2	2	1	0	0	1	1	1	1
C/N	0	0	0	0	0	0	0	1	1	1	1	1	1	2

a: A/D represent the day shift for sub-group A.

b: indicates the certain shift on certain day for nurse #3.

Since the variables d_e^+ and d_n^+ in constraint (7) are dropped, the second goal is dropped. The special requests include: nurse #1 requests days off on t = 12, nurse # 4 requests days off on t = 4 and 11, and nurse # 1 requests an evening shift on t = 7. Referring to Table 1 for the values of O_i^-, H_i, and H_i', the objective function is to minimize the total cost:

$$P_1\Big(\sum_{i=1,3,4,7} \sum_{t=1}^{13} d_{it}^{+} + \sum_{i=5} \sum_{t=7}^{13} d_{it}\Big) + P_3\Big(\sum_{i=1}^{7} O_i^- dr_i^+ + \sum_{i=1}^{7} \big(H_i(X_{iR7}+X_{iR14}) + H_i'(X_{iR1}+X_{iR8}+X_{iR9})\big) - X_{7E7} - 32.19X_{1R12} - 24.06X_{4R4} - 24.06X_{4R11}\Big)$$

Table 3. The assignments for each nurse*.

Nurse	**	Sun	Mon	Tue	Wen	Thu	Fri	Sat	Sun	Mon	Tue	Wen	Thu	Fri	Sat
HN	-	R	D	D	D	D	D	D	R	R	D	D	D	D	D
#1	7D	R	R	D	D	R	D	D	E	E	R	D	R^{R}	E	R
#2	1N	N	N	R	N	N	N	N	N	N	R	R	N	N	N
#3	5N	N	R	D	E	R	R	D	D	D	E	E	R	N	N
#4	1D	D	D	N	R^{R}	D	D	R	E	E	E	R^{R}	D	D	D
#5	0R	E	E	E	E	E	E	R	D	D	D	E	R	R	D
#6	0R	N	N	N	N	N	N	R	R	R	N	N	N	N	R
#7	0R	R	D	R	D	D	R	E^{E}	R	R	D	D	E	R	E
#8	3E	N	R	D	D	E	R	D	E	E	E	R	E	E	E
#9	6N	R	N	N	N	N	R	N	N	N	N	N	R	R	R
#10	1E	E	E	R	R	E	R	D	D	R	D	D	D^{D}	R	D
#11	2E	D	D	R	R	D	E	E	R	E^{E}	R	R	D	D	D
#12	7D	R	E	E	E	R	D	D	R	R	R	D	D	D	D
#13	5D	D	R	D	D	D	N	R	D^{D}	D	E^{E}	E	E	E	R
#14	2D	D	D	E	E	R	D	N	R	D	D	E	R	D	E
#15	2D	E	R	E	E	R	R	R	D	R	N	N	N	R^{R}	D
#16	2D	D	D	R	R	E	E	E	E	N	N	R	E	E	N
#17	0R	E	R	D	D	R	E	E	R	D^{D}	D	E	N	N	N
#18	0R	E	R	R	E	R	E	R	N	N	R	D	D	E^{E}	N
#19	0R	R	E	R	D	E	E	E	N	N	R^{R}	D	D	D^{D}	N
#20	7N	R	N	N	N	N	N	R	R	R	N	N	N	N	R
#21	1E	E	E	E	R	D	D	D	D	D	R^{R}	R^{R}	R^{R}	D^{D}	E
#22	3D	D	D	D	R	D	D	E	E	R	D	E	E	R^{R}	E

***: D represents day shift, E represents evening shift, N represents night shift, R represents the day off.**
**: indicates the number of consecutive working days and shift on last day during the previous planning period.

The number of consecutive working days and the working shift on last day in the previous period are also considered, as shown in Table 3. The illustrated problem is solved using LINDO/v 5.0 on PC/486. Instead of sequentially using LINDO three times for solving each sub-problem, P_1 is set to 9999, P_2 is 99, and P_3 is 1. Hence, each sub-problem can be solved using LINDO only one time. Since the objective of the model is to find a quite good solution for nurse scheduling, the IPTOL (the tolerance to optimal solution of corresponding linear problem) is set to 0.002. It takes 3 to 5 minutes to solve the integer GP model for sub-group A and B. It takes about 20 minutes to solve the problem for sub-group C. Table 3 gives the results of the nurse scheduling. In Table 3, all the superscript characters indicate the requests from nurses, and all the requests are satisfied.

5. Conclusions

An integer goal programming model is proposed to assign the working shift of each day for the nurses in a 2-week planning period. It is time-consuming to solve the integer GP problem for scheduling 22 nurses with 1232 zero-one decision variables. Therefore, the 22 nurses in a unit are separated into 3 sub-groups. The nurse scheduling is determined by solving the three sub-groups separately with much fewer time. The algorithm of sub-grouping is the key factor affecting the efficiency of the above integer GP model. How many nurses should form a sub-group and how to assign the workload on each shift on each day should be the next step to improve the efficiency of the above integer GP model.

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Multiple Criteria Vendor Selection: A Case Study

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Abstract: In this article, vendor selection in the hydraulic pump gear division of a manufacturing company is explained. Division wants to find appropriate vendors and amount to buy from them while minimizing cost and maximizing quality and delivery reliability. Visual Interactive Goal Programming (VIG) is used as a decision support system. Interaction with decision makers during model building and solution process is discussed.

1 Introduction

The primary considerations in vendor selection research have been the criteria used in evaluating vendor performance and the decision models used in vendor selection. Decision models are used to quantify vendor performance with respect to relevant criteria and to select vendors based on their performance in the critical areas.

While price, quality, technical service, delivery reliability, and delivery lead time are the five primary criteria used in vendor selection, recent changes in manufacturing technology have introduced new factors that affect buyer-supplier relations. Lyons, Krachenberg and Henke (1990) recognize five strategic moves by original equipment manufacturers (OEMs) that necessitate a re-examination of buyer-supplier relations. OEMs are moving toward more cross functional team decision making, supply base rationalization, longer-term contracts and relationships, more outsourcing of professional and staff functions and increased acquisition of components and subassemblies rather than individual parts. Supply base rationalization reduces the total number of suppliers a firm deals with, resulting in an increased reliance on the remaining suppliers. This move reinforces the need for accurate vendor selection methods.

Deciding what criteria are useful in evaluating vendor performance is important. However, choosing a decision method that uses the criteria to perform the actual vendor analysis is also critical to the vendor selection process. Weber, Current and Benton (1991) group quantitative approaches into three categories. These are weighted point models, statistical/probabilistic models and mathematical programming models. Buffa & Jackson (1983) and Chaudhry, et al. (1991) use a conventional pre-emptive goal programming to allocate order quantity between vendors. Goal programming takes vendor selection a step further than the traditional methods. However, it does have some limitations.

We used Visual Interactive Goal Programming (VIG) for our vendor selection problem. VIG alleviates most of the problems decision making teams might face in using conventional goal programming. It has been implemented in a variety of problems such as pricing decisions, input-output models for emergency management, and media selection (Kananen, et al., 1990, Korhonen & Wallenius, 1989). However, it has not yet been used in purchasing decisions. The availability of simple to use personal computer software should encourage the use of this decision support system by purchasing managers or teams (it certainly encouraged our team.).

Paper consists of seven sections. Introduction explains the purpose of the study. Visual Interactive Goal Programming approach to vendor selection problems is presented next. Supplier selection procedure of the company is discussed next. Section four describes purchasing process of the division. Model formulation both for academicians and practitioners follows system description. Results and interactions with the decision makers are summarized in section six. Paper concludes with a discussion of the implementation problems in using MCDM techniques.

2 Visual Interactive Goal Programming Approach to Vendor Selection

Pareto Race is described in detail in Korhonen and Wallenius (1988). Please see Wierzbicki (1986) also for a method of generating efficient solutions. VIG is a decision support system based on PARETO RACE, which helps the manager in identifying his/her best compromised solution. On a display, users see the values of the goals to be optimized in numeric form and as bar graphs whose lengths are dynamically changing as they travel on the efficient surface (see figure 3). Both constraints and goals are formulated similarly. Constraints are inflexible goals. Flexible goals delineate the goals of the decision maker. VIG helps to find the best possible value for flexible goals. During a solution process if some goals are given as inflexible, there may not be a feasible solution. Yet VIG still gives the current achievement(aspiration) levels for these goals. By changing the status of the goal (relaxing the goal) from "inflexible" to "flexible," a non-dominated solution may be obtained. If the solution is still infeasible one should continue relaxing inflexible goals.

Once flexible goals, target values of each of them, and (rough) upper and lower limits are determined; method finds a reference direction offering a preferable change and projects it on the set of efficient solutions, and identifies an efficient solution. Decision maker(s) may accept the solution. If s/he does not, software provides alternative efficient solutions and the values of flexible goals corresponding to each of them.

3 Supplier Selection Procedure

Company has a very elaborate supplier screening process. The procedure includes supplier self-assessment questionnaire, on-site audit, first article approval or statistical evidence, and third party assessments such as ISO-9000 which may be used in combinations suitable to individual circumstances.

Rejections are recorded on a Vendor Reject Form and processed per the company's Vendor Inspection Procedure. A supplier Quality Notice form will be sent to the supplier if the situation requires special corrective action. The supplier is given 60 days or sooner to respond. Corrective action relating to a supplier's system is requested as a result of a supplier audit, or when a certified supplier falls below the performance levels set for the "A" classification. The supplier rating system measures delivery and quality performance of suppliers. Delivery is considered "on time" based on the due date and delivery window specified by purchasing. Delivery rating specifies percentage of "on time" delivery. Suppliers are classified into four groups based on their quality and delivery rating:

Rating	**Quality**	**Delivery**
A- Certified	99.5	95
B- Preferred	97	90
C- Approved	95	75
D- Not Rated	<95	<75

Minimum of 12 months' history is needed to demonstrate that parts can be used in production without routine incoming inspection. The supplier must meet the "A" rating for the entire period.

If performance falls below the standards for the "A" rating, a Corrective Action Plan is required from the supplier. Routine incoming inspection is established until the corrective action is effective. If Performance does not return to certifiable levels within a reasonable period, the certification is canceled. Certified parts are subject to periodic random audit inspection to verify conformance to requirements.

4 Description of the System

Division procures castings from five suppliers who have successfully passed the screening process explained in the previous section. Supplier 3 and supplier 5 had to be considered because of high demand on castings. Procurement planning is for 4 families of products that the division produces. Supplier capacities allocated for these products for the first four suppliers are given as 180, 100, 125, 80 tons/month respectively. Supplier 5 can produce as much as is required by the company. Required castings are classified into three groups such as gray iron, compacted graphite and ductile iron. Production capability of different vendors varies. These data are summarized in table 1, under **Production capability matrix**.

Table 1. Production Capability Matrix

	Tonnage Tons/ Month	*Casting Type* Gray	Compacted Graphite	Ductile	Quality (%)	Delivery Reliability (%)
Supplier 1	180	**	**		99	95
Supplier 2	100	**	**	**	99	90
Supplier 3	125	**		**	96	65
Supplier 4	80	**		**	99	90
Supplier 5	∞			**	90	35

*** Can produce this type of iron*

Table 2. Production Data

FAMILIES	SERIES	DISTIBU TION (%)	WEIGHT (LBS/UNIT)	TYPE	DEMAND (UNITS/ WEEK)
S	15	1	11.00	GRAY	35.00
	33	63	18.60	GRAY	2,205.00
	36	23	23.70	GRAY	805.00
	7	13	29.00	GRAY	455.00
P	15	1	5.00	GRAY	35.00
	33	63	9.50	GRAY	2,205.00
	36	23	17.50	GRAY	805.00
	7	13	18.00	GRAY	455.00
G	15	1	8.10	GRAY	87.50
	33	63	8.00	C.GRAPH.	5,512.50
	36	23	11.00	C.GRAPH.	2,012.50
	7	13	20.00	GRAY	1,137.50
B	15	1	14.80	GRAY	17.50
	33	63	25.75	C.GRAPH.	1,102.50
	36	23	42.00	C.GRAPH.	402.50
	7	13	38.50	DUCTILE	227.50

Four families consist of the same four series. Product families, series within each family, distribution (%) of the series, weights, type of castings, and demand are given in Table 2. The names of the castings are coded per request of the company. This table completes most of the system description and data except net price which cannot be given for confidentiality reasons. Weekly demand for all series are 17, 500 units. Distributions of this demand among 4 families are 20%, 20%, 50%, and 10% respectively. There are no minimum and/or maximum order quantities, specified directly, for each supplier. Since company implements just in time (JIT) purchasing strategy, they cannot build enough safety stocks now.

5 Model Formulation

We would like to find amounts to buy from different vendors so as to maximize the quality and delivery reliability of each series while minimizing total cost of goods purchased subject to capacity and demand constraints. Therefore we have defined x_{ijk} as our decision variables. Here x_{ijk} denotes the amount of castings of family i, series j to be bought from supplier k. Since our decision makers would like to minimize total price paid to vendors while maximizing quality and delivery reliability of each family of the products, we have defined them as flexible goals.

We defined

$$\min \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{5} p_{ijk} x_{ijk} \tag{4}$$

as total price paid to vendors. Average quality of the casting of family i bought from different suppliers defined as

$$\max \sum_{j=1}^{4}\sum_{k=1}^{5} (q_{ijk} / D_i) x_{ijk} \geq Q_i, \quad \textit{for all } i. \tag{5}$$

Average delivery reliability(rating) for the same family is defined as

$$\max \sum_{j=1}^{4}\sum_{k=1}^{5} (r_{ijk} / D_i) x_{ijk} \geq R_i, \quad \textit{for all } i. \tag{6}$$

Capacity constraints for different suppliers are

$$\sum_{i=1}^{4}\sum_{j=1}^{4} a_{ij} x_{ijk} \leq C_k, \quad k = 1,2,3,4. \tag{7}$$

Since company implements JIT purchasing strategy, castings supplied from different vendors should be equal to demand

$$\sum_{k=1}^{5} x_{ijk} = X_{ij}, \quad \textit{for all } i \textit{ and } j. \tag{8}$$

$$x_{ijk} \geq 0, \quad \textit{for all } i, j, \textit{and } k. \tag{9}$$

Where

p_{ijk} = Unit price charged for product in family i series j bought from supplier k.
q_{ijk} = Quality rating for product in family i series j bought from supplier k.

D_i = Total demand for family i, i=1,2,3,4.
Q_i = Desired quality for product family i (given by the decision making team), i = *1,2,3,4.*
r_{ijk} = Delivery rating for product in family *i* series *j* bought from supplier *k.*
R_i = Desired average delivery rating product family i (given by the decision making team), i = *1,2,3,4.*
a_{ij} = Weight of each casting in family *i* series *j.*
C_k = Capacity of supplier *k.*
X_{ij} = Demand for product in family *i* series *j.*

5.1 Data Input Into Decision Support System

Decision variables, flexible and inflexible goals were named such that they could be easily recognized by the decision making team. For reasons of confidentiality, coded names are given (see figure 1): S15-1 shows *s* family, 15 series, bought from supplier 1. The first 5 rows of figure 1 delineate cost, average quality of *s*, *p*, *g* and quality of *b* bought. The next 4 rows represent average delivery reliability of each family bought. The immediate 4 rows show capacity consumed in 4 suppliers. There is no capacity restriction for supplier 5. The next 16 row delineates product demand. (Only parts of them are shown in figure 1.) Total *s* bought should be equal to total *s* required for the planning period. At this stage we assumed weekly planning horizon. Figure 2 shows part of the flexible and inflexible goals. Flexible goals are the goals we want to maximize/minimize. Inflexible goals delineate the constraints of the system.

Target values for each goal and constraint are given in figure 2. Row 10 defines that capacity used by supplier 1 cannot exceed 90,000 lb. per week. Since company follows just in time purchasing policy total amount each sub-series bought from different suppliers should be exactly equal to demand.

6 Solution

Cost, quality and delivery reliability are taken as flexible goals. There was no feasible solution, although supplier 5 has no capacity restrictions.

Since with the existing capacity of suppliers for these four series we cannot reach any feasible solution, we should either find a way to increase the capacity of existing suppliers or we should find new suppliers if we want to keep the existing goals and the restrictions same. Supplier 1 is a subdivision of our firm. So it was decided that first we can ask them whether they can increase the share of allocated capacity. It was agreed that they can increase it up to 220 tons per month. Solution is given in table 3, and in figure 3. Figure 3 gives current achievements for each (flexible) goal. It shows that we can minimize cost at $124,000 per week while maximizing quality of *s* at 98.52%, quality of *p* at 98.39%, quality of *g* at 98.58%, and quality of *b* at 98.58%. Delivery reliabilities are maximized at 88.08%, 88.08%, 88.08% and 89.99% consecutively. Table 3 shows the current

Editing Matrix Coefficients

	S15-1	S15-2	S15-3	S15-4	S33-1	S33-2
COST	5.9200000	7.6960000	6.8080000	7.1040000	7.7300000	10.049000
QS	0.0282857	0.0282857	0.0274286	0.0282857	0.0282857	0.0282857
QP	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
QG	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
QB	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
DRELS	0.0271428	0.0257143	0.0185714	0.0257143	0.0271428	0.0257143
DRELP	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
DRELG	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
DRELB	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
CAP1	11.000000	0.0000000	0.0000000	0.0000000	18.600000	0.0000000
CAP2	0.0000000	11.000000	0.0000000	0.0000000	0.0000000	18.600000
CAP3	0.0000000	0.0000000	11.000000	0.0000000	0.0000000	0.0000000
CAP4	0.0000000	0.0000000	0.0000000	11.000000	0.0000000	0.0000000
DS15	1.0000000	1.0000000	1.0000000	1.0000000	0.0000000	0.0000000
DS33	0.0000000	0.0000000	0.0000000	0.0000000	1.0000000	1.0000000
DS36	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
DS7	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

F10:Exit to Menu

Fig. 1. Decision Variables, Flexible and Inflexible Goals and Technological Coefficients (Partially exhibited)

Editing the Types and Aspiration Levels of Goals

Names	Types	Given Values	Current Values
COST	<=	175000.00	0.0000000
QS	>=	98.000000	0.0000000
QP	>=	98.000000	0.0000000
QG	>=	98.000000	0.0000000
QB	>=	98.000000	0.0000000
DRELS	>=	90.000000	0.0000000
DRELP	>=	90.000000	0.0000000
DRELG	>=	90.000000	0.0000000
DRELB	>=	90.000000	0.0000000
CAP1	<=	90000.000	0.0000000
CAP2	<=	50000.000	0.0000000
CAP3	<=	62500.000	0.0000000
CAP4	<=	40000.000	0.0000000
DS15	==	35.000000	0.0000000
DS33	==	2205.0000	0.0000000
DS36	==	805.00000	0.0000000

F10:Exit to Menu

Fig. 2. Target Values of Each Goal (Partially exhibited)

achievements for goals and constraints, and delineates amount to buy from each supplier to minimize cost while maximizing quality and delivery reliability of each series. We should buy all s15 from supplier 1 (S15-1 = 35 units), while procuring 1450 s33 from supplier 1, 755 from supplier 4. We should split the s36 orders between supplier 3 and 4. (S36-3 = 111, S36-4 = 694). Table 3 gives current achievement for each goal, difference between target values and current achievements and amount to be bought from different suppliers.

Our decision making team was ready to accept this solution although we emphasized that this is not necessarily an optimal but one of the efficient solutions. They did not ask us to present alternative efficient solutions. We searched for it for our own curiosity. Among the goals the quality of the *p-family* is considered the least satisfying. We started Pareto Race, provided alternative solutions on the screen for improved third goal. We stopped the search as soon as the quality of *p-family* reached almost the same level as the quality level of other families. At this level the quality of p could only be increased if the team accepted further decrease in the quality of *s*. This solution improved delivery reliability of *b* from 89.99% to 90.06%.

Alternative efficient solutions were not significantly different from each other. This may explain the initial attitude of the team in accepting the initial solution without searching for alternative efficient solutions.

7 Implementation Challenges in Multiple Criteria Methods / Software

There are quite a few practical problems which are multiple criteria in nature. Therefore it would be very advantageous to solve these problems using MCDM methods. However, application seems to be a challenging task for a variety of reasons.

First of all, practitioners, even the students are spoiled by current popular software which are menu driven, very easy to use, and very robust to errors. Therefore they may lose the confidence, if they cannot find similar features in the MCDM software they use. On the other hand, it is very difficult to find an MCDM software with such features and it is not fair to expect it either if we consider current state of art in multiple criteria decision making decision support systems. Currently main interest of the authors who develop this kind of software is to write a decision support system to solve the problems, and interacting with the decision makers in a user friendly way rather than providing fancy features.

Second difficulty arises in problem identification. Our experience is such that generally, the analyst is the one who identifies the problem. The problem identification generally requires that the analyst lives in problem environment for a while. Our current study was identified during a faculty internship in a hydraulic gear pump division of a manufacturing company. It is difficult to expect academicians to spend extensive time in the decision making environment. On the other hand it is difficult for decision makers who are not aware of MCDM methods to identify clearly the multiple criteria nature of the problems they face.

Another difficulty is in implementation. First of all, after all the trouble in developing the model, application may not be implemented for a variety of reasons. One of them is priority setting in decision making environments. Often times, decision makers are busy in fire fighting rather than planning, and improving their decision.

Table 3. Results With Increased Supplier 1 Capacity (110,000 lbs/week) Allocated to the Divisions Requirements.

Rownames	type	Goal	Current Level	Difference
COST	≤*	175,000.00	124,151.89	-50,848.11
QS	≥*	98.00	98.51	0.51
QP	≥*	98.00	98.38	0.39
QG	≥*	98.00	98.58	0.58
QB	≥*	98.00	98.58	0.58
DRELS	≥*	90.00	88.08	-1.92
DRELP	≥*	90.00	88.08	-1.92
DRELG	≥*	90.00	88.08	-1.92
DRELB	≥*	90.00	89.99	-0.01
CAP1	≤	110,000.00	110,000.00	0.00
CAP2	≤	50,000.00	50,000.00	0.00
CAP3	≤	62,500.00	61,079.88	-1,420.13
CAP4	≤	40,000.00	40,000.00	0.00
DS15	=	35.00	35.00	0.00
DS33	=	2,205.00	2205.00	0.00
DS36	=	805.00	805.00	0.00
DS7	=	455.00	455.00	0.00
DP15	=	35.00	35.00	0.00
DP33	=	2,205.00	2,205.00	0.00
DP36	=	805.00	805.00	0.00
DP7	=	455.00	455.00	0.00
DG15	=	87.50	87.50	0.00
DG33	=	5,512.50	5,512.50	0.00
DG36	=	2,012.50	2,012.50	0.00
DG7	=	1,137.50	1,137.50	0.00
DB15	=	17.50	17.50	0.00
DB33	=	1,102.50	1,102.50	0.00
DB36	=	402.50	402.50	0.00
DB7	=	227.50	227.50	0.00

Amount to be bought from different suppliers.

S15-1	35.00	S33-1	1,449.50	S33-4	755.50
S36-3	111.03	S36-4	693.97	S7-3	455.00
P15-1	35.00	P33-1	2,205.00	P36-3	262.12
P36-4	542.88	P7-3	455.00	G15-3	87.50
G33-1	3,146.94	G33-2	2,365.56	G36-1	2,012.50
G7-3	1,137.50	B15-3	17.50	B33-1	552.19
B33-2	550.31	B36-2	402.50	B7-3	227.50

* *Signifies flexible goals.*

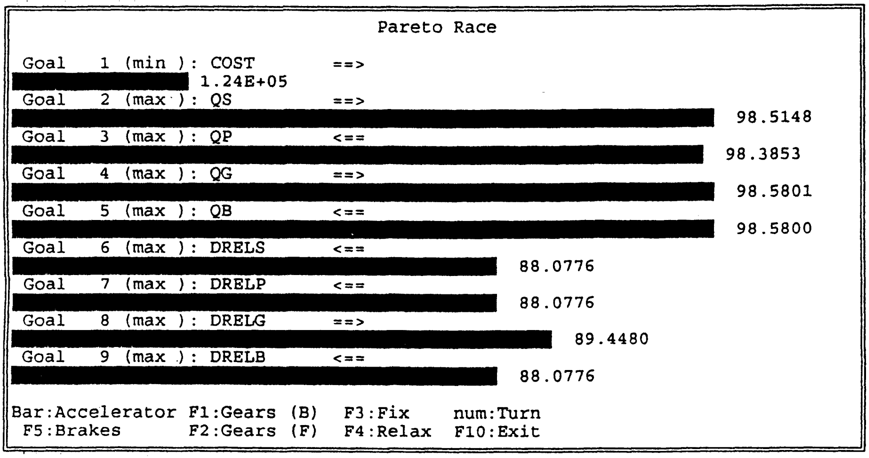

Fig. 3. A Sample Solution.

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Scheduling of Unit Processing Time Jobs on a Single Machine

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Abstract: In this paper we address the scheduling problem of unit processing time jobs on a single machine considering number of tardy jobs and a measure of earliness. We consider total absolute lateness, total earliness and maximum earliness, as earliness measures.

We provide a simple rule to solve the total earliness and maximum earliness problems in a unit execution time single machine environment.

Unit processing times allow for the formulation of the scheduling problem as an assignment problem. When there are two criteria the formulation includes side constraints. We formulate the problem and solve it to generate all efficient solutions. We provide computational results on a set of problems.

Key words: Bicriteria, scheduling, single machine

1. Introduction

Decision makers in a scheduling environment usually consider several measures of performance that are often conflicting (Panwalkar, et al. [10]). Traditionally, scheduling research addressed single criterion problems. The multicriteria scheduling research concentrated on single machine bicriteria problems (see, for example, [4], [7], [9], [1], [6]). Two papers in the literature; Chen and Bulfin [2] and De et al [3], specifically considered the single machine environment where jobs have unit, or equivalently, equal, processing times. Criteria considered in these papers are some combinations of flowtime, tardiness, number of tardy jobs, the weighted counterparts of these measures and maximum tardiness. Chen and Bulfin indicated that unit processing times allow an assignment model to be used to solve the single-criterion single-machine scheduling problem. They developed procedures for finding efficient solutions to some bicriteria problems. They do not report any computational results. Later De et al. [3] clarified some points about the procedures proposed by Bulfin and Chen.

In this paper, we consider the problem of scheduling unit processing time jobs on a single machine for three bicriteria problems. In all three problems, one of the criteria is related with earliness and the other criterion is the number of tardy jobs. That is, we penalize a job if it is finished earlier than its due date as well as when it is late. Earliness related measures have not been considered much in the scheduling literature. It has been assumed that a job that is finished early is delivered directly to the customer. However, customers are not willing to accept early orders and this means carrying finished goods inventory for the manufacturer. Earliness has become a widely used criterion with the increasing awareness brought by the Just-In-Time philosophy. In this paper, we give a brief background for the problems we consider and then define them in Section 2. We then exploit the special structure of the problems considered and discuss some analytical results. We give the details of the procedure to generate all efficient solutions in Section 3. Finally in Section 4, we present some computational results.

2. Problem Definition

Let n be the number of jobs to be processed starting at time zero. Let p_i and d_i denote the processing time and the due date of job i, respectively for i=1,...,n. Let C_i be the completion time of job i for a given schedule. Then, $L_i=C_i-d_i$ is its lateness, $T_i=\max(0,L_i)$ is its tardiness (positive lateness) and $E_i=\max(0,-L_i)$ is its earliness (negative lateness). $T_{max}=\max\{T_i,0\}$ is the maximum tardiness, $E_{max}=\max\{E_i,0\}$ is the maximum earliness, $n_T=\sum U_i$ is the total number of tardy jobs (where $U_i=1$ if $T_i>0$ and 0 otherwise), and $\sum ABL=\sum|L_i|$ is the total absolute lateness of a schedule.

In this paper, we consider three single machine problems with the following two criteria;

I. $\sum ABL$, n_T

II. $\sum E_i$, n_T

III. E_{max}, n_T

Following the three field notation of scheduling, these problems can be represented as:

I. $1|p_i=1|\sum ABL, n_T$

II. $1|p_i=1|\sum E_i, n_T$

III. $1|p_i=1|E_{max}, n_T$

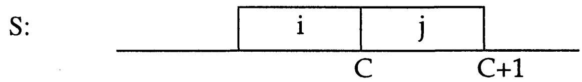
where "1" in the first field indicates a single machine, and the third field specifies the optimality criteria. Any special characteristic of jobs to be processed, e.g. jobs having unit processing times, is represented in the second field.

We assume that inserted idle time is not allowed which could be desirable when considering earliness related criteria.

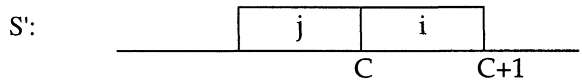
Number of tardy jobs problem, i.e., $1 | p_i = 1 | n_T$, can be solved in polynomial time. Given a set of jobs, Moore's algorithm constructs the schedule that minimizes the number of tardy jobs [8]. Here we present two new results concerning earliness related measures.

Theorem 1: The EDD rule (i.e. , ordering the jobs in nondecreasing order of their due dates) minimizes total earliness.

Proof: Consider a schedule S in which job i precedes job j and $d_i > d_j$.



Consider schedule S' which is the same as schedule S except job i and job j are interchanged.



Jobs other than jobs i and j are not affected by the interchange since $p_i=p_j=1$. Hence we only need to consider the earliness of jobs i and j. Let E(X) be the total earliness of jobs i and j in schedule X. Three cases need to be considered:

Case 1: $d_j < d_i \leq C$

$E(S) = E(S') = 0$

Case 2: $d_j \leq C < d_i$

$E(S) = d_i - C + 0$ $\qquad$ $E(S') = d_i - C - 1 + 0$

$\Rightarrow \quad E(S) > E(S')$

Case 3: $C < d_j < d_i$

$E(S) = d_i - C + d_j - C - 1$ $\qquad$ $E(S') = d_j - C + d_i - C - 1$

$\Rightarrow \quad E(S) = E(S')$

Therefore, $E(S) \geq E(S')$.

QED

Remark: The EDD rule provides the optimal solution for $1 | p_i=1 | \Sigma T_i, \Sigma E_i$ problem.

The remark directly follows the theorem and the fact that the EDD rule solves the total tardiness problem (see Chen and Bulfin [2]).

Theorem 2: The EDD rule minimizes the maximum earliness.

Proof is similar to the previous one and is again by adjacent pairwise interchange.

Due to the above two theorems and other previous results, when handled individually, all criteria considered, except the total absolute lateness, are known to be solved in polynomial time. On the other hand, the bicriteria problem is more complex. Fortunately, unit processing times allow for the formulation of the scheduling problem as an assignment problem. We next give the general formulation of the problems.

Consider the bicriteria problem of minimizing total absolute lateness and number of tardy jobs. Let one of the criteria, say total absolute lateness, appear in the objective function and specify a limit on the other one through a constraint.

Define

$$x_{ij} = \begin{cases} 1 & \text{if job } i \text{ is assigned to position } j \text{ in the schedule, } i=1,\dots,n,\ j=1,\dots,n. \\ 0 & \text{otherwise} \end{cases}$$

$$N_{ij} = \begin{cases} 1 & \text{if the due date of job } i \text{ is greater than its completion time when assigned to position } j \text{, i.e., } d_i > j \\ 0 & \text{otherwise} \end{cases}$$

L_{ij} = absolute lateness value of job i when it is assigned to position j

$L_{ij} = | d_i - j |$

NT = maximum number of tardy jobs allowed.

The model is

$$\begin{aligned} \text{Min} \quad & \sum\sum L_{ij}x_{ij} & & \\ \text{st.} \quad & \sum x_{ij} = 1 & j=1,\dots,n \quad & (1) \\ & \sum x_{ij} = 1 & i=1\dots,n \quad & (2) \\ & \sum\sum N_{ij}\, x_{ij} \leq NT & & (3) \\ & x_{ij} = 0 \text{ or } 1 & & \end{aligned}$$

Constraint sets (1) and (2) are common in all problems. The model minimizes one of the criteria while an upper limit is imposed on the other through a constraint. Due to the special structure of unit processing time problems, a limit on a value of some criteria can be expressed as a single constraint. If we solve the above problem for all possible NT values, we obtain all efficient solutions for the bicriteria problem.

For the $1|p_i=1|\Sigma E_i, n_T$ problem, the objective function and the side constraint can be written as

$$\begin{aligned} \text{Min} \quad & \sum\sum E_{ij}\, x_{ij} \\ \text{st.} \quad & \sum\sum N_{ij}\, x_{ij} \leq NT \end{aligned}$$

where E_{ij} is the earliness value of job i if it is assigned to position j, i.e., $E_{ij} = \max\{0, d_i - j\}$, i=1,...,n, j=1,...,n.

The model is different for $1|p_i=1|E_{max}, n_T$ problem where we have n+1 side constraints instead of one. The objective function and the side constraints for that problem are as follows.

$$
\begin{array}{ll}
\text{Min} & E_{max} \\
\text{st.} & \sum E_{ij}\, x_{ij} \leq E_{max} \qquad i=1,\dots,n \\
 & \sum\sum N_{ij}\, x_{ij} \leq NT
\end{array}
$$

In the next section we discuss our procedure for generating all efficient solutions.

3. Generating Efficient Solutions

A schedule S is efficient with respect to criteria C_1 and C_2 if there does not exist a schedule S' with

$$C_1(S') \leq C_1(S)$$

and

$$C_2(S') \leq C_2(S)$$

with at least one of the above holding as a strict inequality. If there exists such a schedule, S', then we say that S' dominates S and S is an inefficient schedule.

To generate all efficient solutions we first find the range of values for each criterion. We solve two single criterion scheduling problems each having one of the criteria, i.e., we solve two-assignment problems and find the minimum values of the corresponding criteria. Apparently, a schedule minimizing one of the criteria gives an upper bound for the other criterion to be considered. Then we solve the problem

$$
\begin{array}{ll}
\text{Min} & C_1(x) \\
\text{st.} & \sum x_{ij} = 1 \\
 & \sum x_{ij} = 1 \\
 & C_2(x) \leq k \\
 & x_{ij} = 0,1
\end{array}
$$

where k is the upper bound value of criterion C_2 minus one. Then we solve the same problem consecutively, by decreasing the value of k by one each time if the constraint holds as an equality or by decreasing the value of the left hand side obtained in the previous solution by one if the constraint holds as strict inequality. The procedure is repeated until the minimum value of C_2 is reached. Each solution of the above problem yields an efficient schedule. When the procedure terminates we will have all efficient solutions for the bicriteria problem.

4. Experimental Design and Computational Results

The approach proposed by Fisher [5] is used for due date generation. Due dates are generated as a function of three parameters; total processing time, P, average tardiness factor which is a measure of the proportion of jobs that are expected to be tardy in an arbitrary sequence, τ, and due date range, R.

We conducted extensive runs for several values of R and τ to test the effects of these parameters on problem difficulty. We observed that, for R less than 0.5 and τ less than 0.4 there is only one efficient solution in most cases. The number of efficient solutions increases when R and τ values increase.

Based on these preliminary results two different values for R and two different values for τ are chosen resulting in the following four different data sets.

	τ = 0.6	τ = 0.9
0.6	Set I	Set II
0.9	Set III	Set IV

Five replications are run for each data set. The problem size is chosen as 50 jobs except for the $1 \mid p_i=1 \mid E_{max}, n_T$ problem. For the $1 \mid p_i=1 \mid E_{max}, n_T$ problem where we have (n+1) side constraints rather than a single one, we choose the problem size as 10 as it is a more difficult problem. The number of efficient solutions for each problem type and data set are given in Table 1.

Table 4.1 Average number of efficient solutions

Problem Type	Data Set I	II	III	IV
$\Sigma ABL, n_T$	4.5	3.9	10.1	6
$\Sigma E, n_T$	4.6	4.6	11	6.2
Emax, n_T	1.2	1.2	2.2	1.6

There are n! (i.e., 50! for the first two and 10! for the last problem type) different possible sequences for the problems considered. Compared to those values the average number of efficient solution values are drastically few. No significant effect of the tardiness factor on the number of efficient solutions is observed. On the other hand the problems seemed to have more efficient solutions for bigger due date ranges.

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Linear Goal Programming Model for Managing Balance Sheet of a Commercial Bank

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Abstract. This paper develops the deterministic dynamic model to support the management of the bank balance sheet. It takes into account all main managerial goals and legal requirements and its planning horizon is unlimited. For numerical test we adopt the linear goal programming method. Parameters and initial values were chosen so they fit the situation in a Polish commercial bank. We found that this kind of model can be a useful aid in the strategic planning process of a bank.

Keywords. Linear goal programming, dynamic programming, bank management

1 Introduction

Serving as financial intermediary is the traditional activity of a bank. A variety of different sources form bank's liabilities and collected funds are allocated in a number of various assets. This is a quite complex process and managing a bank's balance sheet is one of the most important issues in strategic plans of a bank.

Commercial banks operate in a rapidly changing regulatory and economic environment. During the last decades interest and exchange rates became highly volatile [7]. The result is that the bank's management faces a problem of several conflicting goals instead of one goal such as profit maximization. The majority of these goals are connected with keeping different kinds of risks within reasonable limits.

Optimization methods appear to be a useful tool in supporting strategic decisions in a bank. A significant number of models have been presented in the literature [1, 3-6, 8-12]. These which use MCDM methods are most popular. Some of them employ interactive procedures combined with multiple goal programming [9, 11], some extend the planning horizon over one period [8, 10, 12]. A number of models have been applied in actual banks [6, 8, 12].

This paper presents a deterministic dynamic model with multiple goals which can be used in bank's balance sheet management in many period planning horizon. We take into account only main balance sheet categories in single currency, though our model can be easily extended to cover more variables.

Commercial banking sector in Poland was rebuilt in 1989 as a result of political and economic changes in Eastern Europe. As a consequence the need for modern management methods adequate for the market economy has arisen. Our model was designed to fit an average Polish commercial bank. It comprises all main managerial goals and legal requirements as well. For the numerical test we used interest rates at the level appearing in Poland. Initial values and parameters were fit to Polish reality, too [2].

2 Model Specification

We consider a finite and deterministic dynamic model with time horizon including T planning periods. The bank balance sheet items form a vector-state Y_t which depends only on the previous state and decision vector X_t . It can be formally expressed as $Y_{t+1} = F(Y_t, X_t)$ for $t=0,\dots,T-1$.

Table 2.1 contains definitions of all components of Y and X vectors and accompanying parameters (for simplicity time index t is omitted). The important feature of our model is that new securities, loans and deposits which started in any period s preceding given period, are kept as the separate components of the vector-state. This allows us to keep track of every balance sheet variable with its intrinsic values like the interest rate and the maturity schedule.

Table 2.1. Model variables and parameters.

Vector-state components (index t is omitted)	
y^1	Cash
$y^{2,s}$	Securities purchased in period s
$y^{3,s}$	Loans started in period s
$y^{4,s}$	Deposits taken in period s
y^5	Equity at the end of period t
Decision vector components	
x^i_t	initial value of i-variable started in period t
$x^{i,s}_t$	amount of i,s-variable terminated in period t
Parameters	
p	required reserves ratio
W	capital adequacy ratio
w_i	asset risk weight factors
m_i	market and/or technical limits

The nonzero elements of the transition matrix are ($t=1,\dots,T-1;\ s=1,\dots,t;\ \delta^s_t$ *is the Cronecker* δ-*function*):

$$y_{t+1}^{i,s} = x_t^i \delta_t^s + (c_t^{i,s} y_t^{i,s} - x_t^{i,s})(1 - \delta_t^s) \ , \tag{1}$$

for securities, loans and deposits *(i=2,3,4)*.

Every element has two complementary components: the first one stands for the initial value of a balance sheet variable started in period t, the second represents changes in a balance sheet variable in consecutive periods, while:

$c^{i,s}{}_t$ - means the quote which does not mature in period *t*,

$x^{i,s}{}_t$ - stands for amount of given variable which is terminated in period *t*.

For equity (*i=5*):

$$y_{t+1}^5 = y_t^5 + \sum_i \sum_{s=0}^{t} \alpha^i (r^{i,s} y_t^{i,s} - \gamma^i(s,t) x_t^{i,s}) \ , \tag{2}$$

where factor α gives an appropriate sign for interest bearing balance sheet items ($\alpha^2 = \alpha^3 = -\alpha^4 = 1$) and is zero for the others ($\alpha^1 = \alpha^5 = 0$), $r^{i,t}$ - interest rates. Factors γ give capital gain/loss connected with $x^{i,s}{}_t$, - the amounts of instruments terminated before maturity. They are given by the equation:

$$\gamma^i(s,t) = \frac{r^{i,t} - r^{i,s}}{r^{i,t} + 1} \tag{3}$$

and were calculated from the present value formula.

The only remaining variable - *cash* - is determined by the basic balance sheet equation. This equation and the other constraints are as follows:

Required reserves ratio:

$$y_t^1 \geq p \sum_{s=0}^{t} y_t^{4,s} \ . \tag{4}$$

Required capital adequacy ratio:

$$y_t^5 \geq W \sum_i \sum_{s=0}^{t} w_i y_t^{i,s} \ . \tag{5}$$

Basic balance sheet equation:

$$y_t^1 + \sum_{s=0}^{t} (y_t^{2,s} + y_t^{3,s}) = \sum_{s=0}^{t} y_t^{4,s} + y_{t-1}^5 \ . \tag{6}$$

Market and/or technical limits:

$$x_t^i \le m_t^i. \tag{7}$$

A remark has to be made on decision vector components $x^{i,s}_t$. For securities it simply means the bank's decision to sell in period t $x^{2,s}_t$, portion of securities bought in period s, before their original maturity. For loans, $x^{3,s}_t$ is the portion of loans started in period s which are repaid in period t, before their original maturity; similarly $x^{4,s}_t$ means the portion of deposits withdrawn earlier. However, the two last items are not bank's decisions but rather external parameters. They are matched together because of formal similarity - in all cases they mean the same: termination before original maturity. This allows us to write down one general formula for every interest bearing balance sheet variable.

We assume, as usual, that all decisions are taken at the beginning of any period and it results in the 'new' value of equity at the end of a given period. So if we like to be consistent we have to balance the t period variables with equity from period t-1(see eq. (6)).

3 An Example

Various objective schemes can be incorporated into a framework presented in the previous chapter. For the purpose of numerical test we chose the linear goal programming method. We minimize a linear combination of the over/under achievement variables (for all periods; all weight factors were equal to one) in order to:
(1) minimize reserves ratio,
(2) maximize capital adequacy ratio,
(3) maximize potential business volume,
(4) maximize the sum of discounted equities.

Table 3.1. Parameters for calculations.

Period:	Initial	1	2	3
r^2	0.40	0.36	0.32	0.27
r^3	0.45	0.41	0.38	0.33
r^4	0.35	0.32	0.30	0.25
m^3		500	650	800
m^4		900	1100	1350

For all periods: p=0.15; W=0.08; w_2= 0.1; w_3= 1; c^2=c^4=0.4; c^3=0.5; $x^{3,s}_t$= $x^{4,s}_t$= 0.

We made calculations for 3 planning periods. All values of the parameters are presented in Table 3.1. As it was mentioned in the *Introduction* we have chosen interest rates at the level typical for the Polish financial market in years 1992-94. They are high because of the high inflation rate in Poland and they go down with declining inflation. Capital adequacy ratio is set according to the Polish central bank requirements. The remainder of parameters and initial balance sheet items have been estimated so they fit the situation in a large Polish commercial bank. They are based on the averaged data of the Polish banking system [2].

The resulting values of the model are included in Table 3.2. In a defined economic environment the model maximizes deposit volume and satisfies all of the loan demand. The rest of available funds is used for the purchase of additional securities. The model suggests not to sell any securities before maturity in any period, although it means capital gain in the declining sequence of interest rates.

Table 3.2. Numerical results

		Multiple objectives				Single objective			
	Period:	Initial	1	2	3	Initial	1	2	3
Variable	Started in period								
Cash		180	207	248	302	180	162	161	189
Securities	Initial	400	160	64	25	400	160	0	0
	1		253	101	40		0	0	0
	2			325	130			0	0
	3				423				0
	Total	400	413	490	618	400	160	0	0
Loans	Initial	720	360	180	90	720	360	180	90
	1		500	250	125		500	250	125
	2			650	325			650	325
	3				800				800
	Total	720	860	1080	1340	720	860	1080	1340
Deposits	Initial	1200	480	192	77	1200	480	192	77
	1		900	360	144		602	241	96
	2			1100	440			637	255
	3				1350				832
	Total	1200	1380	1652	2011	1200	1082	1070	1260
Equity		100	166	250	367	100	170	269	406
Capital adequacy ratio		13.2	11.1	14.7	17.8	13.2	11.4	15.7	20.1

We verified what happened when we reduced our model to a linear programming model. The only objective was to maximize equity. The solution is essentially different from those of multiple goal programming model. The volume of deposits does not reach the upper bound. Although the bank is better off, its balance sheet has an improper structure from the liquidity point of view. The bank is to sell in period 2 the remainder of securities initially held in its portfolio and not to purchase any new securities in the planning horizon.

4 Summary

The mathematical programming models have been proven to be a fruitful method in managing a balance sheet of a bank. Even simple models seem to be useful in a bank's practice, especially in the emerging market economies. They can be easier accepted by practitioners and they are very flexible for application in various situations. For example, during examination of a single period model we found that the solution strongly depends on the way in which objectives are chosen and placed on the priority ladder. It makes enough room for effective co-operation of analysts and management.

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Local Tax Planning with AHP and Delphi*

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Abstract. A tax model for the city of Richmond is developed and tested, in close cooperation with local public leaders and with tax experts. To construct and test the model, *Expert Choice*, a commercially available decision support system based on the analytic hierarchy process is used. A Delphi approach is used to have tax experts compare the alternative tax plans with respect to specific decision criteria.

Key words. AHP, Delphi, tax planning, public administration, application

1 Introduction

1.1 Revenue Planning in the Public Sector

When public officials are faced with the prospects of ever-increasing demand for services, an increase in the cost of providing these services, and an existing revenue base that does not increase in line with demand, they may pursue any or all of these possible strategies [1]:

Cut services. This requires the ability to identify priorities, to rank them and to make reductions in a rational manner.

Increase organizational productivity. This challenge, perhaps new to many in the public sector, is extremely valuable in a time of fiscal retrenchment. (See Wooldridge [5] for a description of productivity improvement strategies).

Expand the use of existing financial resources or identify new resources to supplement existing ones.

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This paper focuses on the third strategy – the enhancement of a public organization's revenue base through a systematic approach to revenue planning.

1.2 Criteria for Assessing Revenue Sources

According to Pechman [2], taxation has three basic goals: (1) to transfer resources from the private to the public sector; (2) to distribute the cost of government fairly by income class and among people in approximately the same economic circumstance; and (3) to promote economic growth. Rostvold [3] includes four maxims of a socially acceptable revenue system:

1) compatibility with the social values that govern the general social process;
2) fiscally adequate, broadly based, stable in yield, and balanced in fiscal incidence;
3) administratively simple, economic to administer, with clearly defined tax bases and tax-rate structures; and
4) minimally adverse effects on economic productivity, resource allocation, and the levels of employment, income, and output.

Two other desirable characteristics are administrative feasibility or enforceability (the system does not impose taxes impossible to enforce) and social and political acceptability.

Based on the literature, it was decided to use the following seven criteria in this project:

1) administrative feasibility
2) social and political acceptability
3) horizontal equity
4) vertical equity
5) stability
6) regulatory neutrality
7) compatibility with strategic plan.

Administrative feasibility refers to the economics of administration, the collection costs and the compliance costs. A "good" revenue source is administratively simple, with a clearly defined tax base and tax-rate schedule, is enforceable and displays certainty in its application. That is, taxpayers understand its regulations and can predict their tax burden. Compliance costs can take the form of payments to lawyers and accountants, maintenance of special tax departments by large corporations, or use of the taxpayer's own time and effort for keeping records and preparing tax returns. To be *socially and politically acceptable*, a revenue source must be consistent with notions of fair play. The system cannot be considered too onerous when compared with the generalized perception of benefits derived from services.

Horizontal equity implies that those individuals who are similarly situated (i.e. have the same income) are treated equally. *Vertical equity* is concerned with the "proper" division of the tax burden among people with different abilities to pay. Revenue *stability* for state and local governments may be considered in two different contexts: (1) as a criterion concerning the fluctuations in total income of residents and in involuntary unemployment, induced by changes in tax policies, and (2) in terms of the stability of tax revenues to state and local governments when external changes affect state and local economies. A revenue source is *regulatory neutral*, if it has little or no distorting impact on private production and consumption decisions. *Compatibility with strategic plan* requires tax analysts to anticipate the consequences of each tax in their revenue program and of the overall tax structure, so that if a public policy is directed to one goal, a new tax policy is not directed to an opposing goal.

1.3 Multi-Attribute Decision Support Systems and AHP

Decision support systems (DSS's) are interactive, computer-based information systems, that help managers make decisions in complex and ill-structured decision situations. DSS's help managers structure the available information and thus aid in the derivation of a satisfactory decision. Multiple criteria decision making is an approach to decision analysis that explicitly recognizes that decisions are based on multiple criteria. In *multi-attribute decision making* (MADM), the decision problem is formulated as a choice between a finite number of decision alternatives, where each alternative is characterized by a set of characteristics or *attributes* (criteria). Alternatives that are less desirable in all attributes than some other alternative are called *dominated*, and may be eliminated from the set of candidate decisions. A *multi-attribute decision support system* (MADSS) will help managers choose the most desirable alternative, based on the given attributes. In this paper we describe the application of a specific MADSS for public sector tax planning, based on the *analytic hierarchy process* or *AHP* developed by Saaty [4].

Given a number of alternatives and a set of attributes, the basic approach of AHP is as follows (see Fig. 1): An overall objective (however vague) is defined and the attributes are evaluated according to their relative impact on the overall objective by pair wise comparisons. Similarly, each alternative is evaluated according to its relative contribution with respect to each attribute, also by pair wise comparisons. The evaluations of the alternatives with respect to each attribute and the evaluation of the attributes are then combined to get a utility measure for each alternative, i.e. a measure which indicates the degree to which each alternative satisfies or contributes to the overall objective.

Each alternative, each attribute, and the overall objective is an *element* of the *hierarchy* consisting of three level. The overall objective constitutes the highest level (level 1), the n attributes make up the intermediate level (level 2), and the k alternatives make up the lowest level (level 3). However, the analytic hierarchy process does not limit us to three levels. For example, each of the n attributes may be characterized by a set of lower level attributes, which in turn could be characterized by a set of still lower level attributes, and so on. The lowest level attributes is always linked to the set of alternatives, which always constitute the lowest level of the hierarchy. Not every element of a level need necessarily be linked to each of the elements of the adjacent levels. If every element is linked to each of the elements of the adjacent levels, the hierarchy is called *complete*.

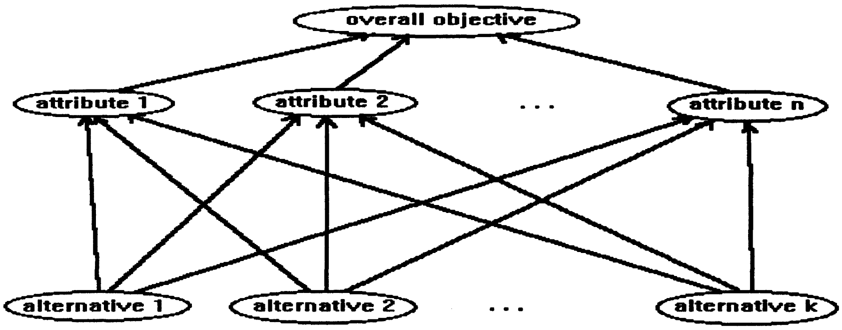

Figure 1

2 Multicriteria Approach to Tax Planning

2.1 Description of the Tax Alternatives

The following four tax options were identified by City of Richmond officials to generate an additional $1,000,000 in revenue:

Real Estate Tax. An increase of 1.19 cents is needed. Real estate tax is levied and collected each year on the assessed value of real property in the City. All real estate is assessed at 100% of fair market value. Reassessments are done each year. The current tax rate is $1.445 per $100 assessed value of real estate. Federal, state, and local government properties, and properties belonging to churches or approved not-for-profit organizations are exempt. Also, tax relief is given to those persons not less than 65 years of age with income not exceeding $17,000 and financial worth not exceeding $75,000. Relief varies from 80% to 11% depending on income and worth. 2,816 individuals are expected to participate in this program for eligible relief of approximately $1,300,000. A one time rate increase would require minimal computer program changes. As the rate increases, delinquencies could conceivably increase, causing a greater tax enforcement need. The primary cost of this would be an increase in personnel and possibly costs of litigation.

Personal Property Tax. An increase of 12.14 cents is needed. Personal property tax is levied and collected on tangible personal property, including boats used for pleasure, mobile homes, automobiles, and furniture and equipment used in business. The current tax rate is $3.70 per $100 assessed value. Automobile values are based on the National Auto Dealers Association average trade-in value. Furniture and equipment is assessed at a depreciated scale based on year of purchase. A one time rate increase would require minimal computer program changes. Since the value of furniture and fixtures, contributing to approximately one third of this revenue source, is assigned by the individual paying the tax, auditing is required as one component of collection. As delinquencies rise, there would be a need for additional audits of businesses within the City.

Local Sales Tax. An increase of .04 cents is needed. The current rate is 4.5 cents per dollar of taxable sales, of which one cent is remitted back to the locality in which the sale occurred. The tax is sent by local businesses to the Virginia Department of Taxation in the month following the actual sales. The local portion is then sent back to the locality in the following month, thus producing a two months lag between sale and receipt of the tax. A rate increase would require legislative changes, since the City is currently at the cap imposed by the State. An increase may result in delinquencies, and failure to pay by some establishments could result in increased State audits or lost tax dollars. A rate increase would not directly result in higher administrative costs to the locality.

Machinery and Tools Tax. This is a tax imposed on the value of machinery and tools used in the manufacturing process. An increase of 15.84 cents is needed. The current rate is $2.30 per $100 assessed value. The value of machinery and tools is depreciated based on age. Previous year purchases are assessed at 90%, two year old purchases at 80%, three year old purchases at 70%, four year old at 60%, five year old at 50%, six year old at 40%, and idle equipment at 10%. There are currently approximately 400 businesses which pay the machinery and tools tax. The three largest manufacturers account for 66% of the total tax payments, with the largest manufacturer contributing 50% of the total tax collected. Bills are mailed March 1 and the tax is due by May 1. Payments not received by this date are deemed delinquent and are charged a penalty of 10% and interest is accrued at 10% per annum. A rate increase would require minimal computer program changes. The value of machinery and tools is provided by the manufacturing business and thus requires audits on occasion. A rate increase could conceivably encourage delayed payments and thus increase audit costs.

2.2 The Hierarchy Model

Using the seven criteria described in the introduction of this paper, the hierarchy depicted in Figure 2 can be constructed. With this model, tax experts will be asked to compare each of the four tax alternatives with each of the other tax alternatives, as to their impact on each of the seven criteria, using a scale of 1 to 9. Thus, if plan 1 is clearly preferred over plan 2 with respect to administrative feasibility, a rating of nine is given. If plan 1 is only slightly preferred over plan 2, a rating of three is given. A rating of one indicates indifference, and reciprocals are used if plan 2 is preferred over plan 1 (i.e. a rating of 1/7 indicates that plan two is much more feasible than plan one). Thus seven 4×4 matrices of comparison ratings are obtained, one for each of the seven criteria. The main diagonal in each of these matrices consists of 1's, and the entries in the lower half of each matrix are the reciprocals of the entries in the upper half.

Similarly then, the decision makers (members of the Richmond City Council) are asked to compare each of the seven decision criteria as to their importance to achieving the best possible tax plan. The same scale of one to nine is used, and a 7×7 comparison matrix is derived.

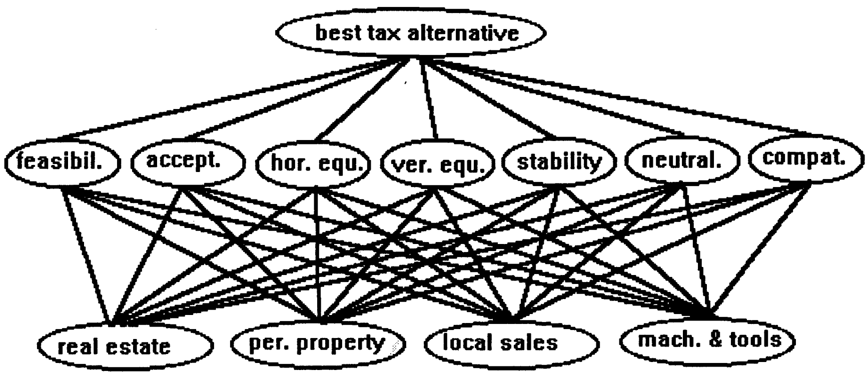

Figure 2

2.3 Solving the Model

Solving the model essentially consists of two separate activities: Evaluating the four tax alternatives with respect to each of the seven criteria, and comparing the seven criteria with respect to their impact on what will be considered the best tax alternative. *Expert Choice*, a commercially available package for selecting among multiple alternatives is utilized (Expert Choice Inc.,4922 Ellsworth Avenue, Pittsburgh, Pennsylvania, USA).

2.3.1 Comparing the Criteria

Evaluating the criteria is primarily a political matter and thus is the prerogative of the Richmond City Council in our approach. A hierarchy model based on the top part of Figure 2 was constructed, that would allow for the combining of the individual comparisons of the Council members into a group result.

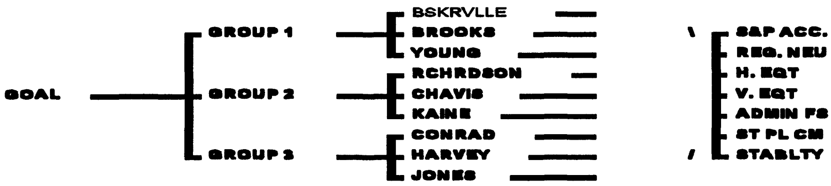

Figure 3

Because of *Expert Choice*'s limitation to a maximum of seven alternatives at each level, the nine Council members were placed into three groups of three people each, as seen in Figure 3. Each Councilor solves his or her own separate model which consists of comparing the seven criteria. The results are automatically combined into a single matrix by assuming equal weights for each group and each Council member, and using *Expert Choice* to solve the hierarchy. "Goal" here refers to the overall objective of selecting the best tax alternative. Appointments were made with individual Council members and the authors met with them to solve their part of the hierarchy using *Expert Choice* implemented on a note book computer.

Expert Choice allows for the comparisons to be made by specifying preferences on a nine point scale, as discussed above, but it also allows decision makers to make the comparisons using a verbal reference mode, or via a graphical mode with preferences indicated by the lengths of bars or the sizes of pie slices. Most Council members chose the verbal mode first, but then, when shown the graphic equivalents, several of them modified their decisions.

2.3.2 Evaluating the Tax Alternatives

An instrument was created and sent to the tax experts that asks the respondents to pair wise compare the four tax alternatives with respect to each of the seven criteria, as shown in the example (for the social and political acceptability criterium) below:

With respect to **social and political acceptability**:

		Reasoning
plan 1 is preferred to plan 2 ___	plan 2 is preferred to plan 1 ___	___
plan 1 is preferred to plan 3 ___	plan 3 is preferred to plan 1 ___	___
plan 1 is preferred to plan 4 ___	plan 4 is preferred to plan 1 ___	___
plan 2 is preferred to plan 3 ___	plan 3 is preferred to plan 2 ___	___
plan 2 is preferred to plan 4 ___	plan 4 is preferred to plan 2 ___	___
plan 3 is preferred to plan 4 ___	plan 4 is preferred to plan 3 ___	___

The respondents were asked to pair wise compare the four plans and to indicate a preference on a scale of 1 to 9, where 1 indicates *indifference*, 3 indicates *moderate preference*, 5 indicates *strong preference*, 7 indicates *very strong preference*, and 9 indicates *extreme preference*. The preference value is to be entered behind the corresponding preference statement. (The reversed statement would have no number entered, i.e. there would be one entry only per line.)

A *Delphi* approach is used in which the responses will be made available anonymously to all of the tax experts, who will then be asked if they wish to revise their original answers. This process may be repeated, if there is still great divergence in the second round results. The final results will be entered by the authors into an *Expert Choice* model of the complete hierarchy as given in Figure 2.

3 Conclusion

The project described in this paper is still ongoing. To date, we have been able to get five of the nine City councilors to complete their evaluation of the tax criteria. At least three of the remaining four have indicated their willingness to work with us, but we have not yet been able to schedule successful meetings with them The results for the five that completed their evaluations were highly divergent, as seen in Figure 4 below.

The divergence actually stems from mainly two Council members, who each focussed on one specific criterium as the dominating one. One Member saw everything as political and focussed on Social and Political Applicability, the other felt that Compliance with Strategic Plan was of overwhelming importance.

No results have been obtained yet from the tax experts, though the participants seem to be eager to take part in this project. The results from the project will be

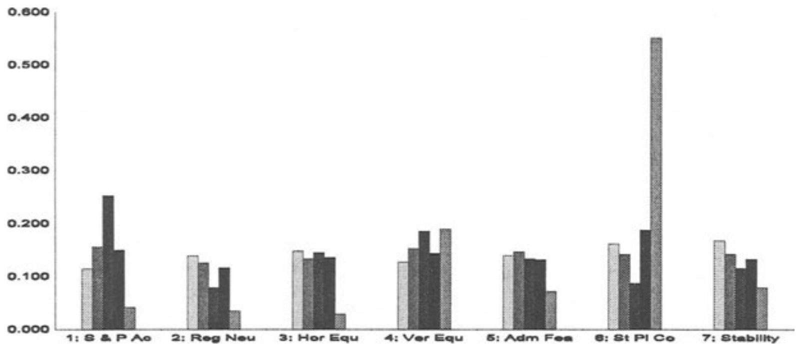

Figure 4

discussed with the City Council members. Though this may not immediately and directly effect the selection of tax options by the City Council, it will demonstrate the applicability of formal analysis tools in this type of decision making. It is anticipated that this will lead to future cooperation of the City of Richmond in formal decision analysis projects, some of which may eventually lead to actual implementation of results.

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