Numerical Analysis Day 9: QR Decomposition

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Wish to write A as the product $A = QR$ of a *rotation* Q and an upper triangular matrix R. Note that A need not be square, but it should have at least as many rows as columns.

$$A = \begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{pmatrix}$$

1 Gram-Schmidt Orthogonalization

Our first method is Gram-Schmidt Orthogonalization: we build an orthogonal matrix Q one column at a time, and keep track of our actions to build R.

Step 1: normalize first column by dividing out $r_{11} = \| v_1 \|_2$

$$y_1 = q_1 = \frac{v_1}{\| v_1 \|_2} = \frac{(1,2,2)}{\sqrt{1+4+4}} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Step 2: To find the second unit vector, compute y_2 and normalize to get q_2.

$$y_2 = v_2 - q_1 q_1^T v_2 = [-4,3,2]^T - (2/3)[1,2,2]^T = [-\frac{14}{3}, \frac{5}{3}, \frac{2}{3}]^T$$

$$q_2 = \frac{y_2}{\| y_2 \|_2} = [-\frac{14}{15}, \frac{1}{3}, \frac{2}{15}]^T$$

The algorithm tracks $r_{12} = q_1^T v_2 = 2$ and $r_{22} = \| v_2 \|_2 = 5$, and constructs Q and R.

Step 3: Find the third column in Q. Start with an (arbitrary) vector $v_3 = [1,0,0]^T$.

Define $y_3 = v_3 - q_1 q_1^T v_3 - q_2 q_2^T v_3 =$

$$= [1,0,0]^T - \left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]^T \frac{1}{3} - \left[-\frac{14}{15}, \frac{1}{3}, \frac{2}{15}\right]^T \left(-\frac{14}{15}\right) = \frac{2}{225}[2,10,-11]^T$$

Put the pieces together to get

$$\begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{pmatrix} = QR = \frac{1}{15}\begin{pmatrix} 5 & -14 & 2 \\ 10 & 5 & 10 \\ 10 & 2 & -11 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}$$

2 Householder Reflectors

Householder Reflectors are a simpler and more stable way to derive the QR decomposition.

Assume $\| x \|_2 = \| w \|_2$. Then $w - x$ and $w + x$ are perpendicular.

Proof: $(w-x)^T(w+x) = w^Tw - x^Tw + w^Tx - x^Tx = \| w \|_2 - \| x \|_2 = 0$

We use the vector $v = w - x$ to make the projection matrix P and the reflector $H = I - 2P$.

$$P = \frac{vv^T}{v^Tv}$$

In our example, $x = [1,2,2]^T$ and $w = [3,0,0]^T$ and H_1 is

$$H_1 = I - 2P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{12}\begin{pmatrix} 4 & -4 & -4 \\ -4 & 4 & 4 \\ -4 & 4 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$H_1 A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & -3 \\ 0 & -4 \end{pmatrix}$$

Next, move the vector $\hat{x} = [-3,-4]^T$ to $\hat{w} = [5,0]^T$. Define H_2 as

$$H_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{10}\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{pmatrix}$$

$$H_2 H_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & -0.8 & 0.6 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 5 \\ 0 & 0 \end{pmatrix} = R$$

Since $H_i H_i = I$, we see that $A = H_1 H_2 R$, so $Q = H_1 H_2$.