Numerical Analysis Day 1

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I'm going to try handing out a single page that combines notes and in-class questions. Let me know if you find this helpful or distracting.

1 Floating Point Representation

In Floating Point, we store numbers as $mantisa \times base^{exponent}$ IEEE 754 double precision uses 53 bits to store the mantisa, and 11 bits to store the exponent.

Base:
Largest possible Exponent:
Largest possible Number:
Smallest possible Exponent:
Smallest non-zero positive Number:

The *Relative Error* when approximating x with the computed value x_c is

$$\frac{|x_c - x|}{x}$$

What is the Relative Error if $x = 4321$ and $x_c = 4320$? When $x_c = 4329$?

What is *Machine Epsilon* for IEEE 754 double precision?

What is the conjugate of $\sqrt{a} - \sqrt{b}$?

2 Geometric Series

The series below converges if $|r| < 1$

$$S = 1 + r + r^2 + r^3 + \cdots$$

$$S = 1/(1 - r)$$

Compute:
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \cdots$
$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \cdots$
$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} \cdots$

3 Taylor Series

We can estimate a function $f(x)$ near a point x_0 with the partial sum
$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_o)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_o)^n$
with an error term of $\dfrac{1}{(n+1)!}f^{(n+1)}(c)(x - x_o)^{n+1}$ for some $c \in [x_0, x]$

3.1 Application

Define $f(x) = x^3 + x^2 + 6$. Compute

	expression	at 0	at 1
$f'(x)$	$3x^2 + 2x$		
$f''(x)$			
$f'''(x)$			
$f''''(x)$			

Using above, what is the Taylor Series for $f(x)$ evaluated at 0?
What is the Taylor Series for $f(x)$ evaluated at 1?

What is the Taylor Series expansion for e^x? For $\cos(x)$? For $\sin(x)$?

3.2 Ratio Test

The *Ratio Test* looks at the limit L of terms in the series $a_0 + a_1 + a_2 + \ldots$
$L = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$
if $L < 1$, the series converges absolutely.

For which values of x does this series converge?

$$a(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots$$

4 Approximation

If we use the first two terms of the Taylor Series for e^x at 0, we have the approximation

$e^x \approx 1 + x$

What is the error in this estimate on the interval $[0, 1]$?

Derive the equation for the straight line $p(x)$ that includes the points $(0, 1)$ and $(1, e)$
Estimate the greatest difference between $p(x)$ and e^x on the interval $[0, 1]$

Can you come up with a better line to approximate e^x on the interval $[0, 1]$?