Numerical Analysis Day 6: Interpolation

Jeff Parker

October 9, 2012

We will be interpolating three sets of points:
$A = \{(1,3),(2,1),(3,2)\}$ $B = \{(1,3),(2,1),(4,2)\}$ $C = \{(1,3),(2,2),(3,1)\}$

1 Lagrange Interpolation

One you understand the scheme, it is easy to convince yourself that it works.

For each point (x_i, y_i) we devise a polynomial that is zero at all other x_js.
$p_1 = \frac{3}{2}(x-2)(x-3)$ is zero when $x = 2$ or $x = 3$ and $p_1(1) = y_1 = 3$.
$p_2 = -(x-1)(x-3)$ is zero when $x = 1$ or $x = 3$ and $p_2(2) = y_2 = 1$.
$p_3 = (x-1)(x-2)$ is zero when $x = 1$ or $x = 2$ and $p_3(3) = y_3 = 2$.
$P(x) = p_1(x) + p_2(x) + p_3(x)$
General form for three points:

$$P(x) = y_1\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Weaknesses: The form is not easy to work with.
We cannot use Horner's method to evaluate.
A change to x_n invalidates the whole series. Changing x_3 changes all the coefficients.
Strengths: easy to prove that it works.
An alternative, the barycentric form, avoids many of the weaknesses above.

2 Newton Divided Differences

We write the polynomial as a series of polynomials: a constant, a linear, a quadratic, etc. Each partial sum interpolates a subsequence of the points.

$p_1(x) = 3$

$p_2(x) = 3 - 2(x-1)$. Note that the linear term is zero at x_1

$p_3(x) = 3 - 2(x-1) + \frac{3}{2}(x-1)(x-2)$. Note quadratic term is zero at x_1 and x_2.

The following table provides the coefficients of the terms above.

x	$f(x)$		
1	3		
2	1	-2	
3	2	1	$\frac{3}{2}$

$p_3(x) = 3 - 2(x-1) + \frac{3}{2}(x-1)(x-2)$.

We plug in the points from data set B for a second example.

x	$f(x)$		
1	3		
2	1	-2	
4	2	1/2	$\frac{5}{6}$

$p_3(x) = 3 - 2(x-1) + \frac{5}{6}(x-1)(x-2)$. Note that the first two terms are unchanged

We plug in the points from data set C for our last example.

x	$f(x)$		
1	3		
2	2	-1	
3	1	-1	0

$p_3(x) = 3 - (x-1) = 4 - x$. Note that the quadratic term is 0.

3 Runge Phenomenon

The interpolation error formula is

$$f(x) - P(x) = (x-x_1)(x-x_2)\cdots(x-x_n)\frac{f^{(n)}(c)}{n!}$$

for some c between the smallest and largest of $(x_1, x_2, \ldots x_n)$

When we use a large number of equally spaced points, the error term grows large relative to the error with Chebyshev nodes. If we can pick the interpolation points, we will do better by using (translates of) Chebyshev zeros due to the Chebyshev minimality property.