Numerical Analysis Day 4

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September 25th 2012

We will be looking at the system of equations

$$x + y = 3 \qquad\qquad 3x - 4y = 2 \tag{1}$$

with solution $(x, y) = (2, 1)$.

1 Gaussian Substitution

Fill in the blanks

1	1	3
3	4	2

1	1	3
0		

2 Using Matrix Algebra

$$\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

$$\frac{1}{ad - bc} \begin{pmatrix} b & -d \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3 Lower Upper (LU) Factorization

Goal is to rewrite A as product LU. Solve for c_1, c_2, then x, y.

$$\begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

4 Metrics

Given $x = (x_1, x_2, x_3, \cdots, x_n)$, and a matrix A, we define

$\| x \|_1 = |x_1| + |x_2| + |x_3| + \cdots + |x_n|$

$\| x \|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$

$\| x \|_\infty = max(x_1, x_2, x_3, \cdots x_n)$

$\| A \|_\infty = max\ absolute\ row\ sum$

5 Condition

Let x_c be an approximate solution to $Ax = b$.

The **backward** error is $\| b - Ax_c \|_\infty$

The **forward** error is $\| x - x_c \|_\infty$

The **relative backward** error of is $\frac{\|b - Ax_c\|_\infty}{\|b\|_\infty}$

The **relative forward** error of is $\frac{\|x - x_c\|_\infty}{\|x\|_\infty}$

The **error magnification factor** error is the ratio

$$\frac{relative\ forward\ error}{relative\ backward\ error} = \frac{\frac{\|x - x_c\|_\infty}{\|x\|_\infty}}{\frac{\|b - Ax_c\|_\infty}{\|b\|_\infty}}$$

The **condition number** of a matrix A is the largest possible error magnification factor.

For given A and b, we look for x and x_c that give the largest error magnification factor.

Easier way to find the condition number of a matrix than searching for x, x_c:

$$cond(A) = \| A \|_\infty \| A^{-1} \|_\infty$$